Lectures Notes

BV functions and sets of finite perimeter

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Introduction

♯ *Geometric Measure Theory* is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces generalizing techniques from differential geometry through measure theoretic arguments. The theory of *functions of bounded variations* and *sets of finite perimeter* is one of the core topics of Geometric Measure Theory, since it deals with extension of the classical notion of Sobolev functions and regular surfaces.

The 1-Laplace operator and BV as a natural extension of $W^{1,1}$

In the Calculus of Variation, the *Direct Method* is a general way of proving the existence of a minimizer for a given functional. More precisely, let X be a topological space and $F : X \to (-\infty, +\infty]$ be a functional. We are interested in finding a minimizer of F in X; that is, a $u \in X$ such that $F(u) \le F(v)$ for any $v \in X$. Assume that

$$m := \inf\{F(v) \,:\, v \in X\} > -\infty.$$

This ensure the existence of a minimizing sequence $\{v_j\}$; that is, a sequence of elements $v_j \in X$ such that $F(v_j) \to m$. Then, the Direct Method consists in the following steps:

(1) show that $\{v_j\}$ admits a converging subsequence $\{v_{j_k}\}$ and $u \in X$ such that $v_{j_k} \to u$, with respect to a the topology of X;

(2) show that F is (sequentially) lower semicontinuous with respect to the topology of X; that is, if $z_j \to z_0$ in X, then
$$F(z_0) \le \liminf_{j \to +\infty} F(z_j).$$

If these properties hold true, we can conclude that u is a minimizer of F. Indeed, we have

$$m = \lim_{k\to+\infty} F(v_{j_k}) \ge \liminf_{k\to+\infty} F(v_{j_k}) \ge F(u) \ge m,$$

from which we immediately conclude that $F(u) = \min\{F(v) \,:\, v \in X\}$.

This method is fundamental in proving the existence of solutions to minimization problems related to boundary value problems. Let us consider for instance the classical Dirichlet problem for the Laplace equation on an open set Ω with C^1-smooth boundary:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$. It is possible to see this system as the Euler-Lagrange equations for the functional
$$F(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega fu\,dx$$
defined on the space

$$X = W_0^{1,2}(\Omega) := \{u \in L^2(\Omega) \,:\, Du \in L^2(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\};$$

♯These notes have been written for the course of *BV Functions and Sets of Finite Perimeter* held in the Department of Mathematics of the Hamburg Universität. The main references are the books [**?**, **?**, **?**]. Please write an email to giovanni.comi@uni-hamburg.de if you have corrections, comments, suggestions or questions.

that is, the space of 2-summable weakly differentiable Sobolev functions with zero trace on $\partial\Omega$[♯]. As customary, we denote by Du the weak gradient of u. Thanks to Poincaré inequality, we can prove that

$$\inf\{F(u) \,:\, u \in W_0^{1,2}(\Omega)\} > -\infty.$$

Hence, we can find the solution looking for minizers of F through the Direct Method: let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence. It is possible to show that $\{u_j\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, which is an Hilbert space, and in particular reflexive: as a consequence, there exists a subsequence $\{u_{j_k}\}$ converging to some $u \in W_0^{1,2}(\Omega)$ with respect to the weak topology. In addition, F is lower semicontinuous with respect to the weak topology, and so we infer the existence of a solution for the minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|Du|^2 - f u\, dx \,:\, u \in W_0^{1,2}(\Omega)\right\}.$$

It seems natural now to wonder if we could substitute the exponent 2 with any $p \in (1,\infty)$. Thanks to the Poincaré inequality and the reflexivity of the L^p-spaces for $p \in (1,\infty)$, it is indeed possible to show that, for any $f \in L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$, the problem

$$\min\left\{\int_\Omega \frac{1}{p}|Du|^p - f u\, dx \,:\, u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution, where

$$W_0^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, Du \in L^p(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

The minimizers to this problem solves the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) =: \Delta_p u$ is the p-Laplace operator.

