Lectures Notes

BV functions and sets of finite perimeter

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Introduction

♯ *Geometric Measure Theory* is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces generalizing techniques from differential geometry through measure theoretic arguments. The theory of *functions of bounded variations* and *sets of finite perimeter* is one of the core topics of Geometric Measure Theory, since it deals with extension of the classical notion of Sobolev functions and regular surfaces.

The 1-Laplace operator and BV as a natural extension of $W^{1,1}$

In the Calculus of Variation, the *Direct Method* is a general way of proving the existence of a minimizer for a given functional. More precisely, let X be a topological space and $F : X \to (-\infty, +\infty]$ be a functional. We are interested in finding a minimizer of F in X; that is, a $u \in X$ such that $F(u) \le F(v)$ for any $v \in X$. Assume that

$$m := \inf\{F(v) \,:\, v \in X\} > -\infty.$$

This ensure the existence of a minimizing sequence $\{v_j\}$; that is, a sequence of elements $v_j \in X$ such that $F(v_j) \to m$. Then, the Direct Method consists in the following steps:

(1) show that $\{v_j\}$ admits a converging subsequence $\{v_{j_k}\}$ and $u \in X$ such that $v_{j_k} \to u$, with respect to a the topology of X;

(2) show that F is (sequentially) lower semicontinuous with respect to the topology of X; that is, if $z_j \to z_0$ in X, then
$$F(z_0) \le \liminf_{j \to +\infty} F(z_j).$$

If these properties hold true, we can conclude that u is a minimizer of F. Indeed, we have

$$m = \lim_{k \to +\infty} F(v_{j_k}) \ge \liminf_{k \to +\infty} F(v_{j_k}) \ge F(u) \ge m,$$

from which we immediately conclude that $F(u) = \min\{F(v) \,:\, v \in X\}$.

This method is fundamental in proving the existence of solutions to minimization problems related to boundary value problems. Let us consider for instance the classical Dirichlet problem for the Laplace equation on an open set Ω with C^1-smooth boundary:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$. It is possible to see this system as the Euler-Lagrange equations for the functional
$$F(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega f u \, dx$$

defined on the space

$$X = W_0^{1,2}(\Omega) := \{u \in L^2(\Omega) \,:\, Du \in L^2(\Omega; \mathbb{R}^n), u = 0 \text{ on } \partial\Omega\};$$

♯These notes have been written for the course of *BV Functions and Sets of Finite Perimeter* held in the Department of Mathematics of the Hamburg Universität. The main references are the books [1, 6, 12]. Please write an email to giovanni.comi@uni-hamburg.de if you have corrections, comments, suggestions or questions.

that is, the space of 2-summable weakly differentiable Sobolev functions with zero trace on $\partial\Omega^\sharp$. As customary, we denote by Du the weak gradient of u. Thanks to Poincaré inequality, we can prove that

$$\inf\{F(u) \,:\, u \in W_0^{1,2}(\Omega)\} > -\infty.$$

Hence, we can find the solution looking for minizers of F through the Direct Method: let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence. It is possible to show that $\{u_j\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, which is an Hilbert space, and in particular reflexive: as a consequence, there exists a subsequence $\{u_{j_k}\}$ converging to some $u \in W_0^{1,2}(\Omega)$ with respect to the weak topology. In addition, F is lower semicontinuous with respect to the weak topology, and so we infer the existence of a solution for the minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|Du|^2 - fu\,dx \,:\, u \in W_0^{1,2}(\Omega)\right\}.$$

It seems natural now to wonder if we could substitute the exponent 2 with any $p \in (1,\infty)$. Thanks to the Poincaré inequality and the reflexivity of the L^p-spaces for $p \in (1,\infty)$, it is indeed possible to show that, for any $f \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the problem

$$\min\left\{\int_\Omega \frac{1}{p}|Du|^p - fu\,dx \,:\, u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution, where

$$W_0^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, Du \in L^p(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

The minimizers to this problem solves the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) =: \Delta_p u$ is the p-Laplace operator.

The next logical step is to consider also the case $p = 1$: for a given $f \in L^\infty(\Omega)$, we want to find a function u which realizes

$$\inf\left\{\int_\Omega |Du| - fu\,dx \,:\, u \in W_0^{1,1}(\Omega)\right\} =: m, \tag{0.0.1}$$

where

$$W_0^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

If we assume $\|f\|_{L^\infty(\Omega)}$ to be sufficiently small, we can again employ the Poincaré inequality to prove that $m \in (-\infty, +\infty]$. Hence, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{1,1}(\Omega)$ such that

$$\lim_{j\to+\infty} \int_\Omega |Du_j| - fu_j\,dx = m.$$

However, in this case we cannot argue as above in the case $p > 1$, since, in general this does *not* imply that the existence of a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ weakly converging to some $u \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |Du| - fu\,dx = m.$$

The reason for this lies in the fact that $L^1(\Omega)$ is not reflexive, and actually it is not the topological dual of any separable space. However, $L^1(\Omega)$ is contained in the space of finite Radon measures on Ω, $\mathcal{M}(\Omega)$, and this space can be see as the dual of the space of continuous functions vanishing on the boundary of Ω, $C_0(\Omega)$.

This fact suggests the definition of a space which contains the Sobolev space $W^{1,1}(\Omega)$ and which, although not reflexive, enjoys the property that bounded sets are weakly* compact: the space of *functions with bounded variation*,

$$BV(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n)\}.$$

$^\sharp$We refer to [5, Chapter 5] and to [6, Chapter 4] for a detailed account on Sobolev spaces.

It is not difficult to prove that the total variation of the Radon measure Du over Ω is indeed lower semicontinuous with respect to the weak* converge of the gradient measures. This indicates that the correct space where to look solutions to (0.0.1) is the space of functions with bounded variation with zero trace$^\flat$,

$$BV_0(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

Finally, it is relevant to mention the fact that the minimizers to (0.0.1) solve the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) =: \Delta_1 u$ is the 1-Laplace operator, which is non trivially defined on nonsmooth functions because of the highly degenerate term $\frac{\nabla u}{|\nabla u|}$.

Minimal area problems and sets of finite perimeter

Other historically relevant problems from the Calculus of Variation are the minimal area problems, among which the most famous example is the *Euclidean isoperimetric problem*: find the possibly unique set with minimal surface area among those with fixed volume. This problem is also called "Dido's problem", referring to the ancient myth on the foundation of Carthago. As recounted in Virgil's *Aeneids*, queen Dido and her group of Phoenician exiles reached the coast of Tunisia, where she asked the Gaetulian king Iarbas for a small bit of land for a refuge, only as much land as could be encompassed by an oxhide; that is, the skin of an ox. The king naturally agreed. Dido then cut the oxhide into fine strips with which she could encircle a large semicircular region (with a side on the sea). In other words, to the ancient Mediterranean civilization it was probably known that the circle was the figure in the plane with minimal surface length among those with a fixed area.

In order to rephrase this problem in mathematical terms, we denote by $|F|$ the n-dimensional volume of a set $F \subset \mathbb{R}^n$ (hence, its Lebesgue measure $\mathscr{L}^n(F)$) and by $\sigma_{n-1}(\partial F)$ its surface area (under the assumption the ∂F is regular enough). Then we look for the set which realizes

$$\inf\{\sigma_{n-1}(\partial F) \,:\, \partial F \in \mathcal{R}, |F| = k\} =: \gamma_k,$$

where $\mathcal{R}$ is a class of sufficiently smooth surfaces and $k > 0$. The Direct Method in this case consists in considering a minimizing sequence of sets F_j such that

$$\partial F_j \in \mathcal{R}, \quad |F_j| = k \quad \text{and} \quad \sigma_{n-1}(\partial F_j) \to \gamma_k, \tag{0.0.2}$$

and then in trying to prove the convergence (possibly up to subsequences) to some limit set E such that

$$\partial E \in \mathcal{R}, \quad |E| = k \quad \text{and} \quad \sigma_{n-1}(\partial E) = \gamma_k.$$

In order to achieve this result, some compactness property in the family of sets satisfying (0.0.2) is required. In addition, the surface measure σ_{n-1} need to be a lower semicontinuous with respect to the chosen convergence of sets, in the sense that

$$\sigma_{n-1}(\partial E) \le \liminf_{j\to+\infty} \sigma_{n-1}(\partial F_j)$$

if $F_j \to E$ in a suitable sense. However, these compactness and lower semicontinuity properties in general fail to be true in family of sets with regular topological boundary. In addition, we notice that the topological boundary is very unstable under modification of a set by Lebesgue negligible sets. For instance, let

$$E_1 = B(0,1) \quad \text{and} \quad E_2 = B(0,1) \cup (\partial B(0,2) \cap \mathbb{Q}^n).$$

It is plain to see that $|E_1 \Delta E_2| = 0$, so that these two sets are equivalent with respect to the Lebesgue measure, and so they have the same volume. However, their topological boundary, which are smooth surfaces, are very different:

$$\partial E_1 = \partial B(0,1) \quad \text{and} \quad \partial E_2 = \partial B(0,1) \cup \partial B(0,2).$$

$^\flat$It can be proved that the trace of a function with bounded variation is well defined on any C^1-regular surface, as in the Sobolev case.

The need of ruling out these problems and of recovering a notion of compactness and a lower semicontinuity property for the surface area is one of the main reasons for the birth of Geometric Measure Theory. This theory concerns methods to study the geometric properties of rough, irregular sets from a measure theoretic point of view. In this course we shall see how to exploit this approach to give a meaningful notion of surface area for an irregular set and to define a suitable class of sets for which we can apply the Direct Method of the Calculus of Variation in order to deal with minimal area problems: the *sets of finite perimeter*. Broadly speaking, the notion of set of finite perimeter extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to minimal area problems and other similar geometric variational minimization problems. More precisely, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ is its characteristic function χ_E is a function with locally bounded variation.

Chapter 1

Notions of abstract Measure Theory

1.1 General measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*; that is, the collection of all subsets of X.

Definition 1.1.1 (Measures)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ if $A \subset \bigcup_{k=1}^{\infty} A_k$ (σ-subadditivity),

is called a measure.

It should be noticed that in the literature a mapping as the one in Definition 1.1.1 is also called an *outer measure*, while the name of measure is used to denote the restriction of the mapping to the family of measurable set (see Definition 1.1.4 below). We shall nevertheless follow the notation of [6], in order to be able to assign a measure even to nonmeasurable sets.

Remark 1.1.2. Thanks to σ-subadditivity, any measure is not decreasing; that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition 1.1.3 (Restriction of a measure)**.** If $Y \subset X$, the *restriction of* μ *to* Y, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$ for any $A \subset X$.

Definition 1.1.4 (μ-measurable sets)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark 1.1.5. This definition is meaningful, since the italian mathematician *Giuseppe Vitali* proved in 1905 that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathscr{L}^1$-measurable [17]. For a modern presentation of his construction, we refer to [12, Section I.1.2].

Definition 1.1.6 (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if the following conditions hold:

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for any $A \in \mathfrak{F}$ we have $X \setminus A \in \mathfrak{F}$,

(3) for any countable family of sets $\{A_i\}_{i \in I}$ such that $A_i \in \mathfrak{F}$ for any $i \in I$ we have have

$$\bigcup_{i \in I} A_i \in \mathfrak{F}.$$

Theorem 1.1.7. *Given any measure μ on X, the family of μ-measurable sets forms a σ-algebra.*

Theorem 1.1.8. *Let μ be a measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu(A_j).$$

We list now some relevant definitions.

Definition 1.1.9.

(1) Given any $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.

(2) The *Borel σ-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel σ-algebra are called *Borel sets.*

(3) A measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (i.e. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem 1.1.10. *Let μ be a Radon measure on $\mathbb{R}^n$. Then we have*

(1) $\mu(A) = \inf\{\mu(U) \,:\, U \supset A,\, U$ *open*$\}$ *for all* $A \subseteq \mathbb{R}^n$ *(outer regularity),*

(2) $\mu(B) = \sup\{\mu(K) : K \subset B,\, K$ *compact*$\}$ *for all μ-measurable sets B* *(inner regularity).*

Theorem 1.1.11 (Carathéodory's criterion)**.** *Let μ be a measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ such that* $\operatorname{dist}(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

then μ is a Borel measure.

Not any Borel regular measure is a Radon measure. However, it is possible to obtain a Radon measure as a restriction of a Borel regular one, as stated in the followin theorem.

Theorem 1.1.12. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable such that $\mu(A) < +\infty$, then $\mu \llcorner A$ is a Radon measure.*

Example 1.1.13 (Dirac delta)**.** Let $X \neq \emptyset$. For any $x_0 \in X$ we define the *Dirac*[♯] *measure centered in* x_0 by setting

$$\delta_{x_0}(A) := \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A, \end{cases}$$

for any set $A \in \mathcal{P}(X)$. It is easy to check that any set A is δ_{x_0}-measurable. In addition, in the case $X = \mathbb{R}^n$ we can show that δ_{x_0} is indeed a Radon measure, for all $x_0 \in \mathbb{R}^n$.

Example 1.1.14 (The counting measure)**.** Let $X \neq \emptyset$. For any $E \subset X$ we define the *counting measure* by setting

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

It is again possible to show that any set is $\#$-measurable. In the case $X = \mathbb{R}^n$, this measure is Borel regular, but *not* a Radon measure, since it is clearly not locally finite.

[♯]Named after Paul Adrien Maurice Dirac (1902-1984), English theoretical physicist who shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory". He actually introduced the so-called *Dirac delta function* as a "convenient notation" in his influential 1930 book *The Principles of Quantum Mechanics.* The name "delta function" was chosen since such measure acts like a continuous analogue of the discrete Kronecker delta

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Indeed, for any sequence $\{a_j\}_{j \in \mathbb{Z}}$, we have

$$\sum_{j=-\infty}^{\infty} a_j \delta_{ij} = a_i,$$

and, analogously (see Example 1.3.2), for any $x_0 \in \mathbb{R}$ and any bounded function $f : \mathbb{R} \to \mathbb{R}$, the Dirac delta satisfies the property

$$\int_{-\infty}^{+\infty} f(y)\delta(x_0 - y)\, dy = \int_{-\infty}^{\infty} f(y)\, d\delta_{x_0}(y) = f(x_0).$$

Example 1.1.15 (The Lebesgue measure)**.** The well-known *Lebesgue measure* on $\mathbb{R}^n$ is defined by

$$\mathscr{L}^n(A) := \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes} \right\},$$

where $\mathscr{L}^n(Q_i) = l(Q_i)^n$ and $l(Q_i)$ is the side length of the cube Q_i. It is possible to show that in one dimension we have

$$\mathscr{L}^1(A) = \inf \left\{ \sum_{ij=1}^{\infty} \operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R} \right\}$$

and that we can characterize $\mathscr{L}^n$ in an alternative way as

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}} = \mathscr{L}^{n-1} \times \mathscr{L}^1.$$

1.2 The Hausdorff measure

Definition 1.2.1 (Hausdorff content)**.** Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *α-dimensional Hausdorff content* of A as

$$\mathscr{H}^\alpha_\delta(A) := \inf \left\{ \sum_{j \in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2} \right)^\alpha \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taken over all the (at most countable) *δ-coverings* $\{C_j\}_{j \in I}$ of A, and we set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2} + 1\right)}.$$

We notice that $\mathscr{H}^\alpha_\delta(A)$ is a non-decreasing function in δ, so that we can take the limit as $\delta \searrow 0$ and it always exists in the extended real numbers. This justifies the following definition.

Definition 1.2.2 (Hausdorff measure)**.** For any $A \subset \mathbb{R}^n$ and $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of A as

$$\mathscr{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathscr{H}^\alpha_\delta(A) = \sup_{\delta > 0} \mathscr{H}^\alpha_\delta(A).$$

Roughly speaking, we take the limit as $\delta \searrow 0$ since it forces the coverings to follow the local geometry of the set A. Indeed, the key idea behind the definition of the Hausdorff measure is that it should be able to capture the properties of thin sets in $\mathbb{R}^n$ (in particular, Lebesgue negligible sets). As we shall see in the following, if $\alpha = k \in \{1, \ldots, n-1\}$, then $\mathscr{H}^k$ agrees with the k-dimensional surface area on sufficiently regular sets, as for instance k-dimensional planes.

It is not too difficult to prove that, as a consequence of Carathéodory's criterion, Theorem 1.1.11, any Borel set is $\mathscr{H}^\alpha$-measurable, for any $\alpha \geq 0$.

Theorem 1.2.3 (Hausdorff measures are Borel regular)**.** *$\mathscr{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem 1.2.4 (Basic properties of the Hausdorff measure)**.** *The following statements hold true:*

(1) $\mathscr{H}^0 = \#$;

(2) $\mathscr{H}^1 = \mathscr{H}^1_\delta = \mathscr{L}^1$ on $\mathbb{R}$, for any $\delta > 0$;

(3) $\mathscr{H}^\alpha \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$;

(4) $\mathscr{H}^\alpha(\lambda A) = \lambda^\alpha \mathscr{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$;

(5) $\mathscr{H}^\alpha(L(A)) = \mathscr{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$, we have $\mathscr{H}^0_\delta(\{x\}) = 1$ for every $x \in \mathbb{R}^n$ and $\delta > 0$. Indeed,

$$\omega_0 \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 = 1,$$

which implies $\mathscr{H}^0_\delta(\{x\}) \geq 1$, and, on the other hand, we can clearly cover the singleton only with itself. Hence, $\mathscr{H}^0(\{x\}) = 1$ for every $x \in \mathbb{R}^n$. Since $\mathscr{H}^0$ is a Borel measure, it is σ-additive on Borel sets, so that

$$\mathscr{H}^0(A) = \sum_{x \in A} \mathscr{H}^0(\{x\}) = \#A,$$

for any finite or countable set A. Finally, if A is incountable, then A contains a countable set B, and so $\mathscr{H}^0(A) \geq \mathscr{H}^0(B) = +\infty$.

(2) We estimate the Lebesgue measure $\mathscr{L}^1$ from both sides by the Hausdorff measure. Since $\omega_1 = 2 = |(-1,1)|$, for any $\delta > 0$ we first get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\leq \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j, \operatorname{diam} C_j \leq \delta \right\} = \mathscr{H}^1_\delta(A), \end{aligned}$$

Now, we define a partition of $\mathbb{R}$ by setting $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$. These are intervals of diameter δ, so that, for every $j \in I$, we get

$$\operatorname{diam}(C_j \cap J_{k,\delta}) \leq \delta. \tag{1.2.1}$$

In addition, we have

$$\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \leq \operatorname{diam} C_j, \tag{1.2.2}$$

since $\{J_{k,\delta}\}_{k \in \mathbb{Z}}$ is a partition $\mathbb{R}$ of essentially disjoint intervals, because $\#(J_{k,\delta} \cap J_{m,\delta}) \leq 1$ for any $k \neq m$. Therefore, by (1.2.2) we get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap J_{k,\delta} \right\}. \end{aligned}$$

We set now $C_j \cap J_{k,\delta} =: \widetilde{C}_{i_{j,k}}$, by relabeling the indexes sets I and $\mathbb{Z}$ to an index set $\widetilde{I}$. Thanks to (1.2.1), we have $\operatorname{diam}(\widetilde{C}_i) \leq \delta$ and so we get

$$\mathscr{L}^1(A) \geq \inf \left\{ \sum_{j \in \widetilde{I}} \widetilde{C}_i \mid A \subset \bigcup_{i \in \widetilde{I}} \widetilde{C}_i, \operatorname{diam} \widetilde{C}_i \leq \delta \right\} \geq \mathscr{H}^1_\delta(A).$$

All in all, we get $\mathscr{L}^1 = \mathscr{H}^1_\delta$ for any $\delta > 0$, from which it easily follows $\mathscr{L}^1 = \mathscr{H}^1$ on $\mathbb{R}$.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. It is easy to see that, for any fixed $m \in \mathbb{N}$, Q can be covered by m^n smaller cubes Q_i with side length $\frac{1}{m}$. Clearly, we have $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$. Therefore, we obtain

$$\mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha},$$

from which we deduce that, since $n < \alpha$,

$$\mathscr{H}^\alpha(Q) = \lim_{m\to\infty} \mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} \lim_{m\to\infty} m^{n-\alpha} = 0.$$

Thus, the claim easily follows, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes and $\mathscr{H}^\alpha$ is σ-subadditive.

The proofs of (4) and (5) are left as an exercise. □

Lemma 1.2.5. *Let $A \subset \mathbb{R}^n$ and $\delta_0 > 0$ such that $\mathscr{H}^\alpha_{\delta_0}(A) = 0$, then we have $\mathscr{H}^\alpha(A) = 0$.*

Proof. Since the Hausdorff content is non-increasing in δ, we have $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_\delta(A)$ for any $\delta > 0$. In particular, this means that $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_{\delta_0}(A) = 0$, so that, for every $\varepsilon > 0$, there exists a countable family of sets $\{C_j\}_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

In particular, the second condition immediately implies

$$\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon.$$

Hence, we have $\mathscr{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$, and $\delta_\varepsilon \searrow 0$ if and only if $\varepsilon \searrow 0$. This implies the claim $\mathscr{H}^\alpha(A) = 0$. □

Proposition 1.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < +\infty$, then $\mathscr{H}^t(A) = 0$.

(2) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a countable family of subsets $\{C_j\}_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathscr{H}^s_\delta(A) + 1 \le \mathscr{H}^s(A) + 1.$$

From this, it follows that

$$\begin{aligned} \mathscr{H}^t_\delta(A) \le \sum_{j\in I} \omega_t \left(\frac{\operatorname{diam} C_j}{2}\right)^t &= \frac{\omega_t}{\omega_s} 2^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathscr{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0, \end{aligned}$$

which implies the claim $\mathscr{H}^t(A) = 0$.

(2) If by contradiction $\mathscr{H}^s(A) < \infty$, then by (1) if follows that $\mathscr{H}^r(A) = 0$ for all $r > s$ and in particular for $r = t$, which is clearly absurd.

□

Definition 1.2.7. We call the *Hausdorff dimension*[♯] of a set $A \subset \mathbb{R}^n$ the number

$$\dim_{\mathscr{H}}(A) := \inf\{\alpha \ge 0 : \mathscr{H}^\alpha(A) = 0\}.$$

Remark 1.2.8. Let $\alpha = \dim_{\mathscr{H}}(A)$. Then one has

$$\mathscr{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and} \qquad \mathscr{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha. \tag{1.2.3}$$

The first part of (1.2.3) follows clearly from the definition of the Hausdorff dimension. The second, instead, can be proved by contradiction. Suppose by contradiction that $\mathscr{H}^t(A) < \infty$ for some $t < \alpha$, then, by the Proposition 1.2.6, we have $\mathscr{H}^r(A) = 0$ for all $r > t$. This implies

$$\alpha = \inf\left\{\beta \ge 0 : \mathscr{H}^\beta(A) = 0\right\} \le t < \alpha,$$

which is clearly absurd.

It should be noticed that, in general, $\mathscr{H}^\alpha(A)$ can be any number in $[0, +\infty]$.

[♯]The interested reader may find a detailed exposition on Hausdorff's and other related concepts of dimension in the monograph [7].

We state now an important result on the equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure, whose proof we postpone to the end of the section.

Theorem 1.2.9. *$\mathscr{H}^n_\delta = \mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$, for any $\delta > 0$.*

Remark 1.2.10. As a consequence of Theorem 1.2.9, we see that $\mathscr{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Indeed, it is not bounded on some compact sets. Take for example the closed unit ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that

$$\mathscr{H}^n(\overline{B(0,1)}) = \mathscr{L}^n(\overline{B(0,1)}) = \omega_n \in (0, \infty)$$

and so, by Proposition 1.2.6, $\mathscr{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

Even though $\mathscr{H}^\alpha$ is not a Radon measure for $\alpha \in [0, n)$, it is possible to show that its restriction to some suitable Borel set is indeed a Radon measure.

Proposition 1.2.11. *If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^\alpha(E) < +\infty$, then $\mathscr{H}^\alpha \llcorner E$ is a Radon measure.*

Proof. It is a simple consequence of Theorem 1.1.12. □

We investigate now the behaviour of the Hausdorff measure under the action of Lipschitz and Hölder functions. We recall first the definition of such family of functions.

Definition 1.2.12 (Lipschitz and Hölder functions)**.** Let $E \subset \mathbb{R}^n$.

(1) We say that $f : E \to \mathbb{R}^m$ is *Lipschitz continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y| \quad \text{for any } x, y \in E. \tag{1.2.4}$$
The smallest constant for which (1.2.4) holds is called the *Lipschitz constant* of f on E and it is denoted by $\mathrm{Lip}(f, E)$.

(2) We say that $f : E \to \mathbb{R}^m$ is *locally Lipschitz continuous* on E if, for all compact sets $K \subset E$, f is Lipschitz continuous on K.

(3) Let $\gamma \in (0, 1)$. We say that $f : E \to \mathbb{R}^m$ is *γ-Hölder continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y|^\gamma \quad \text{for any } x, y \in E. \tag{1.2.5}$$

(4) We say that $f : E \to \mathbb{R}^m$ is *locally γ-Hölder continuous* on E if, for all compact sets $K \subset E$, f is γ-Hölder continuous on K.

From this point on, we shall refer to Lipschitz continuous and Hölder continuous functions simply as Lipschitz and Hölder functions.

