Problem 3-44: Viscous Diffusion Project
Mech 270 Viscous Flow

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1 Diffusion Equation

The two-dimensional unsteady linear viscous diffusion relation, equation 3-250 in the textbook,[1] is given by:

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Here, $u = u(x, y, t)$ is the fluid velocity in the z-direction and ν is the kinematic viscosity. The equation can be applied to when the flow is entirely in the z-direction and when there is an absence of pressure gradient and convective acceleration. It applies to unsteady flow through infinitely long ducts (or far away from the ends of the duct) if no pressure gradient is induced along the flow direction. Note that this equation is equivalent to the two-dimensional heat conduction equation with no internal heat source.

It will be useful to non-dimensionalize all our variables. Define non-dimensional velocity and lengths as:

$$u^* = \frac{u}{U_0}, \;\; x^* = \frac{x}{L}, \;\; y^* = \frac{y}{L}$$

Here, U_0 is some reference velocity, maybe the maximum velocity. L is some reference length, maybe the radius of the duct. For reference time, we could choose $t_0 = L/U_0$, but this relates length and velocity along two different directions. The more useful reference time would instead be $t_0 = L^2/\nu$. This essentially gives an estimate on "how long it takes diffusion effects to travel across the length L" and so this is more relevant. Our non-dimensionalized time variable is therefore:

$$t^* = \frac{t\nu}{L^2}$$

We are ready to non-dimensionalize our diffusion equation:

$$\frac{\nu U_0}{L^2}\frac{L^2}{\nu U_0}\frac{\partial u}{\partial t} = \nu\frac{U_0}{L^2}\left(\frac{L^2}{U_0}\frac{\partial^2 u}{\partial x^2} + \frac{L^2}{U_0}\frac{\partial^2 u}{\partial y^2}\right)$$

[1] White, Frank M. *Viscous Fluid Flow.* 3rd edition. New York, NY. 2006.

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

Interestingly, non-dimensionalizing the diffusion equation gives the same result as assuming the kinematic viscosity is unity. From here on, unless otherwise specified, the asterisks for non-dimensional variables will be omitted. Our non-dimensional two-dimensional unsteady linear viscous diffusion equation is then:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1}$$

2 The Explicit Numerical Algorithm

Our explicit numerical algorithm will use finite difference approximations to model equation (1). Assume that at time $t = 0$, the velocity $u(x, y, 0)$ is known everywhere. Also assume $u(x, y, t)$ is known for all t at the boundaries (usually no-slip at the walls). We will use a square mesh with equal x and y spacings:

$$\Delta x = \Delta y = h$$

The spacing in time will be Δt. The x-location will be indexed by the integer m, the y-location will be indexed by the integer n, and time will be indexed by the integer j. Thus the velocity u at a particular point in time and space is denoted by:

$$u^j_{m,n}$$

Suppose the velocity is known everywhere at some time j. Equation (1) can be numerically approximated as:

$$\frac{u^{j+1}_{m,n} - u^j_{m,n}}{\Delta t} = \frac{u^j_{m+1,n} + u^j_{m-1,n} + u^j_{m,n+1} + u^j_{m,n-1} - 4u^j_{m,n}}{h^2}$$

(We used equation 3-245 from the textbook.) This can be rewritten as:

$$u^{j+1}_{m,n} = \sigma\big(u^j_{m+1,n} + u^j_{m-1,n} + u^j_{m,n+1} + u^j_{m,n-1}\big) + \big(1 - 4\sigma\big)u^j_{m,n} \tag{2}$$

where $\sigma = \Delta t/h^2$ is a dimensionless mesh-size parameter. This is our explicit numerical algorithm. We can use equation (2) to "march forward" starting with time $t = 0$.

Note that we need all coefficients to be positive for numerical stability. Thus we need:

$$\sigma \leq \frac{1}{4}$$

In other words, we require the stability condition given by:

$$\Delta t \leq \frac{h^2}{4} \tag{3}$$

3 Example with Rectangular Duct Flow

Consider a rectangular duct with width $2a$ and height $2b$. The (dimensional) x and y coordinates will vary between $-a \leq x \leq a$ and $-b \leq y \leq b$. Given an induced pressure gradient in the z-direction, $-d\hat{p}/dz$, the fully developed Poiseuille flow that results in the duct is given by equation 3-48 in the textbook:

$$u(x,y) = \frac{16a^2}{\mu\pi^3}\Big(-\frac{d\hat{p}}{dz}\Big)\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\,\frac{\cos(n\pi x/2a)}{n^3}\left[1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right]$$

(I actually did the math myself and verified this equation.) We will non-dimensionalize the equation with reference length $L = a$ and reference velocity $U_0 = \dfrac{16a^2}{\mu\pi^3}\Big(-\dfrac{d\hat{p}}{dz}\Big)$. We will also non-dimensionalize the duct width and height:

$$a^* = \frac{a}{a} = 1, \quad b^* = \frac{b}{a}$$

The non-dimensional equation for fully developed Poiseuille flow becomes (omitting all asterisks):

$$u(x,y) = \sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\,\frac{\cos(n\pi x/2)}{n^3}\left[1-\frac{\cosh(n\pi y/2)}{\cosh(n\pi b/2)}\right] \tag{4}$$

where $-1 \leq x \leq 1$ and $-b \leq y \leq b$. For our example, we'll use (non-dimensional) $b = 2$. At time $t = 0$, we'll assume the velocity flow follows equation (4) as the pressure gradient is suddenly removed. The velocity must always be zero at the walls:

$$u(\pm 1, y) = u(x, \pm b) = 0$$

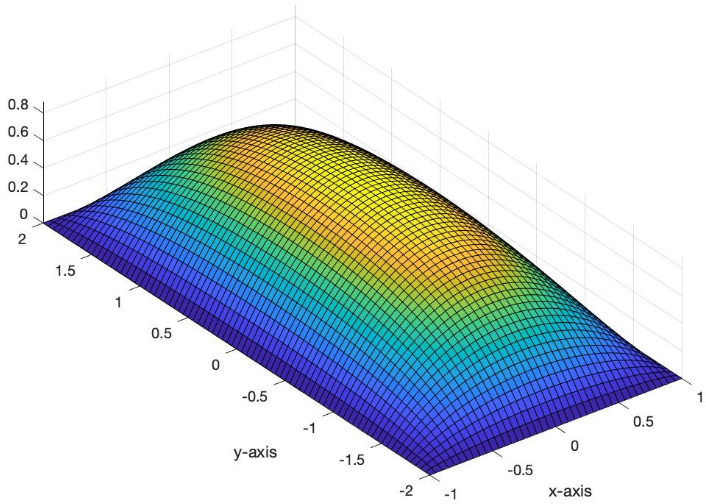
We expect the velocity to decay to zero everywhere. We'll let $h = \dfrac{1}{20}$ and $\Delta t = \dfrac{1}{4000}$. This way, σ is given by:

$$\sigma = \frac{\Delta t}{h^2} = \frac{1}{10} \leq \frac{1}{4}$$

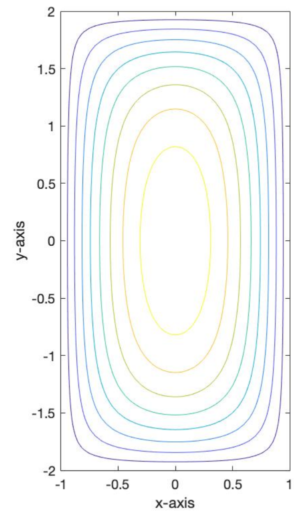
and so we can expect numerical stability when we apply our explicit numerical algorithm given by equation (2).

The algorithm was implemented using MATLAB. All code is attached in the Appendix.

