Quantum Computing

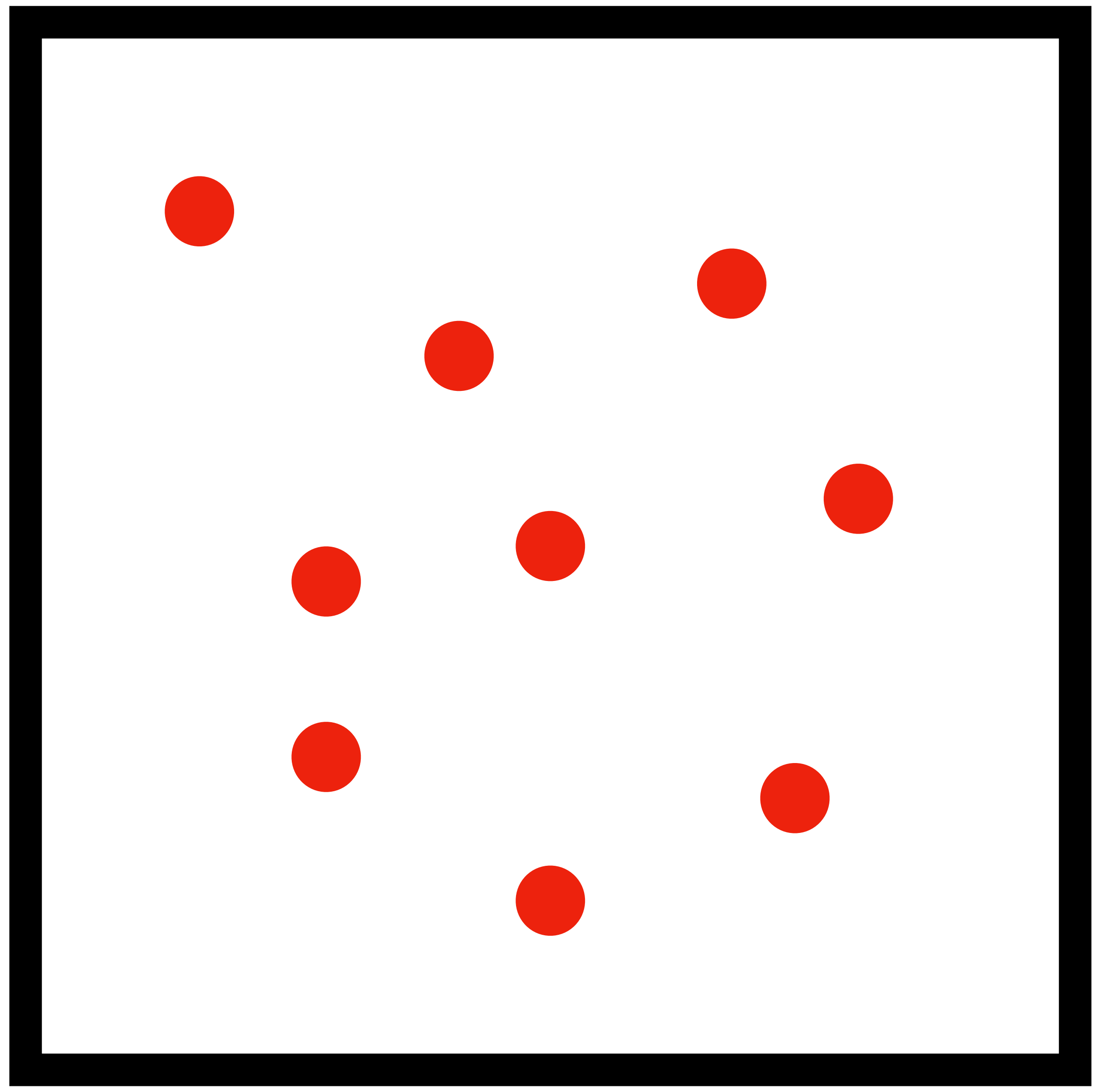
Algorithms and Complexity

Hakim Rachidi, Göttingen SoSe 25

Motivation

Complexity of Quantum

Classical



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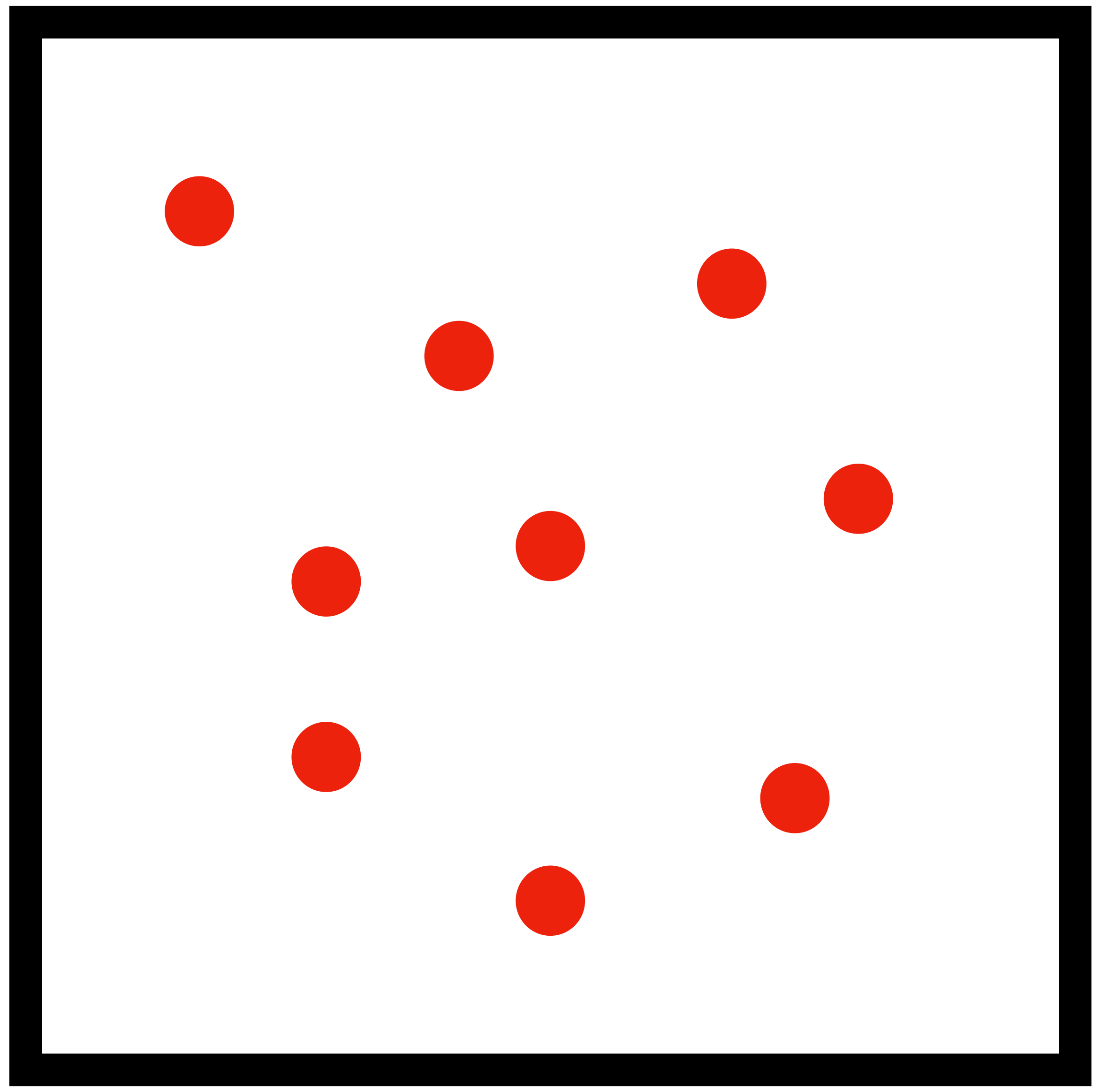

Position $q_i \in \mathbb{R}^f$

Momentum $p_i \in \mathbb{R}^f$

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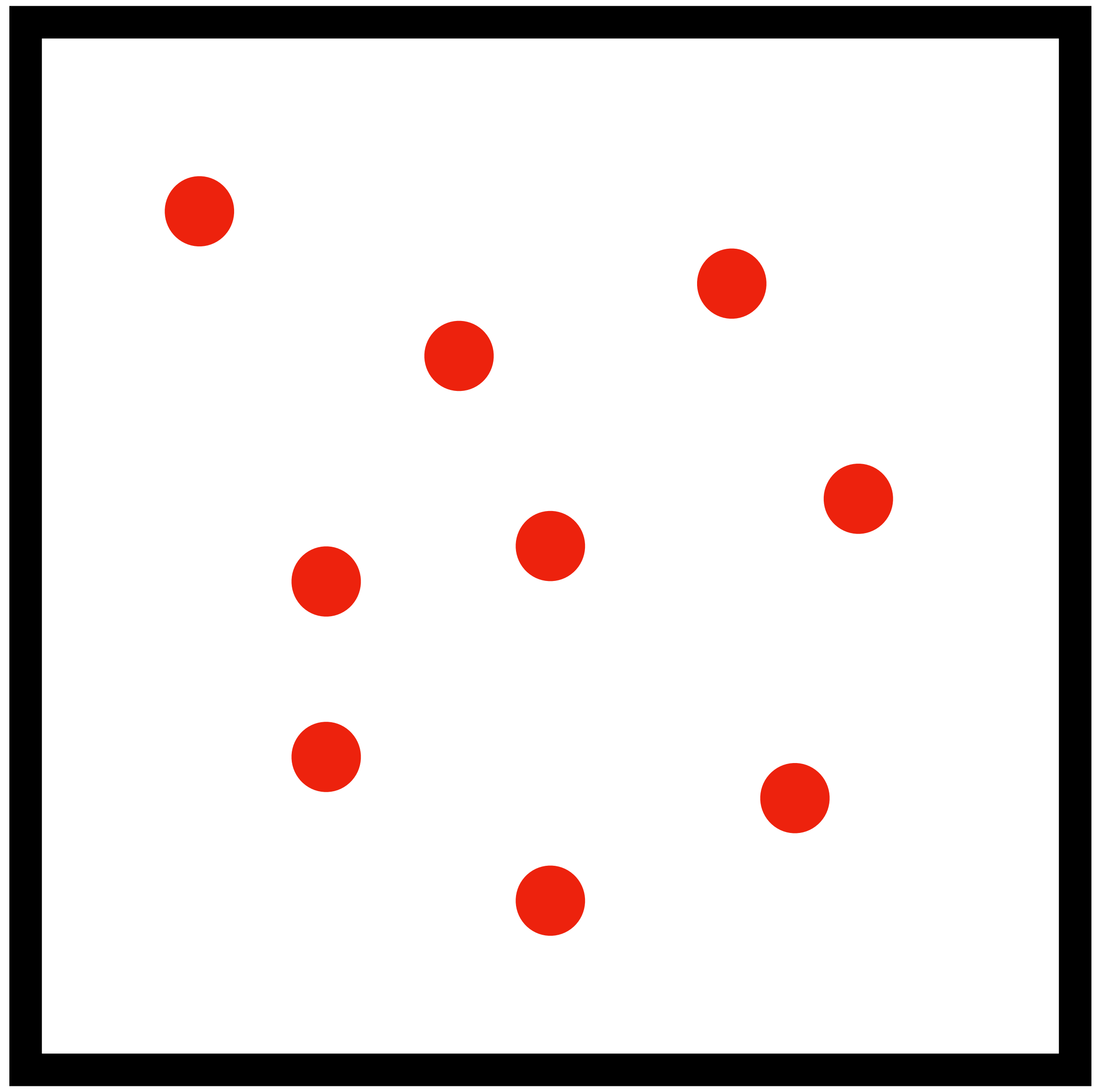

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$$\boxed{\dim \Gamma = 2fN}$$

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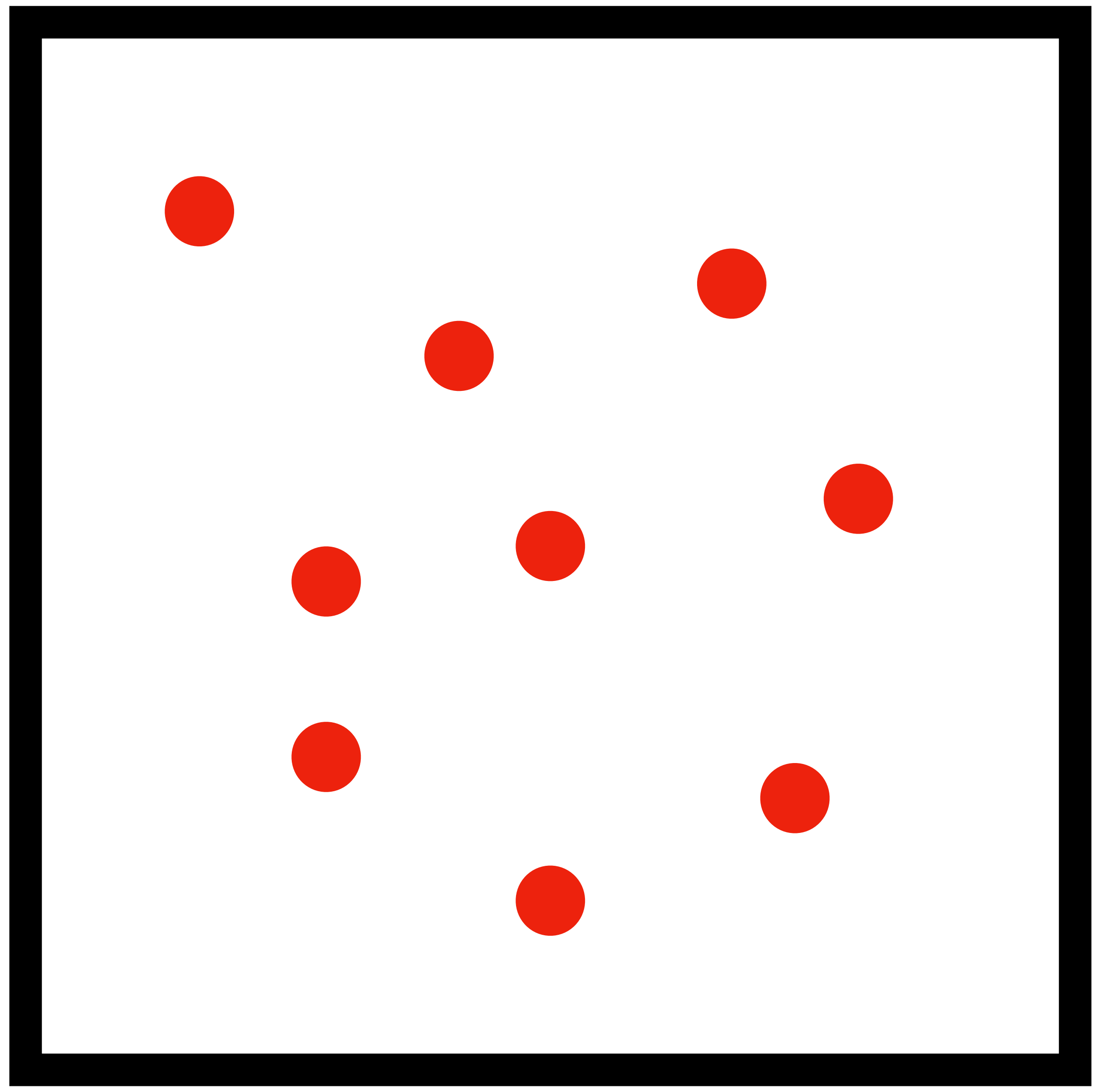

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$|\psi_1\rangle\,|\psi_2\rangle\,|\psi_3\rangle\,|\psi_4\rangle\,|\psi_5\rangle$

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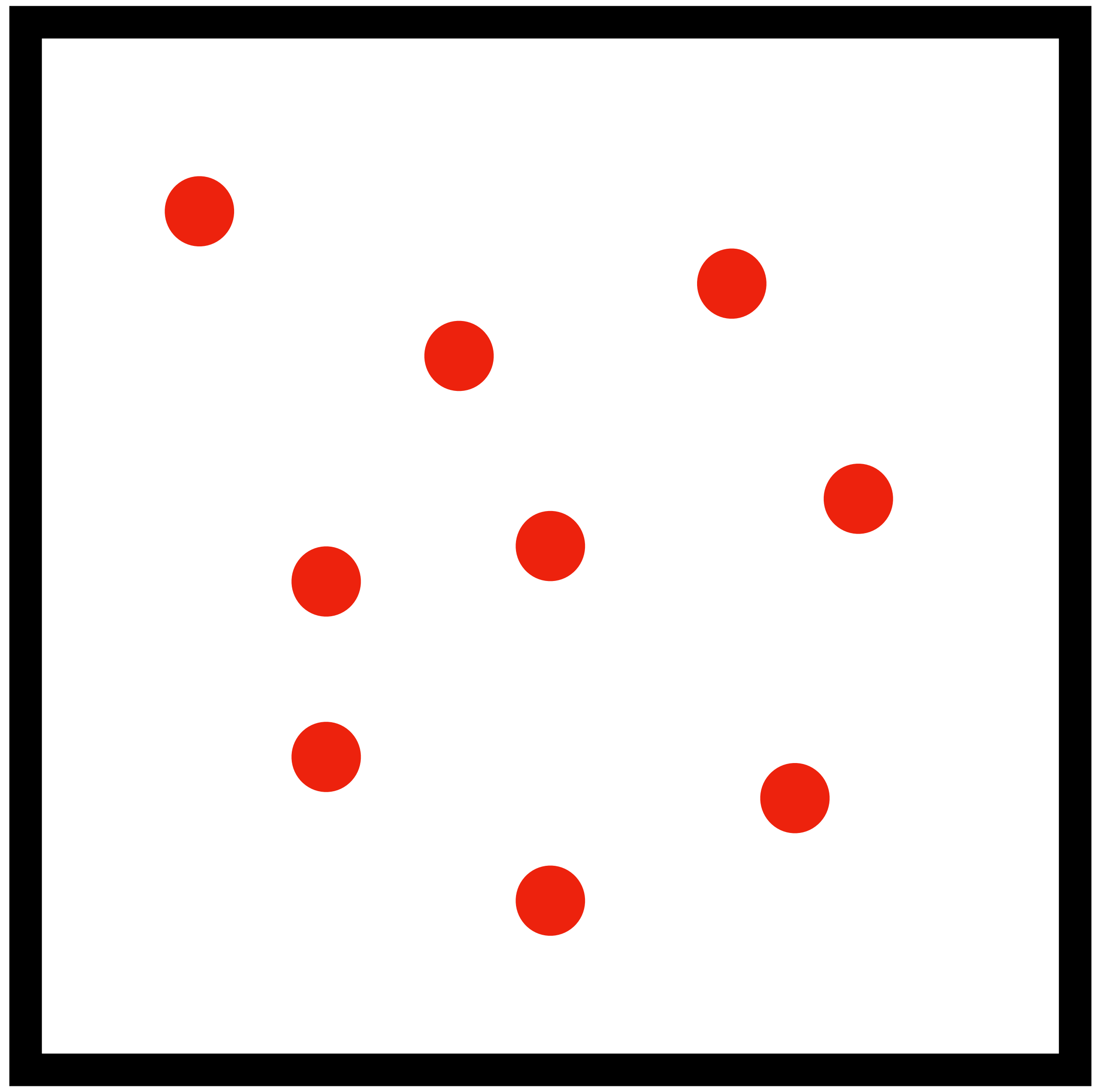

