Lectures Notes

BV functions and sets of finite perimeter

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Chapter 1

Introduction?

1.0 Motivation

Let's consider

$$\inf\left\{\int_\Omega |\nabla u| dx \,:\, u \in W^{1,1}(\Omega), \|u\|_{L^1} = K > 0\right\} =: m_K,$$

where $W^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega, \mathbb{R}^n)\}$ is the Sobolev space. Then there exists a sequence $(u_j \in W^{1,1}(\Omega))_{j\in\mathbb{N}}$ such that $\|u_j\|_{L^1(\Omega)} = K$ and $\|\nabla u_j\|_{L^1(\Omega)} \to m_K$ for $j \to \infty$. Now, in general this does *not* imply that there is a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ which will convergence even only weakly to an $v \in W^{1,1}(\Omega)$ with $\|v\|_{L^1} = K$ and $\|\nabla v\|_{L^1} = m_k$.

The reason for this is essentially because L^1 is not a (topological) dual of any space, though it is contain in one.

Lacks details

Another example is the ***Isoperimetric problem***:

$$\min\{\sigma_{n-1}(\partial F) \,:\, F \text{ with some regularity}, |F| = K > 0\} =: \gamma_k,$$

where $|F| =: \mathcal{L}^n(F)$ denotes the n-dimensional Lebesgue measure.

1.1 Measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*, that is, the collection of all subsets of X.

Definition ((outer) measure)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \leq \sum_{k=1}^\infty \mu(A_k)$ if $A \subset \bigcup_{k=1}^\infty A_k$ (σ-subadditivity)

is called an (outer) measure.

Remark. The (outer) measure is not decreasing, that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition (Restriction of a measure)**.** If $Y \subset X$, the *restriction of* μ *to* Y, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$.

Definition (μ-measurable)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark. This definition is meaningful since *Vitali* found that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathcal{L}^1$-measurable.

Definition (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if holds

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for $A \in \mathfrak{F}$ also $X \setminus A \in \mathfrak{F}$,

(3) for a family $(A_i \in \mathfrak{F})_{i \in I}$ we have have $\bigcup_{i \in I} A_i \in \mathfrak{F}$.

Theorem. *Let μ be a (outer) measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a (at most) countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu(A_j).$$

Definition. Here we collect some important definitions

(1) Let $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.[♯]

(2) The *Borel-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel-algebra are called *Borel sets*.

(3) A (outer) measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A (outer) measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (e.g. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure*.

Theorem. *Let μ be a Radon measure on $\mathbb{R}^n$. We have*

(1) for all $A \subseteq \mathbb{R}^n$ holds $\mu(A) = \inf\{\mu(U) : U \supset A,\, U$ open$\}$ (outer regularity),

(2) for all μ-measurable sets B holds $\mu(B) = \sup\{\mu(K) : K \subset B,\, K$ compact$\}$ (inner regularity).

Theorem (Carathéodory's criteria)**.** *Let μ be a (outer) measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ that satisfy $\operatorname{dist}(A, B) > 0$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$, then μ is a Borel measure.*

Examples.

(1) For $x \in \mathbb{R}^n$ we can define that *dirac measure* by

$$\delta_x(A) := \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

This is in fact a Radon measure.

(2) We define the *counting measure* by

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure (since it is clearly not locally finite).

(3) The also have the well-known *Lebesgue measure* defined by

$$\mathcal{L}^n(A) := \inf\left\{\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes}\right\},$$

where $\mathcal{L}^n(Q_i)$ is equal to the side length of the cubes Q_i to the n-th power. In particular, we have

$$\mathcal{L}^1(A) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R}\right\}$$

and so we can characterize

$$\mathcal{L}^n = \underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n-\text{times}} = \mathcal{L}^{n-1} \times \mathcal{L}^1.$$

[♯] Here $\mathfrak{C} = $ \mathfrak{C}

(4) (**Hausdorff measure**) Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *Hausdorff α-dimensional content of A* as

$$\mathcal{H}^\alpha_\delta(A) := \inf \left\{ \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2} \right)^\alpha \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taking over all the (at most countable) coverings $(C_j \subset \mathbb{R}^n)_{j\in I}$ of A, and set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

