Topological densities in  
Einstein-scalar-Gauss-Bonnet gravity  
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Abstract  
The present work is devoted to studying the background dynamical evolution  
of a scalar field in Einstein-Gauss-Bonnet gravity in maximally symmetric space-  
time. This study is useful for giving meaning to the presence of two Gauss-Bonnet  
vacua, instead of using the spherically symmetric bubbles of the ”true” vacuum  
expand in the ”false” vacuum. The theory admits two possible effective cosmo-  
logical constants, which lead to two maximally symmetric vacuum solutions. The  
first solution corresponds to the dynamics of dark energy. When there is matter,  
the second solution describes dark matter. In Einstein-Gauss-Bonnet gravity, we  
establish the expression of the topological mass spectrum which depends on the  
golden ratio and its inverse. In the Schwarzschild limit, the topological density  
correspondsto the standard model radiation energy density. We find the mass loss  
rate which gives the evolution of mass over time.  
1 Introduction  
The observations of the rotation of galaxies and gravitational lenses indicate the pres-  
ence of a new mass, called dark matter (DM) hiding in galaxies, which does not interact  
with radiation and matter, but can be detected by its gravitational effect. The ΛCDM  
model is a cosmological model, parametrized by a cosmological constant Λ associated  
with cold dark matter. It is often called the standard Big Bang model because it is the  
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1  
3202  
rpA  
1  
]cq-rg[  
1v22300.4032:viXra  
simplest model that accounts for the properties of the cosmos: the large-scale structure  
of the observable universe and the distribution of galaxies, the abundance of light ele-  
ments (hydrogen, helium and lithium) and the expansion of the universe. ΛCDM model  
assumes that general relativity theory correctly describes gravity on a cosmological scale.  
However, theΛCDM model presents several problems, such asthecosmological constant,  
fine-tuning problem, and the problem of cosmic coincidence [1]. Recent developments of  
non-trivial extension of Lovelock theory, namely Einstein-Gauss-Bonnet (EGB) theory  
have been proposed [2], providing new insights into the 4-dimensional theory of gravity  
[3]. Their idea is that before taking the limit D 4, they multiplied the Gauss-Bonnet  
−→  
(GB) term by the factor 1/(D 4). The divergent factor 1/(D 4) is canceled by the  
− −  
vanishing GB contributions, which leads to a theory of gravity with only two dynamical  
degrees of freedom, which is in contradiction with Lovelock theorem [4] which describes  
the gravity at D 5. However, it was shown in several papers that the idea of the limit  
≥  
D 4 is not clearly defined, as well as the absence of proper action [5, 6, 7, 8]. It  
−→  
was explicitly confirmed by a direct product D-dimensional spacetime or by adding a  
counter term, before taking the limit D 4, which can be seen as a class of Horndeski  
−→  
theory [9] but with 2 + 1-dofs. Although the EGB gravity is currently debatable, the  
spherically symmetric black hole solution is still meaningful and worthy of study [10].  
The 4D symmetrical static and spherical black hole solution in EGB gravity were ob-  
tained [11], also, solutions of static and spherically symmetric compact stars [12, 13, 14].  
Many researchers have studied the mass-radius profile and the maximum mass in EGB  
gravity [15, 16]. Consequently, it is possible to describe the matter inside compact ob-  
jects (COs) and the dynamical evolution of the matter at high density and the behavior  
of violent events. The Einstein-scalar-Gauss-Bonnet (EsGB) gravity [17], are the class of  
classical scalar-tensor theories that have second-order EOMs, as a special case the Horn-  
deski gravity [18]. The EGB gravity admits two maximally symmetric vacuum solutions  
as the Einstein vacuum in α 0, and the Gauss-Bonnet vacuum in α = 0 [19, 20]. Pre-  
→ 6  
vious studies [21] including the effect of varying the cosmological constant, showed that  
the correspondence between ordinary thermodynamic systems and black hole mechanics  
would be completed to include a variable cosmological constant. The bubble nucleation  
probability depends on the curvature coupling of the Higgs fields, which is a renormal-  
izable parameter of the Standard Model (SM) in curved spacetime [22]. In the effective  
field theory, the thermal and quantum fluctuations to overcome the barrier are charac-  
terized by the decrease of the vacuum. The bubble nucleation in thermal fluctuations  
can be described in terms of Euclidean time coordinate by instantons [23]. The action of  
the Coleman-de Luccia instanton determines the rate of vacuum decay [24]. The classic  
solutions for switching from a false vacuum to a true vacuum are called instantons. We  
2  
suppose that the Boltzmann constant k , the reduced Planck constant ~ and light speed  
B  
c, are such that: k = ~ = c = 1.  
