Lectures Notes

BV functions and sets of finite perimeter

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Introduction

♯ *Geometric Measure Theory* is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces generalizing techniques from differential geometry through measure theoretic arguments. The theory of *functions of bounded variations* and *sets of finite perimeter* is one of the core topics of Geometric Measure Theory, since it deals with extension of the classical notion of Sobolev functions and regular surfaces.

The 1-Laplace operator and BV as a natural extension of $W^{1,1}$

In the Calculus of Variation, the *Direct Method* is a general way of proving the existence of a minimizer for a given functional. More precisely, let X be a topological space and $F : X \to (-\infty, +\infty]$ be a functional. We are interested in finding a minimizer of F in X; that is, a $u \in X$ such that $F(u) \leq F(v)$ for any $v \in X$. Assume that

$$m := \inf\{F(v) \,:\, v \in X\} > -\infty.$$

This ensure the existence of a minimizing sequence $\{v_j\}$; that is, a sequence of elements $v_j \in X$ such that $F(v_j) \to m$. Then, the Direct Method consists in the following steps:

(1) show that $\{v_j\}$ admits a converging subsequence $\{v_{j_k}\}$ and $u \in X$ such that $v_{j_k} \to u$, with respect to a the topology of X;

(2) show that F is (sequentially) lower semicontinuous with respect to the topology of X; that is, if $z_j \to z_0$ in X, then
$$F(z_0) \leq \liminf_{j \to +\infty} F(z_j).$$

If these properties hold true, we can conclude that u is a minimizer of F. Indeed, we have

$$m = \lim_{k \to +\infty} F(v_{j_k}) \geq \liminf_{k \to +\infty} F(v_{j_k}) \geq F(u) \geq m,$$

from which we immediately conclude that $F(u) = \min\{F(v) \,:\, v \in X\}$.

This method is fundamental in proving the existence of solutions to minimization problems related to boundary value problems. Let us consider for instance the classical Dirichlet problem for the Laplace equation on an open set Ω with C^1-smooth boundary:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$. It is possible to see this system as the Euler-Lagrange equations for the functional
$$F(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega f u \, dx$$
defined on the space

$$X = W_0^{1,2}(\Omega) := \{u \in L^2(\Omega) \,:\, Du \in L^2(\Omega; \mathbb{R}^n), u = 0 \text{ on } \partial\Omega\};$$

♯These notes have been written for the course of *BV Functions and Sets of Finite Perimeter* held in the Department of Mathematics of the Hamburg Universität. The main references are the books [?, ?, ?]. Please write an email to giovanni.comi@uni-hamburg.de if you have corrections, comments, suggestions or questions.

that is, the space of 2-summable weakly differentiable Sobolev functions with zero trace on $\partial\Omega$[♯]. As customary, we denote by Du the weak gradient of u. Thanks to Poincaré inequality, we can prove that

$$\inf\{F(u) \,:\, u \in W_0^{1,2}(\Omega)\} > -\infty.$$

Hence, we can find the solution looking for minizers of F through the Direct Method: let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence. It is possible to show that $\{u_j\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, which is an Hilbert space, and in particular reflexive: as a consequence, there exists a subsequence $\{u_{j_k}\}$ converging to some $u \in W_0^{1,2}(\Omega)$ with respect to the weak topology. In addition, F is lower semicontinuous with respect to the weak topology, and so we infer the existence of a solution for the minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|Du|^2 - fu\,dx \,:\, u \in W_0^{1,2}(\Omega)\right\}.$$

It seems natural now to wonder if we could substitute the exponent 2 with any $p \in (1,\infty)$. Thanks to the Poincaré inequality and the reflexivity of the L^p-spaces for $p \in (1,\infty)$, it is indeed possible to show that, for any $f \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the problem

$$\min\left\{\int_\Omega \frac{1}{p}|Du|^p - fu\,dx \,:\, u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution, where

$$W_0^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, Du \in L^p(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

The minimizers to this problem solves the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) =: \Delta_p u$ is the p-Laplace operator.

The next logical step is to consider also the case $p = 1$: for a given $f \in L^\infty(\Omega)$, we want to find a function u which realizes

$$\inf\left\{\int_\Omega |Du| - fu\,dx \,:\, u \in W_0^{1,1}(\Omega)\right\} =: m, \tag{0.0.1}$$

where

$$W_0^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

If we assume $\|f\|_{L^\infty(\Omega)}$ to be sufficiently small, we can again employ the Poincaré inequality to prove that $m \in (-\infty, +\infty]$. Hence, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{1,1}(\Omega)$ such that

$$\lim_{j\to+\infty} \int_\Omega |Du_j| - fu_j\,dx = m.$$

However, in this case we cannot argue as above in the case $p > 1$, since, in general this does *not* imply that the existence of a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ weakly converging to some $u \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |Du| - fu\,dx = m.$$

The reason for this lies in the fact that $L^1(\Omega)$ is not reflexive, and actually it is not the topological dual of any separable space. However, $L^1(\Omega)$ is contained in the space of finite Radon measures on Ω, $\mathcal{M}(\Omega)$, and this space can be see as the dual of the space of continuous functions vanishing on the boundary of Ω, $C_0(\Omega)$.

This fact suggests the definition of a space which contains the Sobolev space $W^{1,1}(\Omega)$ and which, although not reflexive, enjoys the property that bounded sets are weakly* compact: the space of *functions with bounded variation*,

$$BV(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n)\}.$$

[♯]We refer to [?, Chapter 5] and to [?, Chapter 4] for a detailed account on Sobolev spaces.

It is not difficult to prove that the total variation of the Radon measure Du over Ω is indeed lower semicontinuous with respect to the weak* converge of the gradient measures. This indicates that the correct space where to look solutions to (0.0.1) is the space of functions with bounded variation with zero trace[♭],

$$BV_0(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

Finally, it is relevant to mention the fact that the minimizers to (0.0.1) solve the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) =: \Delta_1 u$ is the 1-Laplace operator, which is non trivially defined on nonsmooth functions because of the highly degenerate term $\frac{\nabla u}{|\nabla u|}$.

Minimal area problems and sets of finite perimeter

Other historically relevant problems from the Calculus of Variation are the minimal area problems, among which the most famous example is the *Euclidean isoperimetric problem*: find the possibly unique set with minimal surface area among those with fixed volume. In mathematical terms, if we denote by $|F|$ the n-dimensional volume of a set $F \subset \mathbb{R}^n$ (hence, its Lebesgue measure $\mathscr{L}^n(F)$) and by $\sigma_{n-1}(\partial F)$ its surface area (under the assumption the ∂F is regular enough), we are looking for the set which realizes

$$\inf \{\sigma_{n-1}(\partial F) \,:\, \partial F \in \mathcal{R}, |F| = k\} =: \gamma_k,$$

where $\mathcal{R}$ is a class of sufficiently smooth surfaces and $k > 0$. The Direct Method now consists in considering a minimizing sequence of sets F_j such that

$$\partial F_j \in \mathcal{R}, \quad |F_j| = k \quad \text{and} \quad \sigma_{n-1}(\partial F_j) \to \gamma_k, \tag{0.0.2}$$

and then in trying to prove the convergence (possibly up to subsequences) to some limit set E such that

$$\partial E \in \mathcal{R}, \quad |E| = k \quad \text{and} \quad \sigma_{n-1}(\partial E) = \gamma_k.$$

In order to achieve this result, some compactness property in the family of sets satisfying (0.0.2) is required. In addition, the surface measure σ_{n-1} need to be a lower semicontinuous with respect to the chosen convergence of sets, in the sense that

$$\sigma_{n-1}(\partial E) \le \liminf_{j\to+\infty} \sigma_{n-1}(\partial F_j)$$

if $F_j \to E$ in a suitable sense. However, these compactness and lower semicontinuity properties in general fail to be true in family of sets with regular topological boundary. In addition, we notice that the topological boundary is very unstable under modification of a set by Lebesgue negligible sets. For instance, let

$$E_1 = B(0,1) \quad \text{and} \quad E_2 = B(0,1) \cup (\partial B(0,2) \cap \mathbb{Q}^n).$$

It is plain to see that $|E_1 \Delta E_2| = 0$, so that these two sets are equivalent with respect to the Lebesgue measure, and so they have the same volume. However, their topological boundary, which are smooth surfaces, are very different:

$$\partial E_1 = \partial B(0,1) \quad \text{and} \quad \partial E_2 = \partial B(0,1) \cup \partial B(0,2).$$

The need of ruling out these problems and of recovering a notion of compactness and a lower semicontinuity property for the surface area is one of the main reasons for the birth of Geometric Measure Theory. This theory concerns methods to study the geometric properties of rough, irregular sets from a measure theoretic point of view. In this course we shall see how to exploit this

[♭] It can be proved that the trace of a function with bounded variation is well defined on any C^1-regular surface, as in the Sobolev case.

approach to give a meaningful notion of surface area for an irregular set and to define a suitable class of sets for which we can apply the Direct Method of the Calculus of Variation in order to deal with minimal area problems: the *sets of finite perimeter*. Broadly speaking, the notion of set of finite perimeter extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to minimal area problems and other similar geometric variational minimization problems. More precisely, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ is its characteristic function χ_E is a function with locally bounded variation.

Chapter 1

Notions of abstract Measure Theory

1.1 General measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*; that is, the collection of all subsets of X.

Definition 1.1.1 (Measures)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ if $A \subset \bigcup_{k=1}^{\infty} A_k$ (σ-subadditivity),

is called a measure.

It should be noticed that in the literature a mapping as the one in Definition 1.1.1 is also called an *outer measure*, while the name of measure is used to denote the restriction of the mapping to the family of measurable set (see Definition 1.1.4 below). We shall nevertheless follow the notation of [**?**], in order to be able to assign a measure even to nonmeasurable sets.

Remark 1.1.2. Thanks to σ-subadditivity, any measure is not decreasing; that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition 1.1.3 (Restriction of a measure)**.** If $Y \subset X$, the *restriction of μ to Y*, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$ for any $A \subset X$.

Definition 1.1.4 (μ-measurable sets)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark 1.1.5. This definition is meaningful, since the italian mathematician *Giuseppe Vitali* proved in 1905 that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathscr{L}^1$-measurable [**?**]. For a modern presentation of his construction, we refer to [**?**, Section I.1.2].

Definition 1.1.6 (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if the following conditions hold:

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for any $A \in \mathfrak{F}$ we have $X \setminus A \in \mathfrak{F}$,

(3) for any countable family of sets $\{A_i\}_{i \in I}$ such that $A_i \in \mathfrak{F}$ for any $i \in I$ we have have

$$\bigcup_{i \in I} A_i \in \mathfrak{F}.$$

Theorem 1.1.7. *Given any measure μ on X, the family of μ-measurable sets forms a σ-algebra.*

Theorem 1.1.8. *Let μ be a measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu\left(A_j\right).$$

We list now some relevant definitions.

Definition 1.1.9.

(1) Given any $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.

(2) The *Borel σ-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel σ-algebra are called *Borel sets.*

(3) A measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (i.e. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem 1.1.10. *Let μ be a Radon measure on $\mathbb{R}^n$. Then we have*

(1) $\mu(A) = \inf\{\mu(U) : U \supset A,\, U$ open$\}$ for all $A \subseteq \mathbb{R}^n$ (outer regularity),

(2) $\mu(B) = \sup\{\mu(K) : K \subset B,\, K$ compact$\}$ for all μ-measurable sets B (inner regularity).

Theorem 1.1.11 (Carathéodory's criterion)**.** *Let μ be a measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ such that* $\operatorname{dist}(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

then μ is a Borel measure.

Not any Borel regular measure is a Radon measure. However, it is possible to obtain a Radon measure as a restriction of a Borel regular one, as stated in the followin theorem.

Theorem 1.1.12. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Example 1.1.13 (Dirac delta)**.** For $x \in \mathbb{R}^n$ we define the *Dirac*$^\sharp$ *measure centered in x* by setting

$$\delta_x(A) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

It is easy to check that this is indeed a Radon measure. In addition, any set in $\mathbb{R}^n$ is δ_x-measurable.

Example 1.1.14 (The counting measure)**.** We define the *counting measure* by setting

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure, since it is clearly not locally finite.

$^\sharp$Named after Paul Adrien Maurice Dirac (1902-1984), English theoretical physicist who shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory". He actually introduced the so-called *Dirac delta function* as a "convenient notation" in his influential 1930 book *The Principles of Quantum Mechanics.* The name "delta function" was chosen since it works like a continuous analogue of the discrete Kronecker delta

$$\delta_{i\,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Indeed, for any sequence $\{a_j\}_{j\in\mathbb{Z}}$, we have

$$\sum_{j=-\infty}^{\infty} a_j \delta_{i\,j} = a_i,$$

and, analogously, for any $x \in \mathbb{R}^n$ and any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Dirac delta satisfies the property

$$\int_{-\infty}^{+\infty} f(y)\delta(x-y)\, dy = \int_{-\infty}^{\infty} f(y)\, d\delta_x(y) = f(x).$$

Example 1.1.15 (The Lebesgue measure)**.** The well-known *Lebesgue measure* is defined by

$$\mathscr{L}^n(A) := \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes} \right\},$$

where $\mathscr{L}^n(Q_i) = (l(Q_i)^n$ and $l(Q_i)$ is the side length of the cube Q_i. It is actually possible to show that in one dimension we have

$$\mathscr{L}^1(A) = \inf \left\{ \sum_{ij=1}^{\infty} \operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R} \right\}$$

and that we can characterize $\mathscr{L}^n$ in an alternative way as

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}} = \mathscr{L}^{n-1} \times \mathscr{L}^1.$$

1.2 The Hausdorff measure

Definition 1.2.1 (Hausdorff content)**.** Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *α-dimensional Hausdorff content* of A as

$$\mathscr{H}^\alpha_\delta(A) := \inf \left\{ \sum_{j \in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2} \right)^\alpha \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taken over all the (at most countable) *δ-coverings* $\{C_j\}_{j \in I}$ of A, and we set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2} + 1\right)}.$$

We notice that $\mathscr{H}^\alpha_\delta(A)$ is a non-decreasing function in δ, so that we can take the limit as $\delta \searrow 0$ and it always exists in the extended real numbers. This justifies the following definition.

Definition 1.2.2 (Hausdorff measure)**.** For any $A \subset \mathbb{R}^n$ and $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of A as

$$\mathscr{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathscr{H}^\alpha_\delta(A) = \sup_{\delta > 0} \mathscr{H}^\alpha_\delta(A).$$

Roughly speaking, we take the limit as $\delta \searrow 0$ since it forces the coverings to follow the local geometry of the set A. Indeed, the key idea behind the definition of the Hausdorff measure is that it should be able to capture the properties of thin sets in $\mathbb{R}^n$ (in particular, Lebesgue negligible sets). As we shall see in the following, if $\alpha = k \in \{1, \dots, n-1\}$, then $\mathscr{H}^k$ agrees with the k-dimensional surface area on sufficiently regular sets, as for instance k-dimensional planes.

It is not too difficult to prove that, as a consequence of Carathéodory's criterion, Theorem 1.1.11, any Borel set is $\mathscr{H}^\alpha$-measurable, for any $\alpha \geq 0$.

Theorem 1.2.3 (Hausdorff measures are Borel regular)**.** *$\mathscr{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem 1.2.4 (Basic properties of the Hausdorff measure)**.** *The following statements hold true:*

(1) $\mathscr{H}^0 = \#$;

(2) $\mathscr{H}^1 = \mathscr{H}^1_\delta = \mathscr{L}^1$ on $\mathbb{R}$, for any $\delta > 0$;

(3) $\mathscr{H}^\alpha \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$;

(4) $\mathscr{H}^\alpha(\lambda A) = \lambda^\alpha \mathscr{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$;

(5) $\mathscr{H}^\alpha(L(A)) = \mathscr{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$, we have $\mathscr{H}^0_\delta(\{x\}) = 1$ for every $x \in \mathbb{R}^n$ and $\delta > 0$. Indeed,

$$\omega_0 \left(\frac{\operatorname{diam}(C_j)}{2}\right)^0 = 1,$$

which implies $\mathscr{H}^0_\delta(\{x\}) \geq 1$, and, on the other hand, we can clearly cover the singleton only with itself. Hence, $\mathscr{H}^0(\{x\}) = 1$ for every $x \in \mathbb{R}^n$. Since $\mathscr{H}^0$ is a Borel measure, it is σ-additive on Borel sets, so that

$$\mathscr{H}^0(A) = \sum_{x \in A} \mathscr{H}^0(\{x\}) = \#A,$$

for any finite or countable set A. Finally, if A is incountable, then A contains a countable set B, and so $\mathscr{H}^0(A) \geq \mathscr{H}^0(B) = +\infty$.

