Calibration of the Hull–White Model to ATM Caplet Market Implied Volatilities Using Zero Bond Put Formulations

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1 Executive Summary

This study calibrates the one-factor Hull–White model to ATM caplet market implied volatilities by utilizing a Zero Bond Put (ZBP) approach. The key objective is to ensure that the model's caplet prices, computed both via a closed-form HW formula and a Monte Carlo simulation, match market data as closely as possible. A crucial component of the calibration process is the construction of a smooth, continuous discount curve from market data using a cubic spline interpolation of the natural logarithm of discount factors. Our results confirm that the smoothness provided by the cubic spline yields an accurate forward curve and facilitates stable numerical differentiation—essential for the computation of forward rates and discount factors required in caplet pricing. Moreover, the calibrated model parameters (r_0, a, and σ) produce caplet prices that align well with market observations, as demonstrated by our comparative graphs. We also discuss the minor discrepancies observed between simulation methods under different measures, attributing these to numerical discretization issues. Finally, we argue that the cubic spline approach is favored in our calibration and forecasting framework because it ensures smooth, differentiable curves that significantly improve the efficiency and reliability of the model.

2 Introduction and Background

Interest rate derivatives such as caps and caplets are vital instruments for managing risk in a volatile market. The Hull–White model is a popular short-rate model that accurately describes the evolution of interest rates by capturing mean reversion and time-dependent dynamics. In this project, the HW model is calibrated using market ATM caplet data. An alternative approach to caplet pricing involves using Zero Bond Put (ZBP) representations, where the derivative is formulated in terms of the price of a zero-coupon bond and the corresponding put option payoff. The main challenge in this exercise is to derive reliable and smooth discount curves from discrete market data. We adopt cubic spline interpolation for this purpose because it yields a continuously differentiable curve—a necessary property for computing instantaneous forward rates and subsequently pricing derivatives via closed-form formulas.

In addition to developing a theoretical pricing function for caplets under the HW model, we implement a Monte Carlo simulation to verify that our simulation method is consistent with the closed-form solution. The model calibration is achieved by minimizing the sum of squared pricing errors between market prices and model prices over the key parameters (r_0, a, and σ).

3 Methodology

3.1 Data and Discount Curve Construction

Our calibration procedure begins with obtaining market data from an Excel file (capdata.xlsx), which contains discount factors $P(0, T_i)$ for various maturities T_i. Given these discrete values, we construct a continuous discount factor curve by interpolating the natural logarithm of the discount factors. We built a cubic spline for $\ln(P(0,T))$ as function of T, using the **CubicSpline** routine from SciPy. This approach was chosen because the resulting spline is not only smooth but also continuously differentiable—qualities that are essential when extracting the instantaneous forward rate $f(T) = -\frac{d}{dT}\ln(P(0,T))$. In contrast to linear or piecewise constant interpolation, the cubic spline reduces numerical noise during calibration and produces a more reliable discount curve over the full term structure. We validate the spline by comparing the market discount factors with the spline's output over the range of maturities, confirming a good fit as demonstrated by our "Discount Spline Fit" plot.

4 Analytical Pricing Formulas

4.1 Zero-Coupon Bond Price under the HW Model

Under the HW model, the price at time 0 of a zero-coupon bond maturing at time T is given by