B  
We begin in section 2 with a discussion of the equation of motion for a coupling between  
a scalar field and the model. the chameleon mechanism andthe scalaron mass areinvesti-  
gated in section 3. Section 4 is devoted to the two branches of solutions for the effective  
cosmological constant in a maximally symmetric vacuum. In Section 5, we study the  
scalar dark matter in the exterior region of CO by the EsGB and the regions of validity  
of the GB functional coupling. In Section 6, we explore the applications for the 4D  
Einstein-Gauss-Bonnet black hole. Finally, the paper ends in Section 7 by summarizing  
our main results.  
2 Einstein-scalar-Gauss-Bonnet gravity  
We start by the action of Einstein-scalar-Gauss-Bonnet (EsGB) gravity [25, 26] in 4-  
dimensions as  
M2  
= d4x√ g p R+f (φ) + + , (2.1)  
DM m  
S − 2 G L S  
Z (cid:18) (cid:19)  
where S is the matter action, M = c4 2 1018[GeV], R is the Ricci scalar, f (φ)  
m p 8πG ≈ ×  
is a Gauss-Bonnet coupling function, which is ultraviolet (UV) corrections to Einstein  
theory. We define the Gauss-Bonnet invariant [27] as  
= R2 4R Rµν +R Rµνρσ. (2.2)  
µν µνρσ  
G −  
The variation with respect to the field φ gives us the equation of motion for the scalaron  
field  
(cid:3)φ ∂ V (φ)+ ∂ f (φ) = 0. (2.3)  
φ φ  
− G  
The variation of the action over the metric g simplified by the Bianchi identity gives  
µν  
the equations of motion in [25]. In the Jordan frame, the scalar DM Lagrangian reads  
[28]  
1  
= gµν∂ φ∂ φ V (φ) (2.4)  
DM µ ν  
L −2 −  
the kinetic term is invariant under the transformation φ φ of the Z symmetry. The  
2  
→ −  
metric of a spatially flat homogeneous and isotropic universe in FLRW model is given  
by:  
3  
ds2 = dt2 +a2(t) dxi 2 , (2.5)  
−  
i=1  
X(cid:0) (cid:1)  
3  
where a(t) is a dimensionless scale factor, from which we define the Ricci scalar R and  
the GB invariant in FLRW geometry as  
G  
R = 6 2H2+H˙ = 24H2 H˙ +H2 . (2.6)  
G  
(cid:16) (cid:17) (cid:16) (cid:17)  
˙  
We start by considering φ = φ(t). where φ = ∂ φ, f (φ) = ∂ f (φ) and H = a˙/a is the  
t ′ φ  
Hubble parameter. Eq.(2.3) can be written as the Klein Gordon equation in a simple  
form  
φ¨ +3H φ˙ 8f (φ)H H˙ +H2 +V (φ) = 0. (2.7)  
′ ′  
−  
h (cid:16) (cid:17)i  
3 Scalaron mass in EsGB gravity  
Recently, there has been a renewed interest in the relationship between dark matter and  
the scalaron mass (i.e. the mass of fields φ) [31]. We will study later the scalaron mass  
m which will describe the dark matter. To describe the mass of any scalar field, we use  
φ  
the Klein-Gordon equation (cid:3)φ = ∂ V (φ). Note that Eq.(2.3) gives the same form as  
φ eff  
in the last equation. Since in Eq.(2.3), does not depend on φ, in this case, the effective  
G  
potential is evaluated as follows  
V (φ) = V (φ) f (φ) . (3.1)  
eff  
− G  
We notice that the effective potential of the scalaron includes the Gauss-Bonnet coupling  
and the Gauss-Bonnet invariant. In other words, the Gauss-Bonnet term affects the  
potential structure of the scalaron; thus, the scalaron mass depends on the matter’s  
contribution. The particles of the field φ come from the fluctuation around the minimum  
of the effective potential V (φ). The mass of small fluctuations around φ Eq.(??)  
