# Fourier series: basic concepts

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# Square-integrable functions, Definition 1.6

The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$.

A real function $f$ defined on $[0, T]$ is said to be square integrable if $f^2$ is Riemann-integrable, i.e., if the Riemann integral of $f^2$ on $[0, T]$ exists,

$$\int_0^T f(t)^2 \, dt < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

# Inner product spaces, Theorem 1.7

Both $L^2[0, T]$ and $C[0, T]$ are vector spaces. Moreover, if the two functions $f$ and $g$ lie in $L^2[0, T]$ (or in $C[0, T]$), then the product $fg$ is Riemann-integrable (or in $C[0, T]$). Moreover, both spaces are inner product spaces with inner product defined by

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t) g(t)\, dt,$$

and associated norm

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T f(t)^2 dt}.$$

# Fourier series, Definition 1.8

Let $V_{N,T}$ be the subspace of $C[0,T]$ spanned by the set of functions given by

$$\begin{aligned}\mathcal{D}_{N,T} = \{&1, \cos(2\pi t/T), \cos(2\pi 2t/T), \cdots, \cos(2\pi Nt/T),\\ &\sin(2\pi t/T), \sin(2\pi 2t/T), \cdots, \sin(2\pi Nt/T)\}.\end{aligned}$$

The space $V_{N,T}$ is called the *N'th order Fourier space*. The $N$th-order Fourier series approximation of $f$, denoted $f_N$, is defined as the best approximation of $f$ from $V_{N,T}$ with respect to the inner product defined by (1.3).

# Fourier coefficients, Theorem 1.9

The set $\mathcal{D}_{N,T}$ is an orthogonal basis for $V_{N,T}$. In particular, the dimension of $V_{N,T}$ is $2N+1$, and if $f$ is a function in $L^2[0,T]$, we denote by $a_0, \ldots, a_N$ and $b_1, \ldots, b_N$ the coordinates of $f_N$ in the basis $\mathcal{D}_{N,T}$, i.e.

$$f_N(t) = a_0 + \sum_{n=1}^{N} \left( a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T) \right).$$

The $a_0, \ldots, a_N$ and $b_1, \ldots, b_N$ are called the (real) Fourier coefficients of $f$, and they are given by

$$a_0 = \langle f, 1 \rangle = \frac{1}{T} \int_0^T f(t)\, dt,$$

$$a_n = 2\big\langle f, \cos(2\pi nt/T) \big\rangle = \frac{2}{T} \int_0^T f(t) \cos(2\pi nt/T)\, dt \quad \text{for } n \geq 1,$$

$$b_n = 2\langle f, \sin(2\pi nt/T) \rangle = \frac{2}{T} \int_0^T f(t) \sin(2\pi nt/T)\, dt \quad \text{for } n \geq 1.$$

# Convergence of Fourier series Theorem 1.26

Suppose that $f$ is periodic with period $T$, and that

- $f$ has a finite set of discontinuities in each period.
- $f$ contains a finite set of maxima and minima in each period.
- $\int_0^T |f(t)|dt < \infty$.

Then we have that $\lim_{N\to\infty} f_N(t) = f(t)$ for all $t$, except at those points $t$ where $f$ is not continuous.

# Fourier series of the square wave

```
t = linspace(0, T, 100);
y = zeros(size(t));
for n = 1:2:19
    y = y + (4/(n*pi))*sin(2*pi*n*t/T);
end
plot(t,y)
```

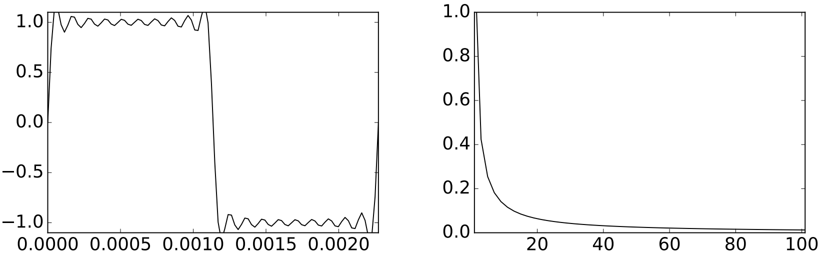


Figure: The Fourier series $(f_s)_{20}$ and the values for the first 100 Fourier coefficients $b_n$.