We can plot the velocity field $u(x, y, t)$ for varying values of (non-dimensional) time t and the corresponding contour plots:

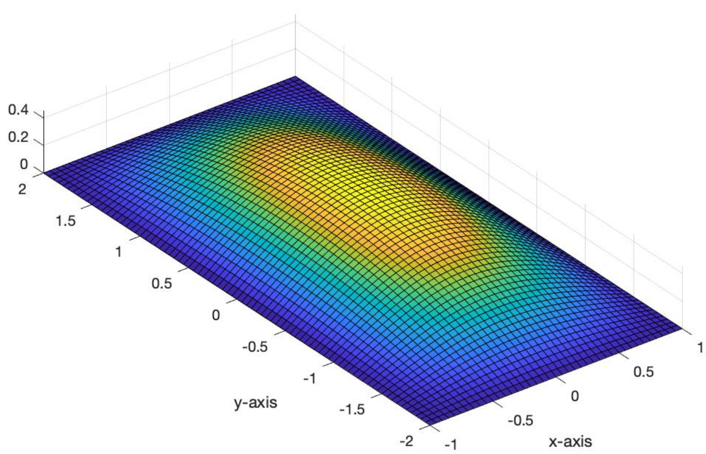


(a) velocity profile

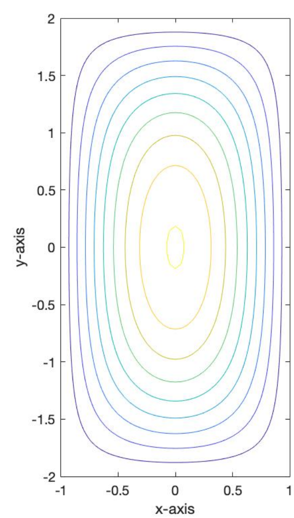


(b) contour plot

$t = 0$

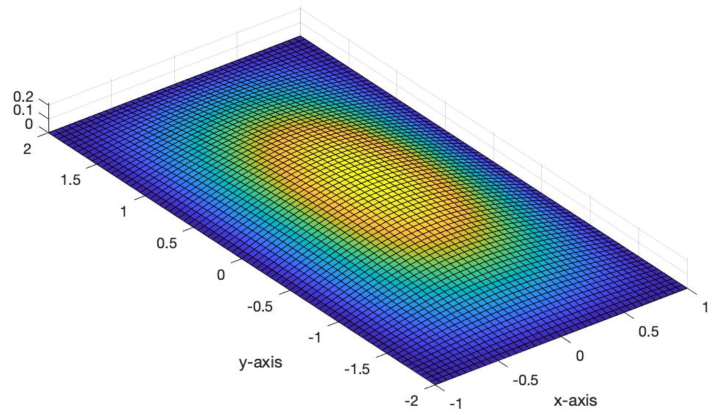


(a) velocity profile

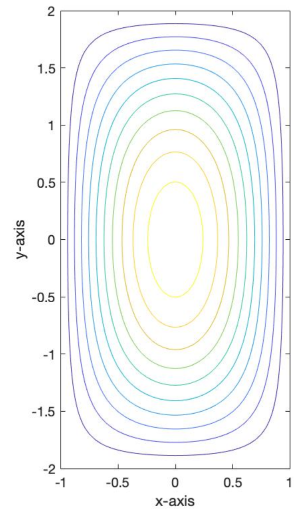


(b) contour plot

$t = 0.25$

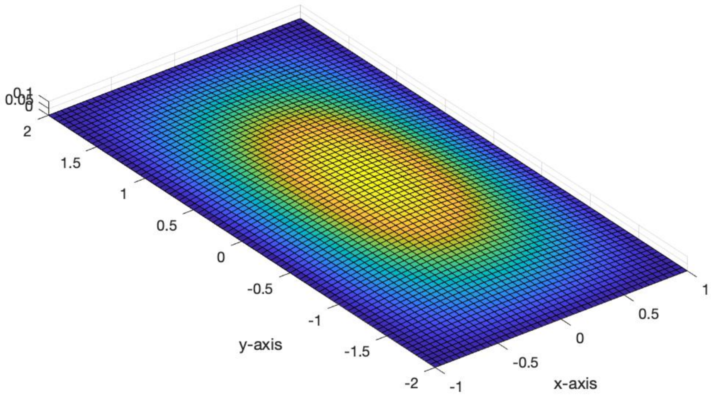


(a) velocity profile

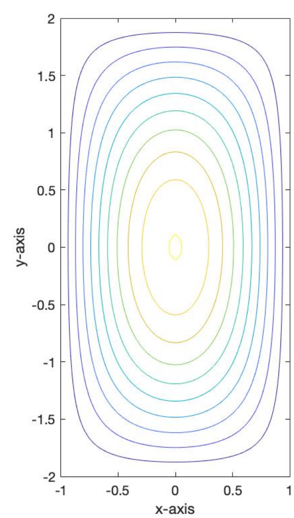


(b) contour plot

$t = 0.5$

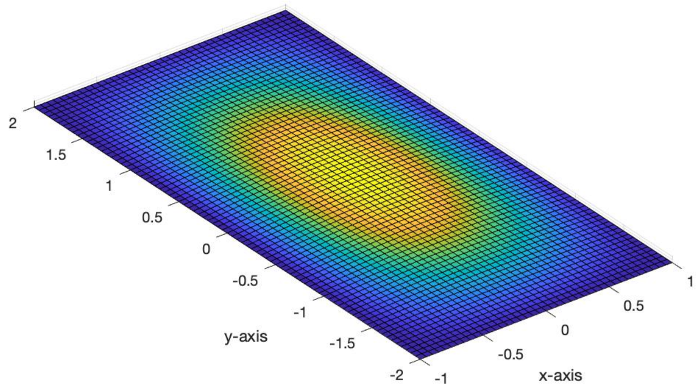


(a) velocity profile

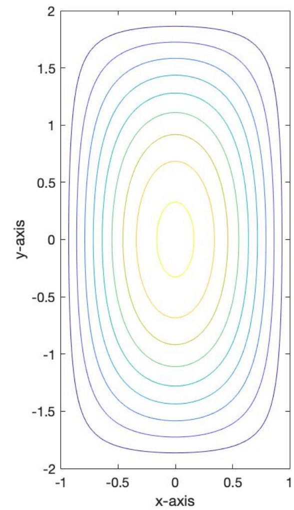


(b) contour plot

$t = 0.75$



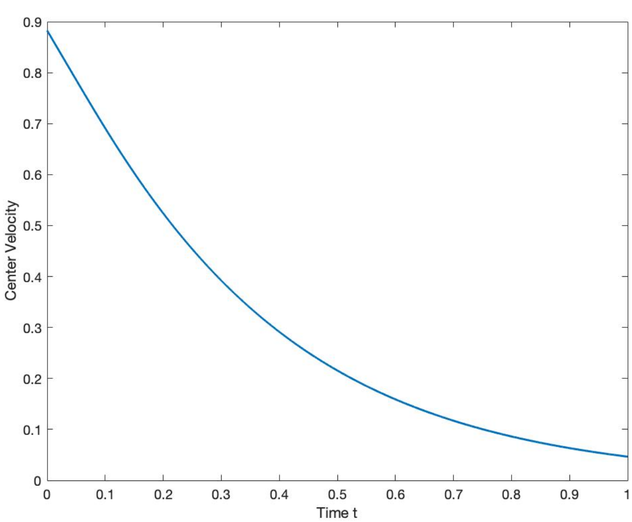
(a) velocity profile



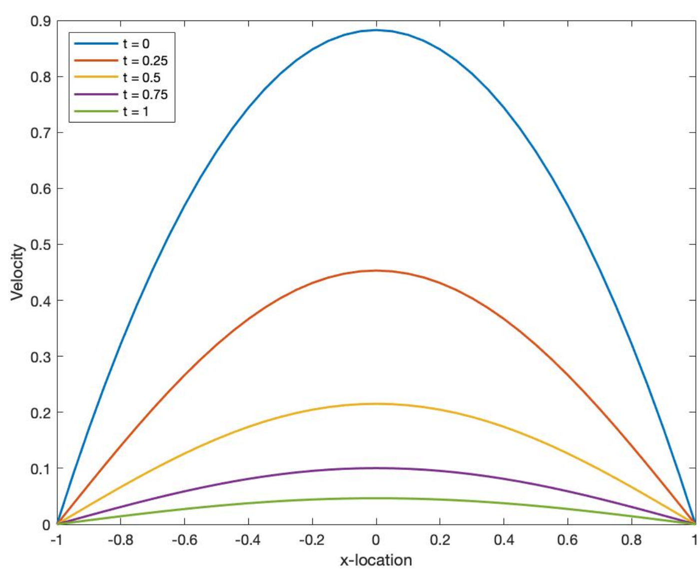
(b) contour plot

$t = 1$

We can also plot the time evolution of the center point as well as the final velocity profile along $y = 0$ at varying times:



