

# Operations on digital sound: digital filter

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Jan 13, 2017

What we will define as *digital filters* is exemplified by the following procedure:

$$z_n = \frac{1}{4}(x_{n-1} + 2x_n + x_{n+1}), \quad \text{for } n = 0, 1, \dots, N-1.$$

# Matrices of filters

Assume that the input vector is periodic with period  $N$ , so that  $x_{n+N} = x_n$ . It is straightforward to show that the output vector  $\mathbf{z}$  is also periodic with period  $N$ .

The filter is also clearly a linear transformation and may therefore be represented by an  $N \times N$  matrix  $S$  that maps the vector  $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})$  to the vector  $\mathbf{z} = (z_0, z_1, \dots, z_{N-1})$ , i.e., we have  $\mathbf{z} = S\mathbf{x}$ .

The elements of  $S$  can be found by row as

$$S = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{pmatrix}.$$

# Circulant Toeplitz matrices

The matrix we just stated is called a circulant Toeplitz matrix. The general definition is as follows and may seem complicated, but is in fact quite straightforward:

**Definition ??** An  $N \times N$ -matrix  $S$  is called a Toeplitz matrix if its elements are constant along each diagonal. More formally,  $S_{k,l} = S_{k+s,l+s}$  for all nonnegative integers  $k, l$ , and  $s$  such that both  $k+s$  and  $l+s$  lie in the interval  $[0, N-1]$ . A Toeplitz matrix is said to be circulant if in addition

$$S_{(k+s) \bmod N, (l+s) \bmod N} = S_{k,l}$$

for all integers  $k, l$  in the interval  $[0, N-1]$ , and all  $s$  (Here mod denotes the remainder modulo  $N$ ).

## More general expression for a filter

$$z_n = \sum_k t_k x_{n-k}.$$

- $\mathbf{x}$  denotes the *input vector*.
- $\mathbf{z}$  the *output vector*.
- $t_k$  denotes the *filter coefficients*.

Assume that  $t_0, t_1, \dots, t_{k_{\max}}$  are the only non-zero filter coefficients.

```
z = zeros(1, N);  
for n = kmax:(N-1)  
    for k = 0:kmax  
        z(n + 1) = z(n + 1) + t(k + 1)*x(n - k + 1);  
    end  
end
```

# Filter in Python

```
z = zeros_like(x)
for n in range(kmax, N):
    for k in range(kmax + 1):
        z[n] += t[k]*x[n - k]
```

## Filters as matrices, Proposition ??

Any operation defined by Equation (??) is a linear transformation which transforms a vector of period  $N$  to another of period  $N$ . It may therefore be represented by an  $N \times N$  matrix  $S$  that maps the vector  $\mathbf{x} = (x_0, x_1, \dots, x_{N-1})$  to the vector  $\mathbf{z} = (z_0, z_1, \dots, z_{N-1})$ , i.e., we have  $\mathbf{z} = S\mathbf{x}$ . Moreover, the matrix  $S$  is a circulant Toeplitz matrix, and the first column  $\mathbf{s}$  of this matrix is given by

$$s_k = \begin{cases} t_k, & \text{if } 0 \leq k < N/2; \\ t_{k-N} & \text{if } N/2 \leq k \leq N-1. \end{cases}$$

In other words, the first column of  $S$  can be obtained by placing the coefficients in (??) with positive indices at the beginning of  $\mathbf{s}$ , and the coefficients with negative indices at the end of  $\mathbf{s}$ .



## Compact notation for filters, Definition ??

Let  $k_{\min}$ ,  $k_{\max}$  be the smallest and biggest index of a filter coefficient in Equation (??) so that  $t_k \neq 0$  (if no such values exist, let  $k_{\min} = k_{\max} = 0$ ), i.e.

$$z_n = \sum_{k=k_{\min}}^{k_{\max}} t_k x_{n-k}.$$

We will use the following compact notation for  $S$ :

$$S = \{t_{k_{\min}}, \dots, t_{-1}, \underline{t_0}, t_1, \dots, t_{k_{\max}}\}.$$

In other words, the entry with index 0 has been underlined, and only the nonzero  $t_k$ 's are listed.  $k_{\max}$  and  $k_{\min}$  are also called the start and end indices of  $S$ . By the length of  $S$ , denoted  $l(S)$ , we mean the number  $k_{\max} - k_{\min}$ .

## Convolution of vectors, Definition ??

By the *convolution* of two vectors  $\mathbf{t} \in \mathbb{R}^M$  and  $\mathbf{x} \in \mathbb{R}^N$  we mean the vector  $\mathbf{t} * \mathbf{x} \in \mathbb{R}^{M+N-1}$  defined by

$$(\mathbf{t} * \mathbf{x})_n = \sum_k t_k x_{n-k},$$

where we only sum over  $k$  so that  $0 \leq k < M$ ,  $0 \leq n - k < N$ .

