Quantum Computing

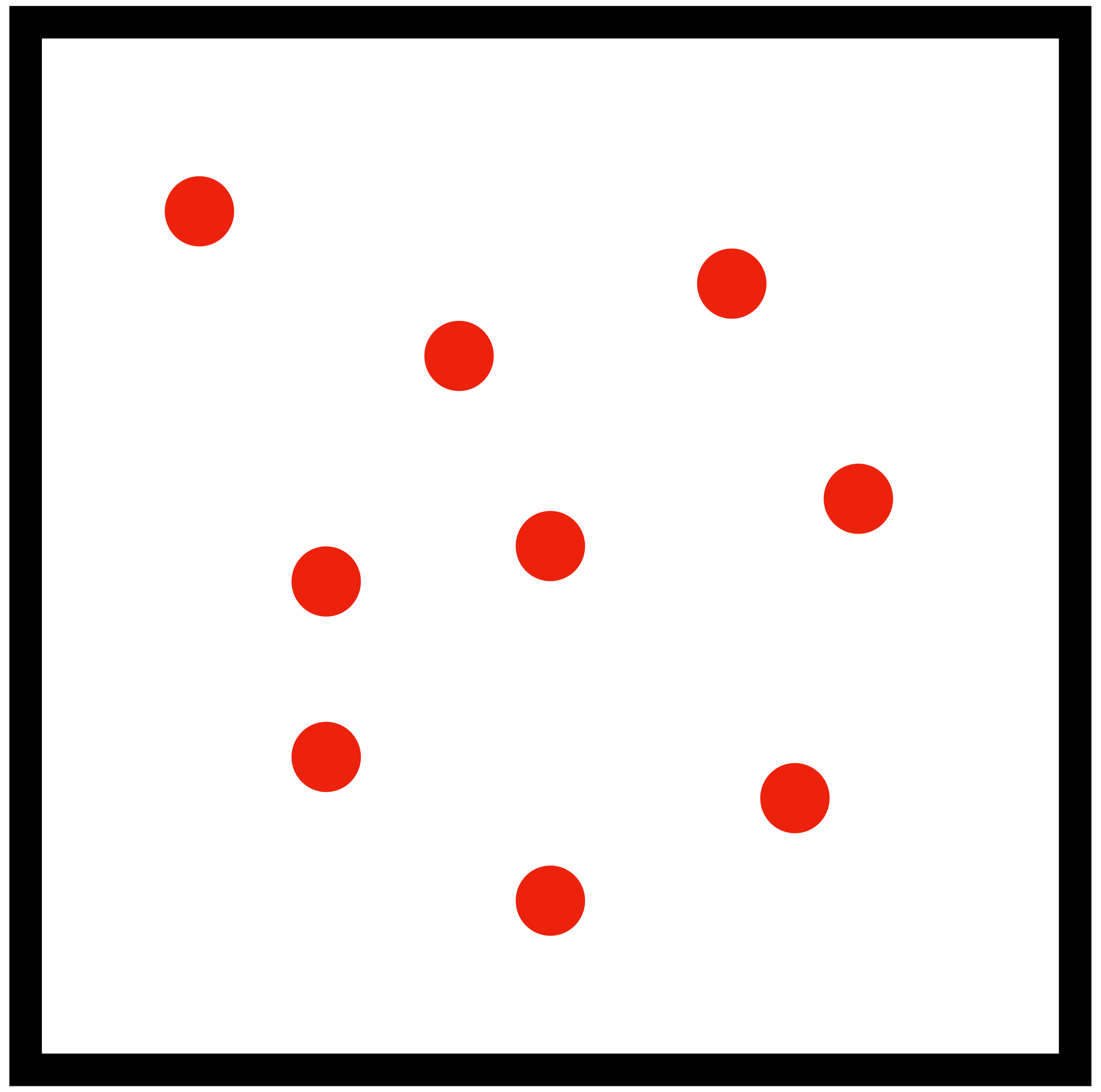
Algorithms and Complexity

Hakim Rachidi, Göttingen SoSe 25

Motivation

Complexity of Quantum

Classical



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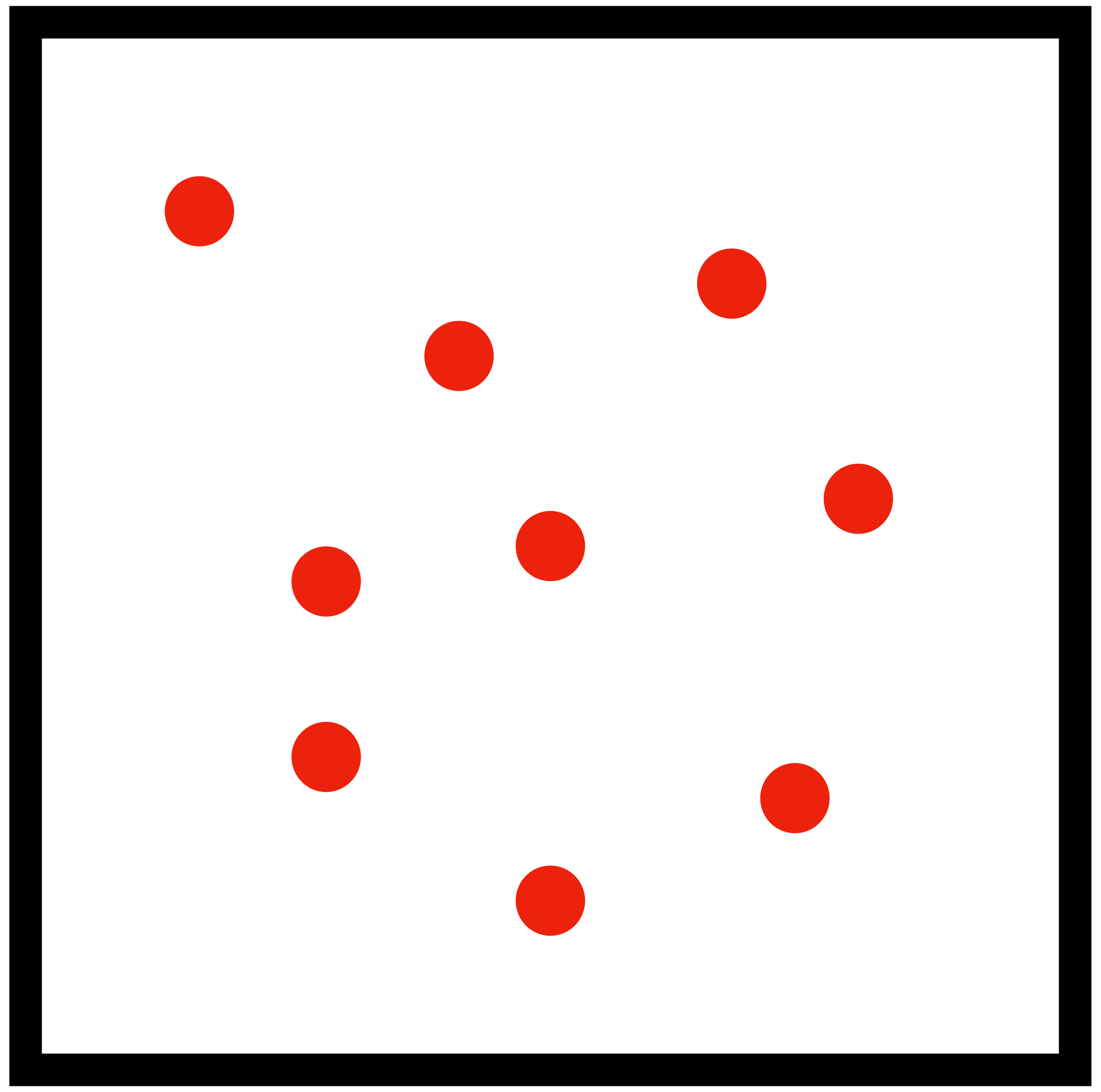

Position $q_i \in \mathbb{R}^f$

Momentum $p_i \in \mathbb{R}^f$

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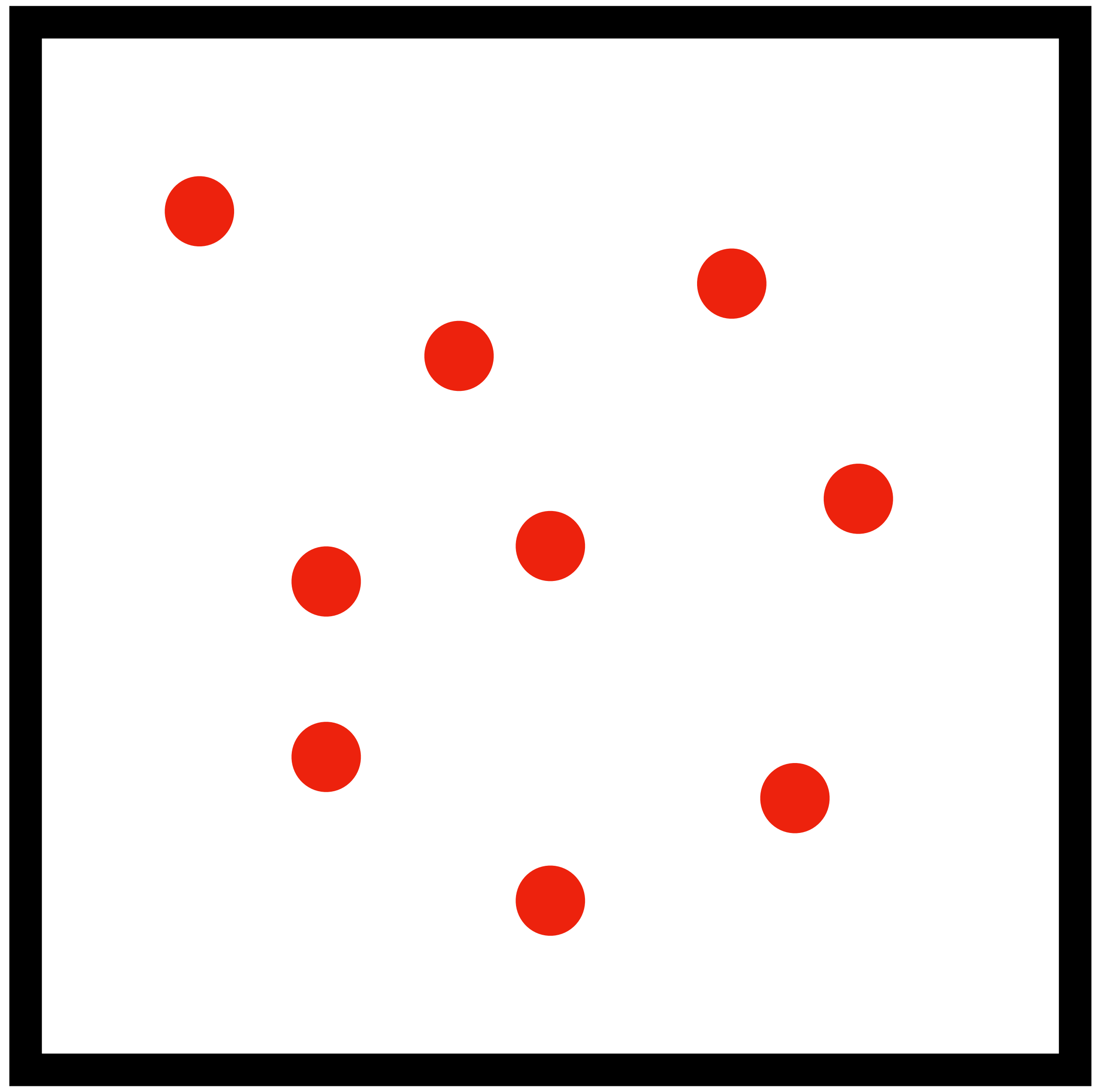

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$$\boxed{\dim \Gamma = 2fN}$$

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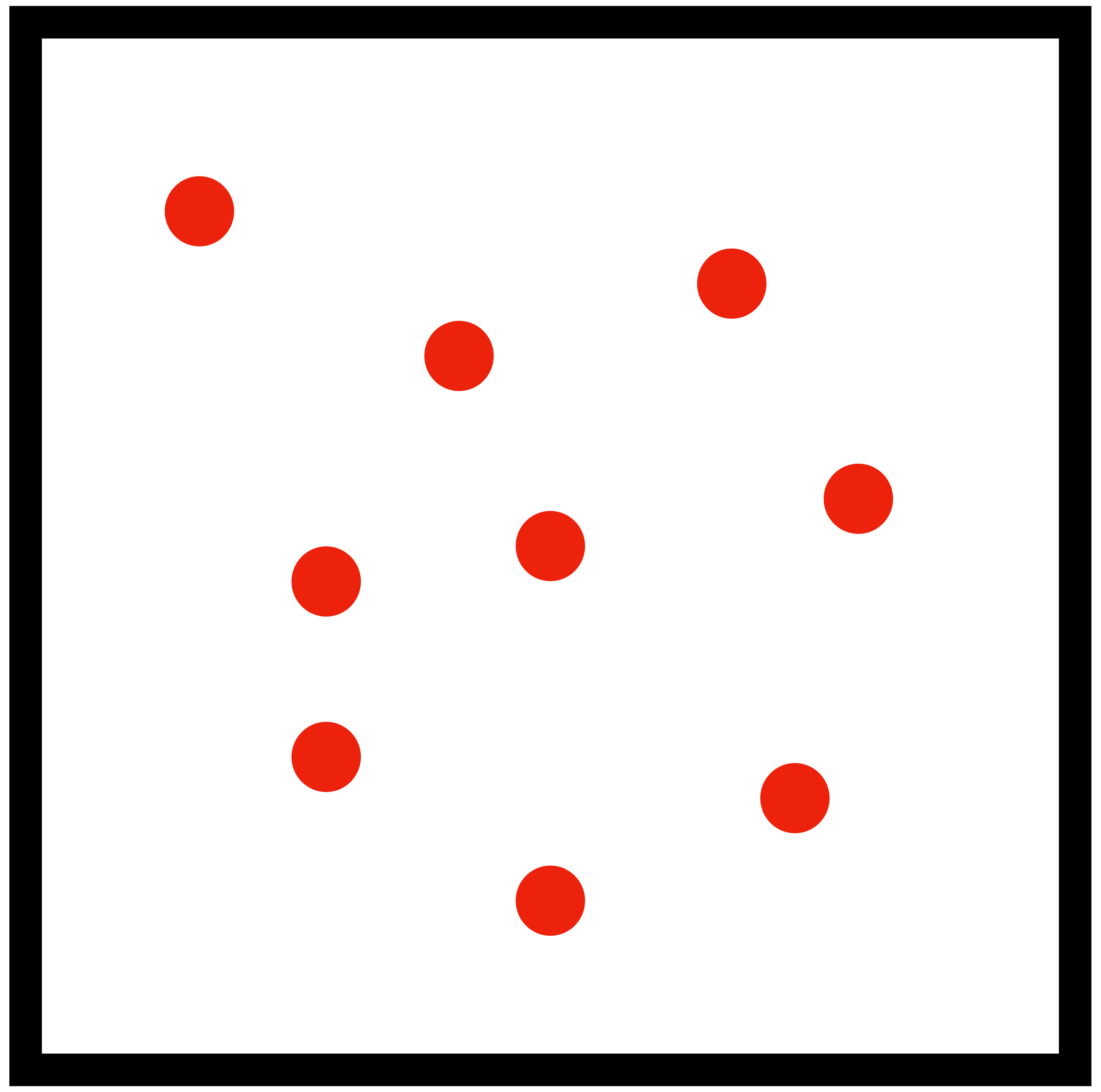

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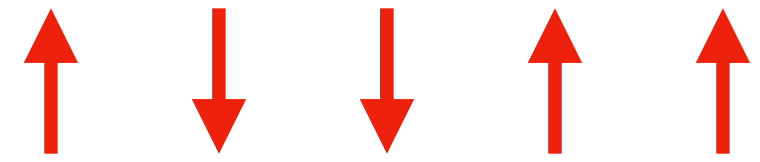

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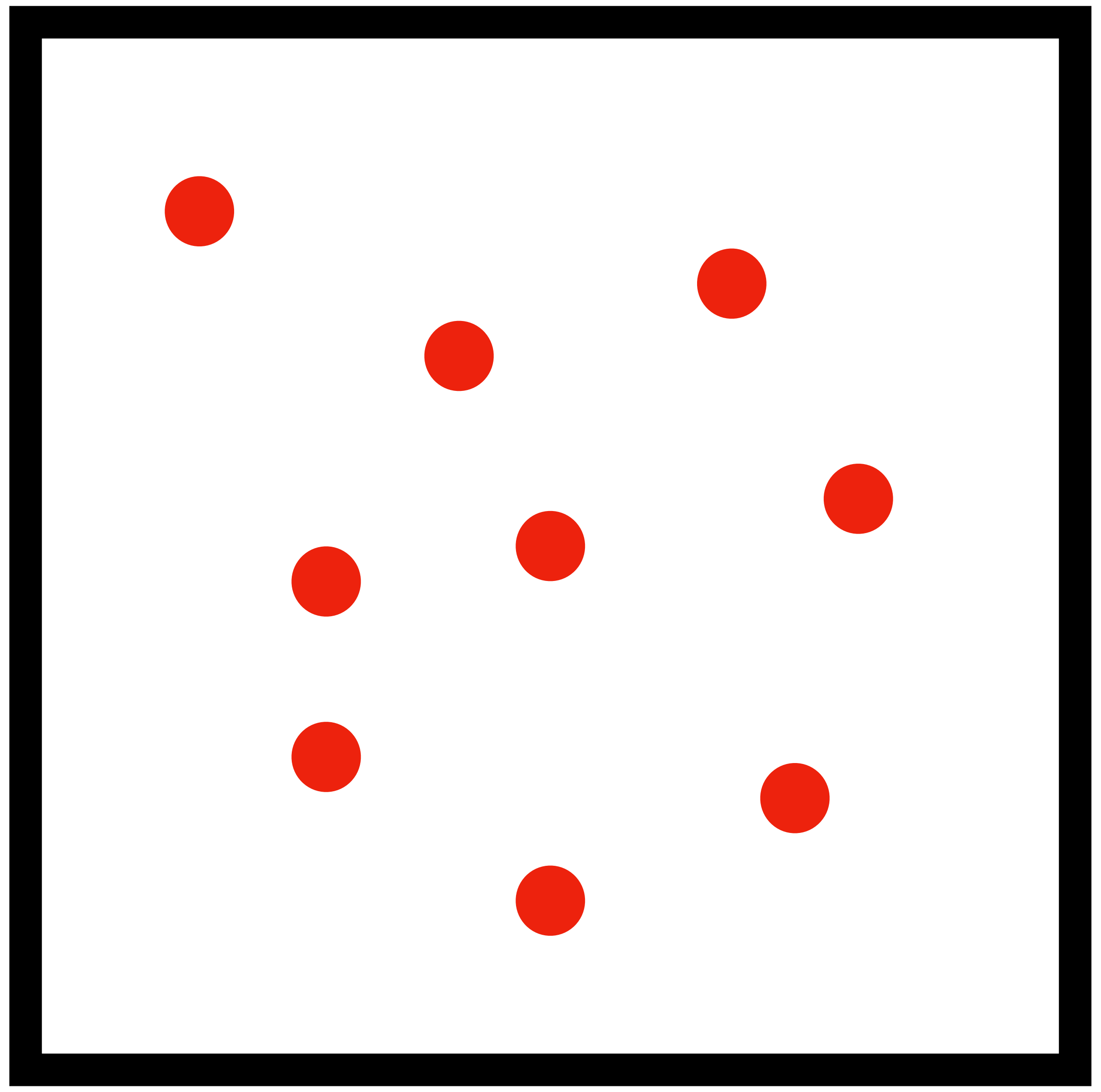