Since $\mathcal{H}^\alpha_\delta$ is a not-increasing function in δ the following limit

$$\mathcal{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathcal{H}^\alpha_\delta(A) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(A)$$

always exists in the extended real numbers. This limit is defined to be the *Hausdorff measure.*

Theorem (Hausdorff measure is Borel regular)**.** *$\mathcal{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem (Basic properties of the Hausdorff measure)**.**

(1) $\mathcal{H}^0 = \#$

(2) $\mathcal{H}^1 = \mathcal{L}^1$ *on* $\mathbb{R}$

(3) $\mathcal{H}^\alpha \equiv 0$ *for all* $\alpha > n$ *in* $\mathbb{R}^n$.

(4) $\mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A)$ *for all* $A \subseteq \mathbb{R}^n$ *and* $\lambda > 0$

(5) $\mathcal{H}^\alpha(L(A)) = \mathcal{H}^\alpha(A)$ *for all affine isometry* $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$ we have

$$\begin{aligned}
\mathcal{H}^0(A) &= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j\in I} \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 \mid A \subset \bigcup_{j\in I\subset\mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta \right\} \\
&= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j\in I} 1 \mid A \subset \bigcup_{j\in I} C_j,\, \operatorname{diam} C_j \leq \delta \right\} \\
&= \begin{cases} \operatorname{card}(A) & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

(2) We estimate the Lebesgue measure $\mathcal{L}^1$ from both sides by the Hausdorff measure: Since $\omega_1 = 2 = |(-1, 1)|$ we first get

$$\begin{aligned}
\mathcal{L}^1(A) &= \inf \left\{ \sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j\in I} C_j \right\} \\
&\leq \inf \left\{ \sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j\in I} C_j,\, \operatorname{diam} C_j \leq \delta \right\} = \mathcal{H}^1_\delta(A),
\end{aligned}$$

which is true for all $\delta > 0$ so we obtained $\mathcal{L}^1(A) \leq \mathcal{H}^1(A)$.
Now, we define a partition of $\mathbb{R}$ by $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$ and first fixed $\delta > 0$. These are intervals of diameter δ so for every $j \in I$ we get $\operatorname{diam}(C_j \cap I_{k,\delta}) \leq \delta$. Also we have

$\sum_{k\in\mathbb{Z}} \operatorname{diam}(C_j \cap I_{k,\delta}) \le \operatorname{diam} C_j$, since $I_{k,\delta}$ are a partition $\mathbb{R}$ of disjoint intervals in k. So we get

$$\mathcal{L}^1(A) = \inf\left\{\sum_{j\in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j\in I} C_j\right\} \ge \inf\left\{\sum_{j\in I}\sum_{k\in\mathbb{Z}} \operatorname{diam}(C_j \cap I_{k,\delta}) \mid A \subset \bigcup_{j\in I}\bigcup_{k\in\mathbb{Z}} C_j \cap I_{k,\delta}\right\}$$

since $\operatorname{diam}(C_j \cap I_{k,\delta}) \le \delta$ and after relabeling the index sets I and $\mathbb{Z}$ to an index set $I^{(k,\delta)}$ this last expressions reads

$$\cdots = \inf\left\{\sum_{j\in I^{(k,\delta)}} C_j^{(k)} \mid A \subset \bigcup_{j\in I^{(k,\delta)}} C_j^{(k)},\ \operatorname{diam} C_j^{(k)} \le \delta\right\} \ge \mathcal{H}^1_\delta.$$

And since this is true for every $\delta > 0$ we arrive at the claim.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. If we consider cubes Q_i with side length $\frac{1}{m}$ for any fixed $m \in \mathbb{N}$, we get $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$ and $Q \subset \bigcup_{i=1}^{m^n} Q_i$. From this we can infer

$$\mathcal{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2}\right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m}\right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha}$$

and read off with the crucial assumptions $n < \alpha$

$$\mathcal{H}^\alpha(Q) = \lim_{m\to\infty} \mathcal{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) = \frac{\omega_\alpha}{2^\alpha} \lim_{m\to\infty} n^{\frac{\alpha}{2}} m^{n-\alpha} = 0.$$

This is the claim, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes.