# Using convolution to compute filters Proposition ??

Assume that  $S$  is a filter on the form

$$S = \{t_{-L}, \dots, \underline{t_0}, \dots, t_L\}.$$

If  $\mathbf{x} \in \mathbb{R}^N$ , then  $S\mathbf{x}$  can be computed as follows:

- Form the vector  $\tilde{\mathbf{x}} = (x_{N-L}, \dots, x_{N-1}, x_0, \dots, x_{N-1}, x_0, \dots, x_{L-1}) \in \mathbb{R}^{N+2L}$ .
- Use the conv function to compute  $\tilde{\mathbf{z}} = \mathbf{t} * \tilde{\mathbf{x}} \in \mathbb{R}^{M+N+2L-1}$ .
- We have that  $S\mathbf{x} = (\tilde{z}_{2L}, \dots, \tilde{z}_{M+N-2})$ .

## Convolution and polynomials Proposition ??

Assume that  $p(x) = a_N x^N + a_{N-1} x_{N-1} + \dots, a_1 x + a_0$  and  $q(x) = b_M x^M + b_{M-1} x_{M-1} + \dots, b_1 x + b_0$  are polynomials of degree  $N$  and  $M$  respectively. Then the coefficients of the polynomial  $pq$  can be obtained by computing  $\text{conv}(a, b)$ .

A linear transformation  $S : \mathbb{R}^N \mapsto \mathbb{R}^N$  is said to be a digital filter, or simply a filter, if, for any integer  $n$  in the range  $0 \leq n \leq N - 1$  there exists a value  $\lambda_{S,n}$  so that

$$S(\phi_n) = \lambda_{S,n}\phi_n,$$

i.e., the  $N$  Fourier vectors are the eigenvectors of  $S$ . The vector of (eigen)values  $\lambda_S = (\lambda_{S,n})_{n=0}^{N-1}$  is often referred to as the (*vector*) *frequency response* of  $S$ .

## The product of two filters is a filter, Corollary ??

The product of two digital filters is again a digital filter. Moreover, all digital filters commute, i.e. if  $S_1$  and  $S_2$  are digital filters,  $S_1 S_2 = S_2 S_1$ .

## Time-invariance, Definition ??

Assume that  $S$  is a linear transformation from  $\mathbb{R}^N$  to  $\mathbb{R}^N$ . Let  $\mathbf{x}$  be input to  $S$ , and  $\mathbf{y} = S\mathbf{x}$  the corresponding output. Let also  $\mathbf{z}$ ,  $\mathbf{w}$  be delays of  $\mathbf{x}$ ,  $\mathbf{y}$  with  $d$  elements (i.e.  $z_n = x_{n-d}$ ,  $w_n = y_{n-d}$ ).  $S$  is said to be *time-invariant* if, for any  $d$  and  $\mathbf{x}$ ,  $S\mathbf{z} = \mathbf{w}$  (i.e.  $S$  sends the delayed input vector to the delayed output vector).

# Characterizations of digital filters, Theorem ??

The following are equivalent characterizations of a digital filter:

- $S = (F_N)^H D F_N$  for a diagonal matrix  $D$ , i.e. the Fourier basis is a basis of eigenvectors for  $S$ .
- $S$  is a circulant Toeplitz matrix.
- $S$  is linear and time-invariant.



# Connection between frequency response and the matrix, Theorem ??

Any digital filter is uniquely characterized by the values in the first column of its matrix. Moreover, if  $\mathbf{s}$  is the first column in  $S$ , the frequency response of  $S$  is given by

$$\lambda_S = \text{DFT}_N \mathbf{s}.$$

Conversely, if we know the frequency response  $\lambda_S$ , the first column  $\mathbf{s}$  of  $S$  is given by

$$\mathbf{s} = \text{IDFT}_N \lambda_S.$$

## Connection between vector- and continuous frequency response, Theorem ??

The function  $\lambda_S(\omega)$  defined on  $[0, 2\pi)$  by

$$\lambda_S(\omega) = \sum_k t_k e^{-ik\omega},$$

where  $t_k$  are the filter coefficients of  $S$ , satisfies

$$\lambda_{S,n} = \lambda_S(2\pi n/N) \text{ for } n = 0, 1, \dots, N-1$$

for any  $N$ . In other words, regardless of  $N$ , the vector frequency response lies on the curve  $\lambda_S$ .

# Higher and lower frequencies

**Observation ?? (Plotting the frequency response):** When plotting the frequency response on  $[0, 2\pi)$ , angular frequencies near 0 and  $2\pi$  correspond to low frequencies, angular frequencies near  $\pi$  correspond to high frequencies

**Observation ?? (higher and lower frequencies):** When plotting the frequency response on  $[-\pi, \pi)$ , angular frequencies near 0 correspond to low frequencies, angular frequencies near  $\pm\pi$  correspond to high frequencies.

