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Port-Hamiltonian Formulation of Rigid-Body Attitude Control

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Abstract: The aim of this paper is to present a port-Hamiltonian (pH) formulation of the rigid-body attitude control problem, therefore enhancing the set of available tools for its modeling and control. First, a pH formulation of both dynamics and kinematics equations is presented. Second, a standard energy-balancing passivity-based controller (EB-PBC) is used for set-point tracking. Third, the controlled system is endowed with a dynamical extension to achieve set-point tracking without measuring the angular velocities. As a conclusive remark, it is showed under specific assumptions that these three results can be achieved regardless the coordinate representation in use. Additional examples follow to motivate the adoption of the pH formulation.

Keywords: Attitude control, port-Hamiltonian systems, nonlinear control systems

1. INTRODUCTION

The attitude control problem, see e.g., (Bullo and Lewis, 2004; Chaturvedi et al., 2011), is the well-known problem of asymptotically stabilizing a rigid spacecraft about a desired reference orientation using its thrusters. The dynamics of the velocities of a rigid body free to thumble in space are described by Euler's equations of motion. Together with the velocities, it is also required to stabilize the attitude kinematics equations, namely the relation between the velocities and the attitude rates of changes in the chosen attitude parametrization. The literature is vast and spans many decades and applications in aerospace, underwater vehicles, and robotic systems.

In this paper, we approach the attitude control problem from a port-Hamiltonian perspective. Apart from offering a systematic and insightful framework for modelling and analysis of multi-physics systems, port-Hamiltonian systems theory provides a natural starting point for control; see e.g., (van der Schaft and Jeltsema, 2014), (Duindam et al., 2009), and the references therein. Especially in the nonlinear case it is widely recognised that physical properties of the system—such as balance and conservation laws and energy considerations—should be exploited and respected in the design of control laws which are robust and physically interpretable.

The contributions of the paper are outlined as follows. In Section 2, first some background on port-Hamiltonian modeling is provided. Next, is is shown how the well-known dynamical equations of the rigid body can be extended to include the kinematics. Section 3 starts by briefly recalling the essence of passivity-based control (PBC) and subsequently shows how to naturally re-derive the attitude controller proposed by (Bullo and Lewis, 2004). A major drawback of this controller is the necessity of velocity measurements. To overcome this issue, the dynamic extension approach for position-only feedback of (Dirksz et al., 2008) is exploited in Section 4. In Section, it is show that, under specific assumptions, it is possible to achieve the pH formulation, the standard controller, and the extended controller, regardless the choice of the attitude parameterization.

Notation: The 'cross' map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined as

$$v^\times = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^\times := \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. \tag{1}$$

The 'vee' map $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ denotes the inverse operation of 'cross', namely $(a^\times)^\vee = a$. The gradient of a scalar function is denoted with the operator $\nabla_x := \partial/\partial x$. All vectors are considered as column vectors, including the gradient of a scalar function. Hence, the Jacobian of a vector function $h : \mathbb{R}^n \to \mathbb{R}^m$ is denoted as

$$\mathrm{Jac}_h(x) := \begin{bmatrix} \nabla_x^\top h_1(x) \\ \vdots \\ \nabla_x^\top h_m(x) \end{bmatrix}. \tag{2}$$

The notation $\mathrm{tr}(A)$ is used to represent the trace of a square matrix A.

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2. PORT-HAMILTONIAN FORMULATION

2.1 Background on Port-Hamiltonian Systems

Port-Hamiltonian (pH) systems can be seen as the extension of the classical Hamiltonian equations of motion for mechanical systems to a more general category of systems, namely systems arising from the network modeling of physical systems. Fundamental in their formalization is the geometric notion of Dirac structure (Duindam et al. (2009)). In this paper, we consider the class of so-called input-state-output pH system with dissipation of the form:

$$\Sigma : \begin{cases} \dot{x} = [J(x) - D(x)]\nabla_x \mathcal{H}(x) + g(x)u, \\ y = g^\top(x)\nabla_x \mathcal{H}(x), \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ is the corresponding output, $J(x) = -J^\top(x)$ is a skew-symmetric interconnection matrix, $D(x) = D^\top(x) \succeq 0$ a dissipation matrix, and $\mathcal{H}(x)$ the Hamiltonian.

Differentiating the Hamiltonian along the trajectories of (3), we recover the fundamental power-balance

$$\dot{\mathcal{H}} = -\nabla_x^\top \mathcal{H}(x) D(x) \nabla_x \mathcal{H}(x) + y^\top u \leq y^\top u. \tag{4}$$

Hence, pH systems (3) are passive under the additional assumption that $\mathcal{H}(x)$ is bounded from below. Specifically, (3) is conservative (lossless) when $D(x) = 0$.

2.2 PH formulation for Attitude Control

In this subsection, a pH formulation of both the attitude and angular velocities of a rigid-body is presented. The attitude is represented by rotation matrices, namely matrices within the special orthogonal group $\mathcal{SO}(3) = \{R \in \mathbb{R}^{3\times3} \mid RR^\top = I, \det R = 1\}$. A generalization over coordinate representation will be given in Section 5.