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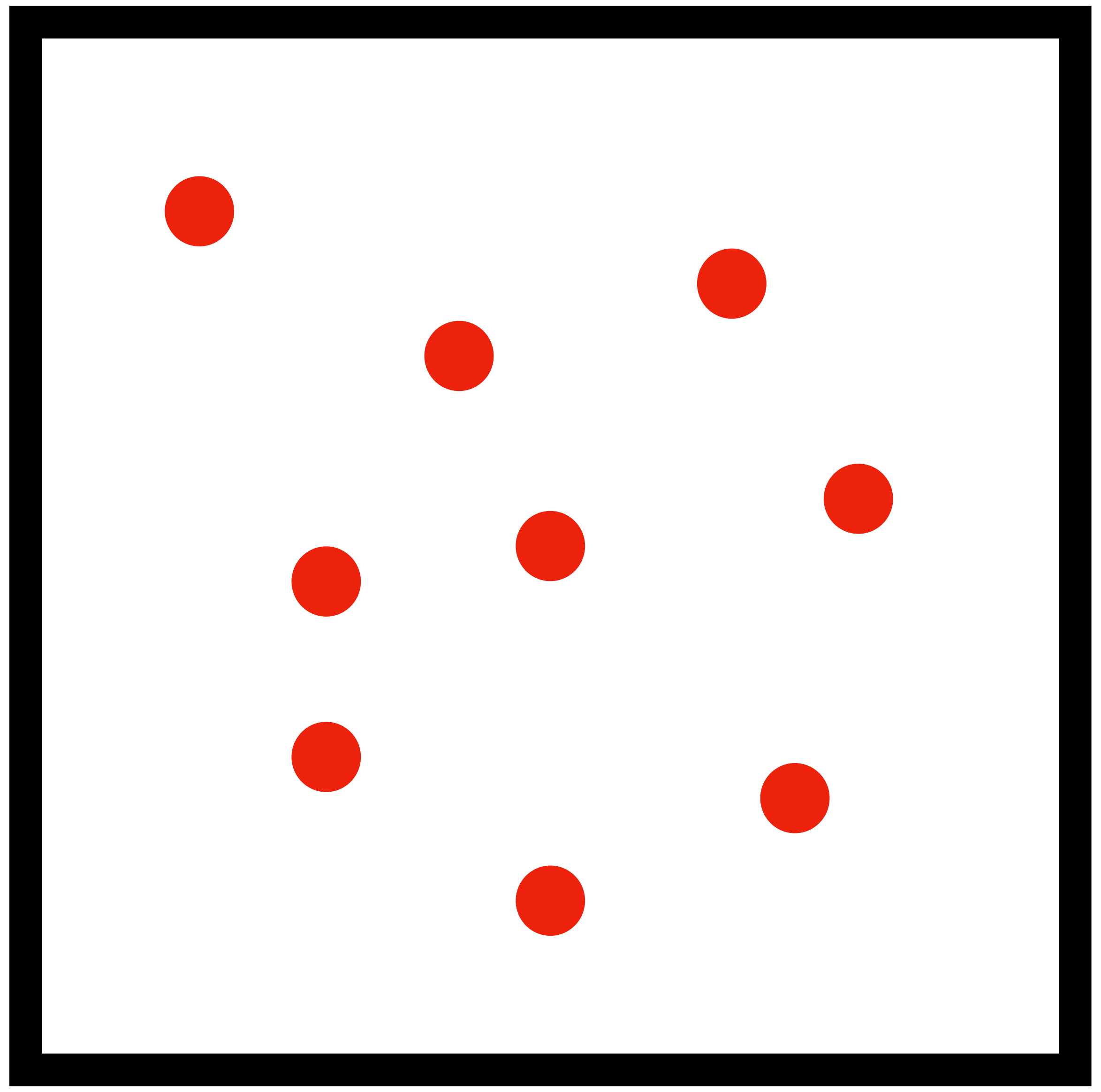

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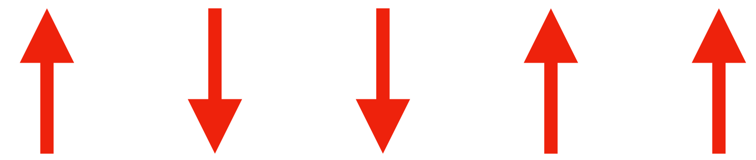

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$$\dim(\mathscr{H}_1 \otimes \mathscr{H}_2) = (\dim \mathscr{H}_1) \cdot (\dim \mathscr{H}_2)$$

$$\boxed{\dim \mathscr{H} = 2^N}$$

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Computational effort scales <u>polynomially</u>

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for $N = 272$

$2^N >$ # Atoms in the visible universe

Computational effort scales
<u>exponentially</u>

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Infeasible

Motivation

Simulating quantum systems

Can we exploit quantum systems to simulate quantum systems?

— Feynmann (1982)

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→Universal Quantum Simulators (Quantum Computers)

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What is complex and what is efficient?

Multiplication

$$113 \cdot 73 = 8249$$

Number of steps <u>linear</u> in the number of digits

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Can we use the complexity of quantum systems to speedup algorithms?

Guiding questions

Outline of the talk

Which quantum algorithms exist? and *How* do they work?

Deutsch-Jozsa, Grover's Algorithm

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Which quantum algorithms exist? and *How* do they work?

Deutsch-Jozsa, Grover's Algorithm

How much faster are they compared to classical counterpart?

Complexity Theory

Fundamentals of Quantum Computing

Quantum Bit

Classical Bit

$$X \in \{0,1\}$$

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$\longrightarrow$

Quantum Bit (Qubit)

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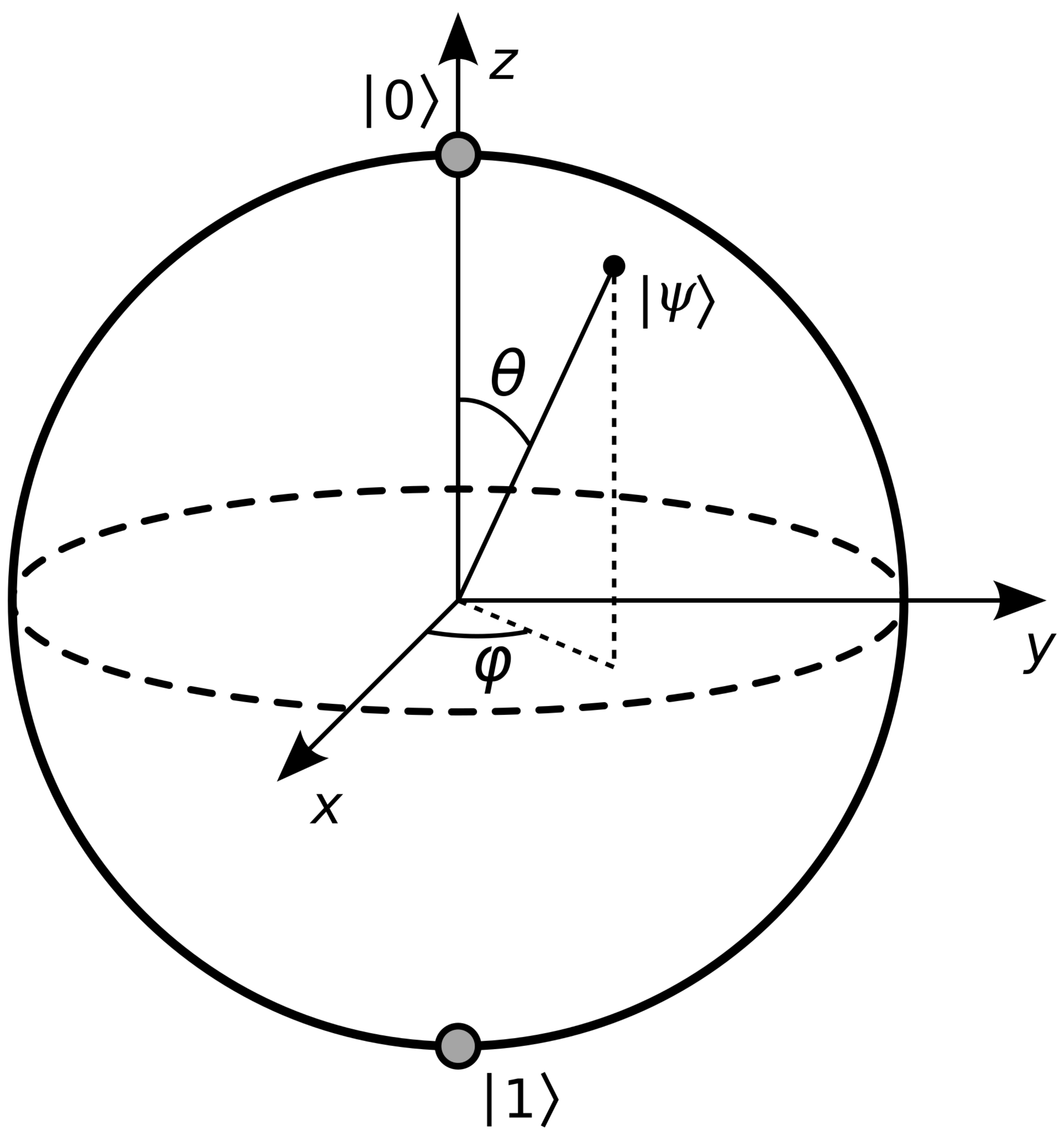

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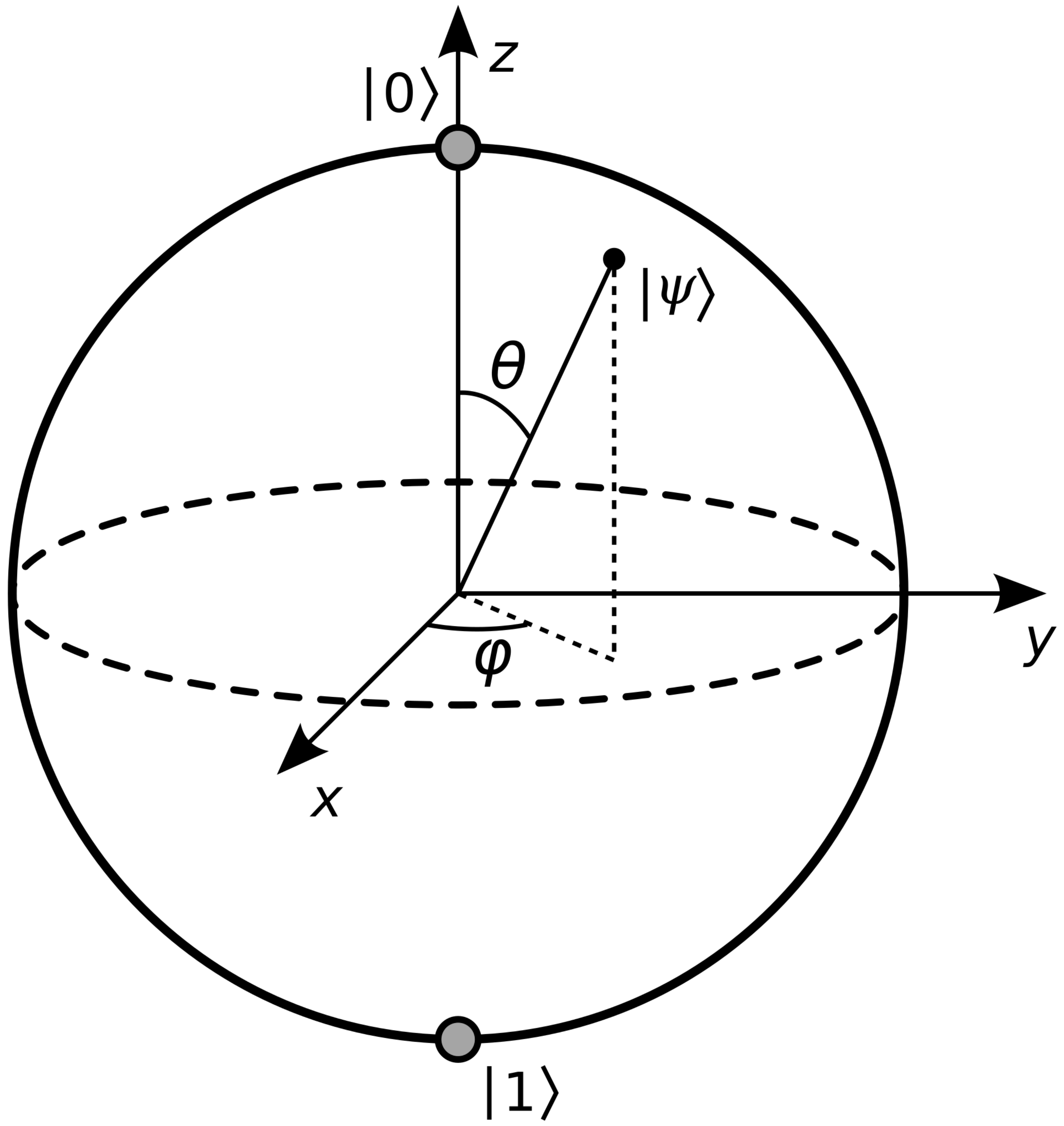

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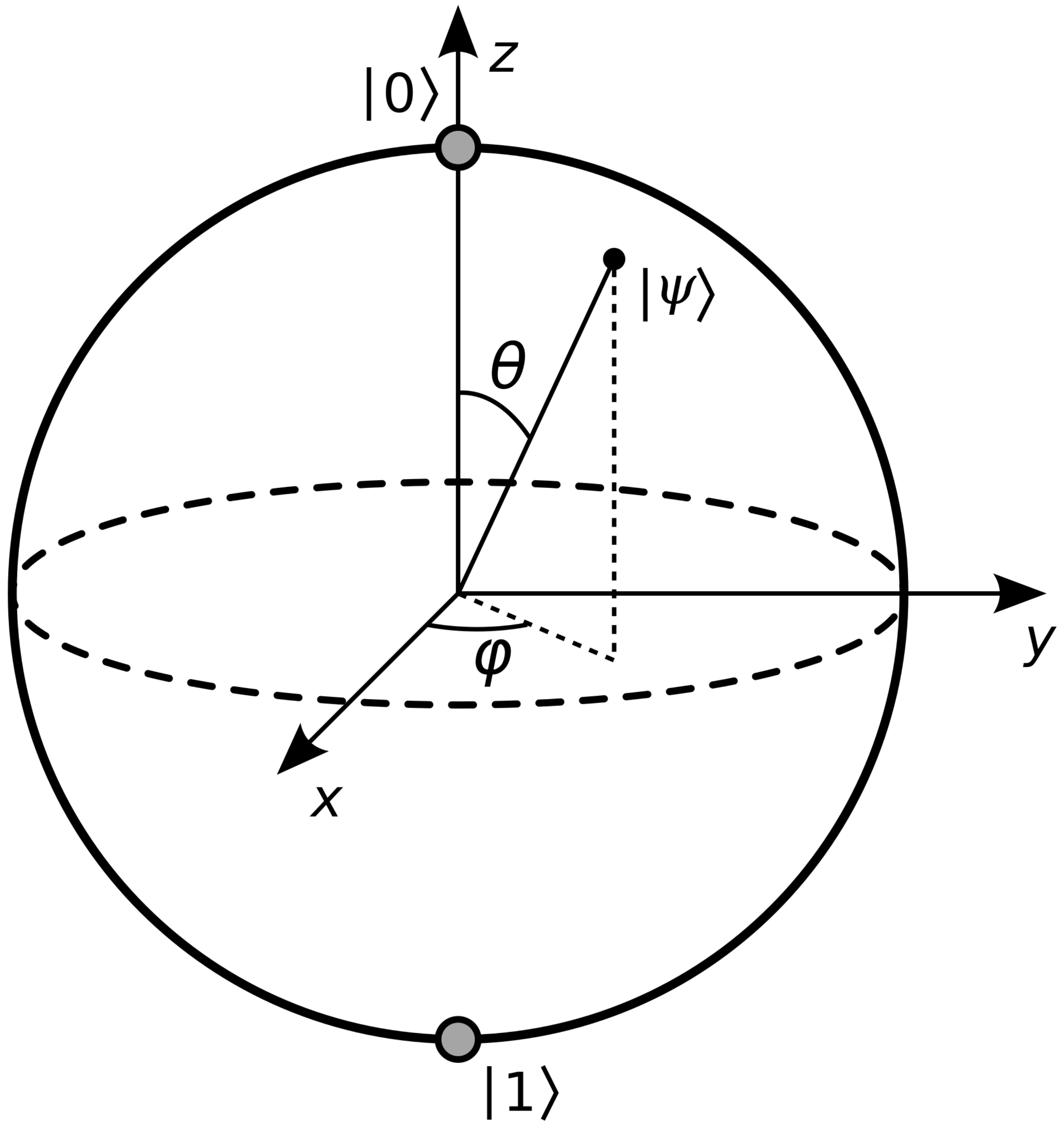
→ measurement yields a classical Bit

Upon measurement only a *finite* amount is attainable



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e.g. $S = 1001$

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We will use the short hand notation

$$|0\rangle \otimes |1\rangle \otimes |0\rangle = |010\rangle$$

Classical logic gates

Processing of classical bits can be described in the language of *gates*

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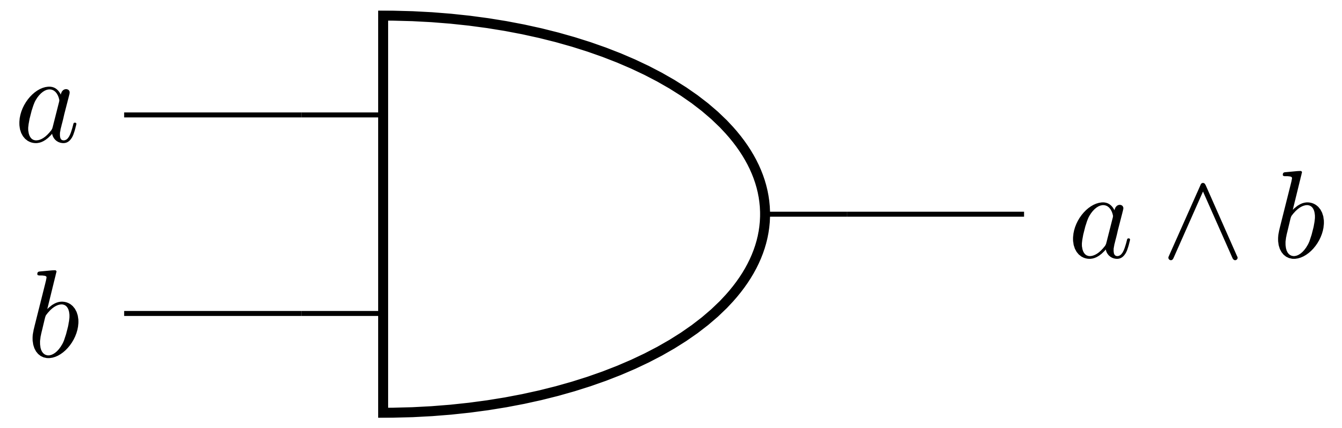

AND Gate

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

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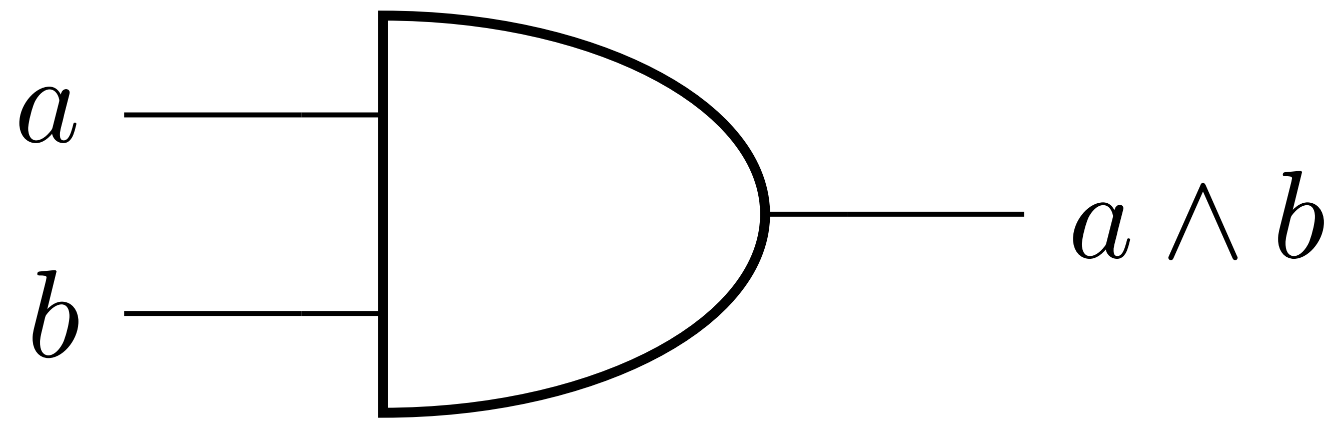