eff min  
give the scalaron mass [55] is determined as  
∂2  
m2 = V (φ ), (3.2)  
φ ∂φ2 eff min  
where V (φ ) as a minimum value of the scalaron effective potential V . Also we  
eff min eff  
have =  
∂ φ2V(φ) ∂ φ2Veff(φ)  
. To make progress, let us express the in a more convenient  
G ∂ φ2f(φ) − ∂ φ2f(φ) G  
form  
∂2V (φ ) m2  
φ min φ  
= . (3.3)  
G ∂2f (φ ) − ∂2f (φ )  
φ min φ min  
For f (φ) = f ekφ and V (φ) = V e kφ[25, 3] we obtain  
0 0 −  
m2  
φ  
f (φ ) = V (φ ) , (3.4)  
min G min − k2  
From Eqs. (3.1, 3.3), we get  
m2  
φ  
V (φ ) = . (3.5)  
eff min k2  
4  
where ∂2 = ∂2/∂φ2 and V as a minimum value of the scalaron effective potential  
φ eff,min  
V . The minimum of the potential at φ = φ should satisfy ∂ V (φ ) = 0. It is  
eff min φ eff min  
shown that scalar fields can explain the abundance of dark matter. The scalaron mass  
change according to the trace of the energy-momentum tensor [33]. The minimum of the  
potential at φ = φ should satisfy ∂ V (φ ) = 0, which give a special value of the  
min φ eff min  
GB invariant as  
V  
= 0 e 4R0φmin (3.6)  
−  
G −f  
0  
also we have  
V (φ) = V (φ) 1+e4R0(φ φmin) (3.7)  
eff −  
If φ = 0, we get V (φ) = 2V cosh(2R(cid:2)φ). Substitutin(cid:3)g Eqs.(??,3.6) into Eq.(3.1)  
min eff 0 0  
then into Eq.(3.2), the scalaron mass can then be expressed as  
m = 2 2V R e R0φmin. (3.8)  
φ 0 0 −  
p  
It is shown that scalar fields can explain the abundance of dark matter. The scalaron  
mass change according to the trace of the energy-momentum tensor [33]. In the large  
curvature limit [34] we have R φ = 1, one finds  
0 min  
R R = 1/φ = M 2ρ, (3.9)  
−→ 0 min p−  
where ρ is the matter matter-energy. We can then express the scalaron mass m as a  
φ  
function of the matter-energy density  
2√2V  
0  
m = ρ. (3.10)  
φ eM2  
p  
The scalaron field φ becomes dynamical in the low energy density environment on the  
cosmological scale. Since the mass m depends on the matter density ρ, thus, the  
φ  
scalaron becomes heavy in the high-density region of matter. This feature is called the  
chameleon mechanism which is one of the screening mechanism in the modified gravity  
[35]. The chameleon mechanisms is defined when the scalaron mass depends on the  
environment surrounding the scalaron field. The scalaron φ is regarded as a dynamical  
dark matter and can be a dark matter candidate. Similar topics had been researched in  
many literatures [35, 36].  
4 Maximally symmetric vacuum solutions  
Recently, there are several papers which study the Particle production induced by vac-  
uumdecay[45]. Followingtheseconcepts, here, wewillinterpret theGauss-Bonnetvacua  
by the production of dark matter particles. When gravity is taken into consideration,  
5  
the vacua are those with maximally symmetric spaces [22]. In the maximally symmetric  
space, the scalar curvature of de Sitter space is given by  
2D  
R = Λ, (4.1)  
D 2  
−  
where Λ is the cosmological constant. In the case of the positive Λ, we have the de Sitter  
solution. In the large limit, we obtain 1/φ = 2D Λ . In 4-dimensional vacuum, the  
min D 2 0  
−  
equation of motion hasa solution that R = Λg , implies that R = 4Λ. The Minkowski  
µν µν  
spacetime as the vacuum of the Λ = 0. The singularity problem [37] corresponds to  
0  
R (the curvature singularity) or φ = 0. From Eq.(3.8), Eq.(3.9) and Eq.(4.1),  
0 → ∞ min  
for e 2R0φmin 1 2R φ and R R one obtain  
− ≈ − 0 min 0 −→  
m  
2φ R2 R+ φ = 0, (4.2)  
min − 2√2V  
0  
which can reduce to  
2  
D D m  
4φ Λ2 Λ+ φ = 0. (4.3)  
min D 2 − D 2 4√2V  
(cid:18) − (cid:19) − 0  
In maximally symmetric vacuum solutions, there are two branches of solutions for the  
effective cosmological constant,  
D 2 4m  
φ  
Λ = − 1 1 φ , (4.4)  
± 8Dφ min ±s − √2V 0 min !  