Exercise 1.2.13. Show that any Lipschitz or γ-Hölder function (for some $\gamma \in (0, 1)$) is indeed continuous.

Exercise 1.2.14. Show that the Lipschitz constant of f on E satisfies

$$\mathrm{Lip}(f, E) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in E, x \neq y \right\}. \tag{1.2.6}$$

This equality can indeed be used as an alternative definition.

Remark 1.2.15. It is easy to notice that Lipschitz functions can be seen as 1-Hölder functions. In addition, for any open set $\Omega \subset \mathbb{R}^n$ and any $\gamma \in [0, 1]$, we can define the space $C^{0,\gamma}(\overline{\Omega}; \mathbb{R}^m)$ of bounded γ-Hölder functions as the set of continuous bounded functions $f : \overline{\Omega} \to \mathbb{R}^m$ for which there exists a constant $C > 0$ such that (1.2.5) holds. If $\gamma = 0$, we have $C^{0,0}(\overline{\Omega}; \mathbb{R}^m) = C^0(\overline{\Omega}; \mathbb{R}^m)$. Such spaces may be equipped with the following norms:

$$\|f\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} + [f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)},$$

where

$$\|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} := \sup_{x\in\overline{\Omega}} |f(x)|,$$

$$[f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\gamma} : x,y\in\overline{\Omega}, x\neq y\right\},$$

for $\gamma\in(0,1]$, while we set $\|f\|_{C^{0,0}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)}$. It is not difficult to see that, for all $\gamma\in[0,1]$, $C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)$ equipped with the norm $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)}$ is a Banach space.

Exercise 1.2.16. Let $\gamma>1$ and $f:\Omega\to\mathbb{R}^m$ be such that there exists a constant $C>0$ such that (1.2.5) holds. Show that f is constant.

Proposition 1.2.17. *Let $\alpha\geq 0$, $A\subset\mathbb{R}^n$.*

(1) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is Lipschitz, then $\mathscr{H}^\alpha(f(A))\leq(\mathrm{Lip}(f))^\alpha\mathscr{H}^\alpha(A)$.

(2) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is γ-Hölder, for some $\gamma\in(0,1)$, then $\mathscr{H}^\alpha(f(A))\leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}(A)$.

Proof. Thanks to Remark 1.2.15, it is enough to prove (2) for any $\gamma\in(0,1]$. Fix $\delta>0$, and take a countable family of sets $\{C_j\}_{j\in I}$ such that $A\subset\bigcup_{j\in I}C_j$ and $\operatorname{diam}C_j\leq\delta$. It is clear that

$$f(A)\subseteq\bigcup_{j\in I}f(C_j).$$

Thanks to (1.2.5), we see that $f(C_j)$ satisfies

$$\operatorname{diam}f(C_j)\leq C\,(\operatorname{diam}C_j)^\gamma\leq C\delta^\gamma,$$

where $C=\mathrm{Lip}(f)$ is $\gamma=1$. Hence, we obtain

$$\mathscr{H}^\alpha_{C\delta^\gamma}(f(A))\leq\sum_{j\in I}\omega_\alpha\left(\frac{\operatorname{diam}f(C_j)}{2}\right)^\alpha\leq\underbrace{\frac{\omega_\alpha}{2^\alpha}\frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}}\sum_{j\in I}\omega_{\alpha\gamma}\left(\frac{\operatorname{diam}C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all δ-coverings $\{C_j\}_{j\in I}$ we get

$$\mathscr{H}^\alpha_{C\delta^\gamma}(g(A))\leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}_\delta(A),$$

where $C_{\alpha,\gamma}=\mathrm{Lip}(f)^\alpha$, if $\gamma=1$. By sending $\delta\searrow 0$ we conclude the proof. □

Example 1.2.18 (Sierpinski triangle[♭]). We provide an example on the estimation of the Hausdorff measure for a set with non integer Hausdorff dimension. Let us construct a self-similar fractal in $\mathbb{R}^2$ in the following way:

1. Take S_0 to be an equilateral triangle with side length 1.
2. Divide each side in half, then connect the three middle points, so that S_0 becomes the union of four congruent equilateral triangles. Then, remove the open triangle in the center and denote by S_1 the union of the three remaining closed triangles with side length 1/2.
3. Now repeat the step in 2. in each one of the three equilateral triangles in S_1 in order to generate nine triangles of side length 1/4 which form S_2.

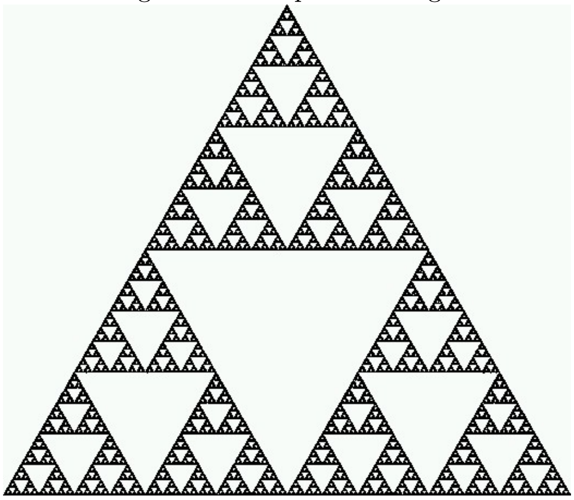
By iterating this procedure k times, we construct the set S_k as the union of 3^k equilateral triangles with side length 2^{-k}. Notice that $S_{k+1}\subset S_k$ and each one of the S_k's is compact and nonempty. Hence, we define the *Sierpinski triangle* as the set

$$S:=\bigcup_{k=0}^\infty S_k,$$

which is therefore compact and nonempty.

[♭]Fractal described by Waclaw Sierpinski (1882-1969) in 1915, [14], and appearing in Italian art from the 11th century [4].

Figure 1.1: Sierpinski triangle



The area of an equilateral triangle of side length l is $\frac{\sqrt{3}}{4}l^2$, so that we have

$$\mathscr{L}^2(S) \leq \mathscr{L}^2(S_k) = 3^k \frac{\sqrt{3}}{4} 4^{-k}$$

for any $k \geq 0$. Therefore, by taking the limit as $k \to +\infty$, we conclude that $\mathscr{L}^2(S) = 0$. We proceed now to estimate the Hausdorff measure of S. We notice that

$$S_k = \bigcup_{j=1}^{3^k} S_{k,j},$$

if we denote by $S_{k,j}$ the j-th equilateral triangle of the k-th iteration step. It is not difficult to see that $\operatorname{diam}(S_{k,j}) = 2^{-k}$. Therefore, since clearly $S \subset S_k$, for any $k \geq 0$, we see that, by choosing $\delta = 2^{-k}$, we obtain the following estimate

$$\mathscr{H}^{\alpha}_{\frac{1}{2^k}}(S) \leq \sum_{j=1}^{3^k} \frac{\omega_\alpha}{2^\alpha} (\operatorname{diam} S_{k,j})^\alpha = \frac{\omega_\alpha}{2^\alpha} 3^k 2^{-k\alpha},$$

which goes to zero for $k \to \infty$ if $\alpha > \frac{\log^3}{\log^2}$. Thus, we can conclude that, for all $\alpha > \frac{\log 3}{\log 2}$, we have $\mathscr{H}^\alpha(S) = 0$, and this yields an upper bound on the Hausdorff dimension of S:

$$\dim_{\mathcal{H}}(S) \leq \frac{\log 3}{\log 2}.$$

Figure 1.2: Sierpinski triangle decoration in a church

We come now to the proof of Theorem 1.2.9, which is crucially based on the two following statements.

Lemma 1.2.19 (Vitali covering property for $\mathscr{L}^n$)**.** *For all open U and for all $\delta > 0$ there exists a family of disjoint closed balls $\{\overline{B_k}\}_{k=1}^\infty$ such that* $\operatorname{diam} B_k < \delta$ *and* $\mathscr{L}^n(U \setminus \bigcup_{k=1}^\infty \overline{B_k}) = 0$.

We postpone the proof of this result to the following chapter, where we shall actually show a more general version of it (Theorem 2.1.5).

Theorem 1.2.20 (Isodiametric inequality). *For all $\mathscr{L}^n$-measurable sets $E \subset \mathbb{R}^n$ we have*

$$|E| \le \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n .$$

Proof of theorem 1.2.9. The proof consists of three steps.

(Step 1) Claim: $\mathscr{L}^n(A) \le \mathscr{H}^n_\delta(A)$ for all $A \subset \mathbb{R}^n$ and for all $\delta > 0$.
Fix $\delta > 0$. Let $\{C_j\}_{j\in I}$: $A \subset \bigcup_{j\in I} C_j$, $\operatorname{diam} C_j \le \delta$. By the σ-subadditivity of the Lebesgue measure, we have

$$\mathscr{L}^n(A) \le \sum_{j=1}^{\infty} \mathscr{L}^n(C_j) \le \sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n ,$$

where in the last inequality we used the *isometric inequality*, Theorem 1.2.20. Taking the infimum over all such δ-coverings $\{C_j\}_{j\in J}$, we obtain the claim

$$\mathscr{L}^n(A) \le \mathscr{H}^n_\delta(A) \qquad \text{for all} \quad \delta > 0.$$

(Step 2) Claim: for all $\delta > 0$, there exists $C_n \ge 1$ such that $\mathscr{H}^n_\delta \le C_n \mathscr{L}^n$.
Notice that for any cube Q we have

$$\mathscr{L}^n(Q) = \left(\frac{\operatorname{diam} Q}{\sqrt{n}}\right)^n .$$

By the definition of the Lebesgue measure we get

$$\begin{aligned}
\mathscr{L}^n(A) &= \inf\left\{\sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j\right\} \\
&= \inf\left\{\sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j,\ \operatorname{diam} Q_j \le \delta\right\} \\
&= \frac{2^n}{\left(\sqrt{n}\right)^n \omega_n} \inf\left\{\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} Q_j}{2}\right)^n \mid A \subset \bigcup Q_j,\ \operatorname{diam} Q_j < \delta\right\} \\
&\ge \frac{1}{C_n} \mathscr{H}^n_\delta(A),
\end{aligned}$$

where in the second equality we used the fact that

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}}, \quad \text{and} \quad \mathscr{L}^1 = \mathscr{H}^1_\delta \quad \text{in } \mathbb{R} \quad \text{for all } \delta > 0.$$

(Step 3) Claim: $\mathscr{H}^n_\delta(A) \le \mathscr{L}^n(A) + \varepsilon$ for any $\varepsilon > 0$.
By the definition of $\mathscr{L}^n$ we see that, for all fixed $\delta, \varepsilon > 0$, there exists a family $\{Q_j\}_{j=1}^\infty$ such that $A \subset \bigcup_{j=1}^\infty Q_j$, $\operatorname{diam} Q_j \le \delta$ and $\sum_{j=1}^\infty \mathscr{L}^n(Q_j) \le \mathscr{L}^n(A) + \varepsilon$. Now, by Lemma 1.2.19, there exists a family $(\overline{B^i_j})_{i=1}^\infty$ of disjoint closed balls such that $\overline{B^i_j} \subset Q_j$ for all ($\operatorname{diam} B^i_j \le \delta$) and

$$\mathscr{L}^n\left(Q_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = \mathscr{L}^n\left(\mathring{Q}_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = 0.$$

Therefore, by Step 2 we also have

$$\mathscr{H}^n_\delta\left(Q_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = 0,$$

from which we deduce that

$$\begin{aligned}
\mathscr{H}^n_\delta(A) &\le \sum_{j=1}^\infty \mathscr{H}^n_\delta(Q_j) = \sum_{j=1}^\infty \mathscr{H}^n_\delta\left(\bigcup_{i=1}^\infty \overline{B^i_j}\right) \\
&= \sum_{j=1}^\infty \sum_{i=1}^\infty \mathscr{H}^n_\delta\left(B^i_j\right) \le \sum_{j=1}^\infty \sum_{i=1}^\infty \underbrace{\omega_n \left(\frac{\operatorname{diam} B^i_j}{2}\right)^n}_{=\mathscr{L}^n\left(\overline{B^i_j}\right)} \\
&= \sum_{j=1}^\infty \mathscr{L}^n\left(\bigcup_{i=1}^\infty \overline{B^i_j}\right) = \sum_{j=1}^\infty \mathscr{L}^n(Q_j) \\
&\le \mathscr{L}^n(A) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this inequality ends the proof. □

Proof of the isodiametric inequality (Theorem 1.2.20). Without loss of generality, we may assume E to be compact. Indeed, notice that $\operatorname{diam} A = \operatorname{diam} \overline{A}$ for any set A, and, if $\operatorname{diam} E = +\infty$, the inequality is trivially true.

Next, observe that, if $E \subset B\left(x, \frac{\operatorname{diam} E}{2}\right)$ for some $x \in \mathbb{R}^n$, then there is nothing to prove. We employ Steiner symmetrization[♭] in order to reduce ourselves to such a case.

Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q : \mathbb{R}^n \to \mathbb{R}$ be the orthogonal projections,

$$p(x) = (x_1, \dots, x_{n-1}), \quad q(x) = x_n,$$

so that

$$x = (p(x), q(x)) \quad \text{and} \quad |x|^2 = |p(x)|^2 + |q(x)|^2.$$

T hen, for any $z \in \mathbb{R}^{n-1}$ we define the *verital section*

$$E_z := \{t \in \mathbb{R} : (z, t) \in E\},$$

and, as a consequence, we introduce the *symmetrization* of E with respect n-th coordinate axis:

$$E^s := \left\{x \in \mathbb{R}^n : |q(x)| \le \frac{\mathscr{L}^1(E_{p(x)})}{2}\right\}.$$

By Fubini's theorem, E_z is $\mathscr{L}^1$-measurable for $\mathscr{L}^{n-1}$-a.e. z, $z \mapsto \mathscr{L}^1(E_z)$ is Lebesgue measurable and so we get

$$|E| = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z) dz = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E^s_z) dz = |E^s|, \tag{1.2.7}$$

where the first equality follows by Fubini's theorem, and the second one is a consequence of the fact that

$$(E^s)_z = \{t \in \mathbb{R} : (z, t) \in E^s\} = \left\{t \in \mathbb{R} : |t| \le \frac{\mathscr{L}^1(E_z)}{2}\right\} = \left[-\frac{\mathscr{L}^1(E_z)}{2}, \frac{\mathscr{L}^1(E_z)}{2}\right].$$

Now we claim that

$$\operatorname{diam} E^s \le \operatorname{diam} E. \tag{1.2.8}$$

In order to prove this, let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned}
&p(m(x)) = p(M(x)) = p(x) \\
&q(m(x)) \le q(z) \le q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x).
\end{aligned}$$

Hence, for all $x, y \in E^s$, we have

$$|q(x) - q(y)| \le \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\},$$

[♭] Introduced in 1838 by Jakob Steiner (1796 - 1863) [16].

in particular, without loss of generality, we can assume that

$$\max\{|q(M(x))-q(m(y))|,|q(M(y))-q(m(x))|\}=|q(M(x))-q(m(y))|,$$

so that

$$|q(x)-q(y)|\le|q(M(x))-q(m(y))|.$$

As a consequence, we see that

$$\begin{aligned}|x-y|^2&=|p(x-y)|^2+|q(x-y)|^2\le|p(M(x))-p(m(y))|^2+|q(M(x))-q(m(y))|^2\\&=|M(x)-m(y)|^2=\max\{|M(x)-m(y)|,|M(y)-m(x)|\}^2\le(\operatorname{diam}E)^2.\end{aligned}$$

This means that $|x-y|\le\operatorname{diam}E$ for all $x,y\in E^s$, which immediately implies (1.2.8).

Given a $\mathscr{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. Hence, if we set $E_0:=E$, $E_i:=(E_{i=1})^i$ with $i\in\{1,2,\dots,n\}$, then, by (1.2.7) we have $|E_n|=|E|$ and $\operatorname{diam}E_n\le\operatorname{diam}E$ by (1.2.8). In addition, we notice that, if $x\in E_n$, then $-x\in E_n$, which implies $E_n\subset B\left(0,\frac{\operatorname{diam}E_n}{2}\right)$. Thus, we conclude that

$$|E|=|E_n|\le\omega_n\left(\frac{\operatorname{diam}E_n}{2}\right)^n\le\omega_n\left(\frac{\operatorname{diam}E}{2}\right)^n.$$

And so we are done! □

1.3 Integration and fundamental convergence theorems

In this section, let $X\neq\emptyset$, and μ be a measure on X. Recall the definition of the *extended real line*

$$\overline{\mathbb{R}}:=[-\infty,\infty].$$

Definition 1.3.1.

(1) A function $u:X\to\overline{\mathbb{R}}$ is *μ-measurable* if the *superlevel set*

$$\{u>t\}:=\{x\in X:u(x)>t\}$$

is μ-measurable for all $t\in\overline{\mathbb{R}}$.

(2) A function $u:X\to\overline{\mathbb{R}}$ is a *μ-simple function* if it is μ-measurable and $u(X)$ is countable; that is,

$$u(x)=\sum_{k=1}^{\infty}u_k\chi_{E_k}(x),$$

for some sequences of real numbers $\{u_k\}$ and of μ-measurable disjoint sets $\{E_k\}$.

(3) If u is a nonnegative μ-simple function, we define

$$\int_X u\,d\mu:=\sum_{t\in u(X)}t\mu(\{u=t\})=\sum_{k=1}^{\infty}u_k\mu(E_k)\in[0,\infty]$$

with the convention that $0\cdot\infty=0$.

(4) We set $u^\pm:=\max\{\pm u,0\}$, so that $u=u^+-u^-$ and $|u|=u^++u^-$. If u is μ-simple and $\int_X u^+d\mu$ or $\int_X u^-d\mu<\infty$, then u is a *μ-integrable simple function*, and we set

$$\int_X u\,d\mu:=\int_X u^+d\mu-\int_X u^-d\mu\in[-\infty,\infty]$$

(5) If $u:X\to\overline{\mathbb{R}}$ is μ-measurable, we define the *upper and lower integrals* of u as

$$\int_X^* u\,d\mu:=\inf\left\{\int_X v\,d\mu\mid v\ge u\ \mu\text{-a.e, }v\ \mu\text{-integrable simple function}\right\}$$

or

$$\int_{*X} u\,d\mu := \sup\left\{\int_X v\,d\mu \mid v \le u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

respectively. If

$$\int_{*X} u\,d\mu = \int_X^* u\,d\mu,$$

then we say that u is *μ-integrable*, and we set

$$\int_X u\,d\mu := \int_X^* u\,d\mu = \int_{*X} u\,d\mu.$$

(6) A measurable function u is *μ-summable* if $|u|$ is μ-integrable and

$$\int_X |u|\,d\mu < \infty.$$

Example 1.3.2 (Integral with respect to the Dirac measure)**.** Let $x_0 \in X$ and $\mu = \delta_{x_0}$ be the Dirac measure centered in x_0, as defined in Example 1.1.13. Notice that any subset in X is δ_{x_0}-measurable, so that any function $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-measurable. Then, any $u : X \to \overline{\mathbb{R}}$ simple function is μ-integrable. Indeed, assuming at first $u : X \to [0, \infty]$, for some sequence of nonnegative real numbers $\{u_k\}$ and a partition $\{E_k\}$ of X, we have

$$u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x),$$

so that

$$\int_X u\,d\delta_{x_0} = \sum_{k=1}^{\infty} u_k \delta_{x_0}(E_k) = u_{k_0},$$

where k_0 satisfies $E_{k_0} \ni x_0$, which implies $u(x_0) = u_{k_0}$. Then, if $u : X \to \overline{\mathbb{R}}$ is simple, then we have either $u^+(x_0) > 0$ or $u^-(x_0) > 0$, so that either

$$\int_X u^-\,d\delta_{x_0} = 0 \quad \text{or} \quad \int_X u^+\,d\delta_{x_0} = 0,$$

respectively. Therefore, we can easily see that

$$\int_X u\,d\delta_{x_0} = u^+(x_0) - u^-(x_0) = u(x_0)$$

for any simple function u. As a consequence, any $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-integrable. Indeed, for any simple function $v \ge u$ and any simple function $w \le u$, we have

$$w(x_0) \le \int_{*X} u\,d\delta_{x_0} \le \int_X^* u\,d\delta_{x_0} \le v(x_0),$$

so that we get

$$\int_{*X} u\,d\delta_{x_0} = \int_X^* u\,d\delta_{x_0} = u(x_0),$$

by choosing

$$v(x) = \begin{cases} u(x_0) & x = x_0, \\ +\infty & x \ne x_0, \end{cases}$$

and

$$w(x) = \begin{cases} u(x_0) & x = x_0, \\ -\infty & x \ne x_0. \end{cases}$$

Thus, we conclude that, for any $u : X \to \overline{\mathbb{R}}$, we have

$$\int_X u\,d\delta_{x_0} = u(x_0),$$

and that u is δ_{x_0}-summable if and only if $|u(x_0)| < \infty$.

We define now general versions of the familiar L^p-function spaces.

Definition 1.3.3.

$L^1(X,\mu) := \{u : X \to \overline{\mathbb{R}} : u \text{ is } \mu\text{-summable}\}$ and we set

$$\|u\|_{L^1(X,\mu)} := \int_X |u|\, d\mu.$$

For any $p \in (1,\infty)$, we define $L^p(X,\mu) := \{u : X \to \overline{\mathbb{R}} : |u|^p \text{ is } \mu\text{-summable}\}$ and we set

$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u|^p\, d\mu\right)^{\frac{1}{p}}.$$

Let $u : X \to \overline{R}$ be μ-measurable. We set

$$\|u\|_{L^\infty(X,\mu)} := \inf\{\lambda > 0 : \mu(\{|u| > \lambda\}) = 0\}.$$

As a consequence, we define $L^\infty(X,\mu) := \{u : X \to \overline{\mathbb{R}} : \|u\|_{L^\infty(X,\mu)} < \infty\}$.

For any $p \in [1,\infty]$, we define the local L^p-spaces as

$$L^p_{\text{loc}}(X,\mu) := \{u : X \to \overline{\mathbb{R}} : \|u\chi_K\|_{L^p(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact}\}.$$

For any $m \in \mathbb{N}$, we set $L^p(X,\mu;\mathbb{R}^m) := \{u : X \to \mathbb{R}^m : |u| \in L^p(X,\mu)\}$, with

$$\|u\|_{L^p(X,\mu;\mathbb{R}^m)} := \||u|\|_{L^p(X,\mu)}, \quad |u(x)| := \sqrt{\sum_{j=1}^m u_j(x)^2},$$

and

$$L^p_{\text{loc}}(X,\mu;\mathbb{R}^m) := \{u : X \to \mathbb{R}^m : \|u\chi_K\|_{L^p(X,\mu;\mathbb{R}^m)} < \infty \text{ for all } K \subset X \text{ compact}\}.$$

In the case $X = \mathbb{R}^n$ and $\mu = \mathscr{L}^n$, we shall set for brevity

$$L^p(\mathbb{R}^n;\mathbb{R}^m) := L^p(\mathbb{R}^n,\mathscr{L}^n;\mathbb{R}^m) \quad \text{and} \quad L^p_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^m) := L^p_{\text{loc}}(\mathbb{R}^n,\mathscr{L}^n;\mathbb{R}^m),$$

for any $p \in [1,\infty]$.

Example 1.3.4 (The spaces $L^p(X,\delta_{x_0})$)**.** In light of Example 1.3.2, we see that, for all $x_0 \in X$ and $p \in [1,\infty]$, the spaces $L^p(X,\delta_{x_0})$ and $L^p_{\text{loc}}(X,\delta_{x_0})$ all coincide with the space

$$\{u : X \to \overline{\mathbb{R}} : |u(x_0)| < +\infty\}.$$

We recall now the simple and very useful Chebyshev's inequality[♮].

Proposition 1.3.5 (Chebyshev's inequality)**.** *Let $u : X \to [0,+\infty]$ be μ-integrable. Then, for all $t > 0$, we have*

$$\int_{\{u \ge t\}} u\, d\mu \ge t\mu(\{u \ge t\}). \tag{1.3.1}$$

Proof. If u is simple, then we have

$$u(x) = \sum_{k=1}^\infty u_k \chi_{E_k}(x),$$

and so

$$\int_{\{u\ge t\}} u\, d\mu = \int_X u\chi_{\{u\ge t\}}\, d\mu \ge t \sum_{k: u_k \ge t} \mu(E_k \cap \{u \ge t\}) = t\mu(\{u \ge t\}).$$

Then the conclusion follows by approximation with simple functions. $\square$

[♮] This inequality is named after the Russian mathematician Pafnuty Chebyshev (1821-1894), whose surname has been transliterated from the Cyrillic Чебышёв in many alternative ways: Chebysheff, Chebychov, Chebyshov, Tchebychev, Tchebycheff, Tschebyschev, Tschebyschef, Tschebyscheff and Chebychev. Actually, this theorem was first stated without proof by his friend and colleague Irénée-Jules Bienaymé (1796-1878) in 1853 [2] and later proved by Chebyshev in 1867 [3]. His student Andrey Markov (1856–1922) provided another proof in his 1884 Ph.D. thesis [13], and for this reason in Probability Theory this inequality is usually called Markov's inequality.

Remark 1.3.6. Let $u : X \to \overline{\mathbb{R}}$ be a μ-integrable function. Then, we have the following simple consequence of the Chebychev inequality (Proposition 1.3.5):

i) if $\int_X |u|\,d\mu < +\infty$, then $\mu(\{|u| = +\infty\}) = 0$;

ii) if $\int_X |u|\,d\mu = 0$, then $u(x) = 0$ for μ-a.e. $x \in X$.