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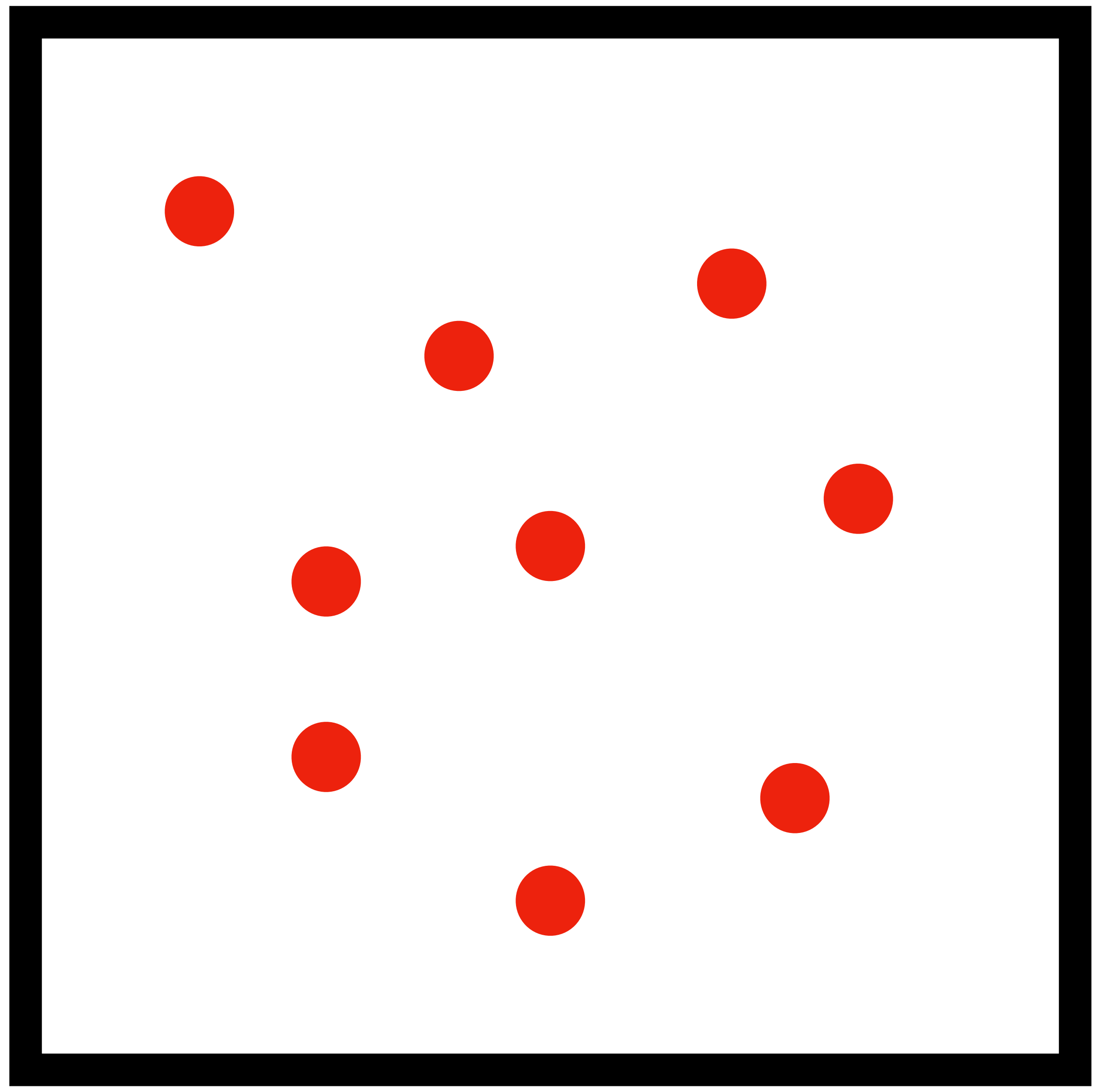

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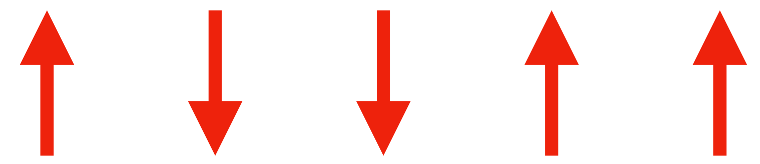

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$$\dim(\mathscr{H}_1 \otimes \mathscr{H}_2) = (\dim \mathscr{H}_1) \cdot (\dim \mathscr{H}_2)$$

$$\boxed{\dim \mathscr{H} = 2^N}$$

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Computational effort scales <u>polynomially</u>

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$$\boxed{\dim \mathscr{H} = 2^N}$$

for $N = 272$

$2^N >$ # Atoms in the visible universe

Computational effort scales
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Feasible

Quantum

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Infeasible

Motivation

Simulating quantum systems

Can we exploit quantum systems to simulate quantum systems?

— Feynmann (1982)

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→Universal Quantum Simulators (Quantum Computers)

Motivation

What is complex and what is efficient?

Multiplication

$$113 \cdot 73 = 8249$$

Number of steps linear in the number of digits

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Can we use the complexity of quantum systems to speedup algorithms?

Guiding questions

Outline of the talk

Which quantum algorithms exist? and *How* do they work?

Deutsch-Jozsa, Grover's Algorithm

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Outline of the talk

Which quantum algorithms exist? and *How* do they work?

Deutsch-Jozsa, Grover’s Algorithm

How much faster are they compared to classical counterpart?

Complexity Theory

Fundamentals of Quantum Computing

Quantum Bit

Classical Bit

$$X \in \{0,1\}$$

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$\longrightarrow$

Quantum Bit (Qubit)

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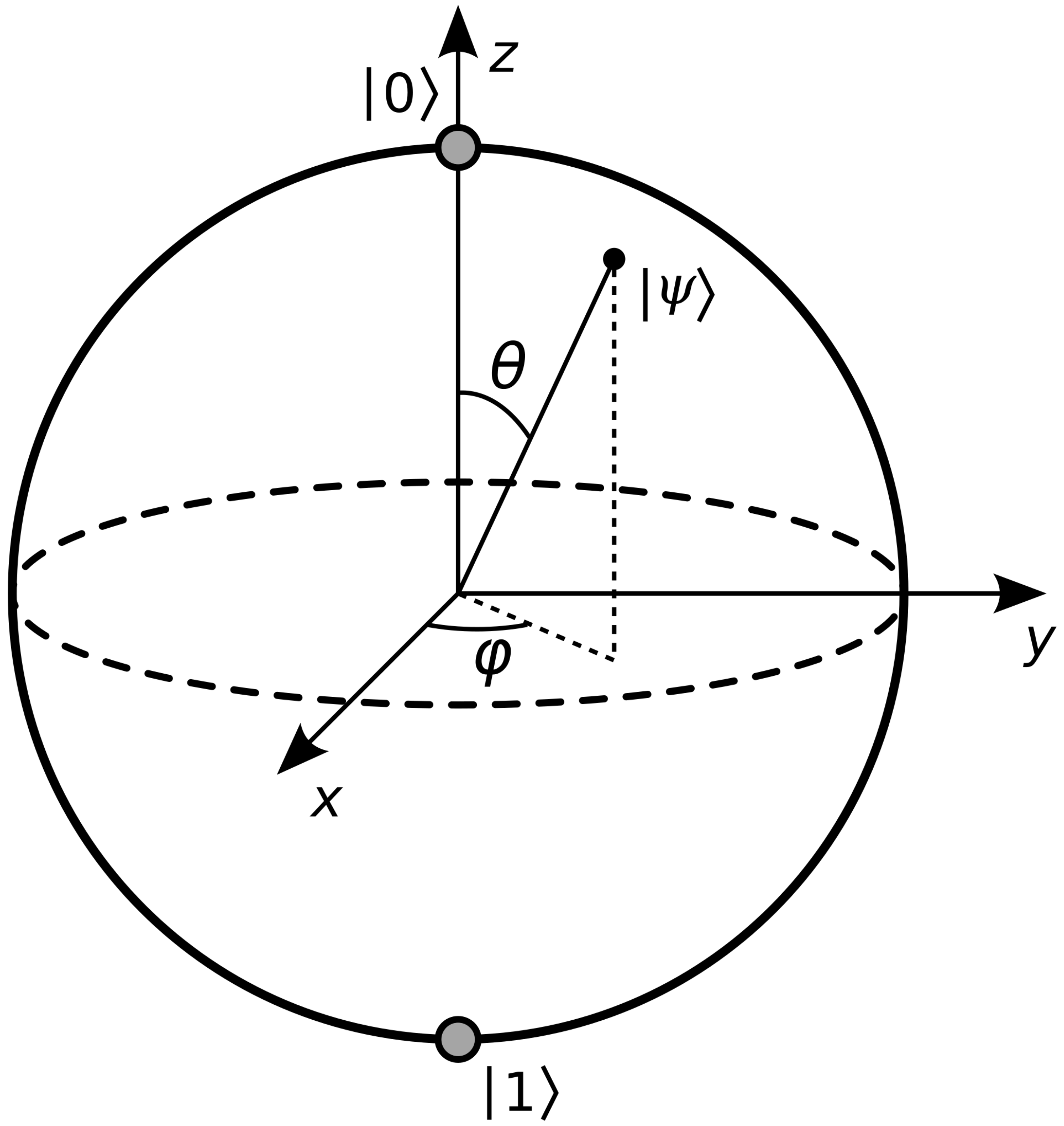

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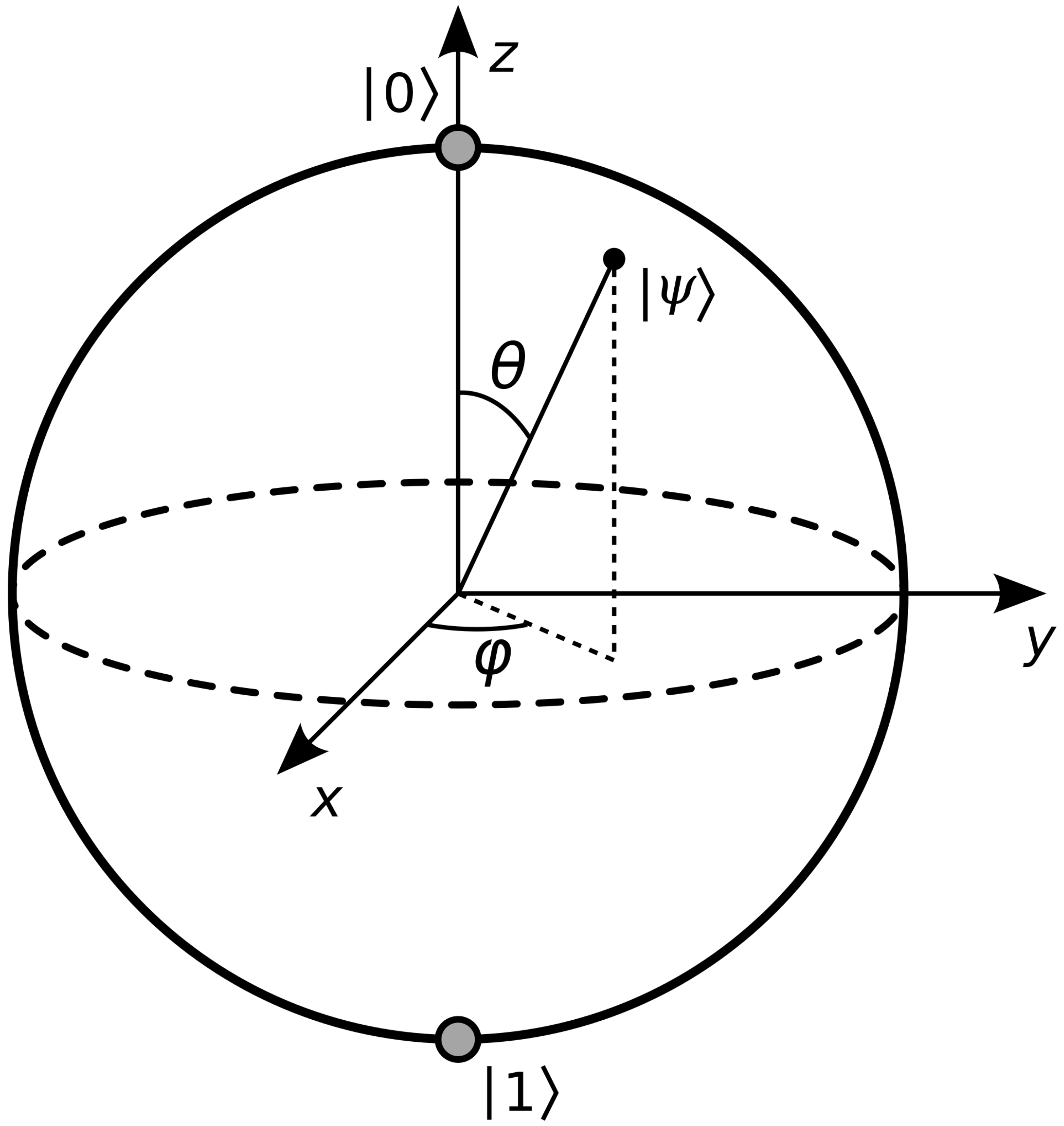


Quantum Bit

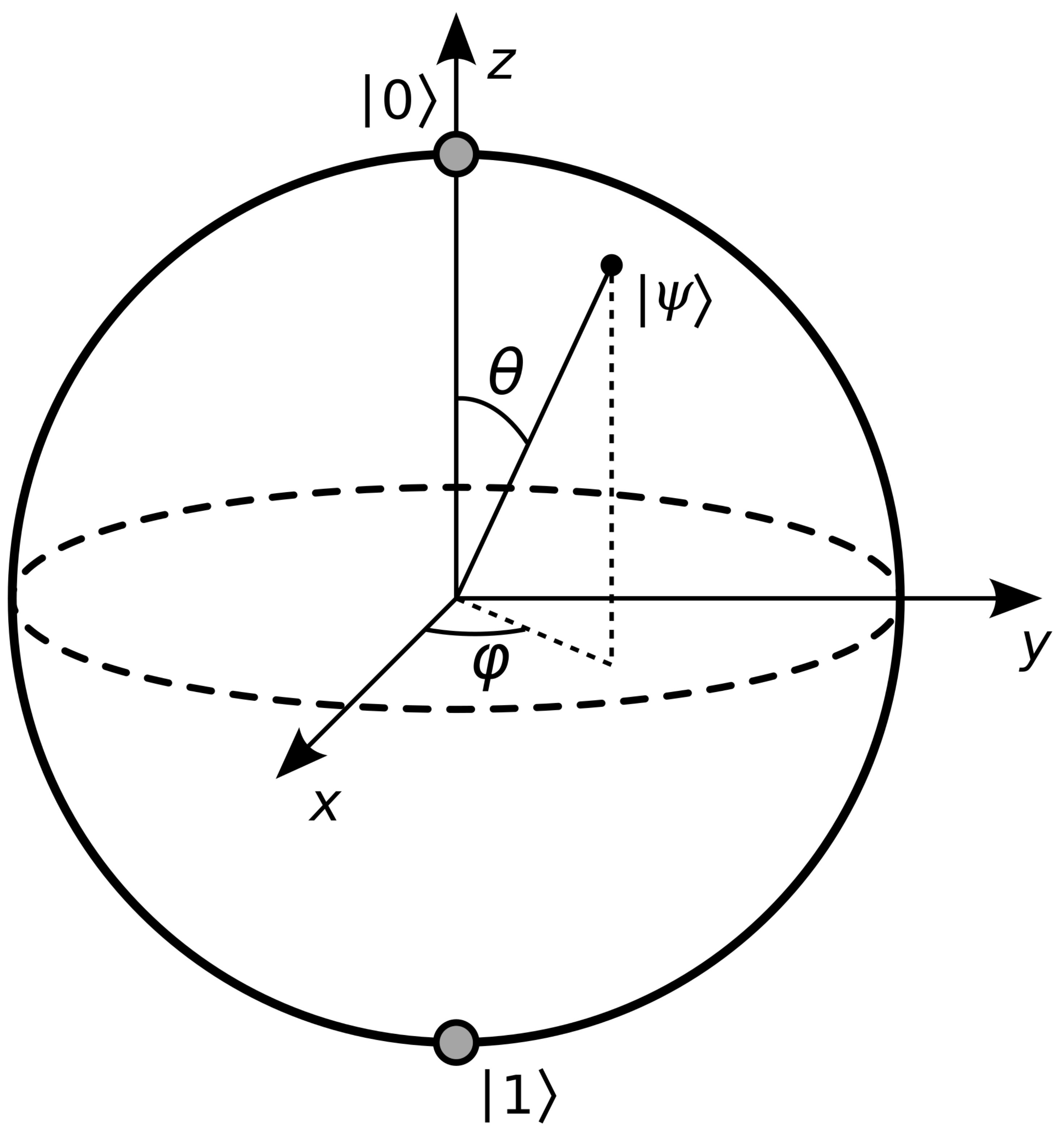
Classical Bit → **Quantum Bit (Qubit)**

$$X \in \{0,1\} \longrightarrow |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathscr{H}$$

In principle Qubit stores *infinite* amount of information

→ measurement yields a classical Bit

Upon measurement only a *finite* amount is attainable



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$$S \in \{0,1\}^n$$

e.g. $S = 1001$

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We will use the short hand notation

$$|0\rangle \otimes |1\rangle \otimes |0\rangle = |010\rangle$$

Classical logic gates

Processing of classical bits can be described in the language of *gates*

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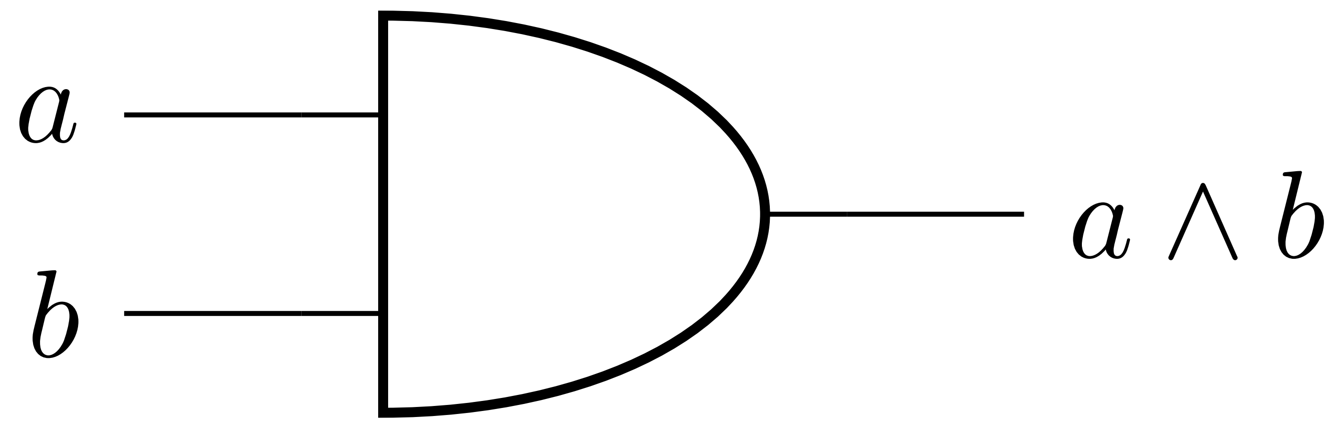

AND Gate

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

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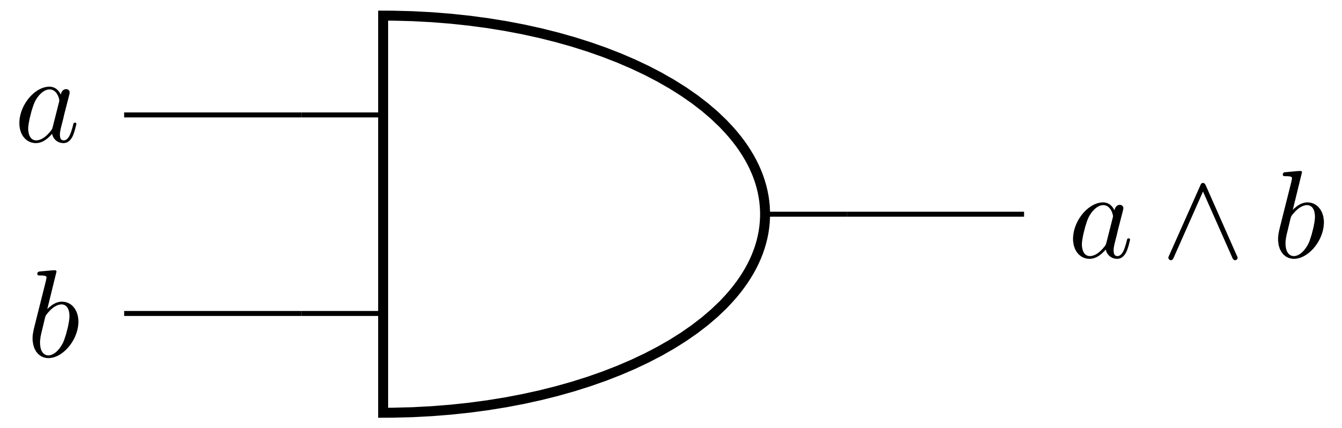

