CS-E4830
Assignment 2
Question 1

Claim.

$$k_c(x_i, x_j) = k(x_i, x_j) - \frac{1}{N}\sum_{p=1}^{N} k(x_p, x_j) - \frac{1}{N}\sum_{q=1}^{N} k(x_i, x_q) + \frac{1}{N^2}\sum_{p=1}^{N}\sum_{q=1}^{N} k(x_p, x_q)$$

Proof.

$$k_c(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$$

$$= \langle \phi(x_i) - \frac{1}{N}\sum_{p=1}^{N} \phi(x_p), \phi(x_j) - \frac{1}{N}\sum_{q=1}^{N} \phi(x_q) \rangle$$

Apply the following inner product properties

1. $\langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$
2. $\langle ax, y \rangle = a\langle x, y \rangle$

$$\rightarrow k_c(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle - \frac{1}{N}\sum_{p=1}^{N} \langle \phi(x_p), \phi(x_j) \rangle - \frac{1}{N}\sum_{q=1}^{N} \langle \phi(x_i), \phi(x_q) \rangle + \frac{1}{N^2}\sum_{p=1}^{N}\sum_{q=1}^{N} \langle \phi(x_p), \phi(x_q) \rangle$$

$$= k(x_i, x_j) - \frac{1}{N}\sum_{p=1}^{N} k(x_p, x_j) - \frac{1}{N}\sum_{q=1}^{N} k(x_i, x_q) + \frac{1}{N^2}\sum_{p=1}^{N}\sum_{q=1}^{N} k(x_p, x_q) \quad \blacksquare$$

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1. The probability that the point $\hat{x}$ belongs to C_1 can be calculated using Bayes' rule

$$p(y=C_1 \mid x=\hat{x}) = \frac{p(x=\hat{x}, C_1)}{p(x=\hat{x}, C_1) + p(x=\hat{x}, C_2)}$$

2. Looking at the figure, we can see that the minimum misclassification error is reached at the union of the two distributions. So, the probability of the minimum misclassification error corresponds to the green and the blue area combined.

$$P(\text{Min. misclassification error}) = \int_{x \in \mathcal{X}} \min(p(x, C_1), p(x, C_2))\, dx$$
$$\leq \int_{x \in \mathcal{X}} p(x, C_1)\, p(x, C_2)^{1/2}\, dx$$

$\blacksquare$

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Question 3

The multi-class desicion function suggested is of form

$$\arg\max_k \; P(Y_i = k \mid X = x_i), \quad k = 1, \dots, K$$

$$\Rightarrow \arg\max_k \; \frac{1}{Z} \exp(\langle w_k, x_i \rangle), \quad k = 1, \dots, K$$

Value of Z for a fixed number of classes can be derived from the following equation, because we know that $\frac{1}{Z} \exp(\langle w_k, x_i \rangle)$ is a probability

$$\sum_{k=1}^{K} \frac{1}{Z} \exp(\langle w_k, x_i \rangle) = 1 \qquad \| \cdot Z$$

$$Z = \sum_{k=1}^{K} \exp(\langle w_k, x_i \rangle)$$

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Question 4

Claim.

$$\mathbb{E}[e^{\lambda\varepsilon}] \leq e^{\frac{\lambda^2}{2}}$$

Proof.

$$\mathbb{E}[e^{\lambda\varepsilon}] = \frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{\lambda^k + (-\lambda)^k}{k!}$$ // Odd k values will zero the expression.

$$= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2k!}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{a \cdot k!}$$ // $a = \prod_{n=k+1}^{2k} n$

$$\leq \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2^k \cdot k!}$$

$$= e^{\frac{\lambda^2}{2}}$$

$\blacksquare$