It is easy to see that (i) follows from (1.3.1), since

$$\int_X |u|\,d\mu \geq \int_{\{|u|=+\infty\}} |u|\,d\mu = \int_{\{|u|=+\infty\}\cap\{|u|\geq t\}} |u|\,d\mu \geq t\mu(\{|u| = +\infty\})$$

for all $t > 0$, since $\{|u| = +\infty\} \cap \{|u| \geq t\} = \{|u| = +\infty\}$. Instead, in order to prove (ii), it is enough to notice that, again by (1.3.1), for any $\varepsilon > 0$ we have

$$0 = \int_X |u|\,d\mu \geq \int_{\{|u|\geq\varepsilon\}} |u|\,d\mu \geq \varepsilon\mu(\{|u| \geq \varepsilon\}),$$

so that

$$\mu\left(\{|u| > 0\}\right) = \mu\left(\bigcup_{k=1}^{\infty}\left\{|u| \geq \frac{1}{k}\right\}\right) \leq \sum_{k=1}^{\infty}\mu\left(\left\{|u| \geq \frac{1}{k}\right\}\right) = 0.$$

We recall now some fundamental convergence results concerning the exchange between limits and integrals.

Theorem 1.3.7 (Monotone convergence theorem)**.** *If $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative μ-measurable functions such that $u_k \leq u_{k+1}$ μ-a.e. on X for all $k \in \mathbb{N}$, then*

$$\lim_{k\to+\infty} \int_X u_k\,d\mu = \int_X \sup_{k\in\mathbb{N}} u_k\,d\mu.$$

If instead $u_k \geq u_{k+1}$ μ-a.e. on X for all $k \in \mathbb{N}$ and $u_1 \in L^1(X, \mu)$, then

$$\lim_{k\to+\infty} \int_X u_k\,d\mu = \int_X \inf_{k\in\mathbb{N}} u_k\,d\mu.$$

Theorem 1.3.8 (Fatou's lemma)**.** *If $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative μ-measurable functions, then*

$$\int_X \liminf_{k\to+\infty} u_k\,d\mu \leq \liminf_{k\to+\infty} \int_X u_k\,d\mu.$$

Theorem 1.3.9 (Dominated convergence theorem)**.** *Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions such that there exist a μ-measurable function u satisfying $u_k(x) \to u(x)$ for μ-a.e. $x \in X$ and $v \in L^1(X, \mu)$ satisfying $|u_k| \leq v$ μ-a.e. on X for all $k \in \mathbb{N}$. Then, we have*

$$\lim_{k\to+\infty} \int_X |u_k - u|\,d\mu = 0,$$

and, in particular,

$$\lim_{k\to+\infty} \int_X u_k\,d\mu = \int_X u\,d\mu.$$

Theorem 1.3.10. *Assume u_k, u are μ-summable functions, for all $k \in \mathbb{N}$, and*

$$\lim_{k\to+\infty} \int_X |u_k - u|\,d\mu = 0.$$

Then there exists a subsequence $\{u_{k_j}\}$ such that $u_{k_j}(x) \to u(x)$ for μ-a.e. $x \in X$.

Proof. Let $\{u_{k_j}\}$ be a subsequence satisfying

$$\sum_{j=1}^{\infty} \int_X |u_{k_j} - u|\,d\mu < \infty.$$

Notice that such a subsequence exists, since

$$a_k := \int_X |u_k - u| \, d\mu \to 0,$$

so that we can select a subsequence a_{k_j} satisfying $a_{k_j} \leq 2^{-j}$ for all $j \in \mathbb{N}$. By the monotone convergence theorem (Theorem 1.3.7), we have

$$\sum_{j=1}^{\infty} \int_X |u_{k_j} - u| \, d\mu = \int_X \sum_{j=1}^{\infty} |u_{k_j} - u| \, d\mu < \infty.$$

Therefore, by Remark 1.3.6, we obtain

$$\sum_{j=1}^{\infty} |u_{k_j}(x) - u(x)| < \infty$$

for μ-a.e. $x \in X$, and thus we conclude that $u_{k_j}(x) \to u(x)$ for μ-a.e. $x \in X$. □

Thanks to the notion of integral with respect to a general measure μ, it is plain to see that the integration of a nonnegative μ-measurable function is itself a measure, so that we can formulate the following definition.

Definition 1.3.11 (Integral measures)**.** If $u : X \to [0, \infty]$ μ-measurable, then we define the *integral measure* $\nu = u\mu$ (or $\mu \llcorner u$) as

$$\nu(A) = \int_A u \, d\mu = \int_X u \chi_A \, d\mu \qquad \text{for all } \mu\text{-measurable } A.$$

1.4 Real and vector valued Radon measures

Through this section, let $\Omega \subset \mathbb{R}^n$ be an open set. We exploit now the concept of integral measure introduced in Definition 1.3.11 to define signed and vector valued Radon measures. In order to avoid ambiguity, from this point on we shall refer to the Radon measure introduced in Definition 1.1.9 as nonnegative Radon measures.

Definition 1.4.1 (Signed Radon measures)**.** Given a nonnegative Radon measure μ on Ω and $f : \Omega \to [-\infty, \infty]$ locally μ-summable. Then we set $\nu := f\mu$ to be the integral measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact}.$$

ν is said to be a *signed Radon measure* on Ω.

Definition 1.4.2 (Vector valued Radon measures)**.** Given a nonnegative Radon measure μ on Ω and $f : \Omega \to \mathbb{R}^m$ is locally μ-summable. Then we set $\nu := f\mu$ to be the vector valued Radon measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact}.$$

ν is said to be a *vector valued Radon measure* on Ω.

Exploiting the inner regularity of the nonnegative Radon measures, it is then possible to extend the signed and vector valued measures defined above to the σ-algebra of Borel sets in Ω.

It is however possible to give an alternative definition of signed and vector valued Radon measures, by means of the concept of total variation.

Definition 1.4.3 (Alternative definition)**.**

i) A nonnegative Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ which is σ-additive and finite on compact sets. We denote the space of such measures by $\mathcal{M}^+_{\text{loc}}(\Omega)$.

ii) A vector valued (real or signed if $m = 1$) Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ which is σ-additive and its total variation $|\mu|$ is finite on compact sets; that is

$$|\mu|(K) := \sup\left\{\sum_{j=1}^{\infty} |\mu(B_j)| : B_j \in \mathcal{B}(\Omega), K = \bigcup B_j, B_j \cap B_i = \emptyset \text{ if } i \neq j, B_j \in \mathcal{B}(\Omega)\right\} < \infty$$

for all compact sets $K \subset \Omega$. The space of such measures is denote by $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$; and by $\mathcal{M}_{\text{loc}}(\Omega)$ if $m = 1$.

iii) We say that a nonnegative Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty)$ is finite if $\mu(\Omega) < \infty$; and we denote by $\mathcal{M}^+(\Omega)$ the space of such measures.

iv) We say that a nonnegative vector-valued Radon measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is finite if $|\mu|(\Omega) < \infty$; and we denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$, and $\mathcal{M}(\Omega)$ if $m = 1$, the space of such measures.

Remarks (Basic facts)**.**

- If $\mu \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$, then $|\mu| \in \mathcal{M}^+_{\text{loc}}(\Omega)$, where

$$|\mu|(B) := \sup\left\{\sum_{j=1}^{\infty} |\mu(B_j)| \mid B = \bigcup B_j, B_j \cap B_i = \emptyset \text{ if } i \neq j, B_j \in \mathcal{B}(\Omega)\right\}$$

  for any Borel set $B \Subset \Omega$. In particular, $\sum \mu(B_j)$ is absolutely convergent for all $\{B_j\}$ Borel partition of a some Borel set $B \Subset \Omega$.

- The total variation is the smallest nonnegative Radon measure ν such that $\nu(B) \geq |\mu(B)|$ for all $B \in \mathcal{B}(\Omega)$.

- If $\mu \in \mathcal{M}(\Omega)$, we define the *positive and negative parts* of μ

$$\mu^+ := \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu^- := \frac{|\mu| - \mu}{2}.$$

  It is easy to notice that $\mu^\pm \geq 0$ and that $\mu = \mu^+ - \mu^-$: this is called the *Jordan decomposition*, and it is unique. In addition, $|\mu| = \mu^+ + \mu^-$.

- The total variation is 1-positively homogeneous and subadditive, and the space $\mathcal{M}(\Omega; \mathbb{R}^m)$ equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega;\mathbb{R}^m)} := |\mu|(\Omega)$$

  is Banach.

We proceed now to prove the equivalence of Definitions 1.4.1 and 1.4.2 with Definition 1.4.3. To this purpose, we need the following preliminary result.

Lemma 1.4.4. *If $\mu \in \mathcal{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu; \mathbb{R}^m)$, then $f\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $|f\mu| = |f|\mu$.*

Proof. We start by proving the inequality

$$|f\mu| \leq |f|\mu. \tag{1.4.1}$$

Let $B \in \mathcal{B}(\Omega)$. It is easy to notice that

$$|(f\mu)(B)| := \left|\int_B f d\mu\right| \leq \int_B |f| d\mu.$$

From this (1.4.1) follows immediately, and this implies $f\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Let then $\varepsilon > 0$ and $D = \{z_h\}_{h\in\mathbb{N}}$ countable dense set in $\mathbb{S}^{m-1}$, let $B \in \mathcal{B}(\Omega)$. We define

$$\sigma(x) := \min\{h \in \mathbb{N} : f(x)z_h \geq (1 - \varepsilon)|f(x)|\}$$

it is Borel measurable. Then, we set

$$B_h := \sigma^{-1}(\{h\}) \cap B,$$

and we notice that

$$B_h \in \mathcal{B}(\Omega),\ B = \bigcup_{h\in\mathbb{N}} B_h \text{ and } B_h \cap B_k = \text{ if } h \neq k.$$

This implies that

$$\int_B |f| d\mu = \sum_{k\in\mathbb{N}} \int_{B_h} |f| d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} \int_{B_h} f z_h d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} |(f\mu)(B_h)| \leq \frac{1}{1-\varepsilon} |f\mu|(B),$$

since

$$\int_{B_h} f z_h d\mu = z_h \int_{B_h} f d\mu \leq \left| \int_{B_h} f d\mu \right| = |(f\mu)(B_h)|.$$

Thus, we obtain the reverse inequality $|f|\mu \leq |f\mu|$ on $\mathcal{B}(\Omega)$. □

Definition 1.4.5. Let μ be a nonnegative measure on Ω.

i) We say that μ is concentrated on a set $E \subset \Omega$ if

$$\mu(\Omega \setminus E) = 0.$$

ii) We call the support of μ, $\operatorname{supp}\mu$, the smallest closed set on which μ is concentrated :

$$\operatorname{supp}(\mu) := \bigcap_{C \text{ closed}, \mu(\Omega\setminus C)=0} C.$$

Exercise 1.4.6. Equivalently, we may characterize the support of a nonnegative Radon measure μ in terms of its behaviour on balls:

$$\operatorname{supp}(\mu) = \{x \in \Omega \mid \mu(B(x,r)) > 0, \forall r > 0 \text{ such that } B(x,r) \subset \Omega\}.$$

Remark 1.4.7. Notice that a nonnegative Radon measure may be concentrated on a set strictly smaller than its support. Indeed, let $\Omega = \mathbb{R}$ and

$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{\frac{1}{k}}.$$

It is clear that μ is concentrated on the set $E = \left\{\frac{1}{k}\right\}_{k\geq 1}$, but $\mu((-r,r)) > 0$ for any $r > 0$, so that $0 \in \operatorname{supp}(\mu)$. In fact, it is not difficult to check that $\operatorname{supp}(\mu) = \{0\} \cup \left\{\frac{1}{k}\right\}_{k\geq 1} = \overline{E}$.

Definition 1.4.8.

1. Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$, $\nu \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$. We say that μ is absolutely continuous with respect to μ, and we write $\nu \ll \mu$, if for all $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$, then $|\nu|(B) = 0$.

2. If $\mu, \nu \in \mathcal{M}^+_{\text{loc}}(\Omega)$, we say that they are mutually singular if there exists $E, F \in \mathcal{B}(\Omega)$ such that $\mu(F) = 0$, $\mu(E) = 0$ and

$$\mu(B) = \mu(B \cap E) \qquad \text{and} \qquad \nu(B) = \nu(B \cap F)$$

for all $B \in \mathcal{B}(\Omega)$ and we write $\mu \perp \nu$. If $\mu, \nu \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$,

$$\mu \perp \nu \iff |\mu| \perp |\mu|.$$

Theorem 1.4.9 (Radon-Nikodym). *Let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$. Then there exist unique measures $\nu^{ac}, \nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m)$ such that $\nu^{ac} \ll \mu$, $\nu^s \perp \mu$ and*

$$\nu = \nu^{ac} + \nu^s. \tag{1.4.2}$$

In addition, there exists a unique measure $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that $\nu^{ac} = f\mu$.
In particular, if $\mu = \mathscr{L}^n$, every $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ can be uniquely decomposed in

$$\mu = f\mathscr{L}^n + \nu^s,$$

for some $f \in L^1(\Omega; \mathbb{R}^n)$ and $\nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m), \nu^s \perp \mathscr{L}^n$.

The decomposition in (1.4.2) is called *Lebesgue decomposition* of the measure ν with respect to μ.

Definition 1.4.10. We say that a property holds $|\mu|$-almost everywhere or for $|\mu|$-almost every x if the set where the property does not hold is $|\mu|$-negligible; that is, it has zero $|\mu|$-measure.

Corollary 1.4.11 (Polar decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. Then there exists a unique $f \in L^1(\Omega, |\mu|;\mathbb{R}^m)$ such that $|f(x)| = 1$ $|\mu|$-a.e and $\mu = f|\mu|$.*

Proof of corollary. Apply Radon-Nikodym theorem (Theorem 1.4.9) to μ and $|\mu|$. We know that $|\mu(B)| \leq |\mu|(B)$ for all $B \in \mathcal{B}(\Omega)$. From this follows $\mu \ll |\mu|$, and so there exists $f \in L^1(\Omega, |\mu|;\mathbb{R}^m)$ such that $\mu = f|\mu|$.
We proved that $|f|\mu|| = |f||\mu|$, hence we obtain

$$|\mu| = |f|\mu|| = |f||\mu| \quad \text{and so} \quad (|f|-1)|\mu| = 0.$$

This means that we have

$$\int_\Omega (|f|-1)d|\mu| = 0,$$

which yields $|f(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$. ☐

Corollary 1.4.12 (Hahn decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega)$, there exists a unique $A \in \mathcal{B}(\Omega)$ (up to $|\mu|$-negligible sets) such that*

$$\mu^+ = \mu \llcorner A \qquad \mu^- = -\mu \llcorner (\Omega \setminus A).$$

Proof. By the polar decomposition, there exists a unique $f \in L^1(\Omega, |\mu|)$ such that $\mu = f|\mu|$ and $f(x) \in \{\pm 1\}$ for $|\mu|$-a.e. $x \in \Omega$. This means that, if we set

$$A := \{f = 1\},$$

we have

$$f(x) = \chi_A - \chi_{\Omega\setminus A}.$$

Thus, we obtain

$$\begin{aligned} \mu^+ &:= \frac{|\mu|+\mu}{2} = \frac{1+\chi_A - \chi_{\Omega\setminus A}}{2}|\mu| = \chi_A|\mu|, \\ \mu^- &:= \frac{|\mu|-\mu}{2} = \frac{1-\chi_A + \chi_{\Omega\setminus A}}{2}|\mu| = \chi_{\Omega\setminus A}|\mu|. \end{aligned}$$

☐

1.5 Duality for Radon measures

Due to the inner and outer regularity, Radon measures are totally characterized by their behaviour on the family of open and compact sets. As a consequence, the integration with respect to a Radon measure can be characterized by its behaviour on the space of continuous functions with compact support.

Definition 1.5.1. We say that $B \Subset \Omega$ if $\overline{B} \subset \Omega$ and it is compact in Ω. The space of *continuous functions with compact support* in Ω is defined as

$$C^0_c(\Omega;\mathbb{R}^m) := \{u \in C^0(\Omega;\mathbb{R}^m) : \operatorname{supp} u \Subset \Omega\},$$

while the space of *continuous functions vanishing on the boundary* of Ω is

$$C^0_0(\Omega;\mathbb{R}^m) := \{u \in C^0(\Omega;\mathbb{R}^m) : \forall \varepsilon > 0\ \exists K \subset \Omega : |u(x)| < \varepsilon \quad \forall x \notin K\}.$$

We equip both spaces with the supremum norm

$$\|u\|_\infty := \sup_{x\in\Omega} |u(x)|.$$

Remark 1.5.2. Notice that $C_0^0(\Omega;\mathbb{R}^m) = \overline{C_c^0(\Omega;\mathbb{R}^m)}^{\|\cdot\|_\infty}$, and that $(C_0^0(\Omega;\mathbb{R}^m), \|\cdot\|_\infty)$ is Banach. On the other hand, $C_c^0(\Omega;\mathbb{R}^m)$ is separable, locally convex, topological vector space with the following topology:

$$\varphi_k \longrightarrow \varphi \text{ in } C_c^0(\Omega;\mathbb{R}^m) \iff \|\varphi_k - \varphi\|_\infty \to 0 \text{ and there exists } K \subset \Omega : \operatorname{supp}\varphi \cup \bigcup_{k\in\mathbb{N}} \operatorname{supp}\varphi_k \subset K.$$

From this point on, to simplify notation, we shall omit the apex 0 to denote the spaces $C^0(\Omega;\mathbb{R}^m), C_0^0(\Omega;\mathbb{R}^m)$ and $C_c^0(\Omega;\mathbb{R}^m)$, and we shall simply write $C(\Omega;\mathbb{R}^m), C_0(\Omega;\mathbb{R}^m)$ and $C_c(\Omega;\mathbb{R}^m)$.

Theorem 1.5.3 (Lusin)**.** *Let μ Borel on Ω and $u : \Omega \to \mathbb{R}$ is μ-measurable, $u \equiv 0$ in $\Omega \setminus E$ with $\mu(E) < \infty$. Then for all $\varepsilon > 0$ there exists $v \in C(\Omega)$ such that $\|v\|_\infty \le \|u\|_\infty$ and*

$$\mu(\{x \in \Omega : v(x) \neq u(x)\}) < \varepsilon.$$

Remark 1.5.4. An equivalent formulation states that, under the additional assumption that μ is finite on Ω, $\mu(\Omega) < \infty$, then there exists a sequence of compact sets $\{K_h\}$ such that

$$\mu\left(\Omega \setminus \bigcup_{h=1}^{\infty} K_h\right) = 0 \qquad \text{and} \qquad u\big|_{K_h} \text{ is continuous.}$$

In other terms, this means that there exists a sequence of functions $\{u_h\} \in C(\Omega)$ such that $u = u_h$ on K_h and $\|u_h\|_\infty \le \|u\|_\infty$.

Proposition 1.5.5. *Let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. Then for all $A \subset \Omega$ open we have*

$$|\mu|(A) = \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c(A;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\}, \tag{1.5.1}$$

with the convention that

$$\int_\Omega \varphi \cdot d\mu := \sum_{j=1}^m \int_\Omega \varphi_j d\mu_j.$$

Proof. Polar decomposition implies that $\mu = f|\mu|$, $|f| = 1$ μ-a.e. So we get

$$\int_\Omega \varphi \cdot d\mu = \int_A \varphi \cdot f d|\mu| \le |\mu|(A).$$

By Lusin theorem, for all $\varepsilon > 0$ there exists $\varphi \in C(A;\mathbb{R}^m)$ such that $\|\varphi\|_\infty \le 1$ and

$$|\mu|\left(\{x \in A : \varphi(x) \neq f(x)\}\right) < \varepsilon.$$

Take $K \subset A$ compact such that $|\mu|(A \setminus K) < \varepsilon$. Construct $\eta \in C_c^\infty(A)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on K, $\tilde\varphi = \varphi\eta \in C_c(A;\mathbb{R}^m)$ and

$$|\mu|\left(\{x : \tilde\varphi(x) \neq f(x)\}\right) \le |\mu|(A \setminus K) + |\mu|\left(\{x : \varphi(x) \neq f(x)\}\right) \le 2$$

to get

$$\int_A \tilde\varphi \cdot d\mu \ge |\mu|(K) - 2\varepsilon$$

and by sending K to A and $\varepsilon \searrow 0$ we arrive at the claim. □

Proposition 1.5.5 shows that, given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$, we can define a linear continuous functional $L_\mu : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu,$$

for any $\varphi \in C_0(\Omega;\mathbb{R}^m)$. In addition, the operatorial norm of L_μ is equal to $|\mu|(\Omega)$, since, by the density of C_c in C_0 with respect to the supremum norm and by (1.5.1), we have

$$\begin{aligned}\|L_\mu\| &:= \sup\{L_\mu(\varphi) : \varphi \in C_0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\} \\ &= \sup\left\{\int_\Omega \varphi \cdot d\mu : \varphi \in C_c(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\} = |\mu|(\Omega).\end{aligned}$$

This suggests that it is possible to characterize $\mathcal{M}(\Omega;\mathbb{R}^m)$ as the dual space of $C_0(\Omega;\mathbb{R}^m)$. In such a way, we gain yields a weaker topology on the space of vector valued Radon measure, and therefore weak* compactness of bounded sequences.

Theorem 1.5.6. (Riesz Representation Theorem) *Let $L : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a continuous linear functional; that is, L is linear and satisfies*

$$\sup\{L(\varphi) : \varphi \in C_0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that

$$L(\varphi) = \int_\Omega \varphi \cdot d\mu, \quad \forall \varphi \in C_0(\Omega;\mathbb{R}^m).$$

Moreover,

$$|\mu|(\Omega) = \sup\{L(\varphi) : \varphi \in C_c(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \leq 1\} = \|L\|.$$

For the proof we refer to [1, Theorem 1.54].

The following corollary is a direct consequence of the global version of the Riesz Representation Theorem.

Corollary 1.5.7. *Let $L : C_c(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\varphi) : \varphi \in C_c(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \leq 1, \operatorname{supp}(\varphi) \subset K\} < \infty,$$

for any compact set $K \subset \Omega$. Then there exists a unique $\mu \in \mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$ such that

$$L(\varphi) = \int_\Omega \varphi \cdot d\mu, \quad \forall \varphi \in C_c(\Omega;\mathbb{R}^m).$$

Thus we can identify any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with a continuous linear functional on $C_0(\Omega;\mathbb{R}^m)$, written as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu, \quad \forall \varphi \in C_0(\Omega;\mathbb{R}^m),$$

so that $\mathcal{M}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_0(\Omega;\mathbb{R}^m)$, under the pairing

$$\langle \varphi, \mu \rangle := \int_\Omega \varphi \cdot d\mu,$$

and with the total variation $|\mu|(\Omega)$ in the role of the dual norm. In fact, this is an alternative way to prove that $\mathcal{M}(\Omega;\mathbb{R}^m)$ equipped with the norm

$$\|\mu\|_{\mathcal{M}(\Omega;\mathbb{R}^m)} := |\mu|(\Omega)$$

is a Banach space. Analogously, $\mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega;\mathbb{R}^m)$. These facts lead us to a notion of weak* convergence for Radon measure.

1.6 Weak* convergence for Radon measures

Definition 1.6.1. Given μ and a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$, we say that μ_k *weak* converges to* μ if

$$\int_\Omega \varphi \cdot d\mu_k \to \int_\Omega \varphi \cdot d\mu, \quad \forall \varphi \in C_0(\Omega;\mathbb{R}^m).$$

If μ and the sequence $\{\mu_k\}$ are in $\mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$, we say that μ_k *locally weak* converges to* μ if

$$\int_\Omega \varphi \cdot d\mu_k \to \int_\Omega \varphi \cdot d\mu, \quad \forall \varphi \in C_c(\Omega;\mathbb{R}^m).$$

We shall write $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$ and in $\mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$, in the case of weak* and local weak* convergence, respectively.

Remark 1.6.2. Weak* convergence of finite Radon measures is equivalent to local weak* convergence with the condition that

$$\sup_k |\mu_k|(\Omega) = C < \infty.$$

Clearly, weak* convergence always implies local weak* convergence. On the other hand, if we suppose that μ_k locally weak* converges to μ, then, given $\psi \in C_0(\Omega;\mathbb{R}^m)$, for any $\varepsilon > 0$ there exists $\varphi \in C_c(\Omega;\mathbb{R}^m)$ such that $\|\psi - \varphi\|_\infty < \varepsilon$ and so

$$\begin{aligned}\left|\int_\Omega \psi \cdot d\mu_k - \int_\Omega \psi \cdot d\mu\right| &\le \left|\int_\Omega (\psi - \varphi) \cdot d\mu_k\right| + \left|\int_\Omega (\psi - \varphi) \cdot d\mu\right| \\ &\quad + \left|\int_\Omega \varphi \cdot d\mu_k - \int_\Omega \varphi \cdot d\mu\right| \\ &\le 2C\varepsilon + \left|\int_\Omega \varphi \cdot d\mu_k - \int_\Omega \varphi \cdot d\mu\right|.\end{aligned}$$

Now,

$$\int_\Omega \varphi \cdot d\mu_k \to \int_\Omega \varphi \cdot d\mu$$

and so, since ε is arbitrary, we obtain the weak* convergence.