Choose an inertial reference frame and a body-fixed frame. Let $R \in \mathcal{SO}(3)$ denote the rotation matrix from the body-fixed frame to the inertial frame, thus describing the attitude of the rigid body with respect to a non-rotating object. Let $\omega \in \mathbb{R}^3$, $\mathcal{I} := \operatorname{diag}(\mathcal{I}_x, \mathcal{I}_y, \mathcal{I}_z) \succ 0$, and $u \in \mathbb{R}^3$ denote the angular velocity vector, the inertia matrix, and the applied control torques of the rigid body with respect to the body-fixed frame, respectively. The equations of motion then read as:

$$\dot{R} = R\omega^\times \quad \text{(kinematics equation)}, \tag{5a}$$

$$\mathcal{I}\dot{\omega} + \omega^\times \mathcal{I}\omega = u \quad \text{(dynamics equation)}. \tag{5b}$$

A pH formulation of the dynamics (5b) has already been presented in the literature (see for instance, Example 3.1.1 in van der Schaft and Jeltsema (2014)) and is given by:

$$\dot{p} = p^\times \nabla_p \mathcal{H}(p) + u, \tag{6}$$

where $p := \mathcal{I}\omega$ denotes the vector of the angular momenta and the Hamiltonian

$$\mathcal{H}(q, p) := \frac{1}{2} p^\top \mathcal{I}^{-1} p \tag{7}$$

corresponds to the total kinetic energy of the rotating rigid-body. Assume now that the elements of matrix R are defined as follows:

$$R := \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}, \tag{8}$$

where R_x, R_y, and R_z denote first, second, and third row of R, respectively. To the end of formulating both equations (5), the displacement vector $q \in \mathbb{R}^9$ is introduced as the flattened version of R, namely

$$q := \operatorname{vec}\{R^\top\} = [R_x \ R_y \ R_z]^\top. \tag{9}$$

Let $r(q) : \mathbb{R}^9 \to \mathbb{R}^{9\times3}$ be the matrix

$$r(q) := \begin{bmatrix} R_x^\times \\ R_y^\times \\ R_z^\times \end{bmatrix}. \tag{10}$$

With the previously introduced notation, the pH system corresponding to equations (5) is then given as

$$\Sigma_A : \underbrace{\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & r(q) \\ -r^\top(q) & p^\times \end{bmatrix}}_{J(x)} \underbrace{\begin{bmatrix} \nabla_q \mathcal{H}(q,p) \\ \nabla_p \mathcal{H}(q,p) \end{bmatrix}}_{\nabla_x \mathcal{H}(x)} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{g(x)} u, \tag{11}$$

with $D(x) = 0$ and passive outputs $y = \nabla_p \mathcal{H}(q,p) = \omega$.

3. STANDARD CONTROLLER

3.1 Background on Passivity-Based Control

Passivity-based control (PBC) is a control methodology that achieves the control objective by rendering a system passive with respect to a desired storage function (Ortega et al. (2008) and Ortega et al. (2001)). The standard assumption in PBC is that, for a general nonlinear system $\dot{x} = f(x,u), y = h(x)$, the energy-balance equation

$$\mathcal{H}\big(x(t)\big) - \mathcal{H}\big(x(0)\big) = \int_0^t u^\top(s) y(s) ds - D(t)$$

is satisfied with $\mathcal{H}(x)$ bounded from below and $D(t)$ being a nonnegative function capturing the dissipation effects. Generally, the point where the open-loop energy $\mathcal{H}(x)$ is minimal is usually not the one of practical interest. Therefore the set-point stabilization objective requires the shaping of the energy function into a new desired one $\mathcal{H}_d(x)$ by means of a state-feedback $u = \beta(x) + v$. In fact, if $\mathcal{H}_d(x)$ has a strict global minimum at x^*, Lyapunov stability of x^* can be proven when $v = 0$. Assigning an energy function with a minimum at the desired equilibrium is usually referred to as *energy shaping*, whereas the modification of the dissipation is called *damping injection* that is achieved by feeding back the passive outputs, i.e.,

$$v = -C(x)y, \tag{12}$$

with $C(x) \succeq \varepsilon I$ for all x and some $\varepsilon > 0$. Damping injection is usually required to infer asymptotic stability under relatively mild additional assumptions (i.e., zero-state detectability [1]).

Energy-balancing (EB-PBC) is a methodology in which the closed-loop energy $\mathcal{H}_d$ is equal to the sum of the energy $\mathcal{H}$ stored in the plant and the energy $\mathcal{H}_{ref}$ supplied by the controller, namely $\mathcal{H}_d := \mathcal{H} + \mathcal{H}_{ref}$. For the pH system (3), EB-PBC aims to find a state-feedback $u = \beta(x) + v$ such that the closed-loop system still retains its pH form:

$$\Sigma_d : \begin{cases} \dot{x} = [J(x) - D(x)]\nabla_x \mathcal{H}_d(x) + g(x)v, \\ y = g^\top(x)\nabla_x \mathcal{H}_d(x), \end{cases} \tag{13}$$

where $\mathcal{H}_d$ has a strict local minimum at x^*, thus rendering (13) Lyapunov stable by exploiting the passivity property

[1] Recall that a system Σ is said to be *zero-state detectable* if $u(t) \equiv 0$ and $y(t) \equiv 0$, for all $t \geq 0$, implies that $\lim_{t\to\infty} x(t) = 0$.