# Fourier series of the triangle wave

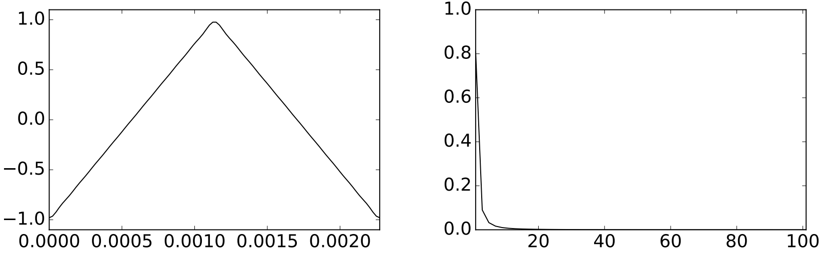


Figure: The Fourier series $(f_t)_{20}$ and the values for the first 100 Fourier coefficients $a_n$.

# Observations

- With $N = 1$, the Fourier series of both the square and the triangle wave are pure tones with frequency $\nu = 1/T$. Largest term.
- $f_s$ is anti-symmetric about 0, and its Fourier series was a sine series.
- $f_t$ is symmetric about 0, and its Fourier series was a cosine series.
- $f_t$ is continuous, and $f_s$ is not.
- The Fourier series of $f_t$ converged faster than that of $f_s$.

# Symmetry and antisymmetry, Theorem 1.10

- If $f$ is antisymmetric about 0 ($f(-t) = -f(t)$ for all $t$), then $a_n = 0$, so the Fourier series is actually a sine-series.
- If $f$ is symmetric about 0 ($f(-t) = f(t)$ for all $t$), then $b_n = 0$, so the Fourier series is actually a cosine-series.

# Complex Fourier basis, Definition 1.11

We define the set of functions

$$\begin{aligned}\mathcal{F}_{N,T} = \{&e^{-2\pi iNt/T}, e^{-2\pi i(N-1)t/T}, \cdots, e^{-2\pi it/T},\\ &1, e^{2\pi it/T}, \cdots, e^{2\pi i(N-1)t/T}, e^{2\pi iNt/T}\},\end{aligned}$$

and call this the order $N$ *complex Fourier basis* for $V_{N,T}$.

# Complex vector spaces and inner products

For general complex functions we extend the definition of the inner product as

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f\,\bar{g}\,dt.$$

The associated norm is

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2 dt}. \tag{1}$$

# Complex Fourier coefficients, Theorem 1.12

We denote by $y_{-N}, \ldots, y_0, \ldots, y_N$ the coordinates of $f_N$ in the basis $\mathcal{F}_{N,T}$, i.e.

$$f_N(t) = \sum_{n=-N}^{N} y_n e^{2\pi i n t/T}.$$

The $y_n$ are called the complex Fourier coefficients of $f$, and they are given by.

$$y_n = \langle f, e^{2\pi i n t/T} \rangle = \frac{1}{T} \int_0^T f(t) e^{-2\pi i n t/T} dt.$$

# Distribution of Fourier coefficients in Example 1.24

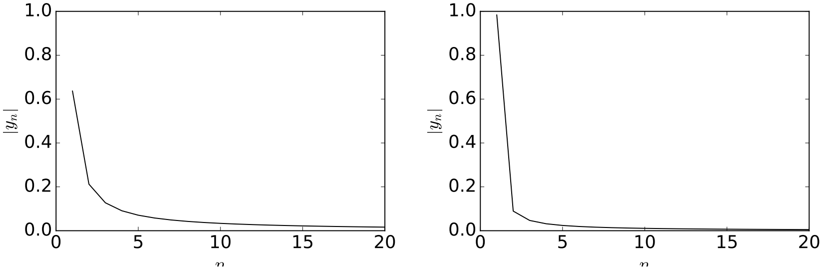


Figure: Plot of $|y_n|$ when $f(t) = e^{2\pi it/T_2}$, and $T_2 > T$. Left: $T/T_2 = 0.5$. Right: $T/T_2 = 0.9$.