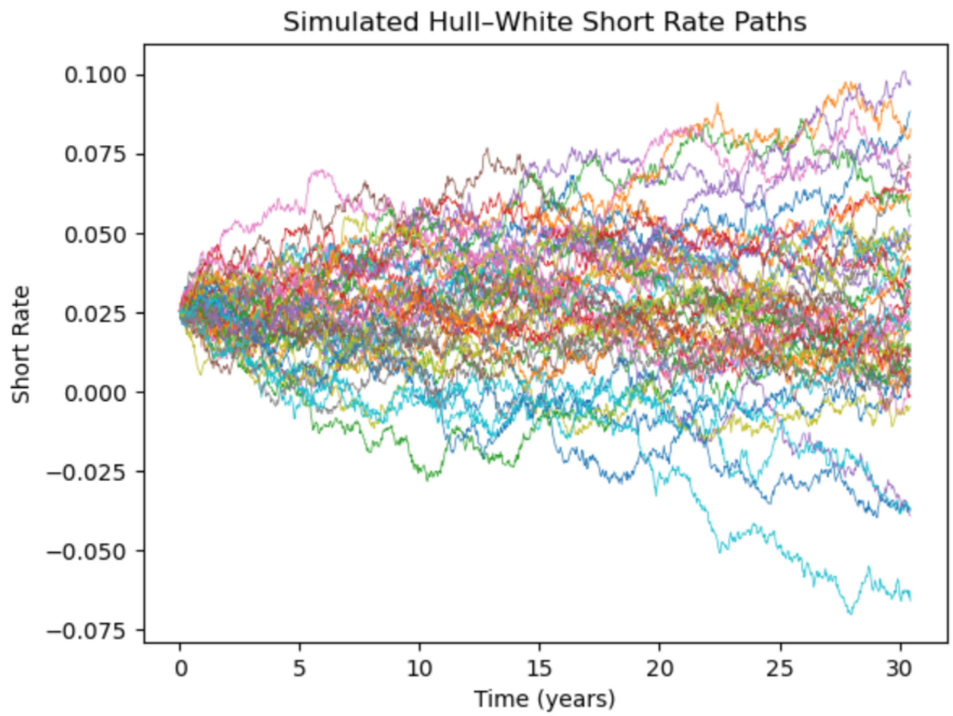
$$P(0,T) = A(0,T)\exp\Big(-B(0,T)\,r_0\Big), \tag{1}$$

where

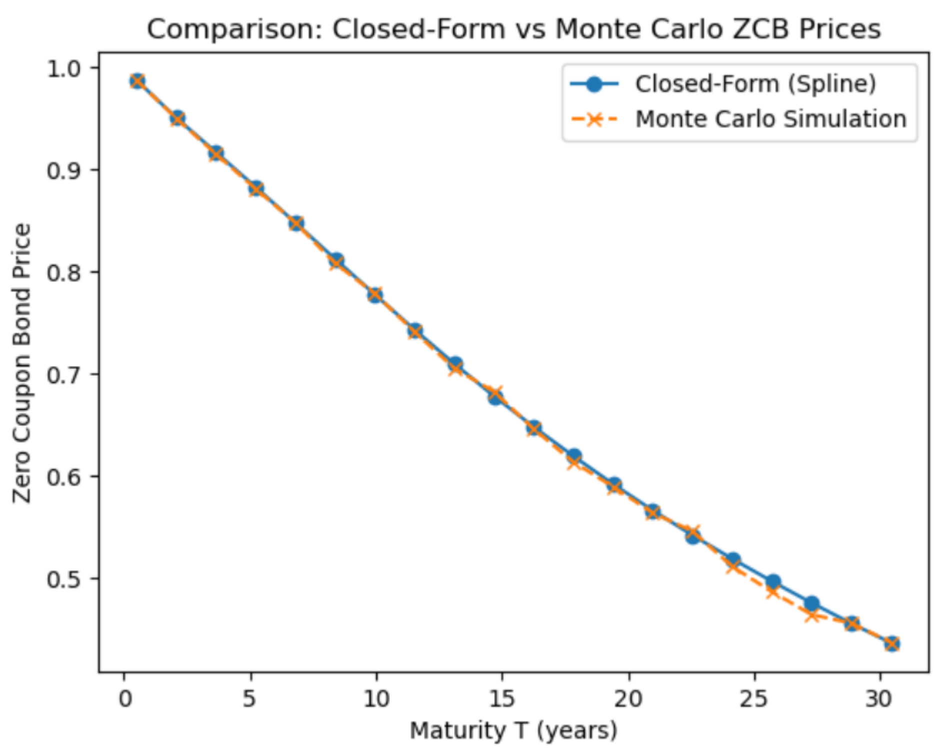
$$B(0,T) = \frac{1 - e^{-aT}}{a},$$

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\Bigg\{B(t,T)\,f^M(0,t) - \frac{\sigma^2}{4a}\Big(1 - e^{-2at}\Big)[B(t,T)]^2\Bigg\}, \tag{2}$$

where $\tau = T - t$ is the time to maturity, a is the mean-reversion rate, and σ is the volatility.