(4). The function $\beta(x)$ achieving this control objective is the solution of the PDE

$$[J(x) - D(x)]\,\nabla_x \mathcal{H}_{ref}(x) = g(x)\beta(x). \tag{14}$$

Additionally, if the closed-loop system (13) is fully damped (for instance by injecting damping as in (12)) and is zero-state detectable, then the equilibrium x^* is locally asymptotically stable. Under the additional assumption of $\mathcal{H}_d(x)$ being radially unbounded (i.e., $\mathcal{H}_d(x) \to \infty$ as $\|x\| \to \infty$), the equilibrium x^* becomes globally asymptotically stable.

3.2 EB-PBC for Attitude Control

Consider the system (5) and its pH formulation (11). Let $R_{ref} \in \mathcal{SO}(3)$ denote the desired attitude configuration for the rigid body as follows

$$R_{ref} := \begin{bmatrix} R_{ref,x} \\ R_{ref,y} \\ R_{ref,z} \end{bmatrix},$$

where $R_{ref,x}, R_{ref,y}$, and $R_{ref,z}$ denote the first, second, and third row of R_{ref}, respectively. The first step of the EB-PBC is to shape the closed-loop energy

$$\mathcal{H}_d(q,p) = \mathcal{H}(q,p) + \mathcal{H}_{ref}(q) \tag{15}$$

such that it has a minimum in $(R^*, p^*) = (R_{ref}, 0)$. Following (Bullo and Lewis, 2004), this is achieved by selecting

$$\mathcal{H}_{ref}(q) := \frac{1}{2}\operatorname{tr}\left[K_p\left(I - R_{ref}^\top R\right)\right], \tag{16}$$

with controller gain $K_p := \operatorname{diag}(k_{px}, k_{py}, k_{pz}) \succ 0$ tuning the steepness of the energy function $\mathcal{H}_{ref}(q)$. Solving the PDE (14) gives the energy-shaping control input:

$$\begin{aligned} u_{es} &= -r^\top(q)\nabla_q \mathcal{H}_{ref}(q) \\ &= -\frac{1}{2}\left[K_p R_{ref}^\top R - R^\top R_{ref} K_p^\top\right]^\vee . \end{aligned} \tag{17}$$

By selecting the damping-injection control input as

$$u_{di} = -K_d \mathcal{I}^{-1} p,$$

with controller gain $K_d := \operatorname{diag}(k_{dx}, k_{dy}, k_{dz}) \succ 0$, the total applied control torque becomes $u = u_{es} + u_{di}$, which, in turn, yields the pH closed-loop system:

$$\Sigma_{A_d} : \begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & r(q) \\ -r^\top(q) & p^\times - K_d \end{bmatrix} \begin{bmatrix} \nabla_q \mathcal{H}_d(q,p) \\ \nabla_p \mathcal{H}_d(q,p) \end{bmatrix}. \tag{18}$$

4. EXTENDED CONTROLLER

4.1 Background on PBC with Only Position Measurements

In (Dirksz et al., 2008), a dynamic extension for position feedback of pH mechanical systems is studied to avoid the injection of damping by means of velocity measurement, thus overcoming the degradation of the derivative action on noisy signals. The underlying idea of (Dirksz et al., 2008) is to make the energy bounce between an Euler-Lagrange (EL) system—the plant—and a virtual EL system—the controller—and to dissipate this energy only when the controller retains it. In this way, dissipation is propagated to the plant.

Let us consider a pH mechanical system Σ as (3). For ease of presentation, it is assumed here that system Σ has been brought to the form Σ_d (13) with desired Hamiltonian $\mathcal{H}_d(x) = \frac{1}{2}p^\top M(q)^{-1}p + V_d(q)$. Note that in (13) the dissipation matrix $D(x)$ is zero due to the prohibition of feeding back the velocities by means of standard damping injection techniques. A virtual mechanical systems Σ_c in the coordinates $x_c = [q_c^\top, p_c^\top]^\top$ is introduced, as follows:

$$\dot x_c = (J_c(x_c) - D_c(x_c))\,\nabla_{x_c}\mathcal{H}_c(x_c), \tag{19}$$

with interconnection matrix $J_c(x_c) = -J_c^\top(x_c)$, damping matrix $D_c(x_c) = D_c^\top(x_c) \succeq 0$, and Hamiltonian $\mathcal{H}_c(x_c) = \frac{1}{2}p_c^\top M_c(q_c)^{-1}p_c + V_c(q_c)$, where the mass matrix $M_c(q_c)$ and the potential energy term $V_c(q_c)$ have to be defined. The two systems Σ_d and Σ_c are coupled together by means a mixed potential energy term $\tilde{\mathcal{V}}_d(q, q_c)$, as in the following. The extended closed-loop system reads as:

$$\tilde\Sigma_d : \begin{bmatrix} \dot x \\ \dot x_c \end{bmatrix} = \begin{bmatrix} J(x) & 0 \\ 0 & J_c(x_c) - D_c(x_c) \end{bmatrix} \begin{bmatrix} \nabla_x \tilde{\mathcal{H}}_d \\ \nabla_{x_c}\tilde{\mathcal{H}}_d \end{bmatrix}, \tag{20}$$

