Lectures Notes

BV functions and sets of finite perimeter

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Introduction

$^\sharp$ *Geometric Measure Theory* is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces generalizing techniques from differential geometry through measure theoretic arguments. The theory of *functions of bounded variations* and *sets of finite perimeter* is one of the core topics of Geometric Measure Theory, since it deals with extension of the classical notion of Sobolev functions and regular surfaces.

The 1-Laplace operator and BV as a natural extension of $W^{1,1}$

In the Calculus of Variation, the *Direct Method* is a general way of proving the existence of a minimizer for a given functional. More precisely, let X be a topological space and $F : X \to (-\infty, +\infty]$ be a functional. We are interested in finding a minimizer of F in X; that is, a $u \in X$ such that $F(u) \leq F(v)$ for any $v \in X$. Assume that

$$m := \inf\{F(v) \,:\, v \in X\} > -\infty.$$

This ensure the existence of a minimizing sequence $\{v_j\}$; that is, a sequence of elements $v_j \in X$ such that $F(v_j) \to m$. Then, the Direct Method consists in the following steps:

(1) show that $\{v_j\}$ admits a converging subsequence $\{v_{j_k}\}$ and $u \in X$ such that $v_{j_k} \to u$, with respect to a the topology of X;

(2) show that F is (sequentially) lower semicontinuous with respect to the topology of X; that is, if $z_j \to z_0$ in X, then
$$F(z_0) \leq \liminf_{j \to +\infty} F(z_j).$$

If these properties hold true, we can conclude that u is a minimizer of F. Indeed, we have

$$m = \lim_{k \to +\infty} F(v_{j_k}) \geq \liminf_{k \to +\infty} F(v_{j_k}) \geq F(u) \geq m,$$

from which we immediately conclude that $F(u) = \min\{F(v) \,:\, v \in X\}$.

This method is fundamental in proving the existence of solutions to minimization problems related to boundary value problems. Let us consider for instance the classical Dirichlet problem for the Laplace equation on an open set Ω with C^1-smooth boundary:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$. It is possible to see this system as the Euler-Lagrange equations for the functional
$$F(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega f u \, dx$$

defined on the space

$$X = W_0^{1,2}(\Omega) := \{u \in L^2(\Omega) \,:\, Du \in L^2(\Omega; \mathbb{R}^n), u = 0 \text{ on } \partial\Omega\};$$

$^\sharp$These notes have been written for the course of *BV Functions and Sets of Finite Perimeter* held in the Department of Mathematics of the Hamburg Universität. The main references are the books [1, 3, 4]. Please write an email to giovanni.comi@uni-hamburg.de if you have corrections, comments, suggestions or questions.

that is, the space of 2-summable weakly differentiable Sobolev functions with zero trace on $\partial\Omega^{\sharp}$. As customary, we denote by Du the weak gradient of u. Thanks to Poincaré inequality, we can prove that

$$\inf\{F(u) \, : \, u \in W_0^{1,2}(\Omega)\} > -\infty.$$

Hence, we can find the solution looking for minizers of F through the Direct Method: let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence. It is possible to show that $\{u_j\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, which is an Hilbert space, and in particular reflexive: as a consequence, there exists a subsequence $\{u_{j_k}\}$ converging to some $u \in W_0^{1,2}(\Omega)$ with respect to the weak topology. In addition, F is lower semicontinuous with respect to the weak topology, and so we infer the existence of a solution for the minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|Du|^2 - fu\,dx \, : \, u \in W_0^{1,2}(\Omega)\right\}.$$

It seems natural now to wonder if we could substitute the exponent 2 with any $p \in (1,\infty)$. Thanks to the Poincaré inequality and the reflexivity of the L^p-spaces for $p \in (1,\infty)$, it is indeed possible to show that, for any $f \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the problem

$$\min\left\{\int_\Omega \frac{1}{p}|Du|^p - fu\,dx \, : \, u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution, where

$$W_0^{1,p}(\Omega) := \{u \in L^p(\Omega) \, : \, Du \in L^p(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

The minimizers to this problem solves the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\nabla u|\nabla u|^{p-2}\right) =: \Delta_p u$ is the p-Laplace operator.

The next logical step is to consider also the case $p = 1$: for a given $f \in L^\infty(\Omega)$, we want to find a function u which realizes

$$\inf\left\{\int_\Omega |Du| - fu\,dx \, : \, u \in W_0^{1,1}(\Omega)\right\} =: m, \tag{0.0.1}$$

where

$$W_0^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

If we assume $\|f\|_{L^\infty(\Omega)}$ to be sufficiently small, we can again employ the Poincaré inequality to prove that $m \in (-\infty, +\infty]$. Hence, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{1,1}(\Omega)$ such that

$$\lim_{j\to+\infty} \int_\Omega |Du_j| - fu_j\,dx = m.$$

However, in this case we cannot argue as above in the case $p > 1$, since, in general this does *not* imply that the existence of a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ weakly converging to some $u \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |Du| - fu\,dx = m.$$

The reason for this lies in the fact that $L^1(\Omega)$ is not reflexive, and actually it is not the topological dual of any separable space. However, $L^1(\Omega)$ is contained in the space of finite Radon measures on Ω, $\mathcal{M}(\Omega)$, and this space can be see as the dual of the space of continuous functions vanishing on the boundary of Ω, $C_0(\Omega)$.

This fact suggests the definition of a space which contains the Sobolev space $W^{1,1}(\Omega)$ and which, although not reflexive, enjoys the property that bounded sets are weakly* compact: the space of *functions with bounded variation*,

$$BV(\Omega) := \{u \in L^1(\Omega) \, : \, Du \in \mathcal{M}(\Omega;\mathbb{R}^n)\}.$$

$\sharp$ We refer to [2, Chapter 5] and to [3, Chapter 4] for a detailed account on Sobolev spaces.

It is not difficult to prove that the total variation of the Radon measure Du over Ω is indeed lower semicontinuous with respect to the weak* converge of the gradient measures. This indicates that the correct space where to look solutions to (0.0.1) is the space of functions with bounded variation with zero trace[♭],

$$BV_0(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

Finally, it is relevant to mention the fact that the minimizers to (0.0.1) solve the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) =: \Delta_1 u$ is the 1-Laplace operator, which is non trivially defined on nonsmooth functions because of the highly degenerate term $\frac{\nabla u}{|\nabla u|}$.

Minimal area problems and sets of finite perimeter

Other historically relevant problems from the Calculus of Variation are the minimal area problems, among which the most famous example is the *Euclidean isoperimetric problem*: find the possibly unique set with minimal surface area among those with fixed volume. In mathematical terms, if we denote by $|F|$ the n-dimensional volume of a set $F \subset \mathbb{R}^n$ (hence, its Lebesgue measure $\mathscr{L}^n(F)$) and by $\sigma_{n-1}(\partial F)$ its surface area (under the assumption the ∂F is regular enough), we are looking for the set which realizes

$$\inf\{\sigma_{n-1}(\partial F) \,:\, \partial F \in \mathcal{R}, |F| = k\} =: \gamma_k,$$

where $\mathcal{R}$ is a class of sufficiently smooth surfaces and $k > 0$. The Direct Method now consists in considering a minimizing sequence of sets F_j such that

$$\partial F_j \in \mathcal{R}, \quad |F_j| = k \quad \text{and} \quad \sigma_{n-1}(\partial F_j) \to \gamma_k, \tag{0.0.2}$$

and then in trying to prove the convergence (possibly up to subsequences) to some limit set E such that

$$\partial E \in \mathcal{R}, \quad |E| = k \quad \text{and} \quad \sigma_{n-1}(\partial E) = \gamma_k.$$

In order to achieve this result, some compactness property in the family of sets satisfying (0.0.2) is required. In addition, the surface measure σ_{n-1} need to be a lower semicontinuous with respect to the chosen convergence of sets, in the sense that

$$\sigma_{n-1}(\partial E) \le \liminf_{j\to+\infty} \sigma_{n-1}(\partial F_j)$$

if $F_j \to E$ in a suitable sense. However, these compactness and lower semicontinuity properties in general fail to be true in family of sets with regular topological boundary. In addition, we notice that the topological boundary is very unstable under modification of a set by Lebesgue negligible sets. For instance, let

$$E_1 = B(0,1) \quad \text{and} \quad E_2 = B(0,1) \cup (\partial B(0,2) \cap \mathbb{Q}^n)\,.$$

It is plain to see that $|E_1 \Delta E_2| = 0$, so that these two sets are equivalent with respect to the Lebesgue measure, and so they have the same volume. However, their topological boundary, which are smooth surfaces, are very different:

$$\partial E_1 = \partial B(0,1) \quad \text{and} \quad \partial E_2 = \partial B(0,1) \cup \partial B(0,2).$$

The need of ruling out these problems and of recovering a notion of compactness and a lower semicontinuity property for the surface area is one of the main reasons for the birth of Geometric Measure Theory. This theory concerns methods to study the geometric properties of rough, irregular sets from a measure theoretic point of view. In this course we shall see how to exploit this

[♭]It can be proved that the trace of a function with bounded variation is well defined on any C^1-regular surface, as in the Sobolev case.

approach to give a meaningful notion of surface area for an irregular set and to define a suitable class of sets for which we can apply the Direct Method of the Calculus of Variation in order to deal with minimal area problems: the *sets of finite perimeter*. Broadly speaking, the notion of set of finite perimeter extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to minimal area problems and other similar geometric variational minimization problems. More precisely, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ is its characteristic function χ_E is a function with locally bounded variation.

Chapter 1

Basic notions of Measure Theory

1.1 General measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*; that is, the collection of all subsets of X.

Definition 1.1.1 (Measures)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ if $A \subset \bigcup_{k=1}^{\infty} A_k$ (σ-subadditivity),

is called a measure.

It should be noticed that in the literature a mapping as the one in Definition 1.1.1 is also called an *outer measure*, while the name of measure is used to denote the restriction of the mapping to the family of measurable set (see Definition 1.1.4 below). We shall nevertheless follow the notation of [3], in order to be able to assign a measure even to nonmeasurable sets.

Remark 1.1.2. Thanks to σ-subadditivity, any measure is not decreasing; that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition 1.1.3 (Restriction of a measure)**.** If $Y \subset X$, the *restriction of μ to Y*, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$ for any $A \subset X$.

Definition 1.1.4 (μ-measurable sets)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark 1.1.5. This definition is meaningful, since the italian mathematician *Giuseppe Vitali* proved in 1905 that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathscr{L}^1$-measurable [5]. For a modern presentation of his construction, we refer to [4, Section I.1.2].

Definition 1.1.6 (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if the following conditions hold:

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for any $A \in \mathfrak{F}$ we have $X \setminus A \in \mathfrak{F}$,

(3) for any countable family of sets $\{A_i\}_{i \in I}$ such that $A_i \in \mathfrak{F}$ for any $i \in I$ we have have

$$\bigcup_{i \in I} A_i \in \mathfrak{F}.$$

Theorem 1.1.7. *Given any measure μ on X, the family of μ-measurable sets forms a σ-algebra.*

Theorem 1.1.8. *Let μ be a measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu\left(A_j\right).$$

We list now some relevant definitions.

Definition 1.1.9.

(1) Given any $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.

(2) The *Borel σ-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel σ-algebra are called *Borel sets.*

(3) A measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (i.e. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem 1.1.10. *Let μ be a Radon measure on $\mathbb{R}^n$. Then we have*

(1) $\mu(A) = \inf\{\mu(U) : U \supset A, U$ open$\}$ for all $A \subseteq \mathbb{R}^n$ (outer regularity),

(2) $\mu(B) = \sup\{\mu(K) : K \subset B, K$ compact$\}$ for all μ-measurable sets B (inner regularity).

Theorem 1.1.11 (Carathéodory's criterion)**.** *Let μ be a measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ such that* $\mathrm{dist}(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

then μ is a Borel measure.

Not any Borel regular measure is a Radon measure. However, it is possible to obtain a Radon measure as a restriction of a Borel regular one, as stated in the followin theorem.

Theorem 1.1.12. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Example 1.1.13 (Dirac delta)**.** For $x \in \mathbb{R}^n$ we define the *Dirac*$^\sharp$ *measure centered in* x by setting

$$\delta_x(A) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

It is easy to check that this is indeed a Radon measure. In addition, any set in $\mathbb{R}^n$ is δ_x-measurable.

Example 1.1.14 (The counting measure)**.** We define the *counting measure* by setting

$$\#(E) = \begin{cases} \mathrm{card}(E) & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure, since it is clearly not locally finite.