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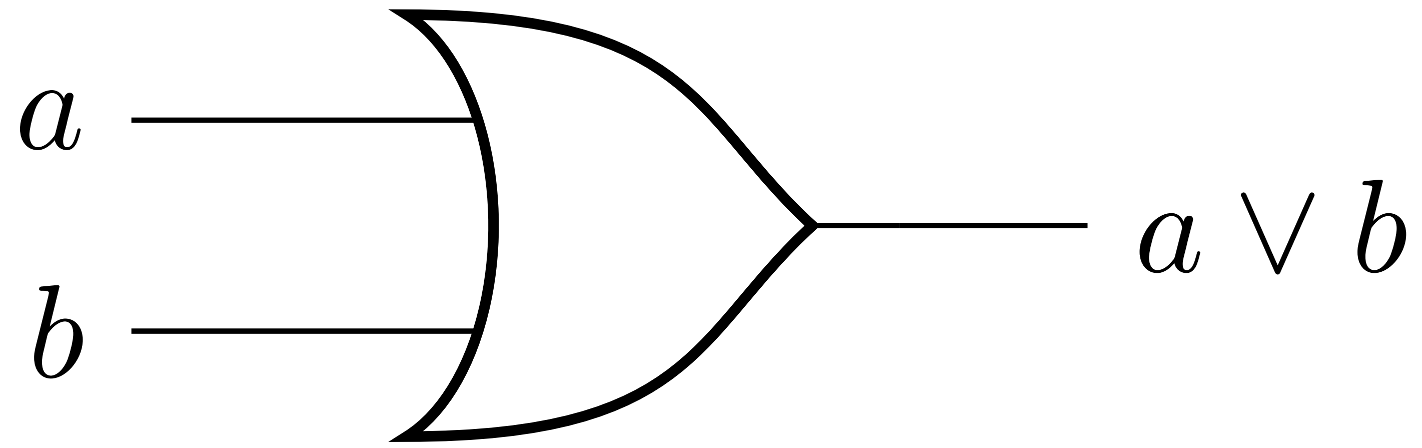

OR Gate

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
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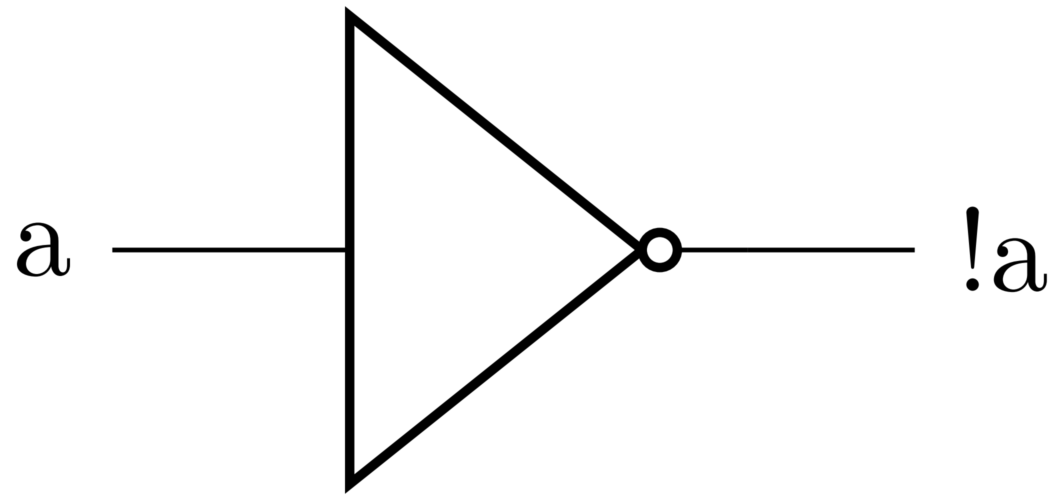

NOT Gate

a	$!a$
0	1
1	0

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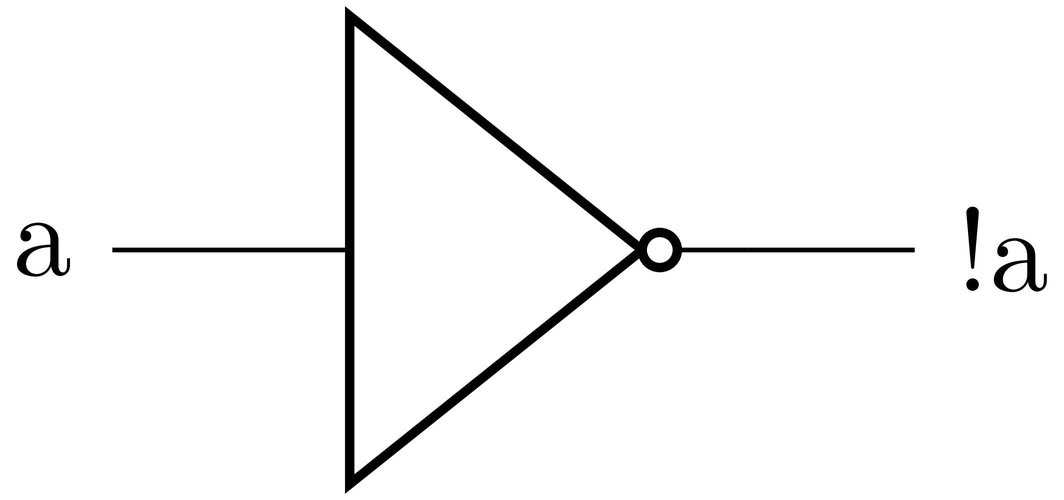

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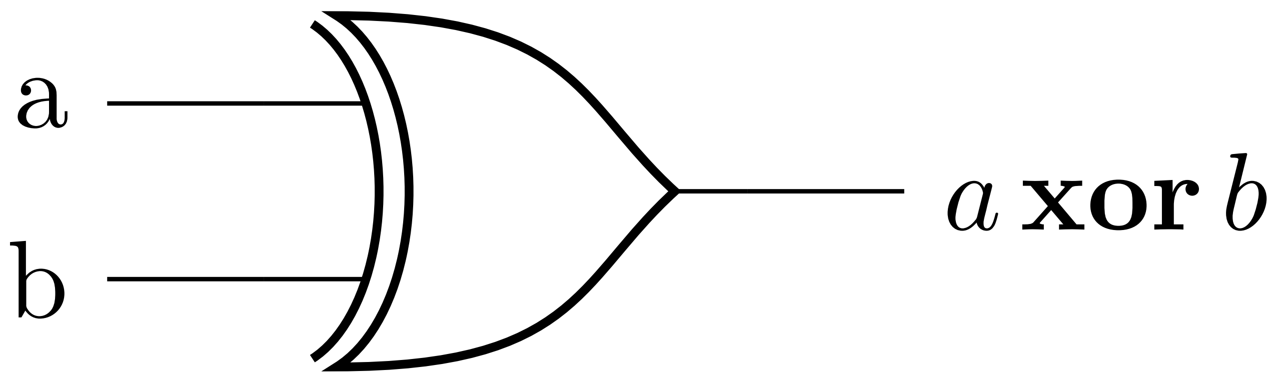

XOR Gate

a	b	a **xor** b
0	0	0
0	1	1
1	0	1
1	1	0

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Since the theory is <u>linear</u> all quantum gates need to be linear

Quantum Gates

Hadamard Gate

$$\alpha|0\rangle + \beta|1\rangle \quad - \boxed{H} - \quad \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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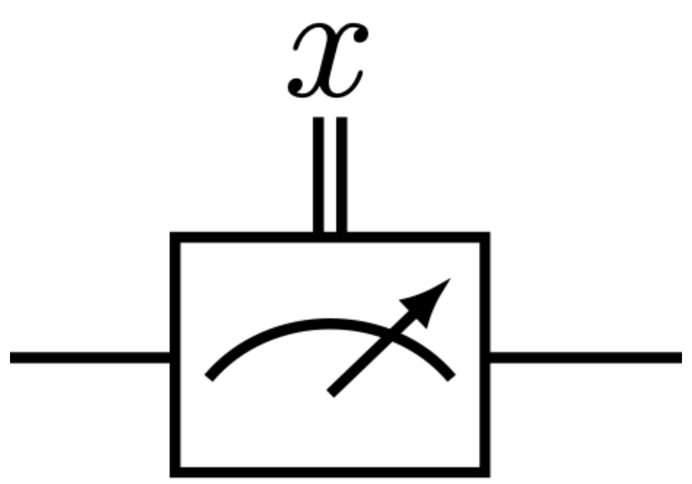
Measurement Gate



$x \in \{-1, 1\}$

$\hat{Z}|\psi\rangle = x|\psi\rangle$

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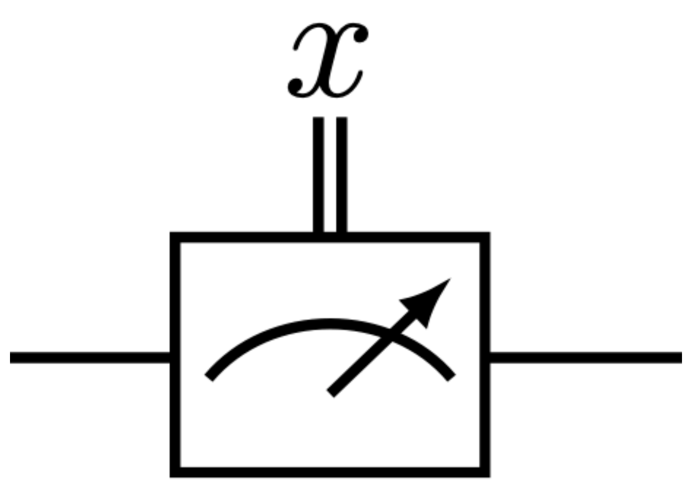

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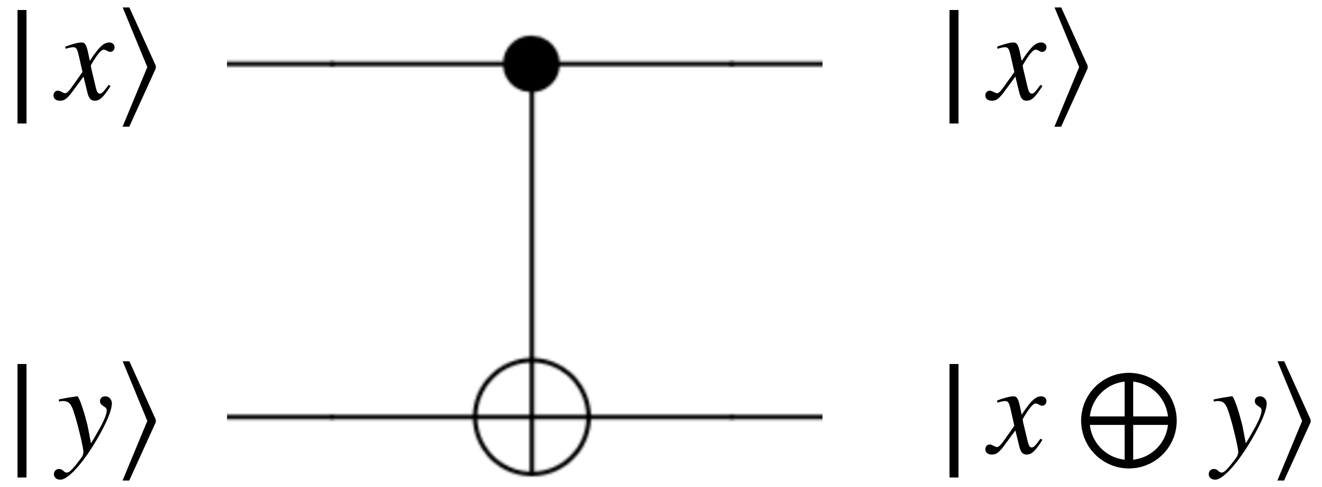
Control Gates

flip the second qubit if the first is $|1\rangle$

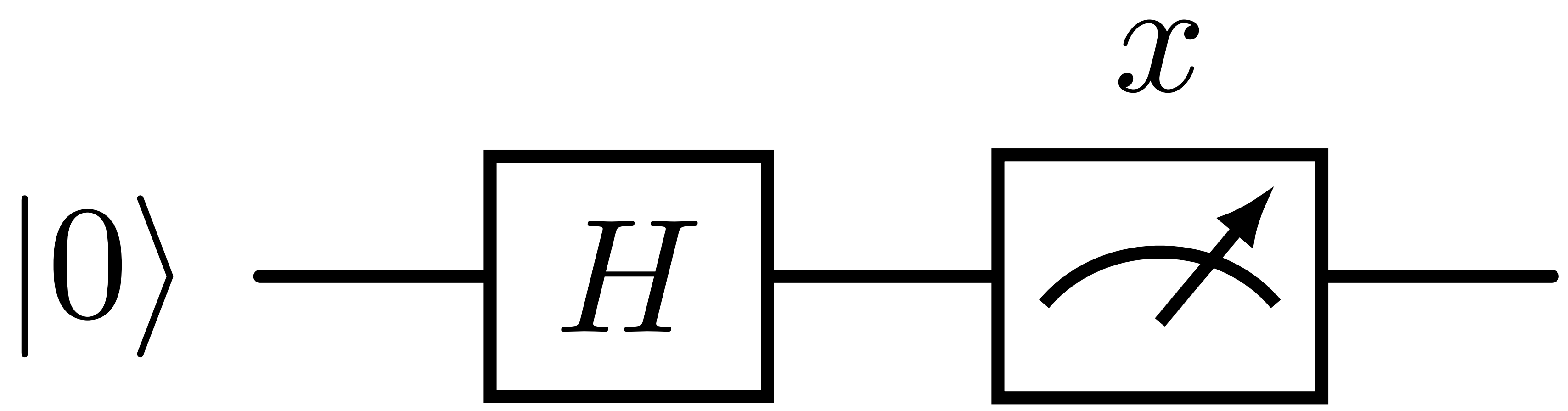


$$0 \oplus 0 = 0$$
$$1 \oplus 0 = 1$$
$$0 \oplus 1 = 1$$
$$1 \oplus 1 = 0$$

Quantum Circuits

Similar to classical circuits, we can connect Quantum gates

Example: Random number generator



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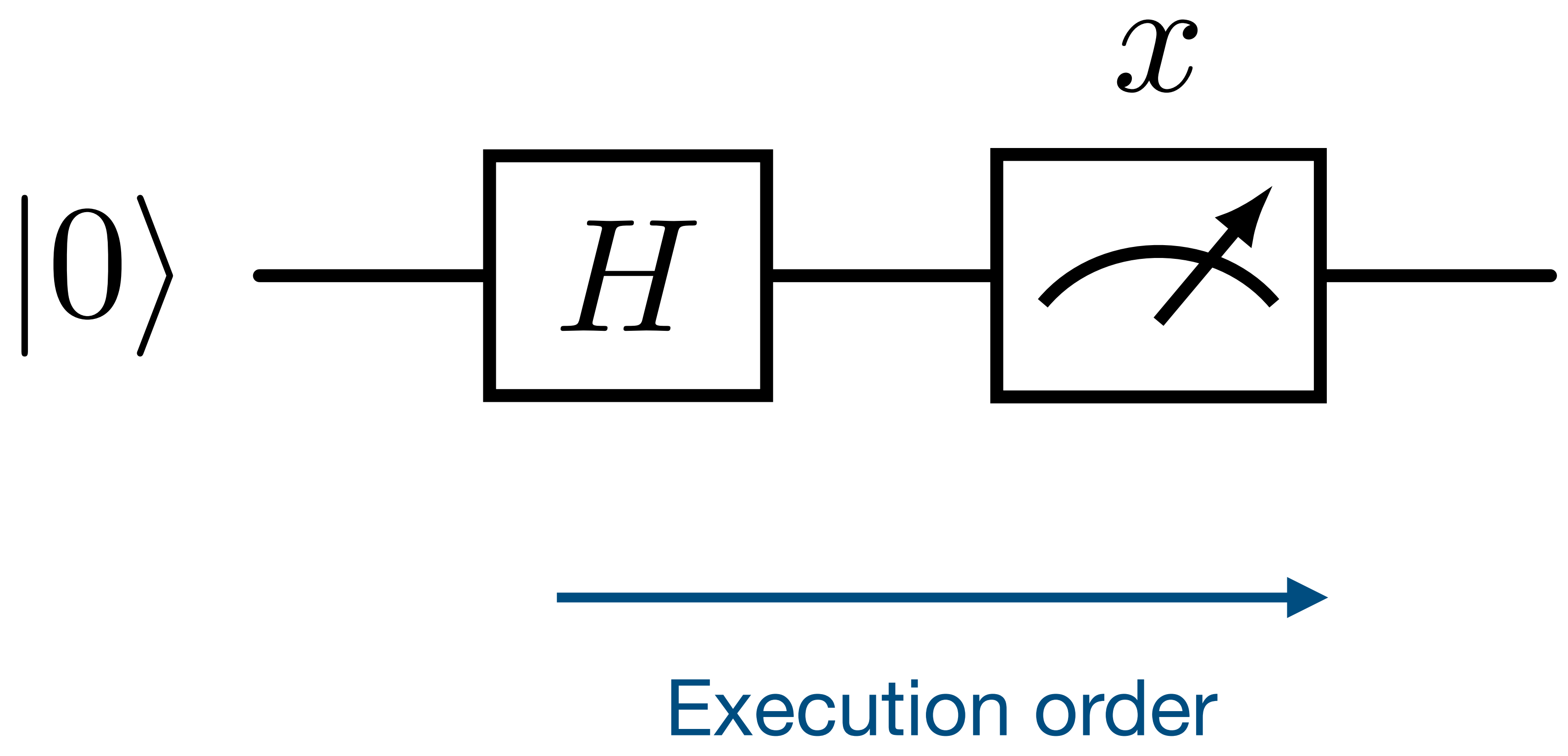

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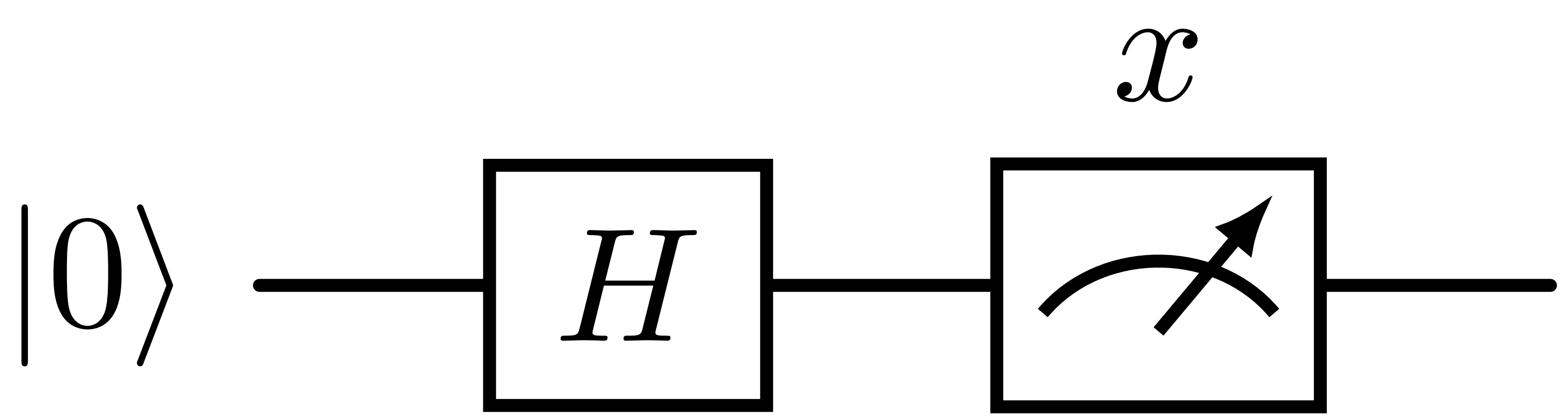