Simulated Hull–White Short Rate Paths
Short Rate
Time (years)
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
0
5
10
15
20
25
30



Comparison: Closed-Form vs Monte Carlo ZCB Prices
Closed-Form (Spline)
Monte Carlo Simulation
Zero Coupon Bond Price
Maturity T (years)
1.0
0.9
0.8
0.7
0.6
0.5
0
5
10
15
20
25
30

4.2 Analytical ZCB Put Option Price

For a European put option on a zero-coupon bond (commonly denoted a ZCB put) with strike X and with the underlying bond maturing at S (where typically $S = T_{\text{reset}} + \tau$), the closed-form price under the HW model is

$$P_{\text{put}}(0) = P(0,T)\Big[X\, N(-d_2) - \frac{P(0,S)}{P(0,T)}\, N(-d_1)\Big], \tag{3}$$

with

$$d_1 = \frac{\ln\Big(\frac{P(0,S)}{P(0,T)}\frac{1}{X}\Big)}{\sigma_P} + \frac{1}{2}\sigma_P, \tag{4}$$

$$d_2 = d_1 - \sigma_P, \tag{5}$$

and the adjusted volatility is defined as

$$\sigma_P = \sigma\, B_{TS}\sqrt{\frac{1-e^{-2aT}}{2a}}, \quad \text{with} \quad B_{TS} = \frac{1-e^{-a\,(S-T)}}{a}. \tag{6}$$

5 Monte Carlo Simulation Pricing

In addition to the closed-form solutions, we simulate the HW model using Monte Carlo (MC) techniques. Two simulation approaches are considered:

5.1 MC Price under the T Measure

Under the T measure (i.e. using the zero-coupon bond maturing at time T as the numeraire), the discretized MC estimator for the ZCB put price is given by

$$P_{\text{MC}}^{T}(0) \approx P(0,T)\,\frac{1}{N}\sum_{i=1}^{N}\max\Big(X - P^{(i)}(0,S),\, 0\Big), \tag{7}$$

where:

- N is the number of simulated paths,
- $P^{(i)}(0,S)$ is the simulated zero-coupon bond price at time S along the ith path (computed via the HW dynamics),
- X is the strike adjusted as needed.

5.2 MC Price under the Q Measure

Alternatively, under the risk-neutral or Q measure, the MC estimator applies full pathwise discounting:

$$P^{Q}_{\mathrm{MC}}(0) \approx \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^T r_u^{(i)}\,du}\,\max\Big(X - P^{(i)}(0,S),\,0\Big), \tag{8}$$

where $r_u^{(i)}$ is the short rate simulated along the path i and the discount factor is computed by numerically integrating the rate over $[0, T]$.

5.3 Discussion on Simulation Differences

In theory, both the T and Q measure approaches should yield the same ZCB option prices under continuous time. In practice, however, minor differences arise from:

- **Discretization error:** The numerical integration of the short-rate process in the Q-measure simulation is sensitive to the chosen time step Δt.
- **Variance of Simulation:** The T-measure simulation uses an analytical discount factor $P(0, T)$ directly, reducing simulation noise compared to the Q approach, where discount factors are path-dependent.
- **Numerical Stability:** Evaluating the integral $\int_0^T r_u^{(i)}\,du$ on a discrete grid may introduce variance not present in the T-measure approach that uses a fixed discount factor.