(2) We estimate the Lebesgue measure $\mathscr{L}^1$ from both sides by the Hausdorff measure. Since $\omega_1 = 2 = |(-1,1)|$, for any $\delta > 0$ we first get

$$\begin{aligned}\mathscr{L}^1(A) &= \inf\left\{\sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j\right\} \\ &\leq \inf\left\{\sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j,\ \operatorname{diam} C_j \leq \delta\right\} = \mathscr{H}^1_\delta(A),\end{aligned}$$

Now, we define a partition of $\mathbb{R}$ by setting $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$. These are intervals of diameter δ, so that, for every $j \in I$, we get

$$\operatorname{diam}(C_j \cap J_{k,\delta}) \leq \delta. \tag{1.2.1}$$

In addition, we have

$$\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \leq \operatorname{diam} C_j, \tag{1.2.2}$$

since $\{J_{k,\delta}\}_{k\in\mathbb{Z}}$ is a partition $\mathbb{R}$ of essentially disjoint intervals, because $\#(J_{k,\delta} \cap J_{m,\delta}) \leq 1$ for any $k \neq m$. Therefore, by (1.2.2) we get

$$\begin{aligned}\mathscr{L}^1(A) &= \inf\left\{\sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j\right\} \\ &\geq \inf\left\{\sum_{j\in I}\sum_{k\in\mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \mid A \subset \bigcup_{j \in I}\bigcup_{k\in\mathbb{Z}} C_j \cap J_{k,\delta}\right\}.\end{aligned}$$

We set now $C_j \cap J_{k,\delta} =: \widetilde{C}_{i_{j,k}}$, by relabeling the indexes sets I and $\mathbb{Z}$ to an index set $\widetilde{I}$. Thanks to (1.2.1), we have $\operatorname{diam}(\widetilde{C}_i) \leq \delta$ and so we get

$$\mathscr{L}^1(A) \geq \inf\left\{\sum_{j\in \widetilde{I}} \widetilde{C}_i \mid A \subset \bigcup_{i \in \widetilde{I}} \widetilde{C}_i,\ \operatorname{diam} \widetilde{C}_i \leq \delta\right\} \geq \mathscr{H}^1_\delta(A).$$

All in all, we get $\mathscr{L}^1 = \mathscr{H}^1_\delta$ for any $\delta > 0$, from which it easily follows $\mathscr{L}^1 = \mathscr{H}^1$ on $\mathbb{R}$.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. It is easy to see that, for any fixed $m \in \mathbb{N}$, Q can be covered by m^n smaller cubes Q_i with side length $\frac{1}{m}$. Clearly, we have $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$. Therefore, we obtain

$$\mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2}\right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m}\right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha},$$

from which we deduce that, since $n < \alpha$,

$$\mathscr{H}^\alpha(Q) = \lim_{m\to\infty} \mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} \lim_{m\to\infty} m^{n-\alpha} = 0.$$

Thus, the claim easily follows, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes and $\mathscr{H}^n$ is σ-subadditive.

The proofs of (4) and (5) are left as an exercise. $\square$

Lemma 1.2.5. *Let $A \subset \mathbb{R}^n$ and $\delta_0 > 0$ such that $\mathscr{H}^\alpha_{\delta_0}(A) = 0$, then we have $\mathscr{H}^\alpha(A) = 0$.*

Proof. Since the Hausdorff content is non-increasing in δ, we have $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_\delta(A)$ for any $\delta > 0$. In particular, this means that $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_{\delta_0}(A) = 0$, so that, for every $\varepsilon > 0$, there exists a countable family of sets $\{C_j\}_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

In particular, the second condition immediately implies

$$\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon.$$

Hence, we have $\mathscr{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$, and $\delta_\varepsilon \searrow 0$ if and only if $\varepsilon \searrow 0$. This implies the claim $\mathscr{H}^\alpha(A) = 0$. $\square$

Proposition 1.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$.

(2) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a countable family of subsets $\{C_j\}_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathscr{H}^s_\delta(A) + 1 \le \mathscr{H}^s(A) + 1.$$

From this, it follows that

$$\begin{aligned} \mathscr{H}^t_\delta(A) \le \sum_{j\in I} \omega_t \left(\frac{\operatorname{diam} C_j}{2}\right)^t &= \frac{\omega_t}{\omega_s} 2^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathscr{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0, \end{aligned}$$

which implies the claim $\mathscr{H}^t(A) = 0$.

(2) If by contradiction $\mathscr{H}^s(A) < \infty$, then by (1) if follows that $\mathscr{H}^r(A) = 0$ for all $r > s$ and in particular for $r = t$, which is clearly absurd.

$\square$

Definition 1.2.7. We call the *Hausdorff dimension*[♯] of a set $A \subset \mathbb{R}^n$ the number

$$\dim_{\mathscr{H}}(A) := \inf\left\{\alpha \ge 0 : \mathscr{H}^\alpha(A) = 0\right\}.$$

Remark 1.2.8. Let $\alpha = \dim_{\mathscr{H}}(A)$. Then one has

$$\mathscr{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and} \qquad \mathscr{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha. \tag{1.2.3}$$

The first part of (1.2.3) follows clearly from the definition of the Hausdorff dimension. The second, instead, can be proved by contradiction. Suppose by contradiction that $\mathscr{H}^t(A) < \infty$ for some $t < \alpha$, then, by the Proposition 1.2.6, we have $\mathscr{H}^r(A) = 0$ for all $r > t$. This implies

$$\alpha = \inf\left\{\beta \ge 0 : \mathscr{H}^\beta(A) = 0\right\} \le t < \alpha,$$

which is clearly absurd.

It should be noticed that, in general, $\mathscr{H}^\alpha(A)$ can be any number in $[0, +\infty]$.

[♯]The interested reader may find a detailed exposition on Hausdorff's and other related concepts of dimension in the monograph [?].

We state now an important result on the equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure, whose proof we postpone to the end of the section.

Theorem 1.2.9. *$\mathscr{H}^n_\delta = \mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$, for any $\delta > 0$.*

Remark 1.2.10. As a consequence of Theorem 1.2.9, we see that $\mathscr{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Indeed, it is not bounded on some compact sets. Take for example the closed unit ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that

$$\mathscr{H}^n(\overline{B(0,1)}) = \mathscr{L}^n(\overline{B(0,1)}) = \omega_n \in (0,\infty)$$

and so, by Proposition 1.2.6, $\mathscr{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

Even though $\mathscr{H}^\alpha$ is not a Radon measure for $\alpha \in [0, n)$, it is possible to show that its restriction to some suitable Borel set is indeed a Radon measure.

Proposition 1.2.11. *If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^\alpha(E) \in (0,\infty)$, then $\mathscr{H}^\alpha \llcorner E$ is a Radon measure.*

Proof. It is a simple consequence of Theorem 1.1.12. □

We investigate now the behaviour of the Hausdorff measure under the action of Lipschitz and Hölder functions. We recall first the definition of such family of functions.

Definition 1.2.12 (Lipschitz and Hölder functions)**.** Let $E \subset \mathbb{R}^n$.

(1) We say that $f : E \to \mathbb{R}^m$ is *Lipschitz continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y| \quad \text{for any } x, y \in E. \tag{1.2.4}$$
The smallest constant for which (1.2.4) holds is called the *Lipschitz constant* of f on E and it is denoted by $\mathrm{Lip}(f, E)$.

(2) We say that $f : E \to \mathbb{R}^m$ is *locally Lipschitz continuous* on E if, for all compact sets $K \subset E$, f is Lipschitz continuous on K.

(3) Let $\gamma \in (0, 1)$. We say that $f : E \to \mathbb{R}^m$ is *γ-Hölder continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y|^\gamma \quad \text{for any } x, y \in E. \tag{1.2.5}$$

(4) We say that $f : E \to \mathbb{R}^m$ is *locally γ-Hölder continuous* on E if, for all compact sets $K \subset E$, f is γ-Hölder continuous on K.

From this point on, we shall refer to Lipschitz continuous and Hölder continuous functions simply as Lipschitz and Hölder functions.

Exercise 1.2.13. Show that any Lipschitz or γ-Hölder function (for some $\gamma \in (0, 1)$) is indeed continuous.

Exercise 1.2.14. Show that the Lipschitz constant of f on E satisfies

$$\mathrm{Lip}(f, E) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in E, x \neq y\right\}. \tag{1.2.6}$$

This equality can indeed be used as an alternative definition.

Remark 1.2.15. It is easy to notice that Lipschitz functions can be seen as 1-Hölder functions. In addition, for any open set $\Omega \subset \mathbb{R}^n$ and any $\gamma \in [0, 1]$, we can define the space $C^{0,\gamma}(\overline{\Omega}; \mathbb{R}^m)$ of bounded γ-Hölder functions as the set of continuous bounded functions $f : \overline{\Omega} \to \mathbb{R}^m$ for which there exists a constant $C > 0$ such that (1.2.5) holds. If $\gamma = 0$, we have $C^{0,0}(\overline{\Omega}; \mathbb{R}^m) = C^0(\overline{\Omega}; \mathbb{R}^m)$. Such spaces may be equipped with the following norms:

$$\|f\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} + [f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)},$$

where

$$\|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} := \sup_{x\in\overline{\Omega}} |f(x)|,$$

$$[f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|^\gamma} : x, y \in \overline{\Omega}, x \neq y \right\},$$

for $\gamma \in (0,1]$, while we set $\|f\|_{C^{0,0}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)}$. It is not difficult to see that, for all $\gamma \in [0,1]$, $C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)$ equipped with the norm $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)}$ is a Banach space.

Exercise 1.2.16. Let $\gamma > 1$ and $f : \Omega \to \mathbb{R}^m$ be such that there exists a constant $C > 0$ such that (1.2.5) holds. Show that f is constant.

Proposition 1.2.17. *Let $\alpha \geq 0$, $A \subset \mathbb{R}^n$.*

(1) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then $\mathscr{H}^\alpha(f(A)) \leq (\mathrm{Lip}(f))^\alpha \mathscr{H}^\alpha(A)$.

(2) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is γ-Hölder, for some $\gamma \in (0,1)$, then $\mathscr{H}^\alpha(f(A)) \leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}(A)$.

Proof. Thanks to Remark 1.2.15, it is enough to prove (2) for any $\gamma \in (0,1]$. Fix $\delta > 0$, and take a countable family of sets $\{C_j\}_{j\in I}$ such that $A \subset \bigcup_{j\in I} C_j$ and $\operatorname{diam} C_j \leq \delta$. It is clear that

$$f(A) \subseteq \bigcup_{j\in I} f(C_j).$$

Thanks to (1.2.5), we see that $f(C_j)$ satisfies

$$\operatorname{diam} f(C_j) \leq C\,(\operatorname{diam} C_j)^\gamma \leq C\delta^\gamma,$$

where $C = \mathrm{Lip}(f)$ is $\gamma = 1$. Hence, we obtain

$$\mathscr{H}^\alpha_{C\delta^\gamma}(f(A)) \leq \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} f(C_j)}{2}\right)^\alpha \leq \underbrace{\frac{\omega_\alpha}{2^\alpha}\frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}} \sum_{j\in I} \omega_{\alpha\gamma}\left(\frac{\operatorname{diam} C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all δ-coverings $\{C_j\}_{j\in I}$ we get

$$\mathscr{H}^\alpha_{C\delta^\gamma}(g(A)) \leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}_\delta(A),$$

where $C_{\alpha,\gamma} = \mathrm{Lip}(f)^\alpha$, if $\gamma = 1$. By sending $\delta \searrow 0$ we conclude the proof. □

Example 1.2.18 (Sierpinski triangle[♭])**.** We provide an example on the estimation of the Hausdorff measure for a set with non integer Hausdorff dimension. Let us construct a self-similar fractal in $\mathbb{R}^2$ in the following way:

1. Take S_0 to be an equilateral triangle with side length 1.
2. Divide each side in half, then connect the three middle points, so that S_0 becomes the union of four congruent equilateral triangles. Then, remove the open triangle in the center and denote by S_1 the union of the three remaining closed triangles with side length 1/2.
3. Now repeat the step in 2. in each one of the three equilateral triangles in S_1 in order to generate nine triangles of side length 1/4 which form S_2.

By iterating this procedure k times, we construct the set S_k as the union of 3^k equilateral triangles with side length 2^{-k}. Notice that $S_{k+1} \subset S_k$ and each one of the S_k's is compact and nonempty. Hence, we define the *Sierpinski triangle* as the set

$$S := \bigcup_{k=0}^{\infty} S_k,$$

[♭]Fractal described by Waclaw Sierpinski in 1915, [?], and appearing in Italian art from the 11th century [?].

which is therefore compact and nonempty. Since the area of an equilateral triangle of side length l is $\frac{\sqrt{3}}{4}l^2$, so that we have

$$\mathscr{L}^2(S) \leq \mathscr{L}^2(S_k) = 3^k \frac{\sqrt{3}}{4} 4^{-k}$$

for any $k \geq 0$, so that, by taking the limit as $k \to +\infty$, we conclude that $\mathscr{L}^2(S) = 0$. We proceed now to estimate the Hausdorff measure of S. We notice that

$$S_k = \bigcup_{j=1}^{3^k} S_{k,j},$$

if we denote by $S_{k,j}$ the j-th equilateral triangle of the k-th iteration step. It is not difficult to see that $\operatorname{diam}(S_{k,j}) = 2^{-k}$. Therefore, since clearly $S \subset S_k$, for any $k \geq 0$, we see that, by choosing $\delta = 2^{-k}$, we obtain the following estimate

$$\mathscr{H}^{\alpha}_{\frac{1}{2^k}}(S) \leq \sum_{j=1}^{3^k} \frac{\omega_\alpha}{2^\alpha} (\operatorname{diam} S_{k,j})^\alpha = \frac{\omega_\alpha}{2^\alpha} 3^k 2^{-k\alpha},$$

which goes to zero for $k \to \infty$ if $\alpha > \frac{\log^3}{\log^2}$. Thus, we can conclude that, for all $\alpha > \frac{\log 3}{\log 2}$, we have $\mathscr{H}^\alpha(S) = 0$, and this yields an upper bound on the Hausdorff dimension of S:

$$\dim_{\mathcal{H}}(S) \leq \frac{\log 3}{\log 2}.$$

We come now to the proof of Theorem 1.2.9, which is crucially based on the two following statements.

Lemma 1.2.19 (Vitali covering property for $\mathscr{L}^n$)**.** *For all open U and for all $\delta > 0$ there exists a family of disjoint closed balls $\{\overline{B_k}\}_{k=1}^\infty$ such that* $\operatorname{diam} B_k < \delta$ *and* $\mathscr{L}^n(U \setminus \bigcup_{k=1}^\infty \overline{B_k}) = 0$.

Theorem 1.2.20 (Isodiametric inequality)**.** *For all $\mathscr{L}^n$-measurable sets $E \subset \mathbb{R}^n$ we have*

$$|E| \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n.$$

Proof of theorem 1.2.9. The proof consists of three steps.

(Step 1) Claim: $\mathscr{L}^n(A) \leq \mathscr{H}^n_\delta(A)$ for all $A \subset \mathbb{R}^n$ and for all $\delta > 0$.
Fix $\delta > 0$. Let $\{C_j\}_{j \in I}$: $A \subset \bigcup_{j \in I} C_j$, $\operatorname{diam} C_j \leq \delta$. By the σ-subadditivity of the Lebesgue measure, we have

$$\mathscr{L}^n(A) \leq \sum_{j=1}^\infty \mathscr{L}^n(C_j) \leq \sum_{j=1}^\infty \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n,$$

where in the last inequality we used the *isometric inequality*, Theorem 1.2.20. Taking the infimum over all such δ-coverings $\{C_j\}_{j \in J}$, we obtain the claim

$$\mathscr{L}^n(A) \leq \mathscr{H}^n_\delta(A) \qquad \text{for all} \quad \delta > 0.$$

(Step 2) Claim: for all $\delta > 0$, there exists $C_n \geq 1$ such that $\mathscr{H}^n_\delta \leq C_n \mathscr{L}^n$.
Notice that for any cube Q we have

$$\mathscr{L}^n(Q) = \left(\frac{\operatorname{diam} Q}{\sqrt{n}}\right)^n.$$

By the definition of the Lebesgue measure we get

$$\begin{aligned}
\mathscr{L}^n(A) &= \inf\left\{\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j)\mid A\subset\bigcup Q_j\right\}\\
&= \inf\left\{\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j)\mid A\subset\bigcup Q_j,\ \operatorname{diam}Q_j\le\delta\right\}\\
&= \frac{2^n}{(\sqrt{n})^n\,\omega_n}\inf\left\{\sum_{j=1}^{\infty}\omega_n\left(\frac{\operatorname{diam}Q_j}{2}\right)^n\mid A\subset\bigcup Q_j,\ \operatorname{diam}Q_j<\delta\right\}\\
&\ge \frac{1}{C_n}\mathscr{H}^n_\delta(A),
\end{aligned}$$

where in the second equality we used the fact that

$$\mathscr{L}^n=\underbrace{\mathscr{L}^1\times\cdots\times\mathscr{L}^1}_{n-\text{times}},\quad\text{and}\quad\mathscr{L}^1=\mathscr{H}^1_\delta\quad\text{in }\mathbb{R}\quad\text{for all }\delta>0.$$

(Step 3) Claim: $\mathscr{H}^n_\delta(A)\le\mathscr{L}^n(A)+\varepsilon$ for any $\varepsilon>0$.
By the definition of $\mathscr{L}^n$ we see that, for all fixed $\delta,\varepsilon>0$, there exists a family $\{Q_j\}_{j=1}^{\infty}$ such that $A\subset\bigcup_{j=1}^{\infty}Q_j$, $\operatorname{diam}Q_j\le\delta$ and $\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j)\le\mathscr{L}^n(A)+\varepsilon$. Now, by Lemma 1.2.19, there exists a family $(\overline{B^i_j})_{i=1}^{\infty}$ of disjoint closed balls such that $\overline{B^i_j}\subset Q_j$ for all ($\operatorname{diam}B^i_j\le\delta$) and

$$\mathscr{L}^n\left(Q_j\setminus\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)=\mathscr{L}^n\left(\mathring{Q}_j\setminus\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)=0.$$

Therefore, by Step 2 we also have

$$\mathscr{H}^n_\delta\left(Q_j\setminus\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)=0,$$

from which we deduce that

$$\begin{aligned}
\mathscr{H}^n_\delta(A)&\le\sum_{j=1}^{\infty}\mathscr{H}^n_\delta(Q_j)=\sum_{j=1}^{\infty}\mathscr{H}^n_\delta\left(\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)\\
&=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\mathscr{H}^n_\delta\left(B^i_j\right)\le\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\underbrace{\omega_n\left(\frac{\operatorname{diam}B^i_j}{2}\right)^n}_{=\mathscr{L}^n\left(\overline{B^i_j}\right)}\\
&=\sum_{j=1}^{\infty}\mathscr{L}^n\left(\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)=\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j)\\
&\le\mathscr{L}^n(A)+\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this inequality ends the proof. □

Proof of the isodiametric inequality (Theorem 1.2.20). Without loss of generality, we may assume E to be compact. Indeed, notice that $\operatorname{diam}A=\operatorname{diam}\overline{A}$, and, if $\operatorname{diam}E=+\infty$, the inequality is trivially true.