In the limit where 2m φ √2V , the two branches are given by  
φ min ≪ 0  
D 2 m φ  
Λ = − 1 φ min , (4.5)  
+  
4Dφ − √2V  
min (cid:18) 0 (cid:19)  
D 2 m  
φ  
Λ = − . (4.6)  
− 4D √2V  
0  
Using Eq. (3.10) and 1/φ = 2D Λ , the above equations can be further rewritten as  
min D 2 0  
−  
Λ 2φ ρ φ ρ  
Λ = 0 1 min ; Λ = Λ min . (4.7)  
+ 2 − eM2 − 0 eM2  
(cid:18) p (cid:19) p  
We notice that Λ = Λ0 Λ . In particular, the large curvature limit 1/4 = Λ φ , we  
+ 2 − − 0 min  
have  
Λ ρ 1 ρ  
0  
= Λ +Λ ; Λ = ; Λ = = , (4.8)  
2 + − − 4eM2 0 4φ 4M2  
p min p  
or equivalently  
(e 2) Λ ρ  
0  
Λ = − Λ ; Λ = ; Λ = (4.9)  
+ 2e 0 − e 0 4M2  
p  
Let us now comment on the two solutions above. The cosmological constant Λ is  
+  
equivalent to that found by [26] which is proportional to the mass of the scalar field  
6  
(chameleonic dark matter [33]). If ρ decreases over time, the value of Λ increases to  
+  
reach Λ . On the other hand, the second branch Λ depends on the matter density. We  
0  
−  
notice that Λ > 0 and Λ > 0, instead of studying the false vacuum forming inside  
+  
−  
the true vacuum in the bubble geometry [39]. The two roots (Λ ,Λ ) can represent  
+  
−  
two faces of the same true vacuum. In a vacuum (without matter), the chameleon  
mechanism will be zero, which corresponds to dark energy. The chameleon mechanism  
appears when there is the matter Eq.(3.10), the vacua are those with Λ . The vacuum  
−  
Λ will be spontaneously produced with ordinary matter fields. The Gauss-Bonnet  
−  
vacuum becomes a chameleon if there is matter, which solves the problem of the Gauss-  
Bonnet vacuum suffering from perturbative ghost instability [19]. This justifies that the  
matter has an impact on the vacuum, is that the two vacua are separated by a domain  
wall, composed of ordinary matter in the thin wall approximation [39]. In particular,  
the large curvature limit 1/φ = R = 4Λ Eq.(4.9), must satisfy Λ = ρ/4M2 and  
min 0 0 0 p  
Λ = ρ/4eM2, we can show that Λ + Λ = Λ , which corresponds to Λ /Λ  
− p + − 0 − 0 ≈  
0,368 36,8% and Λ /Λ 0,132 13,2%. The percentage Λ /Λ 36,8% is  
+ 0 0  
≡ ≈ ≡ − ≡  
close to the density of matter and dark matter in the universe [43]. It is interesting to  
note that 50% = (36,8%+13,2%) of space-time gets a mass in a region occupied by  
(Λ ,Λ ). In this case, we introduce the parameter X which explains the lack of 50% of  
+  
−  
Λ : Λ = Λ +Λ +X, i.e. X/Λ = 0,5 50%. So, we can associate the parameter X  
0 0 + 0  
− ≡  
to dark energy.  
Energy content parameter pourcentage  
Matter+Dark matter Λ Λ /Λ 36,8%  
0  
− − ≡  
Dark energy Λ +X (Λ +X)/Λ = 63,2%  
+ + 0  
Universe Λ 100%  
0  
Table 1: Numerical estimate of the percentages of ordinary matter, dark matter and  
dark energy in the universe [43].  
We notice that the dark energy is described by the parameter Λ and an unknown  
+  
parameter X table (2).  