Execution order

$$|0\rangle \xrightarrow{\hat{H}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

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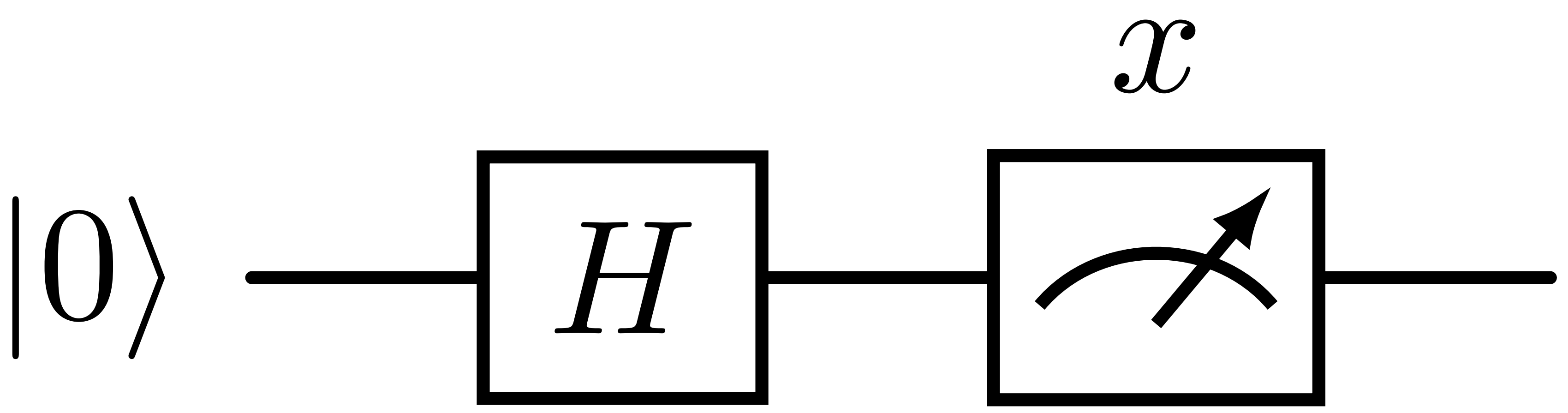

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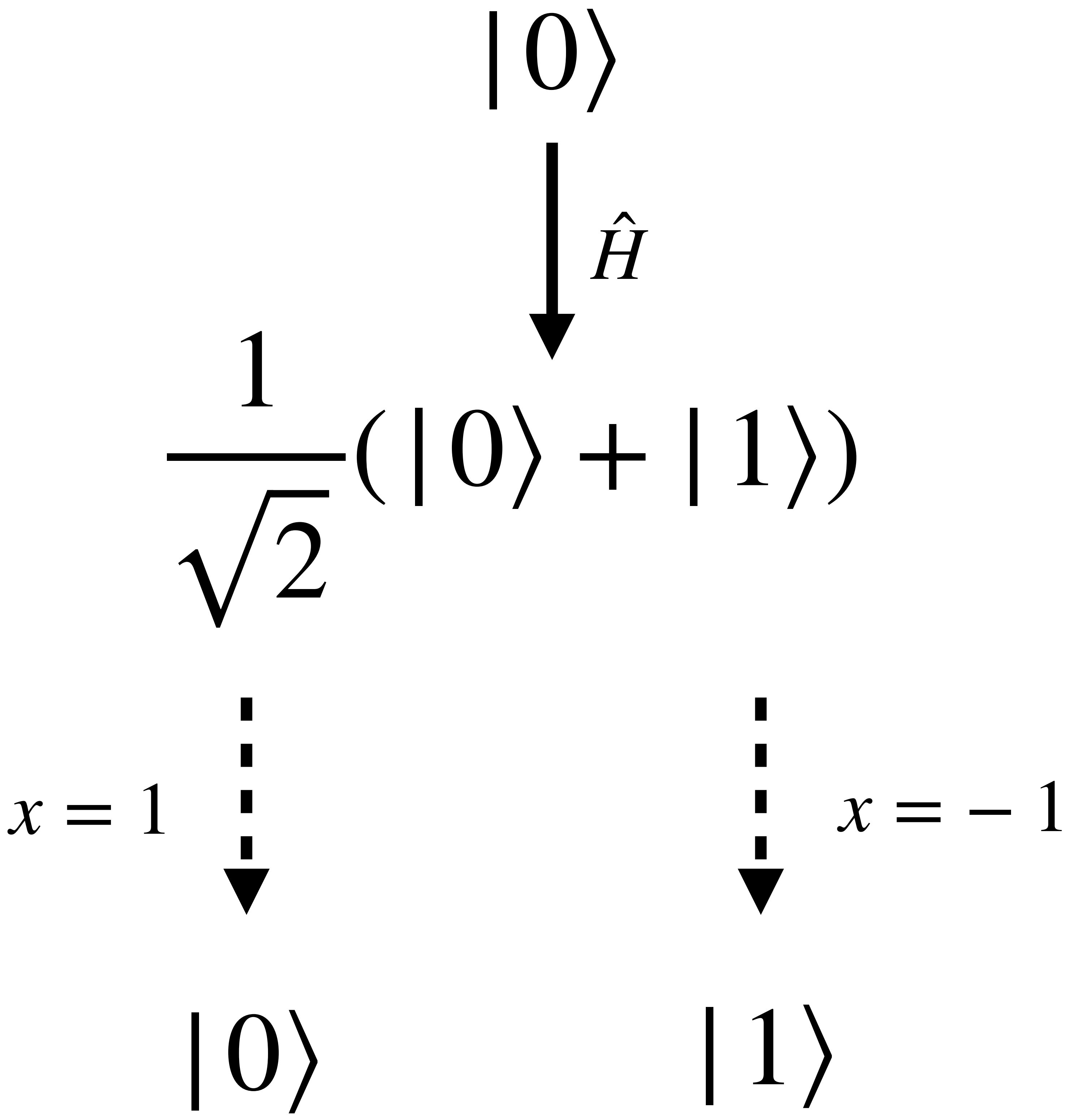

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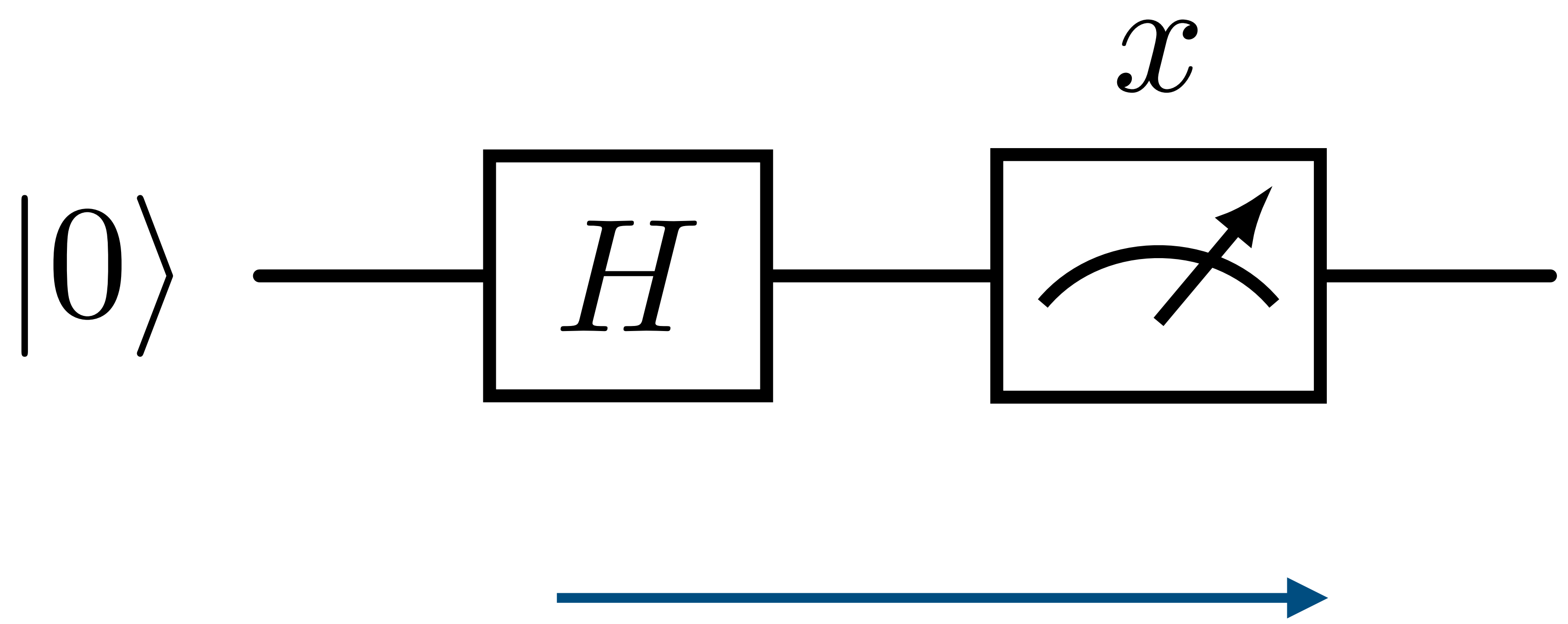

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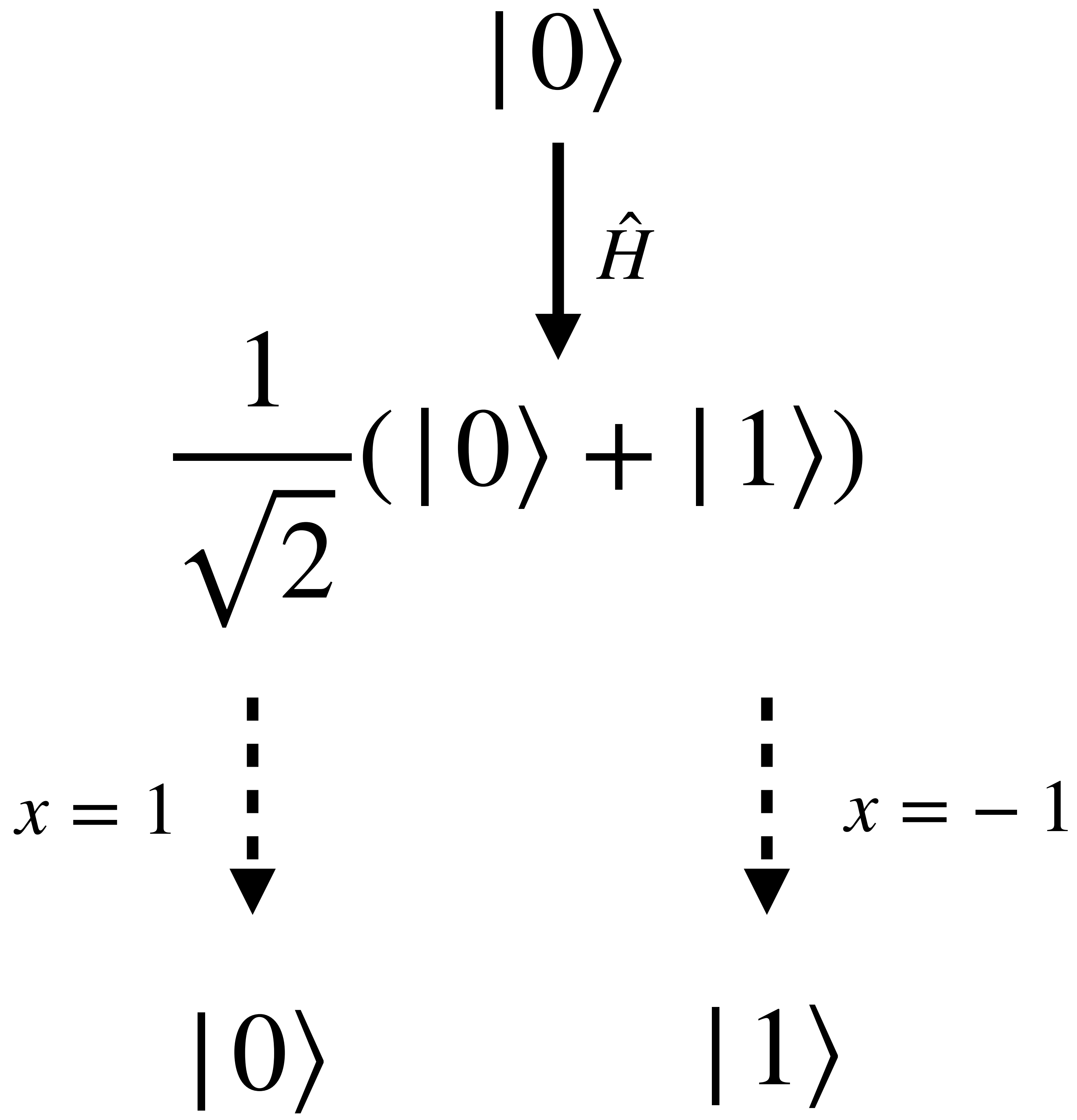

Result of measurement is intrinsically random → sampling necessary

No Cloning Theorem

Suppose $\hat{U} : \mathscr{H} \to \mathscr{H}$ clones the first qubit into second

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Therefore: Quantum circuits cannot split a qubit into two

Deutsch-Jozsa Algorithm

A coin is considered *real* if it has distinct sides …

Deutsch-Jozsa Algorithm

A coin is considered *real* if it has distinct sides …

… and otherwise *fake*

Deutsch-Jozsa Algorithm

real

fake

Classical: we need check both sides → <u>two</u> measurements

Deutsch-Jozsa Algorithm

real

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Classical: we need check both sides → <u>two</u> measurements

Quantum: <u>one</u> measurement sufficient (Deutsch-Jozsa Algorithm)

Deutsch-Jozsa Algorithm

Let $f : \{0,1\} \to \{0,1\}$ be a binary function. We want to know whether

real coin

f is *balanced* ($f(0) \neq f(1)$)

fake coin

f is *constant* ($f(0) = f(1)$)

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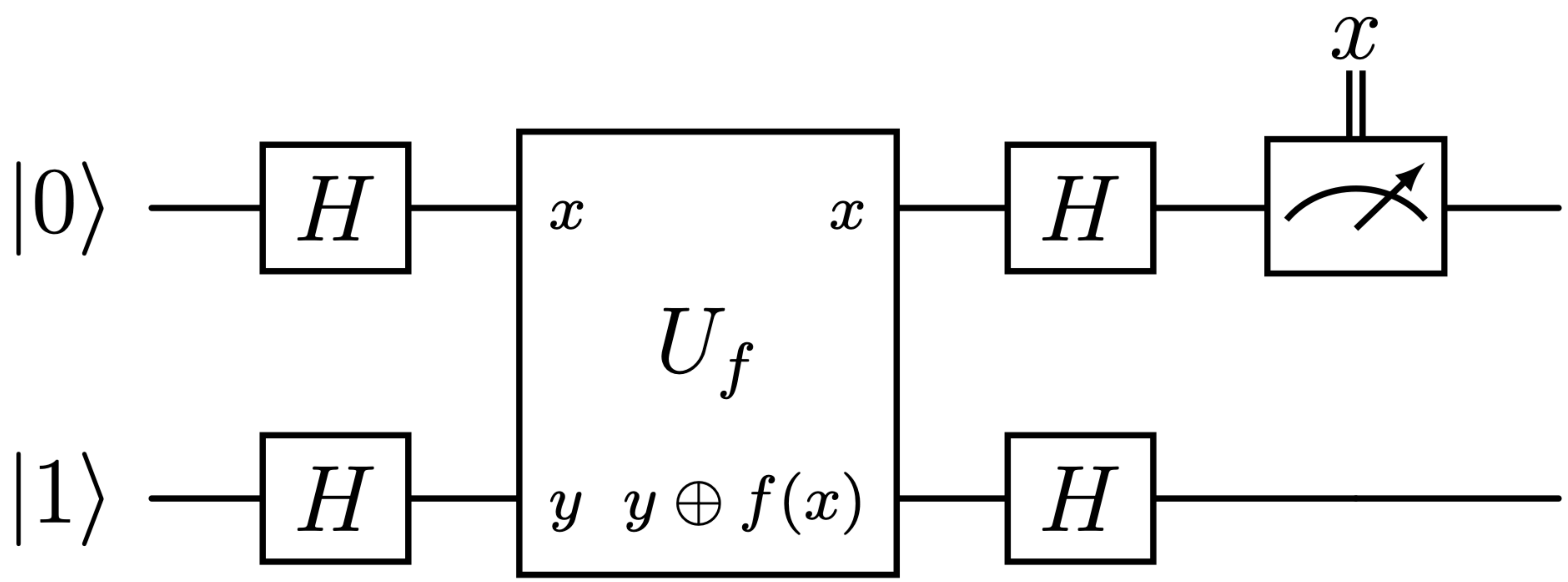

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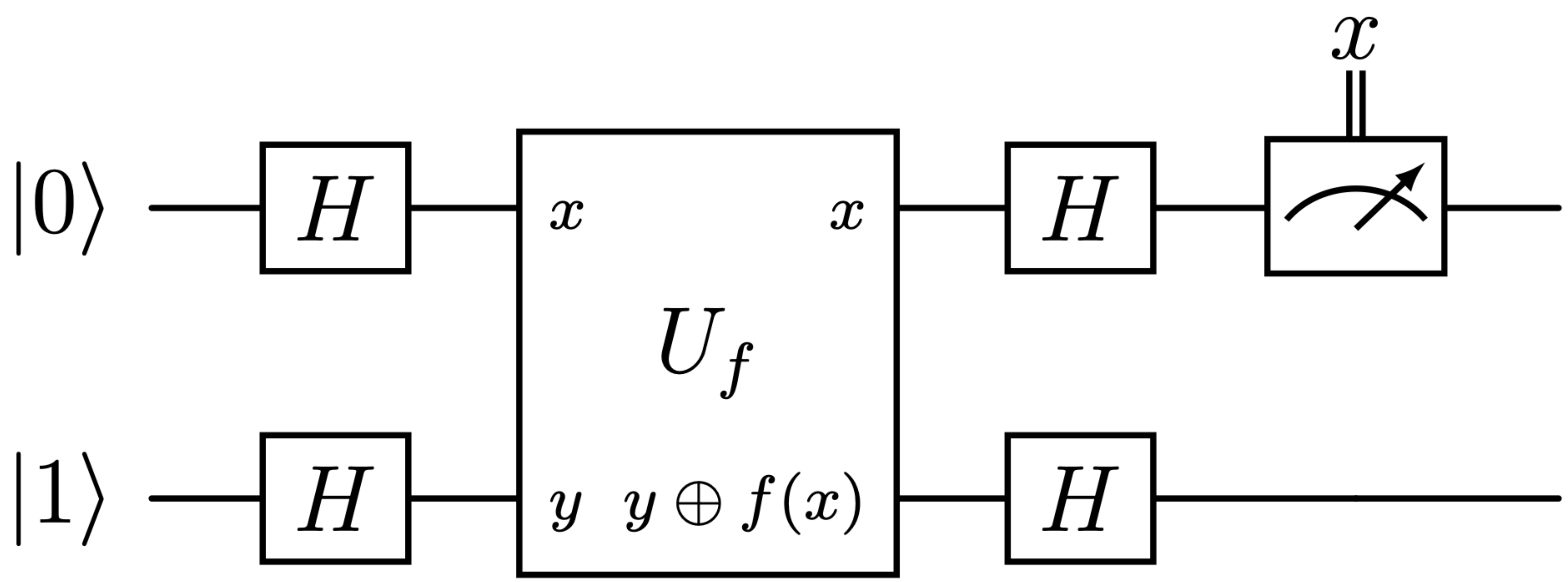