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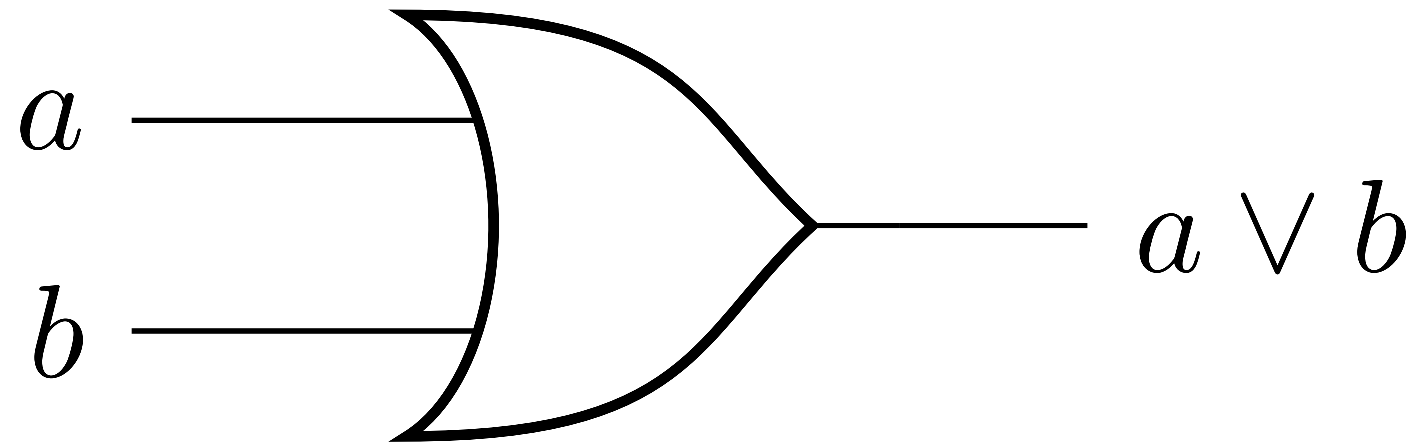

OR Gate

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

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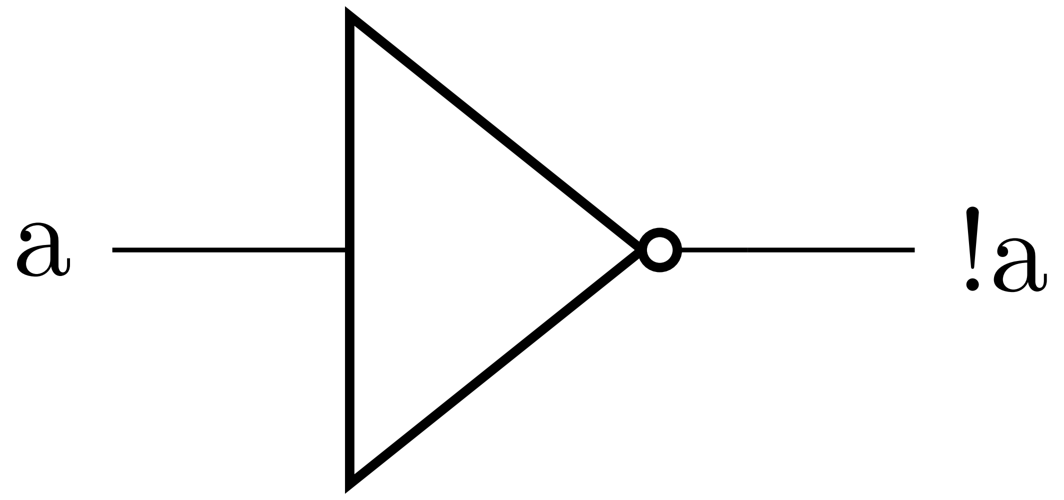

NOT Gate

a	$!a$
0	1
1	0

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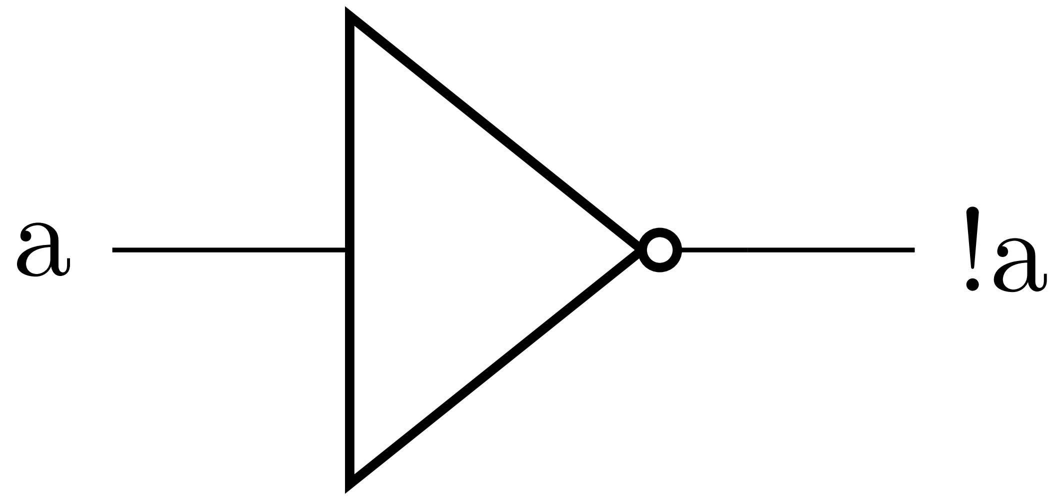

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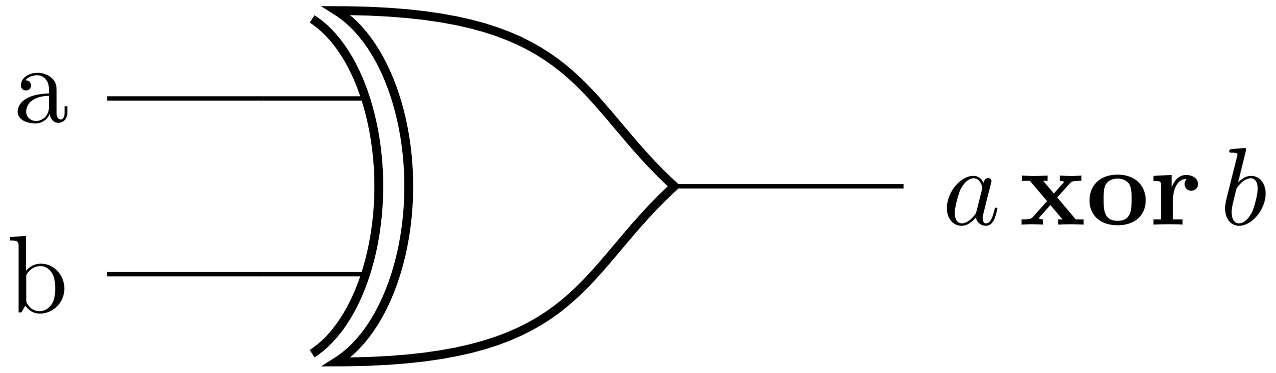

XOR Gate

a	b	a **xor** b
0	0	0
0	1	1
1	0	1
1	1	0

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Since the theory is <u>linear</u> all quantum gates need to be linear

Quantum Gates

Hadamard Gate

$$\alpha|0\rangle + \beta|1\rangle \quad -\boxed{H}- \quad \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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X, Y, Z Gates

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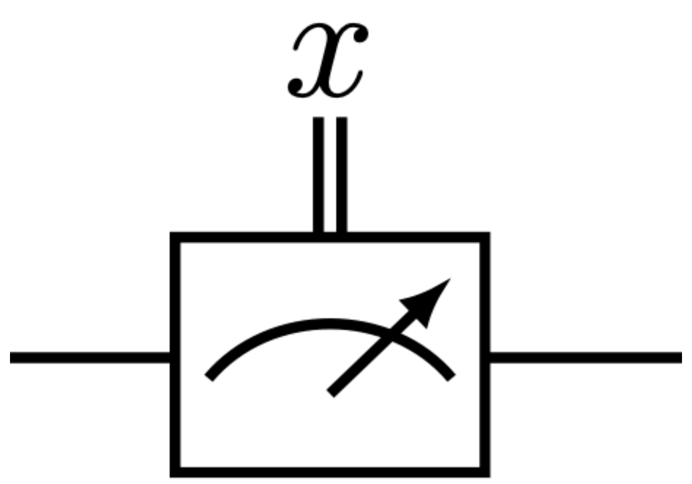
$$\alpha|0\rangle + \beta|1\rangle \;\xrightarrow{\;\boxed{Z}\;}\; \alpha|0\rangle - \beta|1\rangle$$

Quantum Gates

Measurement Gate



$$x \in \{-1,1\}$$

$$\hat{Z}|\psi\rangle = x|\psi\rangle$$

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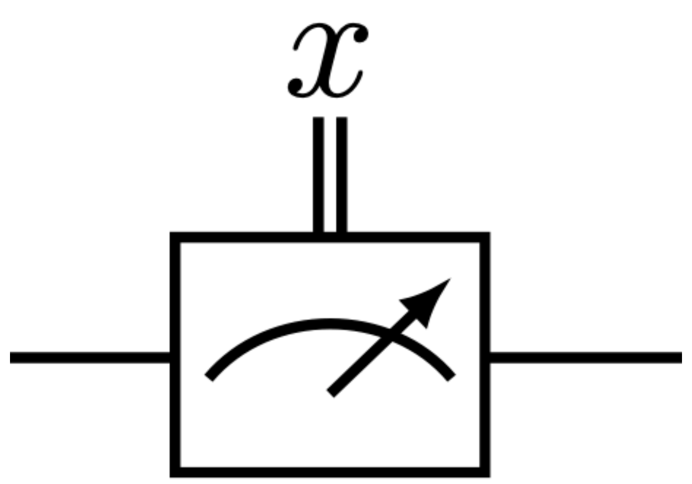

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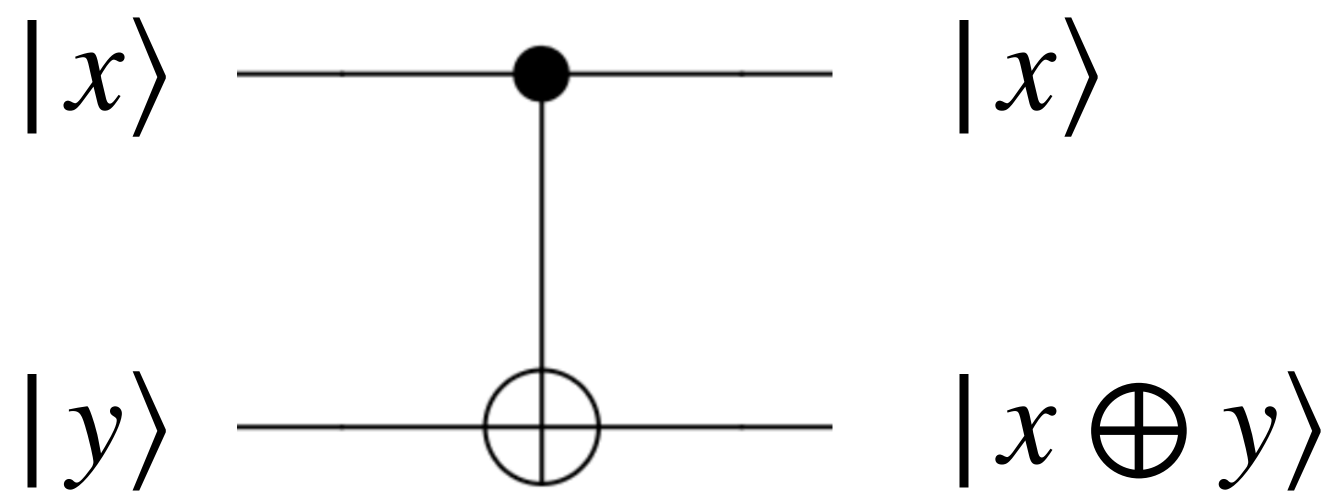
Control Gates

flip the second qubit if the first is $|1\rangle$

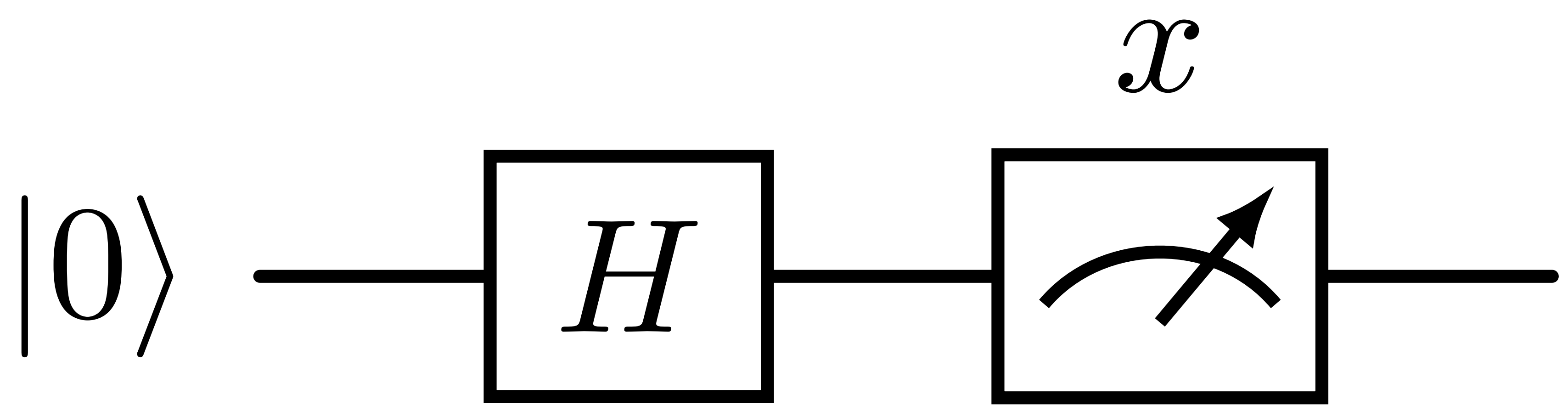


$0 \oplus 0 = 0$
$1 \oplus 0 = 1$
$0 \oplus 1 = 1$
$1 \oplus 1 = 0$

Quantum Circuits

Similar to classical circuits, we can connect Quantum gates

Example: Random number generator



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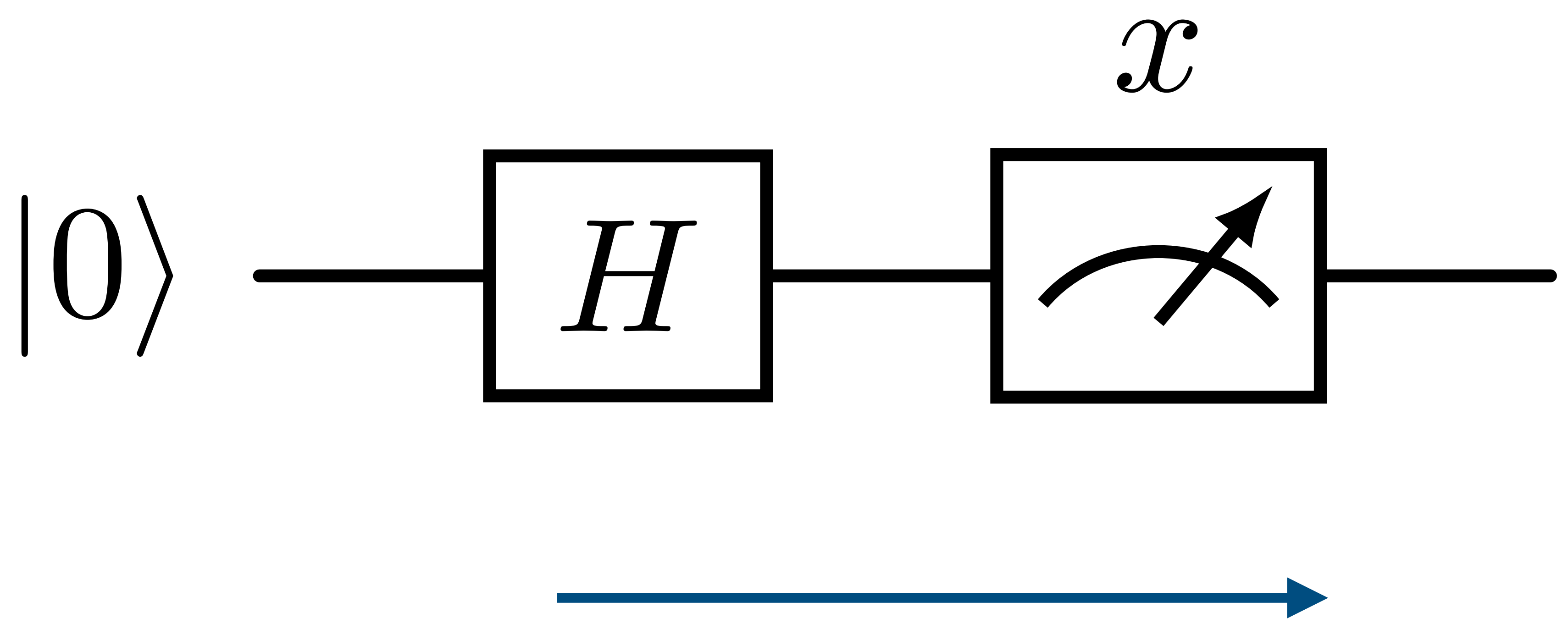

Execution order

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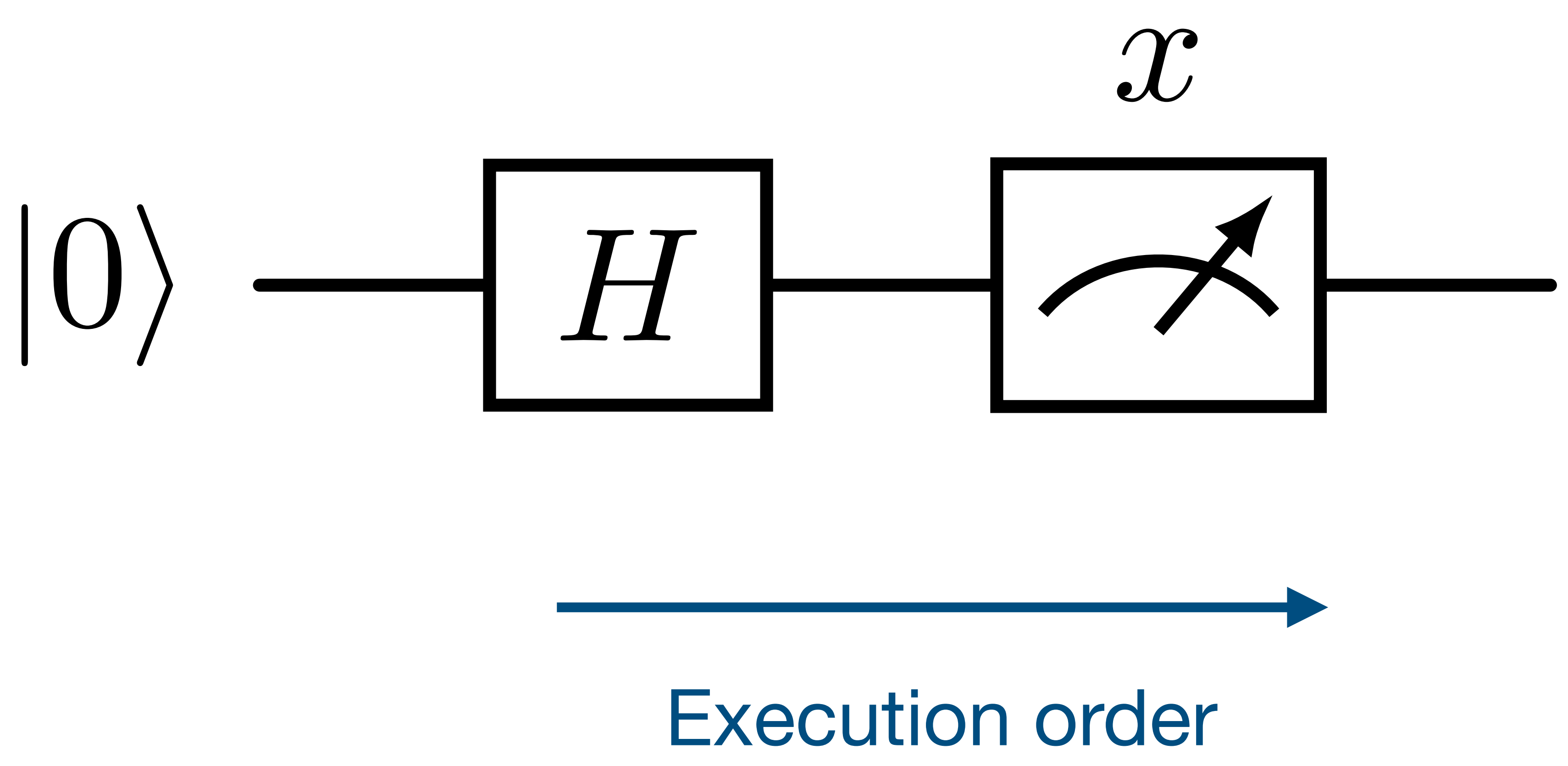

$$|0\rangle \xrightarrow{\hat{H}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