$^\sharp$Named after Paul Adrien Maurice Dirac (1902-1984), English theoretical physicist who shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory". He actually introduced the so-called *Dirac delta function* as a "convenient notation" in his influential 1930 book *The Principles of Quantum Mechanics*. The name "delta function" was chosen since it works like a continuous analogue of the discrete Kronecker delta

$$\delta_{i\,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Indeed, for any sequence $\{a_j\}_{j\in\mathbb{Z}}$, we have

$$\sum_{j=-\infty}^{\infty} a_j \delta_{i\,j} = a_i,$$

and, analogously, for any $x \in \mathbb{R}^n$ and any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Dirac delta satisfies the property

$$\int_{-\infty}^{+\infty} f(y)\delta(x-y)\,dy = \int_{-\infty}^{\infty} f(y)\,d\delta_x(y) = f(x).$$

Example 1.1.15 (The Lebesgue measure)**.** The well-known *Lebesgue measure* is defined by

$$\mathscr{L}^n(A) := \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes} \right\},$$

where $\mathscr{L}^n(Q_i) = (l(Q_i)^n$ and $l(Q_i)$ is the side length of the cube Q_i. It is actually possible to show that in one dimension we have

$$\mathscr{L}^1(A) = \inf \left\{ \sum_{ij=1}^{\infty} \operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R} \right\}$$

and that we can characterize $\mathscr{L}^n$ in an alternative way as

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}} = \mathscr{L}^{n-1} \times \mathscr{L}^1.$$

1.2 The Hausdorff measure

Definition 1.2.1 (Hausdorff content)**.** Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *α-dimensional Hausdorff content* of A as

$$\mathscr{H}^{\alpha}_{\delta}(A) := \inf \left\{ \sum_{j \in I} \omega_{\alpha} \left(\frac{\operatorname{diam} C_j}{2} \right)^{\alpha} \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taken over all the (at most countable) *δ-coverings* $\{C_j\}_{j \in I}$ of A, and we set

$$\omega_{\alpha} := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2} + 1\right)}.$$

We notice that $\mathscr{H}^{\alpha}_{\delta}(A)$ is a non-decreasing function in δ, so that we can take the limit as $\delta \searrow 0$ and it always exists in the extended real numbers. This justifies the following definition.

Definition 1.2.2 (Hausdorff measure)**.** For any $A \subset \mathbb{R}^n$ and $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of A as

$$\mathscr{H}^{\alpha}(A) := \lim_{\delta \searrow 0} \mathscr{H}^{\alpha}_{\delta}(A) = \sup_{\delta > 0} \mathscr{H}^{\alpha}_{\delta}(A).$$

Roughly speaking, we take the limit as $\delta \searrow 0$ since it forces the coverings to follow the local geometry of the set A. Indeed, the key idea behind the definition of the Hausdorff measure is that it should be able to capture the properties of thin sets in $\mathbb{R}^n$ (in particular, Lebesgue negligible sets). As we shall see in the following, if $\alpha = k \in \{1, \ldots, n-1\}$, then $\mathscr{H}^k$ agrees with the k-dimensional surface area on sufficiently regular sets, as for instance k-dimensional planes.

It is not too difficult to prove that, as a consequence of Carathéodory's criterion, Theorem 1.1.11, any Borel set is $\mathscr{H}^{\alpha}$-measurable, for any $\alpha \geq 0$.

Theorem 1.2.3 (Hausdorff measures are Borel regular)**.** *$\mathscr{H}^{\alpha}$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem 1.2.4 (Basic properties of the Hausdorff measure)**.** *The following statements hold true:*

(1) $\mathscr{H}^0 = \#$;

(2) $\mathscr{H}^1 = \mathscr{H}^1_{\delta} = \mathscr{L}^1$ on $\mathbb{R}$, for any $\delta > 0$;

(3) $\mathscr{H}^{\alpha} \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$;

(4) $\mathscr{H}^{\alpha}(\lambda A) = \lambda^{\alpha} \mathscr{H}^{\alpha}(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$;

(5) $\mathscr{H}^{\alpha}(L(A)) = \mathscr{H}^{\alpha}(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$, we have $\mathscr{H}^0_\delta(\{x\}) = 1$ for every $x \in \mathbb{R}^n$ and $\delta > 0$. Indeed,

$$\omega_0 \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 = 1,$$

which implies $\mathscr{H}^0_\delta(\{x\}) \geq 1$, and, on the other hand, we can clearly cover the singleton only with itself. Hence, $\mathscr{H}^0(\{x\}) = 1$ for every $x \in \mathbb{R}^n$. Since $\mathscr{H}^0$ is a Borel measure, it is σ-additive on Borel sets, so that

$$\mathscr{H}^0(A) = \sum_{x \in A} \mathscr{H}^0(\{x\}) = \#A,$$

for any finite or countable set A. Finally, if A is incountable, then A contains a countable set B, and so $\mathscr{H}^0(A) \geq \mathscr{H}^0(B) = +\infty$.

(2) We estimate the Lebesgue measure $\mathscr{L}^1$ from both sides by the Hausdorff measure. Since $\omega_1 = 2 = |(-1,1)|$, for any $\delta > 0$ we first get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\leq \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j,\, \operatorname{diam} C_j \leq \delta \right\} = \mathscr{H}^1_\delta(A), \end{aligned}$$

Now, we define a partition of $\mathbb{R}$ by setting $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$. These are intervals of diameter δ, so that, for every $j \in I$, we get

$$\operatorname{diam}(C_j \cap J_{k,\delta}) \leq \delta. \tag{1.2.1}$$

In addition, we have

$$\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \leq \operatorname{diam} C_j, \tag{1.2.2}$$

since $\{J_{k,\delta}\}_{k \in \mathbb{Z}}$ is a partition $\mathbb{R}$ of essentially disjoint intervals, because $\#(J_{k,\delta} \cap J_{m,\delta}) \leq 1$ for any $k \neq m$. Therefore, by (1.2.2) we get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap J_{k,\delta} \right\}. \end{aligned}$$

We set now $C_j \cap J_{k,\delta} =: \widetilde{C}_{i_{j,k}}$, by relabeling the indexes sets I and $\mathbb{Z}$ to an index set $\widetilde{I}$. Thanks to (1.2.1), we have $\operatorname{diam}(\widetilde{C}_i) \leq \delta$ and so we get

$$\mathscr{L}^1(A) \geq \inf \left\{ \sum_{j \in \widetilde{I}} \widetilde{C}_i \mid A \subset \bigcup_{i \in \widetilde{I}} \widetilde{C}_i,\, \operatorname{diam} \widetilde{C}_i \leq \delta \right\} \geq \mathscr{H}^1_\delta(A).$$

All in all, we get $\mathscr{L}^1 = \mathscr{H}^1_\delta$ for any $\delta > 0$, from which it easily follows $\mathscr{L}^1 = \mathscr{H}^1$ on $\mathbb{R}$.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. It is easy to see that, for any fixed $m \in \mathbb{N}$, Q can be covered by m^n smaller cubes Q_i with side length $\frac{1}{m}$. Clearly, we have $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$. Therefore, we obtain

$$\mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha},$$

from which we deduce that, since $n < \alpha$,

$$\mathscr{H}^\alpha(Q) = \lim_{m\to\infty} \mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} \lim_{m\to\infty} m^{n-\alpha} = 0.$$

Thus, the claim easily follows, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes and $\mathscr{H}^n$ is σ-subadditive.

The proofs of (4) and (5) are left as an exercise. □

Lemma 1.2.5. *Let $A \subset \mathbb{R}^n$ and $\delta_0 > 0$ such that $\mathscr{H}^\alpha_{\delta_0}(A) = 0$, then we have $\mathscr{H}^\alpha(A) = 0$.*

Proof. Since the Hausdorff content is non-increasing in δ, we have $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_\delta(A)$ for any $\delta > 0$. In particular, this means that $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_{\delta_0}(A) = 0$, so that, for every $\varepsilon > 0$, there exists a countable family of sets $\{C_j\}_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

In particular, the second condition immediately implies

$$\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon.$$

Hence, we have $\mathscr{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$, and $\delta_\varepsilon \searrow 0$ if and only if $\varepsilon \searrow 0$. This implies the claim $\mathscr{H}^\alpha(A) = 0$. □

Proposition 1.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$.

(2) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a countable family of subsets $\{C_j\}_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathscr{H}^s_\delta(A) + 1 \le \mathscr{H}^s(A) + 1.$$

From this, it follows that

$$\begin{aligned}\mathscr{H}^t_\delta(A) \le \sum_{j\in I} \omega_t \left(\frac{\operatorname{diam} C_j}{2}\right)^t &= \frac{\omega_t}{\omega_s} 2^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathscr{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0,\end{aligned}$$

which implies the claim $\mathscr{H}^t(A) = 0$.

(2) If by contradiction $\mathscr{H}^s(A) < \infty$, then by (1) if follows that $\mathscr{H}^r(A) = 0$ for all $r > s$ and in particular for $r = t$, which is clearly absurd.

□

Definition 1.2.7. We call the *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ the number

$$\dim_{\mathscr{H}}(A) := \inf\left\{\alpha \ge 0 : \mathscr{H}^\alpha(A) = 0\right\}.$$

Remark 1.2.8. Let $\alpha = \dim_{\mathscr{H}}(A)$. Then one has

$$\mathscr{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and} \qquad \mathscr{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha. \tag{1.2.3}$$

The first part of (1.2.3) follows clearly from the definition of the Hausdorff dimension. The second, instead, can be proved by contradiction. Suppose by contradiction that $\mathscr{H}^t(A) < \infty$ for some $t < \alpha$, then, by the Proposition 1.2.6, we have $\mathscr{H}^r(A) = 0$ for all $r > t$. This implies

$$\alpha = \inf\left\{\beta \ge 0 : \mathscr{H}^\beta(A) = 0\right\} \le t < \alpha,$$

which is clearly absurd.

It should be noticed that, in general, $\mathscr{H}^\alpha(A)$ can be any number in $[0, +\infty]$.

We state now an important result on the equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure, whose proof we postpone to the end of the section.

Theorem 1.2.9. *$\mathscr{H}^n_\delta = \mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$, for any $\delta > 0$.*

Remark 1.2.10. As a consequence of Theorem 1.2.9, we see that $\mathscr{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Indeed, it is not bounded on some compact sets. Take for example the closed unit ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that $0 < \mathscr{H}^n(\overline{B(0,1)}) < \infty$ and so, by Proposition 1.2.6, $\mathscr{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

Even though $\mathscr{H}^\alpha$ is not a Radon measure for $\alpha \in [0, n)$, it is possible to show that its restriction to some suitable Borel set is indeed a Radon measure.

Proposition 1.2.11. *If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^\alpha(E) \in (0, \infty)$, then $\mathscr{H}^\alpha \llcorner E$ is a Radon measure.*

Proof. It is a simple consequence of Theorem 1.1.12. □

We investigate now the behaviour of the Hausdorff measure under the action of Lipschitz and Hölder functions. We recall first the definition of such family of functions.

Definition 1.2.12 (Lipschitz and Hölder functions)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set.

(1) We say that $f : \Omega \to \mathbb{R}^m$ is *Lipschitz continuous* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y| \quad \text{for any } x, y \in \Omega. \tag{1.2.4}$$

The smallest constant for which (1.2.4) holds is called the *Lipschitz constant* of f, denoted by $\mathrm{Lip}(f)$ and alternatively characterized by

$$\mathrm{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} \mid x, y \in \Omega, x \neq y \right\}. \tag{1.2.5}$$

(2) Let $\gamma \in (0, 1)$. We say that $f : \Omega \to \mathbb{R}^m$ is *γ-Hölder continuous* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\gamma \quad \text{for any } x, y \in \Omega. \tag{1.2.6}$$

From this point on, we shall refer to Lipschitz continuous and Hölder continuous functions simply as Lipschitz and Hölder functions.

Exercise 1.2.13. Show that any Lipschitz or γ-Hölder function (for some $\gamma \in (0, 1)$) is indeed continuous.

Remark 1.2.14. Lipschitz functions can be seen as 1-Hölder functions. Indeed, for any open set $\Omega \subset \mathbb{R}^n$ and any $\gamma \in [0, 1]$, we can define the space $C^{0,\gamma}(\Omega; \mathbb{R}^m)$ of γ-Hölder functions as the set of continuous functions $f : \Omega \to \mathbb{R}^m$ for which there exists a constant $C > 0$ such that (1.2.6) holds. If $\gamma = 0$, we have $C^{0,0}(\Omega; \mathbb{R}^m) = C^0(\Omega; \mathbb{R}^m)$.

Exercise 1.2.15. Let $\gamma > 1$ and $f : \Omega \to \mathbb{R}^m$ be such that there exists a constant $C > 0$ such that (1.2.6) holds. Show that f is constant.

Proposition 1.2.16. *Let $\alpha \geq 0$, $A \subset \mathbb{R}^n$.*

(1) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then $\mathscr{H}^\alpha(f(A)) \leq (\mathrm{Lip}(f))^\alpha \mathscr{H}^\alpha(A)$.

(2) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is γ-Hölder, for some $\gamma \in (0, 1)$, then $\mathscr{H}^\alpha(f(A)) \leq C_{\alpha,\gamma} \mathscr{H}^{\alpha\gamma}(A)$.