□

Lemma. Let $A \subset \mathbb{R}^n$ and $\delta > 0$ such that $\mathcal{H}^\alpha_\delta = 0$, then follows that $\mathcal{H}^\alpha(A) = 0$.

Proof. Since the Hausdorff content is non-increasing in δ, we have $0 = \mathcal{H}^\alpha_\delta(A) \ge \mathcal{H}^\alpha_\infty(A)$. That convergence means that for every $\varepsilon > 0$ there exists a (at most countable) family of subsets $(C_j)_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

For this to be true the diameter of every C_j must be controlled by $\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon$. So we have $\mathcal{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$ and as $\delta_\varepsilon \searrow 0$ so will $\varepsilon \searrow 0$. But this is the claim $\mathcal{H}^\alpha(A) = 0$. □

Proposition. Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.

(1) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.

(2) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a (at most countable) family of subsets $(C_j)_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathcal{H}^s_\delta(A) + 1 \le \mathcal{H}^s(A) + 1.$$

From this follows

$$\begin{aligned}\mathcal{H}^t_\delta(A) &\le \sum_{j\in I} \omega_t \left(\operatorname{diam} C_j\right)^t = \frac{\omega_t}{\omega_s} s^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \left(\operatorname{diam} C_j\right)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathcal{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0,\end{aligned}$$

which is the claim $\mathcal{H}^t(A) = 0$.

(2) If by contradiction $\mathcal{H}^s(A) < \infty$, then by (1) if follows that $\mathcal{H}^r(A) = 0$ for all $r > s$ and such in particular $r = t$.

$\square$

Definition. We call the *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$

$$\dim_{\mathcal{H}}(A) := \inf\left\{\alpha \geq 0 : \mathcal{H}^\alpha(A) = 0\right\}.$$

Remark. Let $\alpha = \dim_{\mathcal{H}}(A)$. Then one has

@GC please check and edit

$$\mathcal{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and also} \qquad \mathcal{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha,$$

where the first equality is clear from the definition of the Hausdorff dimension and the second fact follows like this: Suppose by contradiction $\mathcal{H}^t(A) < \infty$ for some $t < \alpha$, then by the above proposition we have $\mathcal{H}^r(A) = 0$ for all $r > t$. But this means it even holds for all $r \in (t, \alpha)$ and this is a contradiction to the fact that α is indeed the infimum.

@GC Is this the right argument?

In particular $\mathcal{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Take for example the closed unit Ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that $0 < \mathcal{H}^n(\overline{B(0,1)}) < \infty$ and so $\mathcal{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^\alpha(E) \in (0, \infty)$, then $\mathcal{H}^\alpha \llcorner E$ is a Radon measure. Indeed in general we have the following

Theorem. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Theorem. $\mathcal{H}^n = \mathcal{L}^n$

Proposition. Let $\alpha \geq 0$, $A \subset \mathbb{R}^n$.

(1) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then $\mathcal{H}^\alpha(f(A)) \leq (\mathrm{Lip}(f))^\alpha \mathcal{H}^\alpha(A)$.

(2) If $g : \mathbb{R}^n \to \mathbb{R}^m$ is γ-Hölder$^\sharp$, then $\mathcal{H}^\alpha(g(A)) \leq C_{\alpha,\gamma} \mathcal{H}^\alpha(A)$.