Next, observe that, if $E\subset B(x,\frac{\operatorname{diam}E}{2})$ for some $x\in\mathbb{R}^n$, then there is nothing to prove. We employ Steiner symmetrization[♭] in order to reduce ourselves to such a case.

Decompose $\mathbb{R}^n=\mathbb{R}^{n-1}\times\mathbb{R}^1$ and let $p:\mathbb{R}^n\to\mathbb{R}^{n-1}$, $q:\mathbb{R}^n\to\mathbb{R}$ be the orthogonal projections,

$$p(x)=(x_1,\dots,x_{n-1}),\quad q(x)=x_n,$$

[♭] Introduced in 1838 by Jakob Steiner [?].

so that

$$x = (p(x), q(x)) \quad \text{and} \quad |x|^2 = |p(x)|^2 + |q(x)|^2.$$

T hen, for any $z \in \mathbb{R}^{n-1}$ we define the *verital section*

$$E_z := \{t \in \mathbb{R} : (z,t) \in E\},$$

and, as a consequence, we introduce the *symmetrization* of E with respect n-th coordinate axis:

$$E^s := \left\{ x \in \mathbb{R}^n : |q(x)| \leq \frac{\mathscr{L}^1(E_{p(x)})}{2} \right\}.$$

By Fubini's theorem, E_z is $\mathscr{L}^1$-measurable for $\mathscr{L}^{n-1}$-a.e. z, $z \mapsto \mathscr{L}^1(E_z)$ is Lebesgue measurable and so we get

$$|E| = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z^s)dz = |E^s|, \tag{1.2.7}$$

where the first equality follows by Fubini's theorem, and the second one is a consequence of the fact that

$$(E^s)_z = \{t \in \mathbb{R} : (z,t) \in E^s\} = \left\{ t \in \mathbb{R} : |t| \leq \frac{\mathscr{L}^1(E_z)}{2} \right\} = \left[-\frac{\mathscr{L}^1(E_z)}{2}, \frac{\mathscr{L}^1(E_z)}{2} \right].$$

Now we claim that

$$\operatorname{diam} E^s \leq \operatorname{diam} E. \tag{1.2.8}$$

In order to prove this, let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned} &p(m(x)) = p(M(x)) = p(x) \\ &q(m(x)) \leq q(z) \leq q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x). \end{aligned}$$

Hence, for all $x, y \in E^s$, we have

$$|q(x) - q(y)| \leq \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\},$$

in particular, without loss of generality, we can assume that

$$\max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\} = |q(M(x)) - q(m(y))|,$$

so that

$$|q(x) - q(y)| \leq |q(M(x)) - q(m(y))|.$$

As a consequence, we see that

$$\begin{aligned} |x-y|^2 &= |p(x-y)|^2 + |q(x-y)|^2 \leq |p(M(x)) - p(m(y))|^2 + |q(M(x)) - q(m(y))|^2 \\ &= |M(x) - m(y)|^2 = \max\{|M(x) - m(y)|, |M(y) - m(x)|\}^2 \leq (\operatorname{diam} E)^2. \end{aligned}$$

This means that $|x - y| \leq \operatorname{diam} E$ for all $x, y \in E^s$, which immediately implies (1.2.8).

Given a $\mathscr{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. Hence, if we set $E_0 := E$, $E_i := (E_{i=1})^i$ with $i \in \{1, 2, \ldots, n\}$, then, by (1.2.7) we have $|E_n| = |E|$ and $\operatorname{diam} E_n \leq \operatorname{diam} E$ by (1.2.8). In addition, we notice that, if $x \in E_n$, then $-x \in E_n$, which implies $E_n \subset B\left(0, \frac{\operatorname{diam} E_n}{2}\right)$. Thus, we conclude that

$$|E| = |E_n| \leq \omega_n \left(\frac{\operatorname{diam} E_n}{2} \right)^n \leq \omega_n \left(\frac{\operatorname{diam} E}{2} \right)^n.$$

And so we are done! $\square$

1.3 Integration and Radon measures

In this section, let $X \neq \emptyset$, and μ be a measure on X. Recall the definition of the *extended real line*

$$\overline{\mathbb{R}} := [-\infty, \infty].$$

Definition 1.3.1.

(1) A function $u : X \to \overline{\mathbb{R}}$ is *μ-measurable* if the *superlevel set*

$$\{u > t\} := \{x \in X : u(x) > t\}$$

is μ-measurable for all $t \in \overline{\mathbb{R}}$.

(2) u is a *μ-simple function* if it is μ-measurable and $u(X)$ is countable; that is,

$$u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x),$$

for some sequences of real numbers $\{u_k\}$ and of μ-measurable disjoint sets $\{E_k\}$.

(3) If u is a non-negative μ-simple function, we define

$$\int_X u \, d\mu := \sum_{t \in u(X)} t\mu(\{u = t\}) = \sum_{k=1}^{\infty} u_k \mu(E_k) \in [0, \infty]$$

with the convention that $0 \cdot \infty = 0$.

(4) We set $u^{\pm} := \max\{\pm u, 0\}$, so that $u = u^+ - u^-$ and $|u| = u^+ + u^-$. If u is μ-simple and $\int_X u^+ d\mu$ or $\int_X u^- d\mu < \infty$, then we define

$$\int_X u \, d\mu := \int_X u^+ d\mu - \int_X u^- d\mu \in [-\infty, \infty] \tag{1.3.1}$$

If u satisfies (1.3.1), then it is called *μ-integrable simple function.*

(5) If u is μ-measurable, we define the *upper and lower integrals* of u as

$$\int_X^* u \, d\mu := \inf \left\{ \int_X v \, d\mu \mid v \geq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function} \right\}$$

or

$$\int_{*X} u \, d\mu := \sup \left\{ \int_X v \, d\mu \mid v \leq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function} \right\}$$

respectively. If

$$\int_{*X} u \, d\mu = \int_X^* u \, d\mu,$$

then u is *μ-integrable.*

(6) A measurable function u is *μ-summable* if $|u|$ is μ-integrable and

$$\int_X |u| \, d\mu < \infty.$$

Example 1.3.2 (Integral with respect to the Dirac measure)**.** Let $x_0 \in X$ and $\mu = \delta_{x_0}$. Notice that any subset in X is δ_{x_0}-measurable, so that any function $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-measurable. Then, any $u : X \to \overline{\mathbb{R}}$ simple function is μ-integrable. Indeed, assuming at first $u : X \to [0, \infty]$, for some sequence of nonnegative real numbers $\{u_k\}$ and a partition $\{E_k\}$ of X, we have

$$u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x),$$

so that

$$\int_X u\,d\delta_{x_0} = \sum_{k=1}^{\infty} u_k \delta_{x_0}(E_k) = u_{k_0},$$

where k_0 satisfies $E_{k_0} \ni x_0$, which implies $u(x_0) = u_{k_0}$. Therefore, we can easily see that

$$\int_X u\,d\delta_{x_0} = u^+(x_0) - u^-(x_0) = u(x_0)$$

for any simple function u. As a consequence, any $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-integrable. Indeed, for any simple function $v \geq u$ and any simple function $w \leq u$, we have

$$w(x_0) \leq \quad {}_*\!\!\int_X u\,d\delta_{x_0} \leq \int_X^* u\,d\delta_{x_0} \leq v(x_0),$$

so that we get

$${}_*\!\!\int_X u\,d\delta_{x_0} = \int_X^* u\,d\delta_{x_0} = u(x_0),$$

by choosing

$$v(x) = \begin{cases} u(x_0) & x = x_0, \\ +\infty & x \neq x_0, \end{cases}$$

and

$$w(x) = \begin{cases} u(x_0) & x = x_0, \\ -\infty & x \neq x_0. \end{cases}$$

Thus, we conclude that, for any $u : X \to \overline{\mathbb{R}}$, we have

$$\int_X u\,d\delta_{x_0} = u(x_0),$$

and that u is δ_{x_0}-summable if and only if $|u(x_0)| < \infty$.

We define now general versions of the familiar L^p-function spaces.

Definition 1.3.3. Let $p \in (1, \infty)$.

$L^1(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mu\text{-summable}\right\}$ and we set

$$\|u\|_{L^1(X,\mu)} := \int_X |u|\,d\mu.$$

$L^1_{\text{loc}}(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^1(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact}\right\}.$

$L^p(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} \mid |u|^p \text{ is } \mu\text{-summable}\right\}$ and we set

$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u|^p\,d\mu\right)^{\frac{1}{p}}.$$

$L^p_{\text{loc}}(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^p(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact}\right\}.$

Let $u : X \to \overline{R}$ be μ-measurable. We set

$$\|u\|_{L^\infty(X,\mu)} := \inf\{\lambda > 0 : \mu(\{|u| > \lambda\}) = 0\}.$$

As a consequence, we define $L^\infty(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} : \|u\|_{L^\infty(X,\mu)} < \infty\right\}$.

$L^\infty_{\text{loc}}(X, \mu) := \left\{u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^\infty(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact}\right\}.$

In the case $X = \mathbb{R}^n$ and $\mu = \mathscr{L}^n$, we shall set for brevity $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \mathscr{L}^n)$, for any $p \in [1, \infty]$.

Definition 1.3.4 (Integral measures)**.** If $u : X \to [0, \infty]$ μ-measurable, then we define the *integral measure* $\nu = u\mu$ (or $\mu \llcorner u$) as

$$\nu(A) = \int_A u\,d\mu = \int_X u\chi_A\,d\mu \qquad \text{for all } \mu\text{-measurable } A.$$

1.4 Real and vector valued Radon measures

Through this section, let $\Omega \subset \mathbb{R}^n$ be an open set. We exploit now the concept of integral measure introduced in Definition 1.3.4 to define signed and vector valued Radon measures. In order to avoid ambiguity, from this point on we shall refer to the Radon measure introduced in Definition 1.1.9 as nonnegative Radon measures.

Definition 1.4.1 (Signed Radon measures)**.** Given a non-negative Radon measure μ on Ω and $f : \Omega \to [-\infty, \infty]$ locally μ-summable. Then we set $\nu := f\mu$ to be the integral measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact.}$$

ν is said to be a *signed Radon measure* on Ω.

Definition 1.4.2 (Vector valued Radon measures)**.** Given a non-negative Radon measure μ on Ω and $f : \Omega \to \mathbb{R}^m$ is locally μ-summable. Then we set $\nu := f\mu$ to be the vector valued Radon measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact.}$$

ν is said to be a *vector valued Radon measure* on Ω.

Definition 1.4.3 (Alternative approach)**.**

- A non-negative Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ which is σ-additive and finite on compact sets. We denote the space of such measures by $\mathcal{M}^+_{\mathrm{loc}}(\Omega)$.
- A vector valued (real or signed if $m = 1$) Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ which is σ-additive and its total variation $|\mu|$ is finite on compact sets; that is

$$|\mu|(K) := \sup \left\{ \sum_{j=1}^{\infty} |\mu(B_j)| \mid K = \bigcup_j B_j, B_i \cap B_i = \emptyset \text{ if } i \neq j \right\} < \infty \text{ for all } K \text{ compact in } \Omega.$$

  The space of such measures is denote by $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$; and by $\mathcal{M}_{\mathrm{loc}}(\Omega)$ if $m = 1$.
- We say that a non-negative Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty)$ is finite if $\mu(\Omega) < \infty$; and we denote by $\mathcal{M}^+(\Omega)$ the space of such measures.
- We say that a non-negative vector-valued Radon measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is finite if $|\mu|(\Omega) < \infty$; and we denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$, and $\mathcal{M}(\Omega)$ if $m = 1$, the space of such measures.

Remarks (Basic facts)**.**

- If $\mu \in \mathcal{M}_{loc}(\Omega, \mathbb{R}^m)$, then $|\mu| \in \mathcal{M}^+_{loc}(\Omega)$, where

$$|\mu|(B) := \sup \left\{ \sum_{j=1}^{\infty} |\mu(B_j)| \mid B = \bigcup B_j, B_j \cap B_i = \emptyset \, i \neq j, B_j \in \mathcal{B}(\Omega) \right\}$$

  for any $B \Subset \Omega$. In particular, $\sum \mu(B_j)$ is absolutely convergent for all $\{B_j\}$ partition of a some set $B \Subset \Omega$.
- The total variation is the smallest non-negative Radon measure ν such that $\nu(B) \geq |\mu(B)|$ for all $B \in \mathcal{B}(\Omega)$.
- If $\mu \in \mathcal{M}(\Omega)$, we define the *positive and negative parts* of μ

$$\mu^+ := \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu^- := \frac{|\mu| - \mu}{2}.$$

  It is easy to notice that $\mu^\pm \geq 0$ and that $\mu = \mu^+ - \mu^-$, which is the *Jordan decomposition*, and it is unique. In addition, $|\mu| = \mu^+ + \mu^-$.

Lemma 1.4.4. *If $\mu \in \mathcal{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu; \mathbb{R}^m)$, then $f\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $|f\mu| = |f|\mu$.*

Proof. Let $B \in \mathcal{B}(\Omega)$.

- It is easy to notice that
$$|(f\mu)(B)| := |\int_B f d\mu| \leq \int_B |f| d\mu.$$
From this it follows immediately that $|f\mu| \leq |f|\mu$, which implies $f\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$.

- Let $\varepsilon > 0$ and $D = \{z_h\}_{h\in\mathbb{N}}$ countable dense set in $\mathbb{S}^{m-1}$, let $B \in \mathcal{B}(\Omega)$. We define
$$\sigma(x) := \min\{h \in \mathbb{N} : f(x)z_h \geq (1-\varepsilon)|f(x)|\}$$
it is Borel measurable. Then, we set
$$B_h := \sigma^{-1}(\{h\}) \cap B,$$
and we notice that
$$B_h \in \mathcal{B}(\Omega),\ B = \bigcup_{h\in\mathbb{N}} B_h \text{ and } B_h \cap B_k = \text{ if } h \neq k.$$
This implies that
$$\int_B |f| d\mu = \sum_{k\in\mathbb{N}} \int_{B_h} |f| d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} \int_{B_h} f z_h d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} |(f\mu)(B_h)| \leq \frac{1}{1-\varepsilon} |f\mu|(B),$$
since
$$\int_{B_h} f z_h d\mu = z_h \int_{B_h} f d\mu \leq \left| \int_{B_h} f d\mu \right| = |(f\mu)(B_h)|$$

□

Definition 1.4.5. Let μ be a non-negative measure on Ω.

- We say that μ is concentrated on a set $E \subset \Omega$ if
$$\mu(\Omega \setminus E) = 0.$$

- We call the support of μ, $\operatorname{supp}\mu$, the smallest closed set on which μ is concentrated :
$$\operatorname{supp}(\mu) := \bigcap_{C \text{ closed}, \mu(\Omega\setminus C)=0} C.$$

Exercise 1.4.6. Equivalently, we may characterize the support of a non-negative Radon measure μ in terms of its behaviour on balls:
$$\operatorname{supp}(\mu) = \{x \in \Omega \mid \mu(B(x,r)) > 0, \forall r > 0 \text{ such that } B(x,r) \subset \Omega\}$$

Remark 1.4.7. Notice that a non-negative Radon measure may be concentrated on a set strictly smaller than its support. Indeed, let $\Omega = \mathbb{R}$ and
$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{\frac{1}{k}}.$$
It is clear that μ is concentrated on the set $E = \left\{\frac{1}{k}\right\}_{k\geq 1}$, but $\mu((-r,r)) > 0$ for any $r > 0$, so that $0 \in \operatorname{supp}(\mu)$. In fact, it is not difficult to check that $\operatorname{supp}(\mu) = \{0\} \cup \left\{\frac{1}{k}\right\}_{k\geq 1} = \overline{E}$.

Definition 1.4.8. 1. Let $\mu \in \mathcal{M}^+(\Omega)$, $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$. We say that μ is absolutely continuous with respect to μ, and we write $\nu \ll \mu$, if for all $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$, then $|\nu|(B) = 0$.

2. If $\mu, \nu \in \mathcal{M}^+(\Omega)$, we say that they are mutually singular if there exists $E, F \in \mathcal{B}(\Omega)$ such that $\mu(F) = 0$, $\mu(E) = 0$ and
$$\mu(B) = \mu(B \cap E) \qquad \text{and} \qquad \nu(B) = \nu(B \cap F)$$
for all $B \in \mathcal{B}(\Omega)$ and we write $\mu \perp \nu$. If $\mu, \nu \in \mathcal{M}(\Omega, \mathbb{R}^m)$,
$$\mu \perp \nu \iff |\mu| \perp |\mu|$$

Theorem 1.4.9 (Radon-Nikodym)**.** *Let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$. Then there exist unique measures $\nu^{ac}, \nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m)$ such that $\nu^{ac} \ll \mu$, $\nu^s \perp \mu$ and*

$$\nu = \nu^{ac} + \nu^s. \tag{1.4.1}$$

In addition, there exists a unique measure $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that $\nu^{ac} = f\mu$.
In particular, if $\mu = \mathscr{L}^n$, every $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ can be uniquely decomposed in

$$\mu = f\mathscr{L}^n + \nu^s,$$

for some $f \in L^1(\Omega; \mathbb{R}^n)$ and $\nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m), \nu^s \perp \mathscr{L}^n$.