with $\tilde{\mathcal{H}}_d(x, x_c) = \frac{1}{2}p^\top M^{-1}(q)\,p + \frac{1}{2}p_c^\top M_c^{-1}(q_c)\,p + \tilde{\mathcal{V}}_d(q, q_c)$. Note that the mixed energy term $\tilde{\mathcal{V}}_d(q, q_c)$ accounts for the reshaping of $V_d(q)$ and $V_c(q_c)$, thus for the coupling between Σ_d and Σ_c. Define the equilibrium as $(x^\star, x^\star) = (q_d, 0, q_{cd}, 0)$ with q_d, q_{cd} being the desired reference positions for (q, q_c), respectively. The conditions for the closed-loop system $\tilde\Sigma_d$ to achieve asymptotic stability of $(x^\star, x^\star)$ are reported in the following theorem.

Theorem 1. (Dirksz et al., 2008, Proposition 1) If

- (C1) the Hamiltonian $\tilde{\mathcal{H}}_d(q, p, q_c, p_c)$ has its strict minimum at $(q_d, 0, q_{cd}, 0)$ and hence satisfies
$$\nabla_q \tilde{\mathcal{H}}_d(q_d, 0, q_{cd}, 0) = 0;$$
- (C2) the PDE matching condition
$$g^\top(x)J(x)\left[\nabla_x \tilde{\mathcal{H}}_d(x, x_c) - \nabla_x \mathcal{H}(x)\right] = 0, \tag{21}$$
is fulfilled;
- (C3) for $\nabla_{q_c}\tilde{\mathcal{V}}_d(q, q_{cd}) = 0$ it holds that q is constant,

then the closed-loop system (20) is asymptotically stable, and the energy-shaping input takes the form:

$$u_{es} = a(x)J(x)\left[\nabla_x \tilde{\mathcal{H}}_d(x, x_c) - \nabla_x \mathcal{H}(x)\right], \tag{22}$$

with $a(x) := \left(g^\top(x)g(x)\right)^{-1} g^\top(x)$.

4.2 Attitude Control with only position measurements

Following the ideas provided in Subsection 4.1, a dynamic extension for (11) is designed in order to stabilize the attitude of the rigid body without directly using velocity measurements. To this end, assume that the energy shaping control input (17) has been applied to (11) to yield the closed-loop system (18) with $K_d = 0$. Consider now the virtual mechanical system (19) as another rotating rigid body with position coordinates q_c and momenta p_c. The coordinates $q_c \in \mathbb{R}^9$ are assumed to be the flattened version of a rotation matrix $R_c \in \mathcal{SO}(3)$ in a similar fashion as in Subsection 2.2 for q and R. This virtual mechanical system and system Σ_{A_d} are coupled together as in (20) through a new desired Hamiltonian $\tilde{\mathcal{H}}_d$. By selecting $\tilde{\mathcal{H}}_d$ as a function of (q, p, q_c) only, the dynamics of p_c are automatically discarded, thus further simplifying the design. Moreover, by setting $J_c(q_c) = 0$, the following closed-loop system is obtained

$$\begin{bmatrix} \dot x \\ \dot q_c \end{bmatrix} = \begin{bmatrix} J(x) & 0 \\ 0 & -D_c(q_c) \end{bmatrix} \begin{bmatrix} \nabla_x \tilde{\mathcal{H}}_d(x, q_c) \\ \nabla_{q_c}\tilde{\mathcal{H}}_d(x, q_c) \end{bmatrix}, \tag{23}$$

where $x := (q^\top, p^\top)^\top$ is the state of Σ_{A_d} and $D_c(q_c) \succeq 0$ is specified later in this Subsection. We are now going to properly design $\tilde{\mathcal{H}}_d(x, q_c)$ and $D_c(q_c)$ in order to verify the conditions of Theorem 1, thus concluding the asymptotic stability of the controller. (C1) Define

$$\tilde{\mathcal{H}}_d(q, p, q_c) := \underbrace{\mathcal{H}(q, p) + \mathcal{H}_{ref}(q)}_{\mathcal{H}_d(q,p)} + \mathcal{H}_c(q, q_c), \tag{24}$$

with $\mathcal{H}_{ref}(q, p)$ as in (16) and

$$\mathcal{H}_c(q, q_c) := \frac{1}{2} \operatorname{tr}\left[K_c \left(I - R_c^\top R\right)\right], \tag{25}$$

with $K_c = \operatorname{diag}(k_{cx}, k_{cy}, k_{cz}) \succ 0$. Note that $\tilde{\mathcal{H}}_d(q, p, q_c)$ has a strict local minimum at the equilibrium:

$$(R^*, p^*, R_c^*) = (R_d, 0, R_d). \tag{26}$$

(C2) Matching condition (21) indeed holds true and yields the control input (22)

$$u = -\frac{1}{2}\big[K_p R_d^\top R - R^\top R_d K_p^\top + K_c R_c^\top R - R^\top R_c K_c^\top\big]^\vee. \tag{27}$$