7  
5 Functional coupling and Barrow entropy  
In this section, we explain in detail how to construct the equation of motion of the  
Einstein-Gauss-Bonnet gravity. We begin by reviewing the the 4D EsGB action  
M2 1  
= p d4x√ g R+f (φ) gµν∂ φ∂ φ V (φ) + ,  
µ ν m  
S 2 − G − 2 − S  
Z (cid:18) (cid:19)  
where M = 1/(8πG ) = 1.221 1019GeV is the reduced Planck mass, R is the Ricci  
p N  
×  
scalar, S is the matter action and f (φ) is a functional coupling of the scalar field φ. In  
m  
the above equation (µ,ν) = (0,1,2,3). We define the GB term as  
R2 4R Rµν +R Rµνρσ. (5.1)  
µν µνρσ  
G ≡ −  
The Kretschmann scalar is R Rµνρσ. The variation with respect to the field φ gives  
µνρσ  
us the equation of motion for the scalar field  
(cid:3)φ = ∂ V (φ), (5.2)  
φ eff  
where (cid:3) µ and the effective potential is  
µ  
≡ ∇ ∇  
V (φ) = V (φ) f (φ) . (5.3)  
eff  
− G  
Varying the action (??) over the metric g , we obtain the following equations of motion:  
µν  
1 1  
Gµν + µν +f (φ) µν + µν gµνV (φ) = κ2Tµν,  
K H 2 Tφ − eff 2  
(cid:2) (cid:3)  
where the Einstein tensor is Gµν = Rµν 1gµνR, the matter stress tensor is Tµν =  
− 2  
2 δ m. On the other hand the µν and µν are given by  
S  
−√ gδgµν K H  
−  
1  
µν = 4[Gµν(cid:3) + R µ ν +(gµνRρσ Rµρνσ) ,  
ρ σ  
K 2 ∇ ∇ − ∇ ∇  
Rνρ µ +Rµρ ν]f (φ) (5.4)  
ρ ρ  
− ∇ ∇ ∇ ∇  
µν = 2RµρστRν RRµν  
H ρστ −  
1  
+ Rµ Rνρ RµρστRν . (5.5)  
2 ρ − ρστ  
The tensor µν represents an operator which acts on f (φ). The energy-momentum  
K  
tensor for the scalar field is  
1  
µν = µφ νφ gµν φ ρφ. (5.6)  
Tφ ∇ ∇ − 2 ∇ρ ∇  
The stress tensor for anisotropic compact object is given as  
Tµν = (ρ+P )uµuν +P gµν +(P P )χµχν, (5.7)  
t t t  
−  
8  
with energy density ρ = ρ(r) c2ρ(r), transverse pressure P (r) and radial pressure  
CO ≡ t  
P(r) of the homogeneously distributed matter in the compact object (CO), where uµ is  
the four-velocity of the fluid and χµ is the unit space-like vector in the radial direction.  
In limit α 0, one can see that this is equivalent to the original form given by  
−→  
the Bekenstein-Hawking entropy. One can check that for D = 2, the topological term  
characterized by a vanishes Lovelock coupling α. The Barrow entropy [59] is a new black  
hole entropy which is given by  
1+δ/2  
A  
S = π , (5.8)  
A  
(cid:18) 0(cid:19)  
where 0 δ 1, A is the black hole horizon area and A is the Planck area [56]. When  
0  
≤ ≤  
δ = 0, the area law is restored, i.e. S = A (where A = 4G) while δ = 1 represents the  
4G 0  
most intricate and fractal structure of the horizon.  
The form the functional f (φ) can take f (φ) = e γφ [60], where γ is a constant, which  
−  
corresponds to EGB gravity coupled with dilaton that arises as a low-energy limit of  
the string theory [61]. Using the same principle as this entropy, we assume that for the  
compact object (or black hole) exterior, there is a presence of scalar fields φ, while inside  
the compact object is replaced by the Gauss-Bonnet (GB) coupling α as  
coupling constant=f (φ), CO exterior (D = 4)  
. (5.9)  
coupling constant= α , CO interior (D 4)  
D 4 →  
−  
The GB coupling α is measured in km2. In the CO exterior interior, we have rescaled  
the coupling constant α α/(D 4). The negative (positive) α leads to a decrease  
→ −  
(increase) of the CO radius and the maximum mass [63]. If α < 0 the solution is still  
the anti-de Sitter (AdS) space, if α > 0 the solution is the de Sitter (dS) space [62].  
We investigate in detail the impact of the Gauss-Bonnet coupling on the properties of  
an anisotropic compact object, such as mass, radius and the factor of compactness.  
Considering the limit D 4, and it has an effect on gravitational dynamics in 4D.  
→  
Additionally, at the CO boundary (r = R), the GB coupling must be continuous, i.e.  
f (φ) α . On the other hand, the function f (φ) describes the star’s exterior region.  