The next logical step is to consider also the case $p = 1$: for a given $f \in L^\infty(\Omega)$, we want to find a function u which realizes

$$\inf\left\{\int_\Omega |Du| - f u\, dx \,:\, u \in W_0^{1,1}(\Omega)\right\} =: m, \tag{0.0.1}$$

where

$$W_0^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

If we assume $\|f\|_{L^\infty(\Omega)}$ to be sufficiently small, we can again employ the Poincaré inequality to prove that $m \in (-\infty,+\infty]$. Hence, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{1,1}(\Omega)$ such that

$$\lim_{j\to+\infty}\int_\Omega |Du_j| - f u_j\, dx = m.$$

However, in this case we cannot argue as above in the case $p > 1$, since, in general this does *not* imply that the existence of a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ weakly converging to some $u \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |Du| - f u\, dx = m.$$

The reason for this lies in the fact that $L^1(\Omega)$ is not reflexive, and actually it is not the topological dual of any separable space. However, $L^1(\Omega)$ is contained in the space of finite Radon measures on Ω, $\mathcal{M}(\Omega)$, and this space can be see as the dual of the space of continuous functions vanishing on the boundary of Ω, $C_0(\Omega)$.

This fact suggests the definition of a space which contains the Sobolev space $W^{1,1}(\Omega)$ and which, although not reflexive, enjoys the property that bounded sets are weakly* compact: the space of *functions with bounded variation*,

$$BV(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n)\}.$$

[♯]We refer to [**?**, Chapter 4] for a detailed account on Sobolev spaces.

It is not difficult to prove that the total variation of the Radon measure Du over Ω is indeed lower semicontinuous with respect to the weak* converge of the gradient measures. This indicates that the correct space where to look solutions to (0.0.1) is the space of functions with bounded variation with zero trace[♭],

$$BV_0(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega; \mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

Finally, it is relevant to mention the fact that the minimizers to (0.0.1) solve the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) =: \Delta_1 u$ is the 1-Laplace operator, which is non trivially defined on nonsmooth functions because of the highly degenerate term $\frac{\nabla u}{|\nabla u|}$.

Minimal area problems and sets of finite perimeter

Other historically relevant problems from the Calculus of Variation are the minimal area problems, among which the most famous example is the *Euclidean isoperimetric problem*: find the possibly unique set with minimal surface area among those with fixed volume. In mathematical terms, if we denote by $|F|$ the n-dimensional volume of a set $F \subset \mathbb{R}^n$ (hence, its Lebesgue measure $\mathscr{L}^n(F)$) and by $\sigma_{n-1}(\partial F)$ its surface area (under the assumption the ∂F is regular enough), we are looking for the set which realizes

$$\inf \{\sigma_{n-1}(\partial F) \,:\, \partial F \in \mathcal{R}, |F| = k\} =: \gamma_k,$$

where $\mathcal{R}$ is a class of sufficiently smooth surfaces and $k > 0$. The Direct Method now consists in considering a minimizing sequence of sets F_j such that

$$\partial F_j \in \mathcal{R}, \quad |F_j| = k \quad \text{and} \quad \sigma_{n-1}(\partial F_j) \to \gamma_k, \tag{0.0.2}$$

and then in trying to prove the convergence (possibly up to subsequences) to some limit set E such that

$$\partial E \in \mathcal{R}, \quad |E| = k \quad \text{and} \quad \sigma_{n-1}(\partial E) = \gamma_k.$$

In order to achieve this result, some compactness property in the family of sets satisfying (0.0.2) is required. In addition, the surface measure σ_{n-1} need to be a lower semicontinuous with respect to the chosen convergence of sets, in the sense that

$$\sigma_{n-1}(\partial E) \leq \liminf_{j \to +\infty} \sigma_{n-1}(\partial F_j)$$

if $F_j \to E$ in a suitable sense. However, these compactness and lower semicontinuity properties in general fail to be true in family of sets with regular topological boundary. In addition, we notice that the topological boundary is very unstable under modification of a set by Lebesgue negligible sets. For instance, let

$$E_1 = B(0,1) \quad \text{and} \quad E_2 = B(0,1) \cup (\partial B(0,2) \cap \mathbb{Q}^n).$$