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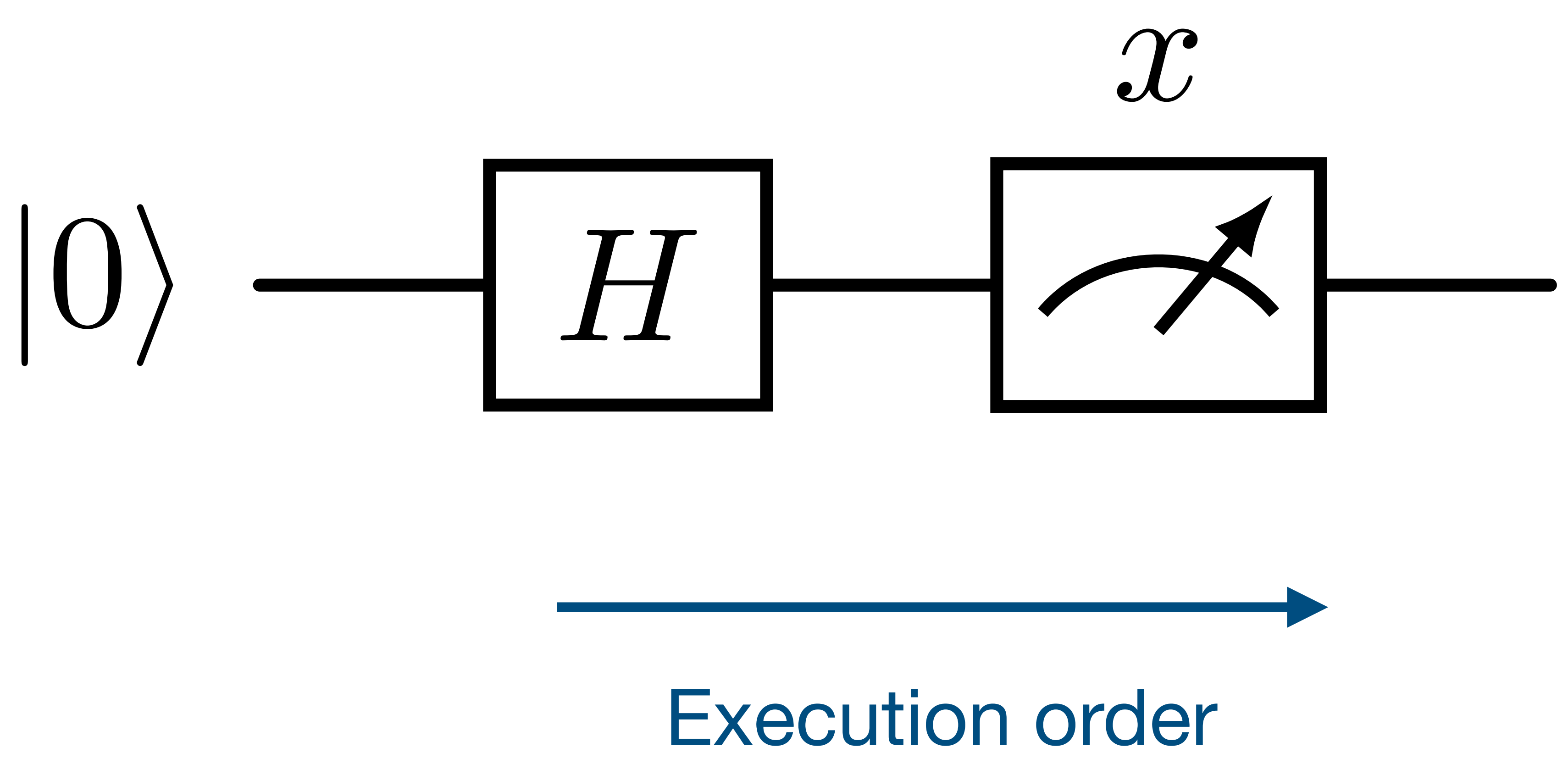

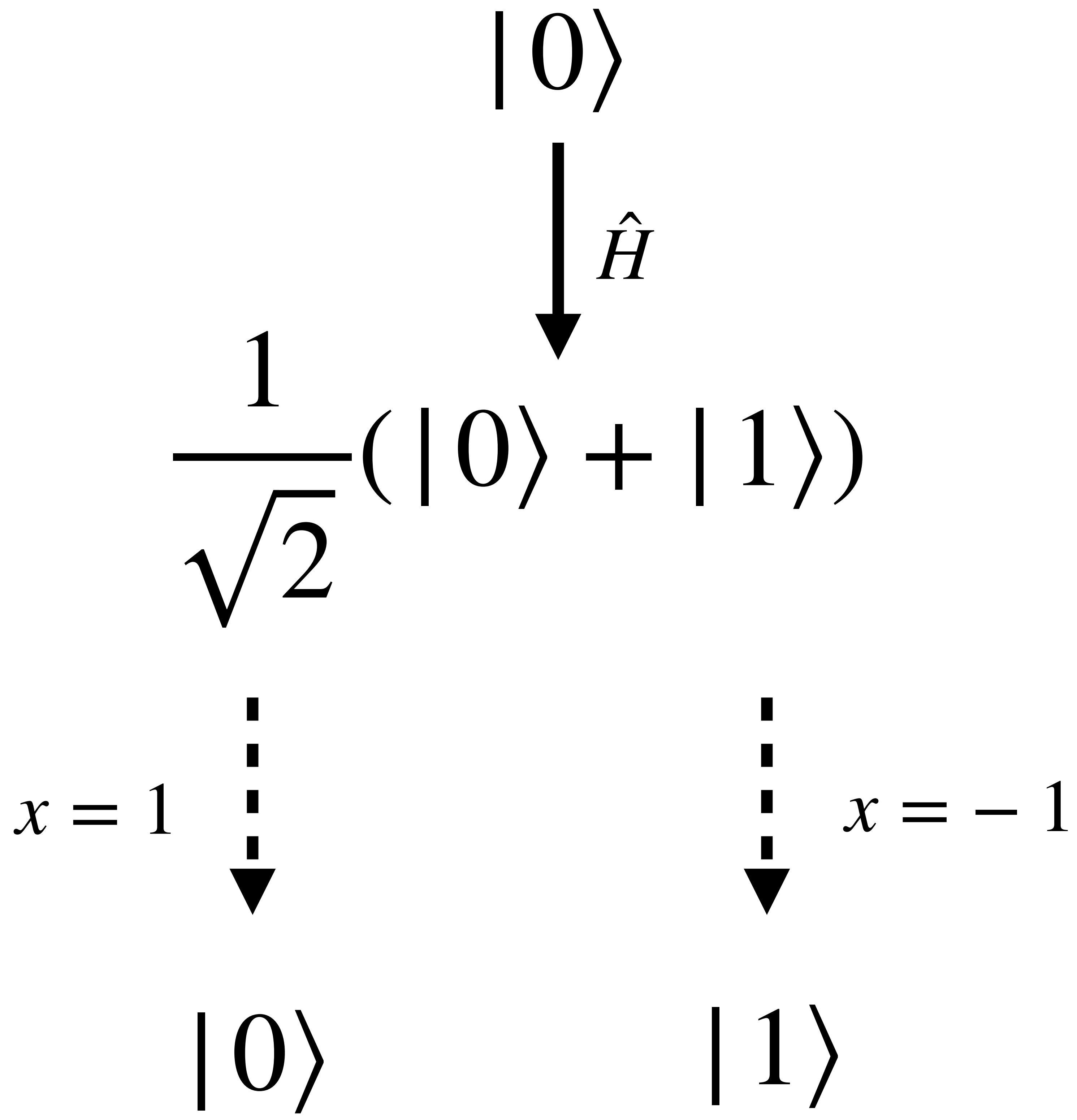

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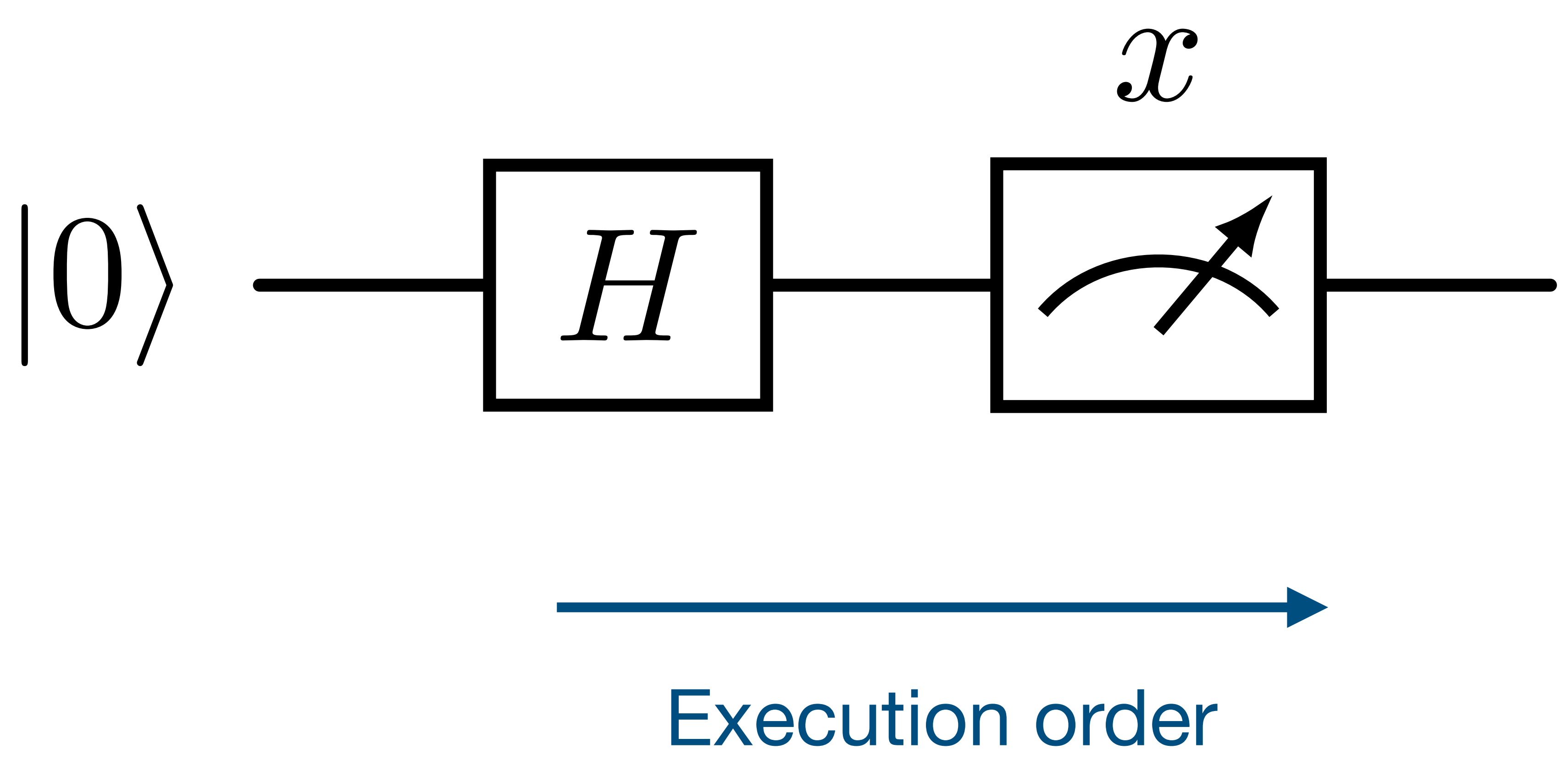

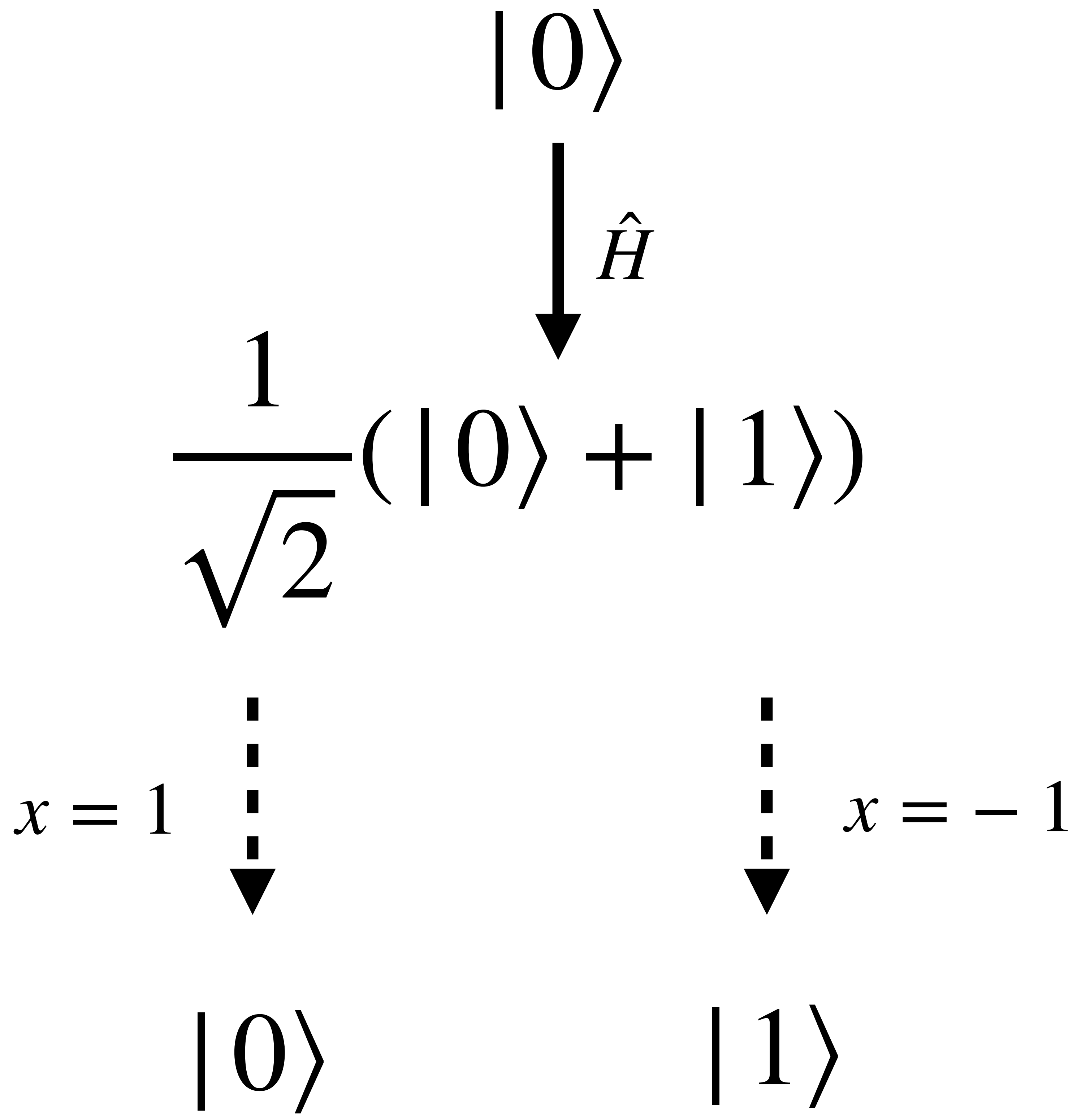

Result of measurement is intrinsically random → sampling necessary

No Cloning Theorem

Suppose $\hat{U} : \mathscr{H} \rightarrow \mathscr{H}$ clones the first qubit into second

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but due to linearity of $\hat{U}$

$$\hat{U}(a|0\rangle + b|1\rangle) \otimes |0\rangle = a\hat{U}|00\rangle + b\hat{U}|10\rangle =$$

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Therefore: Quantum circuits cannot split a qubit into two

Deutsch-Jozsa Algorithm

A coin is considered *real* if it has distinct sides …

Deutsch-Jozsa Algorithm

A coin is considered *real* if it has distinct sides …

… and otherwise *fake*

Deutsch-Jozsa Algorithm

fake

real

Classical: we need check both sides → <u>two</u> measurements

Deutsch-Jozsa Algorithm

fake

real

Classical: we need check both sides → <u>two</u> measurements

Quantum: <u>one</u> measurement sufficient (Deutsch-Jozsa Algorithm)

Deutsch-Jozsa Algorithm

Let $f : \{0,1\} \to \{0,1\}$ be a binary function. We want to know whether

real coin

f is *balanced* ($f(0) \neq f(1)$)

fake coin

f is *constant* ($f(0) = f(1)$)

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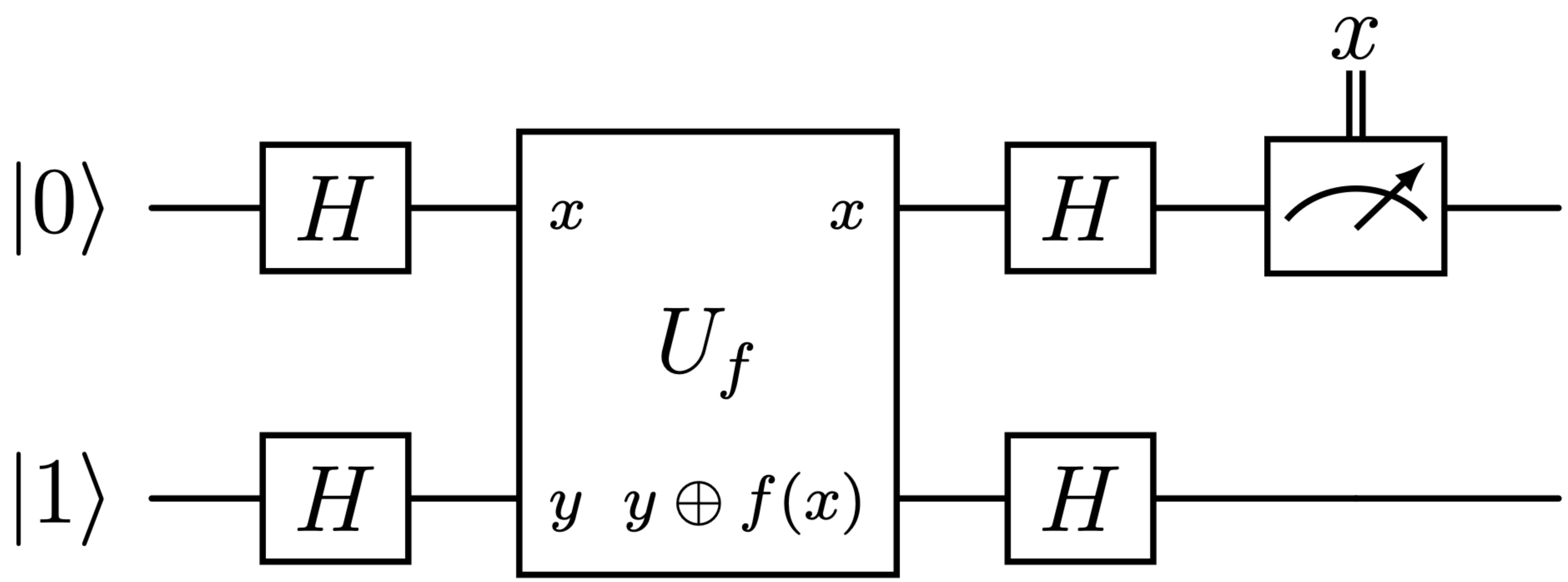