≡ D 4  
−  
To study the equations of motion inside and outside CO, we differentiate between two  
cases:  
In CO interior (D 4) we have:  
→  
1 κ2  
Gµν +α µν + gµν = Tµν. (5.10)  
H 2 G 2  
(cid:18) (cid:19)  
In CO exterior (D = 4) we have:  
1 1  
Gµν + µν +f (φ) µν gµν φ λφ+4V (φ) = κ2Tµν, (5.11)  
λ eff  
K H − 2f (φ) ∇ ∇ 2  
(cid:20) (cid:21)  
(cid:0) (cid:1)  
9  
with g µν = φ λφ. In CO interior (D 4) we have: The GB invariant can be  
µν Tφ −∇λ ∇ →  
greatly simplified to the matter density [64] and using Eq.(5.3) we obtain  
ρ = 4 4 1 φ λφ+V (φ) , CO exterior (D = 4)  
f(φ) G − f(φ) 4∇λ ∇ . (5.12)  
ρ = , CO interior (D 4)  
CO G (cid:0) (cid:1) →  
In this case, the term ρ represents the density of matter in compact objects, and  
CO  
ρ is the density of dark matter surrounding these objects. Note that the relation  
f(φ)  
between ρ and ρ highlight the chameleon dark matter [65, 66]. Formally, at the  
f(φ) CO  
points where the dark matter density ρ equal to ρ . For V (φ) 1 φ λφ,  
DM f(φ) ≈ −4∇λ ∇  
we obtain ρ 4ρ , which is in good agreement with the observation data of the  
DM ≈ CO  
percentages of dark matter and the matter in the Universe [69, 57]: ρ 80% and  
DM ≡  
ρ (ρ ) 20%.  
matter CO ≡  
Next, we assume that φ = φ(t). In cosmological and quintessence behavior [67, 68], the  
energy density ρ and pressure P of the scalar field are given by  
φ φ  
1  
˙2  
ρ = φ +V (φ), (5.13)  
φ 2  
1  
˙2  
P = φ V (φ).  
φ  
2 −  
The quintessence models describe dark energy with a scalar field φ. In this case, ρ  
φ  
and P are respectively, the density and the pressure of the dark energy (DE). The  
φ  
Planck Collaboration [69] provides constraints on the equation of state ω = P /ρ  
φ φ φ ≈  
1.028 0.032. Starting from Eqs. (5.13), we obtain  
− ±  
1  
ρ = 4ρ + P 3ρ . (5.14)  
DM CO f (φ) φ − φ  
(cid:0) (cid:1)  
For small f (φ), the DM chameleon effect vanishes and the DM density depends only on  
P 3ρ /f (φ).  
φ − φ  
The scalar field always sits at the minimum of its effective potential. We assume that a  
(cid:0) (cid:1)  
massive scalar field begins oscillating about a minimum. The mass of small fluctuations  
around φ gives a new scalar field mass as effective mass by m2 = ∂2 V (φ) .  
min eff ∂φ2 eff  
φ=φmin  
From Eqs. (5.3,5.13,5.14) we obtain (cid:12)  
(cid:12)  
(cid:12)  
1 ∂2 ρ  
m2 = ρ P + CO 3ρ P . (5.15)  
eff 2∂φ2 φ − φ ρ 4ρ φ − φ  
(cid:18) DM − CO (cid:19)φ=φmin  
(cid:0) (cid:1)  
From Eq. (5.14), the functional coupling is given by  
P 3ρ  
f (φ) = φ − φ . (5.16)  
ρ 4ρ  
DM − CO  
In the void (ρ = 0,ω 1), we have f (φ) = 4ρ /ρ . In the CO surface, we  
CO φ ∼ − − φ DM  
assume that ρ 0 and P 0, so we get f (φ) 3ρ /4ρ . Since ρ represents  
DM ≈ φ ≈ surface ≈ φ CO φ  
10  
the density of DE according to quintessence, the effect of DE is weak on the CO surface,  
which shows that f (φ) ρ 0. Inside CO, we have (P = ρ = 0), i.e.  
surface ∝ φ → φ φ  
f (φ) = 0, which exactly corresponds with the assumption Eq. (5.9). For this reason,  
(cid:0) (cid:1)  
we exclude f (φ) inside matter, and we replace it with the GB coupling α (see the next  
section).  
6 EGB primordial black holes  
We start by the action of Einstein-Gauss-Bonnet (EGB) gravity [48] in 4-dimensions as  
M2  
= dDx√ g p R 2Λ+f (φ) R2 4R Rµν +R Rµνρσ , (6.1)  
µν µνρσ  
I − 2 − −  
Z (cid:18) (cid:19)  
(cid:0) (cid:1)  
where S is the matter action, M2 =  
c4  
2 1018[GeV], R is the Ricci scalar. We  
m p 8πG ≈ ×  
define the Gauss-Bonnet invariant as = = R2 4R Rµν+R Rµνρσ, where α =  
2 µν µνρσ 2  
G L −  
α/(D 4) is the Gauss-Bonnet coupling have dimensions of [length]2, that represent  
−  
ultraviolet (UV) corrections to Einstein theory. To solving the field equation we obtain  
the black hole solution ds2 = f(r)dt2+ 1 dr2+r2 dθ2 +sin2θdφ2 . Taking the limit  
− f(r)  
D 4, we obtain the exact solution in closed form  
(cid:0) (cid:1)  
−→  
r2 2M 1  
f(r) 1+ 1 1+4α . (6.2)  
≈ 2α −s r3 ± l2  
!  