It is plain to see that $|E_1 \Delta E_2| = 0$, so that these two sets are equivalent with respect to the Lebesgue measure, and so they have the same volume. However, their topological boundary, which are smooth surfaces, are very different:

$$\partial E_1 = \partial B(0,1) \quad \text{and} \quad \partial E_2 = \partial B(0,1) \cup \partial B(0,2).$$

The need of ruling out these problems and of recovering a notion of compactness and a lower semicontinuity property for the surface area is one of the main reasons for the birth of Geometric Measure Theory. This theory concerns methods to study the geometric properties of rough, irregular sets from a measure theoretic point of view. In this course we shall see how to exploit this

[♭]It can be proved that the trace of a function with bounded variation is well defined on any C^1-regular surface, as in the Sobolev case.

approach to give a meaningful notion of surface area for an irregular set and to define a suitable class of sets for which we can apply the Direct Method of the Calculus of Variation in order to deal with minimal area problems: the *sets of finite perimeter*. Broadly speaking, the notion of set of finite perimeter extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to minimal area problems and other similar geometric variational minimization problems. More precisely, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ is its characteristic function χ_E is a function with locally bounded variation.

Chapter 1

Basic notions of Measure Theory

1.1 General measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*; that is, the collection of all subsets of X.

Definition 1.1.1 (Measures)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ if $A \subset \bigcup_{k=1}^{\infty} A_k$ (σ-subadditivity),

is called a measure.

It should be noticed that in the literature a mapping as the one in Definition 1.1.1 is also called an *outer measure*, while the name of measure is used to denote the restriction of the mapping to the family of measurable set (see Definition 1.1.4 below). We shall nevertheless follow the notation of [**?**], in order to be able to assign a measure even to nonmeasurable sets.

Remark 1.1.2. Thanks to σ-subadditivity, any measure is not decreasing; that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition 1.1.3 (Restriction of a measure)**.** If $Y \subset X$, the *restriction of* μ *to* Y, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$ for any $A \subset X$.

Definition 1.1.4 (μ-measurable sets)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark 1.1.5. This definition is meaningful, since the italian mathematician *Giuseppe Vitali* proved in 1905 that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathscr{L}^1$-measurable [**?**]. For a modern presentation of his construction, we refer to [**?**, Section I.1.2].

Definition 1.1.6 (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if the following conditions hold:

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for any $A \in \mathfrak{F}$ we have $X \setminus A \in \mathfrak{F}$,

(3) for any countable family of sets $\{A_i\}_{i \in I}$ such that $A_i \in \mathfrak{F}$ for any $i \in I$ we have have

$$\bigcup_{i \in I} A_i \in \mathfrak{F}.$$

Theorem 1.1.7. *Given any measure μ on X, the family of μ-measurable sets forms a σ-algebra.*

Theorem 1.1.8. *Let μ be a measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a countable disjoint μ-measurable family of subsets of X, then*

$$\mu\left(\bigcup_{j \in I} A_j\right) = \sum_{j \in I} \mu(A_j).$$

We list now some relevant definitions.

Definition 1.1.9.

(1) Given any $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.

(2) The *Borel σ-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel σ-algebra are called *Borel sets.*

(3) A measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (i.e. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem 1.1.10. *Let μ be a Radon measure on $\mathbb{R}^n$. Then we have*

(1) $\mu(A) = \inf\{\mu(U) : U \supset A,\, U$ open$\}$ for all $A \subseteq \mathbb{R}^n$ (outer regularity),

(2) $\mu(B) = \sup\{\mu(K) : K \subset B,\, K$ compact$\}$ for all μ-measurable sets B (inner regularity).

Theorem 1.1.11 (Carathéodory's criterion)**.** *Let μ be a measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ such that* $\operatorname{dist}(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

then μ is a Borel measure.

Not any Borel regular measure is a Radon measure. However, it is possible to obtain a Radon measure as a restriction of a Borel regular one, as stated in the followin theorem.

Theorem 1.1.12. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Example 1.1.13 (Dirac delta)**.** For $x \in \mathbb{R}^n$ we define the *Dirac*[♯] *measure centered in x* by setting

$$\delta_x(A) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

It is easy to check that this is indeed a Radon measure. In addition, any set in $\mathbb{R}^n$ is δ_x-measurable.