The decomposition in (1.4.1) is called *Lebesgue decomposition* of the measure ν with respect to μ.

Definition 1.4.10. We say that a property holds $|\mu|$-almost everywhere or for $|\mu|$-almost every x if the set where the property does not hold is $|\mu|$-negligible; that is, it has zero $|\mu|$-measure.

Corollary 1.4.11 (Polar decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Then there exists a unique $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $|f(x)| = 1$ $|\mu|$-a.e and $\mu = f|\mu|$.*

Proof of corollary. Apply Radon-Nikodym theorem (Theorem 1.4.9) to μ and $|\mu|$. We know that $|\mu(B)| \leq |\mu|(B)$ for all $B \in \mathcal{B}(\Omega)$. From this follows $\mu \ll |\mu|$, and so there exists $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $\mu = f|\mu|$.
We proved that $|f|\mu|| = |f||\mu|$, hence we obtain

$$|\mu| = |f|\mu|| = |f||\mu| \quad \text{and so} \quad (|f| - 1)|\mu| = 0.$$

This means that we have

$$\int_\Omega (|f| - 1) d|\mu| = 0,$$

which yields $|f(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$. □

Corollary 1.4.12 (Hahn decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega)$, there exists a unique $A \in \mathcal{B}(\Omega)$ (up to $|\mu|$-negligible sets) such that*

$$\mu^+ = \mu \llcorner A \qquad \mu^- = -\mu \llcorner (\Omega \setminus A).$$

Proof. By the polar decomposition, there exists a unique $f \in L^1(\Omega, |\mu|)$ such that $\mu = f|\mu|$ and $f(x) \in \{\pm 1\}$ for $|\mu|$-a.e. $x \in \Omega$. This means that, if we set

$$A := \{f = 1\},$$

we have

$$f(x) = \chi_A - \chi_{\Omega \setminus A}.$$

Thus, we obtain

$$\begin{aligned}
\mu^+ &:= \frac{|\mu| + \mu}{2} = \frac{1 + \chi_A - \chi_{\Omega\setminus A}}{2}|\mu| = \chi_A |\mu|, \\
\mu^- &:= \frac{|\mu| - \mu}{2} = \frac{1 - \chi_A + \chi_{\Omega\setminus A}}{2}|\mu| = \chi_{\Omega\setminus A} |\mu|.
\end{aligned}$$

□

1.5 Duality for Radon measures

Another characterization is given via the duality with continuous functions.

Definition 1.5.1. We say that $B \Subset \Omega$ if $\overline{B} \subset \Omega$ and it is compact in Ω.

$$C^0_C(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \operatorname{supp} u \Subset \Omega\}$$

$$C^0_0(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \forall \varepsilon > 0 \ \exists K \subset \Omega : |u(x)| < \varepsilon \quad \forall x \notin K\}$$

$$\|u\|_\infty := \sup_{x \in \Omega} |u(x)|$$

Remark 1.5.2. $C_0^0(\Omega;\mathbb{R}^m) = \overline{C_c^0(\Omega;\mathbb{R}^n)}^{\|\cdot\|_\infty}$, $(C_0^0(\Omega;\mathbb{R}^n), \|\cdot\|_\infty)$ is Banach. C_c^0 is separable, locally convex, topological vector space with the following topology:

$$\varphi_k \longrightarrow \varphi \quad \text{in} \quad C_c^0 \qquad \Longleftrightarrow \qquad \|\varphi_k - \varphi\|_\infty \to 0 \quad \text{and there exists } K \subset \Omega : \operatorname{supp}\varphi \cup \bigcup_{k\in\mathbb{N}} \operatorname{supp}\varphi_k \subset K$$

Theorem 1.5.3 (Lusin)**.** *Let μ Borel on Ω and $u : \Omega \to \mathbb{R}$ is μ-measurable, $u \equiv 0$ in $\Omega \setminus E$ with $\mu(E) < \infty$. Then for all $\varepsilon > 0$ there exists $v \in C^0(\Omega)$ such that $\|v\|_\infty \le \|u\|_\infty$*

$$\mu(\{x \in \Omega : v(x) \neq u(x)\}) < \varepsilon.$$

Remark 1.5.4. An equivalent formulation states that, under the additional assumption $\mu(\Omega) < \infty$, then there exists a sequence of compact sets $\{K_h\}$ such that

$$\mu\left(\Omega \setminus \bigcup_{h=1}^{\infty} K_h\right) = 0 \qquad \text{and} \qquad u\big|_{K_h} \text{ is continuous.}$$

In other terms, this means that there exists a sequence of functions $\{u_h\} \in C^0(\Omega)$ such that $u = u_h$ on K_h and $\|u_h\|_\infty \le \|u\|_\infty$.

Proposition 1.5.5. *Let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. Then for all $A \subset \Omega$ open we have*

$$|\mu|(A) = \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c^0(A;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\}, \tag{1.5.1}$$

with the convention that

$$\int_\Omega \varphi \cdot d\mu := \sum_{j=1}^m \int_\Omega \varphi_j d\mu_j.$$

Proof. Polar decomposition implies that $\mu = f|\mu|$, $|f| = 1$ μ-a.e. So we get

$$\int_\Omega \varphi \cdot d\mu = \int_A \varphi \cdot f d|\mu| \le |\mu|(A).$$

By Lusin theorem, for all $\varepsilon > 0$ there exists $\varphi \in C^0(A;\mathbb{R}^m)$ such that $\|\varphi\|_\infty \le 1$ and

$$|\mu|\left(\{x \in A : \varphi(x) \neq f(x)\}\right) < \varepsilon.$$

Take $K \subset A$ compact such that $|\mu|(A \setminus K) < \varepsilon$. Construct $\eta \in C_c^\infty(A)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on K, $\tilde{\varphi} = \varphi\eta \in C_c^0(A;\mathbb{R}^m)$ and

$$|\mu|\left(\{x : \tilde{\varphi}(x) \neq f(x)\}\right) \le |\mu|(A \setminus K) + |\mu|\left(\{x : \varphi(x) \neq f(x)\}\right) \le 2$$

to get

$$\int_A \tilde{\varphi} \cdot d\mu \ge |\mu|(K) - 2\varepsilon$$

and by sending K to A and $\varepsilon \searrow 0$ we arrive at the claim. $\square$

Proposition 1.5.5 shows that, given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$, we can define a linear continuous functional $L_\mu : C_0^0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu,$$

for any $\varphi \in C_0^0(\Omega;\mathbb{R}^m)$. In addition, the operatorial norm of L_μ is equal to $|\mu|(\Omega)$, since, by the density of C_c^0 in C_0^0 with respect to the supremum norm and by (1.5.1), we have

$$\begin{aligned}\|L_\mu\| &:= \sup\{L_\mu(\varphi) : \varphi \in C_0^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\} \\ &= \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\} = |\mu|(\Omega).\end{aligned}$$

This suggests that it is possible to characterize $\mathcal{M}(\Omega;\mathbb{R}^m)$ as a dual space. In such a way, we gain yields a weaker topology on the space of vector valued Radon measure, and therefore weak* compactness of bounded sequences.

Theorem 1.5.6. (Riesz Representation Theorem) *Let $L : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a continuous linear functional; that is, L is linear and satisfies*

$$\sup\{L(\phi) : \phi \in C_0(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

Moreover,

$$|\mu|(\Omega) = \sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} = \|L\|.$$

For the proof we refer to [**?**, Theorem 1.54].

The following corollary is a direct consequence of the global version of the Riesz Representation Theorem.

Corollary 1.5.7. *Let $L : C_c(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1, \operatorname{supp}(\phi) \subset K\} < \infty,$$

for any compact set $K \subset \Omega$. Then there exists a unique $\mu \in \mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Thus we can identify any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with a continuous linear functional on $C_0(\Omega;\mathbb{R}^m)$, written as

$$L_\mu(\phi) := \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m),$$

and analogously $\mathcal{M}_{\text{loc}}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega;\mathbb{R}^m)$. These facts lead us to a notion of weak* convergence for Radon measure.

1.6 Weak* convergence for Radon measures

Definition 1.6.1. Given a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega)$, we say that μ_k *weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

If $\{\mu_k\}$ and μ are in $\mathcal{M}_{\text{loc}}(\Omega)$, we say that μ_k *locally weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Lemma 1.6.2. *Let $\{\mu_k\} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ be a weak-star convergent sequence, and let μ be its limit. Then we have*

$$\limsup_{k\to+\infty} |\mu_k|(\Omega) < \infty$$

and

$$|\mu|(\Omega) \leq \liminf_{k\to+\infty} |\mu_k|(\Omega).$$

Proof. The first assertion follows from Uniform Boundedness Principle (Banach-Steinhaus Theorem), since $L_{\mu_k}(\phi) \to L_\mu(\phi)$ for each $\phi \in C_0(\Omega;\mathbb{R}^m)$ and therefore $\{L_{\mu_k}(\phi)\}$ is a bounded sequence in $\mathbb{R}$.

The second inequality comes from:

$$|L_{\mu_k}(\phi)| \leq \|\phi\|_\infty |\mu_k|(\Omega)$$

then, passing to the limit we have $|L_\mu(\phi)| \leq \liminf_{k\to+\infty} \|\phi\|_\infty |\mu_k|(\Omega)$ and taking supremum in ϕ yields the result. $\square$

Remark 1.6.3. Weak-star convergence of finite Radon measures is equivalent to local weak-star convergence with the condition that $\sup|\mu_k|(\Omega) = C < \infty$. We observe that, by Lemma 1.6.2, this condition implies $|\mu|(\Omega) \le C$.
Clearly weak-star convergence always implies local weak-star convergence.
On the other hand, if we suppose that μ_k locally weak-star converges to μ, then, given $\psi \in C_0(\Omega;\mathbb{R}^m)$, for any $\epsilon > 0$ there exists $\phi \in C_c(\Omega;\mathbb{R}^m)$ such that $\|\psi - \phi\|_\infty < \epsilon$ and so

$$\begin{aligned}\left|\int_\Omega \psi \cdot d\mu_k - \int_\Omega \psi \cdot d\mu\right| &\le \left|\int_\Omega (\psi - \phi) \cdot d\mu_k\right| + \left|\int_\Omega (\psi - \phi) \cdot d\mu\right| \\ &\quad + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right| \\ &\le 2C\epsilon + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right|.\end{aligned}$$

Now, $\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu$ and so, since ϵ is arbitrary, we obtain weak-star convergence.
Therefore, in what follows, we will always write $\mu_k \overset{*}{\rightharpoonup} \mu$ to denote local weak-star convergence, and, in the case of finite Radon measures, we will also check the condition $\sup|\mu_k|(\Omega) < \infty$.

Remark 1.6.4. Let μ be a positive Radon measure. If $\{A_t\}_{t\in\mathcal{I}}$, where $\mathcal{I}$ is uncountable, is a family of μ-measurable sets in Ω such that their boundaries are disjoint, $\bigcup_{t\in\mathcal{I}} \partial A_t = \Omega$ and for every compact K there exists an uncountable set of indices $\mathcal{J} \subset \mathcal{I}$ such that $K \cap \partial A_t \neq \emptyset$, $\forall t \in \mathcal{J}$, then there exists a countable set $\mathcal{N}$ such that

$$\mu(K \cap \partial A_t) = 0 \quad \forall t \notin \mathcal{N}.$$

We claim that, if such a set $\mathcal{N}$ did not exist, then there would be an uncountable set $\mathcal{Y}$ such that $\mu(K \cap \partial A_t) > \epsilon > 0$, $\forall t \in \mathcal{Y}$. Suppose to the contrary that for each $\epsilon > 0$ the set of t's which satisfy $\mu(K \cap \partial A_t) > \epsilon$ is countable.
We set $\epsilon_j = \frac{1}{j}$ and we have

$$\{t \in \mathcal{I} : \mu(K \cap \partial A_t) \neq 0\} = \bigcup_{j=1}^{+\infty} \left\{t \in \mathcal{I} : \mu(K \cap \partial A_t) > \frac{1}{j}\right\},$$

so this set, being countable union of countable sets, is itself countable, contradicting our assumption. We extract now from $\mathcal{Y}$ a sequence $\{t_j\}$.
By the monotonicity and the σ-additivity, we have

$$\mu(K) \ge \sum_{j=1}^{+\infty} \mu(K \cap \partial A_{t_j}) = +\infty,$$

which is absurd, since μ is a Radon measure. Therefore, such a $\mathcal{Y}$ cannot exist and so $\mathcal{N}$ exists.
In the applications, the sets $\{A_t\}$ will usually be balls $B(x,r)$.

Finally, we state a characterization of nonnegative linear functionals on $C_c^\infty(\Omega)$.

Lemma 1.6.5. *Let $L : C_c^\infty(\Omega) \to \mathbb{R}$ be linear and nonnegative; that is,*

$$L(\phi) \ge 0, \quad \forall \phi \in C_c^\infty(\Omega) \text{ with } \phi \ge 0.$$

Then there exists a positive Radon measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c^\infty(\Omega).$$

Proof. We choose a compact set $K \subset \Omega$ and we select a smooth function $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on K and $0 \le \zeta \le 1$. Then, for any $\phi \in C_c^\infty(\Omega)$ with $\mathrm{supp}(\phi) \subset K$, we set $\psi = \|\phi\|_\infty \zeta - \phi \ge 0$. Therefore, since L is nonnegative, we have $0 \le L(\psi) = \|\phi\|_\infty L(\zeta) - L(\phi)$ and so $L(\phi) \le C\|\phi\|_\infty$, with $C := L(\zeta)$.
L thus may be extended to a linear mapping $\hat{L} : C_c(\Omega) \to \mathbb{R}$ such that, for any compact $K \subset \Omega$,

$$\sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \le 1, \mathrm{supp}(\phi) \subset K\} < \infty.$$

Hence, Corollary 1.5.7 yields the existence of a real Radon measure μ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c(\Omega).$$

By the polar decomposition of measures, $\mu = h|\mu|$, where $|h| = 1$ $|\mu|$-a.e. The fact that L is nonnegative implies that $h = 1$ $|\mu|$-a.e.; that is, μ is a positive Radon measure. □

Theorem 1.6.6 (Criterions for weak$*$ convergence)**.** *Let $\mu_h, \mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$. The following are equivalent*

(1) $\mu_h \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

(2) For all $U \subset \Omega$ open and for all $K \subset \Omega$ compact we have

$$\liminf_{h\to\infty} \mu_h(U) \geq \mu(U), \tag{1.6.1}$$

$$\limsup_{h\to\infty} \mu_h(K) \leq \mu(K). \tag{1.6.2}$$

(3) For all Borel sets $B \Subset \Omega$ such that $\mu(\partial B) = 0$, we have

$$\lim_{h\to\infty} \mu_h(B) = \mu(B). \tag{1.6.3}$$

Proof. (1) ⇒ (2) Let $K \subset U \subset \Omega$ where K is compact and U is open, and choose $\varphi \in C_0(\Omega)$, $\operatorname{supp} \varphi \subset U$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K. By our assumption, we have

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu.$$

Hence, we get

$$\mu(K) \leq \int_\Omega \varphi \, d\mu = \lim_{h\to\infty} \int_\Omega \varphi \, d\mu_h \leq \liminf_{h\to\infty} \mu_h(U),$$

and we deduce (1.6.1) by taking the supremum in $K \subset U$ and using the innner regularity of μ. On the other hand, it also clear that we have

$$\mu(U) \geq \int_\Omega \varphi \, d\mu = \lim_{h\to\infty} \int_\Omega \varphi \, d\mu_h \geq \limsup_{h\to\infty} \mu_h(K),$$

from which we deduce (1.6.2) by taking the infimum in $U \supset K$ and using the outer regularity.