Proof of (2). Fix $\delta > 0$, and take a (at most countable) family of subsets $(C_j)_{j \in I}$ such that $A \subset \bigcup_{j \in I} C_j$ and $\operatorname{diam} C_j \leq \delta$. For $g(C_j)$ we get for all $j \in I$

$$\operatorname{diam} g(C_j) \leq C \left(\operatorname{diam} C_j\right)^\gamma \leq C\delta^\gamma \quad \text{and} \quad g(A) \subseteq \bigcup_{j \in I} g(C_j).$$

Using this, we can infer

$$\mathcal{H}^\alpha_{C\delta^\gamma}(g(A)) \leq \sum_{j \in I} \omega_\alpha \left(\frac{\operatorname{diam} g(C_j)}{2}\right)^\alpha \leq \underbrace{\frac{\omega_\alpha}{2^\alpha} \frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}} \sum_{j \in I} \omega_{\alpha\gamma} \left(\frac{\operatorname{diam} C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all coverings $(C_j)_{j \in I}$ we get

$$\mathcal{H}^\alpha_{C\delta^\gamma}(g(A)) \leq C_{\alpha,\gamma} \mathcal{H}^{\alpha\gamma}_\delta(A)$$

and by sending $\delta \searrow 0$ we arrive at the claim. $\square$

Silyinksi triangle (Wachar Siejinski 1915).
Construction a fractal of triangles in triangles. Specifically take S_k is the union of s^k equilateral triangles with side length 2^{-k}.

$$S = \bigcup_{k=0}^{\infty} S_k \qquad \mathcal{H}^\alpha_{\frac{1}{2^k}} \leq \sum_{j=1}^{3^k} \frac{\omega_\alpha}{2^\alpha} (\operatorname{diam} S_k^j)^\alpha = \frac{\omega_\alpha}{2^\alpha} 3^k 2^{-k\alpha} \searrow 0 \quad as \quad k \to \infty \quad iff \quad \alpha > \frac{\log^3}{\log^2}$$

$$\Rightarrow \mathcal{H}^\alpha(S) = 0 \quad \forall \alpha > \frac{\log 3}{\log 2} \Rightarrow \dim_{\mathcal{H}}(S) \leq \frac{\log 3}{\log 2}$$

Theorem. $\mathcal{H}^n = \mathcal{L}^n$ *on* $\mathbb{R}^n$.

$^\sharp$E.g. $|g(x) - g(y)| \leq C_\gamma |x - y|^\gamma$ for a $\gamma \in (0, 1)$.

Lemma (Vitali covering property for $\mathcal{L}$)**.** For all open U and for all $\delta > 0$ there exists a family of disjunct closed balls $(\overline{B_K})_{k=1}^{\infty}$ such that $\operatorname{diam} B_k < \delta$ and $\mathcal{L}^n(U \setminus \bigcup_{k=1}^{\infty} \overline{B_k}) = 0$

Theorem (isodiametric inequality)**.** *For all $E \subset \mathbb{R}^n$ Lebesgue measurable we have*

$$|E| \le \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n .$$

Proof of first theorem. TODO

Step 2 $\mathcal{L}^n(A) \le \mathcal{H}^n(A)$ for all $A \subset \mathbb{R}^n$ Fix $\delta > 0$. Let $(C_j)_{j\in I}$: $A \subset \bigcup_{j\in I} C_j$, $\operatorname{diam} C_j \le \delta$. From this follows

$$\mathcal{L}^n(A) \le \sum_{j=1}^{\infty} \mathcal{L}^n(C_j) \le \sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n ,$$

where in the last inequality we used the *isometric inequality*. Taking the infimum over all (C_j) we get

$$\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A) \qquad \text{for all} \quad \delta > 0.$$

Step 2 $\mathcal{H}^n_\delta \le C_n \mathcal{L}^n$ for some $C_n \ge 1$

$$\mathcal{L}^n(A) = \inf\left\{\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \mid A \subset \bigcup Q_j\right\} \ge \inf\left\{\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \mid A \subset \bigcup Q_j, \operatorname{diam} Q_j < \delta\right\} = \frac{2^n}{(\sqrt{n})^n \omega_n} \inf\left\{\sum_{j=1}^{\infty} \omega\right.$$