Example 1.1.14 (The counting measure)**.** We define the *counting measure* by setting

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure, since it is clearly not locally finite.

[♯]Named after Paul Adrien Maurice Dirac (1902-1984), English theoretical physicist who shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory". He actually introduced the so-called *Dirac delta function* as a "convenient notation" in his influential 1930 book *The Principles of Quantum Mechanics.* The name "delta function" was chosen since it works like a continuous analogue of the discrete Kronecker delta

$$\delta_{i\,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Indeed, for any sequence $\{a_j\}_{j\in\mathbb{Z}}$, we have

$$\sum_{j=-\infty}^{\infty} a_j \delta_{i\,j} = a_i,$$

and, analogously, for any $x \in \mathbb{R}^n$ and any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Dirac delta satisfies the property

$$\int_{-\infty}^{+\infty} f(y)\delta(x-y)\, dy = \int_{-\infty}^{\infty} f(y)\, d\delta_x(y) = f(x).$$

Example 1.1.15 (The Lebesgue measure)**.** The well-known *Lebesgue measure* is defined by

$$\mathscr{L}^n(A) := \inf\left\{\sum_{i=1}^{\infty}\mathscr{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes}\right\},$$

where $\mathscr{L}^n(Q_i) = (l(Q_i)^n$ and $l(Q_i)$ is the side length of the cube Q_i. It is actually possible to show that in one dimension we have

$$\mathscr{L}^1(A) = \inf\left\{\sum_{ij=1}^{\infty}\operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R}\right\}$$

and that we can characterize $\mathscr{L}^n$ in an alternative way as

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}} = \mathscr{L}^{n-1} \times \mathscr{L}^1.$$

1.2 The Hausdorff measure

Definition 1.2.1 (Hausdorff content)**.** Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *α-dimensional Hausdorff content* of A as

$$\mathscr{H}^\alpha_\delta(A) := \inf\left\{\sum_{j\in I}\omega_\alpha\left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha \mid A \subset \bigcup_{j\in I\subset\mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n\right\},$$

where the infimum is taken over all the (at most countable) *δ-coverings* $\{C_j\}_{j\in I}$ of A, and we set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

We notice that $\mathscr{H}^\alpha_\delta(A)$ is a non-decreasing function in δ, so that we can take the limit as $\delta \searrow 0$ and it always exists in the extended real numbers. This justifies the following definition.

Definition 1.2.2 (Hausdorff measure)**.** For any $A \subset \mathbb{R}^n$ and $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of A as

$$\mathscr{H}^\alpha(A) := \lim_{\delta\searrow 0}\mathscr{H}^\alpha_\delta(A) = \sup_{\delta>0}\mathscr{H}^\alpha_\delta(A).$$

Roughly speaking, we take the limit as $\delta \searrow 0$ since it forces the coverings to follow the local geometry of the set A. Indeed, the key idea behind the definition of the Hausdorff measure is that it should be able to capture the properties of thin sets in $\mathbb{R}^n$ (in particular, Lebesgue negligible sets). As we shall see in the following, if $\alpha = k \in \{1, \dots, n-1\}$, then $\mathscr{H}^k$ agrees with the k-dimensional surface area on sufficiently regular sets, as for instance k-dimensional planes.

It is not too difficult to prove that, as a consequence of Carathéodory's criterion, Theorem 1.1.11, any Borel set is $\mathscr{H}^\alpha$-measurable, for any $\alpha \geq 0$.

Theorem 1.2.3 (Hausdorff measures are Borel regular)**.** *$\mathscr{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem 1.2.4 (Basic properties of the Hausdorff measure)**.** *The following statements hold true:*

(1) $\mathscr{H}^0 = \#$;

(2) $\mathscr{H}^1 = \mathscr{H}^1_\delta = \mathscr{L}^1$ on $\mathbb{R}$, for any $\delta > 0$;

(3) $\mathscr{H}^\alpha \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$;

(4) $\mathscr{H}^\alpha(\lambda A) = \lambda^\alpha\mathscr{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$;