(2) ⇒ (3) Notice that $B = \overset{\circ}{B} \cup (\partial B \cap B)$. Therefore, using $\mu(\partial B) = 0$, we have

$$\begin{aligned}\mu(B) = \mu(\overset{\circ}{B}) + \mu(\partial B \cap B) = \mu(\overset{\circ}{B}) &\leq \liminf_{h\to\infty} \mu_h(\overset{\circ}{B}) \\ &\leq \limsup_{h\to\infty} \mu_h(\overset{\circ}{B}) \leq \limsup_{h\to\infty} \mu_h(\overline{B}) \leq \mu(\overline{B}) = \mu(B).\end{aligned}$$

(3) ⇒ (1) Let $\varepsilon > 0$ and $\varphi \in C^0_c(\Omega)$. We need to prove that

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu.$$

Let us at first assume $\varphi \geq 0$. Choose $0 = t_0 < t_1 < \cdots < t_N := 2\|\varphi\|_\infty$, such that $0 < t_i - t_{i-1} < \varepsilon$ and $\mu(\varphi^{-1}\{t_i\}) = 0$. By Remark 1.6.4, it is always possible to choose such good t_i's. Let $B_i = \varphi^{-1}((t_{i-1}, t_i))$, then $\mu(\partial B_i) = 0$. Hence, by (1.6.3) we have

$$\mu_h(B_i) \to \mu(B_i).$$

In addition, it is easy to notice that

$$\sum_{i=2}^N t_{i-1}\mu_h(B_i) \leq \int_\Omega \varphi \, d\mu_h \leq \sum_{i=2}^N t_i \mu_h(B_i) + t_1 \mu_h(B_0),$$

$$\sum_{i=2}^N t_{i-1}\mu(B_i) \leq \int_\Omega \varphi \, d\mu \leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B_0).$$

Therefore, by the triangle inequality and the subadditivity of the limsup, we have

$$\begin{aligned}\limsup_{h\to+\infty}\left|\int_\Omega \varphi\,d\mu_h-\int_\Omega \varphi\,d\mu\right| &\le \limsup_{h\to+\infty}\left|\int_\Omega \varphi\,d\mu_h-\sum_{i=2}^N t_{i-1}\mu_h(B_i)\right|+\\ &+\left|\sum_{i=2}^N t_{i-1}\mu_h(B_i)-\sum_{i=2}^N t_{i-1}\mu(B_i)\right|+\left|\int_\Omega \varphi\,d\mu-\sum_{i=2}^N t_{i-1}\mu(B_i)\right|\\ &\le \limsup_{h\to+\infty} t_1\mu_h(B_0)+t_1\mu(B_0)=2t_1\mu(B_0)\\ &< \varepsilon\mu(\operatorname{supp}\varphi),\end{aligned}$$

from which we conclude, since ε is arbitrary. Let us now consider the general case of $\varphi:\Omega\to\mathbb{R}$, and consider

$$\psi := \varphi + \|\varphi\|_\infty \eta,$$

for some $\eta\in C_c(\Omega)$ such that $0\le\eta\le 1$ and $\eta\equiv 1$ on $\operatorname{supp}(\varphi)$. It is plain to see that $\psi\ge 0$ and $\psi\in C_c(\Omega)$, so that we have

$$\begin{aligned}\int_\Omega \varphi\,d\mu_h &= \int_\Omega \psi\,d\mu_h-\int_\Omega \|\varphi\|_\infty\eta\,d\mu_h\\ &\to \int_\Omega \psi\,d\mu-\int_\Omega \|\varphi\|_\infty\eta\,d\mu=\int_\Omega \varphi\,d\mu\end{aligned}$$

and this ends the proof.

□

We quote now a useful result about weak* convergence of vector valued Radon measures.

Lemma 1.6.7. *If μ_k and μ are $\mathbb{R}^m$-vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then $|\mu|\le\nu$. Moreover, if a μ-measurable set $E\subset\subset\Omega$ satisfies $\nu(\partial E)=0$, then*

$$\mu(E)=\lim_{k\to+\infty}\mu_k(E).$$

More generally, if $f:\Omega\to\mathbb{R}^m$ is a bounded Borel function with compact support such that the set of its discontinuity points is ν-neglegible, then

$$\lim_{k\to+\infty}\int_\Omega f\cdot d\mu_k=\int_\Omega f\cdot d\mu.$$

Remark 1.6.8. By Remark 1.6.4 and Lemma 1.6.7, we can assert that, if μ_k and μ are positive Radon measures in Ω, for any $x\in\Omega$ and almost every $r\in(0,R)$, with $R=R_x>0$ such that $B(x,R_x)\subset\subset\Omega$, $\mu(\partial B(x,r))=0$ and so, if $\mu_k\overset{*}{\rightharpoonup}\mu$, $\mu_k(B(x,r))\to\mu(B(x,r))$.
Moreover, if μ_k and μ are vector valued Radon measures, $\mu_k\overset{*}{\rightharpoonup}\mu$ and $|\mu_k|\overset{*}{\rightharpoonup}\nu$, then for any $x\in\Omega$ and almost every $r\in(0,R)$, with $R=R_x>0$ such that $B(x,R_x)\subset\subset\Omega$, $\nu(\partial B(x,r))=0$ and $\mu_k(B(x,r))\to\mu(B(x,r))$.

Chapter 2

Basic results from Geometric Measure Theory

2.1 Covering theorems and differentiation of measures

In this section we introduce the coverings and differentiation theorems, fundamental tools of Geometric Measure Theory. The exposition is based mostly on [**?**, Chapter 2].

2.1.1 Covering theorems

We say that a family of sets $\mathcal{F}$ is disjoint if $F \cap F' = \emptyset$ for all $F, F' \in \mathcal{F}$, $F \neq F'$. We notice that, since $\mathbb{R}^n$ is separable, every disjoint family of sets with nonempty interior is at most countable.

Theorem 2.1.1 (Besicovitch covering theorem)**.** *There exists a $\xi_n \in \mathbb{N}$ such that for all families of closed balls $\mathcal{F}$ such that the set $A := \{x \in \mathbb{R}^n \mid \exists \varrho > 0 : \overline{B(x, \varrho)} \in \mathcal{F}\}$ is bounded, there exists at most ξ_n disjoint subfamilies $\mathcal{F}_i \subset \mathcal{F}$ such that*

$$A \subset \bigcup_{i=1}^{\xi_n} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}.$$

Remark 2.1.2. The balls in the statement of Theorem 2.1.1 may be taken to be open.

Theorem 2.1.3 (Consequence of Besicovitch theorem)**.** *Let A be a bounded set and $\varrho : A \to (0, \infty)$. Then there exists $S \subset A$ at most countable such that $A \subset \bigcup_{x \in S} B(x, \varrho(x))$ and such that every point of $\mathbb{R}^n$ belongs to at most ξ_n balls centered in points of S; that is,*

$$\sum_{x \in S} \chi_{B(x, \varrho(x))}(y) \leq \xi_n$$

for all $y \in \mathbb{R}^n$, where ξ_n is the same constant of Theorem 2.1.1.

Proof. Let $\mathcal{F} := \{B(x, \varrho(x)) \mid x \in A\}$ and apply Besicovitch covering theorem (Theorem 2.1.1): there exists $\xi_n \in \mathbb{N}$ and $\mathcal{F}_i, \ldots, \mathcal{F}_{\xi_n}$ disjoint families of open balls (thanks to Remark 2.1.2) such that

$$A \subset \bigcup_{i=1}^{\xi_n} \bigcup_{B \in \mathcal{F}_i} B.$$

Then, we set S to be the set of centers of the balls in the families $\mathcal{F}_i$, for $i \in \{1, \ldots, \xi_n\}$. Clearly, each one of these families is at most countable, being disjoint, so that S is at most countable. This ends the proof. □

We employ now these results to show a covering theorem for Radon measure due to Vitali, which is the general version of Lemma 1.2.19. To this purpose we define the notion of fine covering.

Definition 2.1.4. Let $\mathcal{F}$ be a family of closed balls and $A \subset \mathbb{R}^n$. $\mathcal{F}$ is called a *fine covering* of A if

$$\inf \left\{ \varrho > 0 \mid \overline{B(x, \varrho)} \in \mathcal{F} \right\} = 0 \quad \text{for all } x \in A.$$

Theorem 2.1.5 (Vitali covering theorem)**.** *Let A be a bounded Borel set and $\mathcal{F}$ be a fine covering of A. In any case $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^n)$ there exists $\mathcal{F}' \subset \mathcal{F}$ disjoint such that*

$$\mu\left(A \setminus \bigcup_{\overline{B}\in\mathcal{F}'} \overline{B}\right) = 0$$

Proof. □

2.1.2 Differentiation of Radon and Hausdorff measures

Theorem 2.1.6 (Lebesgue-Besicovitch differentiation theorem)**.** *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$, $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, $\lambda \ll \mu$. Then, for μ-a.e. $x \in \operatorname{supp}\mu$, the limit*

$$D_\mu\lambda(x) := \lim_{\varrho\to 0}\frac{\lambda(B(x,\varrho))}{\mu(B(x,\varrho))} \tag{2.1.1}$$

exists in $\mathbb{R}^m$. In addition, we have $\lambda = D_\mu\lambda\,\mu$.

The limit in (2.1.1), sometimes also denoted by $(d\lambda/d\mu)(x)$, is called the *derivative*, or the *density*, of λ with respect to μ.

Corollary 2.1.7. *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $f \in L^1_{\mathrm{loc}}(\Omega,\mu;\mathbb{R}^m)$. Then we have*

$$f(x) = \lim_{\rho\to 0}\frac{1}{\mu(B(x,\rho))}\int_{B(x,\rho)} f(y)\,d\mu(y)$$

for μ-a.e. $x \in \Omega$.

Proof. Let $\lambda := f\mu$. It is easy to check that $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. By Theorem 2.1.6, we know that, for μ-a.e. $x \in \Omega$, there exists the limit

$$D_\mu\lambda(x) := \lim_{\rho\to 0}\frac{\lambda(B(x,\rho))}{\mu(B(x,\rho))} = \lim_{\rho\to 0}\frac{1}{\mu(B(x,\rho))}\int_{B(x,\rho)} f(y)\,d\mu(y),$$

and it satisfies $\lambda = D_\mu\lambda\,\mu$. Thus, we have $(f - D_\mu\lambda)\,\mu = 0$, which implies $f(x) = D_\mu\lambda(x)$ for μ-a.e. $x \in \Omega$. □

Remark 2.1.8. If we apply Corollary 2.1.7 to the case $\mu = \mathscr{L}^n$ and $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, we obtain the classical version of Lebesgue's differentiation theorem:

$$f(x) = \lim_{\rho\to 0} \fint_{B(x,\rho)} f(y)\,dy = \lim_{\rho\to 0}\frac{1}{\omega_n\rho^n}\int_{B(x,\rho)} f(y)\,dy$$

for $\mathscr{L}^n$-a.e. $x \in \Omega$. In particular, if $f = \chi_E$ for some Lebesgue measurable set E in $\mathbb{R}^n$, we obtain

$$\chi_E(x) = \lim_{\rho\to 0}\frac{|E\cap B(x,\rho)|}{|B(x,\rho)|} = \lim_{\rho\to 0}\frac{|E\cap B(x,\rho)|}{\omega_n\rho^n} \tag{2.1.2}$$

for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Actually, it is not difficult to show that the statement of Corollary 2.1.7 may be refined to a slightly stronger version.

Corollary 2.1.9. *Let $\mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$ and $f \in L^1(\Omega,\mu;\mathbb{R}^m)$. Prove that for μ-a.e. $x \in \Omega$ we have*

$$\lim_{\rho\to 0}\frac{1}{\mu(B(x,\rho))}\int_{B(x,\rho)} |f(y) - f(x)|\,d\mu(y) = 0. \tag{2.1.3}$$

Any point $x \in \Omega$ for which (2.1.3) holds is called *Lebesgue point* (or *approximate continuity point*) of f.

Hint of the proof. It is enough to apply Corollary 2.1.7 to the measures $\nu_q := |f - q|\mu$, for $q \in \mathbb{Q}$, and then to exploit the fact that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. □

We recall that the Hausdorff measures $\mathscr{H}^\alpha$ for $\alpha \in [0, n)$ are Borel regular, but not Radon (see Remark **??**). Hence, the Lebesgue-Besicovitch differentiation theorem (Theorem 2.1.6) does not apply to such measures. Nevertheless, it is possible to define and to study a notion of density with respect to the Hausdorff measures. Such analysis is relevant in many applications, since it is useful to compare a generic Radon measure μ to $\mathscr{H}^\alpha$ for some $\alpha \in [0, n]$, in order to to have some idea of the "dimensionality" of μ.

Definition 2.1.10. Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$ and $\alpha \in [0, n]$. For all $x \in \Omega$, we define the *upper and lower α-dimensional densities* of μ at x as

$$\Theta^*_\alpha(\mu, x) := \limsup_{\rho \to 0} \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha} \quad \text{and} \quad \Theta_{*\alpha}(\mu, x) := \liminf \frac{\mu(B(x,\rho))}{\omega_\alpha \rho^\alpha}.$$

If $\Theta^*_\alpha(\mu, x) = \Theta_{*\alpha}(\mu, x)$, the common value is denoted by $\Theta_\alpha(\mu, x)$. In the case $\mu = \mathscr{H}^\alpha \llcorner E$, for a $\mathscr{H}^\alpha$-measurable set E such that $\mathscr{H}^\alpha(E) < +\infty$, for simplicity we write

$$\Theta^*_\alpha(E, x) := \Theta^*_\alpha(\mathscr{H}^\alpha \llcorner E, x),\ \Theta_{*\alpha}(E, x) := \Theta_{*\alpha}(\mathscr{H}^\alpha \llcorner E, x) \text{ and } \Theta_\alpha(E, x) := \Theta_\alpha(\mathscr{H}^\alpha \llcorner E, x).$$

Recall that, by Proposition **??**, $\mu = \mathscr{H}^\alpha \llcorner E$ is a Radon measure, for any $\mathscr{H}^\alpha$-measurable set E such that $\mathscr{H}^\alpha(E) < +\infty$, so that the definitions of $\Theta^*_\alpha(E, x), \Theta_{*\alpha}(E, x)$ and $\Theta_\alpha(E, x)$ are well posed.

Theorem 2.1.11 (α-dimensional densities of Radon measures)**.** *Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$ and $\alpha \in [0, n]$. Then, for all $B \in \mathcal{B}(\Omega)$ and $t > 0$, the following implications hold:*

$$\Theta^*_\alpha(\mu, x) \geq t \ \forall x \in B \Longrightarrow \ \mu \geq t\mathscr{H}^\alpha \llcorner B, \tag{2.1.4}$$
$$\Theta^*_\alpha(\mu, x) \leq t \ \forall x \in B \Longrightarrow \ \mu \leq 2^\alpha t\mathscr{H}^\alpha \llcorner B. \tag{2.1.5}$$

We illustrate now two useful consequences of Theorem 2.1.11.

Corollary 2.1.12. *Let $\mu \in \mathcal{M}^+_{\text{loc}}(\Omega)$ and $\alpha \in [0, n]$. Then we have*

1. *$\Theta^*_\alpha(\mu, x) < \infty$ for $\mathscr{H}^\alpha$-a.e. $x \in \Omega$,*
2. *if $\mu(B) = 0$ for some $B \in \mathcal{B}(\Omega)$, then $\Theta_\alpha(\mu, x) = 0$ for $\mathscr{H}^\alpha$-a.e. $x \in B$.*

Let us consider the case $\mu = \mathscr{H}^\alpha \llcorner E$, for some $\mathscr{H}^\alpha$-measurable set E in $\mathbb{R}^n$ such that $\mathscr{H}^\alpha(E) < +\infty$. We start by considering the extreme case $\alpha \in \{0, n\}$:

- if $\alpha = n$, thanks to (2.1.2), we know that $\Theta_n(E, x) = \chi_E(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$;
- if $\alpha = 0$, it is easy to check that

$$\Theta_0(E, x) = \lim_{\rho \to 0} \mathscr{H}^0(E \cap B(x, \rho)) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} = \chi_E(x),$$

  since $\mathscr{H}^0(E) < \infty$ implies that E is finite and therefore discrete.

Instead, in the case $\alpha \in (0, n)$, we cannot hope for such a strong result, in general. However, we are able to obtain the following estimates.

Proposition 2.1.13. *Let $\alpha \in [0, n]$ and E be an $\mathscr{H}^\alpha$-measurable set in $\mathbb{R}^n$ such that $\mathscr{H}^\alpha(E) < \infty$. Then we have*

1. *$\Theta_\alpha(E, x) = 0$ for $\mathscr{H}^\alpha$-a.e. $x \notin E$,*
2. *$2^{-\alpha} \leq \Theta^*_\alpha(E, x) \leq 1$ for $\mathscr{H}^\alpha$-a.e. $x \in E$.*

We notice that we do not have any general result on lower bounds for $\Theta_{*\alpha}(E, x)$, and this is the reason why we cannot ensure the existence of the full limit in general. However, as we shall see in the following, in the case $\alpha = k \in \{1, \ldots, n-1\}$ there is a way to characterize the sets for which $\Theta_k(E, x)$ is well defined and equal to 1 for $\mathscr{H}^k$-a.e. $x \in E$.

2.2 Fine properties of Lipschitz functions

We devote this section to the discussion of some properties of Lipschitz functions, which proved to be very useful in the framework of Geometric Measure Theory. The choice of working with Lipschitz functions is due to the fact that such functions have a less rigid structure than C^1-differentiable functions (for instance, extension theorems are much easier to prove, see McShane's lemma), while they enjoy differentiability properties almost everywhere (see Rademacher's theorem).

Lemma 2.2.1 (McShane's lemma)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be a Lipschitz function. Then the function $f^+ : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$f^+(x) := \inf\{f(y) + \mathrm{Lip}(f, E)|x - y| : y \in E\}$$

is Lipschitz and it satisfies $f^+(x) = f(x)$ for all $x \in E$ and $\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, \mathbb{R}^n)$.

Proof. For any $x, z \in \mathbb{R}^n$, by the triangle inequality, we have

$$f^+(x) \leq \inf\{f(y) + \mathrm{Lip}(f, E)(|x - z| + |z - y|) : y \in E\} = f^+(z) + \mathrm{Lip}(f, E)|x - z|.$$

Then, interchanging the role of x and z, we immediately get

$$|f^+(x) - f^+(z)| \leq \mathrm{Lip}(f, E)|x - z|,$$

which implies $\mathrm{Lip}(f^+, \mathbb{R}^n) \leq \mathrm{Lip}(f, E)$. Now, let $x \in E$. It is easy to see that $f^+(x) \leq f(x)$. In order to obtain the reverse inequality, notice that

$$f(x) \leq f(y) + \mathrm{Lip}(f, E)|x - y|$$

for any $y \in E$, since f is Lipschitz on E. By taking the infimum in $y \in E$, we get $f(x) \leq f^+(x)$, so that $f^+(x) = f(x)$ for all $x \in E$. Finally, this identity implies

$$\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, E) \leq \mathrm{Lip}(f^+, \mathbb{R}^n),$$

from which we conclude that $\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, \mathbb{R}^n)$. □

Remark 2.2.2. The extension given in McShane's lemma is the largest extension of f, while, arguing analogously, one can show that the smaller extension is given by

$$f^-(x) := \sup\{f(y) - \mathrm{Lip}(f, E)|x - y| : y \in E\}.$$

It is not difficult to see that McShane's lemma can be extended to vector valued Lipschitz functions by hands; however, in such a way we loose the equality between the Lipschitz constants.