Proof. Thanks to Remark 1.2.14, it is enough to prove (2) for any $\gamma \in (0, 1]$. Fix $\delta > 0$, and take a countable family of sets $\{C_j\}_{j \in I}$ such that $A \subset \bigcup_{j \in I} C_j$ and $\operatorname{diam} C_j \leq \delta$. It is clear that

$$f(A) \subseteq \bigcup_{j \in I} f(C_j).$$

Thanks to (1.2.6), we see that $f(C_j)$ satisfies

$$\operatorname{diam} f(C_j) \leq C \left(\operatorname{diam} C_j\right)^\gamma \leq C\delta^\gamma,$$

where $C = \mathrm{Lip}(f)$ is $\gamma = 1$. Hence, we obtain

$$\mathscr{H}^{\alpha}_{C\delta^{\gamma}}(f(A)) \leq \sum_{j\in I} \omega_{\alpha} \left(\frac{\operatorname{diam} f(C_j)}{2}\right)^{\alpha} \leq \underbrace{\frac{\omega_{\alpha}}{2^{\alpha}} \frac{C^{\alpha}2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}} \sum_{j\in I} \omega_{\alpha\gamma} \left(\frac{\operatorname{diam} C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all δ-coverings $\{C_j\}_{j\in I}$ we get

$$\mathscr{H}^{\alpha}_{C\delta^{\gamma}}(g(A)) \leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}_{\delta}(A),$$

where $C_{\alpha,\gamma} = \mathrm{Lip}(f)^{\alpha}$, if $\gamma = 1$. By sending $\delta \searrow 0$ we conclude the proof. $\square$

Remark 1.2.17 (Sierpinski triangle (Waclaw Sierpinski 1915))**.** One can construct a fractal triangle as follows:

1. Take S_0 to be a equilateral triangle.
2. Divide S_0 evenly into four smaller equilateral triangles. Cut out the triangle in the center.
3. Now do the step in 2. with these three equilateral triangles indefinitely.

So the S_k's are the union of 3^k equilateral triangles with side length 2^{-k}. We define

$$S := \bigcup_{k=0}^{\infty} S_k$$

and compute

$$\mathscr{H}^{\alpha}_{\frac{1}{2^k}}(S) \leq \sum_{j=1}^{3^k} \frac{\omega_{\alpha}}{2^{\alpha}} \left(\operatorname{diam} {S_k}^j\right)^{\alpha} = \frac{\omega_{\alpha}}{2^{\alpha}} 3^k 2^{-k\alpha},$$

which goes to zero for $k \to \infty$ if and only if $\alpha > \frac{\log^3}{\log^2}$. So we can conclude that for all $\alpha > \frac{\log 3}{\log 2}$ we have $\mathscr{H}^{\alpha}(S) = 0$ and with that we found:

$$\dim_{\mathcal{H}}(S) \leq \frac{\log 3}{\log 2}.$$

Next we aim to proof an important characterization for the Hausdorff measure for positive integers:

Theorem 1.2.18. *$\mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$.*

The proof is crucially based on the two following statements.

Lemma 1.2.19 (Vitali covering property for $\mathscr{L}^n$)**.** *For all open U and for all $\delta > 0$ there exists a family of disjoint closed balls $\{\overline{B_k}\}_{k=1}^{\infty}$ such that* $\operatorname{diam} B_k < \delta$ *and* $\mathscr{L}^n(U \setminus \bigcup_{k=1}^{\infty} \overline{B_k}) = 0$.

Theorem 1.2.20 (isodiametric inequality)**.** *For all $\mathscr{L}^n$-measurable sets $E \subset \mathbb{R}^n$ we have*

$$|E| \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n.$$

Proof of theorem 1.2.18. We show the proof in three steps.

(Step 1) To show: $\mathscr{L}^n(A) \leq \mathscr{H}^n(A)$ for all $A \subset \mathbb{R}^n$.
Fix $\delta > 0$. Let $\{C_j\}_{j\in I}$: $A \subset \bigcup_{j\in I} C_j$, $\operatorname{diam} C_j \leq \delta$. From this follows

$$\mathscr{L}^n(A) \leq \sum_{j=1}^{\infty} \mathscr{L}^n(C_j) \leq \sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n,$$

where in the last inequality we used the *isometric inequality* (e.g. theorem 1.2.20). Taking the infimum over all $\{C_j\}$ we arrive at claim

$$\mathscr{L}^n(A) \leq \mathscr{H}^n_{\delta}(A) \qquad \text{for all} \quad \delta > 0.$$

(Step 2) To show: $\mathscr{H}^n_\delta \le C_n \mathscr{L}^n$ for some $C_n \ge 1$.
With the definition of the Lebesgue measure we get

$$\begin{aligned}
\mathscr{L}^n(A) &= \inf\left\{\sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j\right\} \\
&= \inf\left\{\sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j,\, \operatorname{diam} Q_j < \delta\right\} \\
&= \frac{2^n}{\left(\sqrt{n}\right)^n \omega_n} \inf\left\{\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} Q_j}{2}\right)^2 \mid A \subset \bigcup Q_j,\, \operatorname{diam} Q_j < \delta\right\} \\
&\ge \frac{1}{C_n} \mathscr{H}^n_\delta(A),
\end{aligned}$$

where for the second equality we used that

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}}, \qquad \mathscr{L}^1 = \mathscr{H}^1_\delta \quad \text{in } \mathbb{R} \quad \text{for all } \delta > 0, \quad \text{and} \quad \mathscr{L}^n(Q_j) = \left(\frac{\operatorname{diam} Q_j}{\sqrt{n}}\right)^n.$$

(Step 3) To show: $\mathscr{H}^n_\delta(A) \le \mathscr{L}^n(A) + \varepsilon$ for any $\varepsilon > 0$.
By definition of $\mathscr{L}^n$: For all $\delta, \varepsilon > 0$, there exists a family $\{Q_j\}_{j=1}^{\infty}$ such that $A \subset \bigcup_{j=1}^{\infty} Q_j$, $\operatorname{diam} Q_j \le \delta$ and $\sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \le \mathscr{L}^n(A) + \varepsilon$. Now, with lemma 1.2.19, there exists a family $(B^i_j)_{i=1}^{\infty}$ of disjoint closed balls such that $B^i_j \subset Q_j$ for all ($\operatorname{diam} B^i_j \le \delta$) and

$$\mathscr{L}^n\left(\mathring{Q}_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = 0 = \mathscr{L}^n\left(Q_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right).$$

So with step 2 we also have

$$\mathscr{H}^n_\delta\left(Q_j \setminus \bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = 0,$$

from which we can infer that

$$\begin{aligned}
\mathscr{H}^n_\delta(A) &\le \sum_{j=1}^{\infty} \mathscr{H}^n_\delta(Q_j) = \sum_{j=1}^{\infty} \mathscr{H}^n_\delta\left(\bigcup_{i=1}^{\infty} \overline{B^i_j}\right) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mathscr{H}^n_\delta\left(B^i_j\right) \le \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \underbrace{\omega_n \left(\frac{\operatorname{diam} B^i_j}{2}\right)^n}_{=\mathscr{L}^n\left(\overline{B^i_j}\right)} \\
&= \sum_{j=1}^{\infty} \mathscr{L}^n\left(\bigcup_{i=1}^{\infty} \overline{B^i_j}\right) = \sum_{j=1}^{\infty} \mathscr{L}^n(Q_j) \\
&\le \mathscr{L}^n(A) + \varepsilon.
\end{aligned}$$

And since the $\varepsilon > 0$ is arbitrary in (step 2) we arrive at the claim. □

Proof of isodiametric inequality (lemma 1.2.20). If $E \subset B(x, \frac{\operatorname{diam} E}{2})$ for some x then it's trivial. w.l.o.g, E is compact. $\operatorname{diam} A = \operatorname{diam} \overline{A}$. Steiner symmetrization (1838). Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q : \mathbb{R}^n \to \mathbb{R}$ so that $x = (px, qx)$, $q(x) = x_n$

$$\forall z \in \mathbb{R}^{n-1} \quad E_z := \{t \in \mathbb{R} : (z,t) \in E\} \quad \text{vertical section}$$

define

$$E^s := \left\{x \in \mathbb{R}^n : |q(x)| \le \frac{\mathscr{L}^1(E_{p(x)})}{2}\right\}.$$

By Fubini's theorem, E_z is $\mathscr{L}^1$-measurable for $\mathscr{L}^{n-1}$-a.e. z, $z \mapsto \mathscr{L}^1(E_z)$ is Lebesgue measurable and so we get

$$|E| = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z^s)dz = |E^s|,$$

where the first equality follows with Fubini, and the second equal sign is due to

$$(E^s)_z = \{t \in \mathbb{R} : (z,t) \in E^s\} = \left\{t \in \mathbb{R} : |t| \leq \frac{\mathscr{L}^1(E_z)}{2}\right\} = \left[-\frac{\mathscr{L}^1(E_z)}{2}, \frac{\mathscr{L}^1(E_z)}{2}\right].$$

Now we claim

$$\operatorname{diam} E^s \leq \operatorname{diam} E.$$

To proof this, let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned} &p(m(x)) = p(M(x)) = px \\ &q(m(x)) = q(z) \leq q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x). \end{aligned}$$

Let $x, y \in E^s$,

$$|q(x) - q(y)| \leq \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\} \overset{w.l.o.g.}{=} |q(M(x)) - q(m(y))|$$

$$|x-y|^2 = |p(x-y)|^2 + |q(x-y)|^2 \leq \max\{|M(x) - m(y)|, |M(y) - m(x)|\}^2 \leq (\operatorname{diam} E)^2.$$

From this follows $|x - y| \leq \operatorname{diam} E$ for all $x, y \in E^s$.

Given a $\mathscr{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. $E_0 := E$, $E_i := (E_{i=1})^i$ with $i \in \{1, 2, \dots, n\}$. Then $|E_n| = |E|$, $\operatorname{diam} E_n \leq \operatorname{diam} E$ and, if $x \in E_n$, then $-x \in E_n$. From this follows $E_n \subset B\left(0, \frac{\operatorname{diam} E_n}{2}\right)$. And with this we are done! □

1.3 Integration and Radon measures

Let $X \neq \emptyset$, and μ be a measure on X.

Definition 1.3.1.

(1) A function $u : X \to [-\infty, \infty] =: \overline{\mathbb{R}}$ is *μ-measurable* if $\{u > t\} = \{x \in X : u(x) > t\}$ is μ-measurable for all $t \in \mathbb{R}$.

(2) u is a *μ-simple function* if it is μ-measurable and $u(X)$ is countable (that is $u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x)$)

(3) If u is a non-negative μ-simple function, we define

$$\int_X u d\mu := \sum_{t \in u(X)} t\mu(\{u = t\}) = \sum_{k=1}^{\infty} u_k \mu(E_k) \in [0, \infty]$$

where $0 \cdot \infty = 0$.

(4) Set $u^{\pm} := \max\{\pm u, 0\}$, $u = u^+ - u^-$. If u is μ-simple and $\int_X u^+ d\mu$ or $\int_X u^- d\mu < \infty$, then

$$\int_X u d\mu := \int_X u^+ d\mu - \int_X \mu^- d\mu \in [-\infty, \infty]$$

If v satisfies (4), is called μ-integrable simple function.

(5) If u is μ-measurable, we define the upper and lower integrals of u as

$$\int_X^* u d\mu := \inf\left\{\int_X v d\mu \mid v \geq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

or

$$\int_{*X} u d\mu := \sup\left\{\int_X v d\mu \mid v \leq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

respectively. If

$$\int_{*X} u d\mu = \int_X^* u d\mu,$$

then u is μ-integrable.

(6) A measurable function u is *μ-summable* if $|u|$ is μ-integrable and

$$\int_X |u| d\mu < \infty$$

(7) Now we define the following sets

$L^1(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mu\text{-summable}\}$
$L^1_{\text{loc}}(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid u\chi_K \text{ is } \mu\text{-summable for all } K \subset X \text{ compact}\}$
$L^p(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid |u|^p \text{ is } \mu\text{-summable}\}$
$L^1_{\text{loc}}(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid |u|^p\chi_K \text{ is } \mu\text{-summable for all } K \subset X \text{ compact}\}$

Definition 1.3.2. If $u : X \to [0,\infty]$ μ-measurable, then we define $\nu = u\mu$ (or $\mu \llcorner u$) as

$$\nu(A) = \int_A u d\mu = \int_X u\chi_A d\mu \qquad \text{for all} \quad \mu\text{-measurable } A$$

Definition 1.3.3 (Radon measure)**.** Given a Radon measure μ on an open subset $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to [-\infty,\infty]$ locally μ-summable. Then $\nu := f\mu$ defined by

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A\, \mu\text{-measurable}$$

is said to be a *signed Radon measure* on Ω.

Definition 1.3.4 (Radon measure)**.** Given a non-negative Radon measure μ on an open subset $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ is locally μ-summable. Then we set $\nu := f\mu$ the vector valued Radon measure satisfying

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A\, \mu\text{-measurable}$$

is said to be a *signed Radon measure* on Ω.

Definition 1.3.5. (alternative approach). Let $\Omega \subset \mathbb{R}^n$ be open.

- A non-negative Radon measure is a mapping $\mu : \mathcal{B}\Omega \to [0,\infty]$ which is σ-additive and finite compact sets.
- A real (vector valued) Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ which is σ-additive and its total variation $|\mu|$ is finite on compact sets; that is

$$|\mu|(B) := \sup\left\{\sum_{j=1}^{\infty} |\mu(B_j)| \mid K = \bigcup_j B_j, B_i \cap B_i = \emptyset \text{ if } i \neq j\right\} \qquad \text{for all } K \text{ compact in } \Omega.$$

  In particular, $\sum \mu(B_j)$ is absolutely convergent for all $\{B_j\}$ partition of a compact set K.
- We say that a non-negative Radon measure $\mu : \mathcal{B}(\Omega) \to [0,\infty)$ is finite if $\mu(\Omega) < \infty$; and we denote $\mu \in \mathcal{M}^+(\Omega)$.
- We say that a non-negative vector-valued Radon measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is finite if $|\mu|(\Omega) < \infty$; and we denote $\mu \in \mathcal{M}(\Omega,\mathbb{R}^m)$.