(5) $\mathscr{H}^\alpha(L(A)) = \mathscr{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$, we have $\mathscr{H}^0_\delta(\{x\}) = 1$ for every $x \in \mathbb{R}^n$ and $\delta > 0$. Indeed,

$$\omega_0 \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 = 1,$$

which implies $\mathscr{H}^0_\delta(\{x\}) \geq 1$, and, on the other hand, we can clearly cover the singleton only with itself. Hence, $\mathscr{H}^0(\{x\}) = 1$ for every $x \in \mathbb{R}^n$. Since $\mathscr{H}^0$ is a Borel measure, it is σ-additive on Borel sets, so that

$$\mathscr{H}^0(A) = \sum_{x \in A} \mathscr{H}^0(\{x\}) = \#A,$$

for any finite or countable set A. Finally, if A is incountable, then A contains a countable set B, and so $\mathscr{H}^0(A) \geq \mathscr{H}^0(B) = +\infty$.

(2) We estimate the Lebesgue measure $\mathscr{L}^1$ from both sides by the Hausdorff measure. Since $\omega_1 = 2 = |(-1,1)|$, for any $\delta > 0$ we first get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\leq \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j,\ \operatorname{diam} C_j \leq \delta \right\} = \mathscr{H}^1_\delta(A), \end{aligned}$$

Now, we define a partition of $\mathbb{R}$ by setting $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$. These are intervals of diameter δ, so that, for every $j \in I$, we get

$$\operatorname{diam}(C_j \cap J_{k,\delta}) \leq \delta. \tag{1.2.1}$$

In addition, we have

$$\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \leq \operatorname{diam} C_j, \tag{1.2.2}$$

since $\{J_{k,\delta}\}_{k \in \mathbb{Z}}$ is a partition $\mathbb{R}$ of essentially disjoint intervals, because $\#(J_{k,\delta} \cap J_{m,\delta}) \leq 1$ for any $k \neq m$. Therefore, by (1.2.2) we get

$$\begin{aligned} \mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap J_{k,\delta} \right\}. \end{aligned}$$

We set now $C_j \cap J_{k,\delta} =: \widetilde{C}_{i_{j,k}}$, by relabeling the indexes sets I and $\mathbb{Z}$ to an index set $\widetilde{I}$. Thanks to (1.2.1), we have $\operatorname{diam}(\widetilde{C}_i) \leq \delta$ and so we get

$$\mathscr{L}^1(A) \geq \inf \left\{ \sum_{j \in \widetilde{I}} \widetilde{C}_i \mid A \subset \bigcup_{i \in \widetilde{I}} \widetilde{C}_i,\ \operatorname{diam} \widetilde{C}_i \leq \delta \right\} \geq \mathscr{H}^1_\delta(A).$$

All in all, we get $\mathscr{L}^1 = \mathscr{H}^1_\delta$ for any $\delta > 0$, from which it easily follows $\mathscr{L}^1 = \mathscr{H}^1$ on $\mathbb{R}$.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. It is easy to see that, for any fixed $m \in \mathbb{N}$, Q can be covered by m^n smaller cubes Q_i with side length $\frac{1}{m}$. Clearly, we have $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$. Therefore, we obtain

$$\mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha},$$

from which we deduce that, since $n < \alpha$,

$$\mathscr{H}^\alpha(Q) = \lim_{m\to\infty} \mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} \lim_{m\to\infty} m^{n-\alpha} = 0.$$

Thus, the claim easily follows, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes and $\mathscr{H}^n$ is σ-subadditive.

The proofs of (4) and (5) are left as an exercise. □

Lemma 1.2.5. *Let $A \subset \mathbb{R}^n$ and $\delta_0 > 0$ such that $\mathscr{H}^\alpha_{\delta_0}(A) = 0$, then we have $\mathscr{H}^\alpha(A) = 0$.*

Proof. Since the Hausdorff content is non-increasing in δ, we have $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_\delta(A)$ for any $\delta > 0$. In particular, this means that $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_{\delta_0}(A) = 0$, so that, for every $\varepsilon > 0$, there exists a countable family of sets $\{C_j\}_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

In particular, the second condition immediately implies

$$\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon.$$

Hence, we have $\mathscr{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$, and $\delta_\varepsilon \searrow 0$ if and only if $\varepsilon \searrow 0$. This implies the claim $\mathscr{H}^\alpha(A) = 0$. □

Proposition 1.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$.