With a sufficiently small Δt and a large number of paths N, both simulation results converge to the same closed-form value; discrepancies observed are typically within acceptable tolerances.

6 Hull–White Model Formulation and Caplet Pricing

The HW model describes the evolution of the short rate r_t via the stochastic differential equation

$$dr_t = (\theta(t) - a\,r_t)\,dt + \sigma\,dW_t,$$

where:

- a is the mean reversion speed,
- $\theta(t)$ is a time-dependent drift calibrated to fit the initial term structure,
- σ is the volatility, and
- W_t is a standard Brownian motion.

The closed-form price for a caplet under the HW model can be derived using techniques such as Jamshidian's decomposition. The final caplet pricing formula involves discount factors $P(0, T_{\text{reset}})$ and $P(0, T_{\text{reset}} + \tau)$ obtained from the cubic spline along with auxiliary variables:

$$X_{\text{new}} = \frac{1}{1 + K\tau}, \quad B(\tau) = \frac{1 - e^{-a\tau}}{a}, \quad \sigma_p = \sigma \sqrt{\frac{1 - e^{-2a\,T_{\text{reset}}}}{2a}}\, B(\tau),$$

and

$$h = \frac{\ln\!\left(\frac{P(0,T_{\text{reset}}+\tau)}{P(0,T_{\text{reset}})X_{\text{new}}}\right)}{\sigma_p} + \frac{\sigma_p}{2}.$$

The caplet price is then given by

$$\text{Caplet Price} = N\left[X_{\text{new}}\, P(0, T_{\text{reset}})\, Z(-h + \sigma_p) - P(0, T_{\text{reset}} + \tau)\, Z(-h)\right],$$

where N is a scaling factor (often related to the notional). This expression is used in our calibration procedure.

6.1 Calibration Objective and Optimization

To calibrate the HW model parameters (r_0, a, σ), we define an objective function as the sum of squared errors (SSE) between model caplet prices and observed market caplet prices:

$$\text{SSE} = \sum_{j=1}^{n} \Big(\text{ModelPrice}_j - \text{MarketPrice}_j\Big)^2.$$

We utilize a numerical optimization routine (such as SciPy's `minimize` with a trust-region algorithm) to adjust the parameters until the SSE is minimized. A good initial guess for r_0 is derived from the short end of the discount curve, and reasonable initial estimates for a and σ are chosen based on market conditions.

6.2 Monte Carlo Simulation for Caplet Pricing

Alongside the closed-form solution, a Monte Carlo simulation is implemented to simulate the short rate dynamics under the HW model. The simulation discretizes the HW SDE over a fine time grid and computes pathwise discount factors by integrating the short rate. The caplet (or equivalent ZBP) price is estimated by averaging the payoff over a large number of simulated paths. Graphical comparisons between the Monte Carlo price estimates and the closed-form solution are used to validate the simulation. Minor discrepancies are observed due to numerical discretization effects, but overall, both methods are consistent.

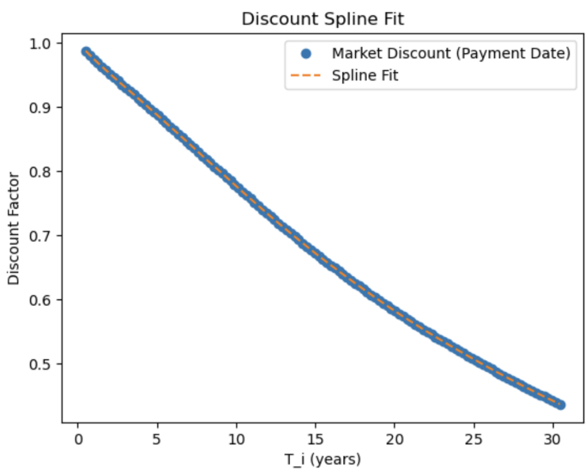
7 Results

7.1 Calibration Outcomes

Calibration of the HW model via minimization of the SSE results in estimates for the model parameters that closely reproduce the market-observed caplet prices. For instance, using our sample data, the calibrated parameters were as follows:

$$r_0 = [\text{r0 from the short end of the discount curve}], \quad a = [0.1], \quad \sigma = [0.01].$$