We now exploit the dual nature of this topology on the space of vector valued Radon measures in order to derive some of its properties from classical results from Functional Analysis. To this purpose we recall the statements of the Uniform Boundedness Principle (or the Banach-Steinhaus theorem) and Banach-Alaoglu theorem.

Theorem 1.6.3 (Banach-Steinhaus). *Let X be a Banach space and Y be a normed space. Let*

$$\mathcal{F} := \{T : X \to Y, T \text{ linear and continuous}\}.$$

If

$$\sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty \text{ for all } x \in X,$$

then

$$\sup_{\substack{T \in \mathcal{F} \\ x \in X, \|x\|_X \le 1}} \|T(x)\|_Y = \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} < \infty,$$

where

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in X, \|x\|_X \le 1} \|T(x)\|_Y,$$

is the operator norm.

Theorem 1.6.4 (Banach-Alaoglu). *Let X be a normed space and X^* be its dual. The closed unit ball of X^* is compact with respect to the weak* topology.*

Lemma 1.6.5. *Let μ and a sequence $\{\mu_k\}$ be in $\mathcal{M}(\Omega;\mathbb{R}^m)$ and assume that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$. Then we have*

$$\sup_{k \in \mathbb{N}} |\mu_k|(\Omega) < \infty. \tag{1.6.1}$$

Proof. The first assertion follows by applying the Uniform Boundedness Principle (Theorem 1.6.3) in the case $X = C_0(\Omega;\mathbb{R}^m), Y = \mathbb{R}$ and $\mathcal{F}$ is the family of linear functionals $L_{\mu_k} : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ defined by

$$L_{\mu_k}(\varphi) := \int_\Omega \varphi \cdot d\mu_k$$

for all $\varphi \in C_0(\Omega;\mathbb{R}^m)$. Since

$$L_{\mu_k}(\varphi) \to L_\mu(\varphi)$$

for each $\varphi \in C_0(\Omega;\mathbb{R}^m)$, we see that $\{L_{\mu_k}(\varphi)\}$ is a bounded sequence in $\mathbb{R}$ for any fixed $\varphi \in C_0(\Omega;\mathbb{R}^m)$: hence, we have

$$\sup_{k \in \mathbb{N}} |L_{\mu_k}(\varphi)| < \infty$$

for all $\varphi \in C_0(\Omega;\mathbb{R}^m)$. Thus, by Theorem 1.6.3, we obtain (1.6.1). □

Theorem 1.6.6 (Weak* compactness). *If $\{\mu_k\}$ is a sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ satisfying*

$$\sup_{k \in \mathbb{N}} |\mu_k|(\Omega) < \infty,$$

then there exists a subsequence $\{\mu_{k_j}\}$ and μ in $\mathcal{M}(\Omega;\mathbb{R}^m)$ such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$.

Proof. Let $C := \sup_{k\in\mathbb{N}} |\mu_k|(\Omega)$ and define

$$\nu_k := \frac{\mu_k}{C}$$

for all $k \in \mathbb{N}$. Hence, it is clear that $|\nu_k|(\Omega) \leq 1$, so that the sequence $\{\nu_k\}$ belongs to the unit ball of $\mathcal{M}(\Omega;\mathbb{R}^m)$. By Banach-Alaoglu theorem (Theorem 1.6.4), the unit ball

$$B_1^{\mathcal{M}} := \{\nu \in \mathcal{M}(\Omega;\mathbb{R}^m) : |\nu|(\Omega) \leq 1\}$$

of $\mathcal{M}(\Omega;\mathbb{R}^m)$ is a compact set with respect to the weak* topology, so that any sequence in $B_1^{\mathcal{M}}$ admits a weakly* converging subsequence $\{\nu_{k_j}\}$. Thus, $\mu_{k_j} = C\nu_{k_j}$ is a weakly* converging subsequence of $\{\mu_k\}$ and this ends the proof. □

By a simple localization argument, it is easy to obtain the following local weak* compactness result.

Corollary 1.6.7 (Local weak* compactness). *Let $\{\mu_k\}$ be a sequence in $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ satisfying*

$$\sup_{k\in\mathbb{N}} |\mu_k|(K) < \infty \text{ for all compact sets } K \subset \Omega.$$

Then there exists a subsequence $\{\mu_{k_j}\}$ and μ in $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ such that $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$.

Another advantage of the duality with the space of continuous functions with compact support lies in the possibility of proving the weak* lower semicontinuity of the total variation.

Proposition 1.6.8 (Weak* lower semicontinuity of the total variation). *Let μ and a sequence $\{\mu_k\}$ in $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ satisfying $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. Then, for any open set $A \Subset \Omega$, we have*

$$|\mu|(A) \leq \liminf_{k\to+\infty} |\mu|(A). \tag{1.6.2}$$

If μ and μ_k are finite on Ω for all $k \in \mathbb{N}$, then (1.6.2) *holds for any open set $A \subset \Omega$.*

Proof. We notice that, if $\mu, \mu_k \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, then $\mu, \mu_k \in \mathcal{M}(\Omega';\mathbb{R}^m)$ for any open set $\Omega' \Subset \Omega$. Hence, without loss of generality, up to restricting ourselves to some open set compactly contained in Ω, we can assume $\mu, \mu_k \in \mathcal{M}(\Omega;\mathbb{R}^m)$ for all $k \in \mathbb{N}$. Let $A \subset \Omega$ be an open set and $\varphi \in C_c(A;\mathbb{R}^m)$ with $\|\varphi\|_\infty \leq 1$. We have

$$\int_A \varphi \cdot d\mu = \int_\Omega \varphi \cdot d\mu = \lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \lim_{k\to+\infty} \int_A \varphi \cdot d\mu_k \leq \liminf_{k\to+\infty} |\mu_k|(A).$$

Then, we take the supremum on such φ and we obtain (1.6.2). □

Remark 1.6.9. We notice that (vector valued) Radon measures can be also seen as (vector valued) distributions of order 0; that is, elements $L \in \mathcal{D}'(\Omega;\mathbb{R}^m)$ satisfying

$$|L(\varphi)| \leq C\|\varphi\|_\infty \text{ for all } \varphi \in C_c^\infty(\Omega;\mathbb{R}^m). \tag{1.6.3}$$

Indeed, it is clear that any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ induces the distribution L_μ defined as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu$$

for any $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$. Notice that

$$|L_\mu(\varphi)| \leq |\mu|(\Omega)\|\varphi\|_\infty,$$

which easily implies that L_μ is a distribution of order 0. On the other hand, if $L \in \mathcal{D}'(\Omega;\mathbb{R}^m)$ satisfies (1.6.3), we can extend it to a continuous linear functional $\widehat{L}$ on $C_0(\Omega;\mathbb{R}^m)$, by setting

$$\widehat{L}(\varphi) := \lim_{k\to+\infty} L(\varphi_k)$$

for all $\varphi \in C_0(\Omega; \mathbb{R}^m)$, where $\varphi_k \in C_c^\infty(\Omega; \mathbb{R}^m)$ satisfies $\|\varphi_k - \varphi\|_\infty \to 0$[♮♮]. Thanks to (1.6.3), we see that $\widehat{L}$ is uniquely defined: indeed, let $\{\varphi_k\}$ and $\{\psi_k\}$ be two sequences in $C_c^\infty(\Omega; \mathbb{R}^m)$ such that

$$\|\varphi_k - \varphi\|_\infty \to 0 \text{ and } \|\psi_k - \varphi\|_\infty \to 0.$$

Hence, we see that

$$\limsup_{k\to+\infty} |L(\varphi_k) - L(\psi_k)| = \limsup_{k\to+\infty} |L(\varphi_k - \psi_k)| \le C \lim_{k\to+\infty} \|\varphi_k - \psi_k\|_\infty = 0,$$

and this ensures the well posedness of $\widehat{L}$. Thus, $\widehat{L}$ is linear continuous functional on $C_0(\Omega; \mathbb{R}^m)$, and so, by Riesz Representation theorem (Theorem 1.5.6), there exists a unique finite vector valued Radon measure μ_L on Ω such that

$$L(\varphi) = \int_\Omega \varphi \cdot d\mu_L \text{ for all } \varphi \in C_0(\Omega; \mathbb{R}^m).$$

In addition, if we have a sequence $\{\mu_k\}$ and a measure μ in $\mathcal{M}(\Omega; \mathbb{R}^m)$ such that

$$\sup_{k\in\mathbb{N}} |\mu_k|(\Omega) < \infty \text{ and } \lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \text{ for all } \varphi \in C_c^\infty(\Omega; \mathbb{R}^m),$$

then $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$. This can be proved arguing analogously as in Remark 1.6.2, and exploiting the density of $C_c^\infty(\Omega; \mathbb{R}^m)$ in $C_0(\Omega; \mathbb{R}^m)$ with respect to the supremum norm.

We state now a characterization of nonnegative linear functionals on $C_c^\infty(\Omega)$, which further analyzes the relation between distributions and Radon measures.

Lemma 1.6.10 (Schwarz's lemma). *Let $L : C_c^\infty(\Omega) \to \mathbb{R}$ be linear and nonnegative; that is,*

$$L(\varphi) \ge 0, \quad \forall \varphi \in C_c^\infty(\Omega) \text{ with } \varphi \ge 0.$$

Then there exists $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ such that

$$L(\varphi) = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Proof. We choose a compact set $K \subset \Omega$ and we select a smooth function $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on K and $0 \le \zeta \le 1$. Then, for any $\varphi \in C_c^\infty(\Omega)$ with $\mathrm{supp}(\varphi) \subset K$, we set $\psi = \|\varphi\|_\infty \zeta - \varphi \ge 0$. Therefore, since L is nonnegative, we have

$$0 \le L(\psi) = \|\varphi\|_\infty L(\zeta) - L(\varphi)$$

and so

$$L(\varphi) \le C\|\varphi\|_\infty,$$

with $C := L(\zeta)$.

Therefore, L may be extended to a linear mapping $\widehat{L} : C_c(\Omega) \to \mathbb{R}$ such that, for any compact $K \subset \Omega$,

$$\sup\{L(\varphi) : \varphi \in C_c(\Omega), \|\varphi\|_\infty \le 1, \mathrm{supp}(\varphi) \subset K\} < \infty.$$

Hence, Corollary 1.5.7 yields the existence of a real Radon measure μ such that

$$L(\varphi) = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in C_c(\Omega).$$

By the polar decomposition of measures (Corollary 1.4.11), $\mu = h|\mu|$, where $|h(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$. The fact that L is nonnegative implies that $h(x) = 1$ for $|\mu|$-a.e. $x \in \Omega$. Thus, we conclude that μ is a positive Radon measure. □

[♮♮] The existence of such sequence follows from the density of $C_c(\Omega; \mathbb{R}^m)$ in $C_0(\Omega; \mathbb{R}^m)$ with respect to the supremum norm and by a smoothing argument.

Remark 1.6.11. Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$, J be an interval in $\mathbb{R}$ and $\{A_t\}_{t\in J}$ be an increasing family of relatively compact sets in Ω, so that they satisfy $\overline{A_s} \subset A_t \Subset \Omega$ for all $s, t \in J, s < t$. Then, we have $\mu(\partial A_t) = 0$ for all but countably many $t \in J$.

By localizing to an open set $\Omega' \Subset \Omega$ such that $\overline{A_t} \subset \Omega'$ for all t in an interval $I \subset J$, we can assume without loss of generality that $\mu(\Omega) < \infty$. Indeed, the sets ∂A_t are pairwise disjoint, hence the set

$$J_k := \left\{ t \in J : \mu(\partial A_t) > \frac{1}{k} \right\}$$

must be finite for any $k \in \mathbb{N}$. To prove this fact, notice that

$$\frac{1}{k} \# J_k < \sum_{t \in J_k} \mu(\partial A_t) = \mu\left(\bigcup_{t\in J_k} \partial A_t \right) \le \mu\left(\bigcup_{t \in J} \partial A_t \right) \le \mu(\Omega) < \infty.$$

Then, it is clear that

$$\{t \in J : \mu(\partial A_t) > 0\} = \bigcup_{k\in\mathbb{N}} J_k,$$

so that the set of $t \in J$ for which $\mu(\partial A_t) > 0$ is at most countable.

The simplest example of an application of this property is the case of families of balls $\{B(x,r)\}_{r\in(0,r_x)}$ for any $x \in \Omega$, where $r_x := \mathrm{dist}(x, \partial\Omega)$.

We state some important equivalent characterizations of the weak* convergence.

Theorem 1.6.12 (Criterions for weak* convergence)**.** *Let $\{\mu_k\}$ be a sequence and μ in $\mathcal{M}^+_{\mathrm{loc}}(\Omega)$. The following are equivalent*

(1) $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

(2) For all $U \subset \Omega$ open and for all $K \subset \Omega$ compact we have

$$\liminf_{k\to+\infty} \mu_k(U) \ge \mu(U), \tag{1.6.4}$$

$$\limsup_{k\to+\infty} \mu_k(K) \le \mu(K). \tag{1.6.5}$$

(3) For all Borel sets $B \Subset \Omega$ such that $\mu(\partial B) = 0$, we have

$$\lim_{k\to+\infty} \mu_k(B) = \mu(B). \tag{1.6.6}$$

Proof. (1) $\Rightarrow$ (2) Let $K \subset U \subset \Omega$ where K is compact and U is open, and choose $\varphi \in C_0(\Omega)$, $\operatorname{supp}\varphi \subset U$, $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on K. By our assumption, we have

$$\int_\Omega \varphi \, d\mu_k \to \int_\Omega \varphi \, d\mu.$$

Hence, we get

$$\mu(K) \le \int_\Omega \varphi \, d\mu = \lim_{k\to+\infty} \int_\Omega \varphi \, d\mu_k \le \liminf_{k\to+\infty} \mu_k(U),$$

and we deduce (1.6.4) by taking the supremum in $K \subset U$ and using the innner regularity of μ. On the other hand, it also clear that we have

$$\mu(U) \ge \int_\Omega \varphi \, d\mu = \lim_{k\to+\infty} \int_\Omega \varphi \, d\mu_k \ge \limsup_{k\to+\infty} \mu_k(K),$$

from which we deduce (1.6.5) by taking the infimum in $U \supset K$ and using the outer regularity.

(2) $\Rightarrow$ (3) Notice that $B = \mathring{B} \cup (\partial B \cap B)$. Therefore, using $\mu(\partial B) = 0$, we have

$$\begin{aligned} \mu(B) = \mu(\mathring{B}) + \mu(\partial B \cap B) = \mu(\mathring{B}) &\le \liminf_{k\to+\infty} \mu_k(\mathring{B}) \\ &\le \limsup_{k\to+\infty} \mu_k(\mathring{B}) \le \limsup_{k\to+\infty} \mu_k(\overline{B}) \le \mu(\overline{B}) = \mu(B). \end{aligned}$$

(3) $\Rightarrow$ (1) Let $\varepsilon > 0$ and $\varphi \in C_c^0(\Omega)$. We need to prove that

$$\int_\Omega \varphi \, d\mu_k \to \int_\Omega \varphi \, d\mu.$$

Let us at first assume $\varphi \geq 0$. Choose

$$0 = t_0 < t_1 < \cdots < t_N := 2\|\varphi\|_\infty,$$

such that $0 < t_i - t_{i-1} < \varepsilon$ and $\mu(\varphi^{-1}\{t_i\}) = 0$. By Remark 1.6.11, it is always possible to choose such good t_i's. Let $B_i = \varphi^{-1}((t_{i-1}, t_i))$, then $\mu(\partial B_i) = 0$. Hence, by (1.6.6) we have

$$\mu_k(B_i) \to \mu(B_i).$$

In addition, it is easy to notice that

$$\sum_{i=2}^N t_{i-1}\mu_k(B_i) \leq \int_\Omega \varphi \, d\mu_k \leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B_0),$$
$$\sum_{i=2}^N t_{i-1}\mu(B_i) \leq \int_\Omega \varphi \, d\mu \leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B_0).$$

Therefore, by the triangle inequality and the subadditivity of the limsup, we have

$$\begin{aligned}
\limsup_{k\to+\infty} \left| \int_\Omega \varphi \, d\mu_k - \int_\Omega \varphi \, d\mu \right| &\leq \limsup_{k\to+\infty} \left| \int_\Omega \varphi \, d\mu_k - \sum_{i=2}^N t_{i-1}\mu_k(B_i) \right| + \\
&+ \left| \sum_{i=2}^N t_{i-1}\mu_k(B_i) - \sum_{i=2}^N t_{i-1}\mu(B_i) \right| + \left| \int_\Omega \varphi \, d\mu - \sum_{i=2}^N t_{i-1}\mu(B_i) \right| \\
&\leq \limsup_{k\to+\infty} t_1\mu_k(B_0) + t_1\mu(B_0) = 2t_1\mu(B_0) \\
&< \varepsilon \mu(\operatorname{supp} \varphi),
\end{aligned}$$

from which we conclude, since ε is arbitrary. Let us now consider the general case of $\varphi : \Omega \to \mathbb{R}$, and consider

$$\psi := \varphi + \|\varphi\|_\infty \eta,$$

for some $\eta \in C_c(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\operatorname{supp}(\varphi)$. It is plain to see that $\psi \geq 0$ and $\psi \in C_c(\Omega)$, so that we have

$$\begin{aligned}
\int_\Omega \varphi \, d\mu_k &= \int_\Omega \psi \, d\mu_k - \int_\Omega \|\varphi\|_\infty \eta \, d\mu_k \\
&\to \int_\Omega \psi \, d\mu - \int_\Omega \|\varphi\|_\infty \eta \, d\mu = \int_\Omega \varphi \, d\mu
\end{aligned}$$

and this ends the proof.

$\square$

Remark 1.6.13. It is possible to improve the implication (1) $\Rightarrow$ (2) of Theorem 1.6.12 to the following statement: if $\mu_k, \mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ for all $k \in \mathbb{N}$ and $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$, then

i) $\displaystyle\liminf_{k\to+\infty} \int_\Omega u \, d\mu_k \geq \int_\Omega u \, d\mu$ for all lower semicontinuous functions $u : \Omega \to [0, +\infty]$,

ii) $\displaystyle\limsup_{k\to+\infty} \int_\Omega v \, d\mu_k \leq \int_\Omega v \, d\mu$ for all upper semicontinuous functions with compact support $v : \Omega \to [0, +\infty)$.

We consider now the case of weakly* convergencing sequences of vector valued Radon measures.

Theorem 1.6.14. *Let a sequence $\{\mu_k\}$ and a measure μ be in $\mathcal{M}(\Omega; \mathbb{R}^m)$. Assume that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$, and that there exists $\nu \in \mathcal{M}^+(\Omega)$ such that $|\mu_k| \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(\Omega)$. Then we have $|\mu| \leq \nu$. Moreover, if a Borel set $E \Subset \Omega$ satisfies $\nu(\partial E) = 0$, then*

$$\mu(E) = \lim_{k\to+\infty} \mu_k(E).$$

Proof. Let $\varphi \in C_0(\Omega;\mathbb{R}^m)$. It is clear that we have

$$\left|\int_\Omega \varphi \cdot d\mu_k\right| \le \int_\Omega |\varphi|\, d|\mu_k|.$$

Therefore, by taking the limit as $k \to +\infty$, we obtain

$$\left|\int_\Omega \varphi \cdot d\mu\right| \le \int_\Omega |\varphi|\, d\nu$$

for any $\varphi \in C_0(\Omega;\mathbb{R}^m)$. Let now $A \subset \Omega$ be an open set: by passing to the supremum in $\varphi \in C_c(A;\mathbb{R}^m)$ with $\|\varphi\|_\infty \le 1$, thanks to Proposition 1.5.5 we get

$$|\mu|(A) \le \nu(A).$$

By the outer regularity of Radon measures, this inequality implies that $|\mu|(B) \le \nu(B)$ for any Borel set B; that is, $|\mu| \le \nu$ in the sense of Radon measures. Let now $E \Subset \Omega$ be a Borel set satisfying $\nu(\partial E) = 0$. Assume at first that $\mathring{E} \ne \emptyset$. For all $\varepsilon > 0$, thanks to the regularity of the Radon measure ν, there exist an open set A and a compact set K such that

$$\overline{A} \subset E \subset \mathring{K} \text{ and } \nu(K \setminus A) \le \varepsilon.$$

For all $\varphi \in C_c(\mathring{K})$ satisfying $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\overline{A}$, and for all $j \in \{1, \dots, n\}$, we have

$$\begin{gathered}
\left|\int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu_k - \mathrm{e}_j \cdot \mu_k(E)\right| \le \int_\Omega |\varphi - \chi_E|\, d|\mu_k| \le |\mu_k|(K \setminus A), \\
\left|\int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu - \mathrm{e}_j \cdot \mu(E)\right| \le \int_\Omega |\varphi - \chi_E|\, d|\mu| \le |\mu|(K \setminus A), \\
\lim_{k \to +\infty} \int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu_k = \int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu,
\end{gathered}$$

where e_j is the j-th coordinate unit vector in $\mathbb{R}^n$. All in all, by the triangle inequality we obtain

$$\begin{aligned}
\limsup_{k \to +\infty} |\mu_k(E) - \mu(E)| &\le \limsup_{k \to +\infty} \sum_{j=1}^n |\mathrm{e}_j \cdot (\mu_k(E) - \mu(E))| \\
&\le \limsup_{k \to +\infty} n|\mu_k|(K \setminus A) + n\nu(K \setminus A) + \sum_{j=1}^n \left|\int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu_k - \int_\Omega \varphi\, \mathrm{e}_j \cdot d\mu\right| \\
&\le 2n\nu(K \setminus A) \le 2n\varepsilon,
\end{aligned}$$

and so the conclusion follows, since $\varepsilon > 0$ is arbitrary. Finally, let $\mathring{E} = \emptyset$. In such case, it is plain to see that $E = E \cap \partial E$, and so

$$|\mu(E)| \le \nu(E) \le \nu(\partial E) = 0,$$

by the hypotheses on E. Thus, we can choose $A = \emptyset$ and repeat the previous steps to obtain the convergence $\mu_k(E) \to 0$. This ends the proof. □

Remark 1.6.15. Under the same hypotheses of Theorem 1.6.14, it is actually possible to prove that, more generally, if $f : \Omega \to \mathbb{R}^m$ is a bounded Borel function with compact support such that the set of its discontinuity points is ν-neglegible, then

$$\lim_{k \to +\infty} \int_\Omega f \cdot d\mu_k = \int_\Omega f \cdot d\mu.$$

Remark 1.6.16. By Remark 1.6.11 and Theorem 1.6.14, we can assert that, if μ_k and μ are positive Radon measures in Ω, then, for any $x \in \Omega$ and $\mathscr{L}^1$-almost every $r \in (0, r_x)$, with $r_x := \mathrm{dist}(x, \partial\Omega)$, we get $\mu(\partial B(x,r)) = 0$ and so, if $\mu_k \xrightharpoonup{*} \mu$, we have

$$\mu_k(B(x,r)) \to \mu(B(x,r)).$$

Moreover, if μ_k and μ are vector valued Radon measures, $\mu_k \xrightharpoonup{*} \mu$ and $|\mu_k| \xrightharpoonup{*} \nu$, then, for any $x \in \Omega$ and $\mathscr{L}^1$-almost every $r \in (0, r_x)$, we get $\nu(\partial B(x,r)) = 0$, and so we have

$$\mu_k(B(x,r)) \to \mu(B(x,r)).$$

Remark 1.6.17. We stress the fact that the inequality $|\mu| \leq \nu$ in Theorem 1.6.14 is strict, in general. Indeed, let $\Omega = \mathbb{R}$ and $\mu_k = \sin(kx)\mathscr{L}^1$. It can be proved that

$$\mu_k \overset{*}{\rightharpoonup} 0 \text{ and } |\mu_k| \overset{*}{\rightharpoonup} \frac{2}{\pi}\mathscr{L}^1 \text{ in } \mathcal{M}_{\mathrm{loc}}(\mathbb{R}).$$

In addition, if we have a sequence of signed Radon measures $\{\mu_k\}$ locally weakly* converging to a signed Radon measure μ, we cannot in general deduce that

$$\mu_k^{\pm} \overset{*}{\rightharpoonup} \mu^{\pm}.$$

Indeed, let Ω be an open set in $\mathbb{R}^n$, $x_0 \in \Omega$ and $\{x_k\}, \{y_k\}$ be two sequences of points in Ω such that $x_k \neq y_k$ for all $k \in \mathbb{N}$, $x_k \to x_0$ and $y_k \to x_0$. Then, it is not difficult to check that the sequence of measures

$$\mu_k = \delta_{x_k} - \delta_{y_k}$$

satisfies

$$\mu_k^+ = \delta_{x_k} \text{ and } \mu_k^- = \delta_{y_k},$$

so that

$$\mu_k \overset{*}{\rightharpoonup} 0, \quad \mu_k^+ \overset{*}{\rightharpoonup} \delta_{x_0} \text{ and } \mu_k^- \overset{*}{\rightharpoonup} \delta_{x_0}$$

in $\mathcal{M}(\Omega)$. Incidentally, we also notice that $|\mu_k| \overset{*}{\rightharpoonup} 2\delta_{x_0} = \nu$, so that $\nu > |\mu|$, where $\mu = 0$ is the weak* limit of μ_k.