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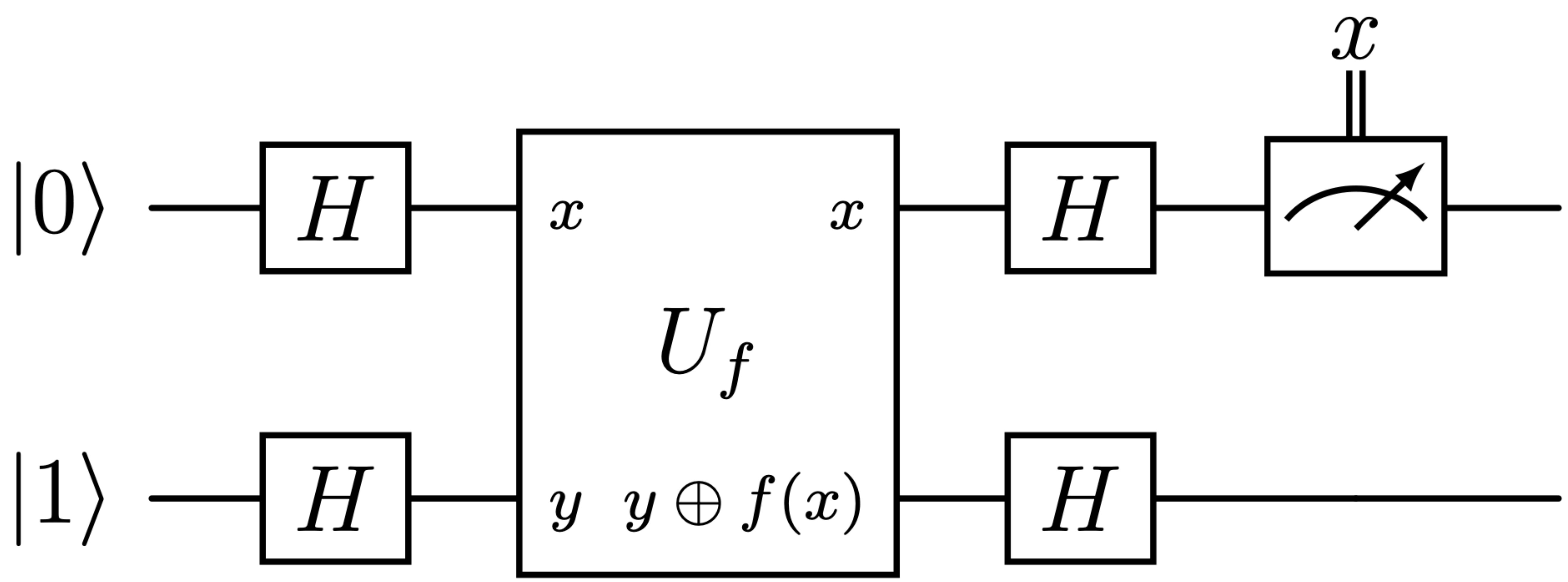


$\underline{x = -1}$: f is balanced
real coin

$\underline{x = 1}$: f is constant
fake coin

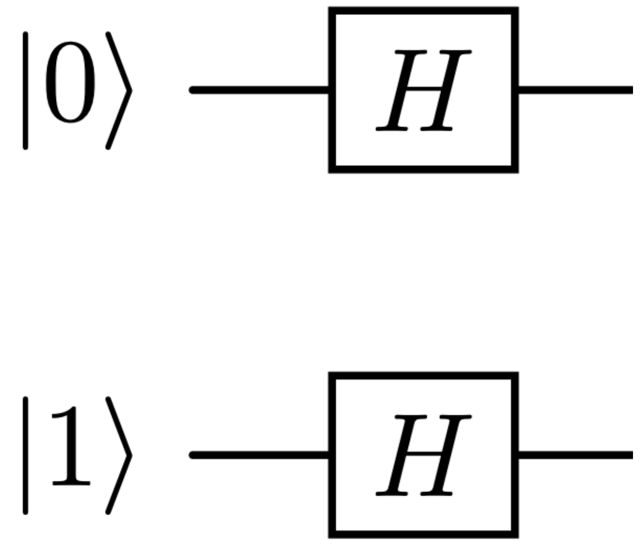
Deutsch-Jozsa Algorithm

$|0\rangle$ —

$|1\rangle$ —

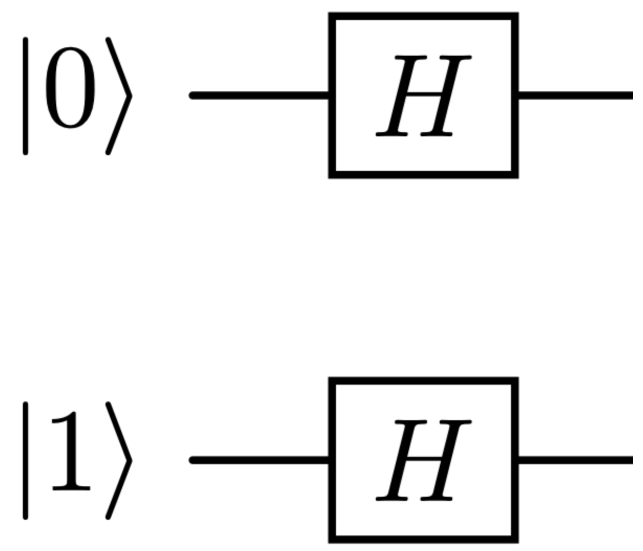
$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

Deutsch-Jozsa Algorithm



$$|\psi\rangle = |0\rangle \otimes |1\rangle \xrightarrow{\hat{H} \otimes \hat{H}} \hat{H}|0\rangle \otimes \hat{H}|1\rangle = \frac{1}{2}\big(|0\rangle + |1\rangle\big)\big(|0\rangle - |1\rangle\big)$$

Deutsch-Jozsa Algorithm



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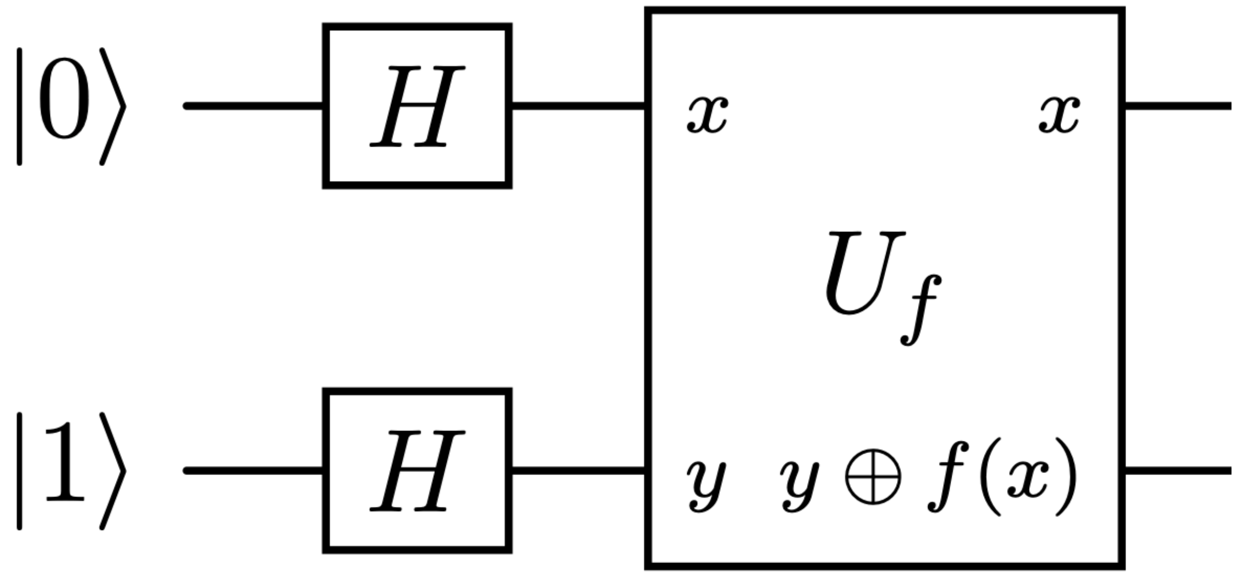
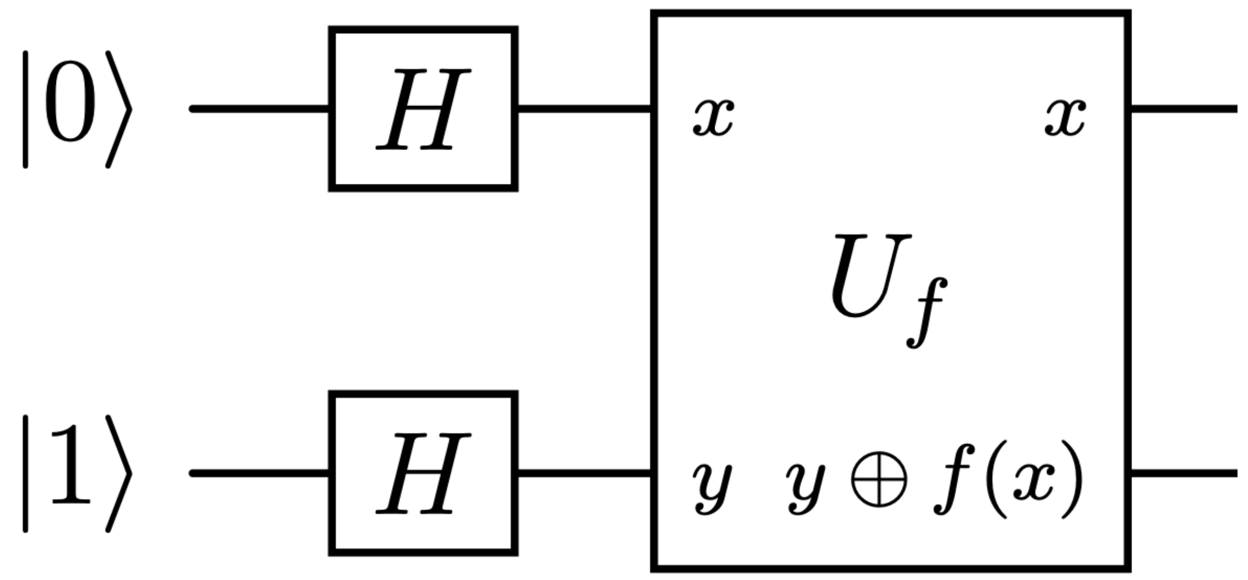
$$= \frac{1}{2}(|00\rangle + |10\rangle - |11\rangle - |01\rangle)$$

Deutsch-Jozsa Algorithm



$$\frac{1}{2}(\,|00\rangle + |10\rangle - |11\rangle - |01\rangle) \xrightarrow{U_f} \frac{1}{2}U_f\big(\,|00\rangle + |10\rangle - |11\rangle - |01\rangle\big)$$

Deutsch-Jozsa Algorithm



$$U_f|xy\rangle = |x(y \oplus f(x))\rangle$$

$$\frac{1}{2}(|00\rangle + |10\rangle - |11\rangle - |01\rangle) \xrightarrow{U_f} \frac{1}{2}U_f\big(|00\rangle + |10\rangle - |11\rangle - |01\rangle\big)$$

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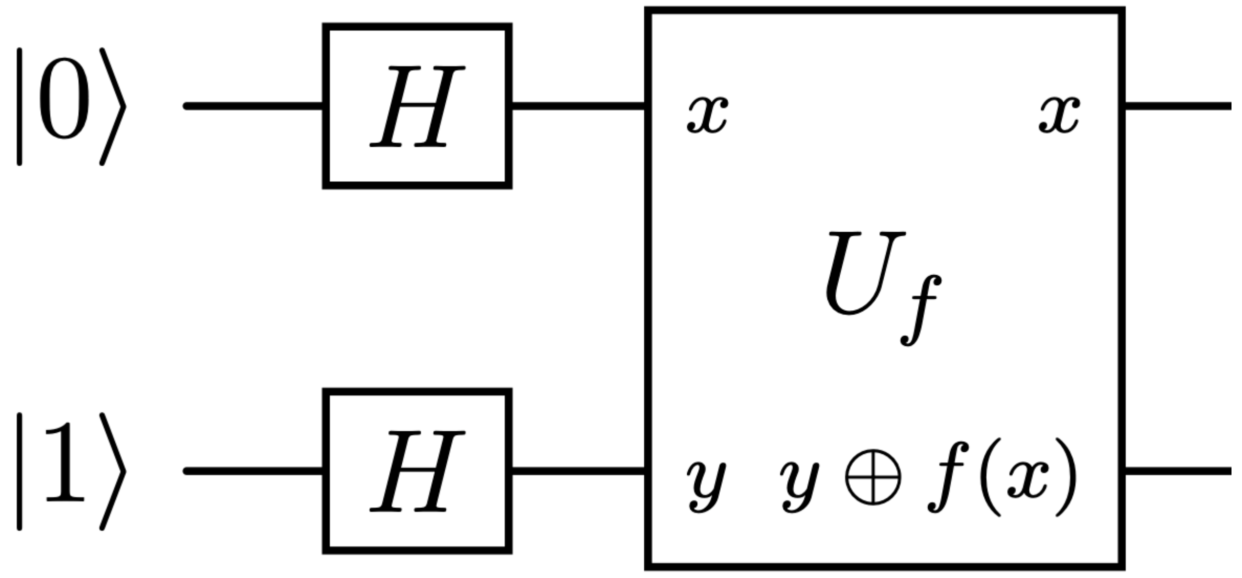

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$$= \frac{1}{2}\big(|0(0 \oplus f(0))\rangle + |1(0 \oplus f(1))\rangle - |1(1 \oplus f(1))\rangle - |0(1 \oplus f(0))\rangle\big)$$

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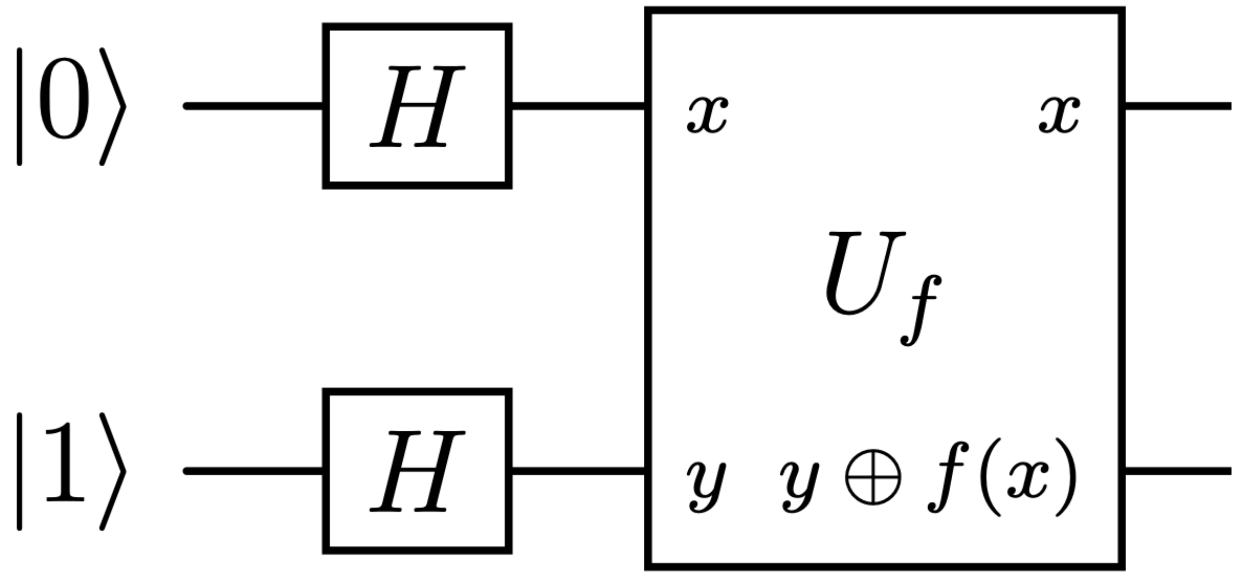

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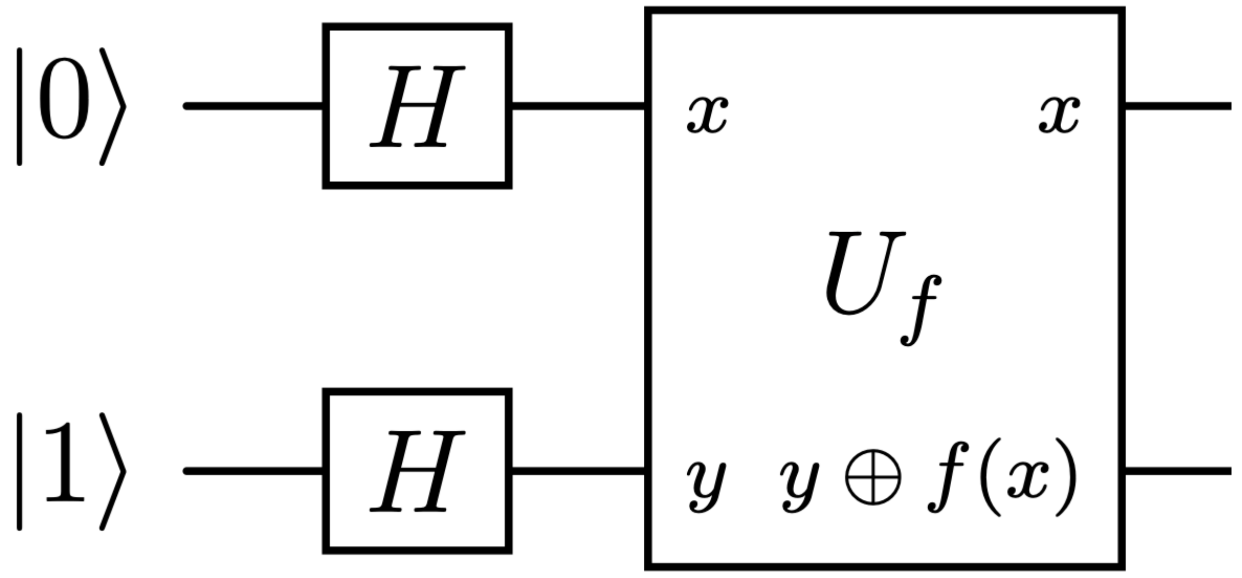

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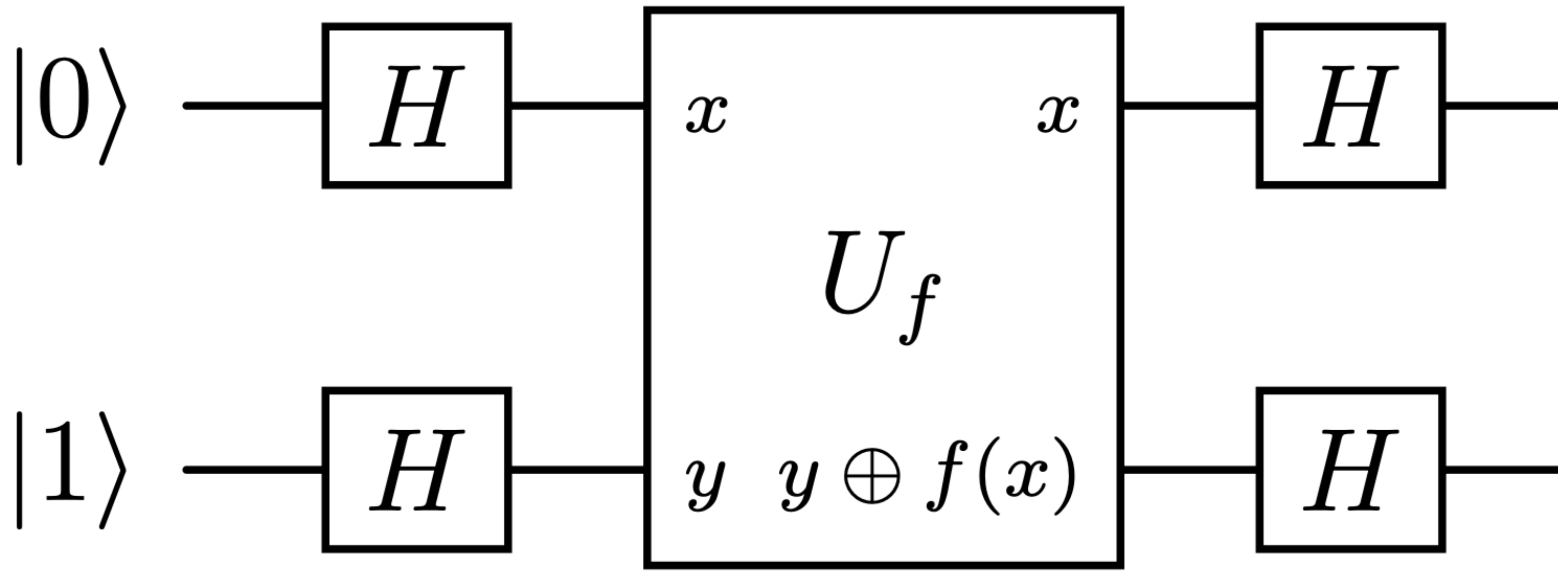