Corollary 2.2.3. *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\widetilde{f} = f$ on E and $\mathrm{Lip}(\widetilde{f}, \mathbb{R}^n) \leq \sqrt{m}\,\mathrm{Lip}(f, E)$.*

Proof. Apply McShane's lemma (Lemma 2.2.1) to each component of f, thus defining

$$\widetilde{f} := (f_1^+, \dots, f_m^+).$$

Then it is easy to see that $\widetilde{f} = f$ on E. As for the Lipschitz constant, notice that

$$|\widetilde{f}(x) - \widetilde{f}(y)|^2 = \sum_{i=1}^{m} |f_i^+(x) - f_i^+(y)|^2 \leq m(\mathrm{Lip}(f, E))^2|x - y|^2.$$

This ends the proof. □

A more refined result was found by Kirszbraun.

Theorem 2.2.4 (Kirszbraun theorem)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g = f$ on E and $\mathrm{Lip}(g, \mathbb{R}^n) = \mathrm{Lip}(f, E)$.*

A practical consequence of these extension results for Lipschitz functions is that we may always assume, without loss of generality, that our Lipschitz maps are defined on the whole space $\mathbb{R}^n$.

We shall now see that, quite surprisingly, the Lipschitz continuity property is enough to ensure differentiability outside of a Lebesgue negligible set. We start by recalling the notion of differentiability.

Definition 2.2.5. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $x \in \mathbb{R}^n$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0.$$

This linear mapping is denoted by $\nabla f(x)$ or $df(x)$.

Theorem 2.2.6 (Rademacher's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable at $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. In particular, $\nabla f(x)$ is well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ and belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m \times \mathbb{R}^n)$, with*

$$\|\nabla f\|_{L^\infty(K;\mathbb{R}^m \times \mathbb{R}^n)} \leq \mathrm{Lip}(f, K)$$

for any compact set K.

An interesting consequence of this result is that the differential of a Lipschitz function vanishes on the level sets of the function.

Theorem 2.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz and $t \in \mathbb{R}$. Then $\nabla f(x) = 0$ for $\mathscr{L}^n$-a.e. $x \in \{f = t\} := \{y \in \mathbb{R}^n : f(y) = t\}$.*

2.3 The area and Gauss–Green formulas

2.3.1 Linear maps and Jacobians

We recall here some standard definitions and facts from linear algebra.

Definition 2.3.1.

i) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if

$$(Ox) \cdot (Oy) = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

ii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^m$ is *symmetric* if

$$x \cdot (Sy) = (Sx) \cdot y$$

for all $x, y \in \mathbb{R}^n$.

iii) Let $A : \mathbb{R}^n \to \mathbb{R}^m$. The *adjoint* of A is the linear map $A^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$x \cdot (A^* y) = (Ax) \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Proposition 2.3.2.

i) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps. Then we have $A^{**} = A$ and $(A \circ B)^* = B^* \circ A^*$.

ii) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear map. Then $S^* = S$.

iii) If $O : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal linear map, then $n \leq m$ and $O^* \circ O = I$ on $\mathbb{R}^n$.

Theorem 2.3.3 (Polar decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.*

i) If $n \le m$, there exists a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$L = O \circ S.$$

ii) If $n \ge m$, there exists a symmetric map $S : \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$L = S \circ O^*.$$

Definition 2.3.4 (Jacobian)**.** Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

i) If $n \le m$, we write $L = O \circ S$ as above, and we define the *Jacobian* of L as

$$\mathbf{J}L := |\det S|.$$

ii) If $n \le m$, we write $L = S \circ O^*$ as above, and we define the *Jacobian* of L as

$$\mathbf{J}L := |\det S|.$$

In the literature, these two different definitions of Jacobian are also called *n-dimensional Jacobian* (or *area factor*), and m-dimensional *coarea factor*, respectively, and are denoted by $\mathbf{J}_n$ and $\mathbf{C}_m$.

Theorem 2.3.5 (Representation of Jacobian)**.**

i) If $n \le m$,

$$\mathbf{J}L = \sqrt{\det(L^* \circ L)}.$$

ii) If $n \ge m$,

$$\mathbf{J}L = \sqrt{\det(L \circ L^*)}.$$

Proof. Let $n \le m$ and $L = O \circ S$, by Theorem 2.3.3. Then we have $L^* = S \circ O^*$, so that

$$L^* \circ L = S \circ O^* \circ O \circ S = S^2,$$

since O is orthogonal and so $O^* \circ O$ is the identity mapping on $\mathbb{R}^n$ (by Proposition 2.3.2). Hence

$$\det(L^* \circ L) = \det S^2 = (\mathbf{J}L)^2.$$

The proof of (ii) is similar. □

Remark 2.3.6. The definition of the Jacobian of L is independent of the choices of O and S, and we have $\mathbf{J}L = \mathbf{J}L^*$.

Proposition 2.3.7 (Cauchy-Binet formula)**.** *If $n \le m$ and $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then*

$$\mathbf{J}L = \sqrt{\sum_B \overset{2}{\det}(B)}$$

where the sum is taken over all $n \times n$ minor of any matrix representation of L.

Let now $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, be a Lipschitz map. By Rademacher's theorem, f is differentiable $\mathscr{L}^n$-a.e. and therefore the gradient matrix

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}$$

is well defined and can be considered a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Definition 2.3.8. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous and x is a differentiability point, we define the *Jacobian* of f as

$$\mathbf{J}f(x) := \mathbf{J}\nabla f(x).$$

Remark 2.3.9. Notice that $\mathbf{J}f \le c_n \operatorname{Lip}(f)^n$.

2.3.2 The area formula

Through this subsection we assume $n \leq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ to be Lipschitz continuous.

Lemma 2.3.10. *Let $A \subset \mathbb{R}^n$ be $\mathscr{L}^n$-measurable. Then*

i) $f(A)$ is $\mathscr{H}^n$-measurable,

ii) the mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is $\mathscr{H}^n$-measurable on $\mathbb{R}^m$ and

$$\int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y) \leq (\mathrm{Lip}(f))^n \mathscr{L}^n(A)$$

Definition 2.3.11. The mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is the *multiplicity function* of f in A.

Remark 2.3.12. It is easy to notice that $\mathscr{H}^0(A \cap f^{-1}(y))$ is equal to the cardinality of the set of

$$\{x \in A : f(x) = y\},$$

so that $f^{-1}(y)$ is finite for $\mathscr{H}^n$-a.e. $y \in \mathbb{R}^m$. In particular, if f is injective, then

$$\mathscr{H}^0(A \cap f^{-1}(y)) = \begin{cases} 1 & y \in f(A), \\ 0 & y \notin f(A). \end{cases}$$

Theorem 2.3.13 (Area formula). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \leq m$. Then, for all $\mathscr{L}^n$-measurable sets $A \subset \mathbb{R}^n$, we have*

$$\int_A \mathbf{J}f(x)\, dx = \int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y). \tag{2.3.1}$$

This means that the $\mathscr{H}^n$-measure of $f(A)$, counting multiplicity, is equal to the integral of the Jacobian of f over A. As an immediate consequence, we deduce a generalization of the classical change of variables formula.

Theorem 2.3.14 (General change of variables). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \leq m$. Then, for all $\mathscr{L}^n$-summable functions $g : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, dx = \int_{\mathbb{R}^m} \left(\sum_{x \in f^{-1}(y)} g(x) \right) d\mathscr{H}^n(y). \tag{2.3.2}$$

Corollary 2.3.15 (Injective maps). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \leq m$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{L}^n$-summable function, and assume that f is injective on the support of g. Then, we have*

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, dx = \int_{f(\mathbb{R}^n)} g(f^{-1}(y))\, d\mathscr{H}^n(y). \tag{2.3.3}$$

Equivalently, if $h : \mathbb{R}^m \to \mathbb{R}$ is such that $h \circ f$ is $\mathscr{L}^n$-summable and f is injective on the support of h, then we have

$$\int_{\mathbb{R}^n} h(f(x))\, \mathbf{J}f(x)\, dx = \int_{f(\mathbb{R}^n)} h(y)\, d\mathscr{H}^n(y). \tag{2.3.4}$$

If $g = \chi_A$ for some $\mathscr{L}^n$-measurable set A, then

$$\mathscr{H}^n(f(A)) = \int_A \mathbf{J}f(x)\, dx. \tag{2.3.5}$$

Remark 2.3.16. Theorem 2.3.14 and Corollary 2.3.15 hold also in the case $g : \mathbb{R}^n \to [0, +\infty]$ is $\mathscr{L}^n$-measurable; however, the left hand sides of (2.3.2) and (2.3.3) may be equal to $+\infty$. In addition, since any Borel function is Lebesgue measurable, Theorem 2.3.14 and Corollary 2.3.15 are valid for all Borel functions $g : \mathbb{R}^n \to \mathbb{R}$ either nonnegative or $\mathscr{L}^n$-summable.

We list here some remarkable applications of the area formula.

Example 2.3.17 (Length of a curve)**.** Let $n = 1, m \geq 1$. Assume $f : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and injective. It is clear that, for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,

$$\mathbf{J}_1 f(t) = |\dot{f}(t)|.$$

Therefore, for any $a, b \in \mathbb{R}, a < b$, the length of a curve $C := f([a, b])$ is given by

$$\mathscr{H}^1(C) = \int_a^b |\dot{f}| \, dt,$$

thanks to (2.3.5).

Example 2.3.18 (Surface area of a graph)**.** Let $n \geq 1$ and $m = n + 1$. Assume $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ as

$$f(x) := (x, g(x)).$$

Then

$$\nabla f(x) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix},$$

and so, by Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J} f = \sqrt{1 + |\nabla g|^2}.$$

For any open set $\Omega \subset \mathbb{R}^n$, we define the graph of g over Ω as

$$\Gamma(g, \Omega) := \{(x, g(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}.$$

Therefore, (2.3.5) yields

$$\mathscr{H}^n(\Gamma(g, \Omega)) = \int_\Omega \sqrt{1 + |\nabla g|^2} \, dx.$$

Example 2.3.19 (Surface area of a parametric hypersurface)**.** Let $n \geq 1$ and $m = n + 1$. Assume $f : \mathbb{R}^n \to \mathbb{R}^{n+1}, f = (f^1, \ldots, f^{n+1})$, is Lipschitz and injective. For any $k \in \{1, \ldots, n + 1\}$, we define

$$\hat{f}_k := (f^1, \ldots, f^{k-1}, f^{k+1}, \ldots, f^{n+1});$$

that is, the vector valued map f without its k-th component. Then, it is not difficult to see that

$$\mathbf{J} \hat{f}_k = |\det \nabla \hat{f}_k|,$$

and so, as a consequence of Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J} f = \sqrt{\sum_{k=1}^{n+1} (\mathbf{J} \hat{f}_k)^2}.$$

Thus, if we define $\Sigma(f, \Omega) := f(\Omega)$, for any open set $\Omega \subset \mathbb{R}^n$ to be a portiong of the parametric hypersurface, (2.3.5) yields

$$\mathscr{H}^n(\Sigma(f, \Omega)) = \int_\Omega \sqrt{\sum_{k=1}^{n+1} (\mathbf{J} \hat{f}_k)^2} \, dx.$$

2.3.3 The Gauss–Green and integration by parts formulas

Chapter 3

BV theory

3.1 Functions of Bounded Variation

Definition 3.1.1. A function $u \in L^1(\Omega)$ is called a *function of bounded variation* if

$$\sup\left\{\int_\Omega u \,\mathrm{div}\phi \, dx : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), \|\phi\|_\infty \leq 1\right\} < \infty.$$

We denote by $BV(\Omega)$ the space of all functions of bounded variation on Ω.
We say that u is *locally of bounded variation*, and we write $u \in BV_{\mathrm{loc}}(\Omega)$, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if $\forall$ open set $W \subset\subset \Omega$,

$$\sup\left\{\int_W u \,\mathrm{div}\phi \, dx : \phi \in C_c^\infty(W;\mathbb{R}^d), \|\phi\|_\infty \leq 1\right\} < \infty.$$

Theorem 3.1.2. (Riesz) *Let $u \in BV_{\mathrm{loc}}(\Omega)$, then there exists a unique $\mathbb{R}^N$-vector valued Radon measure μ such that*

$$\int_\Omega u \,\mathrm{div}\phi \, dx = -\int_\Omega \phi \cdot d\mu \quad \forall \phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Proof. We define the linear functional $L : C_c^1(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$L(\phi) := -\int_\Omega u \,\mathrm{div}\phi \, dx, \quad \text{for} \quad \phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Since $u \in BV_{\mathrm{loc}}(\Omega)$, we have

$$\sup\left\{L(\phi) : \phi \in C_c^\infty(W;\mathbb{R}^N), \|\phi\|_\infty \leq 1\right\} = C(W) < \infty$$

for each open set $W \subset\subset \Omega$, and thus

$$|L(\phi)| \leq C(W)\|\phi\|_\infty \quad \text{for} \quad \phi \in C_c^1(W;\mathbb{R}^N).$$

We fix any compact set $K \subset \Omega$ and then we choose an open set W such that $K \subset W \subset\subset \Omega$. For each $\phi \in C_c(\Omega;\mathbb{R}^N)$ with $\mathrm{supp}(\phi) \subset K$, we choose a sequence $\phi_k \in C_c^1(W;\mathbb{R}^N)$ such that $\phi_k \to \phi$ uniformly on W. Then we define

$$\bar{L}(\phi) := \lim_{k\to+\infty} L(\phi_k).$$

By the continuity of L on $C_c^1(\Omega;\mathbb{R}^N)$ we have that this limit exists and is independent of the choice of the sequence $\{\phi_k\}$ converging to ϕ. Thus $\bar{L}$ uniquely extends to a linear functional

$$\bar{L} : C_c(\Omega;\mathbb{R}^N) \to \mathbb{R}$$

and

$$\sup\left\{\bar{L}(\phi) : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), \|\phi\|_\infty \leq 1, \mathrm{supp}(\phi) \subset K\right\} < \infty$$

for each compact set $K \subset \Omega$. So, by the Riesz Representation Theorem (Corollary 1.5.7), there exists an $\mathbb{R}^N$-vector valued Radon measure μ satisfying

$$\bar{L}(\phi) = -\int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega, \mathbb{R}^N)$$

and so, since $\bar{L}(\phi) = L(\phi)$ for $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, the result follows. □

This means that the distributional derivative Du of a BV function u is an $\mathbb{R}^N$-vector valued Radon measure.
We write $|Du|$ to indicate its total variation, which is a positive Radon measure on Ω.

Remark 3.1.3. $W^{1,1}(\Omega) \subset BV(\Omega)$ and $|Du|(\Omega) = \|Du\|_{L^1(\Omega;\mathbb{R}^N)}$ for $u \in W^{1,1}(\Omega)$.

Theorem 3.1.4. *If $\{u_n\} \subset BV(\Omega)$ is such that $u_n \rightharpoonup u$ in $L^p(\Omega)$ for some $p \in [1, +\infty)$, or weak-star for $p = +\infty$, or in $L^p_{\mathrm{loc}}(\Omega)$. Then $\forall A \subseteq \Omega$ open*

$$|Du|(A) \leq \liminf_{n\to+\infty} |Du_n|(A).$$

Proof. Indeed, we have $\forall \phi \in C_c^\infty(A; \mathbb{R}^N)$

$$\int_A u_n \,\mathrm{div}\phi \, dx \to \int_A u \,\mathrm{div}\phi \, dx$$

and so, by Proposition **??**,

$$\int_A u \,\mathrm{div}\phi \, dx = \lim_{n\to+\infty} \int_A u_n \,\mathrm{div}\phi \, dx \leq \liminf_{n\to+\infty} |Du_n|(A).$$

Taking the supremum over $\phi \in C_c^\infty(A; \mathbb{R}^N)$ with $\|\phi\|_\infty \leq 1$ on the left hand side, we have the claim. □

Remark 3.1.5. $|Du|(\Omega)$ is a seminorm in $BV(\Omega)$. Clearly it is positively homogeneous and we get subadditivity by observing that

$$\int_\Omega (u_1 + u_2)\mathrm{div}\phi \, dx \leq |Du_1|(\Omega) + |Du_2|(\Omega).$$

Theorem 3.1.6. *The space $BV(\Omega)$ endowed with the norm*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $BV(\Omega)$, then it is Cauchy in $L^1(\Omega)$ and so $\exists u \in L^1(\Omega)$ such that $u_n \to u$ in L^1.
By the lower semicontinuity (Theorem 3.1.4), $u \in BV(\Omega)$.
Moreover, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|D(u_k - u_n)|(\Omega) < \epsilon, \forall k, n \geq N$.
So, again by lower semicontinuity, $|D(u_k - u)|(\Omega) \leq \liminf_n |D(u_k - u_n)|(\Omega) < \epsilon$ and from this it follows u_n converges to u in BV norm. □

Theorem 3.1.7. (Meyers-Serrin Approximation theorem)
Let $u \in BV(\Omega)$, then $\exists \{u_n\} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that

1. *$u_n \to u$ in $L^1(\Omega)$*
2. *$|Du_n|(\Omega) \to |Du|(\Omega)$.*

Proof.
Fix $\epsilon > 0$. Given a positive integer m, we set $\Omega_0 = \emptyset$, define for each $k \in \mathbb{N}, k \geq 1$ the sets

$$\Omega_k = \left\{ x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B(0, k+m)$$

and then we choose m such that $|Du|(\Omega \setminus \Omega_1) < \epsilon$.
We define now $\Sigma_k := \Omega_{k+1} \setminus \overline{\Omega}_{k-1}$. Since $\{\Sigma_k\}$ is an open cover of Ω, then there exists a partition of unity subordinate to that open cover; that is, a sequence of functions $\{\zeta_k\}$ such that:

1. $\zeta_k \in C_c^\infty(\Sigma_k)$;
2. $0 \le \zeta_k \le 1$;
3. $\sum_{k=1}^{+\infty} \zeta_k = 1$ on Ω.