Remarks (Basic facts)**.**

- If $\mu \in \mathcal{M}_{loc}(\Omega,\mathbb{R}^m)$, then $|\mu| \in \mathcal{M}^+_{loc}(\Omega)$.

$$|\mu|(B) := \sup\{\sum_{j=1}^{\infty} |\mu(B_j)| \mid B = \bigcup B_j, B_j \cap B_i = \emptyset\, i \neq j, B_j \in \otimes\}$$

- The total variation is the smallest non-negative Radon measure ν such that $\nu(B) \geq |\mu(B)|$ for all $B \in \mathcal{B}(\Omega)$.
- If $\mu \in \mathcal{M}(\Omega)$, we define the positive and negative parts of μ (Jordan decomposition (unique)

$$\mu^+ = \frac{|\mu| + \mu}{2} \qquad \mu^- = \frac{|\mu| - \mu}{2} \qquad \mu^{\pm} \geq 0.$$

Lemma 1.3.6. *If $\mu \in \mathcal{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu; \mathbb{R}^m)$, then $f\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $|f\mu| = |f|\mu$*

Proof. Let $B \in \mathcal{B}(\Omega)$.

- $|f\mu(B)| = |\int_B f d\mu| \leq \int_B |f| d\mu$. From this follows $|f\mu| \leq |f|\mu$ From this follows $f\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$
- Let $\varepsilon > 0$ and $D = \{z_h\}_{h\in\mathbb{N}}$ countable dense set in S^{m-1}, let $B \in \mathcal{B}(\Omega)$. We define

$$\sigma(x) := \min\{h \in \mathbb{N} : f(x)z_h \geq (1-\varepsilon)|f(x)|\}$$

it is Borel measurable. Further define $B_h = \sigma^{-1}(\{h\}) \cap B \in \mathcal{B}(\Omega)$, $B = \bigcup_{h\in\mathbb{N}} B_h$, $B_h \cap B_k =$ if $h \neq k$. From this follows

$$\int_B |f| d\mu = \sum_{k\in\mathbb{N}} \int_{B_h} |f| d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} \int_{B_h} f z_h d\mu \leq \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} |f_\mu|(B_h) \leq \frac{1}{1-\varepsilon} |f_\mu|(B),$$

where we used

$$\int_{B_h} f z_h d\mu = z_h \int_{B_h} f d\mu \leq \left| \int_{B_h} f d\mu \right| = |f_\mu|(B_h)$$

□

Definition 1.3.7. Let μ be a non-negative measure on Ω.

- We say that μ is concentrated on a set $E \subset \Omega$ if

$$\mu(\Omega \setminus E) = 0.$$

- We call the support of μ, $\operatorname{supp} \mu$, the smallest closed set on which μ is concentrated :

$$\operatorname{supp} \mu = \bigcap_{C \text{ closed}, \mu(\Omega \setminus C) = 0} C.$$

Exercise 1.3.8. Equivalently,

$$\operatorname{supp} \mu = \{x \in \Omega \mid \mu(B(x,r)) > 0, \forall B(x,r) \subset \Omega\}$$

@GC: A counterexample?!

Definition 1.3.9. 1. Let $\mu \in \mathcal{M}^+(\Omega)$, $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$. We say that μ is absolutely continuous w.r.t μ, and we write $\nu << \mu$, if for all $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$, then $|\mu|(B) = 0$.

2. If $\mu, \nu \in \mathcal{M}^+(\Omega)$, we say that they are mutually singular if there exists $E, F \in \mathcal{B}(\Omega)$ such that $\mu(F) = 0$, $\mu(E) = 0$ and

$$\mu(B) = \mu(B \cap E) \qquad \text{and} \qquad \nu(B) = \nu(B \cap F)$$

for all $B \in \mathcal{B}(\Omega)$ and we write $\mu \perp \nu$. If $\mu, \nu \in \mathcal{M}(\Omega, \mathbb{R}^m)$,

$$\mu \perp \nu :\iff |\mu| \perp |\mu|$$

Theorem 1.3.10 (Radon-Nikodyn)**.** *Let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$. Then there exists exactly one $\nu^{ac}, \nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m)$ such that $\nu^{ac} << \mu$, $\nu^s \perp \mu$ and $\nu = \nu^{ac} + \nu^s$. In addition, there exists exactly one $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that $\nu^{ac} = f\mu$.*
In particular, if $\mu = \mathscr{L}^n$, every $\mu \in \otimes; \mathbb{R}^\Updownarrow$ can be uniquely decomposed in

$$\mu = f\mathscr{L}^n + \nu^s, \nu^s \perp \mathscr{L}^n \qquad f \in L^1(\Omega; \mathbb{R}^n).$$

Definition 1.3.11. We say that a property holds $|\mu|$-almost everywhere or for $|\mu|$-almost every x if the set where the property does not hold is $|\mu|$-negligable; that is, it has zero $|\mu|$-measure.

Corollary 1.3.12 (Polar decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Then there exists a unique $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $|f(x)| = 1$ $|\mu|$-a.e and $\mu = f|\mu|$.*

Proof of corollary. Apply Radon-Nikodyn to μ and $|\mu|$. We know that $|\mu(B)| \leq |\mu|(B)$ for all $B \in \mathcal{B}(\Omega)$. From this follows $\mu << |\mu|$, and so there exists $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $\mu = f|\mu|$. We proved that $|f|\mu|| = |f||\mu|$, hence we obtain

$$|mu| = |f|\mu|| = |f||\mu| \qquad (|f|-1)|mu| = 0.$$

This means that we have

$$\int_\Omega (|f|-1) d|mu| = 0,$$

which yields $|f(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$. □

Corollary 1.3.13 (Hahn decomposition). *Let $\mu \in \mathcal{M}(\Omega)$, there exists a unique $A \in \mathcal{B}(\Omega)$ (up to $|\mu| - negligible$) such that*

$$\mu^+ = \mu \llcorner A \qquad \mu^- = -\mu \llcorner (\Omega \setminus A).$$

Proof. By the polar decomposition, $\mu = f|\mu|$, $f(x) \in \{\pm 1\}$ for $|\mu|$-a.e. $x \in \Omega$.

$$A := \{f = 1\} \qquad f(x) = \chi_A - \chi_{\Omega \setminus A} \qquad \mu^+ - \mu^- = \mu = \chi_A |\mu| - \chi_{\Omega\setminus A}|\mu|,$$

where we used the Jordan decomposition (which is unique). □

1.4 Duality for Radon measures

Another characterization is given via the duality with continuous functions.

Definition 1.4.1. We say that $B \Subset \Omega$ if $\overline{B} \subset \Omega$ and it is compact in Ω.

$$C^0_C(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \operatorname{supp} u \Subset \Omega\}$$

$$C^0_0(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \forall \varepsilon > 0 \ \exists K \subset \Omega : |u(x)| < \varepsilon \quad \forall x \notin K\}$$

$$\|u\|_\infty := \sup_{x \in \Omega} |u(x)|$$

Remark 1.4.2. $C^0_0(\Omega; \mathbb{R}^m) = \overline{C^0_c(\Omega; \mathbb{R}^n)}^{\|\cdot\|_\infty}$, $(C^0_0(\Omega; \mathbb{R}^n), \|\cdot\|_\infty)$ is Banach. C^0_c is separable, locally convex, topological vector space with the following topology:

$$\varphi_k \longrightarrow \varphi \quad \text{in} \quad C^0_c \qquad :\Longleftrightarrow \qquad \|\varphi_k - \varphi\|_\infty \to 0 \quad \text{and} \quad \text{there exist } K \subset \Omega : \operatorname{supp} \varphi \cup \bigcup_{k \in \mathbb{N}} \operatorname{supp} \varphi_k \subset K$$

Theorem 1.4.3 (Lusin). *Let μ Borel on Ω and $u : \Omega \to \mathbb{R}$ is μ-measurable, $u \equiv 0$ in $\Omega \setminus E$ with $\mu(E) < \infty$. Then for all $\varepsilon > 0$ there exists $v \in C^0(\Omega)$ such that $\|v\|_\infty \leq \|u\|_\infty$*

$$\mu(\{x \in \Omega : v(x) \neq u(x)\}) < \varepsilon.$$

Remark 1.4.4. An equivalent formulation state that, under the additional assumption $\mu(\Omega) < \infty$, then there exists $\{K_h\}$ compact sets such that

$$\mu(\Omega \setminus \bigcup K_l) = 0 \qquad \text{and} \qquad u\big|_{K_h} \text{ is continuous.}$$

There exists $\{u_h\} \in C^0(\Omega)$: $u = u_h$ on K_h and $\|u_h\|_\infty \leq \|u\|_\infty$.

Proposition 1.4.5. *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Then for all $A \subset \Omega$ open we have*

$$|\mu|(A) = \sup \left\{ \int_\Omega \varphi \cdot d\mu \mid \varphi \in C^0_c(A; \mathbb{R}^m), \|\varphi\|_\infty \leq 1 \right\}, \tag{1.4.1}$$

where we understand

$$\int_\Omega \varphi \cdot d\mu := \sum_{j=1}^m \int_\Omega \varphi_j d\mu_j.$$

Proof. Polar decomposition implies that $\mu = f|\mu|$, $|f| = 1$ μ-a.e. So we get

$$\int_\Omega \varphi \cdot d\mu = \int_A \varphi \cdot f d|\mu| \leq |\mu|(A).$$

By Lusin theorem, for all $\varepsilon > 0$ there exists $\varphi \in C^0(A;\mathbb{R}^m)$ such that $\|\varphi\|_\infty \leq 1$ and

$$|\mu| (\{x \in A : \varphi(x) \neq f(x)\}) < \varepsilon.$$

Take $K \subset A$ compact such that $|\mu|(A \setminus K) < \varepsilon$. Construct $\eta \in C_c^\infty(A)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on K, $\tilde{\varphi} = \varphi\eta \in C_c^0(A;\mathbb{R}^m)$ and

$$|\mu| (\{x : \tilde{\varphi}(x) \neq f(x)\}) \leq |\mu|(A \setminus K) + |\mu| (\{x : \varphi(x) \neq f(x)\}) \leq 2$$

to get

$$\int_A \tilde{\varphi} \cdot d\mu \geq |\mu|(K) - 2\varepsilon$$

and by sending K to A and $\varepsilon \searrow 0$ we arrive at the claim. □

Proposition 1.4.5 shows that, given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$, we can define a linear continuous functional $L_\mu : C_0^0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu,$$

for any $\varphi \in C_0^0(\Omega;\mathbb{R}^m)$. In addition, the operatorial norm of L_μ is equal to $|\mu|(\Omega)$, since, by the density of C_c^0 in C_0^0 with respect to the supremum norm and by (1.4.1), we have

$$\begin{aligned} \|L_\mu\| &:= \sup\{L_\mu(\varphi) : \varphi \in C_0^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \leq 1\} \\ &= \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \leq 1\right\} = |\mu|(\Omega). \end{aligned}$$

This suggests that it is possible to characterize $\mathcal{M}(\Omega;\mathbb{R}^m)$ as a dual space. In such a way, we gain yields a weaker topology on the space of vector valued Radon measure, and therefore weak* compactness of bounded sequences.

Theorem 1.4.6. (Riesz Representation Theorem) *Let $L : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a continuous linear functional; that is, L is linear and satisfies*

$$\sup\{L(\phi) : \phi \in C_0(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

Moreover,

$$|\mu|(\Omega) = \sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} = \|L\|.$$

For the proof we refer to [1, Theorem 1.54].

The following corollary is a direct consequence of the global version of the Riesz Representation Theorem.

Corollary 1.4.7. *Let $L : C_c(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1, \operatorname{supp}(\phi) \subset K\} < \infty,$$

for any compact set $K \subset \Omega$. Then there exists a unique $\mu \in \mathcal{M}_{\rm loc}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Thus we can identify any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with a continuous linear functional on $C_0(\Omega;\mathbb{R}^m)$, written as

$$L_\mu(\phi) := \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m),$$

and analogously $\mathcal{M}_{\rm loc}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega;\mathbb{R}^m)$.

This leads us to the following notion.