Hadamard operator

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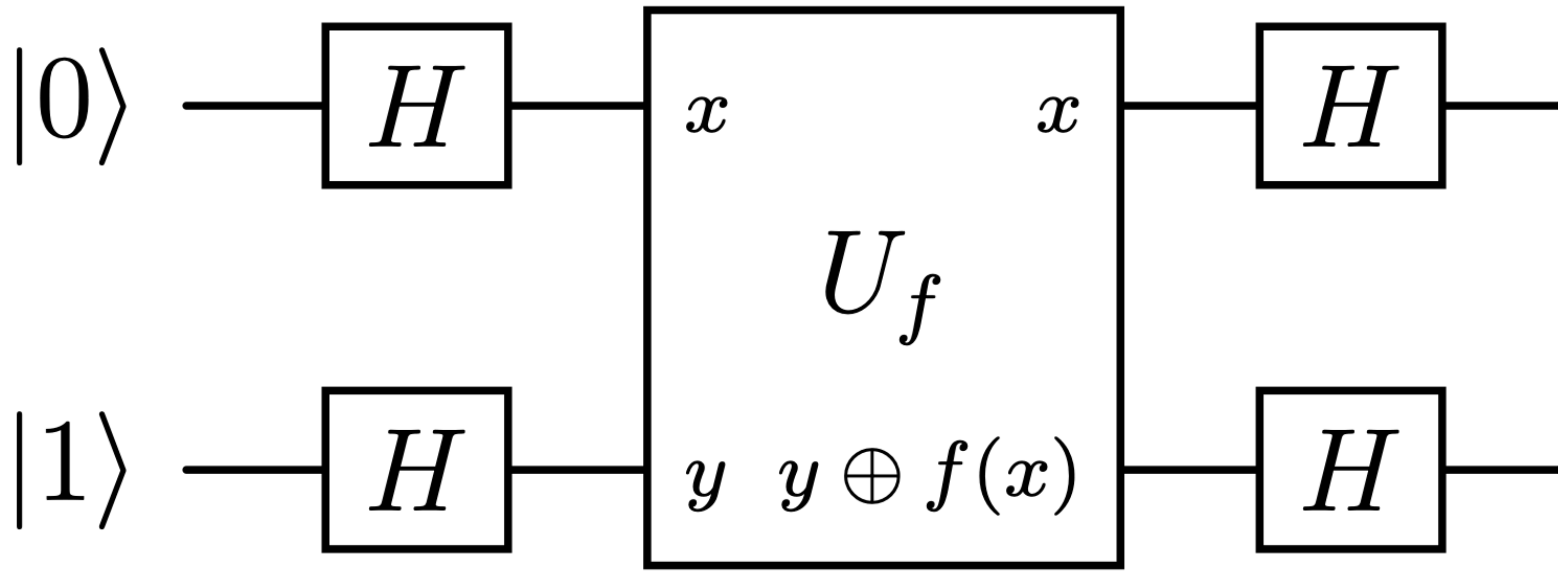

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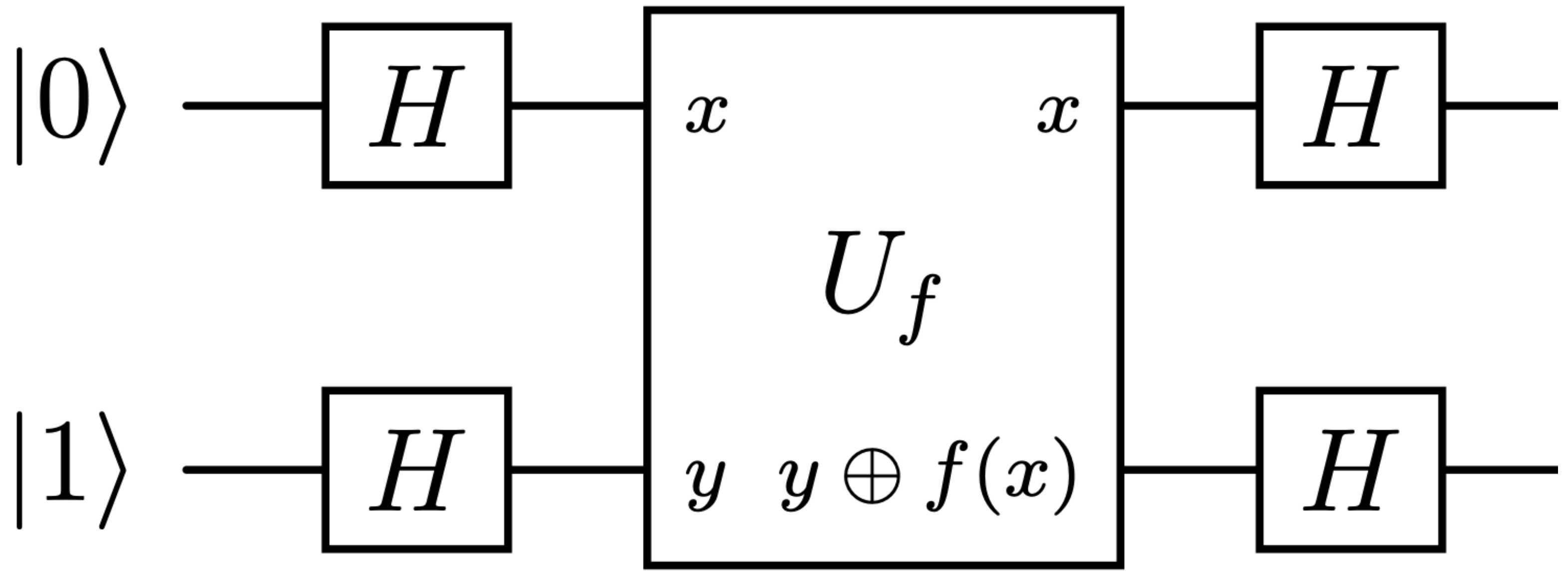

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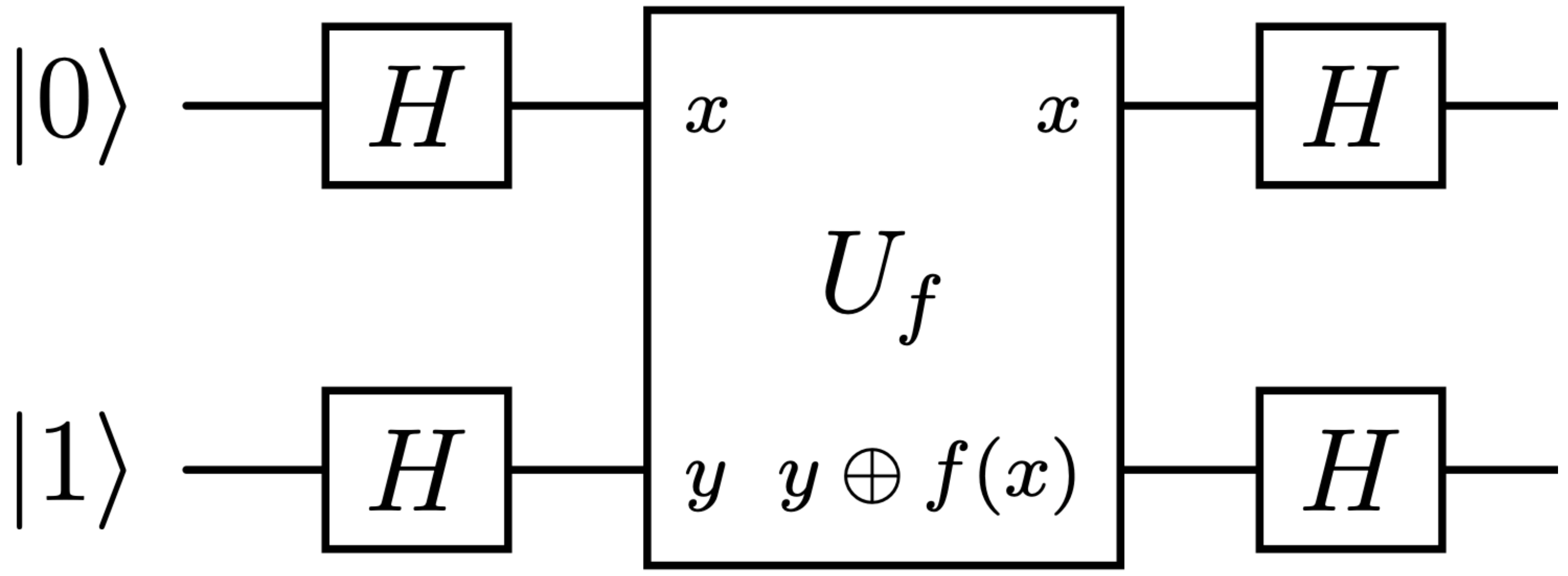

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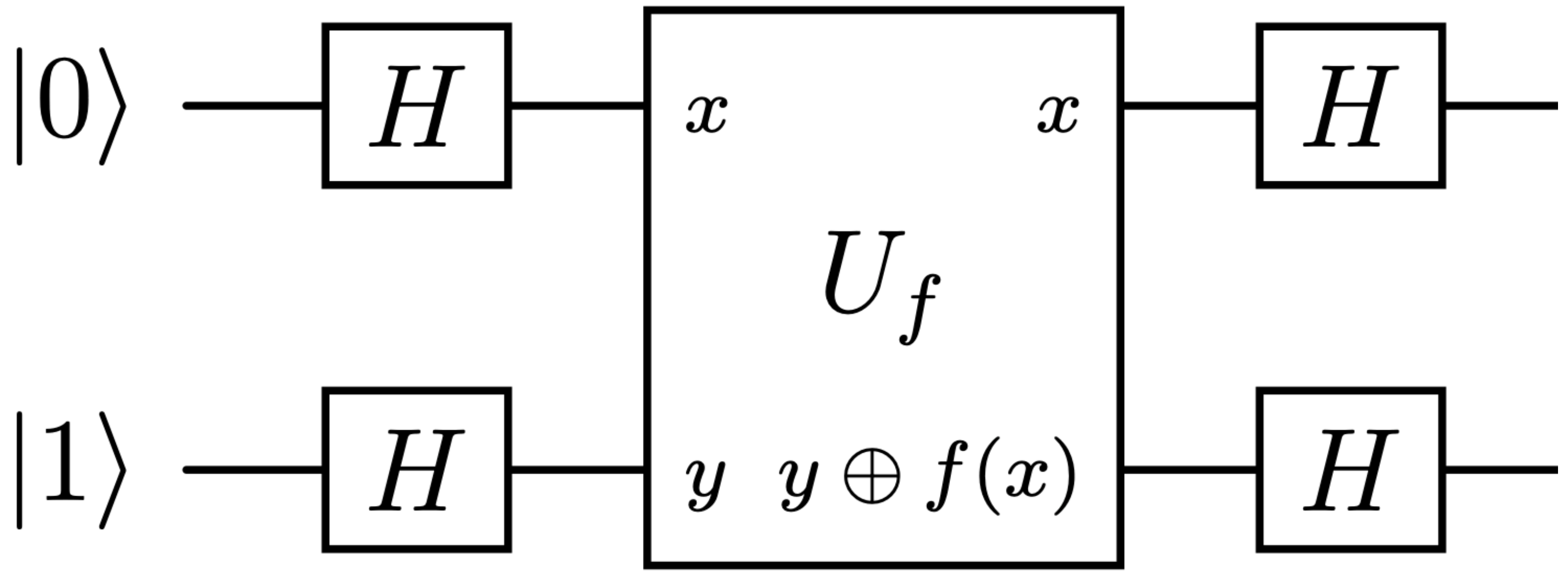

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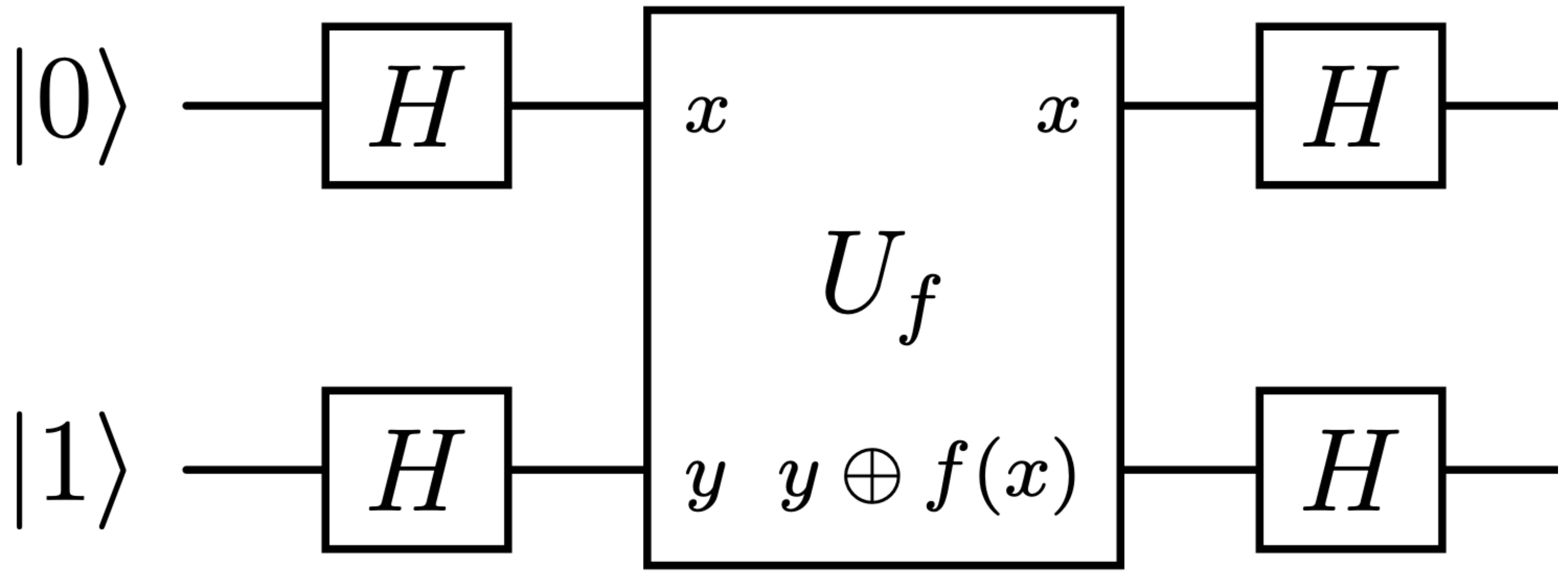

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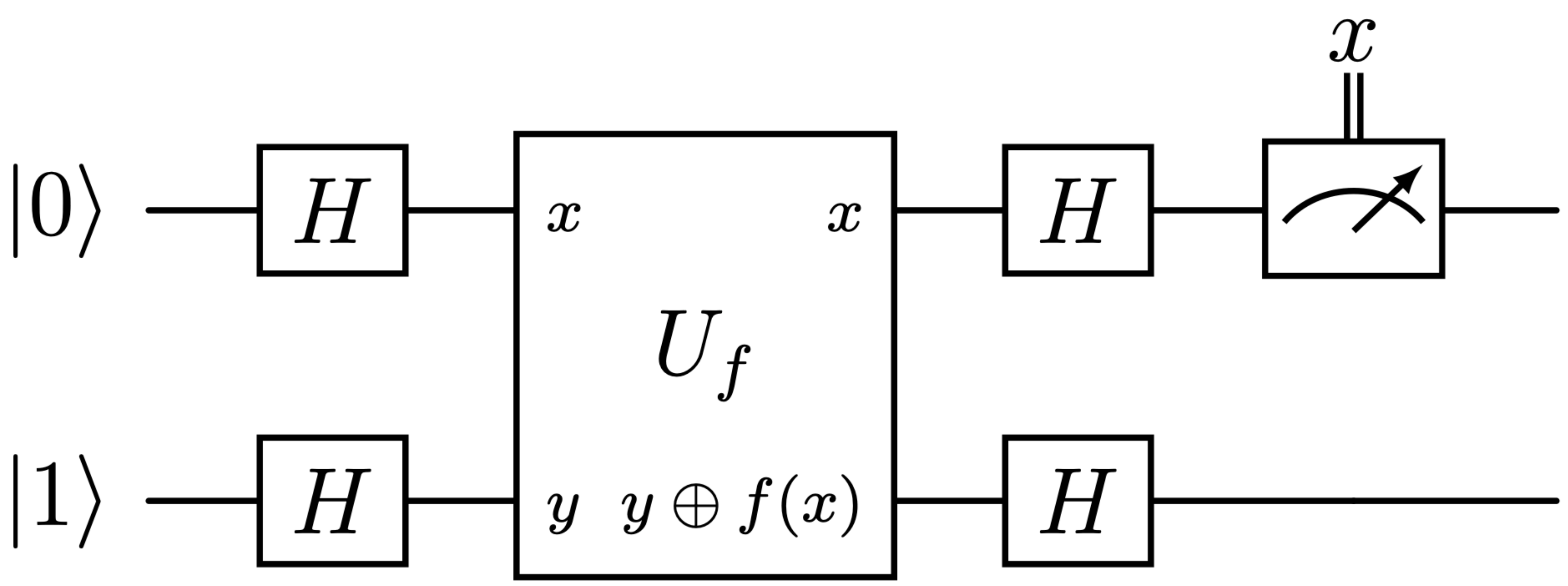