Then we take a standard mollifier ρ and $\forall k$ we choose ϵ_k such that:

$$\mathrm{spt}(\rho_{\epsilon_k} * (u\zeta_k)) \subset \Sigma_k$$

$$\|\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \frac{\epsilon}{2^k}$$

$$\|\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k\|_{L^1(\Omega;\mathbb{R}^N)} < \frac{\epsilon}{2^k}$$

and we define $u_\epsilon = \sum_{k=1}^{+\infty} \rho_{\epsilon_k} * (u\zeta_k)$.
Then $u_\epsilon \in C^\infty$, since locally there are only a finite number of nonzero terms in the sum.
Also, $u_\epsilon \to u$ in $L^1(\Omega)$ since

$$\|u - u_\epsilon\|_{L^1(\Omega)} \le \sum_{k=1}^{+\infty} \|\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \epsilon.$$

Now, since $u_\epsilon \in L^1(\Omega)$, Theorem 3.1.4 implies $|Du|(\Omega) \le \liminf_{\epsilon\to 0} |Du_\epsilon|(\Omega)$.

In order to obtain the reverse inequality, let $\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$, $\|\phi\|_\infty \le 1$. Then

$$\begin{aligned}\int_\Omega u_\epsilon \mathrm{div}\phi dx &= \sum_{k=1}^{+\infty} \int_\Omega \rho_{\epsilon_k} * (u\zeta_k)\mathrm{div}\phi dx = \sum_{k=1}^{+\infty} \int_\Omega u\zeta_k \mathrm{div}(\rho_{\epsilon_k} * \phi) dx \\ &= \sum_{k=1}^{+\infty} \int_\Omega u\mathrm{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx - \sum_{k=1}^{+\infty} \int_\Omega u\nabla\zeta_k \cdot (\rho_{\epsilon_k} * \phi)dx.\end{aligned}$$

Using $\sum_{k=1}^{+\infty} \nabla\zeta_k = 0$ in Ω and the properties of the convolution, this last expression equals

$$\sum_{k=1}^{+\infty} \int_\Omega u\mathrm{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx - \sum_{k=1}^{+\infty} \int_\Omega \phi \cdot (\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k)dx =: I_1^\epsilon + I_2^\epsilon$$

Now, $|\zeta_k(\rho_{\epsilon_k} * \phi)| \le 1$ and each point in Ω belongs to at most three of the sets $\{\Sigma_k\}$. Thus

$$|I_1^\epsilon| \le \left| \int_\Omega u\mathrm{div}(\zeta_1(\rho_{\epsilon_1} * \phi))dx \quad + \quad \sum_{k=2}^{+\infty} \int_\Omega u\mathrm{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx \right| \le$$

$$|Du|(\Omega) + \sum_{k=2}^{+\infty} |Du|(\Sigma_k) \le |Du|(\Omega) + 3|Du|(\Omega \setminus \Omega_1) \le |Du|(\Omega) + 3\epsilon$$

For the second term, we have $|I_2^\epsilon| < \epsilon$ directly from our choice of ϵ_k.
Therefore, after passing to the supremum over ϕ, $|Du_\epsilon|(\Omega) \le |Du|(\Omega)+4\epsilon$, which yields $u_\epsilon \in BV(\Omega)$ and point 2. □

Remark 3.1.8. If $u \in BV(\mathbb{R}^N)$; that is, if Ω is the entire space $\mathbb{R}^N$, then the approximating sequence satisfying properties 1) and 2) of Theorem 3.1.7 is much easier to construct. Indeed, we need just to take $u_\epsilon = u * \rho_\epsilon$, where ρ is a standard symmetric mollifier.
Indeed, $u_\epsilon \to u$ in $L^1(\mathbb{R}^N)$ since $u \in L^1(\mathbb{R}^N)$.
Secondly, we observe that

$$\begin{aligned}\|\nabla u_\epsilon\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} &= \sup\left\{ \int_{\mathbb{R}^N} u_\epsilon(x)\mathrm{div}\phi(x)\,dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \le 1 \right\} \\ = \sup&\left\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(y)\rho_\epsilon(x-y)\mathrm{div}\phi(x)\,dxdy : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \le 1 \right\} \\ = \sup&\left\{ \int_{\mathbb{R}^N} u(y)\mathrm{div}\phi_\epsilon(y)\,dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \le 1 \right\} \le |Du|(\mathbb{R}^N)\end{aligned}$$

and so, by lower semicontinuity of the total variation, $\|\nabla u_\epsilon\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \to |Du|(\mathbb{R}^N)$.
We may fix a sequence $\epsilon_k \to 0$. Theorem 3.1.4 implies that for any open set A $|Du|(A) \leq \liminf_{k\to+\infty} |Du_{\epsilon_k}|(A)$ and we observe that for any compact set K and $\phi \in C_c^\infty(K;\mathbb{R}^N)$, $\|\phi\|_\infty \leq 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} u_{\epsilon_k}(x)\mathrm{div}\phi(x)\,dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathrm{div}\phi(x)u(y)\rho_{\epsilon_k}(x-y)\,dydx \\ &= \int_{\mathbb{R}^N} u(y)\mathrm{div}\phi_{\epsilon_k}(y)\,dy \leq |Du|(K+\overline{B(0,\epsilon_k)})\end{aligned}$$

since $\mathrm{supp}(\phi_{\epsilon_k}) \subset K + \overline{B(0,\epsilon_k)}$. Thus we can take the supremum over ϕ in order to obtain $|Du_{\epsilon_k}|(K) \leq |Du|(K+\overline{B(0,\epsilon_k)})$, which implies $\limsup_{k\to+\infty} |Du_{\epsilon_k}|(K) \leq |Du|(K)$ since K is compact. Hence the sequence of Radon measures $|\nabla u_{\epsilon_k}|\mathscr{L}^n$ satisfies point 2 of Lemma 1.6.7 and so we have point 1 of the same lemma; that is, $|Du_{\epsilon_k}| \overset{*}{\rightharpoonup} |Du|$ in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, since we have shown above that $\sup_k |Du_{\epsilon_k}|(\mathbb{R}^N) \leq |Du|(\mathbb{R}^N) < \infty$, Remark 1.6.3 yields also weak-star convergence in $\mathcal{M}(\mathbb{R}^N)$.

This remark applies also to BV functions with compact support inside Ω, since these are trivially in $BV(\mathbb{R}^N)$. Given $u \in BV(\Omega)$ with compact support, we can indeed extend it to

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is clear that $\hat{u} \in L^1(\mathbb{R}^N)$. If we let $\xi \in C_c^\infty(\Omega)$, $\|\xi\|_\infty \leq 1$ and $\xi = 1$ in a neighborhood of the support of u, then, for any $\phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $\|\phi\|_\infty \leq 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \hat{u}\mathrm{div}\phi\,dx &= \int_\Omega u\mathrm{div}\phi\,dx = \int_\Omega u\mathrm{div}(\xi\phi + (1-\xi)\phi)\,dx \\ &= \int_\Omega u\mathrm{div}(\xi\phi)\,dx \leq |Du|(\Omega),\end{aligned}$$

since $\xi\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$ and $\|\xi\phi\|_\infty \leq 1$.
Taking the supremum over ϕ we obtain $|D\hat{u}|(\mathbb{R}^N) \leq |Du|(\Omega) < \infty$.

Lemma 3.1.9. *Let $u \in BV(\Omega)$ and $\{u_n\}$ which satisfies point 1) and 2) in Theorem 3.1.7. Then, if we define for all Borel sets $B \subset \mathbb{R}^N$ the Radon measures $\mu_n(B) := \int_{B\cap\Omega} \nabla u_n\,dx$ and $\mu(B) := Du(B\cap\Omega)$, we have $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{R}^N)$, i.e.,*

$$\int_{\mathbb{R}^N} \xi \cdot d\mu_n \to \int_{\mathbb{R}^N} \xi \cdot d\mu \quad \forall \xi \in C_c(\mathbb{R}^N;\mathbb{R}^N).$$

Proof.
We define Ω_1 as in the proof of Theorem 3.1.7, so that $|Du|(\Omega\setminus\Omega_1) < \epsilon$ for some $\epsilon > 0$ fixed.
Let η be a smooth cut-off function such that $\eta = 1$ in Ω_1, $0 \leq \eta \leq 1$ and $\mathrm{supp}(\eta) \subset \Omega$.
Let $\xi \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \xi \cdot d\mu_n = \int_{\mathbb{R}^N} \xi \cdot \nabla u_n\,dx = \int_{\mathbb{R}^N} \eta\xi \cdot \nabla u_n\,dx + \int_{\mathbb{R}^N} (1-\eta)\xi \cdot \nabla u_n\,dx.$$

Now, since $u_n \to u$ in $L^1(\Omega)$ and by the Riesz Theorem (Theorem 1.5.7),

$$\begin{aligned}\int_{\mathbb{R}^N} \eta\xi \cdot \nabla u_n\,dx &= -\int_{\mathbb{R}^N} u_n\mathrm{div}(\eta\xi)\,dx \to -\int_{\mathbb{R}^N} u\mathrm{div}(\eta\xi)\,dx \\ &= \int_{\mathbb{R}^N} \eta\xi \cdot dDu = \int_{\mathbb{R}^N} \xi \cdot dDu + \int_{\mathbb{R}^N} (\eta-1)\xi \cdot dDu\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^N} (\eta-1)\xi \cdot dDu\right| \leq \|\xi\|_\infty |Du|(\Omega\setminus\Omega_1) < \epsilon\|\xi\|_\infty.$$

Also,

$$\left|\int_{\mathbb{R}^N} (1-\eta)\xi \cdot \nabla u_n \, dx\right| \le \|\xi\|_\infty |Du_n|(\Omega \setminus \Omega_1) < \epsilon \|\xi\|_\infty,$$

since, by the fact that $|Du_n|(\Omega) \to |Du|(\Omega)$ and by the lower semicontinuity of the total variation,

$$\liminf_{n\to+\infty} |Du_n|(\Omega \setminus \Omega_1) = \liminf_{n\to+\infty} |Du_n|(\Omega) - |Du_n|(\Omega_1) \le$$
$$|Du|(\Omega) - |Du|(\Omega_1) = |Du|(\Omega \setminus \Omega_1) < \epsilon.$$

Therefore, for any $\epsilon > 0$, there exists a n_0 such that, $\forall n \ge n_0$,

$$\left|\int_{\mathbb{R}^N} \xi \cdot d\mu_n - \int_{\mathbb{R}^N} \xi \cdot d\mu\right| \le 2\epsilon \|\xi\|_\infty.$$

Now let $\xi \in C_c(\mathbb{R}^N; \mathbb{R}^N)$, and take its mollification $\xi_\delta = \xi * \rho_\delta$.
$\xi_\delta \to \xi$ uniformly on compact subsets of $\mathbb{R}^N$, in particular on $K := \mathrm{supp}(\eta)$. So

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \xi \cdot d\mu_n - \int_{\mathbb{R}^N} \xi \cdot d\mu\right| \le \left|\int_{\mathbb{R}^N} \eta(\xi - \xi_\delta) \cdot d\mu_n\right| + \left|\int_{\mathbb{R}^N} \eta(\xi - \xi_\delta) \cdot d\mu\right| + \\
&+ \left|\int_{\mathbb{R}^N} \xi_\delta \cdot d\mu_n - \int_{\mathbb{R}^N} \xi_\delta \cdot d\mu\right| + \left|\int_{\mathbb{R}^N} (1-\eta)(\xi - \xi_\delta) \cdot d\mu_n\right| + \\
&+ \left|\int_{\mathbb{R}^N} (1-\eta)(\xi_\delta - \xi) \cdot d\mu\right| \le \|\xi - \xi_\delta\|_{L^\infty(K)} (|Du_n|(\Omega) + |Du|(\Omega)) + \\
&+ \epsilon \|\xi_\delta\|_\infty + 2\|\xi\|_\infty (|Du_n|(\Omega \setminus \Omega_1) + |Du|(\Omega \setminus \Omega_1))
\end{aligned}$$

and, by the estimates already found, we can conclude that, up to choosing a suitable $\delta(\epsilon)$, it is all bounded by $C\epsilon$ for n big enough, thus proving weak-star convergence of measures. □

3.2 Sets of finite perimeter

Definition 3.2.1. A measurable set $E \subset \Omega$ is called a *finite perimeter set* in Ω (or a Caccioppoli set) if $\chi_E \in BV(\Omega)$.
A measurable set $E \subset \mathbb{R}^N$ is said to have *locally finite perimeter* in Ω if $\chi_E \in BV_{\mathrm{loc}}(\Omega)$.

Consequently, $D\chi_E$ is an $\mathbb{R}^N$-vector valued Radon measure on Ω whose total variation is $|D\chi_E|$. By the polar decomposition of measures, there exists a $|D\chi_E|$-measurable function with modulus 1 $|D\chi_E|$-a.e., which we denote by ν_E, such that $D\chi_E = \nu_E |D\chi_E|$.
Unless otherwise stated, from now on E will be a set of locally finite perimeter in Ω.

Example 3.2.2. Any open bounded set $E \subset \Omega$ with $\partial E \in C^2$ is a set of finite perimeter in Ω. Indeed, $\forall \phi \in C_c^\infty(\Omega; \mathbb{R}^N)$ with $\|\phi\|_\infty \le 1$, by the classical Gauss-Green formula we have

$$\begin{aligned}
\int_{\Omega\cap E} \mathrm{div}\phi \, dx &= -\int_{\partial(\Omega\cap E)} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} \\
&\le \int_{\Omega\cap\partial E} |\phi||\nu_E| \, d\mathcal{H}^{N-1} \le \mathcal{H}^{N-1}(\Omega \cap \partial E),
\end{aligned}$$

where ν_E is the interior unit normal. Taking the supremum over ϕ yields $|D\chi_E|(\Omega) \le \mathcal{H}^{N-1}(\Omega \cap \partial E)$.
Therefore, E has finite perimeter and so, for any $\phi \in C_c^\infty(\Omega; \mathbb{R}^N)$,

$$\int_\Omega \chi_E \mathrm{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

This implies that $D\chi_E = \nu_E \, \mathcal{H}^{N-1} \llcorner \partial E$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$, by Riesz Representation Theorem (Theorem 3.1.2), and so $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial E$, which in particular yields

$$|D\chi_E|(\Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial E). \tag{3.2.1}$$

Remark 3.2.3. It can be also shown that every open bounded set with Lipschitz boundary is a set of finite perimeter, with equality (3.2.1) holding, since this is a consequence of the extension theorem for functions of bounded variation (see [EG], Section 5.4). Moreover, any bounded open set Ω satisfying $\mathcal{H}^{N-1}(\partial\Omega) < \infty$ has finite perimeter in $\mathbb{R}^N$ (see [AFP], Proposition 3.62).

Equality (3.2.1) is not valid in general for sets of finite perimeter, as the following example will show.

Example 3.2.4. Let $N \geq 2$, $\{x_i\} = \mathbb{Q}^N \cap [0,1]^N$, $E = \bigcup_{i=0}^{\infty} B(x_i, \epsilon 2^{-i})$, with $\epsilon > 0$ that shall be assigned, and $[0,1]^N \subset \Omega$. We have

$$|E| \leq \sum_{i=0}^{\infty} \omega_N \epsilon^N 2^{-iN} = \frac{\omega_N \epsilon^N}{1 - 2^{-N}}.$$

Since the rational points are dense in $[0,1]^N$, then $\overline{E} = [0,1]^N$ and $\partial E = \overline{E} \setminus E$, since E is open, which implies

$$|\partial E| \geq |\overline{E}| - |E| \geq 1 - \frac{\omega_N \epsilon^N}{1 - 2^{-N}} > 0$$

for ϵ small enough. This implies $\mathcal{H}^{N-1}(\partial E) = \infty$.
Observing that $\partial E \subset \bigcup_{i=0}^{\infty} \partial B(x_i, \epsilon 2^{-i})$, we have

$$\begin{aligned}\int_{\Omega \cap E} \mathrm{div}\phi \, dx &= -\int_{\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} \leq \sum_{i=0}^{\infty} \int_{\partial B(x_i, \epsilon 2^{-i})} |\phi||\nu_E| \, d\mathcal{H}^{N-1} \\ &\leq \sum_{i=0}^{\infty} \mathcal{H}^{N-1}(\partial B(x_i, \epsilon 2^{-i})) = \sum_{i=0}^{\infty} N\omega_N \epsilon^{N-1} 2^{-(N-1)i} = \frac{N\omega_N \epsilon^{N-1}}{1 - 2^{-(N-1)}}.\end{aligned}$$

Thus E is a set of finite perimeter for which $|D\chi_E|(\Omega) \neq \mathcal{H}^{N-1}(\Omega \cap \partial E)$.