Definition 1.4.8. Given a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega)$, we say that μ_k *weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega; \mathbb{R}^m).$$

If $\{\mu_k\}$ and μ are in $\mathcal{M}_{\mathrm{loc}}(\Omega)$, we say that μ_k *locally weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega; \mathbb{R}^m).$$

Lemma 1.4.9. *Let $\{\mu_k\} \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ be a weak-star convergent sequence, and let μ be its limit. Then we have*

$$\limsup_{k \to +\infty} |\mu_k|(\Omega) < \infty$$

and

$$|\mu|(\Omega) \leq \liminf_{k \to +\infty} |\mu_k|(\Omega).$$

Proof. The first assertion follows from Uniform Boundedness Principle (Banach-Steinhaus Theorem), since $L_{\mu_k}(\phi) \to L_\mu(\phi)$ for each $\phi \in C_0(\Omega; \mathbb{R}^m)$ and therefore $\{L_{\mu_k}(\phi)\}$ is a bounded sequence in $\mathbb{R}$.
The second inequality comes from:

$$|L_{\mu_k}(\phi)| \leq \|\phi\|_\infty |\mu_k|(\Omega)$$

then, passing to the limit we have $|L_\mu(\phi)| \leq \liminf_{k \to +\infty} \|\phi\|_\infty |\mu_k|(\Omega)$ and taking supremum in ϕ yields the result. □

Remark 1.4.10. Weak-star convergence of finite Radon measures is equivalent to local weak-star convergence with the condition that $\sup |\mu_k|(\Omega) = C < \infty$. We observe that, by Lemma 1.4.9, this condition implies $|\mu|(\Omega) \leq C$.
Clearly weak-star convergence always implies local weak-star convergence.
On the other hand, if we suppose that μ_k locally weak-star converges to μ, then, given $\psi \in C_0(\Omega; \mathbb{R}^m)$, for any $\epsilon > 0$ there exists $\phi \in C_c(\Omega; \mathbb{R}^m)$ such that $\|\psi - \phi\|_\infty < \epsilon$ and so

$$\begin{aligned}\left| \int_\Omega \psi \cdot d\mu_k - \int_\Omega \psi \cdot d\mu \right| &\leq \left| \int_\Omega (\psi - \phi) \cdot d\mu_k \right| + \left| \int_\Omega (\psi - \phi) \cdot d\mu \right| \\ &\quad + \left| \int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu \right| \\ &\leq 2C\epsilon + \left| \int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu \right|.\end{aligned}$$

Now, $\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu$ and so, since ϵ is arbitrary, we obtain weak-star convergence.
Therefore, in what follows, we will always write $\mu_k \overset{*}{\rightharpoonup} \mu$ to denote local weak-star convergence, and, in the case of finite Radon measures, we will also check the condition $\sup |\mu_k|(\Omega) < \infty$.

We quote now a useful result about weak-star convergence.

Lemma 1.4.11. *Let μ be a Radon measure on Ω, and let $\{\mu_k\}$ be a sequence of Radon measures. If μ_k and μ are positive, then the following are equivalent:*

1. $\mu_k \overset{*}{\rightharpoonup} \mu$.
2. *$\forall A \subset \Omega$ open,*
$$\mu(A) \leq \liminf_{k \to +\infty} \mu_k(A)$$
and $\forall K \subset \Omega$ compact,
$$\mu(K) \geq \limsup_{k \to +\infty} \mu_k(K).$$
3. *$\forall B \subset\subset \Omega$ Borel set with $\mu(\partial B) = 0$,*
$$\lim_{k \to +\infty} \mu_k(B) = \mu(B).$$

If μ_k and μ are $\mathbb{R}^m$-vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then $|\mu| \leq \nu$. Moreover, if a μ-measurable set $E \subset\subset \Omega$ satisfies $\nu(\partial E) = 0$, then

$$\mu(E) = \lim_{k\to+\infty} \mu_k(E).$$

More generally, if $f : \Omega \to \mathbb{R}^m$ is a bounded Borel function with compact support such that the set of its discontinuity points is ν-neglegible, then

$$\lim_{k\to+\infty} \int_\Omega f \cdot d\mu_k = \int_\Omega f \cdot d\mu.$$

Proof. For the second part of the statement and the implication $1 \to 2$ we refer to [AFP], Proposition 1.62. For the two remaining implications, we adapt the proof in [EG], Section 1.9, Theorem 1, where $\Omega = \mathbb{R}^N$.
In order to show that 2 implies 3, we take a Borel set B such that $\overline{B} \subset \Omega$ and $\mu(\partial B) = 0$. Then

$$\mu(B) = \mu(B^\circ) \leq \liminf_{k\to+\infty} \mu_k(B^\circ) \leq \limsup_{k\to+\infty} \mu_k(\overline{B}) \leq \mu(\overline{B}) = \mu(B).$$

Now we suppose that 3 holds and we observe that, since ϕ can be decomposed into its positive and negative parts, we need only to prove 1 for nonnegative functions. We fix $\epsilon > 0$ and $\phi \in C_c(\Omega)$ with $\phi \geq 0$. Let Ω_s be defined as in the proof of Proposition **??**, but for $s \in (1, +\infty)$. By Remark 1.4.12, for all but countable s, we have $\mu(\partial\Omega_s) = 0$. Therefore, there exists s_0 such that $\mathrm{supp}(\phi) \subset \Omega_{s_0}$ and $\mu(\partial\Omega_{s_0}) = 0$. We can choose $0 = t_0 < t_1 < ... < t_N = 2\|\phi\|_\infty$ such that $0 < t_i - t_{i-1} < \epsilon$ and $\mu(\phi^{-1}(\{t_i\})) = 0$ for any $i = 1, ..., N$, by Remark 1.4.12. We set $B_i = \phi^{-1}((t_{i-1}, t_i])$, then $\mu(\partial B_i) = 0$ for $i \geq 2$. Now

$$\sum_{i=2}^{N} t_{i-1}\mu_k(B_i) \leq \int_\Omega \phi \, d\mu_k \leq \sum_{i=2}^{N} t_i\mu_k(B_i) + t_1\mu_k(\Omega_{s_0})$$

and

$$\sum_{i=2}^{N} t_{i-1}\mu(B_i) \leq \int_\Omega \phi \, d\mu \leq \sum_{i=2}^{N} t_i\mu(B_i) + t_1\mu(\Omega_{s_0});$$

and so 3 implies

$$\limsup_{k\to+\infty} \left| \int_\Omega \phi \, d\mu_k - \int_\Omega \phi \, d\mu \right| \leq 2\epsilon\mu(\Omega_{s_0}),$$

which gives 1. □

Remark 1.4.12. Let μ be a positive Radon measure. If $\{A_t\}_{t\in\mathcal{I}}$, where $\mathcal{I}$ is uncountable, is a family of μ-measurable sets in Ω such that their boundaries are disjoint, $\bigcup_{t\in\mathcal{I}} \partial A_t = \Omega$ and for every compact K there exists an uncountable set of indices $\mathcal{J} \subset \mathcal{I}$ such that $K \cap \partial A_t \neq \emptyset$, $\forall t \in \mathcal{J}$, then there exists a countable set $\mathcal{N}$ such that

$$\mu(K \cap \partial A_t) = 0 \quad \forall t \notin \mathcal{N}.$$

We claim that, if such a set $\mathcal{N}$ did not exist, then there would be an uncountable set $\mathcal{Y}$ such that $\mu(K \cap \partial A_t) > \epsilon > 0$, $\forall t \in \mathcal{Y}$. Suppose to the contrary that for each $\epsilon > 0$ the set of t's which satisfy $\mu(K \cap \partial A_t) > \epsilon$ is countable.
We set $\epsilon_j = \frac{1}{j}$ and we have

$$\{t \in \mathcal{I} : \mu(K \cap \partial A_t) \neq 0\} = \bigcup_{j=1}^{+\infty} \left\{ t \in \mathcal{I} : \mu(K \cap \partial A_t) > \frac{1}{j} \right\},$$

so this set, being countable union of countable sets, is itself countable, contradicting our assumption. We extract now from $\mathcal{Y}$ a sequence $\{t_j\}$.
By the monotonicity and the σ-additivity, we have

$$\mu(K) \geq \sum_{j=1}^{+\infty} \mu(K \cap \partial A_{t_j}) = +\infty,$$

which is absurd, since μ is a Radon measure. Therefore, such a $\mathcal{Y}$ cannot exist and so $\mathcal{N}$ exists.
In the following chapters, the sets $\{A_t\}$ will usually be balls $B(x, r)$.

Remark 1.4.13. By Remark 1.4.12 and Lemma 1.4.11, we can assert that, if μ_k and μ are positive Radon measures in Ω, for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\mu(\partial B(x, r)) = 0$ and so, if $\mu_k \overset{*}{\rightharpoonup} \mu$, $\mu_k(B(x, r)) \to \mu(B(x, r))$.
Moreover, if μ_k and μ are vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\nu(\partial B(x, r)) = 0$ and $\mu_k(B(x, r)) \to \mu(B(x, r))$.

Finally, we state a characterization of nonnegative linear functionals on $C_c^\infty(\Omega)$.

Lemma 1.4.14. *Let $L : C_c^\infty(\Omega) \to \mathbb{R}$ be linear and nonnegative; that is,*

$$L(\phi) \geq 0, \quad \forall \phi \in C_c^\infty(\Omega) \text{ with } \phi \geq 0.$$

Then there exists a positive Radon measure $\mu \in \mathcal{M}_{\text{loc}}(\Omega)$ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c^\infty(\Omega).$$

Proof. We choose a compact set $K \subset \Omega$ and we select a smooth function $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on K and $0 \leq \zeta \leq 1$. Then, for any $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset K$, we set $\psi = \|\phi\|_\infty \zeta - \phi \geq 0$. Therefore, since L is nonnegative, we have $0 \leq L(\psi) = \|\phi\|_\infty L(\zeta) - L(\phi)$ and so $L(\phi) \leq C\|\phi\|_\infty$, with $C := L(\zeta)$.
L thus may be extended to a linear mapping $\hat{L} : C_c(\Omega) \to \mathbb{R}$ such that, for any compact $K \subset \Omega$,

$$\sup\{L(\phi) : \phi \in C_c(\Omega; \mathbb{R}^m), \|\phi\|_\infty \leq 1, \text{supp}(\phi) \subset K\} < \infty.$$

Hence, Corollary 1.4.7 yields the existence of a real Radon measure μ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c(\Omega).$$

By the polar decomposition of measures, $\mu = h|\mu|$, where $|h| = 1$ $|\mu|$-a.e. The fact that L is nonnegative implies that $h = 1$ $|\mu|$-a.e.; that is, μ is a positive Radon measure. $\square$

Chapter 2

Area and coarea formulas and the notion of rectifiability

2.1 The area formula

2.2 The coarea formula

2.3 Rectifiability

We start with the definitions of rectifiable set and approximated tangent space.

Definition 2.3.1. Let $k \in [0, n]$ be an integer and let $S \subset \mathbb{R}^n$ be a $\mathscr{H}^k$-measurable set. We say that S is countably k-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$S \subset \bigcup_i f_i(\mathbb{R}^k).$$

Definition 2.3.2. Let $k \in [0, n]$ be an integer, μ be a Radon measure in Ω and $x \in \Omega$. We say that the approximate tangent space of μ is a k-plane π with multiplicity $\theta \in \mathbb{R}$ in x, and we write

$$\mathrm{Tan}^k(\mu, x) = \theta \mathscr{H}^k \llcorner \pi$$

if $r^{-k}\mu_{x,r}$ locally weak* converges to $\theta\mathscr{H}^k \llcorner \pi$ in Ω as $r \to 0$; that is,

$$\lim_{r \to 0} \frac{1}{r^k} \int_\Omega \phi\left(\frac{y - x}{r}\right) d\mu(y) = \int_\pi \phi(y)\, d\mathscr{H}^k(y)$$

for any $\phi \in C_c(\Omega)$.

Chapter 3

BV theory

3.1 Functions of Bounded Variation

Definition 3.1.1. A function $u \in L^1(\Omega)$ is called a *function of bounded variation* if

$$\sup\left\{\int_\Omega u\,\mathrm{div}\phi\,dx : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), \|\phi\|_\infty \leq 1\right\} < \infty.$$

We denote by $BV(\Omega)$ the space of all functions of bounded variation on Ω.
We say that u is *locally of bounded variation*, and we write $u \in BV_{\mathrm{loc}}(\Omega)$, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if $\forall$ open set $W \subset\subset \Omega$,

$$\sup\left\{\int_W u\,\mathrm{div}\phi\,dx : \phi \in C_c^\infty(W;\mathbb{R}^d), \|\phi\|_\infty \leq 1\right\} < \infty.$$

Theorem 3.1.2. (Riesz) *Let $u \in BV_{\mathrm{loc}}(\Omega)$, then there exists a unique $\mathbb{R}^N$-vector valued Radon measure μ such that*

$$\int_\Omega u\,\mathrm{div}\phi\,dx = -\int_\Omega \phi \cdot d\mu \quad \forall \phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Proof. We define the linear functional $L : C_c^1(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$L(\phi) := -\int_\Omega u\,\mathrm{div}\phi\,dx, \quad \text{for} \quad \phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Since $u \in BV_{\mathrm{loc}}(\Omega)$, we have

$$\sup\left\{L(\phi) : \phi \in C_c^\infty(W;\mathbb{R}^N), \|\phi\|_\infty \leq 1\right\} = C(W) < \infty$$

for each open set $W \subset\subset \Omega$, and thus

$$|L(\phi)| \leq C(W)\|\phi\|_\infty \quad \text{for} \quad \phi \in C_c^1(W;\mathbb{R}^N).$$

We fix any compact set $K \subset \Omega$ and then we choose an open set W such that $K \subset W \subset\subset \Omega$. For each $\phi \in C_c(\Omega;\mathbb{R}^N)$ with $\mathrm{supp}(\phi) \subset K$, we choose a sequence $\phi_k \in C_c^1(W;\mathbb{R}^N)$ such that $\phi_k \to \phi$ uniformly on W. Then we define

$$\bar{L}(\phi) := \lim_{k\to+\infty} L(\phi_k).$$

By the continuity of L on $C_c^1(\Omega;\mathbb{R}^N)$ we have that this limit exists and is independent of the choice of the sequence $\{\phi_k\}$ converging to ϕ. Thus $\bar{L}$ uniquely extends to a linear functional

$$\bar{L} : C_c(\Omega;\mathbb{R}^N) \to \mathbb{R}$$

and

$$\sup\left\{\bar{L}(\phi) : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), \|\phi\|_\infty \leq 1, \mathrm{supp}(\phi) \subset K\right\} < \infty$$

for each compact set $K \subset \Omega$. So, by the Riesz Representation Theorem (Corollary 1.4.7), there exists an $\mathbb{R}^N$-vector valued Radon measure μ satisfying

$$\bar{L}(\phi) = -\int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega, \mathbb{R}^N)$$

and so, since $\bar{L}(\phi) = L(\phi)$ for $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, the result follows. □

This means that the distributional derivative Du of a BV function u is an $\mathbb{R}^N$-vector valued Radon measure.
We write $|Du|$ to indicate its total variation, which is a positive Radon measure on Ω.