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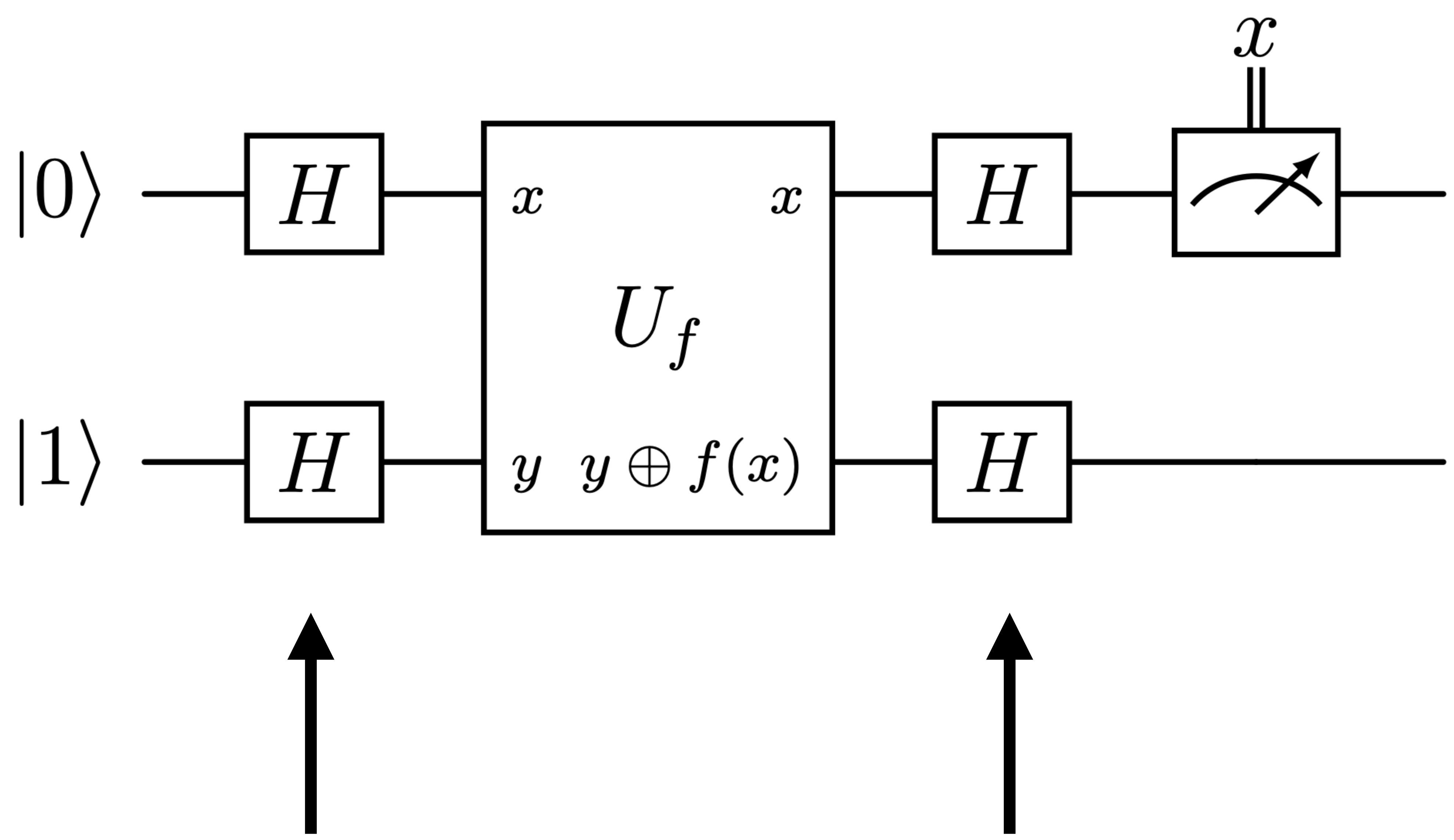
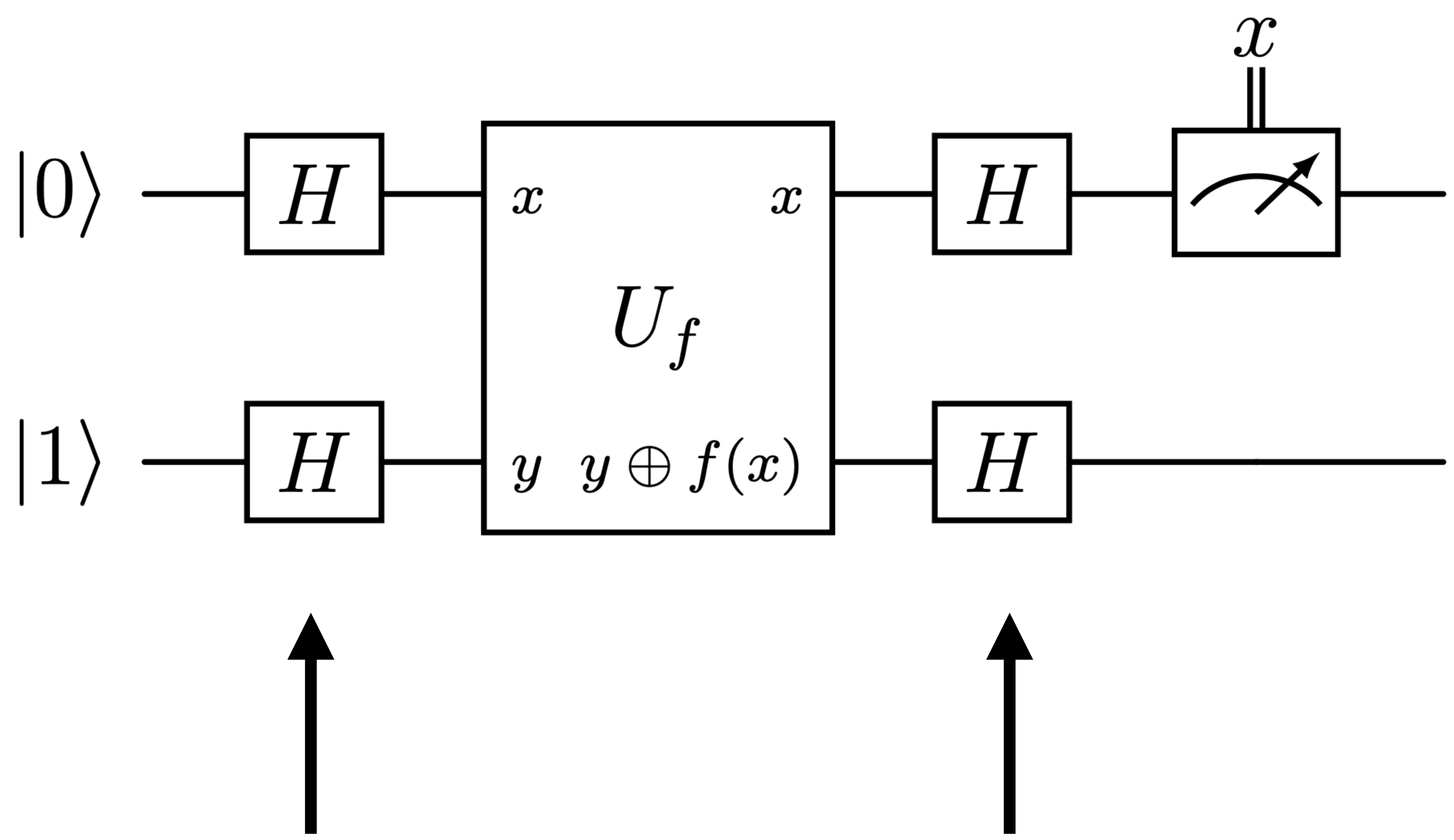


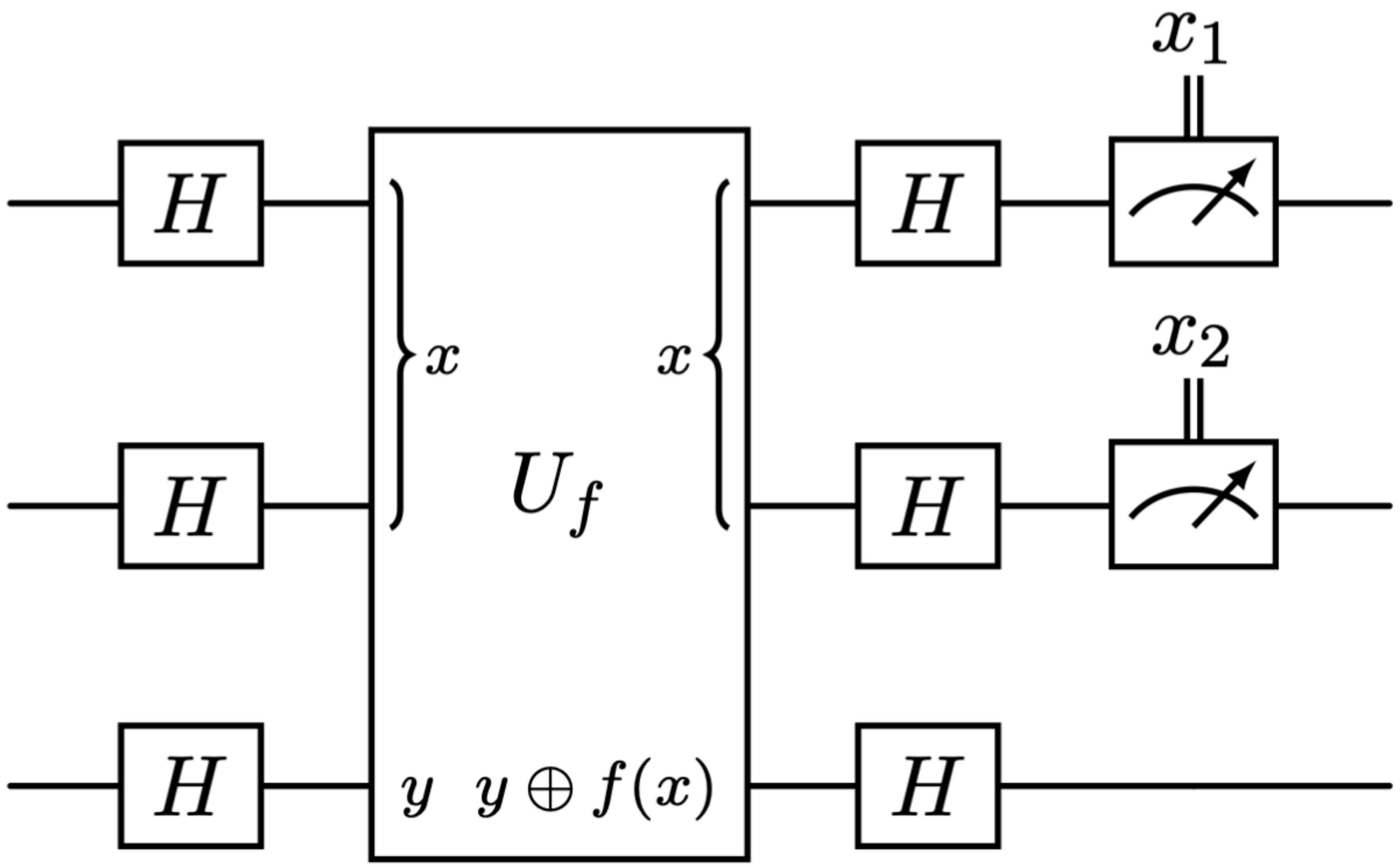
the Hadamard gates are the key *ingredient*

Deutsch-Jozsa Algorithm



the Hadamard gates are the key *ingredient* ⟶ apply f on a superposition of $0, 1$

Deutsch-Jozsa Algorithm



Algorithm extends to

$$f : \{0,1\}^n \to \{0,1\}$$

f is balanced if and only if

$$(x_1, x_2, \ldots, x_n) = (-1, -1, ..., -1)$$

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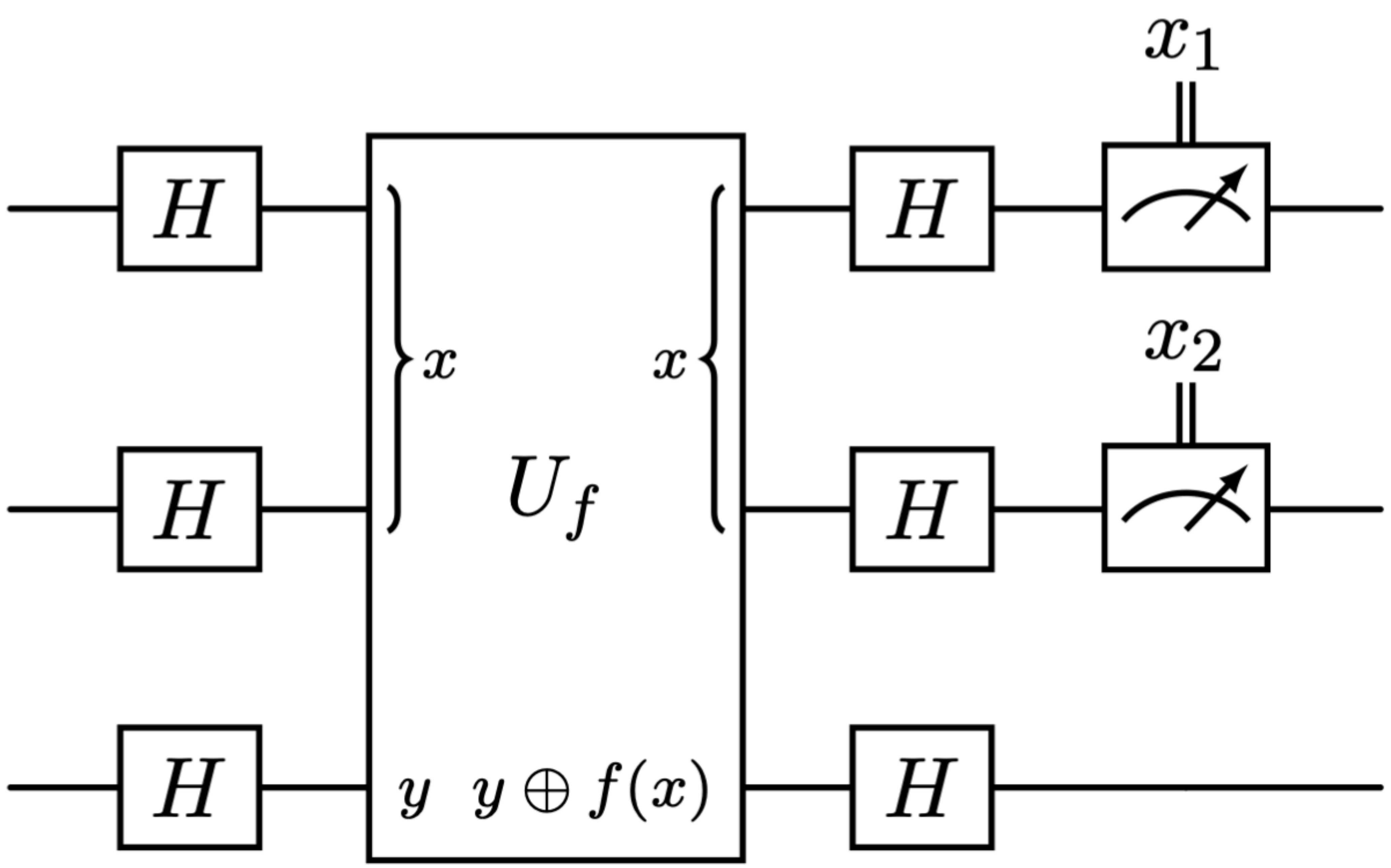

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Classical $2^{\frac{n}{2}} + 1$ ⟶ Quantum 1 step

Complexity Theory

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What is the computational cost of an algorithm?

in space (memory) and time (runtime)

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Landau notation $$\mathcal{O}(f(n)) = \left\{ g : \mathbb{R} \to \mathbb{R} \,\middle|\, \limsup_{n\to\infty} \left| \frac{g(n)}{f(n)} \right| < \infty \right\}$$ set of upper bounds

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NP Solution verifiable in $\mathcal{O}(p(n))$ space and time

BPP **P** with an bounded error probability of at least 2/3

Complexity Class NP

NP Solution verifiable in $\mathcal{O}(p(n))$ space and time

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
7								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

4	8	3	9	2	1	6	5	7
9	6	7	3	4	5	8	2	1
2	5	1	8	7	6	4	9	3
5	4	8	1	3	2	9	7	6
7	2	9	5	6	4	1	3	8
1	3	6	7	9	8	2	4	5
3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

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		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
7								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

solve →

in **EXP**

verify ←

in **P**

4	8	3	9	2	1	6	5	7
9	6	7	3	4	5	8	2	1
2	5	1	8	7	6	4	9	3
5	4	8	1	3	2	9	7	6
7	2	9	5	6	4	1	3	8
1	3	6	7	9	8	2	4	5
3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

where n is the number of empty squares

Complexity Theory

We know that

$\mathbf{L} \subsetneq \mathbf{PSPACE}$

$\mathbf{P} \subsetneq \mathbf{EXP}$

thus

$\mathbf{L} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$

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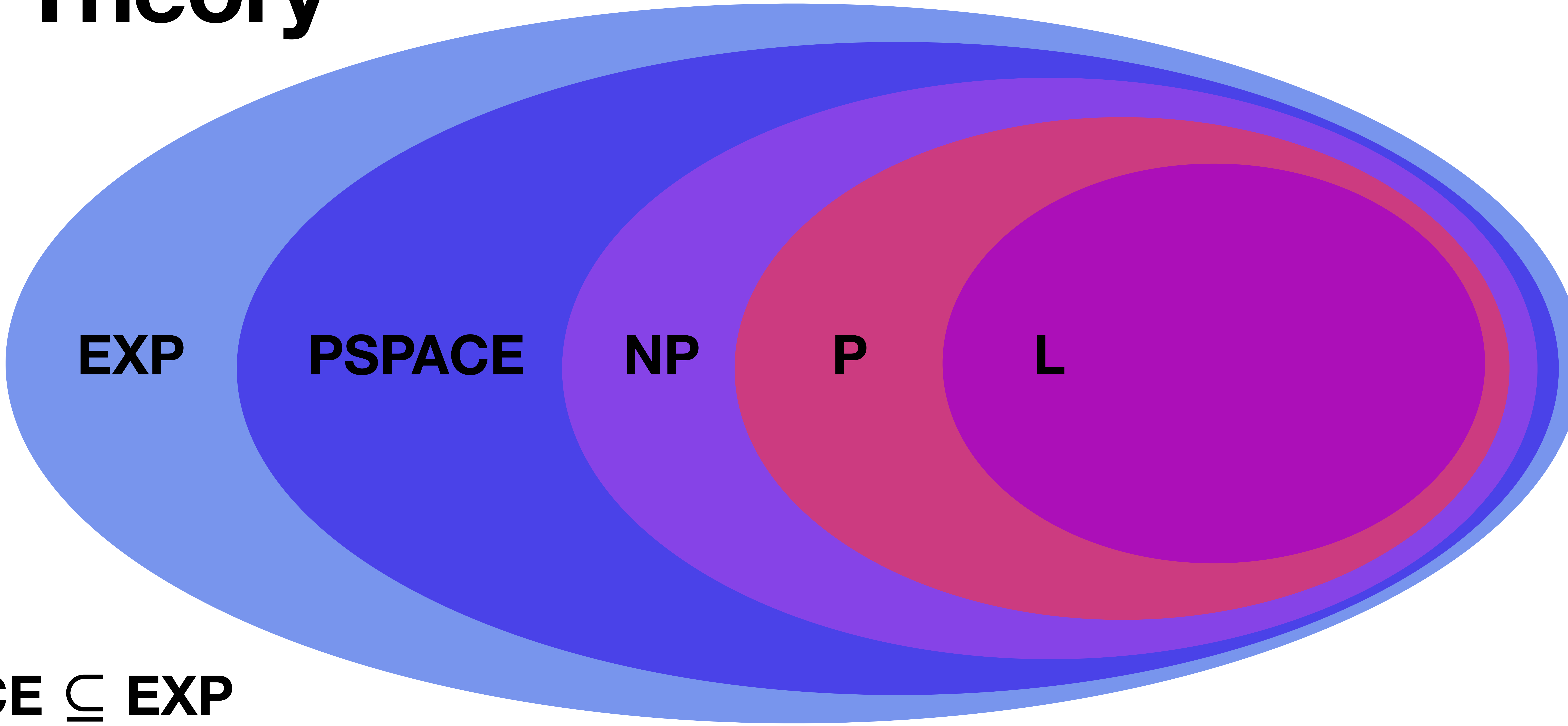

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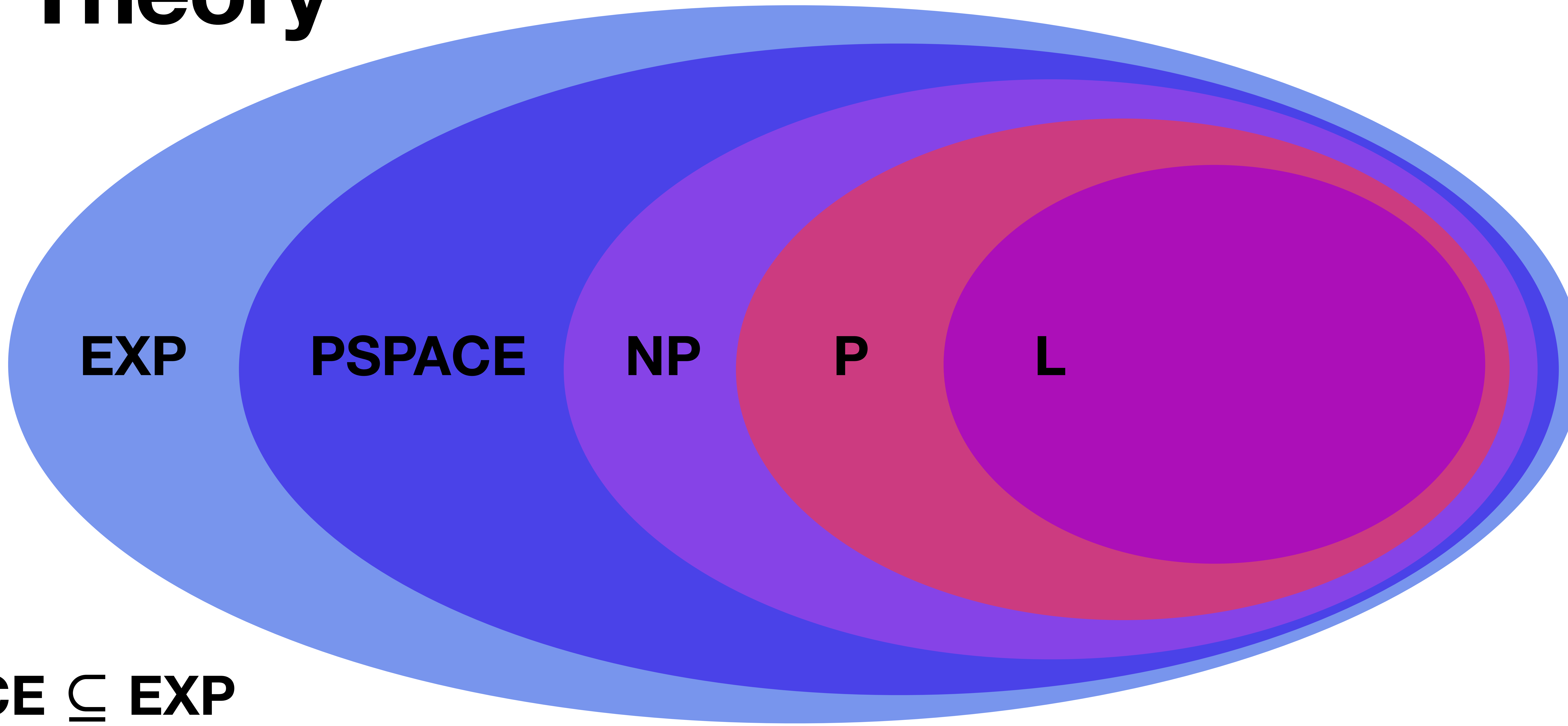

