

Fourier series: basic concepts

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Square-integrable functions, Definition 1.6

The set of continuous, real functions defined on an interval $[0, T]$ is denoted $C[0, T]$.

A real function f defined on $[0, T]$ is said to be square integrable if f^2 is Riemann-integrable, i.e., if the Riemann integral of f^2 on $[0, T]$ exists,

$$\int_0^T f(t)^2 dt < \infty.$$

The set of all square integrable functions on $[0, T]$ is denoted $L^2[0, T]$.

Both $L^2[0, T]$ and $C[0, T]$ are vector spaces. Moreover, if the two functions f and g lie in $L^2[0, T]$ (or in $C[0, T]$), then the product fg is Riemann-integrable (or in $C[0, T]$). Moreover, both spaces are inner product spaces with inner product defined by

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f(t)g(t) dt,$$

and associated norm

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T f(t)^2 dt}.$$

Let $V_{N,T}$ be the subspace of $C[0, T]$ spanned by the set of functions given by

$$\mathcal{D}_{N,T} = \{1, \cos(2\pi t/T), \cos(2\pi 2t/T), \dots, \cos(2\pi Nt/T), \\ \sin(2\pi t/T), \sin(2\pi 2t/T), \dots, \sin(2\pi Nt/T)\}.$$

The space $V_{N,T}$ is called the N 'th order Fourier space. The N th-order Fourier series approximation of f , denoted f_N , is defined as the best approximation of f from $V_{N,T}$ with respect to the inner product defined by (1.3).

Fourier coefficients, Theorem 1.9

The set $\mathcal{D}_{N,T}$ is an orthogonal basis for $V_{N,T}$. In particular, the dimension of $V_{N,T}$ is $2N + 1$, and if f is a function in $L^2[0, T]$, we denote by a_0, \dots, a_N and b_1, \dots, b_N the coordinates of f_N in the basis $\mathcal{D}_{N,T}$, i.e.

$$f_N(t) = a_0 + \sum_{n=1}^N (a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T)).$$

The a_0, \dots, a_N and b_1, \dots, b_N are called the (real) Fourier coefficients of f , and they are given by

$$a_0 = \langle f, 1 \rangle = \frac{1}{T} \int_0^T f(t) dt,$$

$$a_n = 2 \langle f, \cos(2\pi nt/T) \rangle = \frac{2}{T} \int_0^T f(t) \cos(2\pi nt/T) dt \quad \text{for } n \geq 1,$$

$$b_n = 2 \langle f, \sin(2\pi nt/T) \rangle = \frac{2}{T} \int_0^T f(t) \sin(2\pi nt/T) dt \quad \text{for } n \geq 1.$$

Fourier series of the square wave

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t = linspace(0, T, 100);  
y = zeros(size(t));  
for n = 1:2:19  
    y = y + (4/(n*pi))*sin(2*pi*n*t/T);  
end  
plot(t,y)
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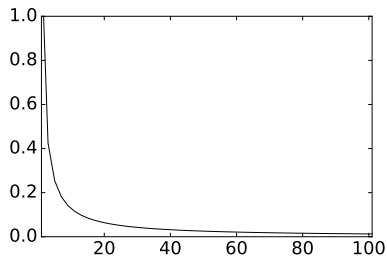
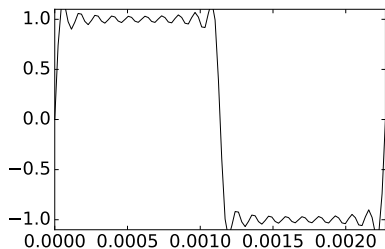


Figure: The Fourier series $(f_s)_{20}$ and the values for the first 100 Fourier coefficients b_n .