Remark 3.1.3. $W^{1,1}(\Omega) \subset BV(\Omega)$ and $|Du|(\Omega) = \|Du\|_{L^1(\Omega;\mathbb{R}^N)}$ for $u \in W^{1,1}(\Omega)$.

Theorem 3.1.4. *If $\{u_n\} \subset BV(\Omega)$ is such that $u_n \rightharpoonup u$ in $L^p(\Omega)$ for some $p \in [1, +\infty)$, or weak-star for $p = +\infty$, or in $L^p_{\mathrm{loc}}(\Omega)$. Then $\forall A \subseteq \Omega$ open*

$$|Du|(A) \le \liminf_{n \to +\infty} |Du_n|(A).$$

Proof. Indeed, we have $\forall \phi \in C_c^\infty(A; \mathbb{R}^N)$

$$\int_A u_n \,\mathrm{div}\phi \, dx \to \int_A u \,\mathrm{div}\phi \, dx$$

and so, by Proposition **??**,

$$\int_A u \,\mathrm{div}\phi \, dx = \lim_{n \to +\infty} \int_A u_n \,\mathrm{div}\phi \, dx \le \liminf_{n \to +\infty} |Du_n|(A).$$

Taking the supremum over $\phi \in C_c^\infty(A; \mathbb{R}^N)$ with $\|\phi\|_\infty \le 1$ on the left hand side, we have the claim. □

Remark 3.1.5. $|Du|(\Omega)$ is a seminorm in $BV(\Omega)$. Clearly it is positively homogeneous and we get subadditivity by observing that

$$\int_\Omega (u_1 + u_2)\mathrm{div}\phi \, dx \le |Du_1|(\Omega) + |Du_2|(\Omega).$$

Theorem 3.1.6. *The space $BV(\Omega)$ endowed with the norm*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $BV(\Omega)$, then it is Cauchy in $L^1(\Omega)$ and so $\exists u \in L^1(\Omega)$ such that $u_n \to u$ in L^1.
By the lower semicontinuity (Theorem 3.1.4), $u \in BV(\Omega)$.
Moreover, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|D(u_k - u_n)|(\Omega) < \epsilon, \forall k, n \ge N$.
So, again by lower semicontinuity, $|D(u_k - u)|(\Omega) \le \liminf_n |D(u_k - u_n)|(\Omega) < \epsilon$ and from this it follows u_n converges to u in BV norm. □

Theorem 3.1.7. (Meyers-Serrin Approximation theorem)
Let $u \in BV(\Omega)$, then $\exists \{u_n\} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that

1. *$u_n \to u$ in $L^1(\Omega)$*
2. *$|Du_n|(\Omega) \to |Du|(\Omega)$.*

Proof.
Fix $\epsilon > 0$. Given a positive integer m, we set $\Omega_0 = \emptyset$, define for each $k \in \mathbb{N}, k \ge 1$ the sets

$$\Omega_k = \left\{ x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B(0, k+m)$$

and then we choose m such that $|Du|(\Omega \setminus \Omega_1) < \epsilon$.
We define now $\Sigma_k := \Omega_{k+1} \setminus \overline{\Omega}_{k-1}$. Since $\{\Sigma_k\}$ is an open cover of Ω, then there exists a partition of unity subordinate to that open cover; that is, a sequence of functions $\{\zeta_k\}$ such that:

1. $\zeta_k \in C_c^\infty(\Sigma_k)$;
2. $0 \leq \zeta_k \leq 1$;
3. $\sum_{k=1}^{+\infty} \zeta_k = 1$ on Ω.

Then we take a standard mollifier ρ and $\forall k$ we choose ϵ_k such that:

$$\text{spt}(\rho_{\epsilon_k} * (u\zeta_k)) \subset \Sigma_k$$

$$\|\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \frac{\epsilon}{2^k}$$

$$\|\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k\|_{L^1(\Omega;\mathbb{R}^N)} < \frac{\epsilon}{2^k}$$

and we define $u_\epsilon = \sum_{k=1}^{+\infty} \rho_{\epsilon_k} * (u\zeta_k)$.
Then $u_\epsilon \in C^\infty$, since locally there are only a finite number of nonzero terms in the sum.
Also, $u_\epsilon \to u$ in $L^1(\Omega)$ since

$$\|u - u_\epsilon\|_{L^1(\Omega)} \leq \sum_{k=1}^{+\infty} \|\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k\|_{L^1(\Omega)} < \epsilon.$$

Now, since $u_\epsilon \in L^1(\Omega)$, Theorem 3.1.4 implies $|Du|(\Omega) \leq \liminf_{\epsilon\to 0} |Du_\epsilon|(\Omega)$.

In order to obtain the reverse inequality, let $\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$, $\|\phi\|_\infty \leq 1$. Then

$$\begin{aligned}\int_\Omega u_\epsilon \text{div}\phi dx &= \sum_{k=1}^{+\infty} \int_\Omega \rho_{\epsilon_k} * (u\zeta_k)\text{div}\phi dx = \sum_{k=1}^{+\infty} \int_\Omega u\zeta_k \text{div}(\rho_{\epsilon_k} * \phi) dx \\ &= \sum_{k=1}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx - \sum_{k=1}^{+\infty} \int_\Omega u\nabla\zeta_k \cdot (\rho_{\epsilon_k} * \phi)dx.\end{aligned}$$

Using $\displaystyle\sum_{k=1}^{+\infty} \nabla\zeta_k = 0$ in Ω and the properties of the convolution, this last expression equals

$$\sum_{k=1}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx - \sum_{k=1}^{+\infty} \int_\Omega \phi \cdot (\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k)dx =: I_1^\epsilon + I_2^\epsilon$$

Now, $|\zeta_k(\rho_{\epsilon_k} * \phi)| \leq 1$ and each point in Ω belongs to at most three of the sets $\{\Sigma_k\}$. Thus

$$|I_1^\epsilon| \leq \left| \int_\Omega u\text{div}(\zeta_1(\rho_{\epsilon_1} * \phi))dx \quad + \quad \sum_{k=2}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi))dx \right| \leq$$

$$|Du|(\Omega) + \sum_{k=2}^{+\infty} |Du|(\Sigma_k) \leq |Du|(\Omega) + 3|Du|(\Omega \setminus \Omega_1) \leq |Du|(\Omega) + 3\epsilon$$

For the second term, we have $|I_2^\epsilon| < \epsilon$ directly from our choice of ϵ_k.
Therefore, after passing to the supremum over ϕ, $|Du_\epsilon|(\Omega) \leq |Du|(\Omega) + 4\epsilon$, which yields $u_\epsilon \in BV(\Omega)$ and point 2. □

Remark 3.1.8. If $u \in BV(\mathbb{R}^N)$; that is, if Ω is the entire space $\mathbb{R}^N$, then the approximating sequence satisfying properties 1) and 2) of Theorem 3.1.7 is much easier to construct. Indeed, we need just to take $u_\epsilon = u * \rho_\epsilon$, where ρ is a standard symmetric mollifier.
Indeed, $u_\epsilon \to u$ in $L^1(\mathbb{R}^N)$ since $u \in L^1(\mathbb{R}^N)$.
Secondly, we observe that

$$\begin{aligned}\|\nabla u_\epsilon\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} &= \sup\left\{ \int_{\mathbb{R}^N} u_\epsilon(x)\text{div}\phi(x)\, dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(y)\rho_\epsilon(x-y)\text{div}\phi(x)\, dxdy : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \leq 1 \right\} \\ &= \sup\left\{ \int_{\mathbb{R}^N} u(y)\text{div}\phi_\epsilon(y)\, dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), \|\phi\|_\infty \leq 1 \right\} \leq |Du|(\mathbb{R}^N)\end{aligned}$$

and so, by lower semicontinuity of the total variation, $\|\nabla u_\epsilon\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \to |Du|(\mathbb{R}^N)$.
We may fix a sequence $\epsilon_k \to 0$. Theorem 3.1.4 implies that for any open set A $|Du|(A) \leq \liminf_{k\to+\infty} |Du_{\epsilon_k}|(A)$ and we observe that for any compact set K and $\phi \in C_c^\infty(K;\mathbb{R}^N)$, $\|\phi\|_\infty \leq 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} u_{\epsilon_k}(x)\mathrm{div}\phi(x)\,dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathrm{div}\phi(x)u(y)\rho_{\epsilon_k}(x-y)\,dydx \\ &= \int_{\mathbb{R}^N} u(y)\mathrm{div}\phi_{\epsilon_k}(y)\,dy \leq |Du|(K+\overline{B(0,\epsilon_k)})\end{aligned}$$

since $\mathrm{supp}(\phi_{\epsilon_k}) \subset K + \overline{B(0,\epsilon_k)}$. Thus we can take the supremum over ϕ in order to obtain $|Du_{\epsilon_k}|(K) \leq |Du|(K+\overline{B(0,\epsilon_k)})$, which implies $\limsup_{k\to+\infty} |Du_{\epsilon_k}|(K) \leq |Du|(K)$ since K is compact. Hence the sequence of Radon measures $|\nabla u_{\epsilon_k}|\mathscr{L}^n$ satisfies point 2 of Lemma 1.4.11 and so we have point 1 of the same lemma; that is, $|Du_{\epsilon_k}| \overset{*}{\rightharpoonup} |Du|$ in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, since we have shown above that $\sup_k |Du_{\epsilon_k}|(\mathbb{R}^N) \leq |Du|(\mathbb{R}^N) < \infty$, Remark 1.4.10 yields also weak-star convergence in $\mathcal{M}(\mathbb{R}^N)$.

This remark applies also to BV functions with compact support inside Ω, since these are trivially in $BV(\mathbb{R}^N)$. Given $u \in BV(\Omega)$ with compact support, we can indeed extend it to

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is clear that $\hat{u} \in L^1(\mathbb{R}^N)$. If we let $\xi \in C_c^\infty(\Omega)$, $\|\xi\|_\infty \leq 1$ and $\xi = 1$ in a neighborhood of the support of u, then, for any $\phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $\|\phi\|_\infty \leq 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \hat{u}\mathrm{div}\phi\,dx &= \int_\Omega u\mathrm{div}\phi\,dx = \int_\Omega u\mathrm{div}(\xi\phi + (1-\xi)\phi)\,dx \\ &= \int_\Omega u\mathrm{div}(\xi\phi)\,dx \leq |Du|(\Omega),\end{aligned}$$

since $\xi\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$ and $\|\xi\phi\|_\infty \leq 1$.
Taking the supremum over ϕ we obtain $|D\hat{u}|(\mathbb{R}^N) \leq |Du|(\Omega) < \infty$.