(a) center point velocity



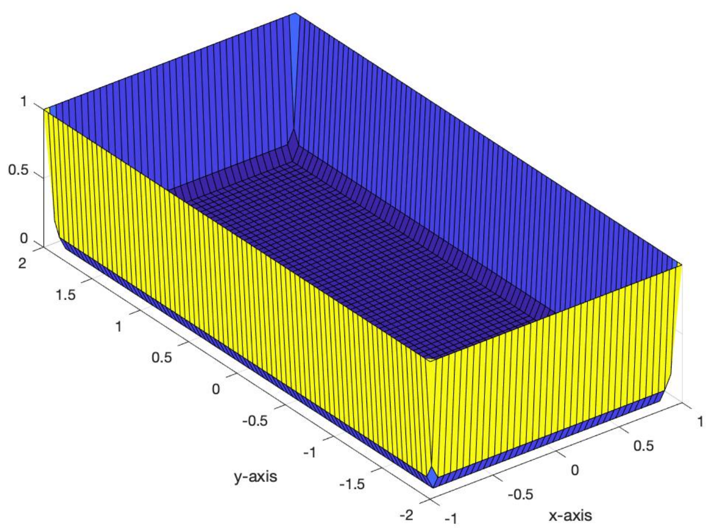
(b) velocity profiles of $y = 0$ at varying times

4 Examples with Moving Walls

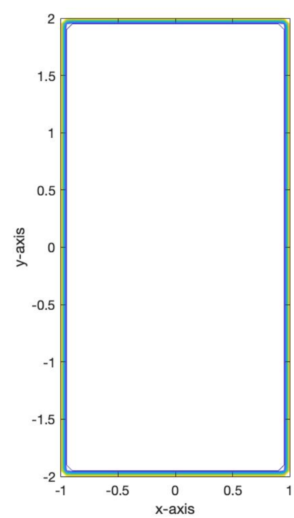
We'll also look at a few examples of rectangular duct flow where the walls also move.

4.1 Four Moving Walls

First, we'll look at an example where at $t = 0$, the velocity is zero everywhere and all four walls begin to move at a constant velocity U_0. We thus expect the (non-dimensionalized) velocity to approach 1 everywhere.