Fourier series of the triangle wave

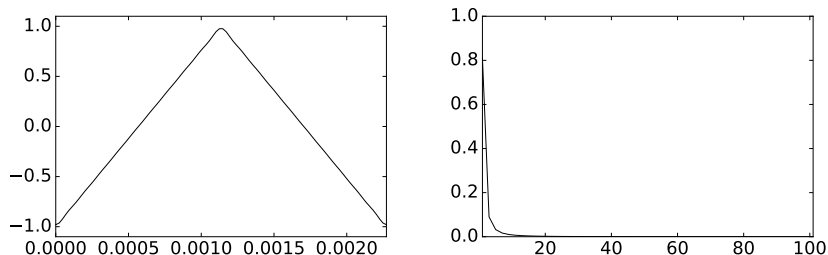


Figure: The Fourier series $(f_t)_{20}$ and the values for the first 100 Fourier coefficients a_n .

- With $N = 1$, the Fourier series of both the square and the triangle wave are pure tones with frequency $\nu = 1/T$. Largest term.
- f_s is anti-symmetric about 0, and its Fourier series was a sine series.
- f_t is symmetric about 0, and its Fourier series was a cosine series.
- f_t is continuous, and f_s is not.
- The Fourier series of f_t converged faster than that of f_s .

Symmetry and antisymmetry, Theorem 1.10

- If f is antisymmetric about 0 ($f(-t) = -f(t)$ for all t), then $a_n = 0$, so the Fourier series is actually a sine-series.
- If f is symmetric about 0 ($f(-t) = f(t)$ for all t), then $b_n = 0$, so the Fourier series is actually a cosine-series.

We define the set of functions

$$\mathcal{F}_{N,T} = \{e^{-2\pi i Nt/T}, e^{-2\pi i (N-1)t/T}, \dots, e^{-2\pi i t/T}, 1, e^{2\pi i t/T}, \dots, e^{2\pi i (N-1)t/T}, e^{2\pi i Nt/T}\},$$

and call this the order N *complex Fourier basis* for $V_{N,T}$.

Complex vector spaces and inner products

For general complex functions we extend the definition of the inner product as

$$\langle f, g \rangle = \frac{1}{T} \int_0^T f \bar{g} dt.$$

The associated norm is

$$\|f\| = \sqrt{\frac{1}{T} \int_0^T |f(t)|^2 dt}. \quad (1)$$

Complex Fourier coefficients, Theorem 1.12

We denote by $y_{-N}, \dots, y_0, \dots, y_N$ the coordinates of f_N in the basis $\mathcal{F}_{N,T}$, i.e.

$$f_N(t) = \sum_{n=-N}^N y_n e^{2\pi i n t / T}.$$

The y_n are called the complex Fourier coefficients of f , and they are given by.

$$y_n = \langle f, e^{2\pi i n t / T} \rangle = \frac{1}{T} \int_0^T f(t) e^{-2\pi i n t / T} dt.$$

Distribution of Fourier coefficients in Example 1.24

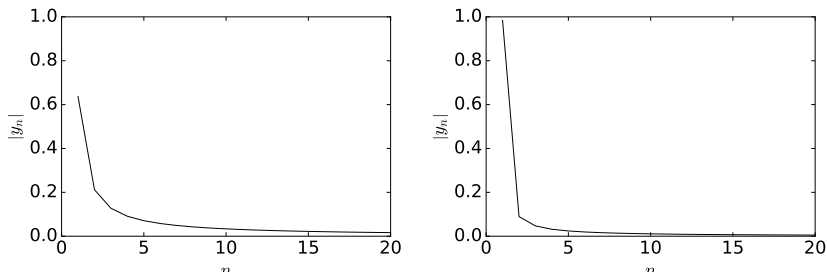


Figure: Plot of $|y_n|$ when $f(t) = e^{2\pi it/T_2}$, and $T_2 > T$. Left: $T/T_2 = 0.5$. Right: $T/T_2 = 0.9$.