We conclude this section by showing that, under some suitable assumptions, the weak* convergence of vector valued Radon measures implies the weak* convergence of the total variations.

Proposition 1.6.18. *Let a sequence $\{\mu_k\}$ and a measure μ be in $\mathcal{M}(\Omega; \mathbb{R}^m)$. Assume that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$, and that $|\mu_k|(\Omega) \to |\mu|(\Omega)$. Then we have $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$ in $\mathcal{M}(\Omega)$.*

Proof. By the weakly* lower semicontinuity of the total variation (Proposition 1.6.8), we have

$$|\mu|(A) \leq \liminf_{k \to +\infty} |\mu_k|(A)$$

for any open set $A \subset \Omega$. Let now $K \subset \Omega$ be a compact set, and let $A = \Omega \setminus K$. Then, we get

$$\begin{aligned} |\mu|(K) = |\mu|(\Omega) - |\mu|(A) &\geq \lim_{k \to +\infty} |\mu_k|(\Omega) - \liminf_{k \to +\infty} |\mu_k|(A) = \limsup_{k \to +\infty} |\mu_k|(\Omega) + \limsup_{k \to +\infty} -|\mu_k|(A) \\ &\geq \limsup_{k \to +\infty} |\mu_k|(\Omega) - |\mu_k|(A) = \limsup_{k \to +\infty} |\mu_k|(K), \end{aligned}$$

by the subadditivity of the limsup. By Theorem 1.6.12, we deduce that $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then, we notice that $\sup_{k \in \mathbb{N}} |\mu_k|(\Omega) < \infty$, since $\{|\mu_k|(\Omega)\}$ is a converging sequence. Thus, by Remark 1.6.2 we conclude that $|\mu_k|$ weakly* converges to $|\mu|$ in $\mathcal{M}(\Omega)$. □

1.7 Mollification of Radon measures

Let us recall that the *convolution* of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$$

for any $x \in \mathbb{R}^n$ for which the integral is well defined. It is easy to notice that, at least formally, the convolution operation is associative and commutative. In addition, it is well known that, under some summability assumptions for the functions f and g, the convolution is well posed: for instance, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ for some $p \geq 1$, then $f * g \in L^p(\mathbb{R}^n)$, with

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

It is also not difficult to see that, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in C^k_c(\mathbb{R}^n)$, for some $k \in \mathbb{N}_0 \cup \{\infty\}$, then $f * g \in C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and its derivatives up to order k are bounded, with

$$\partial^\gamma (f * g) = f * (\partial^\gamma g)$$

for any multi-index $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \le k$. In addition, we have

$$\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g),$$

so that $f * g$ has compact support if both f and g are compactly supported. Moreover, all these properties of the convolution may be extended to the case of functions defined on some open set $\Omega \subset \mathbb{R}^n$, up to some suitable adjustments. The main application of the notion of convolution is the construction of families of regular approximations of a locally summable function. To this purpose, we recall the definition of mollifier$^{\sharp\sharp\sharp}$.

Definition 1.7.1. A function $\rho \in C_c^\infty(B(0,1))$ which satisfies

$$\rho(-x) = \rho(x), \ \rho \ge 0 \text{ and } \int_{\mathbb{R}^n} \rho \, dx = 1$$

is called a *mollifier*. Given a mollifier ρ, we denote by $\{\rho_\varepsilon\}_{\varepsilon>0}$ the family of mollifiers defined by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right).^{\ddagger}$$

It is well known that, given $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for some $p \in [1, \infty)$, and a mollifier ρ, the sequence $\{\rho_\varepsilon * f\}_{\varepsilon>0}$ is smooth and it satisfies $\rho_\varepsilon * f \to f$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, if f is continuous, then $\rho_\varepsilon * f \to f$ uniformly on compact sets of $\mathbb{R}^n$.

In addition, it is a standard result that the convolution operator may be extended to the space of distributions, so that, in particular, we may define the convolution between Radon measures and continuous functions.

Definition 1.7.2. Let $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $g \in C_c(\mathbb{R}^n)$. We denote by $(\mu * g)$ the convolution between f and μ, given by

$$(\mu * g)(x) := \int_\Omega g(x - y) d\mu(y)$$

for all $x \in \mathbb{R}^n$ such that the map $y \mapsto g(x - y)$ is in $C_c(\Omega)$.

In particular, if ρ is a mollifier, then the mollification of a Radon measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, given by

$$(\mu * \rho_\varepsilon)(x) = \int_\Omega \rho_\varepsilon(x - y) d\mu(y),$$

is well defined for all $x \in \Omega^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Theorem 1.7.3 (Properties of the mollifications)**.** *Let $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and ρ be a mollifier. Then $\mu * \rho_\varepsilon \in C^\infty(\Omega^\varepsilon; \mathbb{R}^m)$, with $\partial^\gamma(\mu * \rho_\varepsilon) = \mu * \partial^\gamma \rho_\varepsilon$ for all $\gamma \in \mathbb{N}_0^n$. In addition, we have*

$$(\mu * \rho_\varepsilon)\mathscr{L}^n \overset{*}{\rightharpoonup} \mu, \quad |\mu * \rho_\varepsilon|\mathscr{L}^n \overset{*}{\rightharpoonup} |\mu| \tag{1.7.1}$$

and, for all E Lebesgue measure and $\varepsilon > 0$,

$$\int_E |\mu * \rho_\varepsilon| dx \le |\mu|(E_\varepsilon), \tag{1.7.2}$$

where $E_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, E) \le \varepsilon\}$.

$^{\sharp\sharp\sharp}$The notion of mollifier was introduced in 1944 by Kurt Otto Friedrich (1901-1982) in the groundbreaking paper [9]. The curious origin of this name is recalled by Peter David Lax (1926-) in his commentary on that paper in [10]: *"On English usage Friedrichs liked to consult his friend and colleague, Donald Flanders, a descendant of puritans and a puritan himself, with the highest standard of his own conduct, noncensorious towards others. In recognition of his moral qualities he was called Moll by his friends* (a reference to the character of Moll Flanders, by Daniel Defoe, A/N). *When asked by Friedrichs what to name the smoothing operator, Flanders remarked that they could be named mollifier after himself; Friedrichs was delighted, as on other occasions, to carry this joke into print."*

In addition, the term mollifier is related to the verb 'to mollify', which means 'to smooth over' in a figurative sense. Finally, it should be noticed that the convolution kernels which we now call mollifiers where previously used in 1938 by Sergei Lvovich Sobolev (1908-1989) in the fundamental paper [15], where the Sobolev embedding theorem appeared for the first time.

‡It is easy to notice that the families of mollifiers are a particular type of *approximate identities*.

Proof. The regularity and the differentiability are easy to prove and are left as an exercise. In order to prove the first weak* convergence in (1.7.1), let $\varphi \in C_c(\Omega; \mathbb{R}^m)$ and notice that, by Fubini's theorem, we have

$$\int_\Omega \varphi \cdot (\mu * \rho_\varepsilon) dx = \int_\Omega \int_\Omega \varphi(x) \rho_\varepsilon(x-y) \cdot d\mu(y) dx = \int_\Omega (\varphi * \rho_\varepsilon)(y) \cdot d\mu(y) \to \int \varphi \cdot d\mu(y),$$

where we used $\rho_\varepsilon(x-y) = \rho_\varepsilon(y-x)$ and $\varphi * \rho_\varepsilon \to \varphi$ uniformly. Then, we employ again Fubini's Theorem in order to prove (1.7.2): we have

$$\begin{aligned}\int_E |\rho_\varepsilon * \mu| \, dx &= \frac{1}{\varepsilon^n} \int_E \left| \int_\Omega \rho\left(\frac{x-y}{\varepsilon}\right) d\mu(y) \right| dx \leq \varepsilon^{-n} \int_E \int_\Omega \rho\left(\frac{x-y}{\varepsilon}\right) d|\mu|(y) \, dx \\ &= \varepsilon^{-n} \int_{E_\varepsilon} \int_E \rho\left(\frac{x-y}{\varepsilon}\right) dx \, d|\mu|(y) \leq \varepsilon^{-n} \int_{E_\varepsilon} \int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right) dx \, d|\mu|(y) = |\mu|(E_\varepsilon).\end{aligned}$$

Finally, the weak* convergence of the total variations follows from the weak* convergence of the measure and (1.7.2), and it is left as an exercise. $\square$

Chapter 2

Basic results from Geometric Measure Theory

2.1 Covering theorems and differentiation of measures

In this section we introduce the coverings and differentiation theorems, fundamental tools of Geometric Measure Theory. The exposition is based mostly on [1, Chapter 2].

2.1.1 Covering theorems

We say that a family of sets $\mathcal{F}$ is disjoint if $F \cap F' = \emptyset$ for all $F, F' \in \mathcal{F}$, $F \neq F'$. We notice that, since $\mathbb{R}^n$ is separable, every disjoint family of sets with nonempty interior is at most countable.

Theorem 2.1.1 (Besicovitch covering theorem)**.** *There exists a $\xi_n \in \mathbb{N}$ such that for all families of closed balls $\mathcal{F}$ such that the set $A := \{x \in \mathbb{R}^n \mid \exists \varrho > 0 : \overline{B(x, \varrho)} \in \mathcal{F}\}$ is bounded, there exists at most ξ_n disjoint subfamilies $\mathcal{F}_i \subset \mathcal{F}$ such that*

$$A \subset \bigcup_{i=1}^{\xi_n} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}.$$

Remark 2.1.2. The balls in the statement of Theorem 2.1.1 may be taken to be open.

Theorem 2.1.3 (Consequence of Besicovitch theorem)**.** *Let A be a bounded set and $\varrho : A \to (0, \infty)$. Then there exists $S \subset A$ at most countable such that $A \subset \bigcup_{x \in S} B(x, \varrho(x))$ and such that every point of $\mathbb{R}^n$ belongs to at most ξ_n open balls centered in points of S; that is,*

$$\sum_{x \in S} \chi_{B(x, \varrho(x))}(y) \leq \xi_n$$

for all $y \in \mathbb{R}^n$, where ξ_n is the same constant of Theorem 2.1.1.

Proof. Let $\mathcal{F} := \{B(x, \varrho(x)) \mid x \in A\}$ and apply Besicovitch covering theorem (Theorem 2.1.1): there exists $\xi_n \in \mathbb{N}$ and $\mathcal{F}_i, \dots, \mathcal{F}_{\xi_n}$ disjoint families of open balls (thanks to Remark 2.1.2) such that

$$A \subset \bigcup_{i=1}^{\xi_n} \bigcup_{B \in \mathcal{F}_i} B.$$

Then, we set S to be the set of centers of the balls in the families $\mathcal{F}_i$, for $i \in \{1, \dots, \xi_n\}$. Clearly, each one of these families is at most countable, being disjoint, so that S is at most countable. This ends the proof. □

We employ now these results to show a covering theorem for Radon measure due to Vitali, which is the general version of Lemma 1.2.19. To this purpose we define the notion of fine covering.

Definition 2.1.4. Let $\mathcal{F}$ be a family of closed balls and $A \subset \mathbb{R}^n$. $\mathcal{F}$ is called a *fine covering* of A if

$$\inf \left\{ \varrho > 0 \mid \overline{B(x, \varrho)} \in \mathcal{F} \right\} = 0 \quad \text{for all } x \in A.$$

Theorem 2.1.5 (Vitali covering theorem). *Let A be a bounded Borel set and $\mathcal{F}$ be a fine covering of A. For any $\mu \in \mathcal{M}^+_{\text{loc}}(\mathbb{R}^n)$ there exists $\mathcal{F}' \subset \mathcal{F}$ disjoint such that*

$$\mu\left(A \setminus \bigcup_{\overline{B} \in \mathcal{F}'} \overline{B}\right) = 0$$

Proof. Let $\xi = \xi_n$ be the constant from Besicovitch covering theorem (Theorem 2.1.1), and let

$$\delta := 1 - \frac{1}{2\xi}.$$

It is clear that $\delta \in (0,1)$.

Claim: there exists a finite subfamily $\mathcal{F}_1 \subset \mathcal{F}$ such that

$$\mu\left(A \cap \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B}\right) \geq \frac{1}{2\xi}\mu(A). \tag{2.1.1}$$

In order to prove this claim, we notice that, since $\mathcal{F}$ is a covering of A, there exist disjoint at most countable families $\mathcal{D}_1, \ldots, \mathcal{D}_\xi$ which cover A. In particular, there exists $i \in \{1, \ldots, \xi\}$ such that

$$\mu\left(A \cap \bigcup_{\overline{B} \in \mathcal{D}_i} \overline{B}\right) \geq \frac{1}{\xi}\mu(A).$$

The reason of this lies in the fact that

$$\mu(A) = \mu\left(A \cap \bigcup_{j=1}^{\xi} \bigcup_{\overline{B} \in \mathcal{D}_j} \overline{B}\right) \leq \sum_{j=1}^{\xi} \mu\left(A \cap \bigcup_{\overline{B} \in \mathcal{D}_j} \overline{B}\right).$$

In particular, we can select a finite subfamily $\mathcal{F}_1 \subset \mathcal{D}_i$ such that (2.1.1) holds.

We let now

$$A_1 := A \setminus \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B},$$

and we apply the same procedure to the family

$$\widetilde{\mathcal{F}} := \left\{\overline{B(x,\rho)} \in \mathcal{F} : \overline{B(x,\rho)} \cap \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B} = \emptyset\right\}.$$

Notice that $\widetilde{\mathcal{F}}$ covers A_1, since $\mathcal{F}$ is a fine cover of A. Indeed, if $x \in A_1$, then there exists $\rho > 0$ such that $\overline{B(x,\rho)} \cap \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B} = \emptyset$, being $\mathcal{F}_1$ finite and the balls closed. In this way, we find a disjoint subfamily $\mathcal{F}_2 \subset \widetilde{\mathcal{F}}$ disjoint from $\mathcal{F}_1$ such that

$$\mu\left(A_1 \cap \bigcup_{\overline{B} \in \mathcal{F}_2} \overline{B}\right) \geq \frac{1}{2\xi}\mu(A_1).$$

In addition, we observe that, by (2.1.1),

$$\mu(A_1) = \mu\left(A \setminus \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B}\right) = \mu(A) - \mu\left(A \cup \bigcup_{\overline{B} \in \mathcal{F}_1} \overline{B}\right) \leq \mu(A) - \frac{1}{2\xi}\mu(A) = \delta\mu(A). \tag{2.1.2}$$

By iteration, we construct a collection of finite disjoint family of closed balls $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ and a decreasing family of sets $\{A_k\}_{k \in \mathbb{N}}$, by setting $A_0 = A$ and

$$A_k = A_{k-1} \setminus \bigcup_{\overline{B} \in \mathcal{F}_k} \overline{B}.$$

By (2.1.2), it is easy to see that

$$\mu(A_{k+1}) \leq \delta\mu(A_k),$$

which easily implies

$$\mu(A_k) \leq \delta^k \mu(A). \tag{2.1.3}$$

If we let

$$\mathcal{F}' := \bigcup_{k=1}^{\infty} \mathcal{F}_k,$$

it is easy to see that $\mathcal{F}'$ is countable and disjoint, and satisfies

$$A \setminus \bigcup_{\overline{B} \in \mathcal{F}'} \overline{B} \subset \bigcap_{k=1}^{\infty} A_k.$$

Thus, (2.1.3) implies

$$\mu\left(A \setminus \bigcup_{\overline{B} \in \mathcal{F}'} \overline{B}\right) \leq \lim_{k \to +\infty} \mu(A_k) \leq \mu(A) \lim_{k \to +\infty} \delta^k = 0,$$

and this ends the proof. □

2.1.2 Differentiation of Radon and Hausdorff measures

One of the main applications of the covering theorems from the previous section is the differentiation of measures, which allows us to pass from local properties to global ones.

Theorem 2.1.6 (Lebesgue-Besicovitch differentiation theorem)**.** *Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$, $\lambda \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$, $\lambda \ll \mu$. Then, for μ-a.e. $x \in \operatorname{supp} \mu$, the limit*

$$D_\mu \lambda(x) := \lim_{\varrho \to 0} \frac{\lambda(B(x, \varrho))}{\mu(B(x, \varrho))} \tag{2.1.4}$$

exists in $\mathbb{R}^m$. In addition, we have $\lambda = D_\mu \lambda \, \mu$.

The limit in (2.1.4), sometimes also denoted by $(d\lambda/d\mu)(x)$, is called the *derivative*, or the *density*, of λ with respect to μ.

Corollary 2.1.7. *Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega, \mu; \mathbb{R}^m)$. Then we have*

$$f(x) = \lim_{\rho \to 0} \frac{1}{\mu(B(x, \rho))} \int_{B(x,\rho)} f(y) \, d\mu(y)$$

for μ-a.e. $x \in \Omega$.

Proof. Let $\lambda := f\mu$. It is easy to check that $\lambda \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$. By Theorem 2.1.6, we know that, for μ-a.e. $x \in \Omega$, there exists the limit

$$D_\mu \lambda(x) := \lim_{\rho \to 0} \frac{\lambda(B(x, \rho))}{\mu(B(x, \rho))} = \lim_{\rho \to 0} \frac{1}{\mu(B(x, \rho))} \int_{B(x,\rho)} f(y) \, d\mu(y),$$

and it satisfies $\lambda = D_\mu \lambda \, \mu$. Thus, we have $(f - D_\mu \lambda) \, \mu = 0$, which implies $f(x) = D_\mu \lambda(x)$ for μ-a.e. $x \in \Omega$. □

Remark 2.1.8. If we apply Corollary 2.1.7 to the case $\mu = \mathscr{L}^n$ and $f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$, we obtain the classical version of Lebesgue's differentiation theorem:

$$f(x) = \lim_{\rho \to 0} \fint_{B(x,\rho)} f(y) \, dy = \lim_{\rho \to 0} \frac{1}{\omega_n \rho^n} \int_{B(x,\rho)} f(y) \, dy$$

for $\mathscr{L}^n$-a.e. $x \in \Omega$. In particular, if $f = \chi_E$ for some Lebesgue measurable set E in $\mathbb{R}^n$, we obtain

$$\chi_E(x) = \lim_{\rho \to 0} \frac{|E \cap B(x, \rho)|}{|B(x, \rho)|} = \lim_{\rho \to 0} \frac{|E \cap B(x, \rho)|}{\omega_n \rho^n} \tag{2.1.5}$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Actually, it is not difficult to show that the statement of Corollary 2.1.7 may be refined to a slightly stronger version.

Corollary 2.1.9. *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $f \in L^1(\Omega, \mu; \mathbb{R}^m)$. Prove that for μ-a.e. $x \in \Omega$ we have*

$$\lim_{\rho \to 0} \frac{1}{\mu(B(x,\rho))} \int_{B(x,\rho)} |f(y) - f(x)| \, d\mu(y) = 0. \tag{2.1.6}$$

Any point $x \in \Omega$ for which (2.1.6) holds is called *Lebesgue point* (or *approximate continuity point*) of f.

Hint of the proof. It is enough to apply Corollary 2.1.7 to the measures $\nu_q := |f - q|\mu$, for $q \in \mathbb{Q}$, and then to exploit the fact that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. □

We recall that the Hausdorff measures $\mathscr{H}^\alpha$ for $\alpha \in [0, n)$ are Borel regular, but not Radon (see Remark 1.2.10). Hence, the Lebesgue-Besicovitch differentiation theorem (Theorem 2.1.6) does not apply to such measures. Nevertheless, it is possible to define and to study a notion of density with respect to the Hausdorff measures. Such analysis is relevant in many applications, since it is useful to compare a generic Radon measure μ to $\mathscr{H}^\alpha$ for some $\alpha \in [0, n]$, in order to to have some idea of the "dimensionality" of μ.

Definition 2.1.10. Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $\alpha \in [0, n]$. For all $x \in \Omega$, we define the *upper and lower α-dimensional densities* of μ at x as

$$\Theta^*_\alpha(\mu, x) := \limsup_{\rho \to 0} \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha} \quad \text{and} \quad \Theta_{*\alpha}(\mu, x) := \liminf_{\rho \to 0} \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha}.$$

If $\Theta^*_\alpha(\mu, x) = \Theta_{*\alpha}(\mu, x)$, the common value is denoted by $\Theta_\alpha(\mu, x)$. In the case $\mu = \mathscr{H}^\alpha \llcorner E$, for a $\mathscr{H}^\alpha$-measurable set E such that $\mathscr{H}^\alpha(E) < +\infty$, for simplicity we write

$$\Theta^*_\alpha(E, x) := \Theta^*_\alpha(\mathscr{H}^\alpha \llcorner E, x),\ \Theta_{*\alpha}(E, x) := \Theta_{*\alpha}(\mathscr{H}^\alpha \llcorner E, x) \text{ and } \Theta_\alpha(E, x) := \Theta_\alpha(\mathscr{H}^\alpha \llcorner E, x).$$

Remark 2.1.11. Recall that, by Proposition 1.2.11, $\mu = \mathscr{H}^\alpha \llcorner E$ is a Radon measure, for any $\mathscr{H}^\alpha$-measurable set E such that $\mathscr{H}^\alpha(E) < +\infty$, so that the definitions of $\Theta^*_\alpha(E, x), \Theta_{*\alpha}(E, x)$ and $\Theta_\alpha(E, x)$ are well posed.

In addition, for any $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $\alpha \in [0, n]$, the functions $\Theta^*_\alpha(\mu, \cdot)$ and $\Theta_{*\alpha}(\mu, \cdot)$ are Borel, since the map $x \to \mu(B(x, \rho))$ is lower semicontinuous, for any fixed $\rho > 0$, by Fatou's lemma.

Theorem 2.1.12 (α-dimensional densities of Radon measures)**.** *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $\alpha \in [0, n]$. Then, for all $B \in \mathcal{B}(\Omega)$ and $t > 0$, the following implications hold:*

$$\Theta^*_\alpha(\mu, x) \geq t \ \forall x \in B \Longrightarrow \ \mu \geq t\mathscr{H}^\alpha \llcorner B, \tag{2.1.7}$$
$$\Theta^*_\alpha(\mu, x) \leq t \ \forall x \in B \Longrightarrow \ \mu \leq 2^\alpha t\mathscr{H}^\alpha \llcorner B. \tag{2.1.8}$$

Proof. Without loss of generality, let $t = 1$ and B bounded. Indeed, we may set $\widetilde{\mu} := t\mu$; and, if B is unbounded, we can cover it with countably many bounded sets.

We start by proving (2.1.7). To this purpose, let $B' \in \mathcal{B}(\Omega)$ be a subset of B, $\delta \in (0, 1)$, and $A \supset B'$ be an open bounded set in Ω. Let us consider the open balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset A, 2r_j < \delta$ and

$$\mu(B(x_j, r_j)) \geq (1 - \delta)\omega_\alpha r_j^\alpha.$$

By Theorem 2.1.3, there exists a subfamily $\{B(x_{j_k}, r_{j_k})\}_{k \in \mathbb{N}}$ such that

$$\sum_{k \in \mathbb{N}} \chi_{B(x_{j_k}, r_{j_k})} \leq \xi_n$$

and $B' \subset \bigcup_{k \in \mathbb{N}} B(x_{j_k}, r_{j_k})$. Then, we have

$$\mathscr{H}^\alpha_\delta(B') \leq \sum_{k \in \mathbb{N}} \omega_\alpha r_{j_k}^\alpha \leq \frac{1}{1 - \delta} \sum_{k \in \mathbb{N}} \mu(B(x_{j_k}, r_{j_k})) \leq \frac{\xi_n}{1 - \delta} \mu(A),$$

which easily implies

$$\mathscr{H}^\alpha(B') \leq \xi_n \mu(A) < \infty.$$

This means that $\mathscr{H}^\alpha \llcorner B'$ is a Radon measure, thanks to Proposition 1.2.11. Therefore, we can apply Vitali covering theorem (Theorem 2.1.5) to $\mathscr{H}^\alpha \llcorner B'$ and to the covering of closed balls C in A with diameter less than δ and such that

$$\mu(C) \geq (1-\delta)\omega_\alpha \frac{(\mathrm{diam}(C))^\alpha}{2^\alpha}.$$

Hence, we get a disjoint subfamily $\{C_i\}$, for which we have

$$\mathscr{H}^\alpha_\delta(B') \leq \sum_i \frac{\omega_\alpha}{2^\alpha}(\mathrm{diam}(C_i))^\alpha \leq \frac{1}{1-\delta}\sum_i \mu(C_i) \leq \frac{\mu(A)}{1-\delta}.$$

Thus, we obtain $\mathscr{H}^\alpha(B') \leq \mu(B')$ for any Borel set $B' \subset B$, since A and δ are arbitrary and μ is outer regular (see Theorem 1.1.10). This shows (2.1.7).