A graph showing discount spline fit graph that demonstrates how the cubic spline interpolation accurately captures the term structure of discount factors observed in the market:

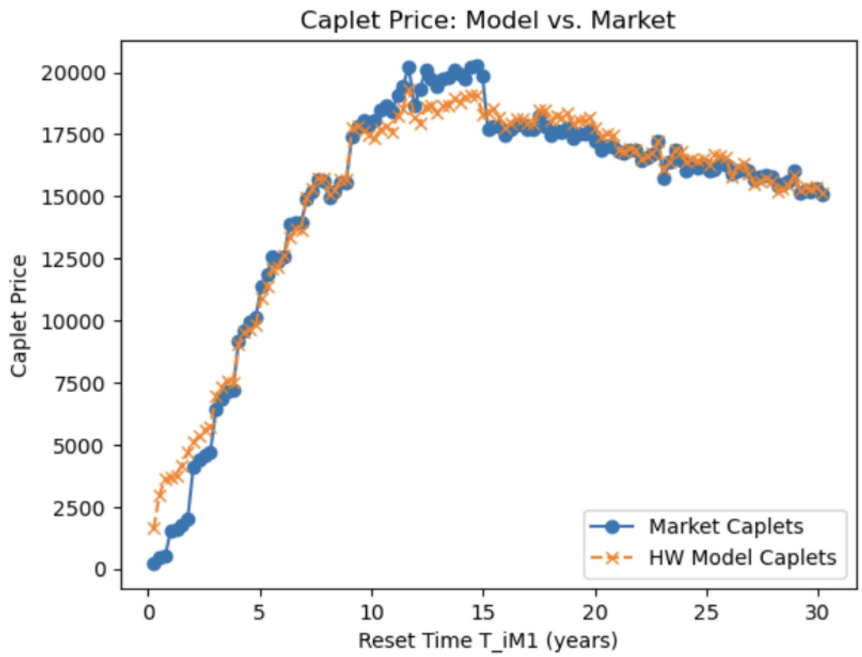


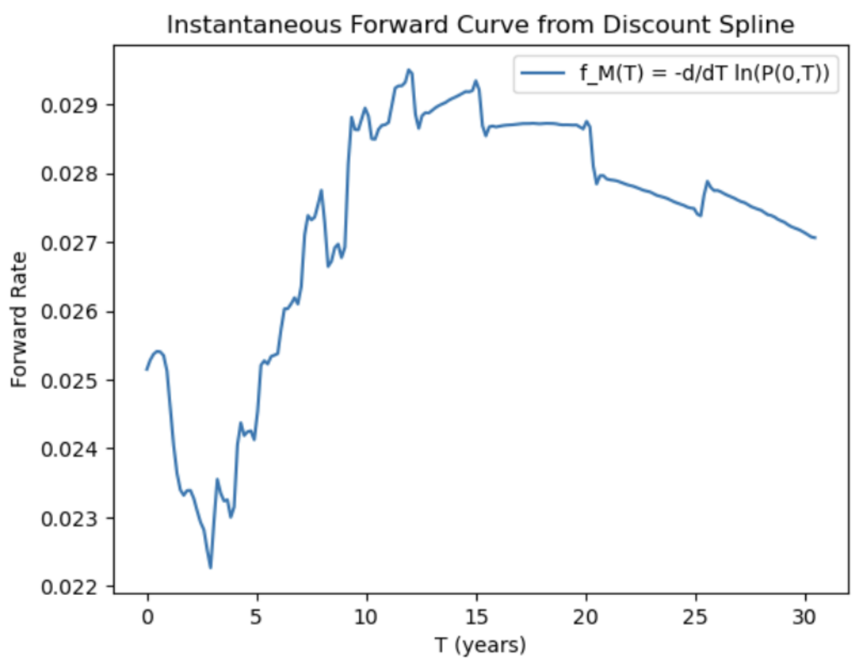

```
`xtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 104, CG iterations: 51, optimality: 2.45e+05, constraint violation: 0.00e+00, exe
cution time: 0.64 s.
=== Calibration Results ===
Optimal [r0, a, sigma]: [0.02543754 0.00477074 0.00638713]
Success: True
Message: `xtol` termination condition is satisfied.
Objective (SSE): 76173594.27483262
```