# Properties of the frequency response, Theorem ??

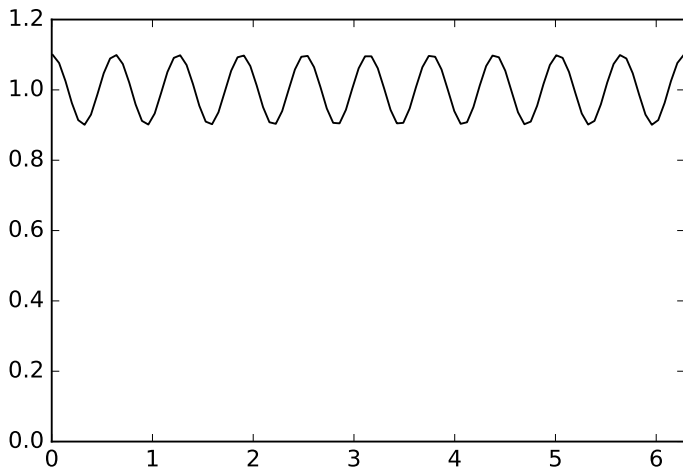
We have that

- The continuous frequency response satisfies  $\lambda_S(-\omega) = \overline{\lambda_S(\omega)}$ .
- If  $S$  is a digital filter,  $S^T$  is also a digital filter. Moreover, if the frequency response of  $S$  is  $\lambda_S(\omega)$ , then the frequency response of  $S^T$  is  $\overline{\lambda_S(\omega)}$ .
- If  $S$  is symmetric,  $\lambda_S$  is real. Also, if  $S$  is antisymmetric (the element on the opposite side of the diagonal is the same, but with opposite sign),  $\lambda_S$  is purely imaginary.
- A digital filter  $S$  is an invertible if and only if  $\lambda_{S,n} \neq 0$  for all  $n$ . In that case  $S^{-1}$  is also a digital filter, and  $\lambda_{S^{-1},n} = 1/\lambda_{S,n}$ .
- If  $S_1$  and  $S_2$  are digital filters, then  $S_1 S_2$  also is a digital filter, and  $\lambda_{S_1 S_2}(\omega) = \lambda_{S_1}(\omega) \lambda_{S_2}(\omega)$ .

# Adding echo to sound 1

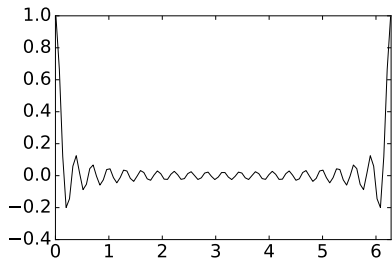
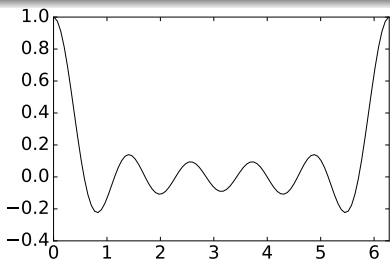
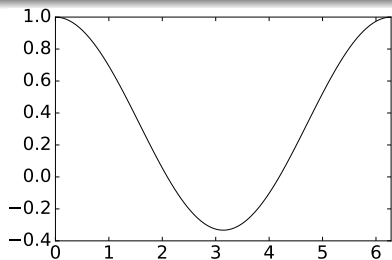
```
[N,nchannels] = size(x);  
z = zeros(N,nchannels);  
z(1:d,:) = x(1:d,:);  
z((d+1):N,:) = x((d+1):N,:)+c*x(1:(N-d),:);
```