We pass now to the proof of (2.1.8). Let $B' \in \mathcal{B}(\Omega)$ be a subset of B, and $\tau > 1$. For any $k \in \mathbb{N}, k \geq 1$ we set

$$B_k := \left\{ x \in B' : \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha} < \tau \ \forall \rho \in \left(0, \frac{1}{k}\right)\right\}.$$

Notice that $B_k \neq \emptyset$, for k large enough, since

$$\limsup_{\rho \to 0} \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha} \leq 1$$

by our assumption. In addition, $B_k \subset B_{k+1}$ and $\bigcup_{k=1}^\infty B_k = B'$. For any k, let $\{C_{i,k}\}$ be a covering of B_k of sets with diameter less than $1/k$ such that

$$\sum_i \omega_\alpha \left(\frac{\mathrm{diam}(C_{i,k})}{2}\right)^\alpha < \mathscr{H}^\alpha_{\frac{1}{k}}(B_k) + \frac{1}{k}. \tag{2.1.9}$$

Without loss of generality, assume $x_i \in B_k \cap C_{i,k}$ and let $C'_{i,k} = B(x_i, 2\rho_{i,k})$, where

$$\rho_{i,k} := \frac{\mathrm{diam}(C_{i,k})}{2}.$$

Therefore, $B_k \subset \bigcup_i C'_{i,k}$ and, by (2.1.9), we have

$$\mu(B_k) \leq \sum_i \mu(C'_{i,k}) \leq \tau \sum_i \omega_\alpha (2\rho_{i,k})^\alpha < \tau\left(2^\alpha \mathscr{H}^\alpha(B') + \frac{1}{k}\right).$$

Then we send $k \to +\infty$ and $\tau \to 1^+$, and this ends the proof. □

We illustrate now two useful consequences of Theorem 2.1.12.

Corollary 2.1.13. *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $\alpha \in [0,n]$. Then we have the following statements:*

1. *$\Theta^*_\alpha(\mu, x) < \infty$ for $\mathscr{H}^\alpha$-a.e. $x \in \Omega$,*
2. *if $\mu(B) = 0$ for some $B \in \mathcal{B}(\Omega)$, then $\Theta_\alpha(\mu, x) = 0$ for $\mathscr{H}^\alpha$-a.e. $x \in B$.*

Proof. We start by showing point 1. Let $D \Subset \Omega$ be a Borel set satisfying

$$D \subset D_\infty := \{x \in \Omega : \Theta^*_\alpha(\mu, x) = \infty\}.$$

Then we have $\Theta^*_\alpha(\mu, x) \geq t$ for any $x \in D$ and $t > 0$, so that, by (2.1.7), we obtain

$$\mu(D) \geq t\mathscr{H}^\alpha(D)$$

for any $t > 0$. Since $\mu(D) < \infty$, this implies $\mathscr{H}^\alpha(D) = 0$. Then, for any $k \in \mathbb{N}, \geq 1$, let

$$D_k := \left\{x \in D_\infty, |x| \leq k, \mathrm{dist}(x, \partial\Omega) \geq \frac{1}{k}\right\}.$$

Recall that $\Theta^*_\alpha(\mu, \cdot)$ is a Borel function (see Remark 2.1.11), so that D_∞ and D_k are Borel sets. It is plain to see that $D_k \subset D_\infty$, $D_k \Subset \Omega$ and $\bigcup_{k=1}^\infty D_k = D_\infty$. Thus, we have

$$\mathscr{H}^\alpha(D_\infty) \le \sum_{k=1}^\infty \mathscr{H}^\alpha(D_k) = 0.$$

As for point 2, let $t > 0$, $B \in \mathcal{B}(\Omega)$ be such that $\mu(B) = 0$, and set

$$E_t := \{x \in B : \Theta^*_\alpha(\mu, x) > t\}.$$

Then, by (2.1.7), we have

$$0 = \mu(B) \ge \mu(E_t) \ge t\mathscr{H}^\alpha(E_t).$$

Since clearly $E_0 = \bigcup_{k=1}^\infty E_{\frac{1}{k}}$, we have

$$\mathscr{H}^\alpha(E_0) \le \sum_{k=1}^\infty \mathscr{H}^\alpha(E_{\frac{1}{k}}) = 0,$$

and this ends the proof. $\square$

Let us consider the case $\mu = \mathscr{H}^\alpha \llcorner E$, for some $\mathscr{H}^\alpha$-measurable set E in $\mathbb{R}^n$ such that $\mathscr{H}^\alpha(E) < +\infty$. We start by considering the extreme values of α:

- if $\alpha = n$, thanks to (2.1.5), we know that $\Theta_n(E, x) = \chi_E(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$;
- if $\alpha = 0$, it is easy to check that

$$\Theta_0(E, x) = \lim_{\rho \to 0} \mathscr{H}^0(E \cap B(x, \rho)) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} = \chi_E(x),$$

  since $\mathscr{H}^0(E) < \infty$ implies that E is finite and therefore discrete.

This means that, for $\alpha \in \{0, n\}$, we have $\Theta_\alpha(E, x) = \chi_E(x)$ for $\mathscr{H}^\alpha$-a.e. $x \in \mathbb{R}^n$. Instead, in the case $\alpha \in (0, n)$, we cannot hope for such a strong result, in general. However, we are able to obtain the following estimates.

Proposition 2.1.14. *Let $\alpha \in [0, n]$ and E be an $\mathscr{H}^\alpha$-measurable set in $\mathbb{R}^n$ such that $\mathscr{H}^\alpha(E) < \infty$. Then we have*

1. $\Theta_\alpha(E, x) = 0$ *for* $\mathscr{H}^\alpha$*-a.e.* $x \notin E$,
2. $2^{-\alpha} \le \Theta^*_\alpha(E, x) \le 1$ *for* $\mathscr{H}^\alpha$*-a.e.* $x \in E$.

Proof. Point 1 follows immediately by applying Corollary 2.1.13 to the Radon measure $\mathscr{H}^\alpha \llcorner E$, since clearly $(\mathscr{H}^\alpha \llcorner E)(\mathbb{R}^n \setminus E) = 0$.

In order to prove point 2, we set

$$B_t := \{x \in E : \Theta^*_\alpha(E, x) \le t\}$$

for $t < 2^{-\alpha}$. By (2.1.8) to the Radon measure $\mathscr{H}^\alpha \llcorner E$, we have

$$\mathscr{H}^\alpha(B_t) = (\mathscr{H}^\alpha \llcorner E)(B_t) \le 2^\alpha t \mathscr{H}^\alpha(B_t),$$

which implies $\mathscr{H}^\alpha(B_t) = 0$. On the other hand, if we set

$$C_t := \{x \in E : \Theta^*_\alpha(E, x) \ge t\}$$

for $t > 1$, and we apply (2.1.7) to $\mathscr{H}^\alpha \llcorner E$, we obtain

$$\mathscr{H}^\alpha(C_t) \ge t\mathscr{H}^\alpha(C_t),$$

which yields $\mathscr{H}^\alpha(C_t) = 0$. $\square$

We notice that we do not have any general result on lower bounds for $\Theta_{*\alpha}(E, x)$, and this is the reason why we cannot ensure the existence of the full limit in general. However, as we shall see in the following, in the case $\alpha = k \in \{1, \dots, n-1\}$ there is a way to characterize the sets for which $\Theta_k(E, x)$ is well defined and equal to 1 for $\mathscr{H}^k$-a.e. $x \in E$.

2.2 Fine properties of Lipschitz functions

We devote this section to the discussion of some properties of Lipschitz functions, which proved to be very useful in the framework of Geometric Measure Theory. The choice of working with Lipschitz functions is due to the fact that such functions have a less rigid structure than C^1-differentiable functions (for instance, extension theorems are much easier to prove, see McShane's lemma), while they enjoy differentiability properties almost everywhere (see Rademacher's theorem).

Lemma 2.2.1 (McShane's lemma)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be a Lipschitz function. Then the function $f^+ : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$f^+(x) := \inf\{f(y) + \mathrm{Lip}(f, E)|x - y| : y \in E\}$$

is Lipschitz and it satisfies $f^+(x) = f(x)$ for all $x \in E$ and $\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, \mathbb{R}^n)$.

Proof. For any $x, z \in \mathbb{R}^n$, by the triangle inequality, we have

$$f^+(x) \le \inf\{f(y) + \mathrm{Lip}(f, E)(|x - z| + |z - y|) : y \in E\} = f^+(z) + \mathrm{Lip}(f, E)|x - z|.$$

Then, interchanging the role of x and z, we immediately get

$$|f^+(x) - f^+(z)| \le \mathrm{Lip}(f, E)|x - z|,$$

which implies $\mathrm{Lip}(f^+, \mathbb{R}^n) \le \mathrm{Lip}(f, E)$. Now, let $x \in E$. It is easy to see that $f^+(x) \le f(x)$. In order to obtain the reverse inequality, notice that

$$f(x) \le f(y) + \mathrm{Lip}(f, E)|x - y|$$

for any $y \in E$, since f is Lipschitz on E. By taking the infimum in $y \in E$, we get $f(x) \le f^+(x)$, so that $f^+(x) = f(x)$ for all $x \in E$. Finally, this identity implies

$$\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, E) \le \mathrm{Lip}(f^+, \mathbb{R}^n),$$

from which we conclude that $\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, \mathbb{R}^n)$. □

Remark 2.2.2. The extension given in McShane's lemma is the largest Lipschitz extension of f preserving the Lipschitz constant. Indeed, let $\tilde{f}$ be another such extension: then, for any $x \in \mathbb{R}^n$ and $y \in E$, we have

$$\tilde{f}(x) \le f(y) + \mathrm{Lip}(E, f)|y - x|,$$

so that, by taking the infimum in $y \in E$, we obtain $\tilde{f}(x) \le f^+(x)$. On the other hand, arguing analogously, one can show that the smaller extension is given by

$$f^-(x) := \sup\{f(y) - \mathrm{Lip}(f, E)|x - y| : y \in E\}.$$

It is not difficult to see that McShane's lemma can be extended to vector valued Lipschitz functions by hands; however, in such a way we loose the equality between the Lipschitz constants.

Corollary 2.2.3. *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\widetilde{f} = f$ on E and $\mathrm{Lip}(\widetilde{f}, \mathbb{R}^n) \le \sqrt{m}\,\mathrm{Lip}(f, E)$.*

Proof. Apply McShane's lemma (Lemma 2.2.1) to each component of f, thus defining

$$\widetilde{f} := (f_1^+, \dots, f_m^+).$$

Then it is easy to see that $\widetilde{f} = f$ on E. As for the Lipschitz constant, notice that

$$|\widetilde{f}(x) - \widetilde{f}(y)|^2 = \sum_{i=1}^{m} |f_i^+(x) - f_i^+(y)|^2 \le m(\mathrm{Lip}(f, E))^2|x - y|^2.$$

This ends the proof. □

A more refined result was found by M. D. Kirszbraun ([8, 2.10.43] and [12, Theorem I.7.2]).

Theorem 2.2.4 (Kirszbraun theorem)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g = f$ on E and $\mathrm{Lip}(g, \mathbb{R}^n) = \mathrm{Lip}(f, E)$.*

A practical consequence of these extension results for Lipschitz functions is that we may always assume, without loss of generality, that our Lipschitz maps are defined on the whole space $\mathbb{R}^n$.

We shall now see that, quite surprisingly, the Lipschitz continuity property is enough to ensure differentiability outside of a Lebesgue negligible set. We start by recalling the notion of differentiability.

Definition 2.2.5. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $x \in \mathbb{R}^n$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0.$$

This linear mapping is denoted by $\nabla f(x)$ or $df(x)$.

Theorem 2.2.6 (Rademacher's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable at $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. In particular, $\nabla f(x)$ is well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ and belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m \times \mathbb{R}^n)$, with*

$$\|\nabla f\|_{L^\infty(K; \mathbb{R}^m \times \mathbb{R}^n)} \leq \mathrm{Lip}(f, K)$$

for any compact set K.

An interesting consequence of this result is that the differential of a Lipschitz function vanishes on the level sets of the function.

Theorem 2.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz and $t \in \mathbb{R}$. Then $\nabla f(x) = 0$ for $\mathscr{L}^n$-a.e. $x \in \{f = t\} := \{y \in \mathbb{R}^n : f(y) = t\}$.*

2.3 The area and Gauss–Green formulas

2.3.1 Linear maps and Jacobians

We recall here some standard definitions and facts from linear algebra.

Definition 2.3.1.

i) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if

$$(Ox) \cdot (Oy) = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

ii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^m$ is *symmetric* if

$$x \cdot (Sy) = (Sx) \cdot y$$

for all $x, y \in \mathbb{R}^n$.

iii) Let $A : \mathbb{R}^n \to \mathbb{R}^m$. The *adjoint* of A is the linear map $A^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$x \cdot (A^* y) = (Ax) \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Proposition 2.3.2.

i) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps. Then we have $A^{**} = A$ and $(A \circ B)^* = B^* \circ A^*$.

ii) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear map. Then $S^* = S$.

iii) If $O : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal linear map, then $n \leq m$ and $O^* \circ O = I$ on $\mathbb{R}^n$.

Theorem 2.3.3 (Polar decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.*

i) If $n \leq m$, there exists a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that
$$L = O \circ S.$$

ii) If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that
$$L = S \circ O^*.$$

Definition 2.3.4 (Jacobian)**.** Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

i) If $n \leq m$, we write $L = O \circ S$ as above, and we define the *Jacobian* of L as
$$\mathbf{J}L := |\det S|.$$

ii) If $n \leq m$, we write $L = S \circ O^*$ as above, and we define the *Jacobian* of L as
$$\mathbf{J}L := |\det S|.$$

In the literature, these two different definitions of Jacobian are also called *n-dimensional Jacobian* (or *area factor*), and m-dimensional *coarea factor*, respectively, and are denoted by $\mathbf{J}_n$ and $\mathbf{C}_m$.

Theorem 2.3.5 (Representation of Jacobian)**.**

i) If $n \leq m$,
$$\mathbf{J}L = \sqrt{\det (L^* \circ L)}.$$

ii) If $n \geq m$,
$$\mathbf{J}L = \sqrt{\det (L \circ L^*)}.$$

Proof. Let $n \leq m$ and $L = O \circ S$, by Theorem 2.3.3. Then we have $L^* = S \circ O^*$, so that
$$L^* \circ L = S \circ O^* \circ O \circ S = S^2,$$
since O is orthogonal and so $O^* \circ O$ is the identity mapping on $\mathbb{R}^n$ (by Proposition 2.3.2). Hence
$$\det (L^* \circ L) = \det S^2 = (\mathbf{J}L)^2.$$
The proof of (ii) is similar. □

Remark 2.3.6. The definition of the Jacobian of L is independent of the choices of O and S, and we have $\mathbf{J}L = \mathbf{J}L^*$.

Proposition 2.3.7 (Cauchy-Binet formula)**.** *If $n \leq m$ and $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then*
$$\mathbf{J}L = \sqrt{\sum_B (\det (B))^2}$$
where the sum is taken over all $n \times n$ minor of any matrix representation of L.

Let now $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, be a Lipschitz map. By Rademacher's theorem (Theorem 2.2.6), f is differentiable $\mathscr{L}^n$-a.e. and therefore the gradient matrix
$$\nabla f(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix} (x)$$
is well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ and can be considered a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Definition 2.3.8. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous and x is a differentiability point, we define the *Jacobian* of f as
$$\mathbf{J}f(x) := \mathbf{J}\nabla f(x).$$

Remark 2.3.9. Notice that $\mathbf{J}f \leq c_n \operatorname{Lip}(f)^n$.

2.3.2 The area formula

Through this subsection we assume $n \le m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ to be Lipschitz continuous.

Lemma 2.3.10. *Let $A \subset \mathbb{R}^n$ be $\mathscr{L}^n$-measurable. Then*

i) $f(A)$ is $\mathscr{H}^n$-measurable,

ii) the mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is $\mathscr{H}^n$-measurable on $\mathbb{R}^m$ and

$$\int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y) \le (\mathrm{Lip}(f))^n \mathscr{L}^n(A)$$

Definition 2.3.11. The mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is the *multiplicity function* of f in A.

Remark 2.3.12. It is easy to notice that $\mathscr{H}^0(A \cap f^{-1}(y))$ is equal to the cardinality of the set of

$$\{x \in A : f(x) = y\},$$

so that $f^{-1}(y)$ is finite for $\mathscr{H}^n$-a.e. $y \in \mathbb{R}^m$. In particular, if f is injective, then

$$\mathscr{H}^0(A \cap f^{-1}(y)) = \begin{cases} 1 & y \in f(A), \\ 0 & y \notin f(A). \end{cases}$$

Theorem 2.3.13 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \le m$. Then, for all $\mathscr{L}^n$-measurable sets $A \subset \mathbb{R}^n$, we have*

$$\int_A \mathbf{J}f(x)\, dx = \int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y). \tag{2.3.1}$$

This means that the $\mathscr{H}^n$-measure of $f(A)$, counting multiplicity, is equal to the integral of the Jacobian of f over A. As an immediate consequence, we deduce a generalization of the classical change of variables formula.

Theorem 2.3.14 (General change of variables)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \le m$. Then, for all $\mathscr{L}^n$-summable functions $g : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, dx = \int_{\mathbb{R}^m} \left(\sum_{x \in f^{-1}(y)} g(x) \right) d\mathscr{H}^n(y). \tag{2.3.2}$$

Corollary 2.3.15 (Injective maps)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \le m$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{L}^n$-summable function, and assume that f is injective on the support of g. Then, we have*

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, dx = \int_{f(\mathbb{R}^n)} g(f^{-1}(y))\, d\mathscr{H}^n(y). \tag{2.3.3}$$

Equivalently, if $h : \mathbb{R}^m \to \mathbb{R}$ is such that $h \circ f$ is $\mathscr{L}^n$-summable and f is injective on the support of h, then we have

$$\int_{\mathbb{R}^n} h(f(x))\, \mathbf{J}f(x)\, dx = \int_{f(\mathbb{R}^n)} h(y)\, d\mathscr{H}^n(y). \tag{2.3.4}$$

If $g = \chi_A$ for some $\mathscr{L}^n$-measurable set A, then

$$\mathscr{H}^n(f(A)) = \int_A \mathbf{J}f(x)\, dx. \tag{2.3.5}$$

Remark 2.3.16. Theorem 2.3.14 and Corollary 2.3.15 hold also in the case $g : \mathbb{R}^n \to [0, +\infty]$ is $\mathscr{L}^n$-measurable; however, the left hand sides of (2.3.2) and (2.3.3) may be equal to $+\infty$. In addition, since any Borel function is Lebesgue measurable, Theorem 2.3.14 and Corollary 2.3.15 are valid for all Borel functions $g : \mathbb{R}^n \to \mathbb{R}$ either nonnegative or $\mathscr{L}^n$-summable.

We list here some remarkable applications of the area formula.

Example 2.3.17 (Length of a curve)**.** Let $n = 1, m \geq 1$. Assume $f : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and injective. It is clear that, for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,

$$\mathbf{J}_1 f(t) = |\dot{f}(t)|.$$

Therefore, for any $a, b \in \mathbb{R}, a < b$, the length of a curve $C := f([a, b])$ is given by

$$\mathscr{H}^1(C) = \int_a^b |\dot{f}|\, dt,$$

thanks to (2.3.5).

Example 2.3.18 (Surface area of a graph)**.** Let $n \geq 1$ and $m = n + 1$. Assume $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ as

$$f(x) := (x, g(x)).$$

Then

$$\nabla f = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix},$$

and so, by Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J} f = \sqrt{1 + |\nabla g|^2}.$$

For any set $G \subset \mathbb{R}^n$, we define the graph of g over G as

$$\Gamma(g, G) := \{(x, g(x)) : x \in G\} \subset \mathbb{R}^{n+1},$$

and we write $\Gamma(g) := \Gamma(g, \mathbb{R}^n)$. Therefore, if G is $\mathscr{L}^n$-measurable, (2.3.5) yields

$$\mathscr{H}^n(\Gamma(g, G)) = \int_G \sqrt{1 + |\nabla g|^2}\, dx,$$

so that $\dim_{\mathscr{H}}(\Gamma(g, G)) = n$ for any $\mathscr{L}^n$-measurable set G. Thus, we see that $\mathscr{H}^n \llcorner \Gamma(g)$ is a locally finite Radon measure on $\mathbb{R}^{n+1}$. In addition, by (2.3.4) and Remark 2.3.16, for any Borel function φ with compact support and any $\mathscr{L}^n$-measurable set G we have

$$\int_{\Gamma(g,G)} \varphi(y)\, d\mathscr{H}^n(y) = \int_G \varphi(x, g(x)) \sqrt{1 + |\nabla g(x)|^2}\, dx. \tag{2.3.6}$$

Example 2.3.19 (Surface area of a parametric hypersurface)**.** Let $n \geq 1$ and $m = n+1$. Assume $f : \mathbb{R}^n \to \mathbb{R}^{n+1}, f = (f^1, \ldots, f^{n+1})$, is Lipschitz and injective. For any $k \in \{1, \ldots, n+1\}$, we define

$$\hat{f}_k := (f^1, \ldots, f^{k-1}, f^{k+1}, \ldots, f^{n+1});$$

that is, the vector valued map f without its k-th component. Then, it is not difficult to see that

$$\mathbf{J} \hat{f}_k = |\det \nabla \hat{f}_k|,$$

and so, as a consequence of Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J} f = \sqrt{\sum_{k=1}^{n+1} (\mathbf{J} \hat{f}_k)^2}.$$

Thus, if we define $\Sigma(f, \Omega) := f(\Omega)$, for any open set $\Omega \subset \mathbb{R}^n$ to be a portiong of the parametric hypersurface, (2.3.5) yields

$$\mathscr{H}^n(\Sigma(f, \Omega)) = \int_\Omega \sqrt{\sum_{k=1}^{n+1} (\mathbf{J} \hat{f}_k)^2}\, dx.$$

2.3.3 The Gauss–Green and integration by parts formulas on regular open sets

We conclude this section by exploiting the area formula to prove the classical Gauss–Green formula on open sets with C^1-regular boundary.

Definition 2.3.20. Let E be an open set in $\mathbb{R}^n$ and $k \in \mathbb{N} \cup \{\infty\}$. We say that E has C^k *boundary* (or *smooth boundary* if $k = \infty$) if for all $x \in \partial E$ there exists $r > 0$ and $\psi \in C^k(B(x,r))$ with $\nabla\psi(y) \neq 0$ for all $y \in B(x,r)$ and

$$B(x,r) \cap E = \{y \in B(x,r) : \psi(y) > 0\},$$
$$B(x,r) \cap \partial E = \{y \in B(x,r) : \psi(y) = 0\}.$$

We define the *inner unit normal to E*, $\nu_E(y)$ for $y \in \partial E$ as

$$\nu_E(y) := \frac{\nabla\psi(y)}{|\nabla\psi(y)|} \quad \text{for all } y \in B(x,r) \cap \partial E.$$

It can be checked that this definition is independent from the choice of ψ and r. Therefore, ν_E can be considered as a vector field defined on the whole of ∂E, and $\nu_E \in C^{k-1}(\partial E; \mathbb{S}^{n-1})$.

Remark 2.3.21. If E is an open set with C^1 boundary, then $\mathscr{H}^{n-1} \llcorner \partial E$ is a (locally finite) Radon measure on $\mathbb{R}^n$. Indeed, we can see this by applying the implicit function theorem. We set

$$p : \mathbb{R}^n \to \mathbb{R}^{n-1} \text{ and } q : \mathbb{R}^n \to \mathbb{R}$$

to be orthogonal projections, so that $x = (p(x), q(x))$, and, for any $s, r > 0$, let

$$D(p(x), s) := \{w \in \mathbb{R}^{n-1} : |p(x) - w| < s\}$$

and

$$C(x,r) := \{y \in \mathbb{R}^n : |p(x-y)| < r, |q(x-y)| < r\}$$

to be a disk in $\mathbb{R}^{n-1}$ and a cylinder in $\mathbb{R}^n$ centered in x, respectively. If $x \in \partial E$ and $r > 0$ satisfy the conditions of Definition 2.3.20, then there exists $s > 0$ and $u \in C^1(D(p(x), s))$ such that $C(x,s) \subset B(x,r)$ and, up to a rotation,

$$\begin{aligned} C(x,s) \cap E &= \{y \in C(x,s) : q(y) > u(p(y))\}, \\ C(x,s) \cap \partial E &= \{y \in C(x,s) : q(y) = u(p(y))\}. \end{aligned} \tag{2.3.7}$$

Then, following the notation of Example 2.3.18, we have

$$\mathscr{H}^{n-1} \llcorner C(x,s) \cap \partial E = \mathscr{H}^{n-1} \llcorner \Gamma(u, D(p(x), s)),$$

so that this is a finite Radon measure, by the area formula (2.3.6). Thus, $\mathscr{H}^{n-1} \llcorner \partial E$ is a locally finite Radon measure in $\mathbb{R}^n$, since we can cover ∂E with countably many cylinders with the properties listed above.

In addition, it is possible to see that the inner unit normal can be written as

$$\nu_E(y) = \frac{(-\nabla u(p(y)), 1)}{\sqrt{1 + |\nabla u(p(y))|^2}} \tag{2.3.8}$$

for any $y \in C(x,s) \cap \partial E$.

Theorem 2.3.22 (Gauss–Green formula)**.** *If E is an open set with C^1 boundary, then for all $\varphi \in C_c^1(\mathbb{R}^n)$ we have*

$$\int_E \nabla\varphi \, dx = -\int_{\partial E} \varphi \, \nu_E \, d\mathscr{H}^{n-1}. \tag{2.3.9}$$

As an immediate consequence of this theorem, we have the following two corollaries: the divergence theorem and the integration by parts formula.