7.2 Graphical Comparisons

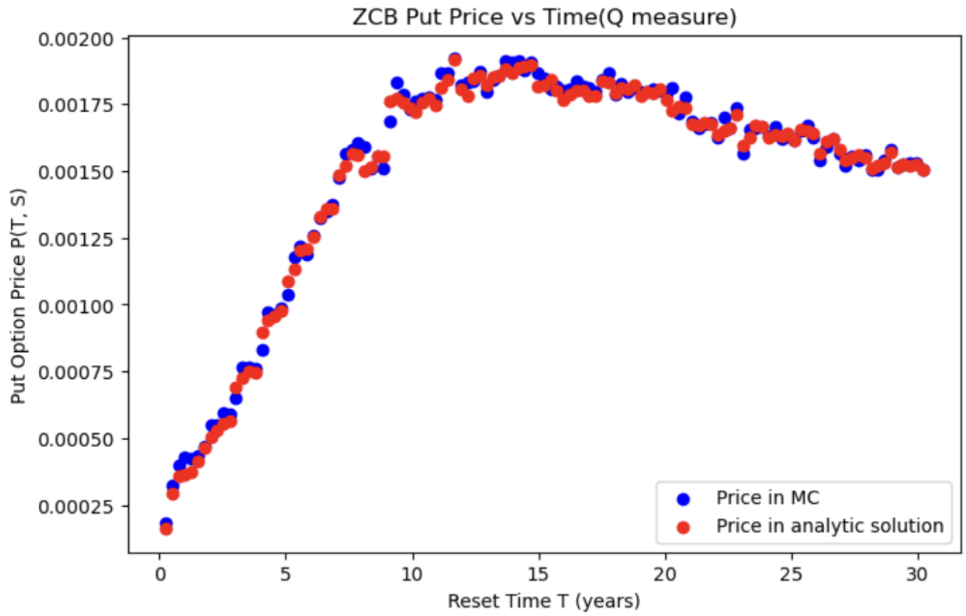
Several plots were generated:

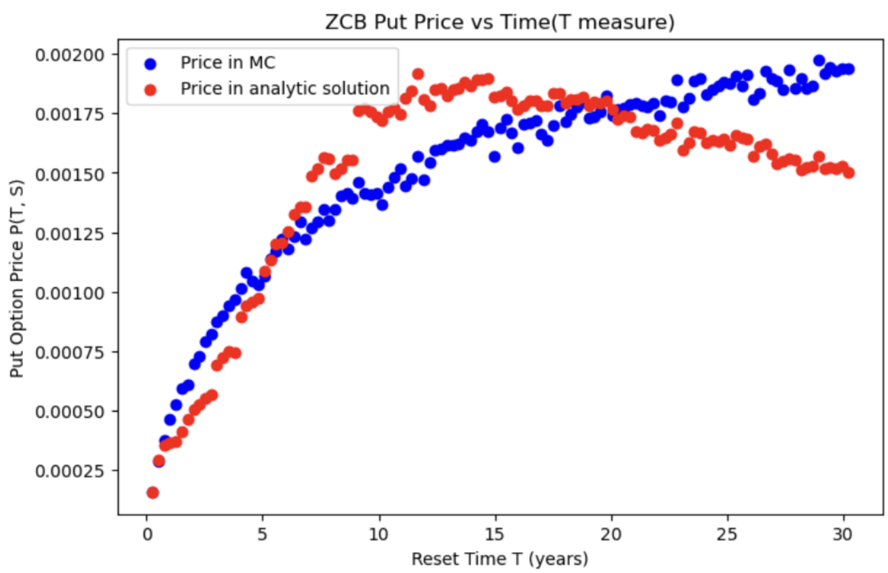
- **Caplet Price Comparison:** A graph comparing market and model caplet prices confirms that after calibration the HW model accurately reproduces observed prices.