Fourier series pairs, Theorem 1.14

we use the notation $f \rightarrow y_n$ to indicate that y_n is the n 'th (complex) Fourier coefficient of $f(t)$. The functions 1 , $e^{2\pi int/T}$, and $\chi_{-a,a}$ have the Fourier coefficients

$$\begin{aligned}1 &\rightarrow \mathbf{e}_0 = (1, 0, 0, 0, \dots,) \\e^{2\pi int/T} &\rightarrow \mathbf{e}_n = (0, 0, \dots, 1, 0, 0, \dots) \\ \chi_{-a,a} &\rightarrow \frac{\sin(2\pi na/T)}{\pi n}.\end{aligned}$$

Fourier series properties, Theorem 1.15

The mapping $f \rightarrow y_n$ is linear: if $f \rightarrow x_n$, $g \rightarrow y_n$, then

$$af + bg \rightarrow ax_n + by_n$$

For all n . Moreover, if f is real and periodic with period T , the following properties hold:

- ❶ $y_n = \overline{y_{-n}}$ for all n .
- ❷ If $f(t) = f(-t)$ (i.e. f is symmetric), then all y_n are real, so that b_n are zero and the Fourier series is a cosine series.
- ❸ If $f(t) = -f(-t)$ (i.e. f is antisymmetric), then all y_n are purely imaginary, so that the a_n are zero and the Fourier series is a sine series.
- ❹ If $g(t) = f(t - d)$ (i.e. g is the function f delayed by d) and $f \rightarrow y_n$, then $g \rightarrow e^{-2\pi ind/T} y_n$.
- ❺ If $g(t) = e^{2\pi idt/T} f(t)$ with d an integer, and $f \rightarrow y_n$, then $g \rightarrow y_{n-d}$.
- ❻ Let d be a number. If $f \rightarrow y_n$, then $f(d + t) = f(d - t)$ for all t if and only if the argument of y_n is $-2\pi nd/T$ for all n .

Convergence of Fourier series 1

Corollary 1.17 If the complex Fourier coefficients of f are y_n and f is differentiable, then the Fourier coefficients of $f'(t)$ are $\frac{2\pi in}{T} y_n$.

Turning this around: the Fourier coefficients of $f(t)$ are $T/(2\pi in)$ times those of $f'(t)$, when f is differentiable. In other words, the Fourier coefficients of a function which is many times differentiable decay to zero very fast.

Observation 1.18 The Fourier series converges quickly when the function is many times differentiable.

Convergence of Fourier series 2

Idea 1.19 Assume that f is continuous on $[0, T)$. Can we construct another periodic function which agrees with f on $[0, T]$, and which is both continuous and periodic (maybe with period different from T)?

If this is possible the Fourier series of the new function could produce better approximations for f . It turns out that the following extension strategy does the job:

Definition 1.20: Let f be a function defined on $[0, T]$. By the *symmetric extension* of f , denoted \check{f} , we mean the function defined on $[0, 2T]$ by

$$\check{f}(t) = \begin{cases} f(t), & \text{if } 0 \leq t \leq T; \\ f(2T - t), & \text{if } T < t \leq 2T. \end{cases}$$

Convergence of Fourier series 3

The following holds:

If f is continuous on $[0, T]$, then \check{f} is continuous on $[0, 2T]$, and $\check{f}(0) = \check{f}(2T)$. If we extend \check{f} to a periodic function on the whole real line (which we also will denote by \check{f}), this function is continuous, agrees with f on $[0, T]$, and is a symmetric function.

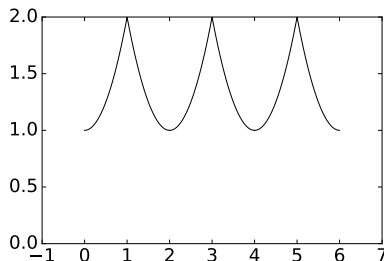
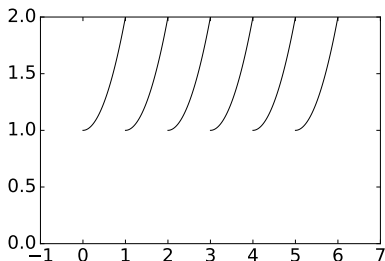


Figure: Two different extensions of f to a periodic function on the whole real line. Periodic extension (left) and symmetric extension (right).