Corollary 2.3.23 (Divergence theorem)**.** *If E is an open set with C^1 boundary, then for all $F \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ we have*

$$\int_E \mathrm{div} F \, dx = -\int_{\partial E} F \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{2.3.10}$$

Corollary 2.3.24 (Integration by parts formula)**.** *If E is an open set with C^1 boundary, then for all $F \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ and $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_E \varphi\, \mathrm{div} F\, dx + \int_E F \cdot \nabla\varphi\, dx = -\int_{\partial E} \varphi\, F \cdot \nu_E\, d\mathscr{H}^{n-1}. \tag{2.3.11}$$

In order to prove Theorem 2.3.22 we need a preliminary result on Lipschitz functions.

Proposition 2.3.25. *If $f \in \mathrm{Lip}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f\, \mathrm{div}\varphi\, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla f\, dx \tag{2.3.12}$$

for all $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Let $\nu \in \mathbb{S}^{n-1}$ and define

$$g_{t,\nu}(x) := \frac{f(x+t\nu) - f(x)}{t}$$

for $t > 0$. It is easy to see that $\|g_{t,\nu}\|_{L^\infty(\mathbb{R}^n)} \le \mathrm{Lip}(f;\mathbb{R}^n)$. By Rademacher's theorem (Theorem 2.2.6), we have that $\nabla f(x)$ exists for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, and it satisfies

$$g_{t,\nu}(x) \to \nabla f(x) \cdot \nu$$

as $t \to 0^+$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. Then, for any $\psi \in C^1_c(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} g_{t,\nu}(x)\, \psi(x)\, dx = \int_{\mathbb{R}^n} \frac{f(x+t\nu) - f(x)}{t}\, \psi(x)\, dx = \int_{\mathbb{R}^n} f(x)\, \frac{\psi(x-t\nu) - \psi(x)}{t}\, dx.$$

Therefore, by taking the limit as $t \to 0^+$, Lebesgue's dominated convergence theorem implies that

$$\int_{\mathbb{R}^n} \nabla f(x) \cdot \nu\, \psi(x)\, dx = \int_{\mathbb{R}^n} f(x)\, \nabla\psi(x) \cdot (-\nu)\, dx.$$

Thus, by taking $\nu = \mathrm{e}_j$ for each $j \in \{1, \ldots, n\}$, we obtain

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}\, \psi\, dx = -\int_{\mathbb{R}^n} f\, \frac{\partial \psi}{\partial x_j}\, dx.$$

Finally, if $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, we conclude that

$$\int_{\mathbb{R}^n} \nabla f \cdot \varphi\, dx = \int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial f}{\partial x_j}\, \varphi_j\, dx = -\sum_{j=1}^n \int_{\mathbb{R}^n} f\, \frac{\partial \varphi_j}{\partial x_j}\, dx = -\int_{\mathbb{R}^n} f\, \mathrm{div}\varphi\, dx.$$

$\square$

Proof of Theorem 2.3.22. Recall the definitions of the projections p, q, the disk $D(p(x), s)$ and the cylinder $C(x, r)$ given in Remark 2.3.21. Given $x \in \partial E$, up to a rotation, we can choose $r, s > 0$ and $u \in C^1(D(p(x), s))$ such that (2.3.7) holds. We claim that

$$\int_E \nabla\varphi\, dx = -\int_{\partial E} \varphi \nu_E\, d\mathscr{H}^{n-1} \tag{2.3.13}$$

for any $\varphi \in C^1_c(C(x, s))$. To show this, let $\delta > 0$ and define a Lipschitz function $f_\delta : \mathbb{R}^n \to \mathbb{R}$ by setting

$$f_\delta(y) = \begin{cases} 1 & \text{if } q(y) > u(p(y)) + \delta, \\ 0 & \text{if } q(y) < u(p(y)) - \delta, \\ \frac{q(y) - u(p(y)) + \delta}{2\delta} & \text{if } |q(y) - u(p(y))| \le \delta, \end{cases}$$

for any $y \in C(x, s)$, and then by extending it to $\mathbb{R}^n$ using McShane's lemma (Lemma 2.2.1). It is easy to see that $f_\delta \to \chi_{C(x,s) \cap E}$ in $L^1(C(x, s))$. Hence, we obtain

$$\int_E \nabla\varphi\, dx = \lim_{\delta \to 0^+} \int_{C(x,s)} f_\delta \nabla\varphi\, dx = -\lim_{\delta \to 0^+} \int_{C(x,s)} \varphi \nabla f_\delta\, dx,$$

where in the last equality we used Proposition 2.3.25. Let

$$F_\delta := \{y \in C(x, s) : |q(y) - u(p(y))| \leq \delta\}.$$

Then, we have $\nabla f_\delta(y) = 0$ for any $y \in C(x, s) \setminus F_\delta$ and

$$\nabla f_\delta(y) = \frac{1}{2\delta}(-\nabla u(p(y)), 1) \ \text{ for any } y \in F_\delta.$$

By Fubini's theorem, we obtain

$$\int_{C(x,s)} \varphi \nabla f_\delta \, dx = \int_{F_\delta} \varphi \nabla f_\delta \, dx = \int_{D(p(x),s)} (-\nabla u(p(y)), 1) \frac{1}{2\delta} \int_{u(z)-\delta}^{u(z)+\delta} \varphi(z, t) \, dt \, dz.$$

Therefore, by exploiting the continuity of φ, we can pass to the limit as $\delta \to 0^+$, and, exploiting (2.3.8) and the area formula (2.3.6), we obtain

$$\begin{aligned}
\int_E \nabla \varphi \, dx &= -\lim_{\delta \to 0^+} \int_{C(x,s)} \varphi \nabla f_\delta \, dx \\
&= -\lim_{\delta \to 0^+} \int_{D(p(x),s)} (-\nabla u(p(y)), 1) \frac{1}{2\delta} \int_{u(z)-\delta}^{u(z)+\delta} \varphi(z, t) \, dt \, dz \\
&= -\int_{D(p(x),s)} (-\nabla u(p(y)), 1) \varphi(z, u(z)) \, dz \\
&= -\int_{D(p(x),s)} \varphi(z, u(z)) \nu_E(z, u(z)) \sqrt{1 + |\nabla u(z)|^2} \, dz \\
&= -\int_{C(x,s) \cap \partial E} \varphi \nu_E \, d\mathscr{H}^{n-1} = -\int_{\partial E} \varphi \nu_E \, d\mathscr{H}^{n-1}.
\end{aligned}$$

We conclude now the proof by a partition of unity argument. Let $\varphi \in C_c^1(\mathbb{R}^n)$ and A be an open bounded set such that $\operatorname{supp} \varphi \cap \partial E \subset A$. By compactness, there exist $\{x_k\}_{k=1}^N \subset A \cap \partial E$ and $\{s_k\}_{k=1}^N \subset (0, +\infty)$ such that

- $C(x_k, s_k) \subset A$,
- $\operatorname{supp} \varphi \cap \partial E \subset \bigcup_{k=1}^N C(x_k, s_k)$,
- $\displaystyle\int_E \nabla(\zeta_k \varphi) \, dx = -\int_{\partial E} \zeta_k \varphi \nu_E \, d\mathscr{H}^{n-1}$ for any $\zeta_k \in C_c^1(C(x_k, s_k))$.

Consider now a partition of unity $\{\zeta_k\}_{k=1}^N$ satisfying

$$\zeta_k \in C_c^1(C(x_k, s_k)), 0 \leq \zeta_k \leq 1 \text{ and } \sum_{k=1}^N \zeta_k(x) = 1 \text{ for any } x \in A.$$

Then, we can find a cutoff function $\zeta_0 \in C_c^1(E)$ such that $0 \leq \zeta_0 \leq 1$ and

$$\sum_{k=0}^N \zeta_k(x) = 1 \text{ for any } x \in E \cap A.$$

It is clear that $\zeta_0 \varphi \in C_c^1(E)$, so that

$$\int_E \nabla(\zeta_0 \varphi) \, dx = 0.$$

Thus, we obtain

$$\int_E \nabla \varphi \, dx = \sum_{k=0}^N \int_E \nabla(\zeta_k \varphi) \, dx = -\sum_{k=1}^N \int_{\partial E} \zeta_k \varphi \nu_E \, d\mathscr{H}^{n-1} = -\int_{\partial E} \varphi \nu_E \, d\mathscr{H}^{n-1}.$$

□

We notice that it is possible to extend Theorem 2.3.22 to a slightly larger family of integration domains.

Definition 2.3.26. We say that an open set E in $\mathbb{R}^n$ has *almost C^1 boundary* if there exists a closed set $M_0 \subset \partial E$ with $\mathscr{H}^{n-1}(M_0) = 0$ and such that $M := \partial E \setminus M_0$ is C^1-regular. M is called the *regular part of ∂E*, while M_0 is the *singular part*.

Theorem 2.3.27. *Let E be an open set with almost C^1 boundary. Then for all $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_E \nabla\varphi \, dx = -\int_M \varphi \, \nu_E \, d\mathscr{H}^{n-1}.$$

Remark 2.3.28. Since the singular part of the boundary of a set with almost C^1 boundary is $\mathscr{H}^{n-1}$-negligible, we see that we actually have

$$\int_E \nabla\varphi \, dx = -\int_{\partial E} \varphi \, \nu_E \, d\mathscr{H}^{n-1}$$

for any $\varphi \in C^1_c(\mathbb{R}^n)$, up to setting $\nu_E \equiv \mathrm{e}_1$ on M_0, for instance.

Chapter 3

BV theory

3.1 Weak derivatives and Sobolev spaces

It is well known that any function $u \in L^1_{\text{loc}}(\Omega)$ induces a distribution L_u, defined by

$$L_u(\varphi) := \int_\Omega \varphi\, u\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. Hence, it is possible to consider the distributional derivatives of L_u of any order $\alpha \in \mathbb{N}_0^n$; that is, the distributions defined by

$$\partial^\alpha L_u(\varphi) := (-1)^{|\alpha|} L_u(\partial^\alpha \varphi)$$

for all $\varphi \in C_c^\infty(\Omega)$. In the particular case in which the distribution $\partial^\alpha L_u$ can be represented by the integration against a locally summable function with respect to the Lebesgue measure, we say that u admits a weak derivative of order α. More precisely, we give the following definition.

Definition 3.1.1 (Weak derivative)**.** Let $u \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. We say that $h_\alpha \in L^1_{\text{loc}}(\Omega)$ is the *weak α-derivative* of u if

$$\int_\Omega u\, \partial^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega \varphi\, h_\alpha\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$, and we write $h_\alpha =: \partial_w^\alpha u$. In particular, we say that $g \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)$ is the *weak gradient* of u if

$$\int_\Omega u\, \text{div}\varphi\, dx = - \int_{\mathbb{R}^n} \varphi \cdot g\, dx$$

for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$, and we write $g =: \nabla_w u$.

It is easy to check that the weak derivatives of any order are uniquely defined. In addition, if a function u admits a classical α-derivative for some $\alpha \in \mathbb{N}_0^n$, then $h_\alpha = \partial^\alpha u$. In addition, in the light of this definition, we see that Proposition 2.3.25 states that the gradient of a Lipschitz function, which is well defined $\mathscr{L}^n$-almost everywhere, also coincides with the weak one.

3.2 Functions of Bounded Variation

Definition 3.2.1. A function $u \in L^1(\Omega)$ is called a *function of bounded variation* if

$$\sup \left\{ \int_\Omega u\, \text{div}\phi\, dx : \phi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\phi\|_\infty \le 1 \right\} < \infty.$$

We denote by $BV(\Omega)$ the space of all functions of bounded variation on Ω.
We say that u is *locally of bounded variation*, and we write $u \in BV_{\text{loc}}(\Omega)$, if $u \in L^1_{\text{loc}}(\Omega)$ and if $\forall$ open set $W \subset\subset \Omega$,

$$\sup \left\{ \int_W u\, \text{div}\phi\, dx : \phi \in C_c^\infty(W; \mathbb{R}^n), \|\phi\|_\infty \le 1 \right\} < \infty.$$

Definition 3.2.2. A measurable set $E \subset \Omega$ is called a *finite perimeter set* in Ω (or a Caccioppoli set) if $\chi_E \in BV(\Omega)$.
A measurable set $E \subset \mathbb{R}^n$ is said to have *locally finite perimeter* in Ω if $\chi_E \in BV_{\text{loc}}(\Omega)$.

Consequently, $D\chi_E$ is an $\mathbb{R}^n$-vector valued Radon measure on Ω whose total variation is $|D\chi_E|$. By the polar decomposition of measures, there exists a $|D\chi_E|$-measurable function with modulus 1 $|D\chi_E|$-a.e., which we denote by ν_E, such that $D\chi_E = \nu_E |D\chi_E|$.
Unless otherwise stated, from now on E will be a set of locally finite perimeter in Ω.

Example 3.2.3. Any open bounded set $E \subset \Omega$ with $\partial E \in C^2$ is a set of finite perimeter in Ω. Indeed, $\forall \phi \in C_c^\infty(\Omega;\mathbb{R}^n)$ with $\|\phi\|_\infty \leq 1$, by the classical Gauss-Green formula we have

$$\begin{aligned}\int_{\Omega\cap E} \text{div}\phi \, dx &= -\int_{\partial(\Omega\cap E)} \phi \cdot \nu_E \, d\mathcal{H}^{n-1} = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1} \\ &\leq \int_{\Omega\cap\partial E} |\phi||\nu_E| \, d\mathcal{H}^{n-1} \leq \mathscr{H}^{n-1}(\Omega\cap\partial E),\end{aligned}$$

where ν_E is the interior unit normal. Taking the supremum over ϕ yields $|D\chi_E|(\Omega) \leq \mathscr{H}^{n-1}(\Omega \cap \partial E)$.
Therefore, E has finite perimeter and so, for any $\phi \in C_c^\infty(\Omega;\mathbb{R}^n)$,

$$\int_\Omega \chi_E \text{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

This implies that $D\chi_E = \nu_E \mathscr{H}^{n-1} \llcorner \partial E$ in $\mathcal{M}(\Omega;\mathbb{R}^n)$, by Riesz Representation Theorem (Theorem 3.2.5), and so $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial E$, which in particular yields

$$|D\chi_E|(\Omega) = \mathscr{H}^{n-1}(\Omega\cap\partial E). \tag{3.2.1}$$

Remark 3.2.4. It can be also shown that every open bounded set with Lipschitz boundary is a set of finite perimeter, with equality (3.2.1) holding, since this is a consequence of the extension theorem for functions of bounded variation (see [EG], Section 5.4). Moreover, any bounded open set Ω satisfying $\mathscr{H}^{n-1}(\partial\Omega) < \infty$ has finite perimeter in $\mathbb{R}^n$ (see [AFP], Proposition 3.62).

Theorem 3.2.5. (Riesz) *Let $u \in BV_{\text{loc}}(\Omega)$, then there exists a unique $\mathbb{R}^n$-vector valued Radon measure μ such that*

$$\int_\Omega u \, \text{div}\phi \, dx = -\int_\Omega \phi \cdot d\mu \quad \forall \phi \in C_c^1(\Omega;\mathbb{R}^n).$$

Proof. We define the linear functional $L : C_c^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$ by

$$L(\phi) := -\int_\Omega u \, \text{div}\phi \, dx, \quad \text{for} \;\; \phi \in C_c^1(\Omega;\mathbb{R}^n).$$

Since $u \in BV_{\text{loc}}(\Omega)$, we have

$$\sup\{L(\phi) : \phi \in C_c^\infty(W;\mathbb{R}^n), \|\phi\|_\infty \leq 1\} = C(W) < \infty$$

for each open set $W \subset\subset \Omega$, and thus

$$|L(\phi)| \leq C(W)\|\phi\|_\infty \;\; \text{for} \;\; \phi \in C_c^1(W;\mathbb{R}^n).$$

We fix any compact set $K \subset \Omega$ and then we choose an open set W such that $K \subset W \subset\subset \Omega$. For each $\phi \in C_c(\Omega;\mathbb{R}^n)$ with $\text{supp}(\phi) \subset K$, we choose a sequence $\phi_k \in C_c^1(W;\mathbb{R}^n)$ such that $\phi_k \to \phi$ uniformly on W. Then we define

$$\bar{L}(\phi) := \lim_{k\to+\infty} L(\phi_k).$$

By the continuity of L on $C_c^1(\Omega;\mathbb{R}^n)$ we have that this limit exists and is independent of the choice of the sequence $\{\phi_k\}$ converging to ϕ. Thus $\bar{L}$ uniquely extends to a linear functional

$$\bar{L} : C_c(\Omega;\mathbb{R}^n) \to \mathbb{R}$$

and

$$\sup\left\{\bar{L}(\phi) : \phi \in C_c^\infty(\Omega;\mathbb{R}^n), \|\phi\|_\infty \leq 1, \mathrm{supp}(\phi) \subset K\right\} < \infty$$

for each compact set $K \subset \Omega$. So, by the Riesz Representation Theorem (Corollary 1.5.7), there exists an $\mathbb{R}^n$-vector valued Radon measure μ satisfying

$$\bar{L}(\phi) = -\int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega, \mathbb{R}^n)$$

and so, since $\bar{L}(\phi) = L(\phi)$ for $\phi \in C_c^1(\Omega, \mathbb{R}^n)$, the result follows. □

This means that the distributional derivative Du of a BV function u is an $\mathbb{R}^n$-vector valued Radon measure.
We write $|Du|$ to indicate its total variation, which is a positive Radon measure on Ω.

Remark 3.2.6. $W^{1,1}(\Omega) \subset BV(\Omega)$ and $|Du|(\Omega) = \|Du\|_{L^1(\Omega;\mathbb{R}^n)}$ for $u \in W^{1,1}(\Omega)$.

Theorem 3.2.7. *If $\{u_n\} \subset BV(\Omega)$ is such that $u_n \rightharpoonup u$ in $L^p(\Omega)$ for some $p \in [1, +\infty)$, or weak-star for $p = +\infty$, or in $L^p_{\mathrm{loc}}(\Omega)$. Then $\forall A \subseteq \Omega$ open*

$$|Du|(A) \leq \liminf_{n\to+\infty} |Du_n|(A).$$

Proof. Indeed, $\forall \phi \in C_c^\infty(A;\mathbb{R}^n)$ we have

$$\int_A u_n \,\mathrm{div}\phi \, dx \to \int_A u \,\mathrm{div}\phi \, dx$$

and so

$$\int_A u \,\mathrm{div}\phi \, dx = \lim_{n\to+\infty} \int_A u_n \,\mathrm{div}\phi \, dx \leq \liminf_{n\to+\infty} |Du_n|(A).$$

Taking the supremum over $\phi \in C_c^\infty(A;\mathbb{R}^n)$ with $\|\phi\|_\infty \leq 1$ on the left hand side, we have the claim. □

Remark 3.2.8. $|Du|(\Omega)$ is a seminorm in $BV(\Omega)$. Clearly it is positively homogeneous and we get subadditivity by observing that

$$\int_\Omega (u_1 + u_2)\mathrm{div}\phi \, dx \leq |Du_1|(\Omega) + |Du_2|(\Omega).$$

Theorem 3.2.9. *The space $BV(\Omega)$ endowed with the norm*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $BV(\Omega)$, then it is Cauchy in $L^1(\Omega)$ and so $\exists u \in L^1(\Omega)$ such that $u_n \to u$ in L^1.
By the lower semicontinuity (Theorem 3.2.7), $u \in BV(\Omega)$.
Moreover, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|D(u_k - u_n)|(\Omega) < \varepsilon, \forall k, n \geq N$.
So, again by lower semicontinuity, $|D(u_k - u)|(\Omega) \leq \liminf_n |D(u_k - u_n)|(\Omega) < \varepsilon$ and from this it follows u_n converges to u in BV norm. □

Theorem 3.2.10. (Meyers-Serrin Approximation theorem)
Let $u \in BV(\Omega)$, then $\exists \{u_n\} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that

1. *$u_n \to u$ in $L^1(\Omega)$*
2. *$|Du_n|(\Omega) \to |Du|(\Omega)$.*

Proof. Fix $\varepsilon > 0$. Given a positive integer m, we set $\Omega_0 = \emptyset$, define for each $k \in \mathbb{N}, k \geq 1$ the sets

$$\Omega_k = \left\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m+k}\right\} \cap B(0, k+m)$$

and then we choose m such that $|Du|(\Omega \setminus \Omega_1) < \varepsilon$.
We define now $\Sigma_k := \Omega_{k+1} \setminus \overline{\Omega}_{k-1}$. Since $\{\Sigma_k\}$ is an open cover of Ω, then there exists a partition of unity subordinate to that open cover; that is, a sequence of functions $\{\zeta_k\}$ such that:

1. $\zeta_k \in C_c^\infty(\Sigma_k)$;
2. $0 \le \zeta_k \le 1$;
3. $\sum_{k=1}^{+\infty} \zeta_k = 1$ on Ω.

Then we take a standard mollifier ρ and $\forall k$ we choose ε_k such that:

$$\operatorname{spt}(\rho_{\varepsilon_k} * (u\zeta_k)) \subset \Sigma_k$$

$$\|\rho_{\varepsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \frac{\varepsilon}{2^k}$$

$$\|\rho_{\varepsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k\|_{L^1(\Omega;\mathbb{R}^n)} < \frac{\varepsilon}{2^k}$$

and we define $u_\varepsilon = \sum_{k=1}^{+\infty} \rho_{\varepsilon_k} * (u\zeta_k)$.
Then $u_\varepsilon \in C^\infty$, since locally there are only a finite number of nonzero terms in the sum.
Also, $u_\varepsilon \to u$ in $L^1(\Omega)$ since

$$\|u - u_\varepsilon\|_{L^1(\Omega)} \le \sum_{k=1}^{+\infty} \|\rho_{\varepsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \varepsilon.$$

Now, since $u_\varepsilon \in L^1(\Omega)$, Theorem 3.2.7 implies $|Du|(\Omega) \le \liminf_{\varepsilon\to 0} |Du_\varepsilon|(\Omega)$.

In order to obtain the reverse inequality, let $\phi \in C_c^\infty(\Omega;\mathbb{R}^n)$, $\|\phi\|_\infty \le 1$. Then

$$\begin{aligned}\int_\Omega u_\varepsilon \operatorname{div}\phi dx &= \sum_{k=1}^{+\infty} \int_\Omega \rho_{\varepsilon_k} * (u\zeta_k)\operatorname{div}\phi dx = \sum_{k=1}^{+\infty} \int_\Omega u\zeta_k \operatorname{div}(\rho_{\varepsilon_k} * \phi) dx \\ &= \sum_{k=1}^{+\infty} \int_\Omega u\operatorname{div}(\zeta_k(\rho_{\varepsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega u\nabla\zeta_k \cdot (\rho_{\varepsilon_k} * \phi) dx.\end{aligned}$$

Using $\displaystyle\sum_{k=1}^{+\infty} \nabla\zeta_k = 0$ in Ω and the properties of the convolution, this last expression equals

$$\sum_{k=1}^{+\infty} \int_\Omega u\operatorname{div}(\zeta_k(\rho_{\varepsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega \phi \cdot (\rho_{\varepsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k) dx =: I_1^\varepsilon + I_2^\varepsilon$$

Now, $|\zeta_k(\rho_{\varepsilon_k} * \phi)| \le 1$ and each point in Ω belongs to at most three of the sets $\{\Sigma_k\}$. Thus

$$|I_1^\varepsilon| \le \left| \int_\Omega u\operatorname{div}(\zeta_1(\rho_{\varepsilon_1} * \phi)) dx \quad + \quad \sum_{k=2}^{+\infty} \int_\Omega u\operatorname{div}(\zeta_k(\rho_{\varepsilon_k} * \phi)) dx \right| \le$$

$$|Du|(\Omega) + \sum_{k=2}^{+\infty} |Du|(\Sigma_k) \le |Du|(\Omega) + 3|Du|(\Omega \setminus \Omega_1) \le |Du|(\Omega) + 3\varepsilon$$

For the second term, we have $|I_2^\varepsilon| < \varepsilon$ directly from our choice of ε_k.

Therefore, after passing to the supremum over ϕ, $|Du_\varepsilon|(\Omega) \le |Du|(\Omega) + 4\varepsilon$, which yields $u_\varepsilon \in BV(\Omega)$ and point 2. $\square$

Remark 3.2.11. If $u \in BV(\mathbb{R}^n)$, then it is much easier to construct the approximating smooth sequence satisfying properties 1) and 2) of Theorem 3.2.10. Indeed, we need just to take $u_\varepsilon = u * \rho_\varepsilon$, where ρ is a standard symmetric mollifier. It is not difficult to show that u_ε strictly converges to u; that is,

i) $\|u_\varepsilon - u\|_{L^1(\mathbb{R}^n)} \to 0$,

ii) $\|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \to |Du|(\mathbb{R}^n)$.

In addition, we have $|\nabla u_\varepsilon| \mathscr{L}^n \overset{*}{\rightharpoonup} |Du|$ in $\mathcal{M}(\mathbb{R}^n)$.

We notice that equality (3.2.1) is not valid in general for sets of finite perimeter, as the following example ([11, Example 1.10]) shows.