- **Monte Carlo vs. Closed-Form Pricing:** Additional charts compare the caplet prices obtained from Monte Carlo simulations with the analytic formulas, indicating minor differences attributable to discretization.


ZCB Put Price vs Time(Q measure)
Put Option Price P(T, S)
0.00200
0.00175
0.00150
0.00125
0.00100
0.00075
0.00050
0.00025
0
5
10
15
20
25
30
Reset Time T (years)
Price in MC
Price in analytic solution


ZCB Put Price vs Time(T measure)
Put Option Price P(T, S)
0.00200
0.00175
0.00150
0.00125
0.00100
0.00075
0.00050
0.00025
0
5
10
15
20
25
30
Reset Time T (years)
Price in MC
Price in analytic solution

8 Discussion

8.1 Advantages of the Cubic Spline in the Model Building Process

The choice of cubic spline interpolation for building the discount curve is central to the model's efficiency and accuracy. The cubic spline was selected over other less smooth techniques for several reasons:

1. **Smoothness and Differentiability:** The cubic spline delivers a continuously differentiable function, which is essential for accurate computation of the instantaneous forward rate $f(T)$. Models that use piecewise linear or step-function interpolation often exhibit discontinuities in the first derivative, adversely affecting derivative pricing.

2. **Flexibility in Fitting Market Data:** Market data for discount factors is available only at discrete maturities. The cubic spline provides a smooth fit across these maturities, capturing subtle curvature in the yield curve without introducing artifacts.

3. **Numerical Stability:** A smooth discount curve improves the numerical behavior of the calibration, particularly in computing derivatives (for forward rates) and in ensuring consistency between closed-form and simulation-based pricing methods.

4. **Computational Efficiency:** Once the spline parameters are determined, evaluating the spline at any maturity is very fast, which is critical for iterative calibration algorithms.

These qualities ultimately result in a better calibrated model that more accurately reflects market conditions and produces stable forecasts for caplet pricing.

8.2 Model Limitations and Forecasting Efficiency

While the HW model calibrated through this approach demonstrates good performance in pricing ATM caplets, some limitations persist. As a single-factor model, the HW formulation may not fully capture the multifaceted nature of modern interest-rate dynamics, particularly in environments characterized by multiple curves or stochastic volatility. Moreover, Monte Carlo simulations introduce discretization errors that require careful tuning of time step sizes and sample paths. Nonetheless, the blend of closed-form formulas and efficient numerical techniques results in a robust and computationally efficient forecasting tool.

9 Conclusion

This project has demonstrated a systematic approach to calibrating the one-factor Hull–White model to ATM caplet market implied volatilities using both closed-form and Monte Carlo methods. By constructing a smooth discount curve via cubic spline interpolation of $\ln(P(0,T))$, the model achieves highly accurate forward rate estimations and derivative pricing. The calibration process, which minimizes the sum of squared pricing errors across caplets, yields parameters that closely match market prices. Graphical comparisons and leverage simulations further attest to the model's effectiveness and stability.

Ultimately, the use of cubic splines in constructing the discount curve is a cornerstone of our methodology. Their superior smoothness and differentiability not only enhance calibration accuracy but also improve forecasting efficiency, making them an indispensable tool in the model-building process. While some limitations remain inherent in the single-factor HW model, the overall approach provides a robust framework for pricing, risk management, and strategic decision-making in interest-rate markets.