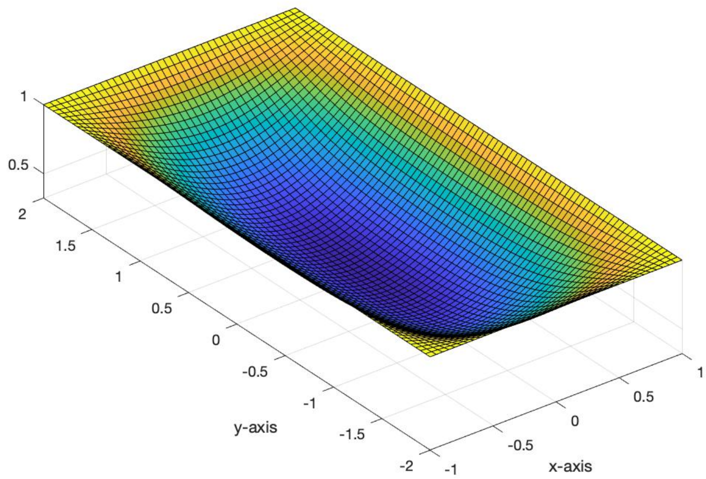


(a) velocity profile

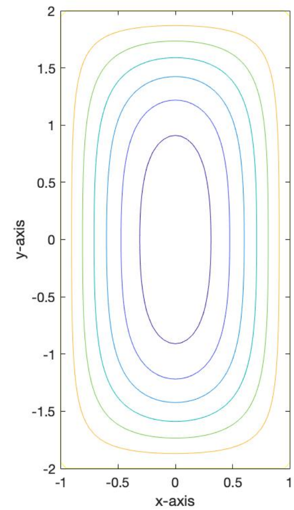


(b) contour plot

$t = 0$

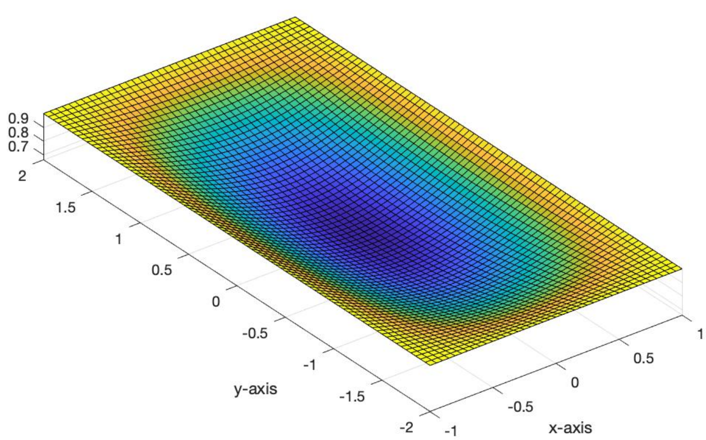


(a) velocity profile

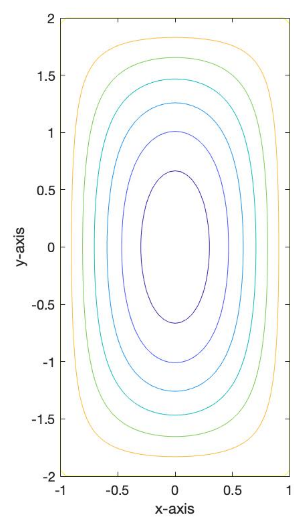


(b) contour plot

$t = 0.25$

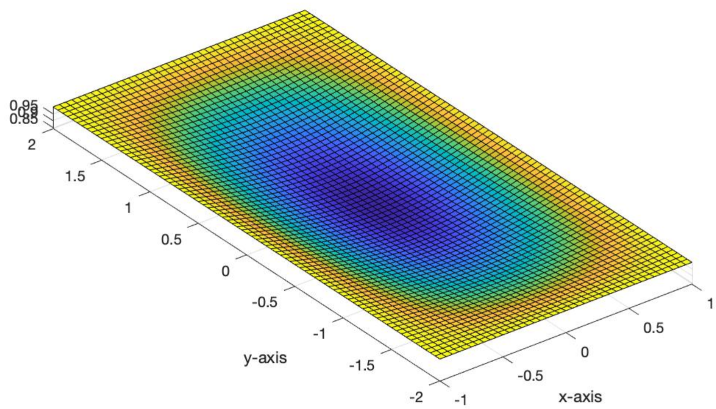


(a) velocity profile

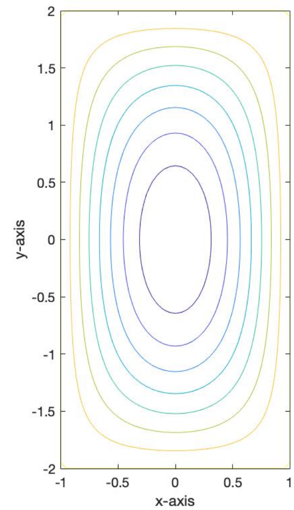


(b) contour plot

$t = 0.5$

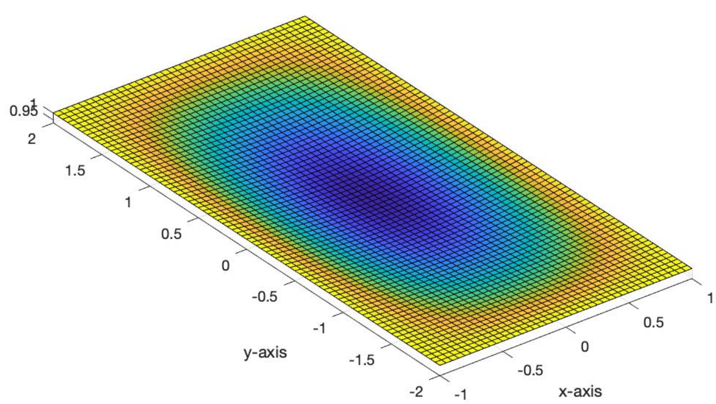


(a) velocity profile

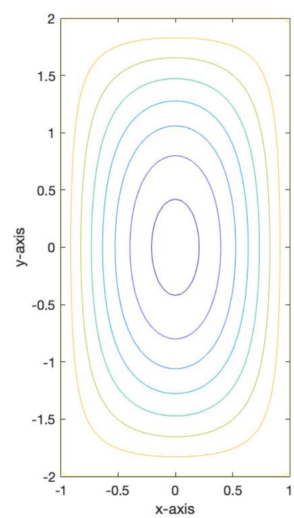


(b) contour plot

$t = 0.75$



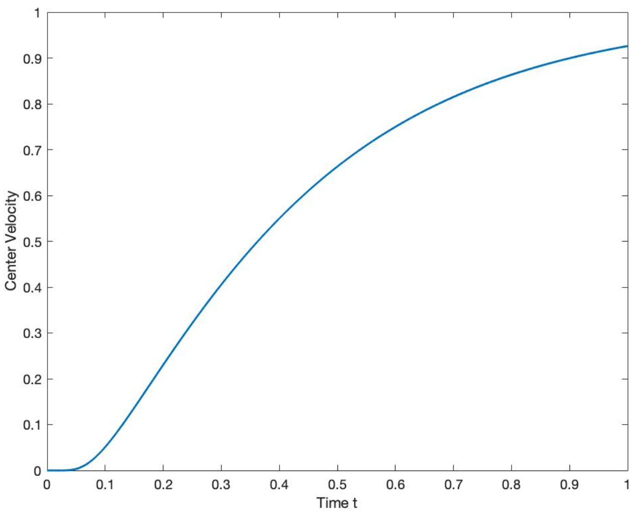
(a) velocity profile



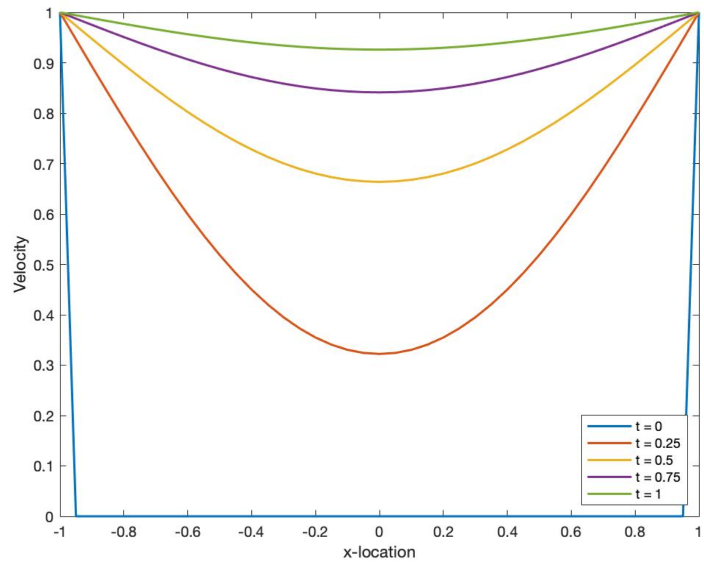
(b) contour plot

$t = 1$

We can also plot the time evolution of the center point as well as the final velocity profile along $y = 0$ at varying times:



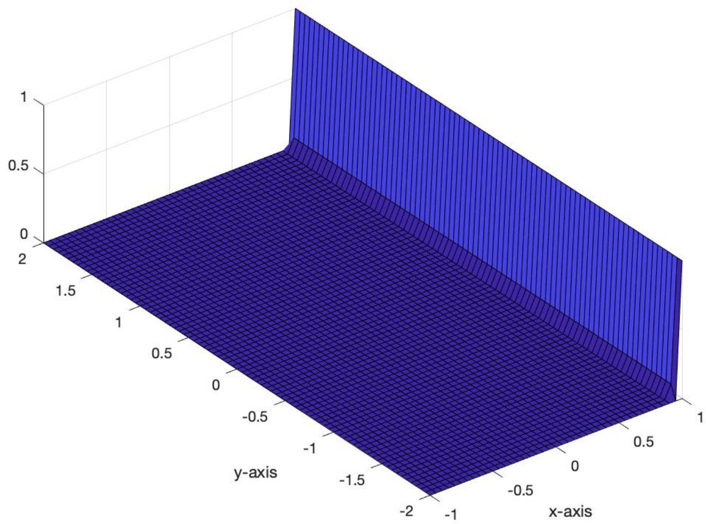
(a) center point velocity



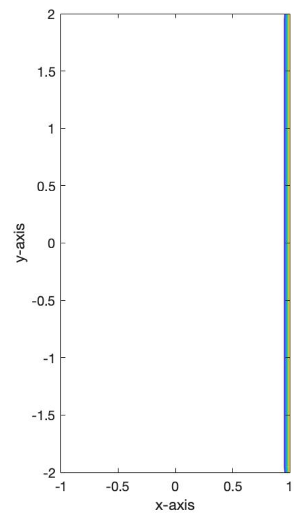
(b) velocity profiles of $y = 0$ at varying times

4.2 One Moving Wall

Next, we'll look at an example where the velocity starts zero everywhere and only the wall at $x = 1$ begins to move with velocity U_0.