Example 3.2.12. Let $n \geq 2$, $\{x_i\}_{i=0}^{\infty} = \mathbb{Q}^n \cap [0,1]^n$, $E = \bigcup_{i=0}^{\infty} B(x_i, \varepsilon 2^{-i})$, with $\varepsilon > 0$ that shall be assigned. We have

$$|E| \leq \sum_{i=0}^{\infty} \omega_n \varepsilon^n 2^{-in} = \frac{\omega_n \varepsilon^n}{1 - 2^{-n}}.$$

Since the rational points are dense in $[0,1]^n$ and E is open, then we have $\overline{E} = [0,1]^n$ and

$$\partial E = \overline{E} \setminus E = [0,1]^n \setminus E.$$

This implies

$$|\partial E| = |\overline{E}| - |E| \geq 1 - \frac{\omega_n \varepsilon^n}{1 - 2^{-n}} > 0$$

for ε small enough. This implies $\mathscr{H}^{n-1}(\partial E) = \infty$, thanks to Proposition 1.2.6.

Thanks to the subadditivity of the perimeter measure, for any $k \in \mathbb{N}$ we have

$$\begin{aligned} P\left(\bigcup_{i=0}^{k} B(x_i, \varepsilon 2^{-i})\right) &\leq \sum_{i=0}^{k} P\left(B(x_i, \varepsilon 2^{-i})\right) = \sum_{i=0}^{k} \mathscr{H}^{n-1}(\partial B(x_i, \varepsilon 2^{-i})) \\ &= \sum_{i=0}^{k} n\omega_n \varepsilon^{n-1} 2^{-(n-1)i} \leq \sum_{i=0}^{\infty} n\omega_n \varepsilon^{n-1} 2^{-(n-1)i} = \frac{n\omega_n \varepsilon^{n-1}}{1 - 2^{-(n-1)}}. \end{aligned}$$

Hence, by the lower semicontinuity of the perimeter, we get

$$P(E) \leq \liminf_{k \to +\infty} P\left(\bigcup_{i=0}^{k} B(x_i, \varepsilon 2^{-i})\right) \leq \frac{n\omega_n \varepsilon^{n-1}}{1 - 2^{-(n-1)}} < \infty.$$

Thus, E is a set of finite perimeter for which $|D\chi_E| \neq \mathscr{H}^{n-1} \llcorner \partial E$.

3.3 The coarea formula

3.3.1 The case $m = 1$

We state now the coarea formula in codimension one, which shows an important connection between BV functions and sets of finite perimeter.

Theorem 3.3.1 (Fleming-Rischel coarea formula)**.** *If $u \in BV(\Omega)$, then for $\mathscr{L}^1$ a.e. $s \in \mathbb{R}$, the set $\{u > s\}$ has finite perimeter in Ω and*

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega) ds.$$

Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega) ds < \infty$, then $u \in BV(\Omega)$. Moreover, for any Borel set $B \subset \Omega$ we have

$$|Du|(B) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(B) ds.$$

Remark 3.3.2. A consequence of Theorem 3.3.1 is that, for any $u \in BV(\Omega)$, $|Du| \ll \mathscr{H}^{n-1}$. Indeed, for any Borel set $B \subset \Omega$ such that $\mathscr{H}^{n-1}(B) = 0$, co-area formula implies $|Du|(B) = 0$.

In the case of a Lipschitz function f

Theorem 3.3.3 (Integration over level sets)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then we have*

$$\int_{\mathbb{R}^n} |\nabla f| \, dx = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}(\{f = t\}) \, dt. \tag{3.3.1}$$

If we assume also that $\operatorname{ess\,inf}|\nabla f| > 0$ *and we let* $g : \mathbb{R}^n \to \mathbb{R}$ *be* $\mathscr{L}^n$*-summable, then for all* $t \in \mathbb{R}$ *we obtain*

$$\int_{\{f>t\}} g\,dx = \int_t^{+\infty} \int_{\{f=s\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds. \tag{3.3.2}$$

In particular,

$$\frac{d}{dt}\int_{\{f>t\}} g\,dx = -\int_{\{f=t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$*-a.e.* $t \in \mathbb{R}$*.*

Proof. It is easy to see that (3.3.1) is a simple consequence of (3.3.3) when $m = 1$. Then, by Theorem 3.3.10, we have

$$\begin{aligned}\int_{\{f>t\}} g\,dx &= \int_{\mathbb{R}^n} \chi_{\{f>t\}} \frac{g}{|\nabla f|}\mathbf{J}f\,dx \\ &= \int_{-\infty}^{+\infty} \int_{\{f=s\}} \chi_{\{f>t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds \\ &= \int_t^{+\infty} \int_{\{f=s\}} \chi_{\{f>t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds.\end{aligned}$$

Then, the final equality follow easily by (3.3.2). □

Lemma 3.3.4. *Let* $u : \Omega \to \mathbb{R}$ *be a Lipschitz function, and let* $A \subset \mathbb{R}^n$ *be a set of measure zero. Then*

$$\mathscr{H}^{n-1}(A \cap u^{-1}(s)) = 0 \quad \text{for} \quad \mathscr{L}^1\text{-a.e. } s \in \mathbb{R}.$$

Proof. It is an immediate consequence of the coarea formula for Lipschitz functions (Theorem 3.3.3):

$$0 = \int_A |\nabla u(x)|dx = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}(A \cap u^{-1}(s))ds.$$

□

We list now some relevant applications of the coarea formula: the polar coordinates' change of variables and the case in which f is the distance function from a compact set.

Theorem 3.3.5 (Polar coordinates)**.** *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be* $\mathscr{L}^n$*-summable. Then*

$$\int_{\mathbb{R}^n} g\,dx = \int_0^{+\infty} \int_{\partial B(0,\rho)} g\,d\mathscr{H}^{n-1}\,d\rho$$

In particular, for any $r > 0$ *and* g *such that* $g\chi_{B(0,r)}$ *is* $\mathscr{L}^n$*-summable, we have*

$$\int_{B(0,r)} g\,dx = \int_0^r \int_{\partial B(0,\rho)} g\,d\mathscr{H}^{n-1}\,d\rho,$$

so that, for $\mathscr{L}^1$*-a.e.* $r > 0$*,*

$$\frac{d}{dr}\int_{B(0,r)} g\,dx = \int_{\partial B(0,r)} g\,d\mathscr{H}^{n-1}.$$

Proof. Apply Theorem 3.3.10 to $f(x) = |x|$, in the case $m = 1$. Then, for all $x \neq 0$, we have

$$\nabla f(x) = \frac{x}{|x|},\ \mathbf{J}f(x) = 1.$$

Finally, the second equality is a consequence of the first, as the third can be derived from the second. □

Theorem 3.3.6 (Integration over the level set of the distance function)**.** *Let $K \subset \mathbb{R}^n$ be a nonempty compact set and set*

$$d(x) := \operatorname{dist}(x, K).$$

Then, for all $0 < a < b$ and all $g : \mathbb{R}^n \to \mathbb{R}$ $\mathscr{L}^n$-summable we have

$$\int_a^b \int_{\{d=t\}} g \, d\mathscr{H}^{n-1} \, dt = \int_{\{a<d\le b\}} g \, dx.$$

In particular,

$$\int_a^b \mathscr{H}^{n-1}(\{d = t\}) \, dt = \mathscr{L}^n(\{a < d \le b\}).$$

Proof. It is enough to prove that $|\nabla d(x)| = 1$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n \setminus K$, and then we need just to apply Theorem 3.3.3. We start by showing the d is Lipschitz. Let $x \in \mathbb{R}^n$: there exists a $c \in K$ such that $|x - c| = d(x)$. By the triangle inequality, we have

$$d(y) - d(x) \le |y - c| - |x - c| \le |x - y|.$$

If we interchange now the roles of x and y, we see that we get

$$|d(y) - d(x)| \le |x - y|,$$

which shows that d is Lipschitz with $\operatorname{Lip}(d) \le 1$. Hence, Rademacher's theorem implies that the function d is differentiable in $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. Let now $x \in \mathbb{R}^n \setminus K$ be such that $\nabla d(x)$ exists. Then we have $|\nabla d(x)| \le 1$. In addition, if we select $c \in K$ as above, we also have

$$d(tx + (1 - t)c) = t|x - c|$$

for all $t \in [0, 1]$ (since the segment is the shortest path). Therefore, by taking a derivative in t, we get

$$|x - c| = \nabla d(x) \cdot (x - c) \le |\nabla d(x)||x - c|,$$

which immediately implies $|\nabla d(x)| \ge 1$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n \setminus K$. Thus, the proof is completed. □

3.3.2 The case $m \ge 1$

In this subsection we assume $n \ge m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ to be Lipschitz.

Lemma 3.3.7. *Let A be $\mathscr{L}^n$-measurable. Then*

i) $A \cap f^{-1}(y)$ is $\mathscr{H}^{n-m}$ measurable for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$.

ii) the mapping $y \to \mathscr{H}^{n-m}(A \cap f^{-1}(y))$ is $\mathscr{L}^m$-measurable, and

$$\int_{\mathbb{R}^m} \mathscr{H}^{n-m}(A \cap f^{-1}(y)) \, dy \le c_{n,m}(\operatorname{Lip}(f))^m \mathscr{L}^n(A).$$

Theorem 3.3.8 (Coarea formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $n \ge m$. Then, for all $\mathscr{L}^n$-measurable sets $A \subset \mathbb{R}^n$, we have*

$$\int_A \mathbf{J}f \, dx = \int_{\mathbb{R}^m} \mathscr{H}^{n-m}(A \cap f^{-1}(y)) \, dy. \tag{3.3.3}$$

Notice that the coarea formula can be seen as a generalized version of Fubini's theorem.

Remark 3.3.9 (Morse-Sard theorem)**.** If we apply the coarea formula to $A = \{\mathbf{J}f = 0\}$, it is immediate to see that

$$\mathscr{H}^{n-m}(\{\mathbf{J}f = 0\} \cap f^{-1}(y)) = 0$$

for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$. This is a weak variant of Morse-Sard theorem, which states that, if $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1 + n - m$, then

$$\{\mathbf{J}f = 0\} \cap f^{-1}(y) = \emptyset$$

for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$.

Theorem 3.3.10. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $n \geq m$. Then, for all $\mathscr{L}^n$-summable functions $g : \mathbb{R}^n \to \mathbb{R}$, we have $g|_{f^{-1}(y)}$ is $\mathscr{H}^{n-m}$-summable for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$, and*

$$\int_{\mathbb{R}^n} g\,\mathbf{J}f\,dx = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} g\,d\mathscr{H}^{n-m}\,dy. \tag{3.3.4}$$

Notice that in the case $m = 1$ we have $\mathbf{J}f = |\nabla f|$, so that we find the previous version.

Remark 3.3.11. Notice that $f^{-1}(y)$ is closed for all $y \in \mathbb{R}^m$, so that it is immediately $\mathscr{H}^{n-m}$-measurable.

3.4 The reduced boundary and the blow-up

Example 3.2.12 shows that the topological boundary of a set of finite perimeter E may be very irregular, so that, if we want to study the properties of $|D\chi_E|$ in relation with the Gauss–Green formula, we should take into consideration suitable subsets of ∂E.

Definition 3.4.1. Let E be a set of locally finite perimeter in Ω, then $x \in \mathscr{F}E$, the *reduced boundary* of E, if $x \in \Omega$ satisfies

1. $|D\chi_E|(B(x,r)) > 0, \forall r > 0$;
2. $\displaystyle\lim_{r\to 0^+} \frac{1}{|D\chi_E|(B(x,r))} \int_{B(x,r)} \nu_E\,d|D\chi_E| = \nu_E(x)$;
3. $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterisation of the reduced boundary, by using the blow-up of the set E around a point of $\mathscr{F}E$.

Definition 3.4.2. For $x \in \mathscr{F}E$ we define the hyperplane

$$H(x) = \{y \in \mathbb{R}^n : \ \nu(x) \cdot (y - x) = 0\}$$

and the half-spaces

$$H^+(x) = \{y \in \mathbb{R}^n : \ \nu(x) \cdot (y - x) \geq 0\},$$
$$H^-(x) = \{y \in \mathbb{R}^n : \ \nu(x) \cdot (y - x) \leq 0\}.$$

Also, for $r > 0$, we set

$$E_r(x) = \{y \in \mathbb{R}^n : \ x + r(y - x) \in E\}$$

Theorem 3.4.3. *If E is a set of finite perimeter in Ω, $x \in \mathscr{F}E$ and $\nu(x) = -\nu_E(x)$, then*

$$\chi_{E_r} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega)$$
$$\chi_{\Omega\setminus E_r} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega)$$

as $r \to 0$.

Proof. See [EG] Section 5.7.2 Theorem 1. □

Formulated in another way, for $r > 0$ small enough, $E \cap B(x,r)$ is approximatively equal to the half ball $H^-(x) \cap B(x,r)$.

Corollary 3.4.4. *If $x \in \mathscr{F}E$ and $\nu(x) = -\nu_E(x)$, then*

1. $\displaystyle\lim_{r\to 0} \frac{1}{r^n}|B(x,r) \cap E \cap H^+(x)| = 0$,
2. $\displaystyle\lim_{r\to 0} \frac{1}{r^n}|(B(x,r) \setminus E) \cap H^-(x)| = 0$.

Proof. We have

$$\frac{1}{r^n}|B(x,r)\cap E\cap H^+(x)| = |B(x,1)\cap E_r\cap H^+(x)| \to |B(x,1)\cap H^-(x)\cap H^+(x)| = 0,$$

by Theorem 3.4.3. Point 2 follows from the same theorem and

$$\begin{aligned}\frac{1}{r^n}|(B(x,r)\setminus E)\cap H^-(x)| &= \frac{1}{r^n}(|B(x,r)\cap H^-(x)| - |B(x,r)\cap E\cap H^-(x)|)\\ &= \frac{\omega_n}{2} - |B(x,1)\cap E_r\cap H^-(x)|\\ &\to \frac{\omega_n}{2} - |B(x,1)\cap H^-(x)| = 0.\end{aligned}$$

□

Using this result, we can give a generalization of the concept of unit interior normal (respectively, unit exterior normal, up to a sign).

Definition 3.4.5. A unit vector $\nu(x) = -\nu_E(x)$ for which property 1 of Corollary 3.4.4 holds is called the *measure theoretic unit exterior normal* to E at x, while, accordingly, $\nu_E(x)$ is called the *measure theoretic unit interior normal* to E at x.

It follows that the measure theoretic unit interior normal ν_E is well defined at least on the reduced boundary.

3.5 Rectifiability and De Giorgi's structure theorem

3.5.1 Rectifiability

We start with the definitions of rectifiable set and approximated tangent space.

Definition 3.5.1. Let $k \in [0,n]$ be an integer and let $S \subset \mathbb{R}^n$ be a $\mathscr{H}^k$-measurable set. We say that S is countably k-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$S \subset \bigcup_i f_i(\mathbb{R}^k).$$

Definition 3.5.2. Let $k \in [0,n]$ be an integer, μ be a Radon measure in Ω and $x \in \Omega$. We say that the approximate tangent space of μ is a k-plane π with multiplicity $\theta \in \mathbb{R}$ in x, and we write

$$\mathrm{Tan}^k(\mu,x) = \theta\mathscr{H}^k \llcorner \pi$$

if $r^{-k}\mu_{x,r}$ locally weak* converges to $\theta\mathscr{H}^k \llcorner \pi$ in Ω as $r \to 0$; that is,

$$\lim_{r\to 0}\frac{1}{r^k}\int_\Omega \phi\left(\frac{y-x}{r}\right)d\mu(y) = \int_\pi \phi(y)\,d\mathscr{H}^k(y)$$

for any $\phi \in C_c(\Omega)$.

3.5.2 De Giorgi's structure theorem

The next theorem shows us that the reduced boundary can be written as a countable union of compact subset of C^1 manifolds, up to a set of Hausdorff dimension at most $n-1$.

Theorem 3.5.3. *Assume E is a set of locally finite perimeter in $\mathbb{R}^n$. Then*

1. *$\mathscr{F}E$ is a $(n-1)$-rectifiable set; that is, there exist a countable family of C^1 manifolds S_k, a family of compact sets $K_k \subset S_k$ and set $\mathcal{N}$ of $\mathscr{H}^{n-1}$-measure zero such that*

$$\mathscr{F}E = \bigcup_{k=1}^{+\infty} K_k \cup \mathcal{N};$$

2. *$\nu_E|_{S_k}$ is normal to S_k;*

3. $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$ *and for* $\mathscr{H}^{n-1}$*-a.e.* $x \in \mathscr{F}E$,

$$\lim_{r \to 0} \frac{|D\chi_E|(B(x,r))}{\omega_{n-1} r^{n-1}} = 1.$$

Proof. See [EG] Section 5.7.3 Theorem 2. □

We introduce now the density of a set at a certain point, in order to select another useful subset of the topological boundary.

Definition 3.5.4. For every $\alpha \in [0,1]$ and every measurable set $E \subset \mathbb{R}^n$, we define

$$E^\alpha := \{x \in \mathbb{R}^n : \Theta_n(E,x) = \alpha\},$$

where

$$\Theta_n(E,x) := \lim_{r \to 0} \frac{|(B(x,r) \cap E)|}{|B(x,r)|}$$

whenever the limit is well defined (see also Definition 2.1.10).

Definition 3.5.5. Referring to Definition 3.5.4,

1. E^1 is called the *measure theoretic interior* of E.

2. E^0 is called the *measure theoretic exterior* of E.

3. The *measure theoretic (or essential) boundary* of E is the set $\partial^* E := \mathbb{R}^n \setminus (E^0 \cup E^1)$.

Remark 3.5.6. It is clear that $E^\circ \subset E^1$ and $\mathbb{R}^n \setminus \overline{E} \subset E^0$. Hence one has

$$\partial^* E \subset \mathbb{R}^n \setminus (E^\circ \cup \mathbb{R}^n \setminus \overline{E}) = \overline{E} \setminus E^\circ = \partial E.$$

Moreover, by the Lebesgue-Besicovitch differentiation theorem (Theorem **??**), $\partial^* E$ has $\mathscr{L}^n$-measure 0, since it is the set of non-Lebesgue points of χ_E.

We further observe that, as in [EG] Section 5.8, it is possibile to define the measure theoretic boundary without using the density of a set.
Indeed the previous definition is equivalent to the following:

Definition ' Let $x \in \mathbb{R}^n$, then $x \in \partial^* E$, the meaure theoretic boundary of E, if the following two conditions hold:

1. $\limsup_{r \to 0} \dfrac{|B(x,r) \cap E|}{r^n} > 0,$

2. $\limsup_{r \to 0} \dfrac{|B(x,r) \setminus E|}{r^n} > 0.$

Theorem 3.5.7. *If* $E \subset \Omega$ *is a set of finite perimeter, then*

$$\mathscr{F}E \subset E^{\frac{1}{2}} \subset \partial^* E, \quad \mathscr{H}^{n-1}(\Omega \setminus (E^0 \cup \mathscr{F}E \cup E^1) = 0.$$

In particular, E *has density either* 0, $\frac{1}{2}$ *or* 1 *at* $\mathscr{H}^{n-1}$*-a.e.* $x \in \Omega$*, and, even if* E *is only locally of finite perimeter,* $\mathscr{H}^{n-1}$*-a.e.* $x \in \partial^* E$ *belongs to* $\mathscr{F}E$*; that is,* $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F}E) = 0$.

Proof. See [EG] Section 5.8 Lemma 1 and [AFP] Theorem 3.61. □

Remark 3.5.8. Since the functions of bounded variations are elements of L^1, they are equivalence class of functions, so that changing the value of any such function on a set of $\mathscr{L}^n$-measure zero does not modify the BV class of the function.
Therefore, this is true also for sets of finite perimeter and we can choose any representative $\widetilde{E}$ for E, which differs only by a set of measure zero, without altering the reduced nor the measure theoretic boundary.

One of the greatest achievements of BV theory is to establish a generalization of the Gauss-Green formula for every set of finite perimeter, though only for differentiable vector fields.

Theorem 3.5.9. (Gauss-Green formula on sets of finite perimeter)
Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then for $\mathscr{H}^{n-1}$ a.e. $x \in \partial^* E$, there is a unique measure theoretic interior unit normal $\nu_E(x)$ such that $\forall \phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ one has

$$\int_E \mathrm{div}\phi\, dx = -\int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

Proof. Since E is a set of locally finite perimeter, $D\chi_E = \nu_E \mathscr{H}^{n-1} \llcorner \mathscr{F}E$ (Theorem 3.5.3), where ν_E is the measure theoretic interior unit normal. Also, Theorem 3.5.7 implies $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F}E) = 0$. Hence, for any $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_\Omega \chi_E \mathrm{div}\phi\, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

□

Remark 3.5.10. Since $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F}E) = 0$ (Theorem 3.5.7), without change, we can integrate on the measure theoretic or on the reduced boundary with respect to the measure $\mathscr{H}^{n-1}$. Since in many practical cases $\partial^* E$ is easier to be determined, Theorem 3.5.9 is often stated in this way. However, since Theorem 3.5.3 states that $|D\chi_E| \ll \mathscr{H}^{n-1} \llcorner \mathscr{F}E$ and the precise representative of χ_E is well defined on $E^1 \cup \mathscr{F}E \cup E^0$ (Lemma 3.5.17 below), in what follows we will always use the reduced boundary in the Gauss-Green formula.

Remark 3.5.11. We also observe that if E is a bounded set of finite perimeter in $\mathbb{R}^n$, then we can drop the assumption on the support of ϕ. Indeed, there exists $R > 0$ such that $\overline{E} \subset B(0,R)$, and so, given $\phi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, we can take $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi = 1$ in $\overline{B(0,R)}$ (which in particular implies $\nabla\varphi = 0$ in E), in order to obtain

$$\begin{aligned}\int_E \mathrm{div}\phi\, dx &= \int_E (\varphi \mathrm{div}\phi + \phi \cdot \nabla\varphi)\, dx = \int_E \mathrm{div}(\phi\varphi)\, dx \\ &= -\int_{\mathscr{F}E} (\phi\varphi) \cdot \nu_E \, d\mathcal{H}^{n-1} = -\int_{\mathscr{F}E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}.\end{aligned}$$

It is also easy to see that if $E \subset\subset \Omega \subset \mathbb{R}^n$, then we can take just $\phi \in C^1(\Omega;\mathbb{R}^n)$.

As in the case of Sobolev functions, it can be shown that for BV functions the precise representative is well defined and it is the limit of the mollified sequence.

Definition 3.5.12. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $a \in \mathbb{R}^n$.
We say that $u_a(x)$ is the approximate limit of u at $x \in \Omega$ restricted to $\Pi_a(x) := \{y \in \mathbb{R}^n : (y-x)\cdot a \geq 0\}$ if, for any $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^n : |u(y) - u_a(x)| \geq \varepsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} = 0$$

Definition 3.5.13. We say that $x \in \Omega$ is a *regular point* of a function $u \in BV(\Omega)$ if there exists a vector $a \in \mathbb{R}^n$ such that the approximate limits $u_a(x)$ and $u_{-a}(x)$ exist. The vector a is called *defining vector.*

Theorem 3.5.14. *Let $u \in BV(\Omega)$. The set of irregular points has $\mathscr{H}^{n-1}$-measure zero.*

Proof. See [VH] Chapter 4 §5.5, or [EG] Section 5.9 Theorem 3. □

Theorem 3.5.15. *Let $u \in BV(\Omega)$ and x be a regular point of u. Then*

1. *If $u_a(x) = u_{-a}(x)$, any $b \in \mathbb{R}^n$ is a defining vector and $u_b(x) = u_a(x)$; that is, x is a point of approximate continuity.*

2. *If $u_a(x) \neq u_{-a}(x)$, then a is unique up to a sign.*

3. *The mollification of u converges to the precise representative u^* at each regular point and $u^*(x) = \frac{1}{2}(u_a(x) + u_{-a}(x))$.*

Proof. See [VH] Chapter 4 §4.4 and Chapter 4 §5.6 Theorem 1, or [EG] Section 5.9 Corollary 1. □

We state now some standard results on the mollification of characteristic functions of sets of finite perimeter.

Remark 3.5.16. By Remark 3.2.11, if E be a set of finite perimeter and $\{\chi_{\delta_k}\}$ denotes the mollification of χ_E, then

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^n)} \leq |D\chi_E|(\mathbb{R}^n)$$

and

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^n)} \to |D\chi_E|(\mathbb{R}^n)$$

We state now some relevant properties of the mollifications of characteristic functions of sets of finite perimeter.

Lemma 3.5.17. *Let $E \subset \Omega$ be a set of locally finite perimeter in Ω and $\rho \in C_c^\infty(B(0,1))$ be a nonnegative radially symmetric mollifier such that $\int_{B(0,1)} \rho \, dx = 1$. Then, the following results hold:*

1. *there is a set $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that, for all $x \in \Omega \setminus \mathcal{N}$, $(\rho_\varepsilon * \chi_E)(x) \to \chi_E^*(x)$ where*
$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1 \\ \frac{1}{2} & \text{if } x \in \mathscr{F}E \; ; \\ 0 & \text{if } x \in E^0 \end{cases} \tag{3.5.1}$$
2. *$\rho_\varepsilon * \chi_E \in C^\infty(\Omega^\varepsilon)$ and $\nabla(\rho_\varepsilon * \chi_E)(x) = (\rho_\varepsilon * D\chi_E)(x)$ for any $x \in \Omega^\varepsilon$;*
3. *one has the following weak* limits in $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^n)$:*
   (a) *$\nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} D\chi_E$;*
   (b) *$\chi_E \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2) D\chi_E$;*
   (c) *$\chi_{\Omega \setminus E} \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2) D\chi_E$;*

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