L $\subseteq$ **P** $\subseteq$ **NP** $\subseteq$ **PSPACE** $\subseteq$ **EXP**

At least one of the inclusions needs to be strict → Open problem

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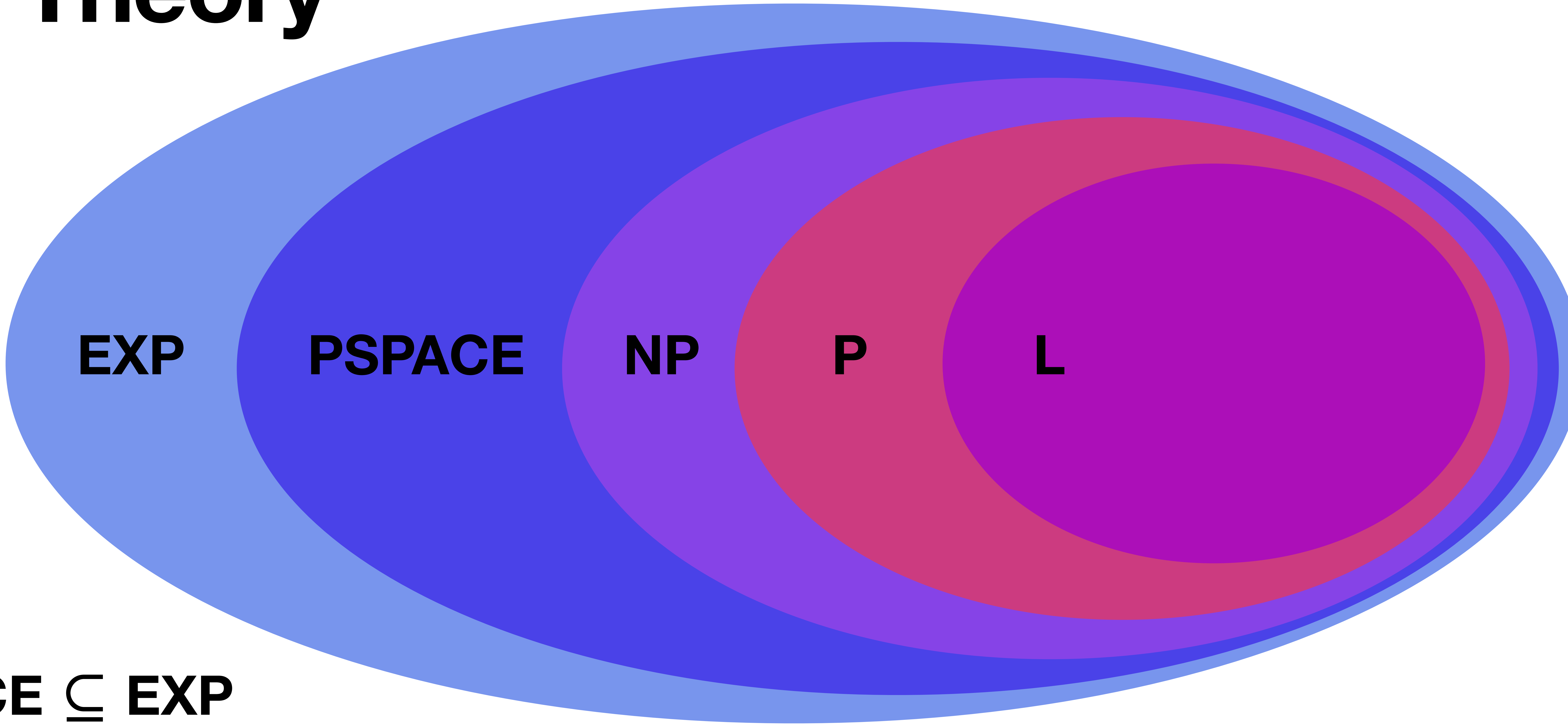

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At least one of the inclusions needs to be strict → Open problem

$\boxed{\mathbf{NP} \stackrel{?}{=} \mathbf{P}}$ Millenium Problem

Quantum complexity classes

BQP $\mathcal{O}(p(n))$ space and time complexity on quantum computer

Bounded error Quantum Probability

Quantum equivalent to BPP

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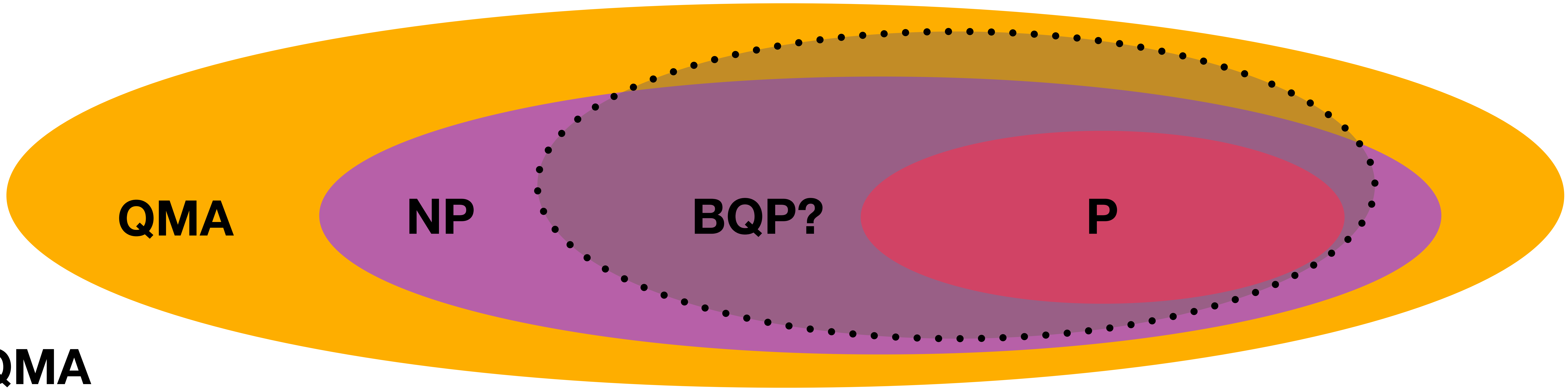

We know that

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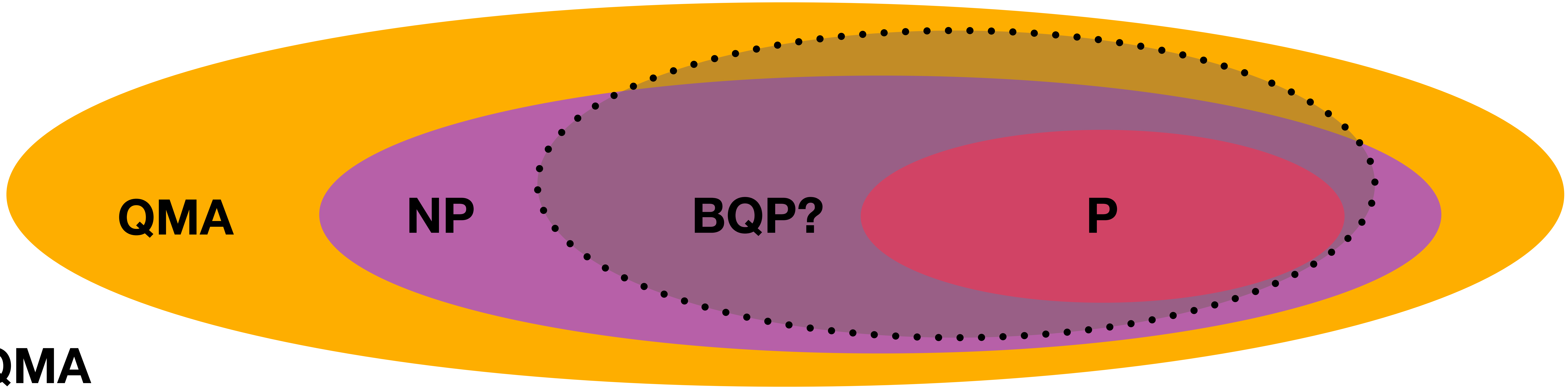

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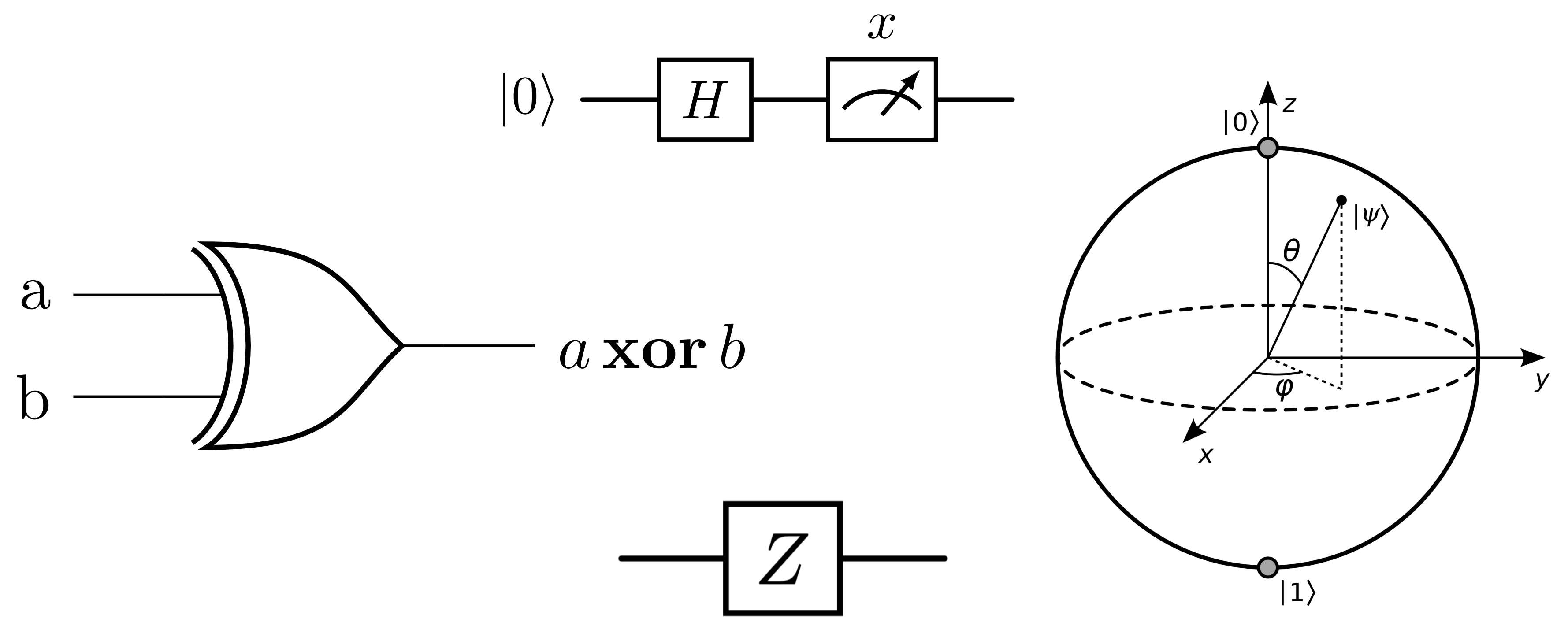
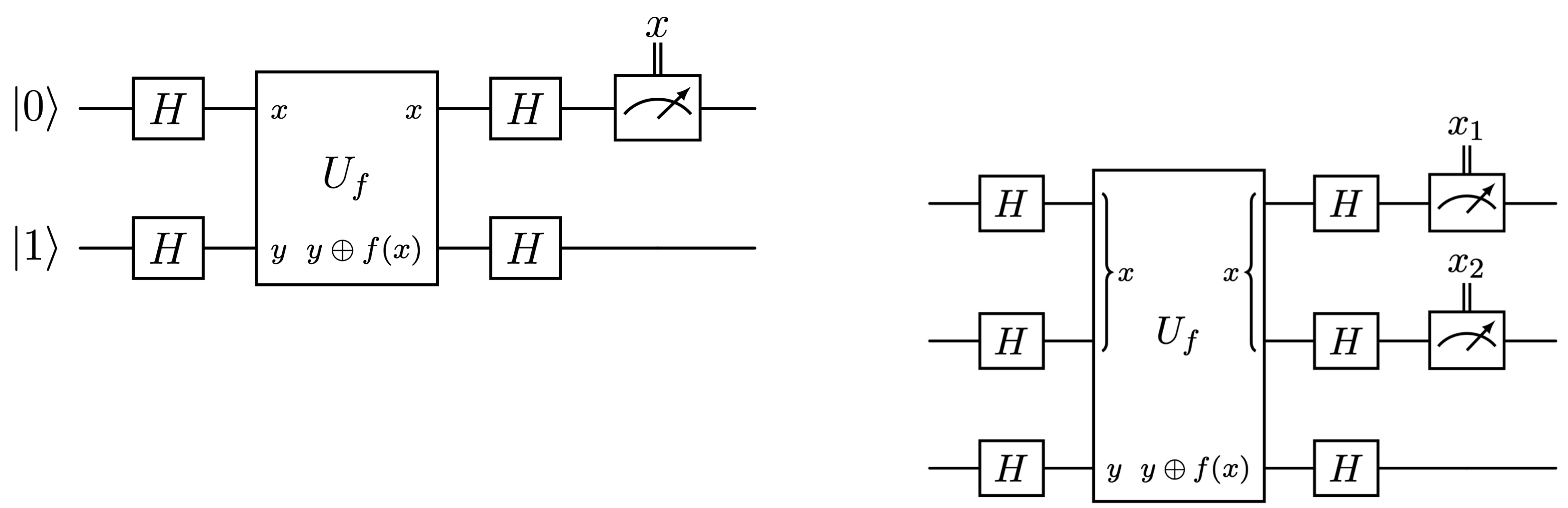
Open problem $\boxed{\textbf{NP} \overset{?}{\subseteq} \textbf{BQP}}$

Summary

Classical and Quantum Circuits

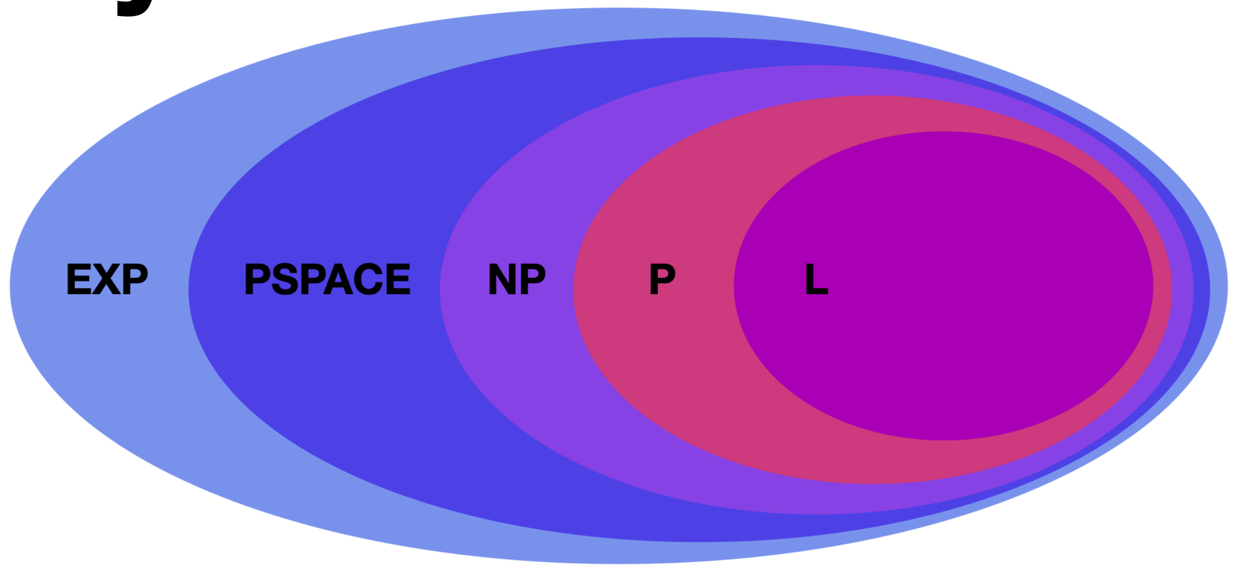


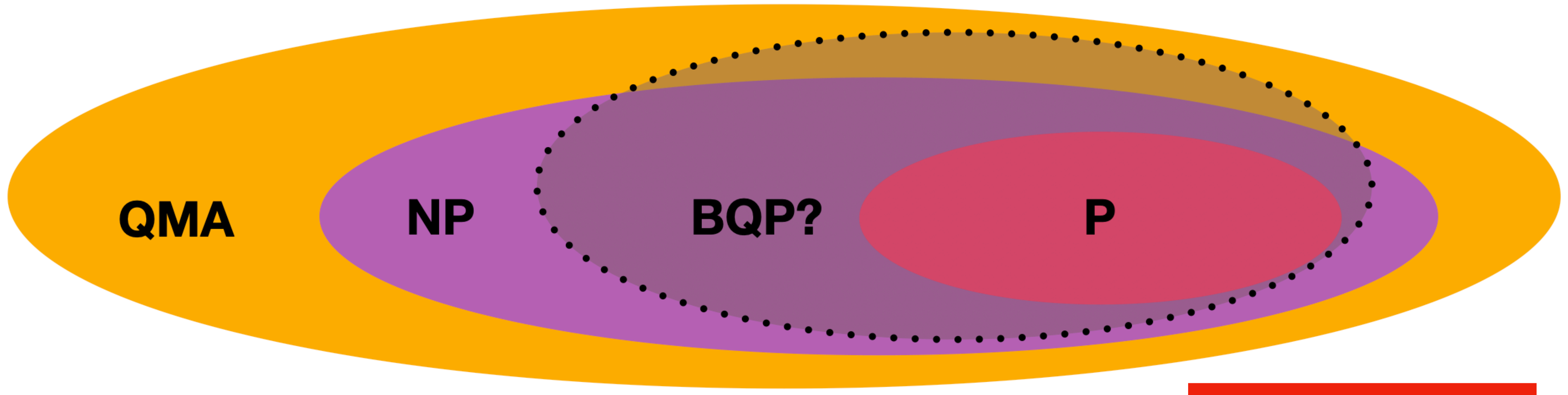
Quantum Algorithms



Complexity theory

$\textbf{NP} \stackrel{?}{=} \textbf{P}$



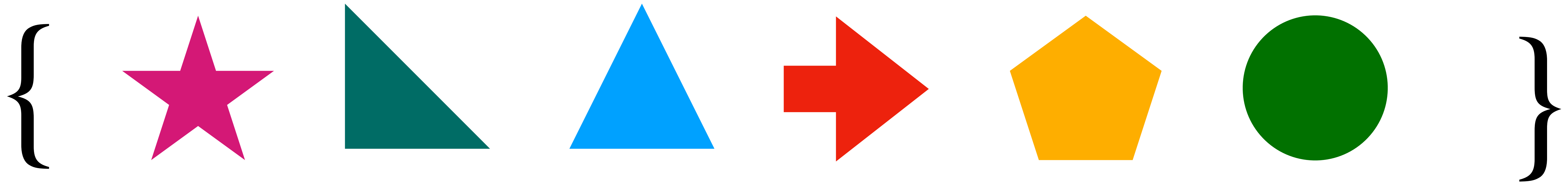


$\textbf{NP} \stackrel{?}{\subseteq} \textbf{BQP}$

Grover’s Algorithm

Classical Search

Searching an item in an unordered list of size N



takes on average $\dfrac{N}{2}$ steps $\rightarrow \mathcal{O}(N)$ in time, thus the problem is in **P**

Quantum Grover's Algorithm is $\mathcal{O}(\sqrt{N})$ in time $\rightarrow$ in **BQP**

Grover's Algorithm

We consider $N = 2$

Solution direction

Initial state $|s\rangle = \frac{1}{\sqrt{2}}(|\omega\rangle + |s'\rangle)$

1. step: $|s\rangle \rightarrow U_\omega |s\rangle$ $U_\omega = \mathbf{Id} - 2|\omega\rangle\langle\omega|$

Reflect the $|\omega\rangle$ - component

2. step: $U_\omega |s\rangle \rightarrow U_s U_\omega |s\rangle$ $U_s = 2|s\rangle\langle s| - \mathbf{Id}$

Reflect around $|s\rangle$ - component