$\underline{x = -1}$: f is balanced
real coin

$\underline{x = 1}$: f is constant
fake coin

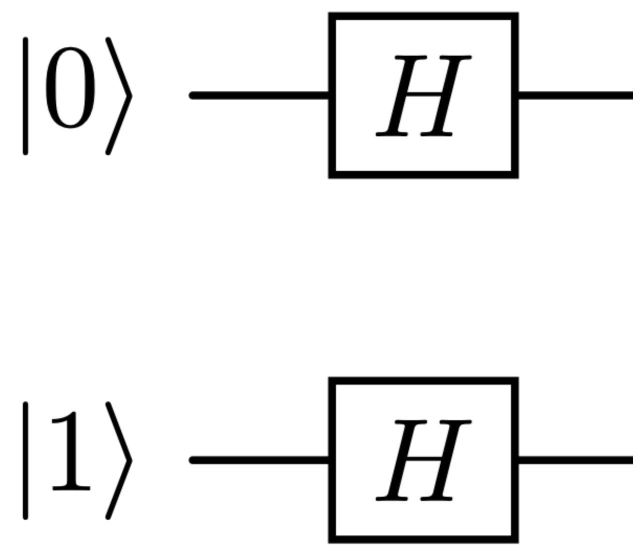
Deutsch-Jozsa Algorithm

$|0\rangle$ —

$|1\rangle$ —

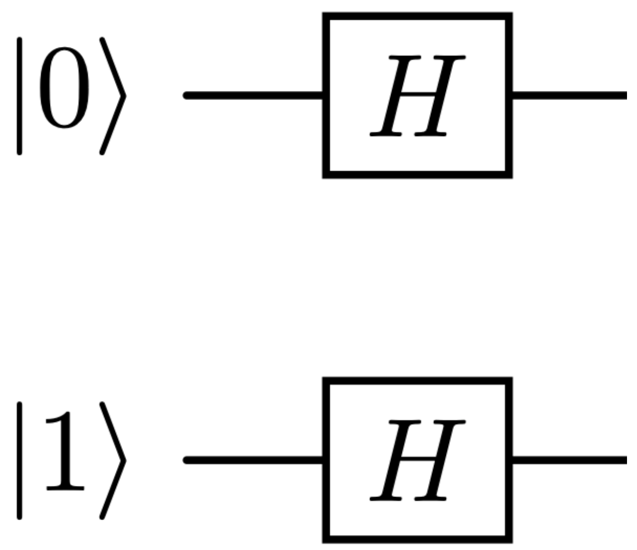
$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

Deutsch-Jozsa Algorithm



$$|\psi\rangle = |0\rangle \otimes |1\rangle \xrightarrow{\hat{H} \otimes \hat{H}} \hat{H}|0\rangle \otimes \hat{H}|1\rangle = \frac{1}{2}\big(|0\rangle + |1\rangle\big)\big(|0\rangle - |1\rangle\big)$$

Deutsch-Jozsa Algorithm



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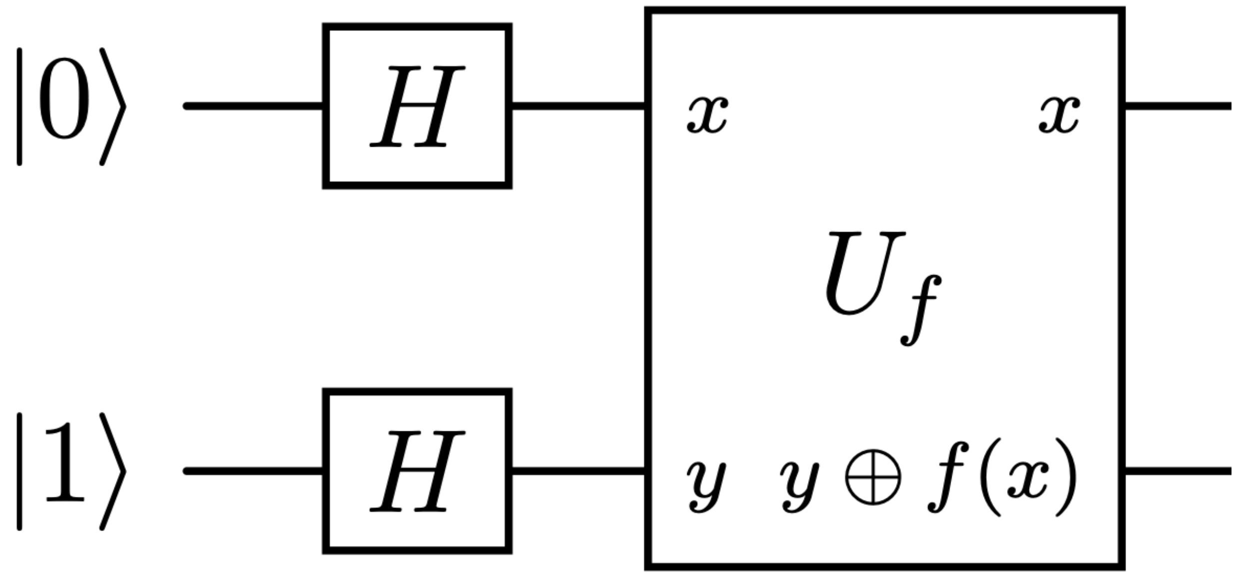
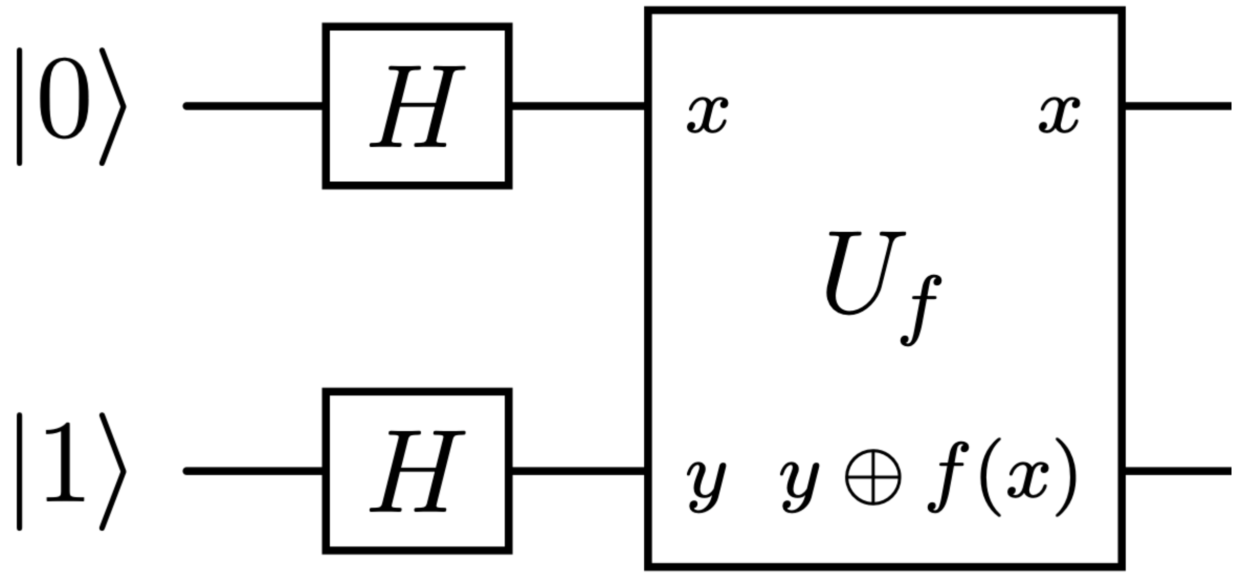
$$= \frac{1}{2}(|00\rangle + |10\rangle - |11\rangle - |01\rangle)$$

Deutsch-Jozsa Algorithm



$$\frac{1}{2}(\,|00\rangle + |10\rangle - |11\rangle - |01\rangle) \xrightarrow{\quad U_f \quad} \frac{1}{2}U_f\big(\,|00\rangle + |10\rangle - |11\rangle - |01\rangle\big)$$

Deutsch-Jozsa Algorithm



$$U_f|xy\rangle = |x(y \oplus f(x))\rangle$$

$$\frac{1}{2}(|00\rangle + |10\rangle - |11\rangle - |01\rangle) \xrightarrow{U_f} \frac{1}{2}U_f\big(|00\rangle + |10\rangle - |11\rangle - |01\rangle\big)$$

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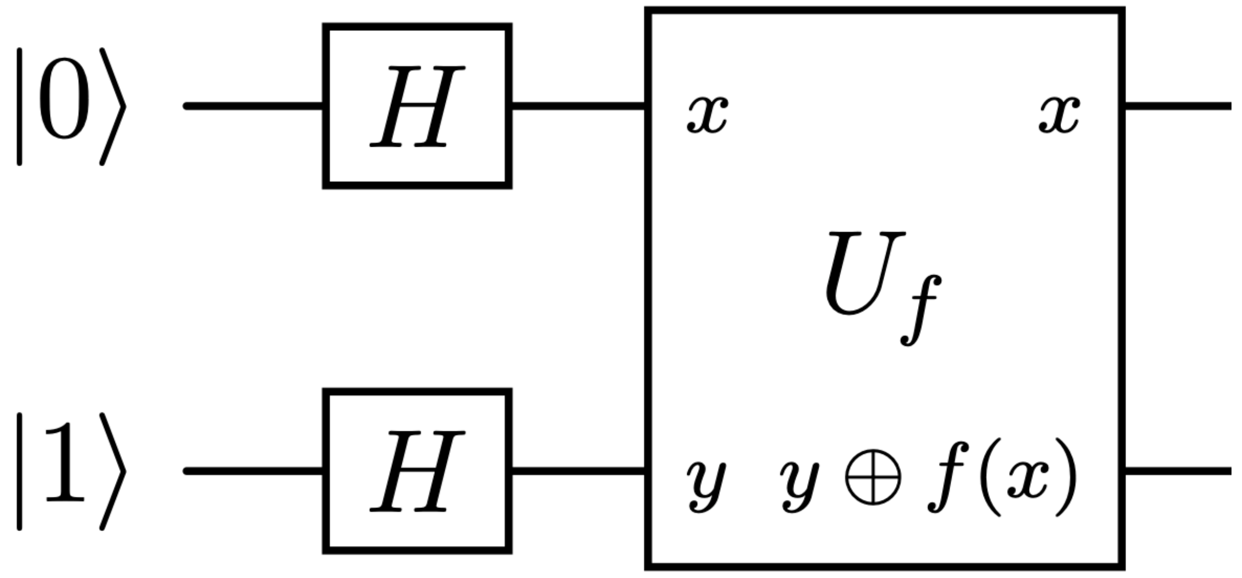

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$$= \frac{1}{2}\big(|0(0 \oplus f(0))\rangle + |1(0 \oplus f(1))\rangle - |1(1 \oplus f(1))\rangle - |0(1 \oplus f(0))\rangle\big)$$

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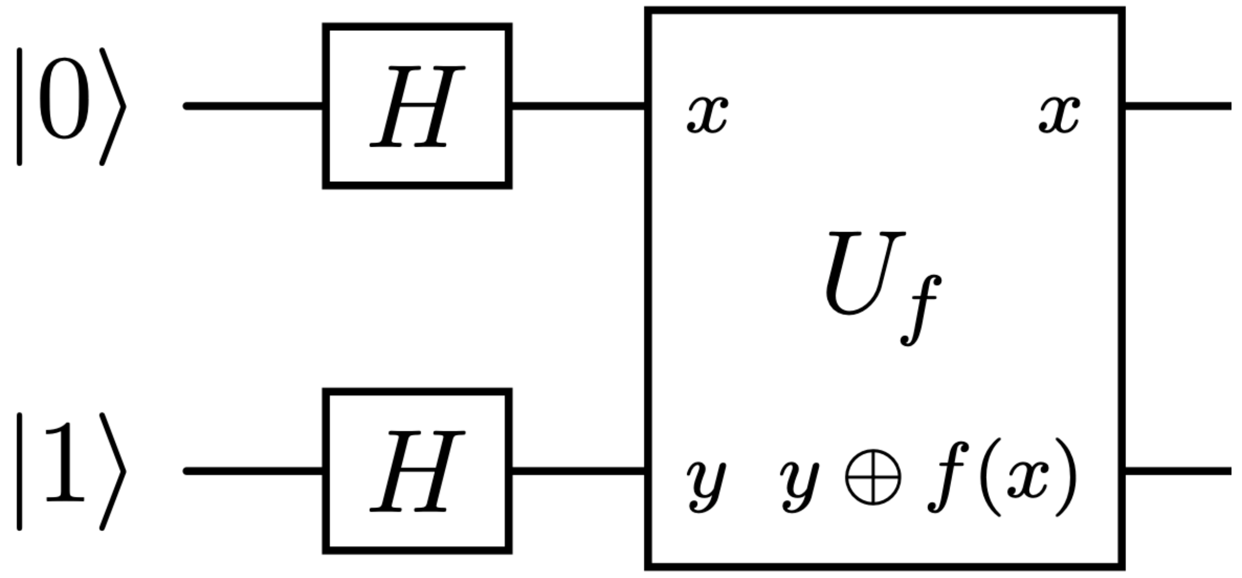

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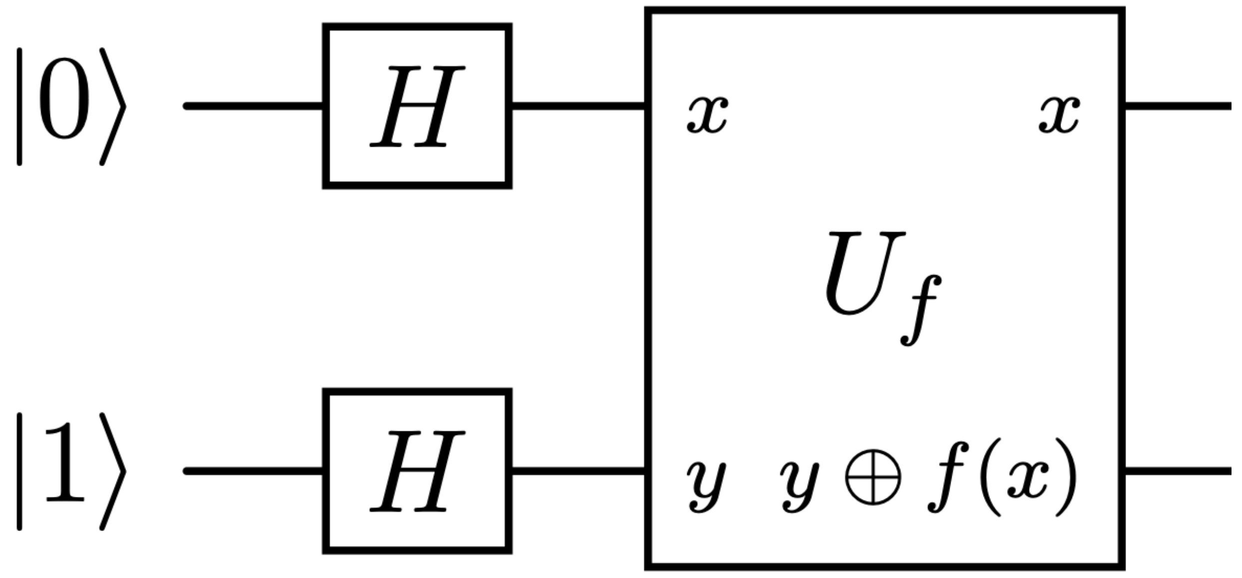

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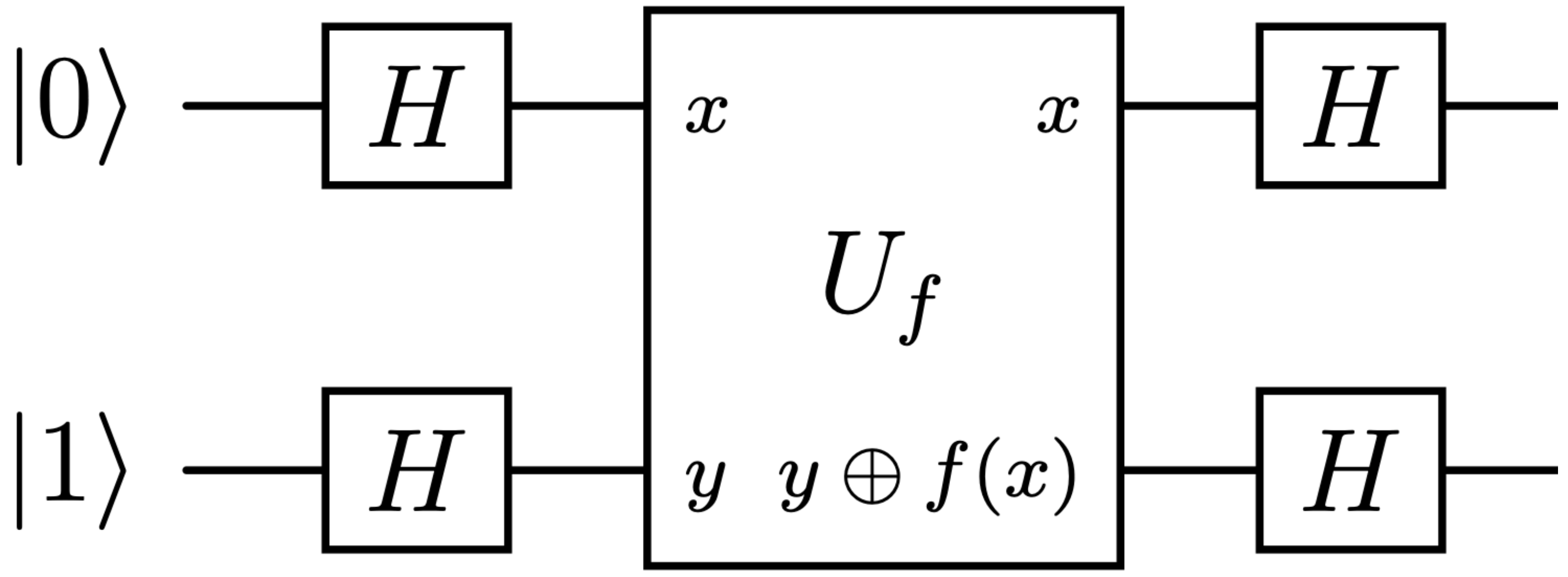

Hadamard operator

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$$\xrightarrow{\hat{H}\otimes\hat{H}}$$

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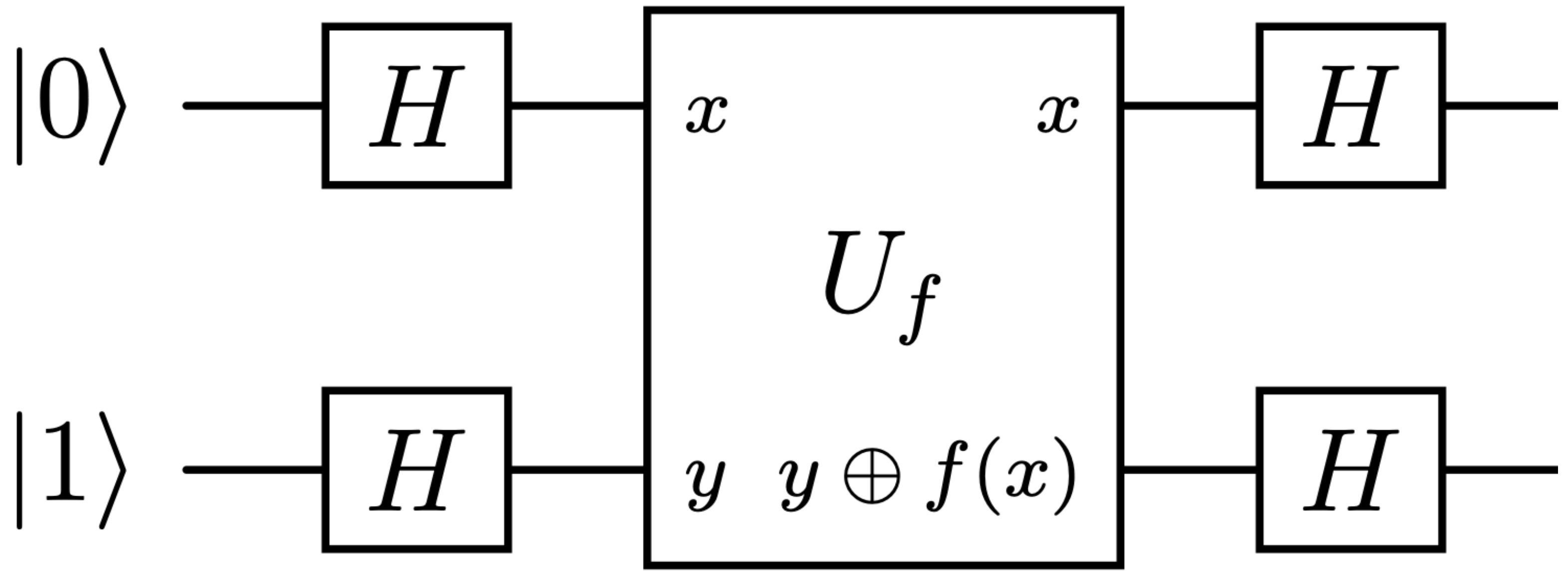