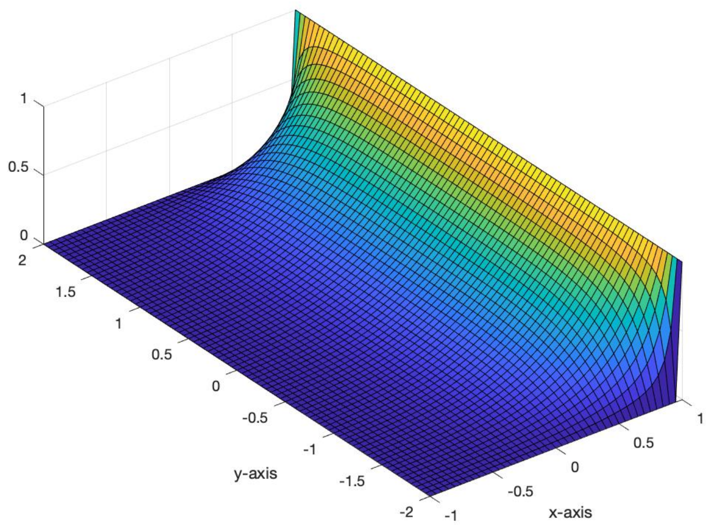


(a) velocity profile

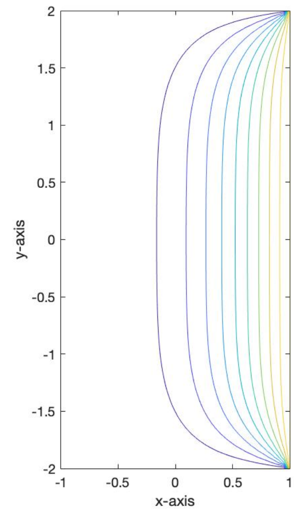


(b) contour plot

$t = 0$

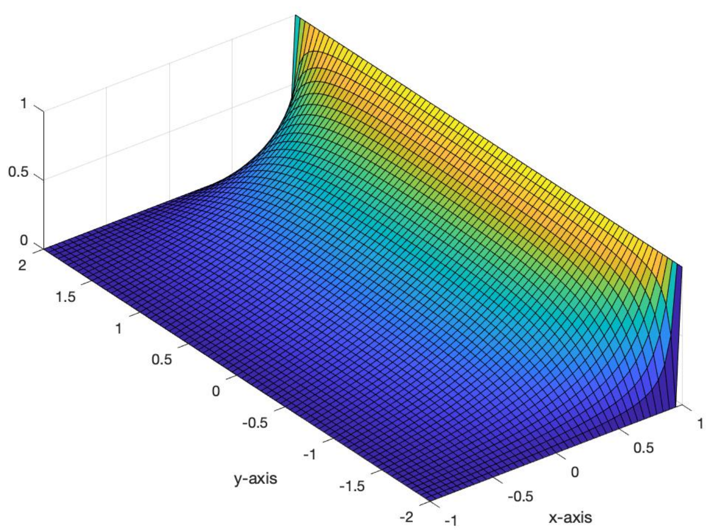


(a) velocity profile

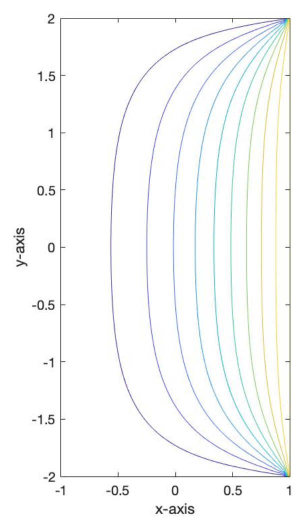


(b) contour plot

$t = 0.25$

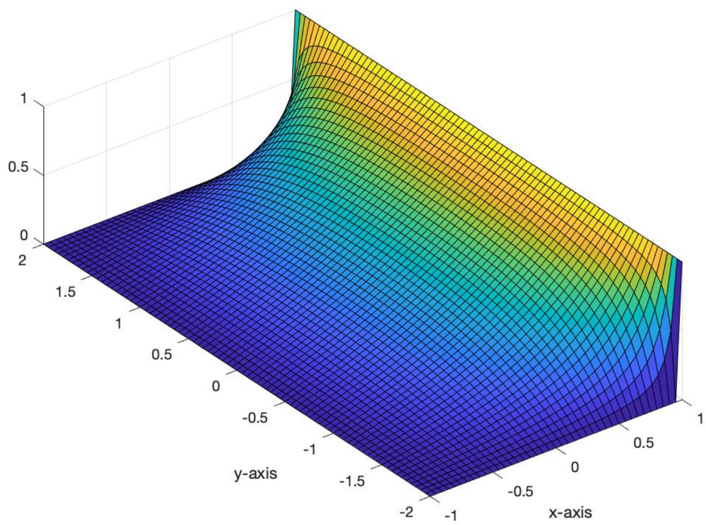


(a) velocity profile

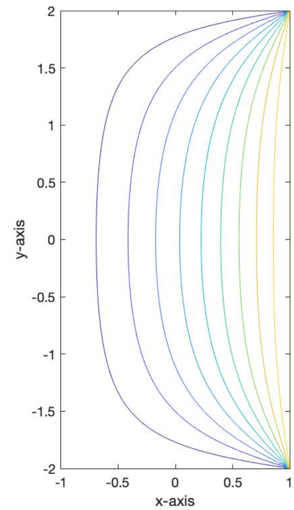


(b) contour plot

$t = 0.5$

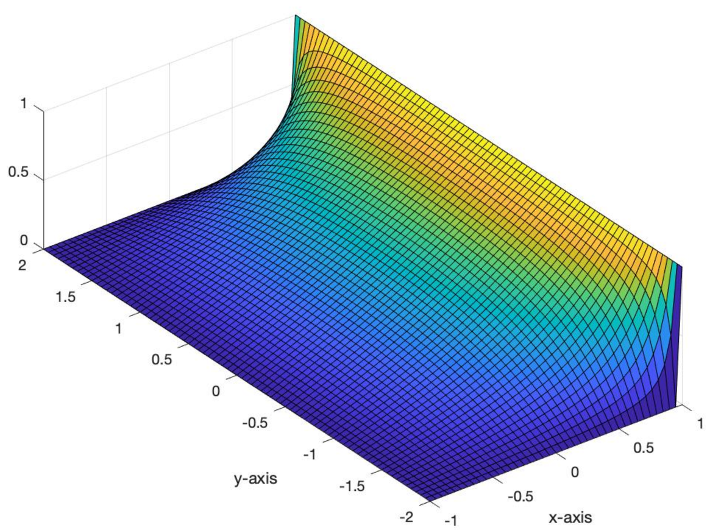


(a) velocity profile

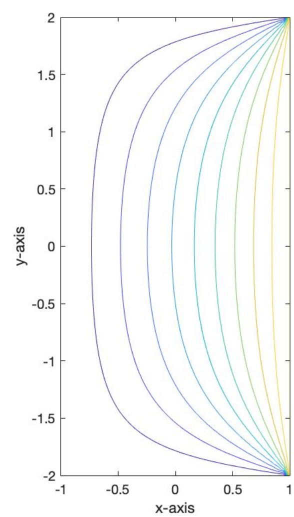


(b) contour plot

$t = 0.75$



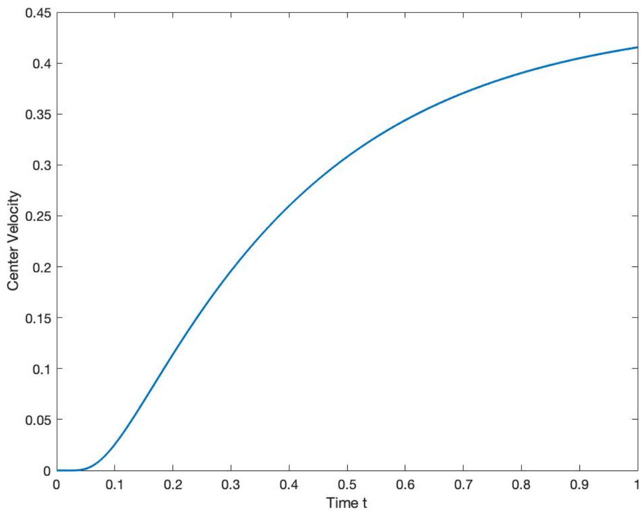
(a) velocity profile



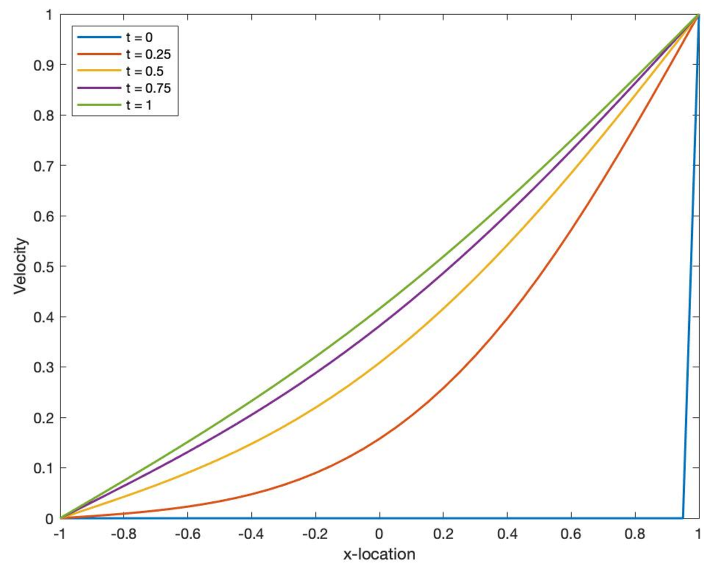
(b) contour plot

$t = 1$

We can also plot the time evolution of the center point as well as the final velocity profile along $y = 0$ at varying times:



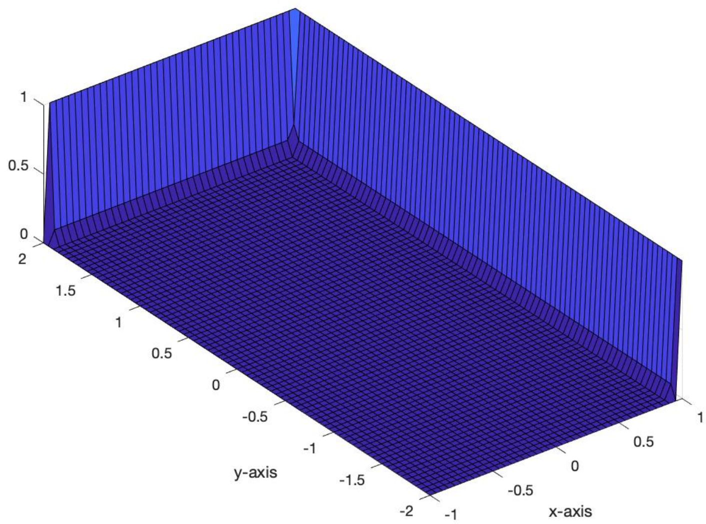
(a) center point velocity



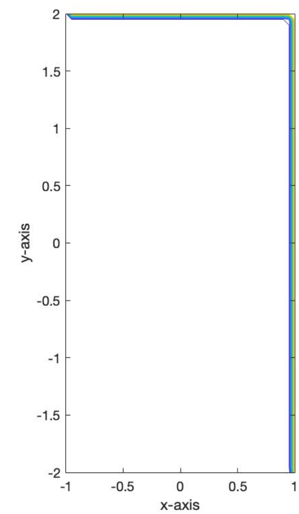
(b) velocity profiles of $y = 0$ at varying times

4.3 Two Adjacent Moving Walls

Next, we'll look at an example where the velocity starts zero everywhere and the walls at $x = 1$ and $y = b$ begin to move with velocity U_0.