Condition (C3) cannot be met because, due to $R \in \mathcal{SO}(3)$,

$$\nabla_{q_c}\mathcal{H}_c(q, q_c) = [k_{cx}R_x \;\; k_{cy}R_y \;\; k_{cz}R_z]^\top$$

cannot simultaneously have all zero components. However, we remark that (C3) is a LaSalle condition ensuring that all trajectories of (20) are trapped into the largest invariant set $\mathbb{M}$ that contains the equilibrium $(q_d, 0, q_{cd})$ only. To overcome this issue, we first consider the time-derivative of (24) along the trajectories of (23). i.e.,

$$\dot{\tilde{\mathcal{H}}}_d = -\nabla_{q_c}^\top \mathcal{H}_c(q, q_c)\, D_c(q_c) \nabla_{q_c}\mathcal{H}_c(q, q_c),$$

and then we select $D_c(q_c)$ in such a way that

- $D_c(q_c)\, \nabla_{q_c}\mathcal{H}_c(q, q_c) \equiv 0$ for all t implies that $\mathbb{M}$ contains $(q_d, 0, q_{cd})$ only;
- $R(t) \in \mathcal{SO}(3)$ for all t along integration.

Both items are automatically satisfied by selecting:

$$D_c(q_c) := r(q_c)\, K_d\, r^\top(q_c) \succ 0.$$

Consequently, the dynamics of q_c becomes

$$\dot{q}_c = -r(q_c) \left(K_d \frac{1}{2}\left[K_c R^\top R_c - R_c^\top R K_c^\top\right]^\vee \right). \tag{28}$$

In Subsection 2.2, $\dot{R} = R\omega^\times$ has been recasted into $\dot{q} = r(q)\omega$. By using the converse procedure, the dynamics (28) turn into

$$\dot{R}_c = \frac{1}{2} R_c \left[K_d \left(K_c R^\top R_c - R_c^\top R K_c^\top\right]^\vee \right)^\times. \tag{29}$$

Finally, the form of the extended controller is given by the control input (27) and the dynamic extension (29).

5. GENERALIZATION OVER COORDINATE REPRESENTATION

A number of coordinate representations, like for instance roll-pitch-yaw (RPY) angles, quaternions, and rotation matrices, serve as parameterizations of the attitude in a three dimensional space. [2] In this Section, we show that, under specific assumptions, it is possible to achieve the presented pH formulation, the standard controller, and the extended controller, regardless the choice of the attitude parameterization.

The following lemma highlights an interesting property of all attitude parameterizations of $\mathcal{SO}(3)$, which will be used in the main result of this section. Specifically, we will show that the kinematics equation is linear in the angular velocity vector ω, regardless of the attitude parameterization in use.

Lemma 2. For any attitude parameterization $q \in \mathbb{R}^{n_q}$ and angular velocity trajectory $\omega(t)$ with $t \in (a, b)$, it is always possible to write:

$$\dot{q}(t) = r(q(t))\omega(t), \tag{30}$$

for some matrix $r(q) : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times 3}$, with $q(t)$ being the attitude trajectory.

Proof. This result stems from two facts: (i) Since $\mathcal{SO}(3)$ is a smooth manifold, any attitude parameterization can be seen as a smooth *coordinate function* $\varphi : \mathcal{U} \to V$ with *coordinate chart* $\mathcal{U} \subset \mathcal{SO}(3)$ and $V \subseteq \mathbb{R}^{n_q}$; (ii) tangent spaces on a smooth manifold relate linearly via a Jacobian matrix. Let $R : (a, b) \to \mathcal{SO}(3)$ be a differentiable attitude trajectory with t varying in (a, b). Consider $f_0 : \mathcal{SO}(3) \to \mathbb{R}^9$ as the flattening map $f_0(\cdot) = \operatorname{vec}\{\cdot\}$. Let $f : \mathcal{SO}(3) \to \mathbb{R}^{n_q}$ be the function mapping R to coordinates q, i.e. $f(R(t)) := q(t)$. By (Baker, 2002, Lemma 7.6) and recalling that $\frac{d}{dt}\operatorname{vec}\{R\} = r(R(t))\omega(t)$, with $r(\cdot)$ as in (10) and (11), it holds that

$$\dot{q}(t) = \operatorname{Jac}_{f f_0^{-1}}(q(t))\; r(R(t))\; \omega(t).$$

With a slight abuse of notation, we can now re-define $r : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times 3}$ as follows

$$r(q) := \operatorname{Jac}_{f f_0^{-1}}(q)\, r\left(f^{-1}(q)\right), \tag{31}$$

thus proving the lemma. Q.E.D.

Now assume that the attitude of the rigid body is parametrized using rotation coordinates $q \in V \subseteq \mathbb{R}^{n_q}$, where a coordinate chart $\mathcal{U}$ and coordinate map f such that $f : \mathcal{U} \to V$ have been defined. Let $q_{ref} \in \mathbb{R}^{n_q}$ denote the desired orientation for the rigid body, with the same parameterization of q. The main result can then be formalized as follows.