This may suggest that for a set of finite perimeter is interesting to consider not the whole topological boundary, but subsets of ∂E instead.

Definition 3.2.5. Let $x \in \Omega$, then $x \in \partial^* E$, the *reduced boundary* of E, if

1. $|D\chi_E|(B(x,r)) > 0, \forall r > 0$;
2. $\lim_{r \to 0} \frac{1}{|D\chi_E|(B(x,r))} \int_{B(x,r)} \nu_E d|D\chi_E| = \nu_E(x)$;
3. $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterisation of the reduced boundary, by using the blow-up of the set E around a point of $\partial^* E$.

Definition 3.2.6. For $x \in \partial^* E$ we define the hyperplane

$$H(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) = 0\}$$

and the half-spaces

$$\begin{aligned}H^+(x) &= \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \geq 0\}, \\ H^-(x) &= \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \leq 0\}.\end{aligned}$$

Also, for $r > 0$, we set

$$E_r(x) = \{y \in \mathbb{R}^N : \ x + r(y - x) \in E\}$$

Theorem 3.2.7. *If E is a set of finite perimeter in Ω, $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

$$\begin{aligned}\chi_{E_r} &\to \chi_{H^-(x)} \text{ in } \ L^1_{loc}(\Omega) \\ \chi_{\Omega \setminus E_r} &\to \chi_{H^-(x)} \text{ in } \ L^1_{loc}(\Omega)\end{aligned}$$

as $r \to 0$.

Proof. See [EG] Section 5.7.2 Theorem 1.

Formulated in another way, for $r > 0$ small enough, $E \cap B(x,r)$ is approximatively equal to the half ball $H^-(x) \cap B(x,r)$.

Corollary 3.2.8. *If $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

1. $\displaystyle\lim_{r\to 0} \frac{1}{r^N} |B(x,r) \cap E \cap H^+(x)| = 0,$

2. $\displaystyle\lim_{r\to 0} \frac{1}{r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0.$

Proof. We have

$$\frac{1}{r^N}|B(x,r) \cap E \cap H^+(x)| = |B(x,1) \cap E_r \cap H^+(x)| \to |B(x,1) \cap H^-(x) \cap H^+(x)| = 0,$$

by Theorem 3.2.7. Point 2 follows from the same theorem and

$$\begin{aligned}\frac{1}{r^N}|(B(x,r) \setminus E) \cap H^-(x)| &= \frac{1}{r^N}(|B(x,r) \cap H^-(x)| - |B(x,r) \cap E \cap H^-(x)|) \\ &= \frac{\omega_N}{2} - |B(x,1) \cap E_r \cap H^-(x)| \\ &\to \frac{\omega_N}{2} - |B(x,1) \cap H^-(x)| = 0.\end{aligned}$$

□

Using this result, we can give a generalization of the concept of unit interior normal (respectively, unit exterior normal, up to a sign).

Definition 3.2.9. A unit vector $\nu(x) = -\nu_E(x)$ for which property 1 of Corollary 3.2.8 holds is called the *measure theoretic unit exterior normal* to E at x, while, accordingly, $\nu_E(x)$ is called the *measure theoretic unit interior normal* to E at x.

It follows that the measure theoretic unit interior normal ν_E is well defined at least on the reduced boundary.
Moreover, the next theorem shows us that the reduced boundary can be written as a countable union of compact subset of C^1 manifolds, up to a set of Hausdorff dimension at most $N-1$.

Theorem 3.2.10. *Assume E is a set of locally finite perimeter in $\mathbb{R}^N$. Then*

1. *$\partial^* E$ is a $(N-1)$-rectifiable set; that is, there exist a countable family of C^1 manifolds S_k, a family of compact sets $K_k \subset S_k$ and set $\mathcal{N}$ of $\mathcal{H}^{N-1}$-measure zero such that*

$$\partial^* E = \bigcup_{k=1}^{+\infty} K_k \cup \mathcal{N};$$

2. *$\nu_E|_{S_k}$ is normal to S_k;*

3. *$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$,*

$$\lim_{r\to 0} \frac{|D\chi_E|(B(x,r))}{\omega_{N-1} r^{N-1}} = 1.$$

Proof. See [EG] Section 5.7.3 Theorem 2.

We introduce now the density of a set at a certain point, in order to select another useful subset of the topological boundary.

Definition 3.2.11. For every $\alpha \in [0,1]$ and every measurable set $E \subset \mathbb{R}^N$, we define

$$E^\alpha := \{x \in \mathbb{R}^N : D(E,x) = \alpha\},$$

where

$$D(E,x) := \lim_{r\to 0} \frac{|(B(x,r) \cap E)|}{|B(x,r)|}.$$

Definition 3.2.12. Referring to Definition 3.2.11,

1. E^1 is called the *measure theoretic interior* of E.
2. E^0 is called the *measure theoretic exterior* of E.
3. The *measure theoretic (or essential) boundary* of E is the set $\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1)$.

Remark 3.2.13. It is clear that $E^\circ \subset E^1$ and $\mathbb{R}^N \setminus \overline{E} \subset E^0$. Hence one has

$$\partial^m E \subset \mathbb{R}^N \setminus (E^\circ \cup \mathbb{R}^N \setminus \overline{E}) = \overline{E} \setminus E^\circ = \partial E.$$

Moreover, by the Lebesgue-Besicovitch differentiation theorem (Theorem **??**), $\partial^m E$ has $\mathcal{L}^N$-measure 0, since it is the set of non-Lebesgue points of χ_E.

We further observe that, as in [EG] Section 5.8, it is possibile to define the measure theoretic boundary without using the density of a set.
Indeed the previous definition is equivalent to the following:

Definition 3.2.12' Let $x \in \mathbb{R}^N$, then $x \in \partial^m E$, the meaure theoretic boundary of E, if the following two conditions hold:

1. $\limsup_{r \to 0} \dfrac{|B(x,r) \cap E|}{r^N} > 0,$
2. $\limsup_{r \to 0} \dfrac{|B(x,r) \setminus E|}{r^N} > 0.$

Theorem 3.2.14. *If $E \subset \Omega$ is a set of finite perimeter, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1) = 0.$$

In particular, E has density either 0, $\frac{1}{2}$ or 1 at $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$, and, even if E is only locally of finite perimeter, $\mathcal{H}^{N-1}$-a.e. $x \in \partial^m E$ belongs to $\partial^* E$; that is, $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$.

Proof. See [EG] Section 5.8 Lemma 1 and [AFP] Theorem 3.61.

Remark 3.2.15. Since the functions of bounded variations are elements of L^1, they are equivalence class of functions, so that changing the value of any such function on a set of $\mathcal{L}^N$-measure zero does not modify the BV class of the function.
Therefore, this is true also for sets of finite perimeter and we can choose any representative $\widetilde{E}$ for E, which differs only by a set of measure zero, without altering the reduced nor the measure theoretic boundary.

One of the greatest achievements of BV theory is to establish a generalization of the Gauss-Green formula for every set of finite perimeter, though only for differentiable vector fields.

Theorem 3.2.16. (Gauss-Green formula on sets of finite perimeter)
Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^m E$, there is a unique measure theoretic interior unit normal $\nu_E(x)$ such that $\forall \phi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$ one has

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

Proof. Since E is a set of locally finite perimeter, $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ (Theorem 3.2.10), where ν_E is the measure theoretic interior unit normal. Also, Theorem 3.2.14 implies $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$. Hence, for any $\phi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$,

$$\int_\Omega \chi_E \mathrm{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}. \; \square$$

Remark 3.2.17. Since $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$ (Theorem 3.2.14), without change, we can integrate on the measure theoretic or on the reduced boundary with respect to the measure $\mathcal{H}^{N-1}$. Since in many practical cases $\partial^m E$ is easier to be determined, Theorem 3.2.16 is often stated in this way. However, since Theorem 3.2.10 states that $|D\chi_E| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$ and the precise representative of χ_E is well defined on $E^1 \cup \partial^* E \cup E^0$ (Lemma 3.2.26 below), in what follows we will always use the reduced boundary in the Gauss-Green formula.

Remark 3.2.18. We also observe that if E is a bounded set of finite perimeter in $\mathbb{R}^N$, then we can drop the assumption on the support of ϕ. Indeed, there exists $R > 0$ such that $\overline{E} \subset B(0,R)$, and so, given $\phi \in C^1(\mathbb{R}^N;\mathbb{R}^N)$, we can take $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi = 1$ in $\overline{B(0,R)}$ (which in particular implies $\nabla\varphi = 0$ in E), in order to obtain

$$\begin{aligned}\int_E \mathrm{div}\phi \, dx &= \int_E (\varphi \mathrm{div}\phi + \phi \cdot \nabla\varphi) \, dx = \int_E \mathrm{div}(\phi\varphi) \, dx \\ &= -\int_{\partial^* E} (\phi\varphi) \cdot \nu_E \, d\mathcal{H}^{N-1} = -\int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.\end{aligned}$$

It is also easy to see that if $E \subset\subset \Omega \subset \mathbb{R}^N$, then we can take just $\phi \in C^1(\Omega;\mathbb{R}^N)$.

As in the case of Sobolev functions, it can be shown that for BV functions the precise representative is well defined and it is the limit of the mollified sequence.

Definition 3.2.19. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $a \in \mathbb{R}^N$.
We say that $u_a(x)$ is the approximate limit of u at $x \in \Omega$ restricted to $\Pi_a(x) := \{y \in \mathbb{R}^N : (y-x)\cdot a \geq 0\}$ if, for any $\epsilon > 0$,

$$\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |u(y) - u_a(x)| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} = 0$$

Definition 3.2.20. We say that $x \in \Omega$ is a *regular point* of a function $u \in BV(\Omega)$ if there exists a vector $a \in \mathbb{R}^N$ such that the approximate limits $u_a(x)$ and $u_{-a}(x)$ exist. The vector a is called *defining vector.*

Example 3.2.21. Let E be a set of finite perimeter, for which we choose the representative $E^1 \cup \partial^m E$ (see Remark 3.2.15), and $u = \chi_E$, then each point in $E^1 \cup E^0 \cup \partial^* E$ is a regular point. If $x \in E^1$, $\forall a \in \mathbb{R}^N$ $(\chi_E)_a(x) = 1$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) = E^0 \cap B(x,r).$$

So,

$$\begin{aligned}\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} &= \lim_{r\to 0} \frac{|E^0 \cap B(x,r)|}{|B(x,r)|} \\ &= 1 - D(E,x) = 0.\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\ &\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \frac{|B(x,r)|}{|B(x,r) \cap \Pi_a(x)|} \\ &= 2\frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

In an analogous way, we show that $\forall x \in E^0$ $(\chi_E)_a(x) = 0$ $\forall a \in \mathbb{R}^N$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) = E \cap B(x,r)$$

and so

$$\begin{aligned}\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} &= \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} \\ &= D(E,x) = 0.\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\ &\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \frac{|B(x,r)|}{|B(x,r) \cap \Pi_a(x)|} \\ &= 2\frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

Now let $x \in \partial^* E$ and a be the measure theoretic interior normal. Then $(\chi_E)_a(x) = 1$ and $(\chi_E)_{-a}(x) = 0$.
Referring to the notation of Corollary 3.2.8, we have $\Pi_a(x) = H^-(x)$ and
$\Pi_{-a}(x) = H^+(x)$, hence $\forall \epsilon > 0$

$$\begin{aligned}&\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\ &= \lim_{r\to 0} \frac{|E^0 \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0\end{aligned}$$

and

$$\begin{aligned}&\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|} \\ &= \lim_{r\to 0} \frac{|E \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |B(x,r) \cap E \cap H^+(x)| = 0.\end{aligned}$$

This shows our claim.

Theorem 3.2.22. *Let $u \in BV(\Omega)$. The set of irregular points has $\mathcal{H}^{N-1}$-measure zero.*

Proof. See [VH] Chapter 4 §5.5, or [EG] Section 5.9 Theorem 3.

Theorem 3.2.23. *Let $u \in BV(\Omega)$ and x be a regular point of u. Then*

1. *If $u_a(x) = u_{-a}(x)$, any $b \in \mathbb{R}^N$ is a defining vector and $u_b(x) = u_a(x)$; that is, x is a point of approximate continuity.*
2. *If $u_a(x) \neq u_{-a}(x)$, then a is unique up to a sign.*
3. *The mollification of u converges to the precise representative u^* at each regular point and $u^*(x) = \frac{1}{2}(u_a(x) + u_{-a}(x))$.*

Proof. See [VH] Chapter 4 §4.4 and Chapter 4 §5.6 Theorem 1, or [EG] Section 5.9 Corollary 1.

Remark 3.2.24. By Theorem **??**, we deduce that condition 1) in Theorem 3.2.23 holds $\mathcal{L}^N$-a.e.

We state now some standard results on the mollification of characteristic functions of sets of finite perimeter.

Remark 3.2.25. By Remark 3.1.8, if E be a set of finite perimeter and $\{\chi_{\delta_k}\}$ denotes the mollification of χ_E, then

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^N)} \leq |D\chi_E|(\mathbb{R}^N)$$

and

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^N)} \to |D\chi_E|(\mathbb{R}^N)$$

We state now some relevant properties of the mollifications of characteristic functions of sets of finite perimeter.

Lemma 3.2.26. *Let $E \subset \Omega$ be a set of locally finite perimeter in Ω and $\rho \in C_c^\infty(B(0,1))$ be a nonnegative radially symmetric mollifier such that $\int_{B(0,1)} \rho\, dx = 1$. Then, the following results hold:*

1. *there is a set $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that, for all $x \in \Omega \setminus \mathcal{N}$, $(\rho_\varepsilon * \chi_E)(x) \to \chi_E^*(x)$ where*
$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1 \\ \frac{1}{2} & \text{if } x \in \mathscr{F}E \ ; \\ 0 & \text{if } x \in E^0 \end{cases} \tag{3.2.2}$$
2. *$\rho_\varepsilon * \chi_E \in C^\infty(\Omega^\varepsilon)$ and $\nabla(\rho_\varepsilon * \chi_E)(x) = (\rho_\varepsilon * D\chi_E)(x)$ for any $x \in \Omega^\varepsilon$;*
3. *one has the following weak* limits in $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$:*

(a) $\nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} D\chi_E$;

(b) $\chi_E \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2)D\chi_E$;

(c) $\chi_{\Omega\setminus E} \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2)D\chi_E$;

We state now the co-area formula, which shows an important connection between BV functions and sets of finite perimeter.

Theorem 3.2.27. (Federer and Fleming co-area fromula)
If $u \in BV(\Omega)$, then for $\mathcal{L}^1$ a.e. $s \in \mathbb{R}$, the set $\{u > s\}$ has finite perimeter in Ω and

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega)ds.$$

Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega)ds < \infty$, then $u \in BV(\Omega)$. Moreover, for any Borel set $B \subset \Omega$ we have

$$|Du|(B) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(B)ds.$$

Dim. (Sketch)

Sia $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$, allora

$$\int_\Omega u \mathrm{div}\phi dx = \int_{u>0} \int_0^{u(x)} \mathrm{div}\phi ds dx - \int_{u<0} \int_{u(x)}^0 \mathrm{div}\phi dx =$$

per Fubini

$$\int_0^{+\infty} \int_\Omega \chi_{\{u>s\}}(x)\mathrm{div}\phi(x)dxds - \int_{-\infty}^0 \int_\Omega (1 - \chi_{\{u>s\}}(x))\mathrm{div}\phi(x)dxds =$$

e, poiché $\int_\Omega \mathrm{div}\phi dx = 0$,

$$\int_{-\infty}^{+\infty} \int_\Omega \chi_{\{u>s\}} \mathrm{div}\phi dxds \leq \int_{-\infty}^{+\infty} Per(\{u > s\}; \Omega)ds$$

Quindi, passando al sup in ϕ al primo membro, si ha $J(u) \leq \int_{-\infty}^{+\infty} Per(\{u > s\}; \Omega)ds$.
Per la disuguaglianza opposta, si procede provandola per funzioni $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$, stabilendo dunque per esse la formula di coarea, poi si prende una successione in tale spazio che converga ad $u \in BV$ come nel teorema di Meyers e Serrin, si prova che $\chi_{\{u_n>s\}} \to \chi_{\{u>s\}}$ in $L^1(\Omega)$ per a.e. $s \in \mathbb{R}$. Quindi si ha $Per(\{u > s\}; \Omega) \leq \liminf_n Per(\{u_n > s\}; \Omega)$ e per Fatou,

$$\int_{-\infty}^{+\infty} Per(\{u > s\}; \Omega)ds \leq \liminf_n \int_{-\infty}^{+\infty} Per(\{u_n > s\}; \Omega)ds = \lim_{n\to+\infty} J(u_n) = J(u)$$

□

Proof. See [EG] Section 5.5 Theorem 1 and [AFP] Theorem 3.40.

Remark 3.2.28. A consequence of Theorem 3.2.27 is that, for any $u \in BV(\Omega)$, $|Du| \ll \mathcal{H}^{N-1}$. Indeed, for any Borel set $B \subset \Omega$ such that $\mathcal{H}^{N-1}(B) = 0$, co-area formula implies $|Du|(B) = 0$.

Lemma 3.2.29. *Let $u : \Omega \to \mathbb{R}$ be a Lipschitz function, and let $A \subset \mathbb{R}^N$ be a set of measure zero. Then*

$$\mathcal{H}^{N-1}(A \cap u^{-1}(s)) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}.$$

Proof. It is an immediate consequence of the classical co-area formula for Lipschitz functions (see [EG], Section 3.4.2 Theorem 1); that is,

$$0 = \int_A |\nabla u(x)|dx = \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(A \cap u^{-1}(s))ds. \square$$