Pertemuan 8:
Teknik Substitusi
(Teknik Integral Fungsi Aljabar Substitusi Trigonometri)

A. Tujuan Pembelajaran

Mahasiswa mampu memahami dan menggunakan materi dasar turunan dalam memecahkan permasalahan integral tak tentu dan integral tentu fungsi aljabar menggunakan metode substitusi trigonometri.

B. Uraian Materi

Untuk integran berbentuk atau mengandung unsur $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, dan $\sqrt{x^2 - a^2}$ maka integralnya dapat diselesaikan dengan memisalkan variabel x dengan suatu fungsi trigonometri berikut:

No.	Bentuk Integran	Pemisalan	Menggunakan Identitas Trigonometri
1	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$ **lalu** $\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1)$ **lalu** $\sin 2\theta = 2(\sin \theta \cos \theta)$
2	$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
3	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Contoh 1a:

Tentukan nilai dari $\int \sqrt{16 - x^2}\, dx$ adalah ….

Misalkan: $x = 4 \sin \theta$ maka $dx = 4 \cos \theta \, d\theta$

Sehingga,

$$\int \sqrt{16 - x^2}\, dx = \int \sqrt{16 - (4 \sin \theta)^2}\, (4 \cos \theta) d\theta$$

$$= \int \sqrt{16 - 16 \sin^2 \theta}\, (4 \cos \theta) d\theta$$

$$= \int \sqrt{16(1 - \sin^2 \theta)}\, (4 \cos \theta) d\theta$$

ingat, $1 - \sin^2 \theta = \cos^2 \theta$, maka

$$= \int \sqrt{16(\cos^2 \theta)}\, (4 \cos \theta) d\theta$$

$$= \int (4 \cos \theta)\, (4 \cos \theta) d\theta$$

$$= 16 \int \cos^2 \theta \, d\theta$$

Ingat, $\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1)$, maka

$$= 16 \int \frac{1}{2} (\cos 2\theta + 1)\, d\theta$$

$$= 8 \int (\cos 2\theta + 1)\, d\theta$$

$$= 8 \left[\tfrac{1}{2} \sin 2\theta + \theta\right] + C$$

Ingat, $\sin 2\theta = 2(\sin \theta \cos \theta)$, maka

$$= 8 \left[\tfrac{1}{2} \cdot 2(\sin \theta \cos \theta) + \theta\right] + C$$

$$= 8 \left(\sin \theta \cdot \sqrt{1 - \sin^2 \theta} + \theta\right) + C$$

Ingat kembali di awal soal: $x = 4 \sin \theta$ maka $\sin \theta = \frac{x}{4}$, sehingga

$$= 8 \left(\sin \theta \cdot \sqrt{1 - \sin^2 \theta} + \theta\right) + C$$

Menjadi:

$$= 8 \left(\frac{x}{4} \cdot \sqrt{1 - \left(\frac{x}{4}\right)^2} + \sin^{-1} \left(\frac{x}{4}\right) \right) + C$$

$$= 8\left(\frac{x}{4}\cdot\sqrt{\frac{16-x^2}{16}}+\sin^{-1}\left(\frac{x}{4}\right)\right)+C$$

$$= 8\left(\frac{x}{4}\cdot\frac{1}{4}\sqrt{16-x^2}+\sin^{-1}\left(\frac{x}{4}\right)\right)+C$$

$$= \frac{x}{2}\sqrt{16-x^2}+8\sin^{-1}\left(\frac{x}{4}\right)+C$$

Contoh 1b:

Tentukan nilai dari $\int x^2\sqrt{9-x^2}\,dx$!

Misal: $x = 3\sin\theta$ maka $dx = 3\cos\theta\,d\theta$

Sehingga

$$\int x^2\sqrt{9-x^2}\,dx = \int (3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}\,(3\cos\theta)d\theta$$