Theorem 3. If, for the given attitude parametrization $q \in V$, there exists an energy function $\Psi(q, q_{ref})$ such that

- $\Psi(q, q_{ref})$ is defined for $q \in \mathcal{Q}_\Psi$ with $\mathcal{Q}_\Psi$ being a neighbourhood of q_{ref};
- $\Psi(q, q_{ref})$ is strictly positive-definite in $\mathcal{Q}_\Psi$ with Ψ being equal to 0 only at $q = q_{ref}$,

then the following statements hold for (6)-(30):

- (T1) it can be formulated as a pH system of the form (3) with $D(x) = 0$;
- (T2) there exists a EB-PBC controller of the form
$$u = -r^\top(q)\, \nabla_q \Psi(q, q_{ref}) - K_d\omega, \tag{32}$$
with $K_d = K_d^\top \succ 0$ a positive definite gain, which renders the equilibrium $(q^*, p^*) = (q_{ref}, 0)$ asymptotically stable in $(\mathcal{Q}_\Psi \cap \mathcal{Q}_r) \times \mathbb{R}^3$;
- (T3) there exists a dynamic controller of the form
$$\dot{q}_c = -r(q_c) K_d\, r^\top(q_c) \nabla_{q_c}\Psi(q, q_c), \tag{33a}$$
$$u = -r^\top(q)\, \left[\nabla_q\Psi(q, q_{ref}) + \nabla_q\Psi(q, q_c)\right], \tag{33b}$$

[2] The reader is referred to (Chaturvedi et al., 2011) for a comprehensive introduction on attitude parameterizations.

which does not use p as an input and renders the equilibrium

$$(q^*, p^*, q_c^*) = (q_{ref}, 0, q_{ref}) \tag{34}$$

asymptotically stable in $(\mathcal{Q}_\Psi \cap \mathcal{Q}_r) \times \mathbb{R}^3 \times (\mathcal{Q}_\Psi \cap \mathcal{Q}_r)$.

Proof. (T1) PH formulation (3) follows immediately by selecting the state, the interconnection and input matrices as in (11), with $q \in V$ and $r(q)$ as given by Lemma 2. (T2) Let the closed-loop Hamiltonian be $\mathcal{H}_d(x) := \mathcal{H}(p) + \mathcal{H}_{ref}(q)$ with $\mathcal{H}_{ref}(q) := \Psi(q, q_{ref})$. Note that $\mathcal{H}_d(x)$ has a strict local minimum at $(q^*, p^*) = (q_{ref}, 0)$. Solving PDE (14) yields $u_{es} := \beta(x) = -r^\top(q)\nabla_q \Psi(q, q_{ref})$ as the energy-shaping part of the input. By means of damping injection $u_{di} = -K_d \nabla_p \mathcal{H}_d(x) = -K_d \mathcal{I}^{-1} p$, we are able to retrieve the EB-PBC control law $u = u_{es} + u_{di}$ as in (32). The proof for (T3) follows the track provided in Dirksz et al. (2008). Consider the EL controller:

$$\dot{q}_c = -D_c(q_c)\nabla_{q_c}\mathcal{H}_c(q_c), \tag{35}$$

with $D_c(q_c)$ and $\mathcal{H}_c(q_c)$ to be defined later in the proof. The closed-loop system then reads as:

$$\begin{bmatrix} \dot{x} \\ \dot{q}_c \end{bmatrix} = \begin{bmatrix} J(x) & 0 \\ 0 & -D_c(q_c) \end{bmatrix} \begin{bmatrix} \nabla_x \tilde{\mathcal{H}}_d(x, q_c) \\ \nabla_{q_c} \tilde{\mathcal{H}}_d(x, q_c) \end{bmatrix}, \tag{36}$$

where the new closed-loop Hamiltonian $\tilde{\mathcal{H}}_d$ is defined as:

$$\tilde{\mathcal{H}}_d(q, p, q_c) := \mathcal{H}(p) + \underbrace{\Psi(q, q_{ref})}_{\mathcal{H}_{ref}(q)} + \underbrace{\Psi(q, q_c)}_{\mathcal{H}_c(q, q_c)}. \tag{37}$$

The conditions of Theorem 1 are now checked to infer the asymptotic stability of equilibrium (34). (C1) $\tilde{\mathcal{H}}_d(q, p, q_c)$ has a strict local minimum at (34) since $\mathcal{H}(p), \Psi(q, q_{ref})$ and $\Psi(q, q_c)$ respectively have a strict local minimum at $p = 0, q = q_{ref}$ and $q_c = q$. (C2) Matching condition (21) holds true since

$$\begin{bmatrix} I_{n_q \times n_q} & 0_{n_q \times 3} \end{bmatrix} \begin{bmatrix} 0 & r(q) \\ -r^\top(q) & p^\times \end{bmatrix} \begin{bmatrix} \nabla_q (\mathcal{H}_{ref} + \mathcal{H}_c) \\ 0 \end{bmatrix} = 0,$$

and solving (22) indeed yields (33b). (C3) Depending on the coordinate representation and corresponding energy function Ψ, condition

$$\nabla_{q_c} \tilde{\mathcal{H}}_d(q, q_c)\Big|_{q, q_{ref}} = 0 \quad \text{at } q = \mathit{const} \text{ only} \tag{38}$$

might not be satisfied. However, this condition arises from a LaSalle argument which is required in order for the asymptotic stability of the closed-loop system to be inferred. Equivalently, it can be required that the largest invariant set $\mathbb{M}$ in