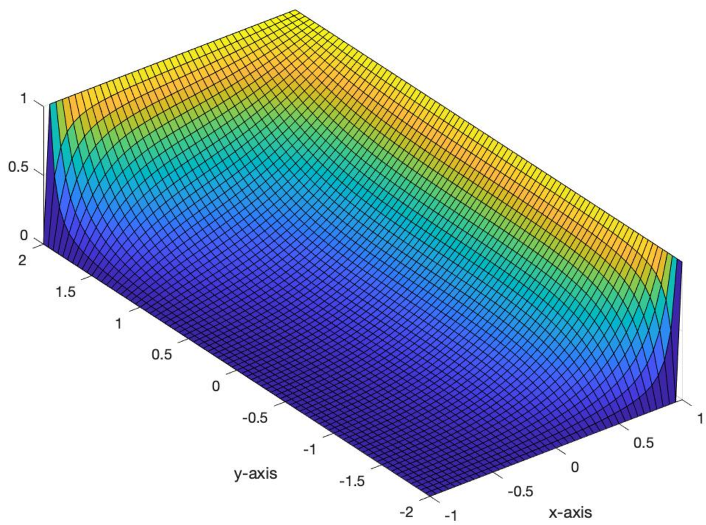


(a) velocity profile

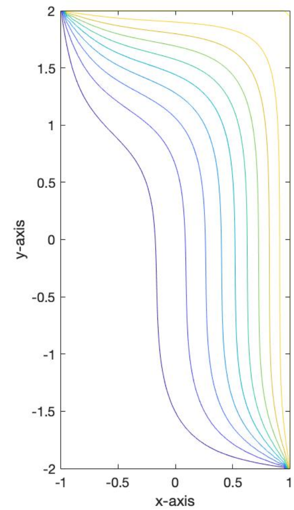


(b) contour plot

$t = 0$

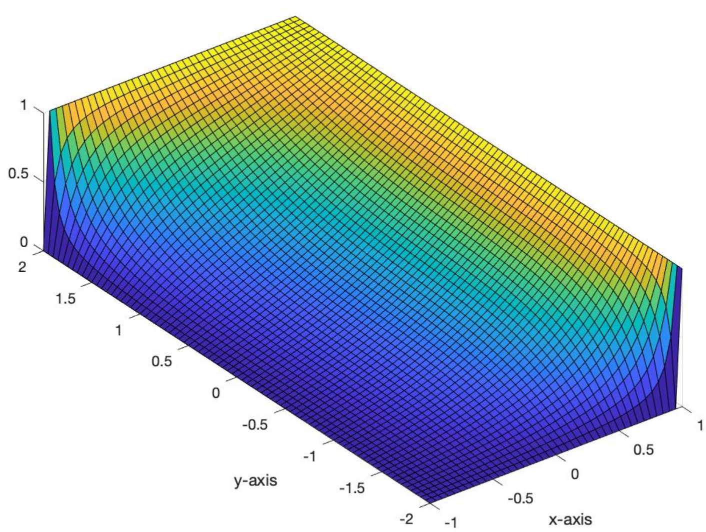


(a) velocity profile

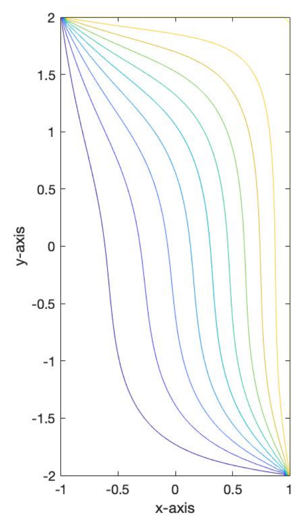


(b) contour plot

$t = 0.25$

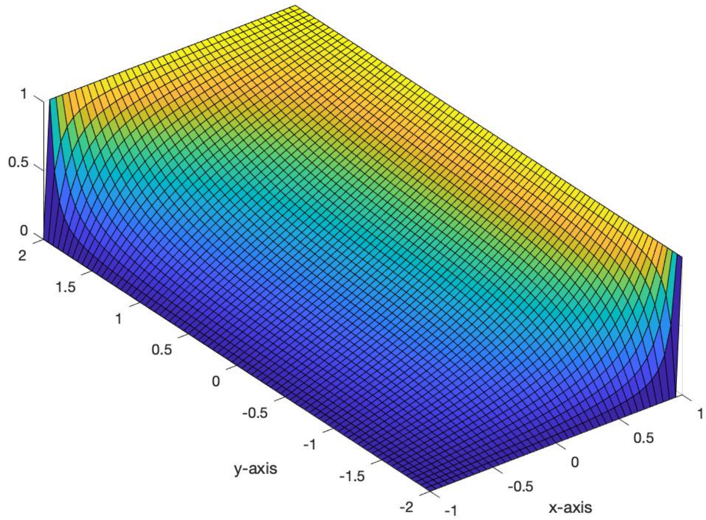


(a) velocity profile

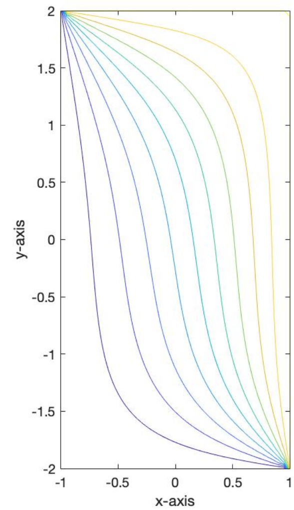


(b) contour plot

$t = 0.5$

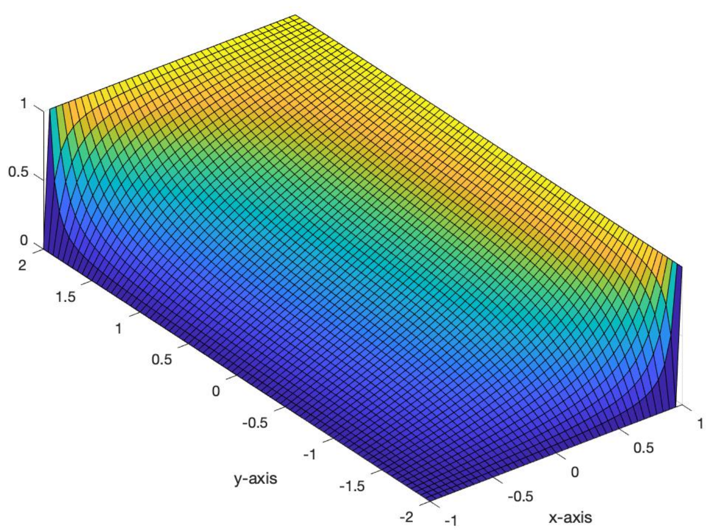


(a) velocity profile

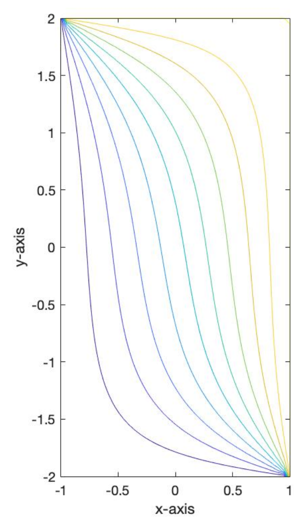


(b) contour plot

$t = 0.75$



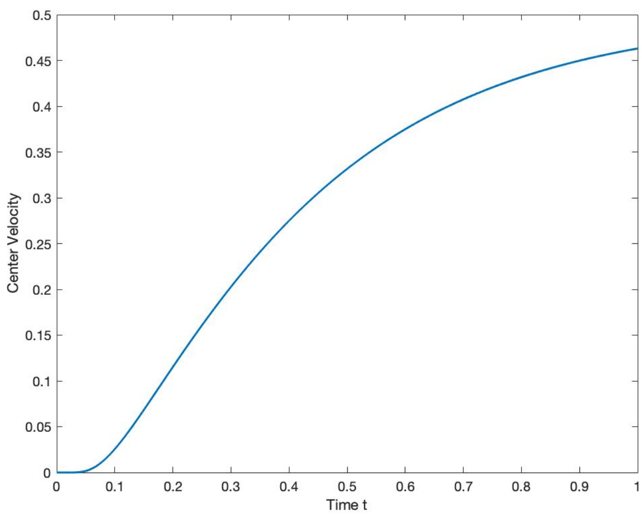
(a) velocity profile



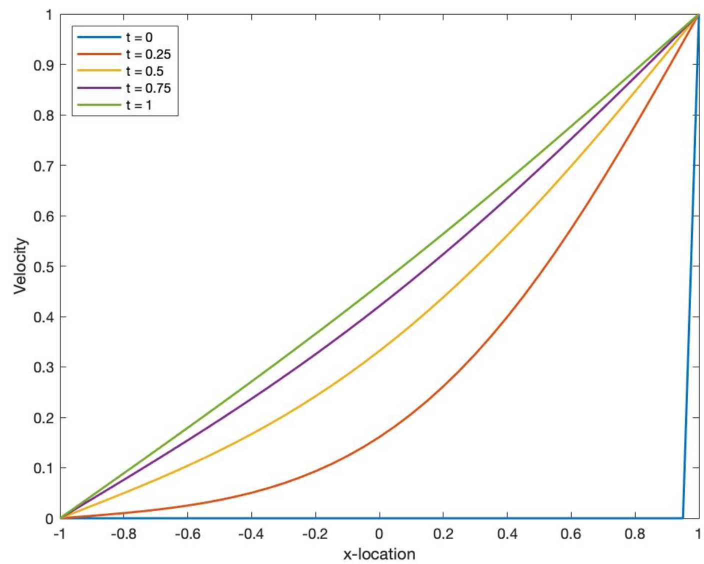
(b) contour plot

$t = 1$

We can also plot the time evolution of the center point as well as the final velocity profile along $y = 0$ at varying times:



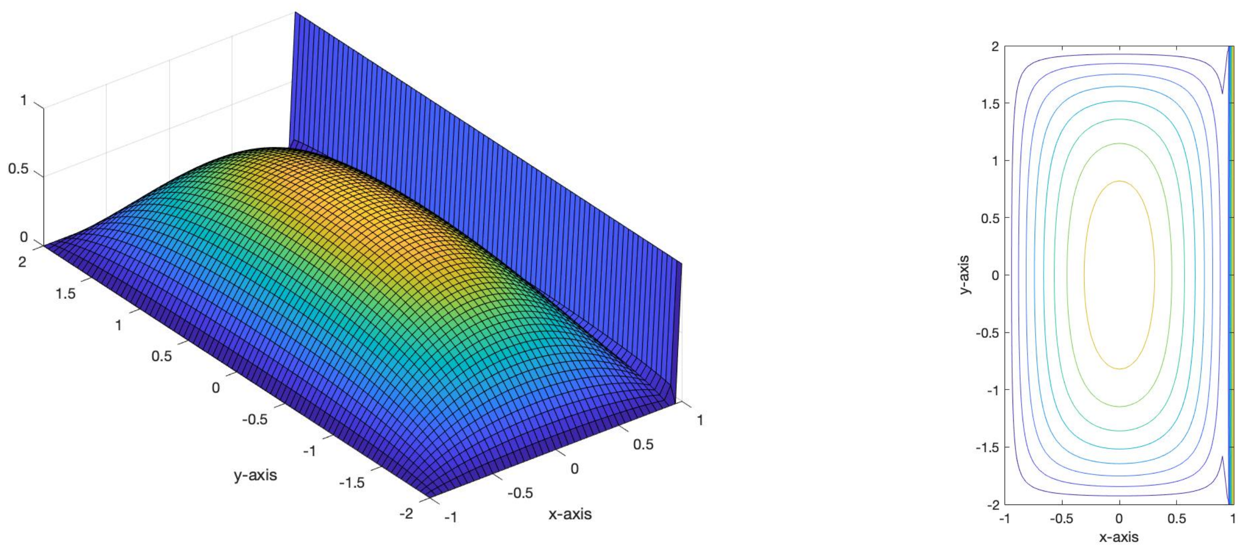
(a) center point velocity



(b) velocity profiles of $y = 0$ at varying times

4.4 One Moving Wall with Poiseuille Flow

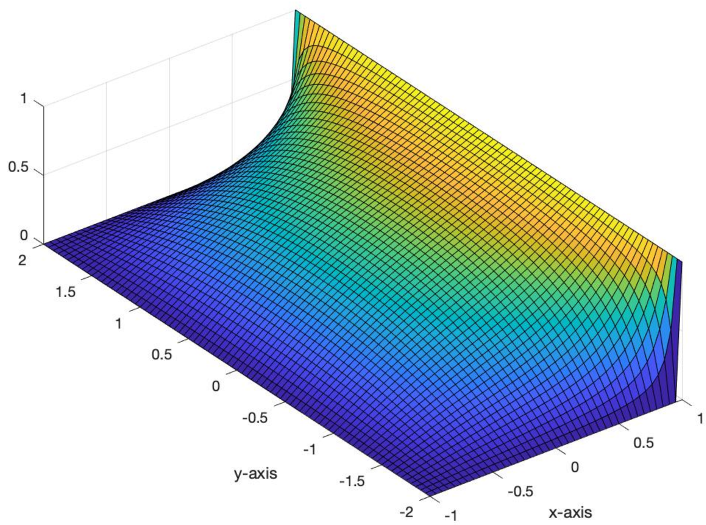
Finally, we'll look at an example where the velocity starts with Poiseuille flow (equation (4)) and the wall at $x = 1$ begins to move with velocity $U_0 = \dfrac{16a^2}{\mu\pi^3}\left(-\dfrac{d\hat{p}}{dz}\right)$ (this is the same reference velocity as that of the Poiseuille flow).