(2) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a countable family of subsets $\{C_j\}_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathscr{H}^s_\delta(A) + 1 \le \mathscr{H}^s(A) + 1.$$

From this, it follows that

$$\begin{aligned}\mathscr{H}^t_\delta(A) \le \sum_{j\in I} \omega_t \left(\frac{\operatorname{diam} C_j}{2}\right)^t &= \frac{\omega_t}{\omega_s} 2^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathscr{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0,\end{aligned}$$

which implies the claim $\mathscr{H}^t(A) = 0$.

(2) If by contradiction $\mathscr{H}^s(A) < \infty$, then by (1) if follows that $\mathscr{H}^r(A) = 0$ for all $r > s$ and in particular for $r = t$, which is clearly absurd.

□

Definition 1.2.7. We call the *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ the number

$$\dim_{\mathscr{H}}(A) := \inf\{\alpha \ge 0 : \mathscr{H}^\alpha(A) = 0\}.$$

Remark 1.2.8. Let $\alpha = \dim_{\mathscr{H}}(A)$. Then one has

$$\mathscr{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and} \qquad \mathscr{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha. \tag{1.2.3}$$

The first part of (1.2.3) follows clearly from the definition of the Hausdorff dimension. The second, instead, can be proved by contradiction. Suppose by contradiction that $\mathscr{H}^t(A) < \infty$ for some $t < \alpha$, then, by the Proposition 1.2.6, we have $\mathscr{H}^r(A) = 0$ for all $r > t$. This implies

$$\alpha = \inf\left\{\beta \ge 0 : \mathscr{H}^\beta(A) = 0\right\} \le t < \alpha,$$

which is clearly absurd.

It should be noticed that, in general, $\mathscr{H}^\alpha(A)$ can be any number in $[0, +\infty]$.

We state now an important result on the equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure, whose proof we postpone to the end of the section.

Theorem 1.2.9. *$\mathscr{H}^n_\delta = \mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$, for any $\delta > 0$.*

Remark 1.2.10. As a consequence of Theorem 1.2.9, we see that $\mathscr{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Indeed, it is not bounded on some compact sets. Take for example the closed unit ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that $0 < \mathscr{H}^n(\overline{B(0,1)}) < \infty$ and so, by Proposition 1.2.6, $\mathscr{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

Even though $\mathscr{H}^\alpha$ is not a Radon measure for $\alpha \in [0, n)$, it is possible to show that its restriction to some suitable Borel set is indeed a Radon measure.

Proposition 1.2.11. *If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^\alpha(E) \in (0, \infty)$, then $\mathscr{H}^\alpha \llcorner E$ is a Radon measure.*

Proof. It is a simple consequence of Theorem 1.1.12. $\square$

We investigate now the behaviour of the Hausdorff measure under the action of Lipschitz and Hölder functions. We recall first the definition of such family of functions.

Definition 1.2.12 (Lipschitz and Hölder functions)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set.

(1) We say that $f : \Omega \to \mathbb{R}^m$ is *Lipschitz continuous* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|x - y| \quad \text{for any } x, y \in \Omega. \tag{1.2.4}$$

The smallest constant for which (1.2.4) holds is called the *Lipschitz constant* of f, denoted by $\mathrm{Lip}(f)$ and alternatively characterized by

$$\mathrm{Lip}(f) := \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} \mid x, y \in \Omega, x \ne y \right\}. \tag{1.2.5}$$

(2) Let $\gamma \in (0, 1)$. We say that $f : \Omega \to \mathbb{R}^m$ is *γ-Hölder continuous* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|x - y|^\gamma \quad \text{for any } x, y \in \Omega. \tag{1.2.6}$$

From this point on, we shall refer to Lipschitz continuous and Hölder continuous functions simply as Lipschitz and Hölder functions.

Exercise 1.2.13. Show that any Lipschitz or γ-Hölder function (for some $\gamma \in (0, 1)$) is indeed continuous.