$$\mathbb{O} := \left\{ \begin{bmatrix} q^\top & p^\top & q_c^\top \end{bmatrix}^\top \mid \dot{\tilde{\mathcal{H}}}_d(q, p, q_c) = 0 \right\}$$

consists of (34) only. This property is indeed satisfied by setting $D_c(q_c) = r(q_c) K_d r^\top(q_c)$ as follows. The closed-loop system reads as

$$\dot{q} = r(q)\mathcal{I}^{-1}p, \tag{39a}$$

$$\dot{p} = -r^\top(q)\nabla_q [\Psi(q, q_{ref}) + \nabla_q \Psi(q, q_c)] + \tilde{p}\mathcal{I}^{-1}p, \tag{39b}$$

$$\dot{q}_c = -r(q_c) K_d r^\top(q_c) \nabla_{q_c} \Psi(q_c, q). \tag{39c}$$

Taking time derivative of the $\tilde{\mathcal{H}}_d$ along the trajectories of (39) yields

$$\dot{\tilde{\mathcal{H}}}_d(q, p, q_c) = -\nu(q, q_c)^\top K_d \, \nu(q, q_c),$$

with $\nu(q, q_c) = r^\top(q_c)\nabla_{q_c}\Psi(q_c, q)$. By hypothesis, the latter equals zero for $q = q_c$ only, namely $\mathbb{O} := \left\{ \begin{bmatrix} q^\top & p^\top & q_c^\top \end{bmatrix}^\top \mid q = q_c \right\}$. Evaluating (39c) in $\mathbb{O}$ gives $\dot{q}_c \equiv 0$ and thus $\dot{q} \equiv 0$. Since $r(q)$ has full column rank, from (39a) it can be inferred that $p \equiv \dot{p} \equiv 0$. Hence, (39b) reduces to

$$0 \equiv -r^\top(q)\nabla_q \Psi(q, q_{ref}),$$

which, in turn, implies $q \equiv q_{ref}$. Therefore, $\mathbb{M} = \{(q_{ref}, 0, q_{ref})\}$ and the equilibrium (34) is asymptotically stable. Q.E.D.

6. APPLICATION

As mentioned in Section 5, a variety of attitude parameterizations are available for rigid-body attitude control. Energy functions and kinematics equations for most common attitude parameterizations are depicted in Table 1.

Two simulations are carried out in order to show the effectiveness of the presented dynamic controller (33). In both simulations, the inertia of the rigid body is chosen as $\mathcal{I} = \operatorname{diag}\{1, 0.8, 1\}$, while initial conditions are chosen as

$$R(0) = \begin{bmatrix} 0 & 0 & -1 \\ \cos\pi/3 & -\sin\pi/3 & 0 \\ -\sin\pi/3 & -\cos\pi/3 & 0 \end{bmatrix}, \quad p(0) = \begin{bmatrix} -5 \\ 4 \\ -3 \end{bmatrix}, \tag{40}$$

and $R_c(0) = I$. The desired attitude is as follows

$$R_{ref} = \begin{bmatrix} -\cos\pi/4 & \sin\pi/4 & 0 \\ \sin\pi/3 & \cos\pi/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{41}$$

In Simulation #1, the attitude parameterization is chosen as $\mathcal{SO}(3)$. To the end of tuning the steepness of the energy function Ψ in Table 1, we use the following $\Psi(q, q_{ref}) = \frac{1}{2} \operatorname{tr} \left[K_p \left(I - R_{ref}^\top R\right)\right]$, with $K_p = \operatorname{diag}\{2.5, 2, 2.5\}$, and $\Psi(q, q_c) = \frac{1}{2} \operatorname{tr} \left[K_c \left(I - R_c^\top R\right)\right]$, with $K_c = \operatorname{diag}\{20, 20, 20\}$. The damping gain is chosen as $K_d = \operatorname{diag}\{0.5, 0.5, 0.5\}$ and the simulation results are shown in Figure 1.

In Simulation #2, the attitude parameterization is chosen as the RPY angles. Again, in order to tune the steepness of the energy function Ψ in Table 1, we use the following $\Psi(q, q_{ref}) = \frac{k_p}{2} e^\top e$, with $k_p = 8$, and $\Psi(q, q_c) = \frac{k_c}{2} e^\top e$, with $k_c = 35$. The damping gain is chosen as $K_d = \operatorname{diag}\{0.2, 0.2, 0.2\}$. The simulation results are shown in Figure 2.

It is interesting to observe that, although if the magnitude of the angular velocities is comparable in both simulations, the control input usage in the RPY case is approximately one order of a magnitude larger than the one in the $\mathcal{SO}(3)$ case. The behaviour of the RPY case is explained by the trajectory approaching the singularities of matrix $r(q)$, namely $\theta \simeq \pm\pi/2$.