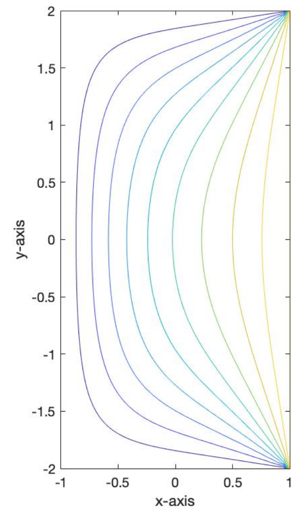
(a) velocity profile

(b) contour plot

$t = 0$

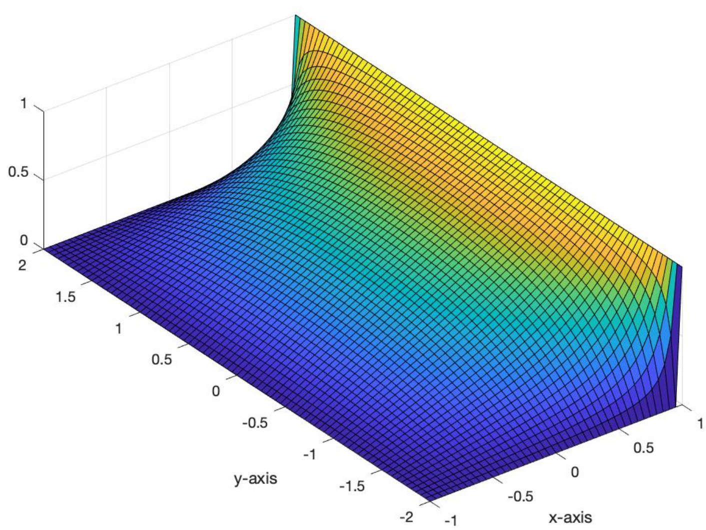


(a) velocity profile

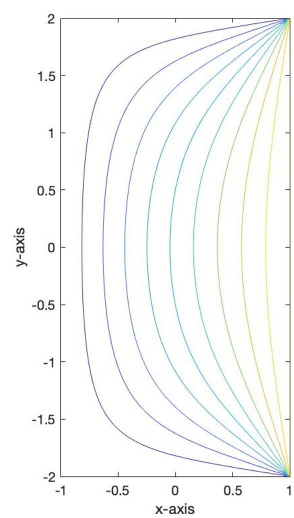


(b) contour plot

$t = 0.25$

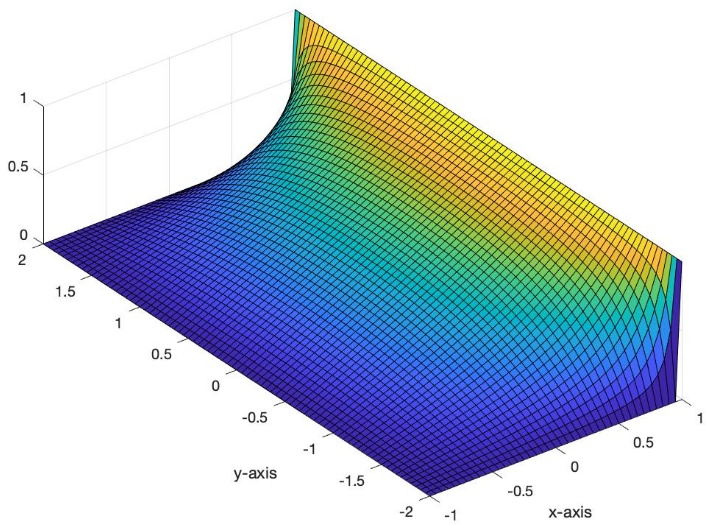


(a) velocity profile

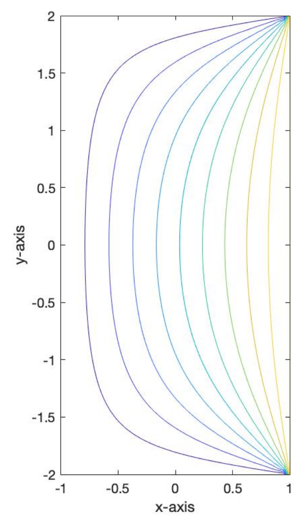


(b) contour plot

$t = 0.5$

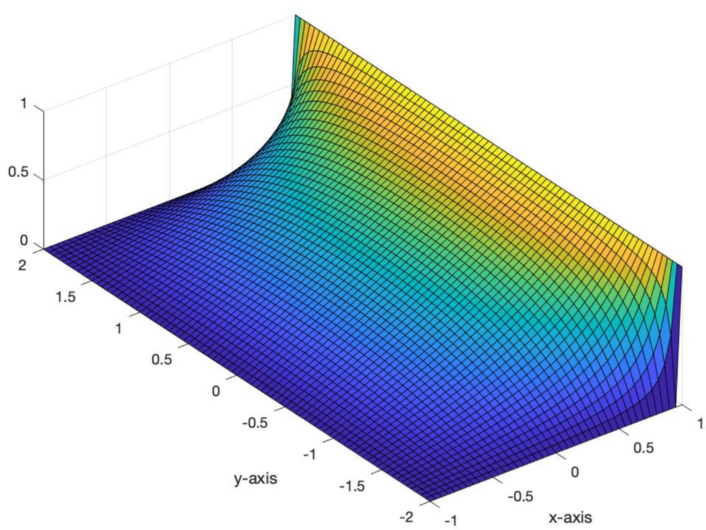


(a) velocity profile

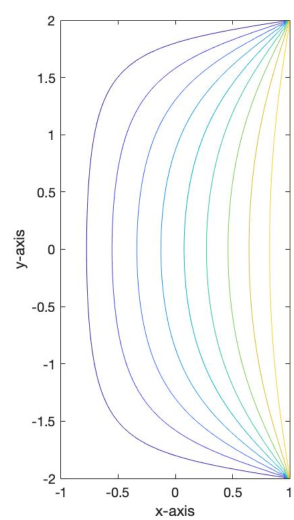


(b) contour plot

$t = 0.75$



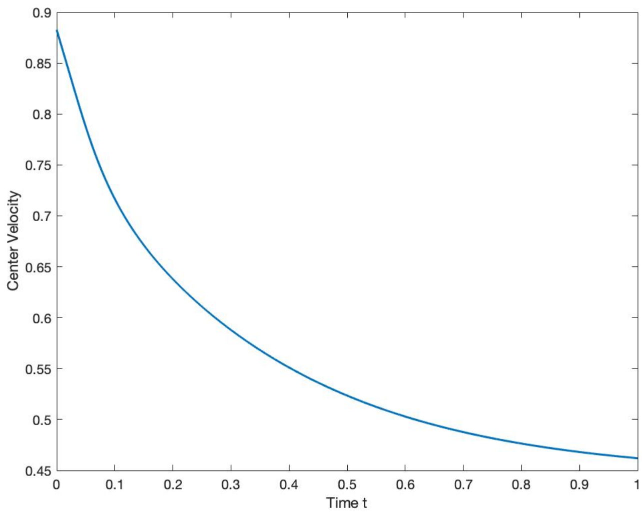
(a) velocity profile



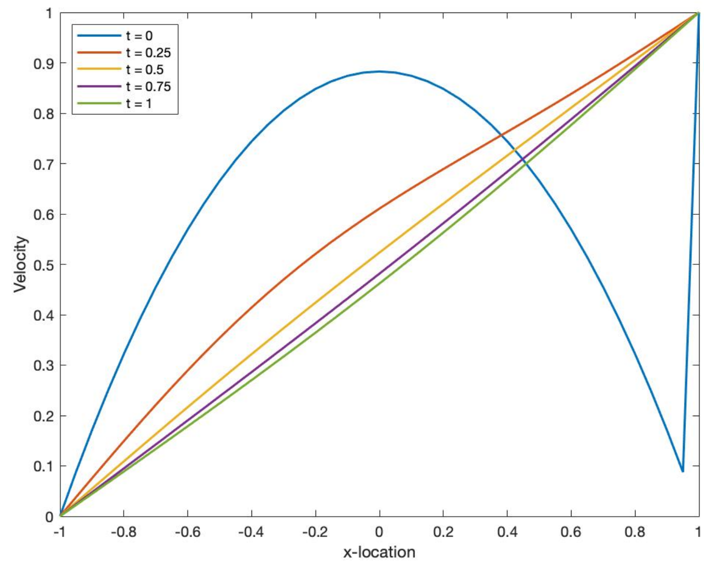
(b) contour plot

$t = 1$

We can also plot the time evolution of the center point as well as the final velocity profile along $y = 0$ at varying times:



(a) center point velocity



(b) velocity profiles of $y = 0$ at varying times

Appendix - Code

All MATLAB code is attached in the following pages.

MATLAB code:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Numerically simulate viscous diffusion described
% by the viscous diffusion equation (equation 3-250)
% using an explicit numerical algorithm.
% All variables are assumed to be non-dimensionalized.

close all
clear all

h = 1/20; % mesh spacing
DeltaT = 1/4000; % time spacing
sigma = DeltaT/h^2; % mesh-size parameter

% We assume the duct width is a = 1.
b = 2; % duct height

[X,Y] = meshgrid(-1:h:1,-b:h:b); % create mesh grid
u = X; % initialize velocity u
[r, c] = size(X); % number of rows and columns

% Implement boundary conditions:
u(1,:) = 0; u(r,:) = 0;
u(:,1) = 0; u(:,c) = 0;

% Implement initial conditions:
for it1 = 2:r-1
    for it2 = 2:c-1
        x = X(it1,it2); y = Y(it1,it2); % extract x, y
        u(it1,it2) = u0(x,y,b);
    end
end

uNew = u; % necessary for updates
T = 1; % total (non-dimensionalized) time of evolution

tArray = 0:DeltaT:T; % array of time progression
uArray = zeros(size(tArray)); % array of velocities at point of interest

% Get initial velocity profile along y = 0:
uInitial = u(round(r/2),:);
```

```
% Implement explicit numerical algorithm:
it0 = 0;
temp = 1;
for t = tArray
   it0 = it0 + 1;
   uArray(it0) = u(round(r/2), round(c/2)); % extract point of interest
   for it1 = 2:r-1
      for it2 = 2:c-1
         x = X(it1,it2); y = Y(it1,it2); % extract x, y
         A = sigma*(u(it1+1,it2) + u(it1-1,it2) + u(it1,it2+1) + u(it1,it2-1));
         B = (1 - 4*sigma)*u(it1,it2);
         uNew(it1,it2) = A + B;
      end
   end
   u = uNew;

   % extract evolution of velocity profile along y = 0:
   if (t > 0.25*T) && (temp == 1)
      u1 = u(round(r/2),:);
      temp = 2;
   end
   if (t > 0.5*T) && (temp == 2)
      u2 = u(round(r/2),:);
      temp = 3;
   end
   if (t > 0.75*T) && (temp == 3)
      u3 = u(round(r/2),:);
      temp = 4;
   end
end
u4 = u(round(r/2),:);



% Plot final results:

% total velocity flow after time T
surf(X,Y,u)
axis equal
xlabel('x-axis')
ylabel('y-axis')

% contour plot
```

```
figure()
contour(X,Y,u)
axis equal
xlabel('x-axis')
ylabel('y-axis')

% evolution of center velocity over time
figure()
plot(tArray, uArray, 'LineWidth',1.5)
xlabel('Time t')
ylabel('Center Velocity')

% evolution of velocity profile along y = 0
figure()
plot(X(round(r/2),:),uInitial, 'LineWidth',1.5)
hold on
plot(X(round(r/2),:),u1, 'LineWidth',1.5)
hold on
plot(X(round(r/2),:),u2, 'LineWidth',1.5)
hold on
plot(X(round(r/2),:),u3, 'LineWidth',1.5)
hold on
plot(X(round(r/2),:),u4, 'LineWidth',1.5)
legend('t = 0','t = 0.25','t = 0.5','t = 0.75','t = 1', 'Location', 'northwest')
xlabel('x-location')
ylabel('Velocity')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Fully developed Poiseuille flow through rectangular duct.
% Will be the initial velocity profile.
% Equation 3-48 of the textbook, non-dimensionalized.
% Assumes a = 1.

function thing = u0(x, y, b)
N = 100; % total number of terms
% If too many terms are used, the cosh() will be undefined.

sum = 0;
for it = 1:2:2*N-1
    neg = (-1)^((it-1)/2);
    C = cos(it*pi*x/2)/(it^3);
    H = 1 - cosh(it*pi*y/2)/cosh(it*pi*b/2);
    sum = sum + neg*C*H;
end
thing = sum;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```