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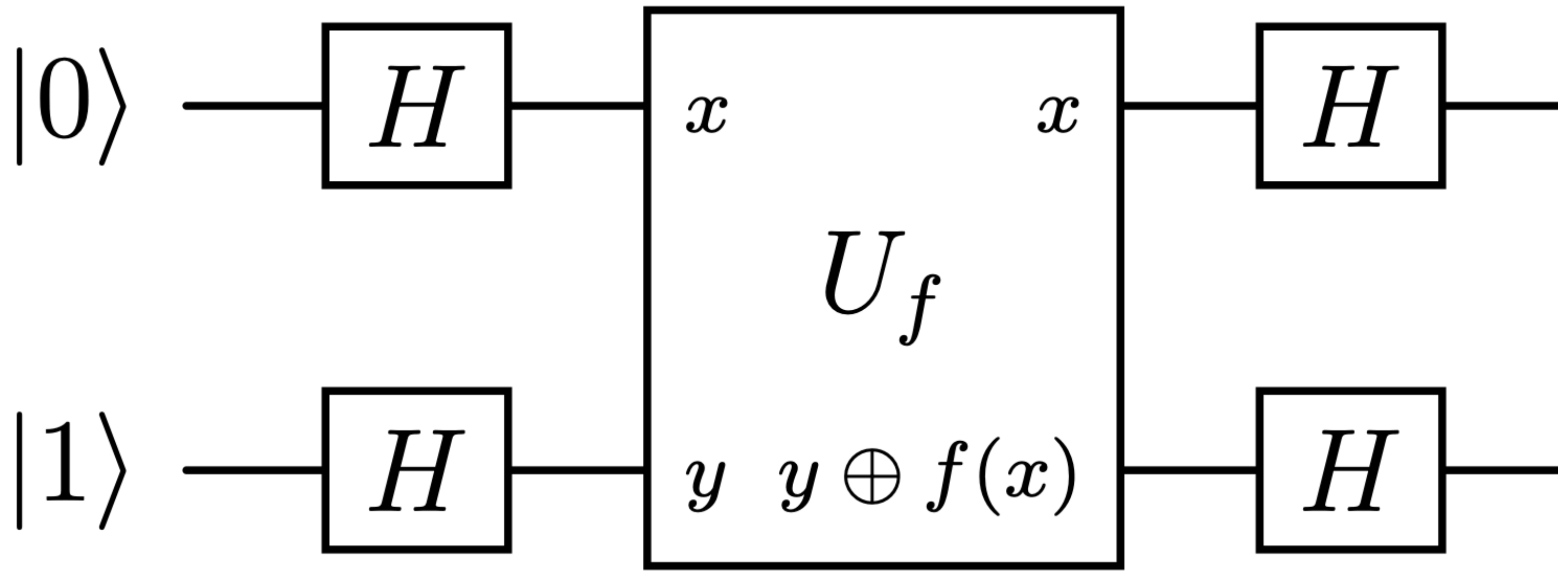

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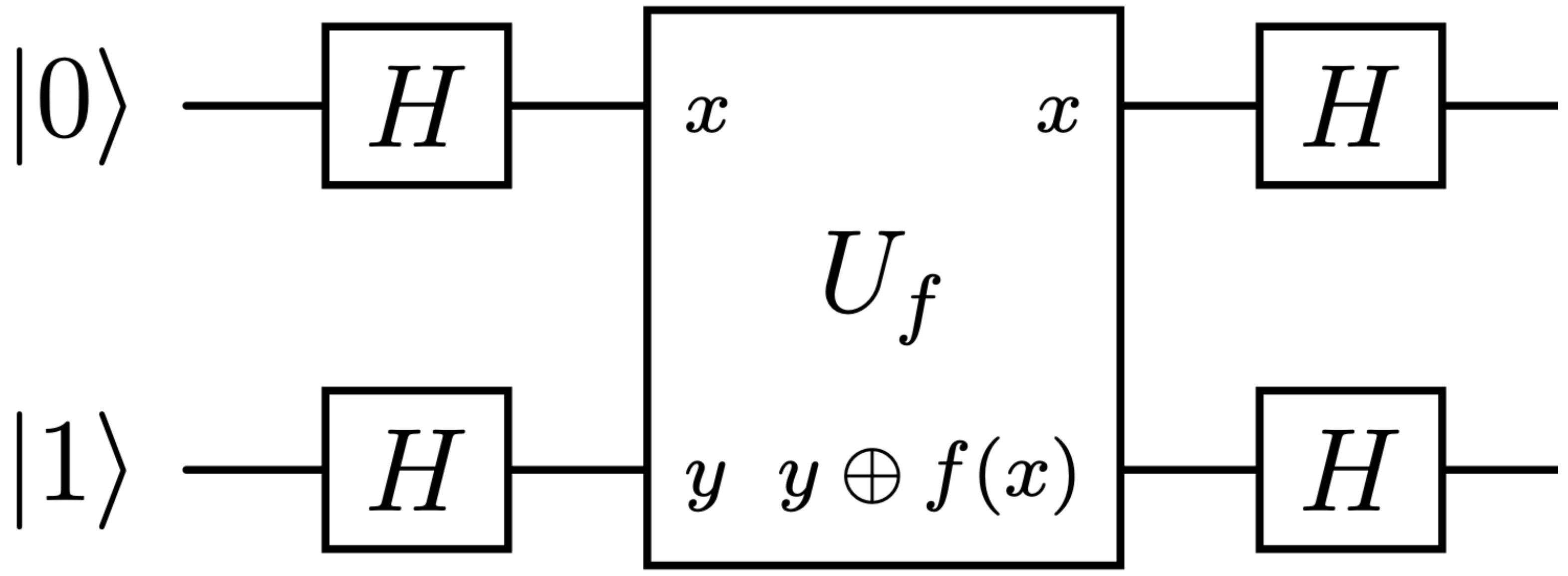

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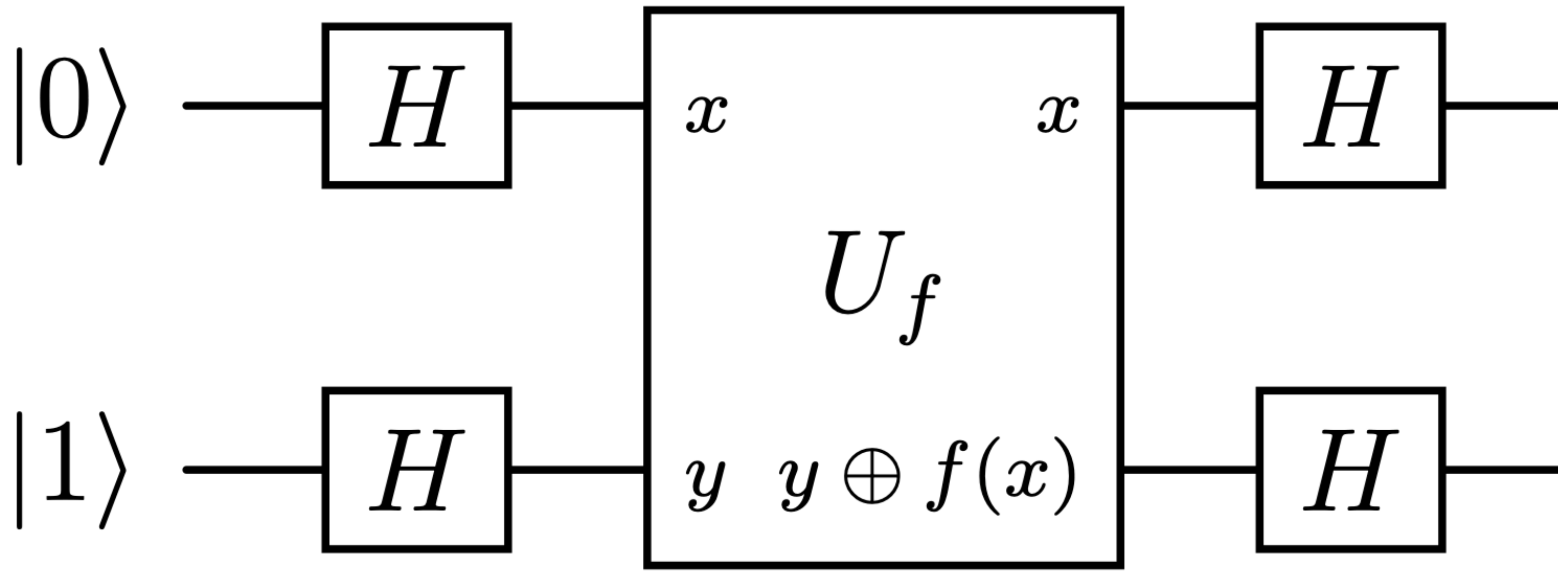

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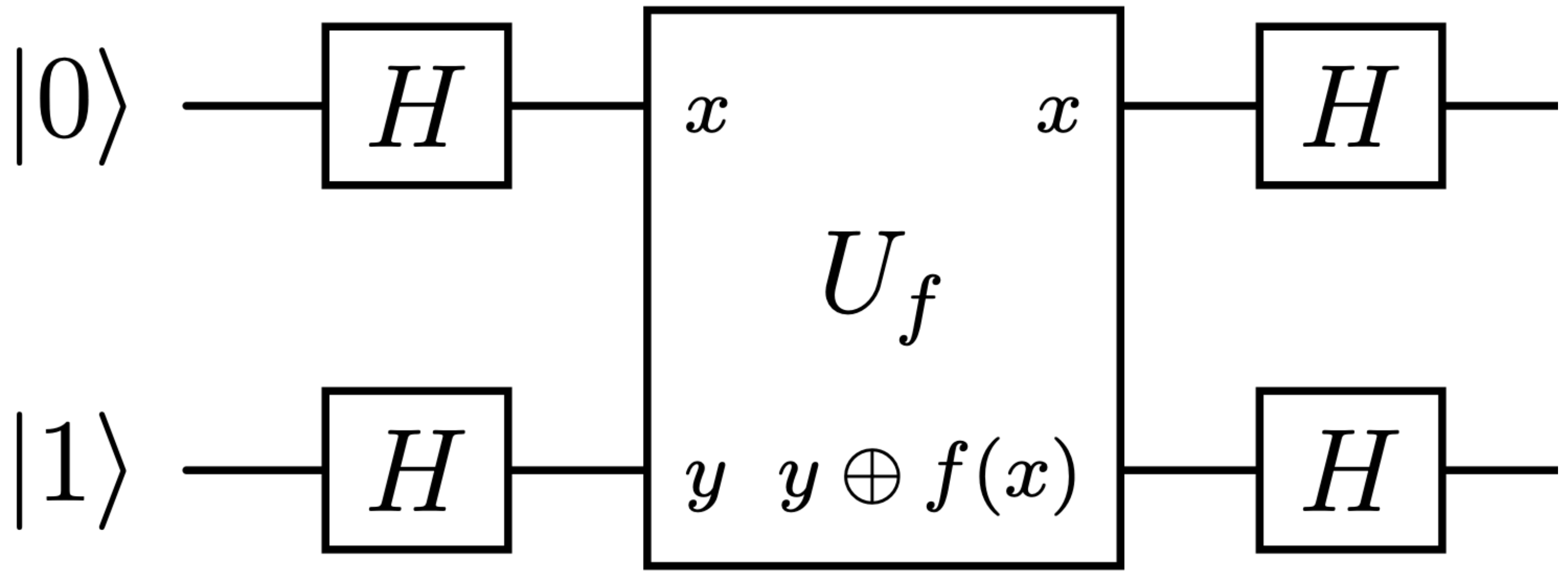

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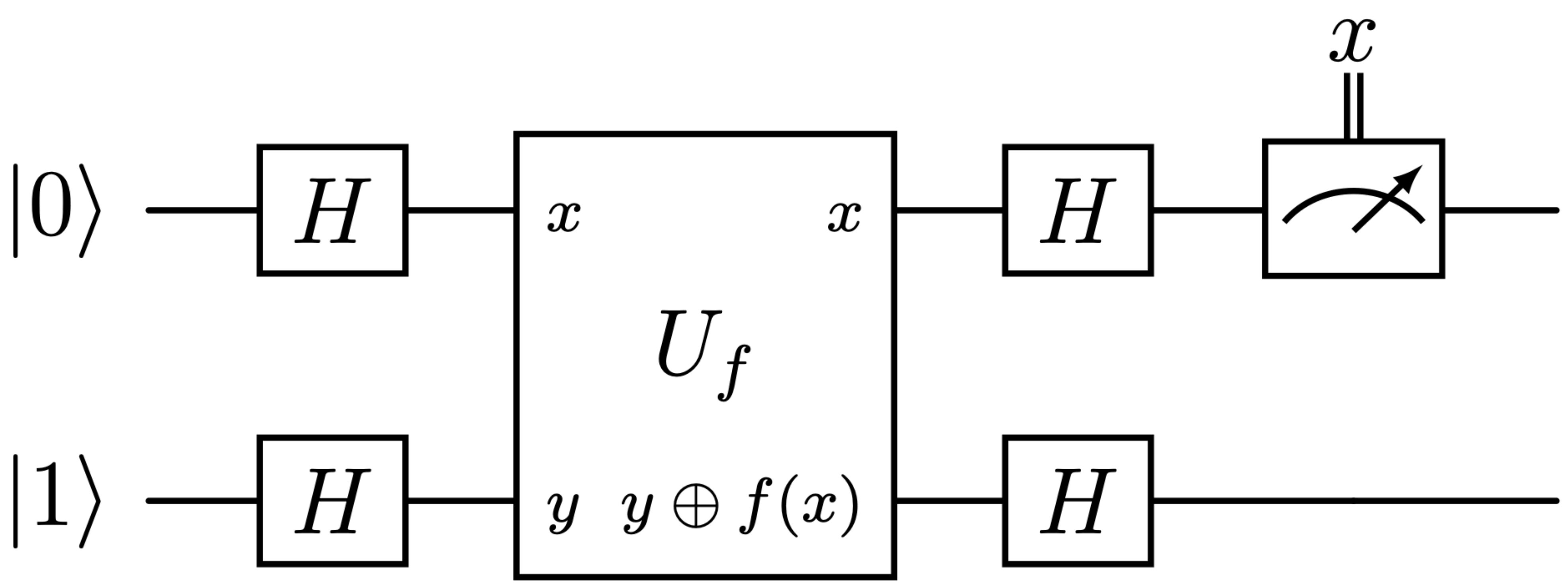

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$$= \begin{cases} \pm\boxed{|0\rangle}\otimes|1\rangle & (x=1) \quad \text{if} \quad f(0)=f(1) \\ \pm\boxed{|1\rangle}\otimes|1\rangle & (x=-1) \quad \text{if} \quad f(0)\neq f(1) \end{cases}$$

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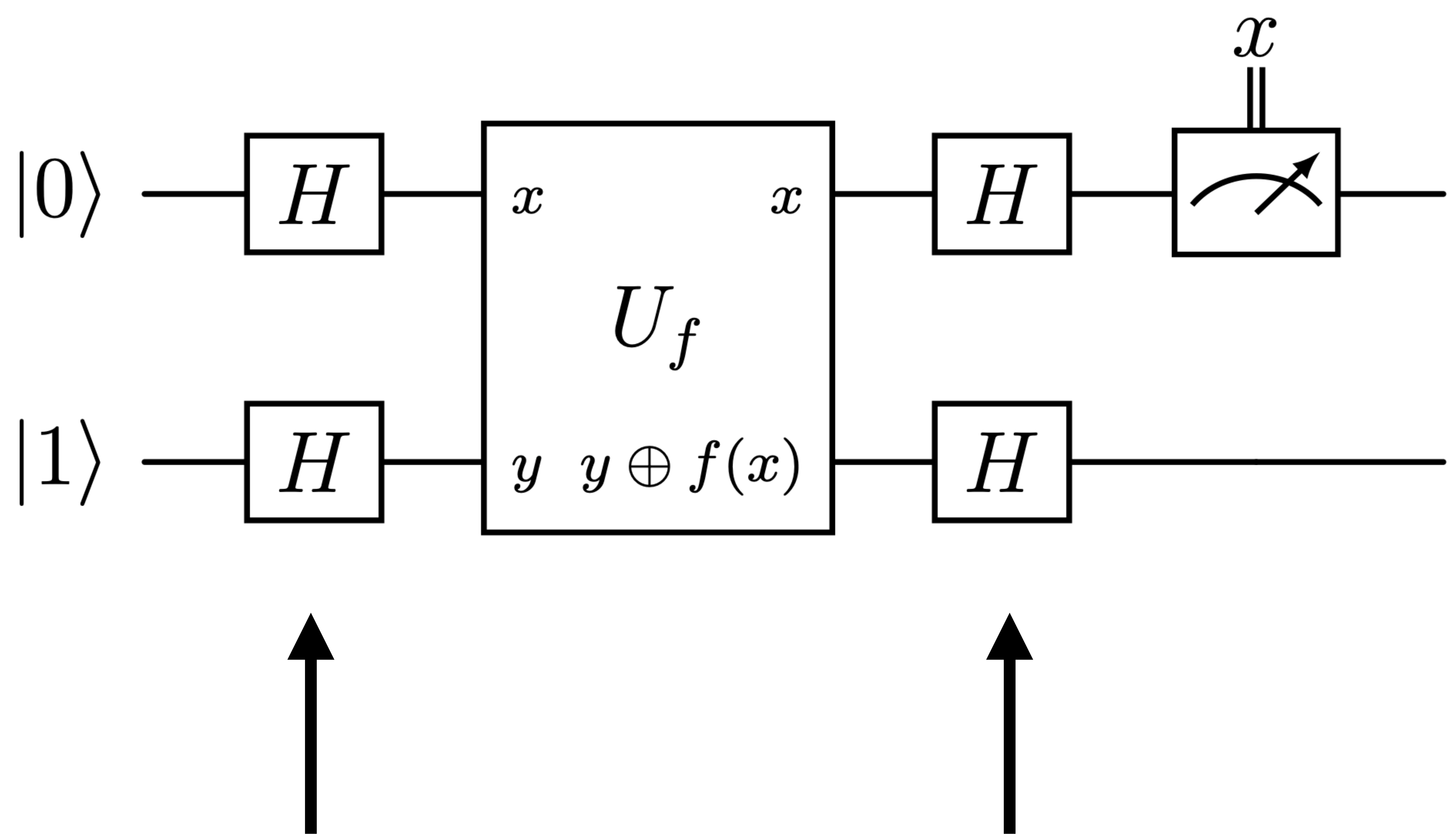
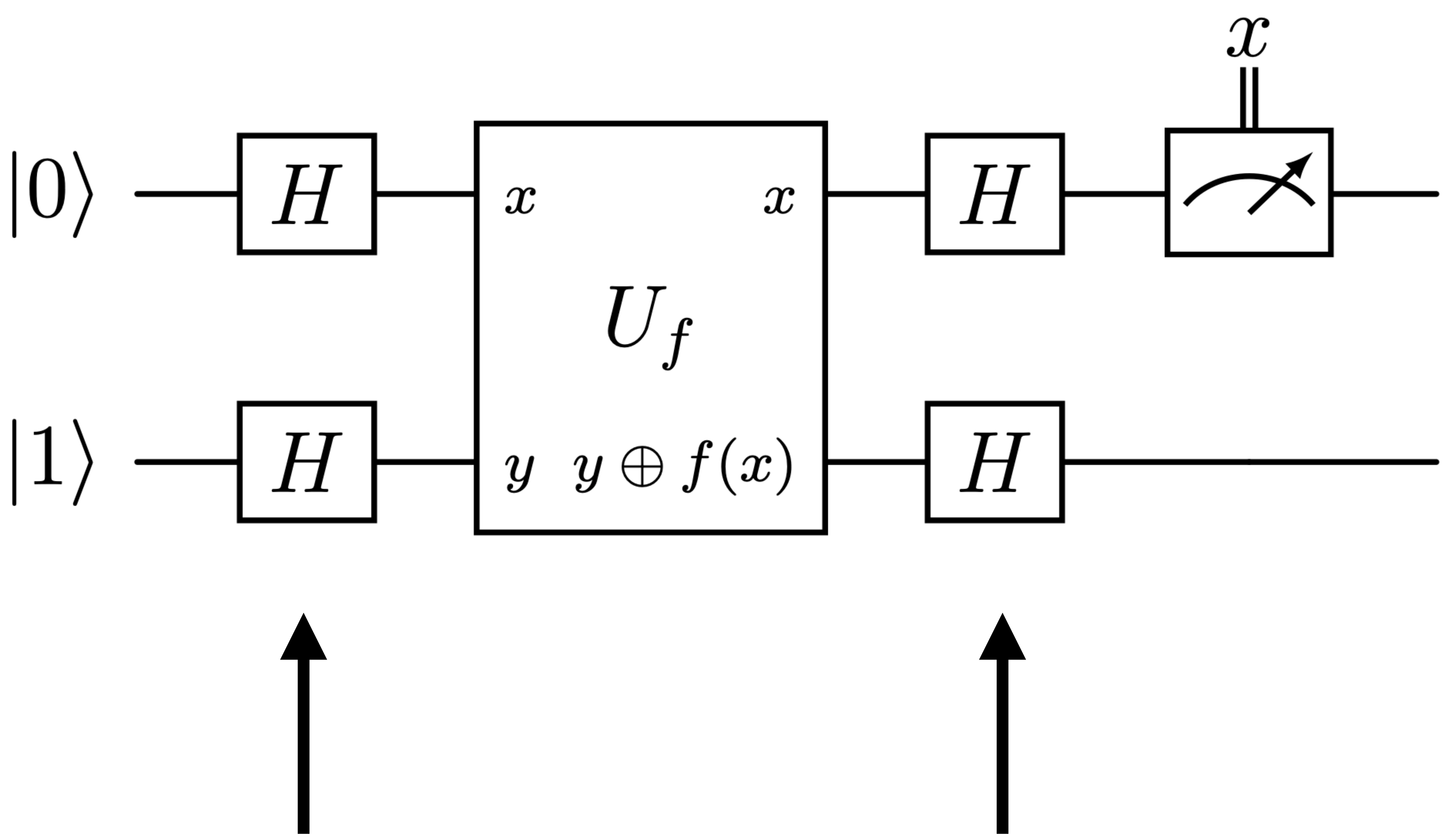