Step 2 By definition of $\mathcal{L}^n$, for all $\delta, \varepsilon > 0$, there exists $(Q_i)_{j=1}^{\infty}$ such that $A \subset \bigcup_{j=1}^{\infty} Q_j$, $\operatorname{diam} Q_j \le \delta$ and $\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \le \mathcal{L}^n(A) + \varepsilon$. There exists $(B^i_j)_{i=1}^{\infty}$ disjoint closed balls such that $B^i_j \subset Q_j$ for all $(\operatorname{diam} B^i_j \subset \delta)$ and $\mathcal{L}^n(\circ Q^j \setminus \bigcup_{i=1}^{\infty} \overline{B}^i_j) = 0 = \mathcal{L}^n(Q_j \setminus \bigcup_{i=1}^{\infty}$

□

Proof of isodiametric inequality. If $E \subset B(x, \frac{\operatorname{diam} E}{2})$ for some x then it's trivial. WLOG, E is compact. $\operatorname{diam} A = \operatorname{diam} \overline{A}$. Steiner symmetrization (1838). Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q : \mathbb{R}^n \to \mathbb{R}$ so that $x = (px, qx)$, $q(x) = x_n$

$$\forall x \in \mathbb{R}^{n-1} \quad E_z := \{t \in \mathbb{R} : (z,t) \in E\} \quad \text{vertical section}$$

define

$$E^s := \left\{x \in \mathbb{R}^n : |q(x)| \le \frac{\mathcal{L}^1(E(p(x)))}{2}\right\}$$

By Fubini's theorem, E_z is $\mathcal{L}^1$-measurable for $\mathcal{L}^{n-1}$-a.e. z, $z \mapsto \mathcal{L}^1(E_z)$ is Lebesgue measurable

$$|E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E^s_z)dz = |E^s|$$

where the first equal sign is a TODO (nebenrechnung), and the second equality follows with Fubini. $\mathcal{L}^1(E^s_z) = \mathcal{L}^1(E_z)$

$E^s_z =$

Now we claim $\operatorname{diam} E^s \le \operatorname{diam} E$. Let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned} &p(m(x)) = p(M(x)) = px \\ &q(m(x)) = q(z) \le q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x). \end{aligned}$$

Let $x, y \in E^s$,

$$|q(x) - q(y)| \le \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\} \overset{WLOG}{=} |q(M(x)) - q(m(y))|$$

$$|x-y|^2 = |p(x-y)|^2 + |q(x-y)|^2 \le \max\{|M(x) - m(y)|, |M(y) - m(x)|\}^2 \le (\operatorname{diam} E)^2.$$

From this follows $|x - y| \le \operatorname{diam} E$ for all $x, y \in E^s$.

Given a $\mathcal{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. $E_0 := E$, $E_i := (E_{i=1})^i$ with $i \in \{1, 2, \dots, n\}$. Then $|E_n| = |E|$, $\operatorname{diam} E_n \le \operatorname{diam} E$ and, if $x \in E_n$, then $-x \in E_n$. From this follows $E_n \subset B\left(0, \frac{\operatorname{diam} E_n}{2}\right)$. And with this we are done! □

1.2 Integration

Let $X \neq \emptyset$, μ be a measure on X.

Definition. (1 A function $u : X \to [-\infty, \infty]$ $\overline{R}$ is μ-measurable if $\{u > t\} = \{x \in X : u(x) > t\}$ is μ-measurable for all $t \in \mathbb{R}$.

(2 u is a μ-simple function is it is μ-measurable and $u(X)$ is countable (that is $u(x) = \sum_{k=1} u_k \chi_{E_k}(x)$)

(3 If u is a non-negative μ-simple function, we define

$$\int_X u d\mu := \sum_{t \in u(X)} t\mu(\{u = t\}) = \sum_{k=1}^{\infty} u_k \mu(E_k) \in [0, \infty]$$

where $0 \cdot \infty = 0$.

(4 Set $u^{\pm} := \max\{\pm u, 0\}$, $u = u^+ - u^-$. If u is μ-simple and $\int_X u^+ d\mu$ or $\int_X u^- d\mu < \infty$, then

$$\int_X u d\mu := \int_X u^+ d\mu - \int_X \mu^- d\mu \in [-\infty, \infty]$$

If v satisfies (4), is called μ-integrable simple function.

(5 If u is μ-measurable, we define the upper and lower integrals of u as

(6 TODO

TODO