(cid:18) (cid:19)  
This last solution could be obtained directly from the derivation done in [49]. In the  
limit r with vanishing black hole charge, we asymptotically obtain the GR  
−→ ∞  
Schwarzschild solution. In the limit α 0, we can recover the Reissner-Nordstr¨om-  
−→  
AdS solution. If α < 0 the solution is still an AdS space, if α > 0 the solution is  
a de Sitter (dS) space, [50]. The solutions show that the event horizon is located at  
R = M √M2 α, where R = R and R are the event horizon and the Cauchy  
H +  
± ± − −  
horizon radius of the EGB black hole [47]. We can express the ADM mass M of the  
black hole in terms of R by solving f(r) = 0 for r = R resulting in  
H H  
l 2R4 +R2 +α  
M = − H H . (6.3)  
2R  
H  
The Hawking temperature of the EBG black hole can be calculated as  
3l 2R4 +R2 α  
T = − H H − . (6.4)  
8παR +4πR3  
H H  
4πR3  
The thermodynamic volume V = H is the conjugate variable to the pressure. The  
3  
parameters V and A are the conjugate quantities of the pressure P and GB coupling  
11  
parameter α. The event horizon in spacetime can be located by solving the metric  
equation: f(r) = 0, and from Eq .(6.2) we obtain  
3 3α  
ρ +P = + , (6.5)  
BH 8πR2 8πR4  
H H  
where M = ρ V, V = 4πr3/3 and P = 3/8πl2. For the limit α 0, we can recover  
BH −→  
the density ρ = 3/16πR2 of the holographic dark energy (DE) [70]. In the spatially flat  
Λ H  
homogeneous and isotropic universe in the FLRW, the (modified) Friedmann equations  
can be obtained [48] H2 +αH4 = 8πGρ+ Λ and (H2 +αH4)H˙ = 4πG(ρ+P). This  
3 3 −  
equation looks like the Eq .(6.5), i.e. can describe the dynamics in space-time associated  
to a black hole. For Gauss–Bonnet branch, we introduce the topological density as  
α  
ρ = ρ(2) = . (6.6)  
α α 16πR4  
H  
From Eq. (6.5) we obtain the Van der Waals equation:  
4παR R  
H H  
(P +ρ ) V = . (6.7)  
BH − 3 2  
(cid:18) (cid:19)  
In the limit α 0 and P ρ, we can recover the ideal gas law. The critical point  
−→ ≫  
occurs when P(V). We note that the Gauss-Bonnet coupling α represente measure of  
the average attraction between particles. Before proceeding further, we note that the  
mass M can be interpreted as a chemical enthalpy [51], which is the total energy of the  
black hole [52] including both the energy PV and the internal energy E and required to  
displace the vacuum energy of its environment. The Hubble horizon mass is connected  
with the Smarr formula as  
R α  
H  
M +PV = + , (6.8)  
H  
2 2R  
H  
4πR3  
where M = Hρ is the Hubble horizon mass. We have also  
H 3  
α  
R 2M . (6.9)  
H  
≈ − 2M  
We can recover the Schwarzschild radius for α 0. This expression gives an interpreta-  
≈  
tion of the difference between Schwarzschild and EGB black holes. From Eqs. (6.8,6.9),  
we obtain the the Smarr formula:  
α 1  
M = M +PV + 1 . (6.10)  
H 4M − 1 α  
(cid:18) − 4M2(cid:19)  
In the limit α 0, we can recover the standard Smarr formula. Let us mention that  
−→  
for the AdS-Schwarzschild limit and using M M , we get PV = α/4M T.  
H  
≈ ∝ − ∝  
This expression gives an interpretation to the term that depends on α, as a solution of  
12  
AdS-GB gravity in the presence of a perfect fluid [53]. This is not valid for the dS-GB  
black holes.  
α 1  
PV = (M M ) 1 . (6.11)  
− H − 4M − 1 α  
(cid:18) − 4M2(cid:19)  
For Gauss-Bonnet branch we obtain 1 4M2 1 α = 1. One of two solutions  
− α − 4M2  
to this equation is the Golden ratio ((cid:16)α > 0) i(cid:17)n(cid:0)de Sitter(cid:1)(dS) space: 4Mα  
2  
= 1+ 2√5 or  
4M2 = 1+√5. If α < 0 the solution is still an AdS space: α = 1 √5 or 4M2 = 1 √5. We  
α 2 4M2 − 2 α − 2  
obtain the spectrum of mass in the Gauss-Bonnet branch is  
√γα  
M = , (6.12)  
α  
2  
withγ = 2 , 1 √5, 2 , 1+√5 . Itcanbewrittenasγ 1.618, 0.618,0.618,1.618 .  