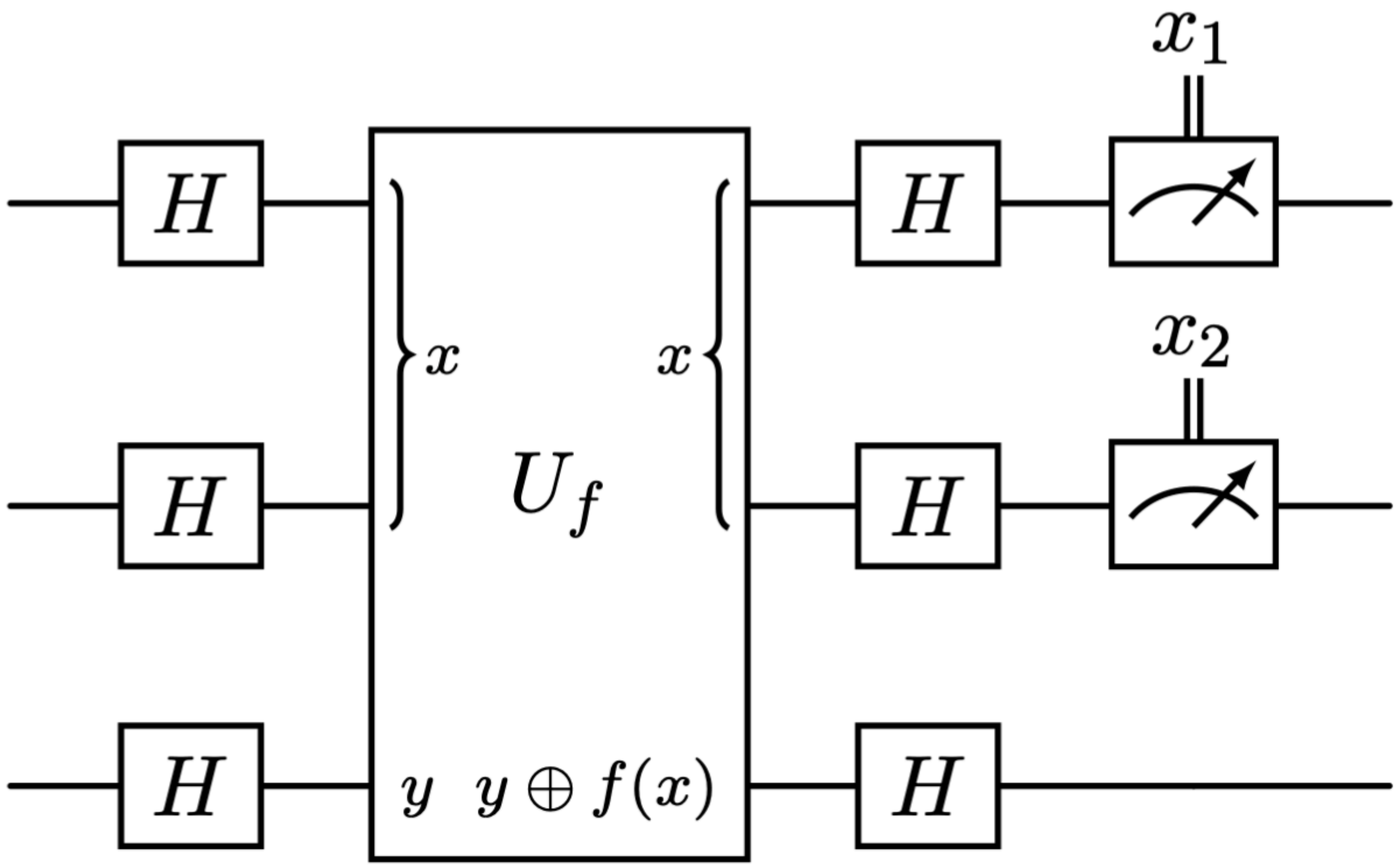
the Hadamard gates are the key *ingredient*

Deutsch-Jozsa Algorithm



the Hadamard gates are the key *ingredient* $\longrightarrow$ apply f on a superposition of $0, 1$

Deutsch-Jozsa Algorithm



Algorithm extends to

$$f: \{0,1\}^n \to \{0,1\}$$

f is balanced if and only if

$$(x_1, x_2, \ldots, x_n) = (-1, -1, \ldots, -1)$$

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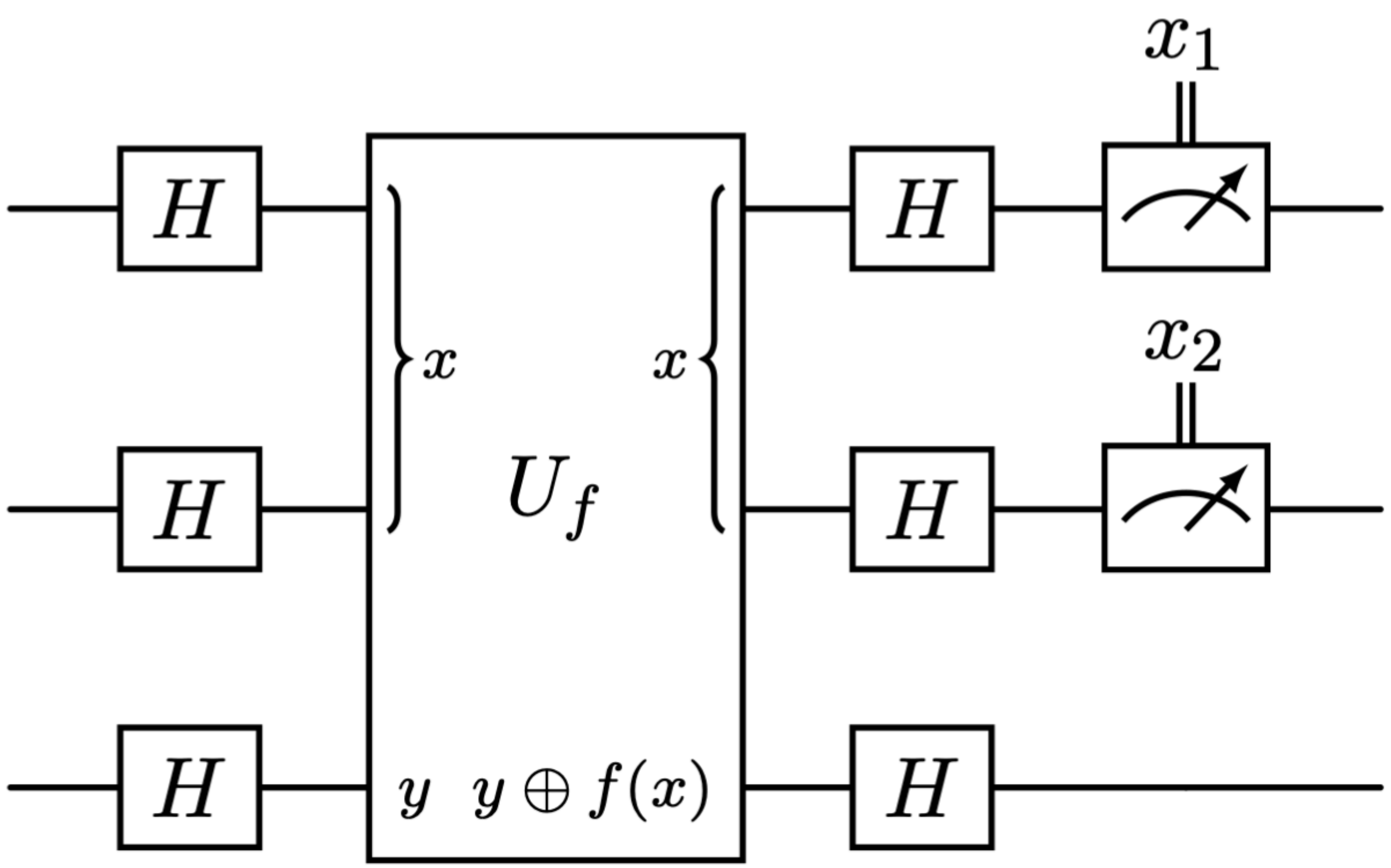

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Classical $2^{\frac{n}{2}} + 1$ → Quantum 1 step

Complexity Theory

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What is the computational cost of an algorithm?

in space (memory) and time (runtime)

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P	$\mathcal{O}(p(n))$ space and time complexity
PSPACE	$\mathcal{O}(p(n))$ space complexity
EXP	$\mathcal{O}(2^{p(n)})$ space and time complexity
NP	Solution verifiable in $\mathcal{O}(p(n))$ space and time
BPP	**P** with an bounded error probability of at least 2/3

Complexity Class NP

NP Solution verifiable in $\mathcal{O}(p(n))$ space and time

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
7								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

4	8	3	9	2	1	6	5	7
9	6	7	3	4	5	8	2	1
2	5	1	8	7	6	4	9	3
5	4	8	1	3	2	9	7	6
7	2	9	5	6	4	1	3	8
1	3	6	7	9	8	2	4	5
3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

Complexity Class NP

NP Solution verifiable in $\mathcal{O}(p(n))$ space and time

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
7								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

solve

in **EXP**

verify

in **P**

4	8	3	9	2	1	6	5	7
9	6	7	3	4	5	8	2	1
2	5	1	8	7	6	4	9	3
5	4	8	1	3	2	9	7	6
7	2	9	5	6	4	1	3	8
1	3	6	7	9	8	2	4	5
3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

where n is the number of empty squares

Complexity Theory

We know that

$\mathbf{L} \subsetneq \mathbf{PSPACE}$

$\mathbf{P} \subsetneq \mathbf{EXP}$

thus

$\mathbf{L} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$

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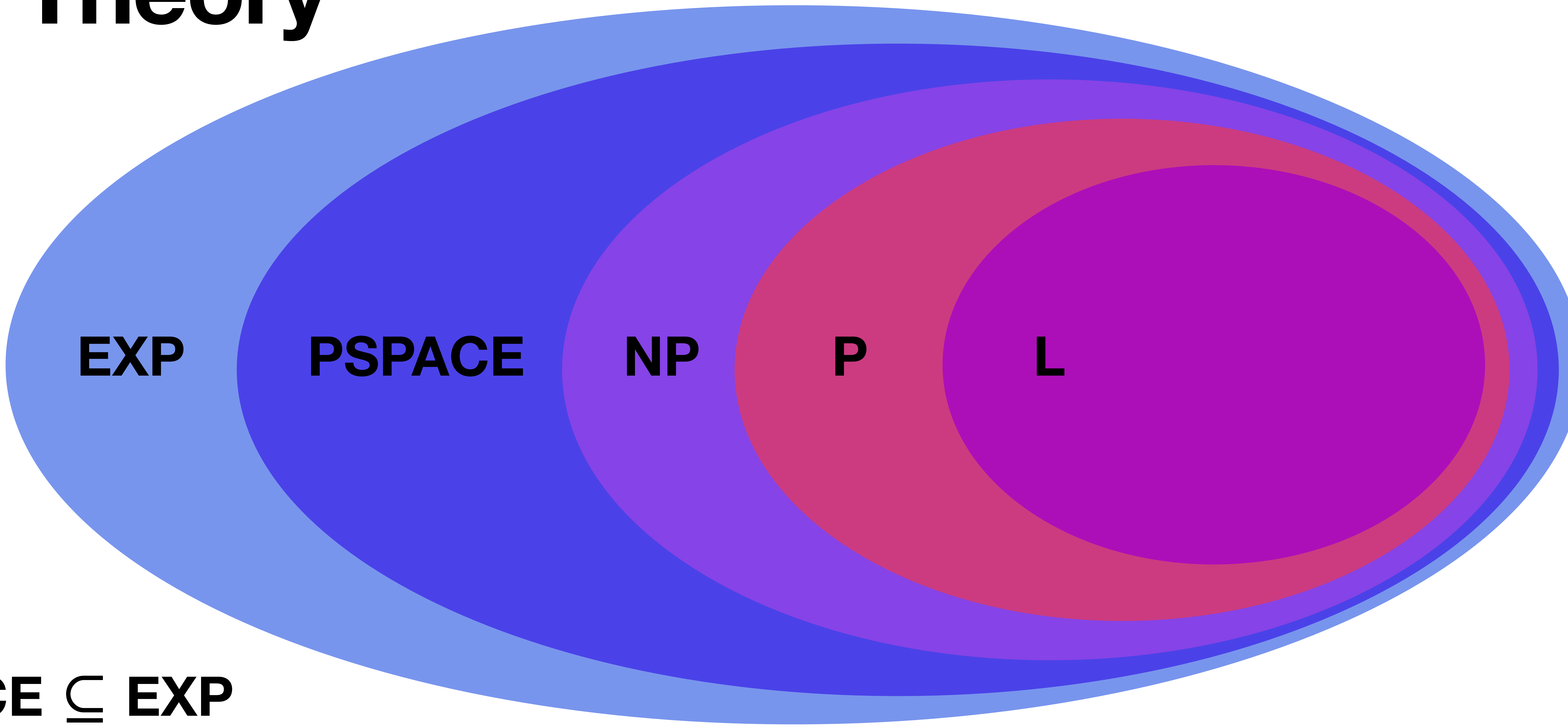

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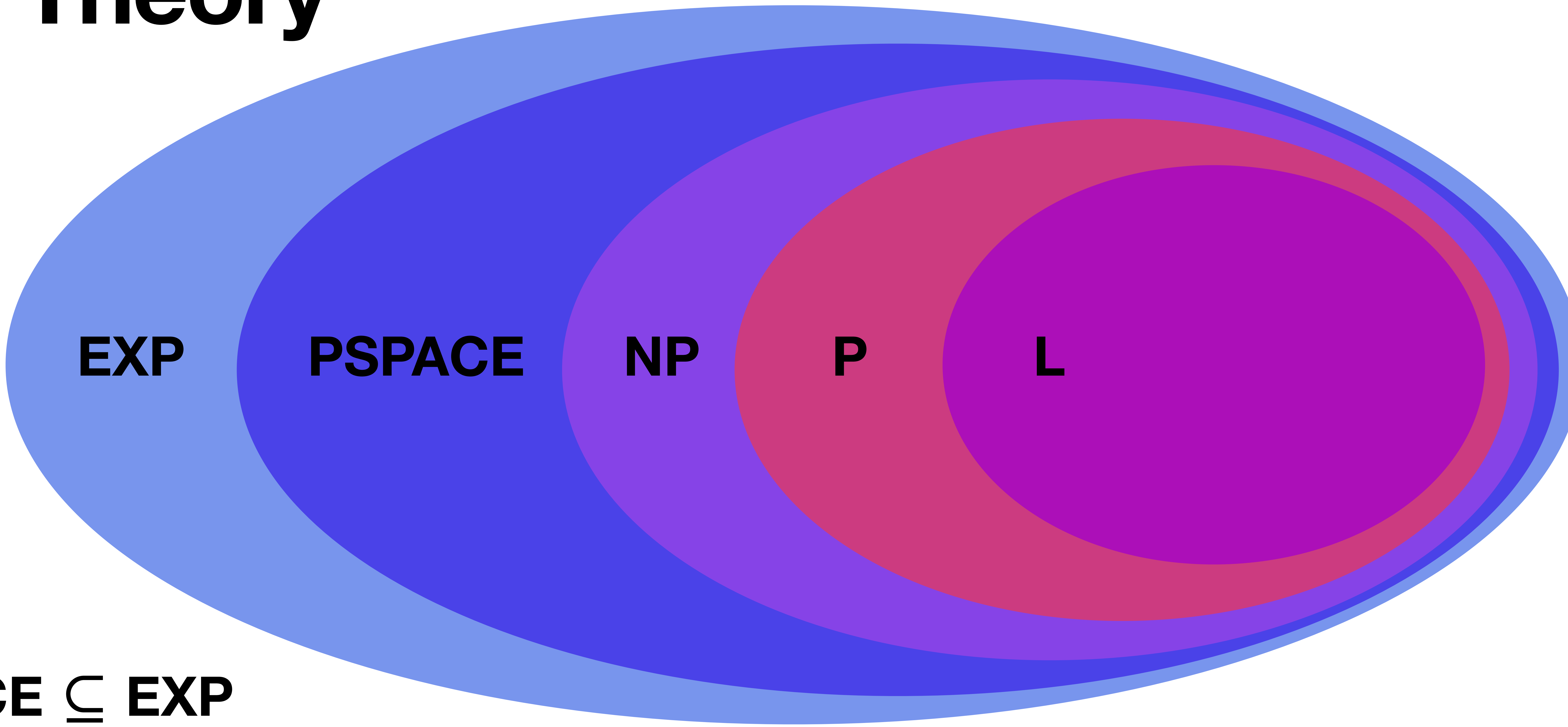

L $\subseteq$ **P** $\subseteq$ **NP** $\subseteq$ **PSPACE** $\subseteq$ **EXP**