Lemma 3.1.9. *Let $u \in BV(\Omega)$ and $\{u_n\}$ which satisfies point 1) and 2) in Theorem 3.1.7. Then, if we define for all Borel sets $B \subset \mathbb{R}^N$ the Radon measures $\mu_n(B) := \int_{B\cap\Omega} \nabla u_n\,dx$ and $\mu(B) := Du(B\cap\Omega)$, we have $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{R}^N)$, i.e.,*

$$\int_{\mathbb{R}^N} \xi\cdot d\mu_n \to \int_{\mathbb{R}^N} \xi\cdot d\mu \quad \forall\xi \in C_c(\mathbb{R}^N;\mathbb{R}^N).$$

Proof.
We define Ω_1 as in the proof of Theorem 3.1.7, so that $|Du|(\Omega\setminus\Omega_1) < \epsilon$ for some $\epsilon > 0$ fixed.
Let η be a smooth cut-off function such that $\eta = 1$ in Ω_1, $0 \leq \eta \leq 1$ and $\mathrm{supp}(\eta) \subset \Omega$.
Let $\xi \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \xi\cdot d\mu_n = \int_{\mathbb{R}^N} \xi\cdot\nabla u_n\,dx = \int_{\mathbb{R}^N} \eta\xi\cdot\nabla u_n\,dx + \int_{\mathbb{R}^N}(1-\eta)\xi\cdot\nabla u_n\,dx.$$

Now, since $u_n \to u$ in $L^1(\Omega)$ and by the Riesz Theorem (Theorem 1.4.7),

$$\begin{aligned}&\int_{\mathbb{R}^N} \eta\xi\cdot\nabla u_n\,dx = -\int_{\mathbb{R}^N} u_n\mathrm{div}(\eta\xi)\,dx \to -\int_{\mathbb{R}^N} u\mathrm{div}(\eta\xi)\,dx \\ &= \int_{\mathbb{R}^N} \eta\xi\cdot dDu = \int_{\mathbb{R}^N} \xi\cdot dDu + \int_{\mathbb{R}^N}(\eta-1)\xi\cdot dDu\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^N}(\eta-1)\xi\cdot dDu\right| \leq \|\xi\|_\infty|Du|(\Omega\setminus\Omega_1) < \epsilon\|\xi\|_\infty.$$

Also,

$$\left|\int_{\mathbb{R}^N}(1-\eta)\xi\cdot\nabla u_n\,dx\right|\leq\|\xi\|_\infty|Du_n|(\Omega\setminus\Omega_1)<\epsilon\|\xi\|_\infty,$$

since, by the fact that $|Du_n|(\Omega)\to|Du|(\Omega)$ and by the lower semicontinuity of the total variation,

$$\liminf_{n\to+\infty}|Du_n|(\Omega\setminus\Omega_1)=\liminf_{n\to+\infty}|Du_n|(\Omega)-|Du_n|(\Omega_1)\leq$$
$$|Du|(\Omega)-|Du|(\Omega_1)=|Du|(\Omega\setminus\Omega_1)<\epsilon.$$

Therefore, for any $\epsilon>0$, there exists a n_0 such that, $\forall n\geq n_0$,

$$\left|\int_{\mathbb{R}^N}\xi\cdot d\mu_n-\int_{\mathbb{R}^N}\xi\cdot d\mu\right|\leq 2\epsilon\|\xi\|_\infty.$$

Now let $\xi\in C_c(\mathbb{R}^N;\mathbb{R}^N)$, and take its mollification $\xi_\delta=\xi*\rho_\delta$.
$\xi_\delta\to\xi$ uniformly on compact subsets of $\mathbb{R}^N$, in particular on $K:=\mathrm{supp}(\eta)$. So

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\xi\cdot d\mu_n-\int_{\mathbb{R}^N}\xi\cdot d\mu\right|\leq\left|\int_{\mathbb{R}^N}\eta(\xi-\xi_\delta)\cdot d\mu_n\right|+\left|\int_{\mathbb{R}^N}\eta(\xi-\xi_\delta)\cdot d\mu\right|+\\&+\left|\int_{\mathbb{R}^N}\xi_\delta\cdot d\mu_n-\int_{\mathbb{R}^N}\xi_\delta\cdot d\mu\right|+\left|\int_{\mathbb{R}^N}(1-\eta)(\xi-\xi_\delta)\cdot d\mu_n\right|+\\&+\left|\int_{\mathbb{R}^N}(1-\eta)(\xi_\delta-\xi)\cdot d\mu\right|\leq\|\xi-\xi_\delta\|_{L^\infty(K)}(|Du_n|(\Omega)+|Du|(\Omega))+\\&+\epsilon\|\xi_\delta\|_\infty+2\|\xi\|_\infty(|Du_n|(\Omega\setminus\Omega_1)+|Du|(\Omega\setminus\Omega_1))\end{aligned}$$

and, by the estimates already found, we can conclude that, up to choosing a suitable $\delta(\epsilon)$, it is all bounded by $C\epsilon$ for n big enough, thus proving weak-star convergence of measures. $\square$

3.2 Sets of finite perimeter

Definition 3.2.1. A measurable set $E\subset\Omega$ is called a *finite perimeter set* in Ω (or a Caccioppoli set) if $\chi_E\in BV(\Omega)$.
A measurable set $E\subset\mathbb{R}^N$ is said to have *locally finite perimeter* in Ω if $\chi_E\in BV_{\mathrm{loc}}(\Omega)$.

Consequently, $D\chi_E$ is an $\mathbb{R}^N$-vector valued Radon measure on Ω whose total variation is $|D\chi_E|$. By the polar decomposition of measures, there exists a $|D\chi_E|$-measurable function with modulus 1 $|D\chi_E|$-a.e., which we denote by ν_E, such that $D\chi_E=\nu_E|D\chi_E|$.
Unless otherwise stated, from now on E will be a set of locally finite perimeter in Ω.

Example 3.2.2. Any open bounded set $E\subset\Omega$ with $\partial E\in C^2$ is a set of finite perimeter in Ω. Indeed, $\forall\phi\in C_c^\infty(\Omega;\mathbb{R}^N)$ with $\|\phi\|_\infty\leq 1$, by the classical Gauss-Green formula we have

$$\begin{aligned}\int_{\Omega\cap E}\mathrm{div}\phi\,dx&=-\int_{\partial(\Omega\cap E)}\phi\cdot\nu_E\,d\mathcal{H}^{N-1}=-\int_{\Omega\cap\partial E}\phi\cdot\nu_E\,d\mathcal{H}^{N-1}\\&\leq\int_{\Omega\cap\partial E}|\phi||\nu_E|\,d\mathcal{H}^{N-1}\leq\mathcal{H}^{N-1}(\Omega\cap\partial E),\end{aligned}$$

where ν_E is the interior unit normal. Taking the supremum over ϕ yields $|D\chi_E|(\Omega)\leq\mathcal{H}^{N-1}(\Omega\cap\partial E)$.
Therefore, E has finite perimeter and so, for any $\phi\in C_c^\infty(\Omega;\mathbb{R}^N)$,

$$\int_\Omega\chi_E\mathrm{div}\phi\,dx=-\int_\Omega\phi\cdot D\chi_E=-\int_{\Omega\cap\partial E}\phi\cdot\nu_E\,d\mathcal{H}^{N-1}.$$

This implies that $D\chi_E=\nu_E\,\mathcal{H}^{N-1}\llcorner\partial E$ in $\mathcal{M}(\Omega;\mathbb{R}^N)$, by Riesz Representation Theorem (Theorem 3.1.2), and so $|D\chi_E|=\mathcal{H}^{N-1}\llcorner\partial E$, which in particular yields

$$|D\chi_E|(\Omega)=\mathcal{H}^{N-1}(\Omega\cap\partial E).\tag{3.2.1}$$

Remark 3.2.3. It can be also shown that every open bounded set with Lipschitz boundary is a set of finite perimeter, with equality (3.2.1) holding, since this is a consequence of the extension theorem for functions of bounded variation (see [EG], Section 5.4). Moreover, any bounded open set Ω satisfying $\mathcal{H}^{N-1}(\partial\Omega) < \infty$ has finite perimeter in $\mathbb{R}^N$ (see [AFP], Proposition 3.62).

Equality (3.2.1) is not valid in general for sets of finite perimeter, as the following example will show.

Example 3.2.4. Let $N \geq 2$, $\{x_i\} = \mathbb{Q}^N \cap [0,1]^N$, $E = \bigcup_{i=0}^{\infty} B(x_i, \epsilon 2^{-i})$, with $\epsilon > 0$ that shall be assigned, and $[0,1]^N \subset \Omega$. We have

$$|E| \leq \sum_{i=0}^{\infty} \omega_N \epsilon^N 2^{-iN} = \frac{\omega_N \epsilon^N}{1 - 2^{-N}}.$$

Since the rational points are dense in $[0,1]^N$, then $\overline{E} = [0,1]^N$ and $\partial E = \overline{E} \setminus E$, since E is open, which implies

$$|\partial E| \geq |\overline{E}| - |E| \geq 1 - \frac{\omega_N \epsilon^N}{1 - 2^{-N}} > 0$$

for ϵ small enough. This implies $\mathcal{H}^{N-1}(\partial E) = \infty$.
Observing that $\partial E \subset \bigcup_{i=0}^{\infty} \partial B(x_i, \epsilon 2^{-i})$, we have

$$\int_{\Omega \cap E} \mathrm{div}\phi \, dx = -\int_{\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} \leq \sum_{i=0}^{\infty} \int_{\partial B(x_i, \epsilon 2^{-i})} |\phi||\nu_E| \, d\mathcal{H}^{N-1}$$
$$\leq \sum_{i=0}^{\infty} \mathcal{H}^{N-1}(\partial B(x_i, \epsilon 2^{-i})) = \sum_{i=0}^{\infty} N\omega_N \epsilon^{N-1} 2^{-(N-1)i} = \frac{N\omega_N \epsilon^{N-1}}{1 - 2^{-(N-1)}}.$$

Thus E is a set of finite perimeter for which $|D\chi_E|(\Omega) \neq \mathcal{H}^{N-1}(\Omega \cap \partial E)$.

This may suggest that for a set of finite perimeter is interesting to consider not the whole topological boundary, but subsets of ∂E instead.

Definition 3.2.5. Let $x \in \Omega$, then $x \in \partial^* E$, the *reduced boundary* of E, if

1. $|D\chi_E|(B(x,r)) > 0, \forall r > 0$;
2. $\lim_{r \to 0} \frac{1}{|D\chi_E|(B(x,r))} \int_{B(x,r)} \nu_E d|D\chi_E| = \nu_E(x)$;
3. $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterisation of the reduced boundary, by using the blow-up of the set E around a point of $\partial^* E$.

Definition 3.2.6. For $x \in \partial^* E$ we define the hyperplane

$$H(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) = 0\}$$

and the half-spaces

$$H^+(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \geq 0\},$$
$$H^-(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \leq 0\}.$$

Also, for $r > 0$, we set

$$E_r(x) = \{y \in \mathbb{R}^N : \ x + r(y - x) \in E\}$$

Theorem 3.2.7. *If E is a set of finite perimeter in Ω, $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

$$\chi_{E_r} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega)$$
$$\chi_{\Omega \setminus E_r} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega)$$

as $r \to 0$.

Proof. See [EG] Section 5.7.2 Theorem 1.

Formulated in another way, for $r > 0$ small enough, $E \cap B(x,r)$ is approximatively equal to the half ball $H^-(x) \cap B(x,r)$.

Corollary 3.2.8. *If $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

1. $\lim_{r \to 0} \frac{1}{r^N} |B(x,r) \cap E \cap H^+(x)| = 0,$
2. $\lim_{r \to 0} \frac{1}{r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0.$

Proof. We have

$$\frac{1}{r^N} |B(x,r) \cap E \cap H^+(x)| = |B(x,1) \cap E_r \cap H^+(x)| \to |B(x,1) \cap H^-(x) \cap H^+(x)| = 0,$$

by Theorem 3.2.7. Point 2 follows from the same theorem and

$$\begin{aligned} \frac{1}{r^N} |(B(x,r) \setminus E) \cap H^-(x)| &= \frac{1}{r^N} (|B(x,r) \cap H^-(x)| - |B(x,r) \cap E \cap H^-(x)|) \\ &= \frac{\omega_N}{2} - |B(x,1) \cap E_r \cap H^-(x)| \\ &\to \frac{\omega_N}{2} - |B(x,1) \cap H^-(x)| = 0. \end{aligned}$$

□

Using this result, we can give a generalization of the concept of unit interior normal (respectively, unit exterior normal, up to a sign).

Definition 3.2.9. A unit vector $\nu(x) = -\nu_E(x)$ for which property 1 of Corollary 3.2.8 holds is called the *measure theoretic unit exterior normal* to E at x, while, accordingly, $\nu_E(x)$ is called the *measure theoretic unit interior normal* to E at x.

It follows that the measure theoretic unit interior normal ν_E is well defined at least on the reduced boundary.
Moreover, the next theorem shows us that the reduced boundary can be written as a countable union of compact subset of C^1 manifolds, up to a set of Hausdorff dimension at most $N-1$.

Theorem 3.2.10. *Assume E is a set of locally finite perimeter in $\mathbb{R}^N$. Then*

1. *$\partial^* E$ is a $(N-1)$-rectifiable set; that is, there exist a countable family of C^1 manifolds S_k, a family of compact sets $K_k \subset S_k$ and set $\mathcal{N}$ of $\mathcal{H}^{N-1}$-measure zero such that*
$$\partial^* E = \bigcup_{k=1}^{+\infty} K_k \cup \mathcal{N};$$
2. *$\nu_E|_{S_k}$ is normal to S_k;*
3. *$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$,*
$$\lim_{r \to 0} \frac{|D\chi_E|(B(x,r))}{\omega_{N-1} r^{N-1}} = 1.$$

Proof. See [EG] Section 5.7.3 Theorem 2.

We introduce now the density of a set at a certain point, in order to select another useful subset of the topological boundary.

Definition 3.2.11. For every $\alpha \in [0,1]$ and every measurable set $E \subset \mathbb{R}^N$, we define

$$E^\alpha := \{x \in \mathbb{R}^N : D(E,x) = \alpha\},$$

where

$$D(E,x) := \lim_{r \to 0} \frac{|(B(x,r) \cap E)|}{|B(x,r)|}.$$

Definition 3.2.12. Referring to Definition 3.2.11,

1. E^1 is called the *measure theoretic interior* of E.
2. E^0 is called the *measure theoretic exterior* of E.
3. The *measure theoretic (or essential) boundary* of E is the set $\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1)$.

Remark 3.2.13. It is clear that $E^\circ \subset E^1$ and $\mathbb{R}^N \setminus \overline{E} \subset E^0$. Hence one has

$$\partial^m E \subset \mathbb{R}^N \setminus (E^\circ \cup \mathbb{R}^N \setminus \overline{E}) = \overline{E} \setminus E^\circ = \partial E.$$

Moreover, by the Lebesgue-Besicovitch differentiation theorem (Theorem **??**), $\partial^m E$ has $\mathcal{L}^N$-measure 0, since it is the set of non-Lebesgue points of χ_E.