−  
1 √5 2 1+√5 2 ≈ {− − }  
−  
This showns that the Gauss-Bonneot coupling represents a topological black hole mass.  
α γ M  
α  
0.5 1.618 0.449  
− −  
0.5 0.618 0.277  
− −  
0.5 0.618 0.277  
0.5 1.618 0.449  
Table 2: Numerical estimate of the values of the PBH mass, according to the frequency  
f and the interval of N between 10 47 and 60. Such as f = 2.561 1033N.  
− × M  
Taking the derivative with respect to time t in both sides of Eq. (6.9), thus the mass  
loss rate of a black hole is obtained as  
dM R˙ α  
H  
1 , (6.13)  
dt ≈ 2 − 2M2  
(cid:16) (cid:17)  
with R˙ = dRH. Using the 4D Stefan-Boltzmann law [54, 58] dM = π2AT4/60, with T  
H dt dt −  
and A are temperature and area of black hole, respectively. Evidently, we have α/M2  
∝  
T4. For the Gauss-Bonnet branch and from Eq. (6.12), we have dM R˙ H (1 2γ).  
dt ≈ 2 −  
Taking the Friedmann equation HR = 1 [70] and from Eq .(6.9) we have  
H  
1/2  
1 1 1  
M (t) = +4α . (6.14)  
± 4H(t) ± 4 H(t)  
(cid:18) (cid:19)  
Focusing on M (t), one can find the condition 1 1 1 4α. For α = 0.5 we plot M  
− H H − ≥ θ2 ±  
vsH Fig. (1, left). UsingH = q/tintheradiation-dominatedera(q = 1/2)forα = 0.5,  
(cid:0) (cid:1)  
−  
we plot M vs t Fig. (1, right). In Fig. (1, right), the domain in red, represents the  
±  
interval where the evolution of the mass with respect to time begins to increase. While,  
13  
Figure 1: Left, curves of M vs H from Eq. (6.14)with α = 0.5. Right, M vs t (dashed  
+  
±  
line) and M vs t (solid line) for the values q = 1/2 and α = 0.5 .  
− −  
before this interval, the mass decreases over time which is due to Hawking evaporation.  
Accordingtothisfigure, twotypesofPBHshaveoppositepropertiessuchastheevolution  
of their masses over time.  
7 Conclusion  
The coupling between the scalar fieldφ andthe4DEinstein-Gauss-Bonnet gravity action  
isstudied. Wehavestudiedthemodeldescribingthecompactstarsin4DEinstein-Gauss-  
Bonnet gravity surrounded by scalar dark matter. We have made a comparison between  
the star interior and the exterior region. In this case, the Gauss-Bonnet coupling de-  
scribes the interior structure of theblack hole, while thecoupling functionf (φ) describes  
the black hole exterior region. We have shown that f (φ) = 0 inside the black hole. By  
solving the equations of motion based on the chameleon mechanism. We refer to φ as  
a chameleon field, since its physical properties, such as its mass, depend sensitively on  
the environment. It can be used to demonstrate the relation between the cosmological  
constant and the matter density. The Gauss-Bonnet vacuum have a chameleon struc-  
ture, and the mass m appears with a small fluctuations. Moreover, we have studied  
φ  
the vacua in maximally symmetric solutions, and we obtain two branches of solutions,  
i.e. there are two such vacua. The first solution corresponds to the dark energy and the  
second represents the chameleonic vacua or dark matter. We have found strong mix-  
ing between the two vacua, which is presented by the chameleon mechanism. We also  
discussed the equation of state parameter in the model of the scalar field φ minimally  
coupled to EGB gravity, also in the maximally symmetric space. The percentages of the  
14  
effective cosmological constants are in good agreement with those of dark matter and  
dark energy in the Universe.  
We have considered 4D Einstein-Gauss-Bonnet gravity. We establish the relationship  
between a topological mass spectrum and the golden ratio. We have obtained the mass  
loss rate, which gives by taking the Friedmann equation of type HR = 1 [70], the evo-  
H  
lution of the mass over time, and the evaporation Hawking time. For such a choice and  
fixed values of Lovelock coupling, we have plotted the mass of PBH versus the Hubble  
parameter and time.  
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