At least one of the inclusions needs to be strict → Open problem

Complexity Theory

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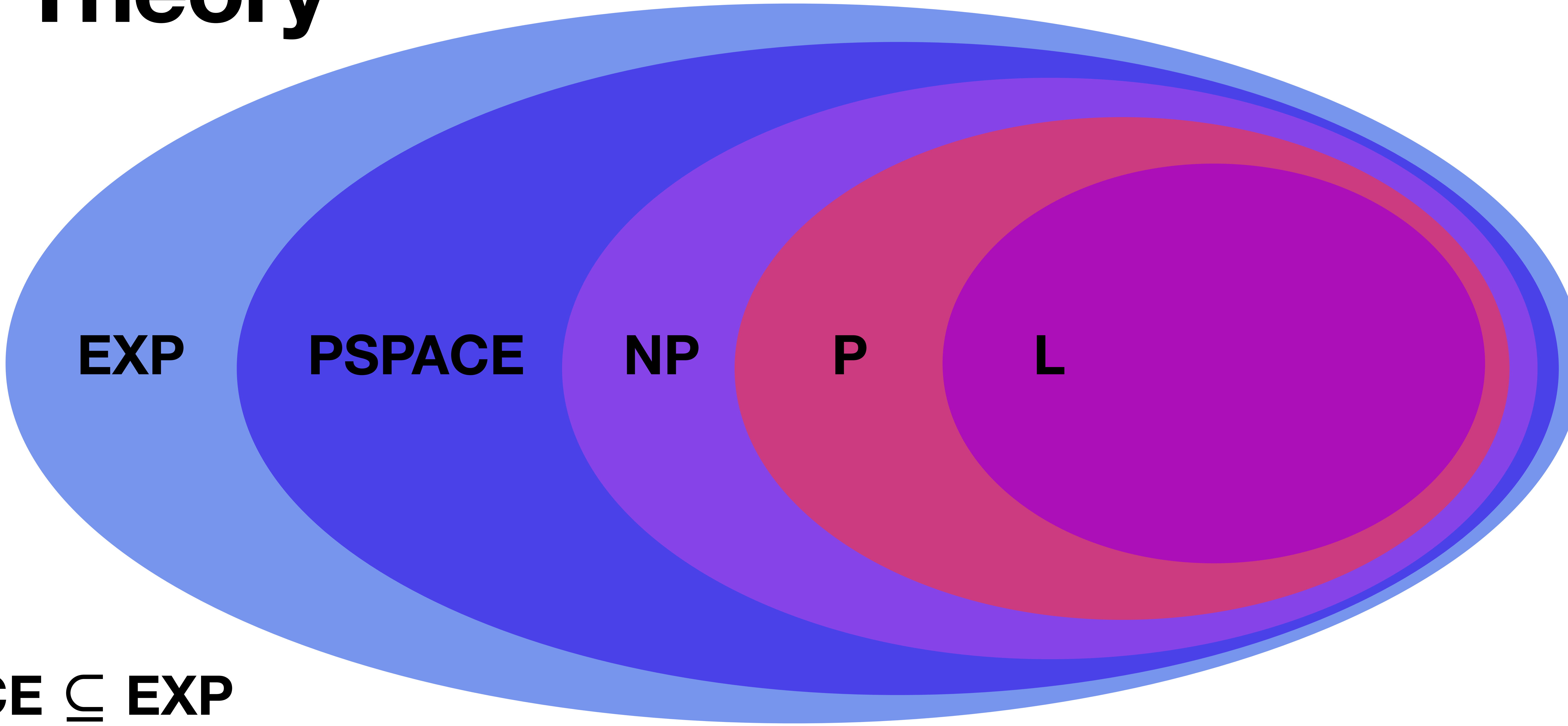

$\mathbf{L} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$

At least one of the inclusions needs to be strict → Open problem

$\boxed{\mathbf{NP} \stackrel{?}{=} \mathbf{P}}$ Millenium Problem

Quantum complexity classes

BQP $\mathcal{O}(p(n))$ space and time complexity on quantum computer

Bounded error Quantum Probability

Quantum equivalent to BPP

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QMA Quantum proof verification with bounded error

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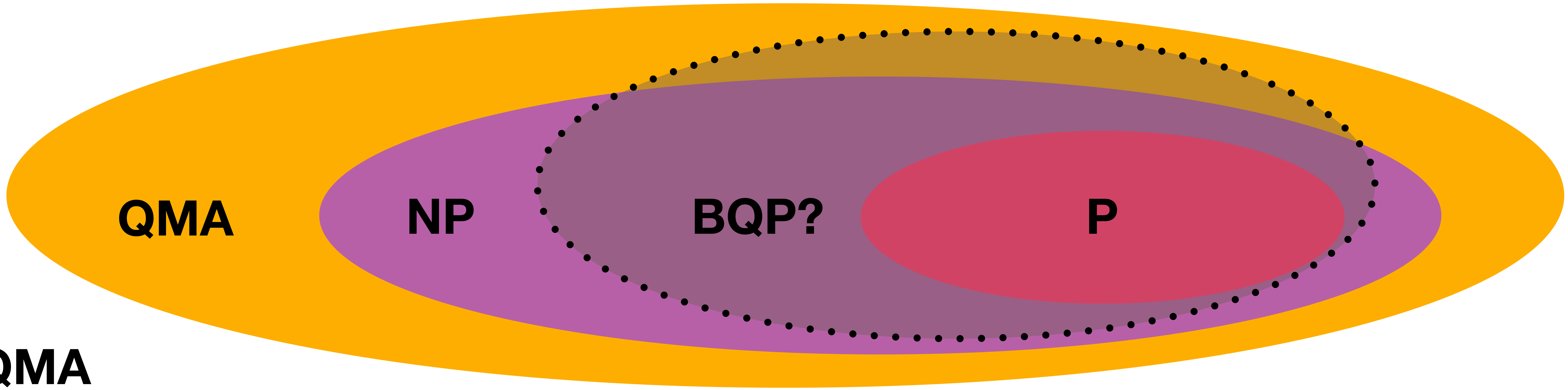

We know that

P $\subseteq$ BQP $\subseteq$ QMA
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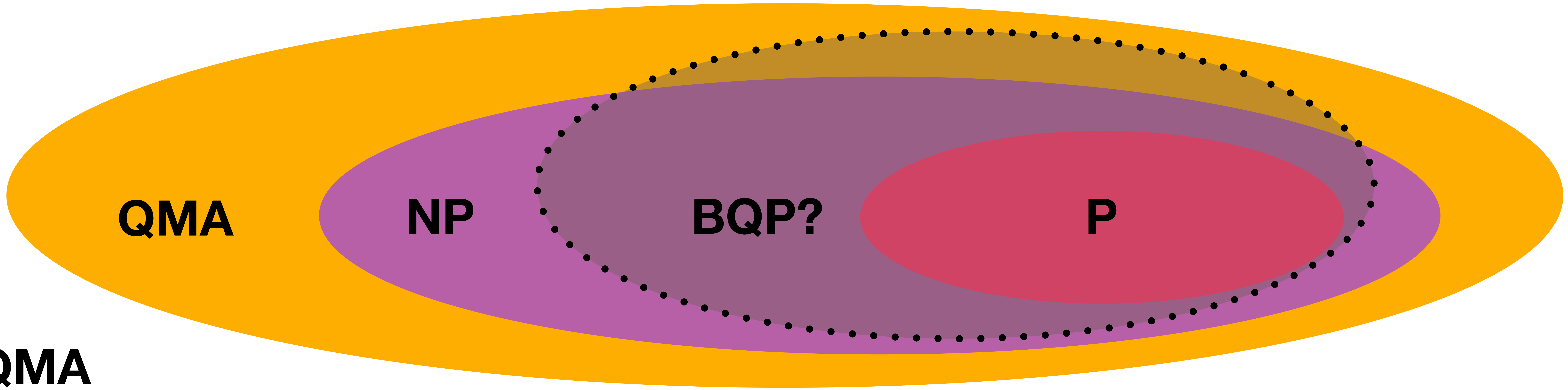

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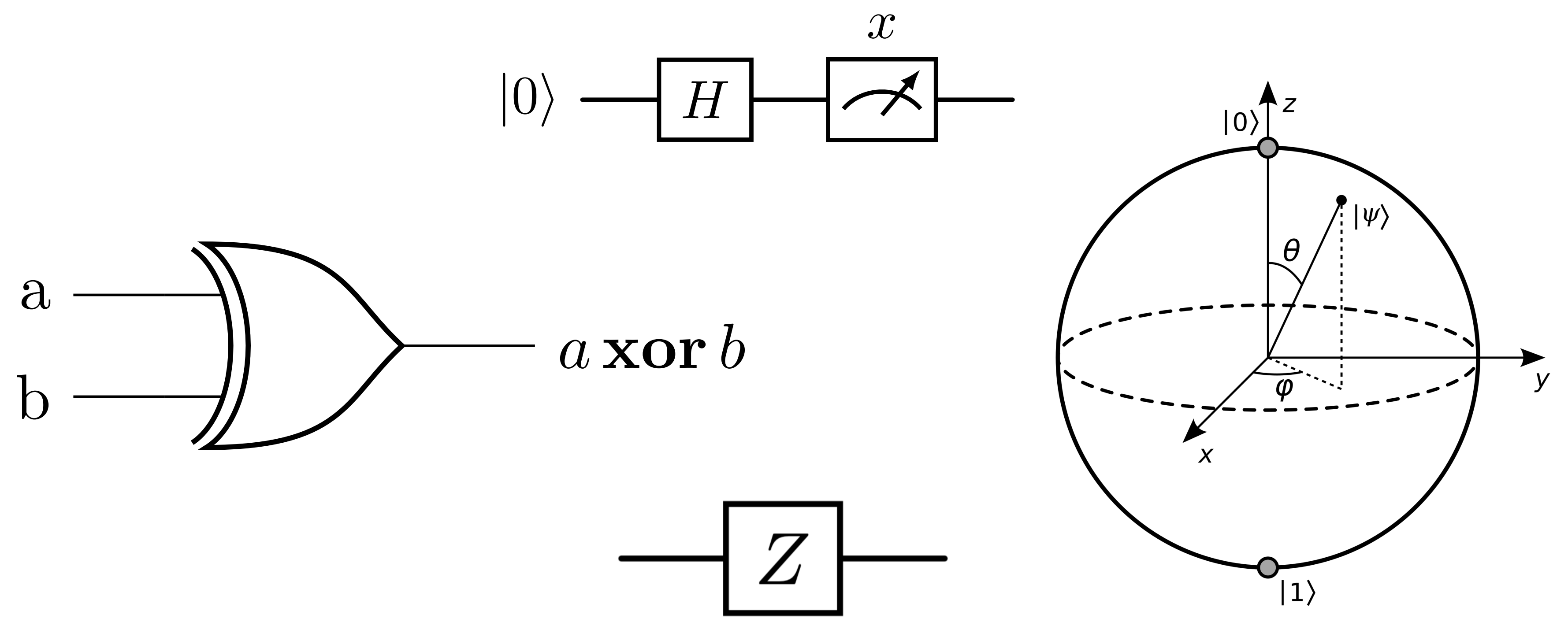
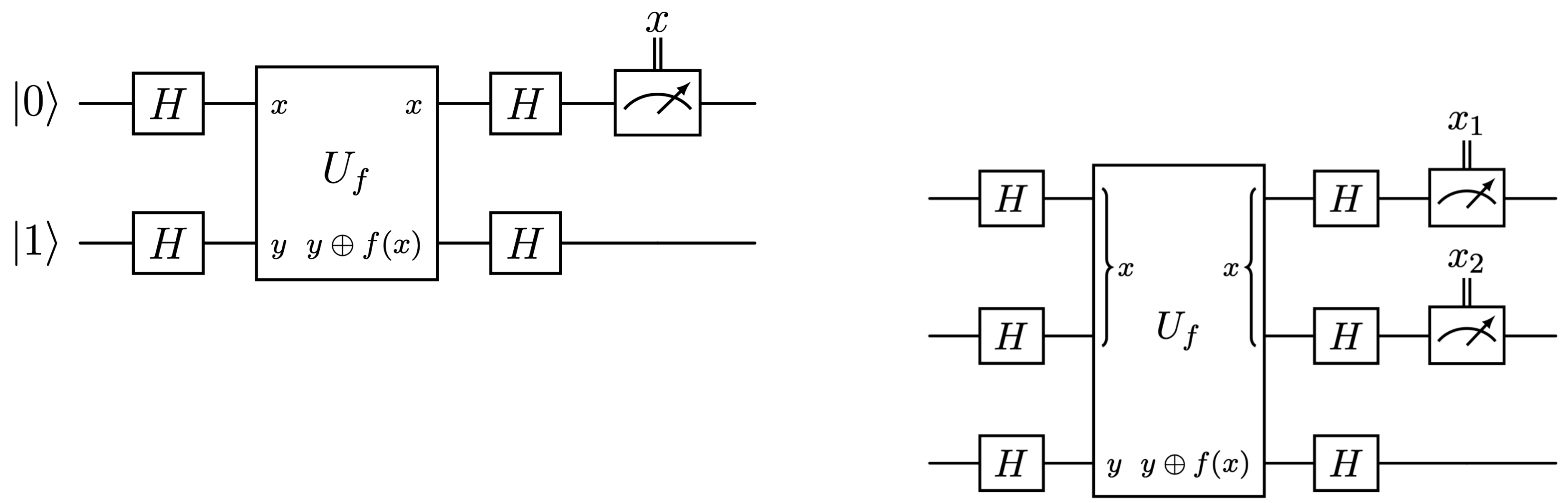
Open problem $\boxed{\textbf{NP} \overset{?}{\subseteq} \textbf{BQP}}$

Summary

Classical and Quantum Circuits

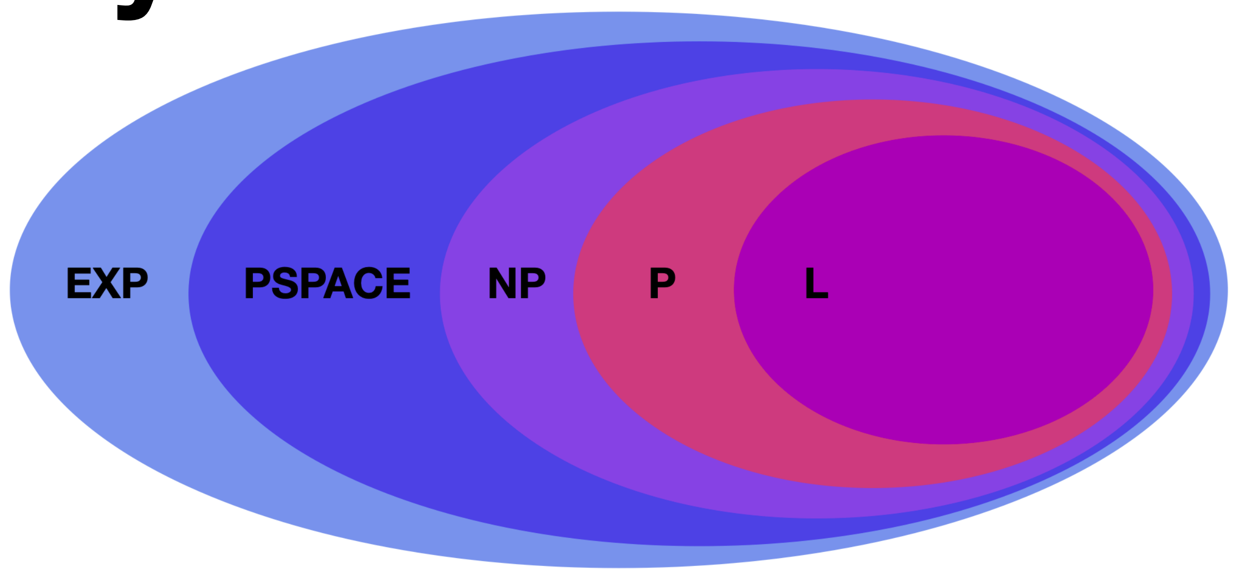


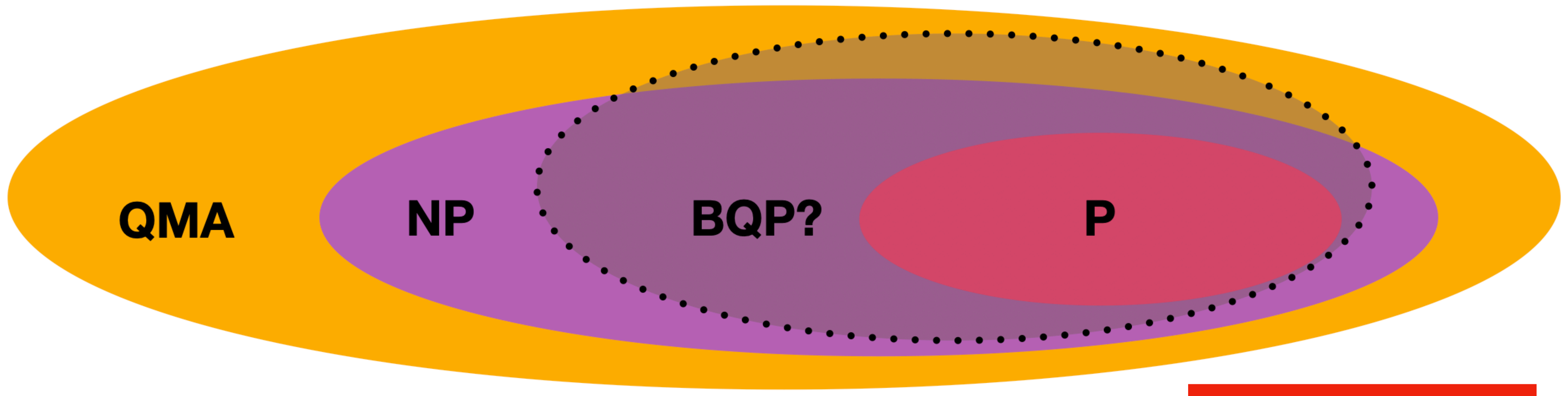
Quantum Algorithms



Complexity theory

$\boxed{\text{NP} \stackrel{?}{=} \text{P}}$





$\boxed{\text{NP} \stackrel{?}{\subseteq} \text{BQP}}$

References

Quantum Computation and Quantum Information, Michael A. Nielsen, Isaac L. Chuang, 2010, Cambridge University Press

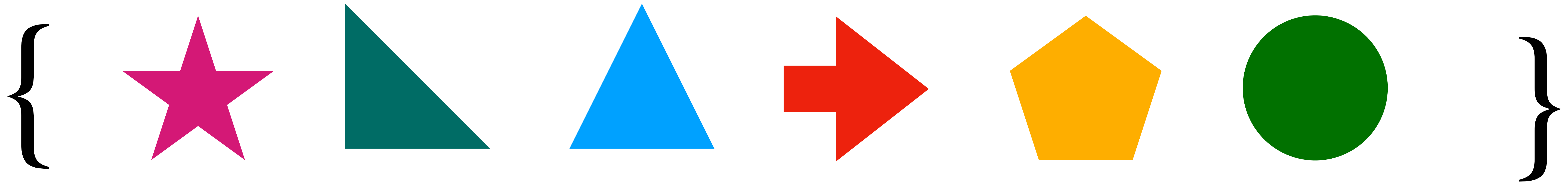
Quantum Computing verstehen, Matthias Homeister, 2014, Springer Verlag

Quantum Computing Lecture Notes, Ronald de Wolf, University of Amsterdam

Grover’s Algorithm

Classical Search

Searching an item in an unordered list of size N



takes on average $\frac{N}{2}$ steps $\rightarrow \mathcal{O}(N)$ in time, thus the problem is in **P**

Quantum Grover's Algorithm is $\mathcal{O}(\sqrt{N})$ in time $\rightarrow$ in **BQP**

Grover's Algorithm

We consider $N = 2$

Solution direction

Initial state $$|s\rangle = \frac{1}{\sqrt{2}}(|\omega\rangle + |s'\rangle)$$

1. step: $|s\rangle \rightarrow U_\omega |s\rangle$ $\quad U_\omega = \mathbf{Id} - 2|\omega\rangle\langle\omega|$

Reflect the $|\omega\rangle$ - component

2. step: $U_\omega |s\rangle \rightarrow U_s U_\omega |s\rangle$ $\quad U_s = 2|s\rangle\langle s| - \mathbf{Id}$

Reflect around $|s\rangle$ - component