$$= \int 9\sin^2\theta\,\sqrt{9-9\sin^2\theta}\,(3\cos\theta)d\theta$$

$$= 9\int \sin^2\theta\,\sqrt{9(1-\sin^2\theta)}\,(3\cos\theta)d\theta$$

$$= 9\int 3\sin^2\theta\,\sqrt{(1-\sin^2\theta)}\,(3\cos\theta)d\theta$$

ingat, $1-\sin^2\theta = \cos^2\theta$, maka

$$= 9\cdot 3\cdot 3\int \sin^2\theta\,\sqrt{\cos^2\theta}\,(\cos\theta)d\theta$$

$$= 81\int \sin^2\theta\,(\cos\theta)\,(\cos\theta)d\theta$$

$$= 81\int \sin^2\theta\cos^2\theta\,d\theta$$

$$= 81\int (\sin\theta\cos\theta)(\sin\theta\cos\theta)\,d\theta$$

Ingat, $\sin\theta\cos\theta = \frac{1}{2}(\sin 2\theta)$, maka

$$= 81\int \frac{1}{2}(\sin 2\theta)\cdot\frac{1}{2}(\sin 2\theta)\,d\theta$$

$$= 81\cdot\frac{1}{4}\int \sin^2 2\theta\,d\theta$$

Ingat, $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$, maka

$$= 81 \cdot \frac{1}{4} \int \frac{1}{2} (1 - \cos 4\theta)\, d\theta$$

$$= 81 \cdot \frac{1}{8} \int (1 - \cos 4\theta)\, d\theta$$

$$= 81 \cdot \frac{1}{8} \left(\theta - \frac{1}{4} \sin 4\theta \right) + C$$

$$= 81 \cdot \frac{1}{8} \left(\theta - \frac{1}{4} \sin 2 \cdot 2\theta \right) + C$$

Ingat, $\sin 2\theta = 2(\sin\theta \cos\theta)$ sehingga $\sin 2 \cdot 2\theta = 2(\sin 2\theta \cos 2\theta)$, maka

$$= 81 \cdot \frac{1}{8} \left[\theta - \frac{1}{4} (2 \sin 2\theta \cos 2\theta) \right] + C$$

$$= 81 \cdot \frac{1}{8} \left[\theta - \frac{1}{2} (\sin 2\theta \cos 2\theta) \right] + C$$

Ingat kembali, $\sin 2\theta = 2(\sin\theta \cos\theta)$ dan $\cos 2\theta = 1 - 2\sin^2\theta$, maka

$$= 81 \cdot \frac{1}{8} \left[\theta - \frac{1}{2} \left(2(\sin\theta \cos\theta) \cdot (1 - 2\sin^2\theta) \right) \right] + C$$

$$= \frac{81}{8} \left[\theta - \left((\sin\theta \cos\theta) \cdot (1 - 2\sin^2\theta) \right) \right] + C$$

Ingat, $\cos\theta = \sqrt{1 - \sin^2\theta}$, maka

$$= \frac{81}{8} \left[\theta - \left(\left(\sin\theta \sqrt{1 - \sin^2\theta} \right) \cdot (1 - 2\sin^2\theta) \right) \right] + C$$

Ingat kembali di awal soal: $x = 3\sin\theta$ maka $\sin\theta = \frac{x}{3}$, sehingga

$$= \frac{81}{8} \left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\left(\frac{x}{3} \sqrt{1 - \left(\frac{x}{3}\right)^2} \right) \cdot \left(1 - 2\left(\frac{x}{3}\right)^2 \right) \right) \right] + C$$

$$= \frac{81}{8} \left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\left(\frac{x}{3} \sqrt{\frac{9 - x^2}{9}} \right) \cdot \left(\frac{9 - 2x^2}{9} \right) \right) \right] + C$$

$$= \frac{81}{8} \left[\sin^{-1}\left(\frac{x}{3}\right) - \left(\left(\frac{x}{3} \cdot \frac{1}{3} \cdot \frac{1}{9} \sqrt{9 - x^2} \right) \cdot (9 - 2x^2) \right) \right] + C$$

$$= \frac{81}{8} \left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{81} \left(\left(\sqrt{9 - x^2} \right) \cdot (9 - 2x^2) \right) \right] + C$$

$$= \frac{81}{8} \sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{8} \left((9 - 2x^2)\sqrt{9 - x^2} \right) + C$$

$$= \frac{1}{8} \left[81 \sin^{-1}\left(\frac{x}{3}\right) - x \left((9 - 2x^2)\sqrt{9 - x^2} \right) \right] + C$$

C. Latihan Soal/Tugas

Tentukan integral fungsi $f(x)$ berikut!

1. $f(x) = \int \sqrt{9+x^2}dx$

2. $f(x) = \int_0^{\pi/4} \sqrt{9+x^2}dx$

D. Ringkasan

Integral fungsi aljabar bentuk tertentu:

$$\boxed{\int \frac{da}{\sqrt{b^2-a^2}} = \sin^{-1}\left(\frac{a}{b}\right) + C}$$

$$\boxed{\int \frac{da}{b^2+a^2} = \frac{1}{b}\tan^{-1}\left(\frac{a}{b}\right) + C}$$

$$\boxed{\int \frac{da}{a\sqrt{a^2-b^2}} = \frac{1}{b}\sec^{-1}\left(\frac{|a|}{b}\right) + C = \frac{1}{b}\cos^{-1}\left(\frac{b}{|a|}\right) + C}$$

E. Daftar Pustaka

Varberg, D., Purcell, E., & Rigdon, S. (2007). *Calculus (9th ed)*. Prentice-Hall.