We further observe that, as in [EG] Section 5.8, it is possibile to define the measure theoretic boundary without using the density of a set.
Indeed the previous definition is equivalent to the following:

Definition 3.2.12' Let $x \in \mathbb{R}^N$, then $x \in \partial^m E$, the meaure theoretic boundary of E, if the following two conditions hold:

1. $\limsup_{r \to 0} \dfrac{|B(x,r) \cap E|}{r^N} > 0$,
2. $\limsup_{r \to 0} \dfrac{|B(x,r) \setminus E|}{r^N} > 0$.

Theorem 3.2.14. *If $E \subset \Omega$ is a set of finite perimeter, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1) = 0.$$

In particular, E has density either 0, $\frac{1}{2}$ or 1 at $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$, and, even if E is only locally of finite perimeter, $\mathcal{H}^{N-1}$-a.e. $x \in \partial^m E$ belongs to $\partial^* E$; that is, $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$.

Proof. See [EG] Section 5.8 Lemma 1 and [AFP] Theorem 3.61.

Remark 3.2.15. Since the functions of bounded variations are elements of L^1, they are equivalence class of functions, so that changing the value of any such function on a set of $\mathcal{L}^N$-measure zero does not modify the BV class of the function.
Therefore, this is true also for sets of finite perimeter and we can choose any representative $\widetilde{E}$ for E, which differs only by a set of measure zero, without altering the reduced nor the measure theoretic boundary.

One of the greatest achievements of BV theory is to establish a generalization of the Gauss-Green formula for every set of finite perimeter, though only for differentiable vector fields.

Theorem 3.2.16. (Gauss-Green formula on sets of finite perimeter)
Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^m E$, there is a unique measure theoretic interior unit normal $\nu_E(x)$ such that $\forall \phi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$ one has

$$\int_E \text{div}\phi \, dx = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

Proof. Since E is a set of locally finite perimeter, $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ (Theorem 3.2.10), where ν_E is the measure theoretic interior unit normal. Also, Theorem 3.2.14 implies $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$. Hence, for any $\phi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$,

$$\int_\Omega \chi_E \text{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}. \ \square$$

Remark 3.2.17. Since $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$ (Theorem 3.2.14), without change, we can integrate on the measure theoretic or on the reduced boundary with respect to the measure $\mathcal{H}^{N-1}$. Since in many practical cases $\partial^m E$ is easier to be determined, Theorem 3.2.16 is often stated in this way. However, since Theorem 3.2.10 states that $|D\chi_E| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$ and the precise representative of χ_E is well defined on $E^1 \cup \partial^* E \cup E^0$ (Lemma 3.2.26 below), in what follows we will always use the reduced boundary in the Gauss-Green formula.

Remark 3.2.18. We also observe that if E is a bounded set of finite perimeter in $\mathbb{R}^N$, then we can drop the assumption on the support of ϕ. Indeed, there exists $R > 0$ such that $\overline{E} \subset B(0, R)$, and so, given $\phi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$, we can take $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi = 1$ in $\overline{B(0, R)}$ (which in particular implies $\nabla\varphi = 0$ in E), in order to obtain

$$\begin{aligned}\int_E \mathrm{div}\phi\, dx &= \int_E (\varphi \mathrm{div}\phi + \phi \cdot \nabla\varphi)\, dx = \int_E \mathrm{div}(\phi\varphi)\, dx \\ &= -\int_{\partial^* E} (\phi\varphi) \cdot \nu_E\, d\mathcal{H}^{N-1} = -\int_{\partial^* E} \phi \cdot \nu_E\, d\mathcal{H}^{N-1}.\end{aligned}$$

It is also easy to see that if $E \subset\subset \Omega \subset \mathbb{R}^N$, then we can take just $\phi \in C^1(\Omega; \mathbb{R}^N)$.

As in the case of Sobolev functions, it can be shown that for BV functions the precise representative is well defined and it is the limit of the mollified sequence.

Definition 3.2.19. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $a \in \mathbb{R}^N$.
We say that $u_a(x)$ is the approximate limit of u at $x \in \Omega$ restricted to $\Pi_a(x) := \{y \in \mathbb{R}^N : (y - x) \cdot a \geq 0\}$ if, for any $\epsilon > 0$,

$$\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^N : |u(y) - u_a(x)| \geq \epsilon\} \cap B(x, r) \cap \Pi_a(x)|}{|B(x, r) \cap \Pi_a(x)|} = 0$$

Definition 3.2.20. We say that $x \in \Omega$ is a *regular point* of a function $u \in BV(\Omega)$ if there exists a vector $a \in \mathbb{R}^N$ such that the approximate limits $u_a(x)$ and $u_{-a}(x)$ exist. The vector a is called *defining vector*.

Example 3.2.21. Let E be a set of finite perimeter, for which we choose the representative $E^1 \cup \partial^m E$ (see Remark 3.2.15), and $u = \chi_E$, then each point in $E^1 \cup E^0 \cup \partial^* E$ is a regular point. If $x \in E^1$, $\forall a \in \mathbb{R}^N$ $(\chi_E)_a(x) = 1$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r) = E^0 \cap B(x, r).$$

So,

$$\begin{aligned}\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} &= \lim_{r \to 0} \frac{|E^0 \cap B(x, r)|}{|B(x, r)|} \\ &= 1 - D(E, x) = 0.\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r) \cap \Pi_a(x)|}{|B(x, r) \cap \Pi_a(x)|} \\ &\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} \frac{|B(x, r)|}{|B(x, r) \cap \Pi_a(x)|} \\ &= 2 \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} \to 0 \text{ as } r \to 0.\end{aligned}$$

In an analogous way, we show that $\forall x \in E^0$ $(\chi_E)_a(x) = 0$ $\forall a \in \mathbb{R}^N$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x, r) = E \cap B(x, r)$$

and so

$$\begin{aligned}\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} &= \lim_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} \\ &= D(E, x) = 0.\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x, r) \cap \Pi_a(x)|}{|B(x, r) \cap \Pi_a(x)|} \\ &\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} \frac{|B(x, r)|}{|B(x, r) \cap \Pi_a(x)|} \\ &= 2 \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} \to 0 \text{ as } r \to 0.\end{aligned}$$

Now let $x \in \partial^* E$ and a be the measure theoretic interior normal. Then $(\chi_E)_a(x) = 1$ and $(\chi_E)_{-a}(x) = 0$.
Referring to the notation of Corollary 3.2.8, we have $\Pi_a(x) = H^-(x)$ and
$\Pi_{-a}(x) = H^+(x)$, hence $\forall \epsilon > 0$

$$\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|}$$
$$= \lim_{r\to 0} \frac{|E^0 \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0$$

and

$$\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|}$$
$$= \lim_{r\to 0} \frac{|E \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |B(x,r) \cap E \cap H^+(x)| = 0.$$

This shows our claim.

Theorem 3.2.22. *Let $u \in BV(\Omega)$. The set of irregular points has $\mathcal{H}^{N-1}$-measure zero.*

Proof. See [VH] Chapter 4 §5.5, or [EG] Section 5.9 Theorem 3.

Theorem 3.2.23. *Let $u \in BV(\Omega)$ and x be a regular point of u. Then*

1. *If $u_a(x) = u_{-a}(x)$, any $b \in \mathbb{R}^N$ is a defining vector and $u_b(x) = u_a(x)$; that is, x is a point of approximate continuity.*

2. *If $u_a(x) \neq u_{-a}(x)$, then a is unique up to a sign.*

3. *The mollification of u converges to the precise representative u^* at each regular point and $u^*(x) = \frac{1}{2}(u_a(x) + u_{-a}(x))$.*

Proof. See [VH] Chapter 4 §4.4 and Chapter 4 §5.6 Theorem 1, or [EG] Section 5.9 Corollary 1.

Remark 3.2.24. By Theorem **??**, we deduce that condition 1) in Theorem 3.2.23 holds $\mathcal{L}^N$-a.e.

We state now some standard results on the mollification of characteristic functions of sets of finite perimeter.

Remark 3.2.25. By Remark 3.1.8, if E be a set of finite perimeter and $\{\chi_{\delta_k}\}$ denotes the mollification of χ_E, then

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^N)} \leq |D\chi_E|(\mathbb{R}^N)$$

and

$$\|\nabla \chi_{\delta_k}\|_{L^1(\mathbb{R}^N)} \to |D\chi_E|(\mathbb{R}^N)$$

We state now some relevant properties of the mollifications of characteristic functions of sets of finite perimeter.

Lemma 3.2.26. *Let $E \subset \Omega$ be a set of locally finite perimeter in Ω and $\rho \in C_c^\infty(B(0,1))$ be a nonnegative radially symmetric mollifier such that $\int_{B(0,1)} \rho \, dx = 1$. Then, the following results hold:*

1. *there is a set $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that, for all $x \in \Omega \setminus \mathcal{N}$, $(\rho_\varepsilon * \chi_E)(x) \to \chi_E^*(x)$ where*
$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1 \\ \frac{1}{2} & \text{if } x \in \mathscr{F}E\ ; \\ 0 & \text{if } x \in E^0 \end{cases} \tag{3.2.2}$$

2. *$\rho_\varepsilon * \chi_E \in C^\infty(\Omega^\varepsilon)$ and $\nabla(\rho_\varepsilon * \chi_E)(x) = (\rho_\varepsilon * D\chi_E)(x)$ for any $x \in \Omega^\varepsilon$;*

3. *one has the following weak* limits in $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^n)$:*

(a) $\nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} D\chi_E$;

(b) $\chi_E \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2)D\chi_E$;

(c) $\chi_{\Omega\setminus E} \nabla(\rho_\varepsilon * \chi_E) \overset{*}{\rightharpoonup} (1/2)D\chi_E$;

We state now the co-area formula, which shows an important connection between BV functions and sets of finite perimeter.

Theorem 3.2.27. (Federer and Fleming co-area fromula)
If $u \in BV(\Omega)$, then for $\mathcal{L}^1$ a.e. $s \in \mathbb{R}$, the set $\{u > s\}$ has finite perimeter in Ω and

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega)ds.$$

Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(\Omega)ds < \infty$, then $u \in BV(\Omega)$. Moreover, for any Borel set $B \subset \Omega$ we have

$$|Du|(B) = \int_{-\infty}^{+\infty} |D\chi_{\{u>s\}}|(B)ds.$$

Dim. (Sketch)

Sia $\phi \in C_c^\infty(\Omega;\mathbb{R}^d)$, allora

$$\int_\Omega u\mathrm{div}\phi dx = \int_{u>0}\int_0^{u(x)} \mathrm{div}\phi ds dx - \int_{u<0}\int_{u(x)}^0 \mathrm{div}\phi dx =$$

per Fubini

$$\int_0^{+\infty}\int_\Omega \chi_{\{u>s\}}(x)\mathrm{div}\phi(x)dxds - \int_{-\infty}^0\int_\Omega (1-\chi_{\{u>s\}}(x))\mathrm{div}\phi(x)dxds =$$

e, poiché $\int_\Omega \mathrm{div}\phi dx = 0$,

$$\int_{-\infty}^{+\infty}\int_\Omega \chi_{\{u>s\}}\mathrm{div}\phi dxds \le \int_{-\infty}^{+\infty} Per(\{u>s\};\Omega)ds$$

Quindi, passando al sup in ϕ al primo membro, si ha $J(u) \le \int_{-\infty}^{+\infty} Per(\{u>s\};\Omega)ds$.
Per la disuguaglianza opposta, si procede provandola per funzioni $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$, stabilendo dunque per esse la formula di coarea, poi si prende una successione in tale spazio che converga ad $u \in BV$ come nel teorema di Meyers e Serrin, si prova che $\chi_{\{u_n>s\}} \to \chi_{\{u>s\}}$ in $L^1(\Omega)$ per a.e. $s \in \mathbb{R}$. Quindi si ha $Per(\{u>s\};\Omega) \le \liminf_n Per(\{u_n>s\};\Omega)$ e per Fatou,

$$\int_{-\infty}^{+\infty} Per(\{u>s\};\Omega)ds \le \liminf_n \int_{-\infty}^{+\infty} Per(\{u_n>s\};\Omega)ds = \lim_{n\to+\infty} J(u_n) = J(u)$$

$\square$

Proof. See [EG] Section 5.5 Theorem 1 and [AFP] Theorem 3.40.

Remark 3.2.28. A consequence of Theorem 3.2.27 is that, for any $u \in BV(\Omega)$, $|Du| \ll \mathcal{H}^{N-1}$. Indeed, for any Borel set $B \subset \Omega$ such that $\mathcal{H}^{N-1}(B) = 0$, co-area formula implies $|Du|(B) = 0$.

Lemma 3.2.29. *Let $u : \Omega \to \mathbb{R}$ be a Lipschitz function, and let $A \subset \mathbb{R}^N$ be a set of measure zero. Then*

$$\mathcal{H}^{N-1}(A \cap u^{-1}(s)) = 0 \quad \textit{for} \quad \mathcal{L}^1\textit{-a.e.}\, s \in \mathbb{R}.$$

Proof. It is an immediate consequence of the classical co-area formula for Lipschitz functions (see [EG], Section 3.4.2 Theorem 1); that is,

$$0 = \int_A |\nabla u(x)|dx = \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(A \cap u^{-1}(s))ds. \square$$

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