# Fourier series pairs, Theorem 1.14

we use the notation $f \to y_n$ to indicate that $y_n$ is the $n$'th (complex) Fourier coefficient of $f(t)$. The functions 1, $e^{2\pi int/T}$, and $\chi_{-a,a}$ have the Fourier coefficients

$$\begin{aligned}
1 &\to \boldsymbol{e}_0 = (1, 0, 0, 0 \ldots, ) \\
e^{2\pi int/T} &\to \boldsymbol{e}_n = (0, 0, \ldots, 1, 0, 0, \ldots) \\
\chi_{-a,a} &\to \frac{\sin(2\pi na/T)}{\pi n}.
\end{aligned}$$

# Fourier series properties, Theorem 1.15

The mapping $f \to y_n$ is linear: if $f \to x_n$, $g \to y_n$, then

$$af + bg \to ax_n + by_n$$

For all $n$. Moreover, if $f$ is real and periodic with period $T$, the following properties hold:

1. $y_n = \overline{y_{-n}}$ for all $n$.
2. If $f(t) = f(-t)$ (i.e. $f$ is symmetric), then all $y_n$ are real, so that $b_n$ are zero and the Fourier series is a cosine series.
3. If $f(t) = -f(-t)$ (i.e. $f$ is antisymmetric), then all $y_n$ are purely imaginary, so that the $a_n$ are zero and the Fourier series is a sine series.
4. If $g(t) = f(t-d)$ (i.e. $g$ is the function $f$ delayed by $d$) and $f \to y_n$, then $g \to e^{-2\pi i n d/T} y_n$.
5. If $g(t) = e^{2\pi i d t/T} f(t)$ with $d$ an integer, and $f \to y_n$, then $g \to y_{n-d}$.
6. Let $d$ be a number. If $f \to y_n$, then $f(d+t) = f(d-t)$ for all $t$ if and only if the argument of $y_n$ is $-2\pi n d/T$ for all $n$.

# Convergence of Fourier series 1

**Corollary 1.17** If the complex Fourier coefficients of $f$ are $y_n$ and $f$ is differentiable, then the Fourier coefficients of $f'(t)$ are $\frac{2\pi in}{T}y_n$.

Turning this around: the Fourier coefficients of $f(t)$ are $T/(2\pi in)$ times those of $f'(t)$, when $f$ is differentiable. In other words, the Fourier coefficients of a function which is many times differentiable decay to zero very fast.

**Observation 1.18** The Fourier series converges quickly when the function is many times differentiable.

# Convergence of Fourier series 2

**Idea 1.19** Assume that $f$ is continuous on $[0, T)$. Can we construct another periodic function which agrees with $f$ on $[0, T]$, and which is both continuous and periodic (maybe with period different from $T$)?

If this is possible the Fourier series of the new function could produce better approximations for $f$. It turns out that the following extension strategy does the job:

**Definition 1.20:** Let $f$ be a function defined on $[0, T]$. By the *symmetric extension* of $f$, denoted $\breve{f}$, we mean the function defined on $[0, 2T]$ by

$$\breve{f}(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq T; \\ f(2T - t), & \text{if } T < t \leq 2T. \end{cases}$$

# Convergence of Fourier series 3

The following holds:

If $f$ is continuous on $[0, T]$, then $\breve{f}$ is continuous on $[0, 2T]$, and $\breve{f}(0) = \breve{f}(2T)$. If we extend $\breve{f}$ to a periodic function on the whole real line (which we also will denote by $\breve{f}$), this function is continuous, agrees with $f$ on $[0, T)$, and is a symmetric function.

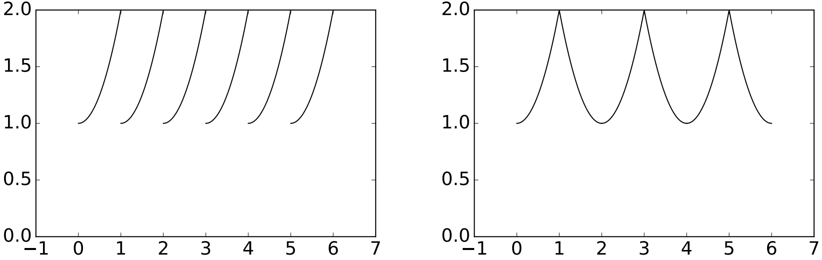


Figure: Two different extensions of $f$ to a periodic function on the whole real line. Periodic extension (left) and symmetric extension (right).