## Adding echo to sound 2



**Figure:** The frequency response of a filter which adds an echo with damping factor  $c = 0.1$  and delay  $d = 10$ .

# Moving average filters



**Figure:** The frequency response of moving average filters with  $L = 1$ ,  $L = 5$ , and  $L = 20$ .

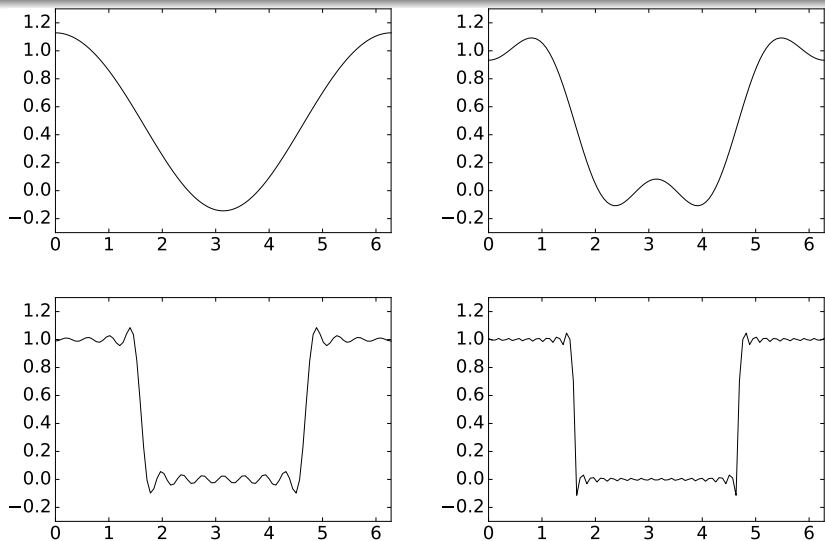
## Lowpass and highpass filters, definition ??

A filter  $S$  is called

- a *lowpass filter* if  $\lambda_S(\omega)$  is large when  $\omega$  is close to 0, and  $\lambda_S(\omega) \approx 0$  when  $\omega$  is close to  $\pi$  (i.e.  $S$  keeps low frequencies and annihilates high frequencies),
- a *highpass filter* if  $\lambda_S(\omega)$  is large when  $\omega$  is close to  $\pi$ , and  $\lambda_S(\omega) \approx 0$  when  $\omega$  is close to 0 (i.e.  $S$  keeps high frequencies and annihilates low frequencies),
- a *bandpass filter* if  $\lambda_S(\omega)$  is large within some interval  $[a, b] \subset [0, 2\pi]$ , and  $\lambda_S(\omega) \approx 0$  outside this interval.

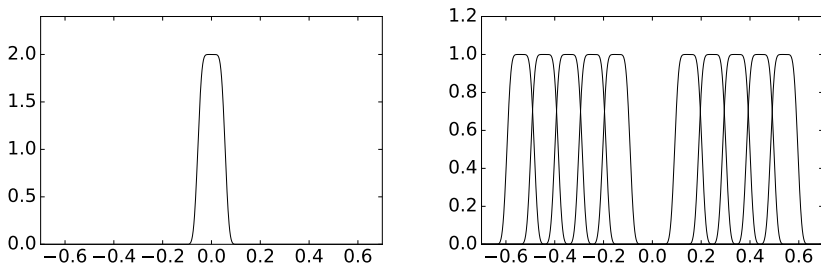


# Dropping filter coefficients in ideal lowpass filters



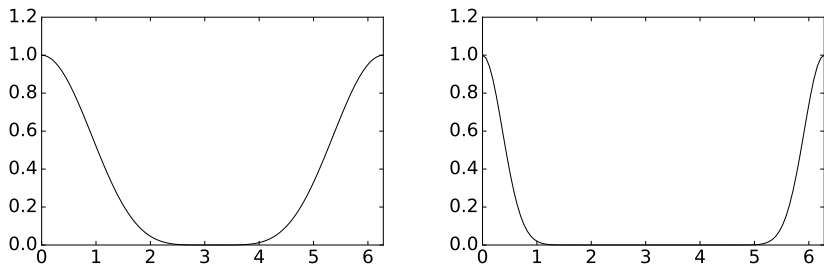
**Figure:** The frequency response which results by including the first  $1/32$ , the first  $1/16$ , the first  $1/4$ , and all of the filter coefficients in the ideal lowpass filter.

# Filters and the MP3 standard



**Figure:** Frequency responses of some filters used in the MP3 standard. The prototype filter is shown left. The other frequency responses at right are simply shifted copies of this.

# Reducing the treble by picking filter coefficients from Pascals triangle



**Figure:** The frequency response of filters corresponding to iterating the moving average filter  $\{1/2, 1/2\}$   $k = 5$  and  $k = 30$  times (i.e. using row  $k$  in Pascal's triangle).

## Reducing treble and bass

**Observation ?? (Reducing the treble):** Let  $\mathbf{x}$  be the samples of a digital sound, and let  $S$  be a filter with coefficients taken from row  $k$  of Pascals triangle. Then  $S\mathbf{x}$  has reduced treble when compared to  $\mathbf{x}$ .

**Observation ?? (Passing between lowpass- and highpass filters):** Assume that  $S_2$  is obtained by adding an alternating sign to the filter coefficients of  $S_1$ . If  $S_1$  is a lowpass filter, then  $S_2$  is a highpass filter. If  $S_1$  is a highpass filter, then  $S_2$  is a lowpass filter.

**Observation ?? (Pascals triangle and reducing the bass):** Let  $\mathbf{x}$  be the samples of a digital sound, and let  $S$  be a filter with filter coefficients taken from row  $k$  of Pascal's triangle, and add an alternating sign to the filter coefficients. Then  $S\mathbf{x}$  has reduced bass when compared to  $\mathbf{x}$ .