7. CONCLUSIONS

We introduced a pH formulation of both dynamics and kinematics equations of the rigid-body attitude control problem. Within the proposed framework, we have presented a standard passivity-based controller and an extended controller which does not make use of angular velocity measurements. By means of both theory and illustrative examples, we have shown that such results are independent of the attitude parameterization in use, thus

Table 1. Energy functions for most common attitude representations

	Roll-Pitch-Yaw	**Unit quaternions**	**Rotation matrices**
Current / desired attitude	$\mathbf{q} := \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \in \mathbb{R}^3$, $\mathbf{q}_d := \begin{bmatrix} \phi_d \\ \theta_d \\ \psi_d \end{bmatrix} \in \mathbb{R}^3$	$\mathbf{q} := \begin{bmatrix} q_4 \\ \mathbf{q}_v \end{bmatrix} \in \mathbb{S}^3$, $\mathbf{q}_d := \begin{bmatrix} q_{d4} \\ \mathbf{q}_{dv} \end{bmatrix} \in \mathbb{S}^3$	$\mathbf{q} :=$ vec R with $R \in \mathcal{SO}(3)$, $\mathbf{q}_d :=$ vec R_d with $R_d \in \mathcal{SO}(3)$
Tracking error	$\mathbf{e} := \mathbf{q} - \mathbf{q}_d$	$\mathbf{z} := \begin{bmatrix} \mathbf{q}_{dv}^\top \mathbf{q}_v + q_4 q_{d4} \\ q_{d4}\mathbf{q}_v - \mathbf{q}_{dv}^\times \mathbf{q}_v - q_4 \mathbf{q}_{dv} \end{bmatrix}$	$\mathbf{e} := \frac{1}{2}\left(R_d^\top R - R^\top R_d\right)^\vee$
Energy function	$\Psi(\mathbf{q}, \mathbf{q}_d) = \frac{1}{2}\mathbf{e}^\top \mathbf{e}$	$\Psi(\mathbf{q}, \mathbf{q}_d) = \frac{1}{2}\mathbf{z}^\top \mathbf{z}$	$\Psi(\mathbf{q}, \mathbf{q}_d) = \frac{1}{2}$ tr $\left(I - R_d^\top R\right)$
Kinematics equation	$r(q) := \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix}$	$r(q) := \begin{bmatrix} -\mathbf{q}_v^\top \\ q_4 I_3 + \mathbf{q}_v^\times \end{bmatrix}$	equation (10)

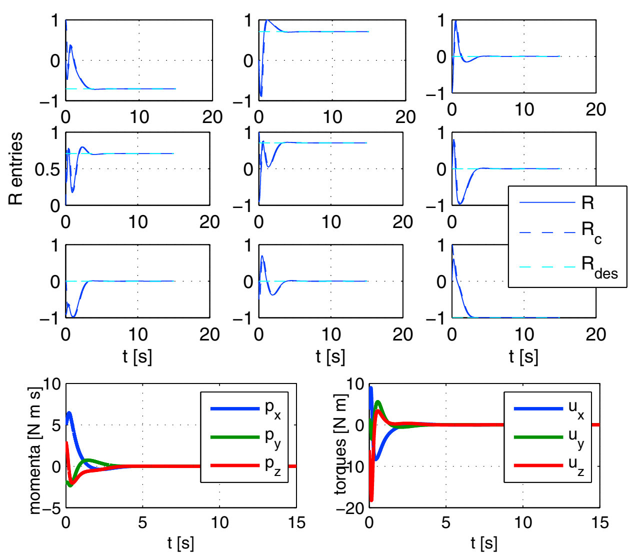

Fig. 1. Controller (33) on the attitude dynamics of the rigid body in $\mathcal{SO}(3)$. Plot 1, from the top: evolution of the entries of R (solid blue) and R_c (dashed blue) versus desired attitude R_d (cyan). Plot 2: evolution of the angular momenta p. Plot 3: evolution of the control inputs u.

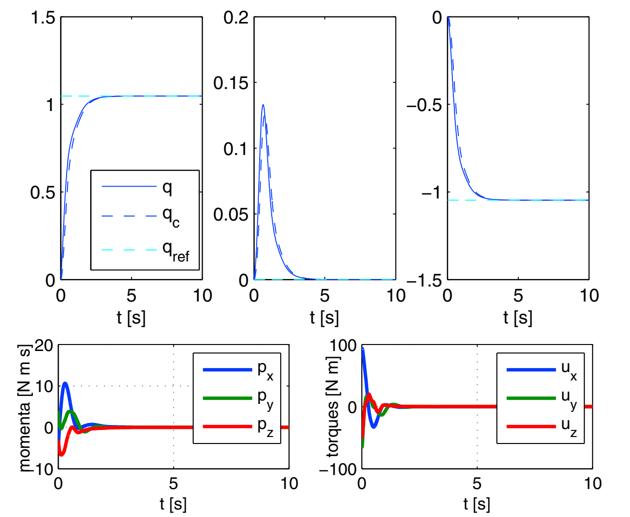

Fig. 2. Controller (33) on the attitude dynamics of the rigid body with RPY angles. Plot 1, from the top: evolution of the RPY angles q (solid blue) and q_c (dashed blue) versus desired attitude R_d (cyan). Plot 2: evolution of the angular momenta p. Plot 3: evolution of the control inputs u.

highlighting that the pH framework is very suitable for tackling the rigid-body attitude control problem. Hopefully, this paper will serve as a starting point for further extension of techniques applying to space manipulators and flexible rigid-body systems.

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