Remark 1.2.14. Lipschitz functions can be seen as 1-Hölder functions. Indeed, for any open set $\Omega \subset \mathbb{R}^n$ and any $\gamma \in [0, 1]$, we can define the space $C^{0,\gamma}(\Omega; \mathbb{R}^m)$ of γ-Hölder functions as the set of continuous functions $f : \Omega \to \mathbb{R}^m$ for which there exists a constant $C > 0$ such that (1.2.6) holds. If $\gamma = 0$, we have $C^{0,0}(\Omega; \mathbb{R}^m) = C^0(\Omega; \mathbb{R}^m)$.

Exercise 1.2.15. Let $\gamma > 1$ and $f : \Omega \to \mathbb{R}^m$ be such that there exists a constant $C > 0$ such that (1.2.6) holds. Show that f is constant.

Proposition 1.2.16. *Let $\alpha \ge 0$, $A \subset \mathbb{R}^n$.*

(1) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then $\mathscr{H}^\alpha(f(A)) \le (\mathrm{Lip}(f))^\alpha \mathscr{H}^\alpha(A)$.

(2) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is γ-Hölder, for some $\gamma \in (0, 1)$, then $\mathscr{H}^\alpha(f(A)) \le C_{\alpha,\gamma} \mathscr{H}^{\alpha\gamma}(A)$.

Proof. Thanks to Remark 1.2.14, it is enough to prove (2) for any $\gamma \in (0, 1]$. Fix $\delta > 0$, and take a countable family of sets $\{C_j\}_{j \in I}$ such that $A \subset \bigcup_{j \in I} C_j$ and $\operatorname{diam} C_j \le \delta$. It is clear that

$$f(A) \subseteq \bigcup_{j \in I} f(C_j).$$

Thanks to (1.2.6), we see that $f(C_j)$ satisfies

$$\operatorname{diam} f(C_j) \le C \left(\operatorname{diam} C_j\right)^\gamma \le C\delta^\gamma,$$

where $C = \operatorname{Lip}(f)$ is $\gamma = 1$. Hence, we obtain

$$\mathscr{H}^\alpha_{C\delta^\gamma}(f(A)) \le \sum_{j\in I}\omega_\alpha\left(\frac{\operatorname{diam} f(C_j)}{2}\right)^\alpha \le \underbrace{\frac{\omega_\alpha}{2^\alpha}\,\frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}}\sum_{j\in I}\omega_{\alpha\gamma}\left(\frac{\operatorname{diam} C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all δ-coverings $\{C_j\}_{j\in I}$ we get

$$\mathscr{H}^\alpha_{C\delta^\gamma}(g(A)) \le C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}_\delta(A),$$

where $C_{\alpha,\gamma} = \operatorname{Lip}(f)^\alpha$, if $\gamma = 1$. By sending $\delta \searrow 0$ we conclude the proof. $\square$

Remark 1.2.17 (Sierpinski triangle (Waclaw Sierpinski 1915))**.** One can construct a fractal triangle as follows:

1. Take S_0 to be a equilateral triangle.
2. Divide S_0 evenly into four smaller equilateral triangles. Cut out the triangle in the center.
3. Now do the step in 2. with these three equilateral triangles indefinitely.

So the S_k's are the union of 3^k equilateral triangles with side length 2^{-k}. We define

$$S := \bigcup_{k=0}^\infty S_k$$

and compute

$$\mathscr{H}^\alpha_{\frac{1}{2^k}}(S) \le \sum_{j=1}^{3^k}\frac{\omega_\alpha}{2^\alpha}\left(\operatorname{diam} {S_k}^j\right)^\alpha = \frac{\omega_\alpha}{2^\alpha}3^k 2^{-k\alpha},$$

which goes to zero for $k \to \infty$ if and only if $\alpha > \frac{\log^3}{\log^2}$. So we can conclude that for all $\alpha > \frac{\log 3}{\log 2}$ we have $\mathscr{H}^\alpha(S) = 0$ and with that we found:

$$\dim_{\mathcal{H}}(S) \le \frac{\log 3}{\log 2}.$$

Next we aim to proof an important characterization for the Hausdorff measure for positive integers:

Theorem 1.2.18. *$\mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$.*

The proof is crucially based on the two following statements.

Lemma 1.2.19 (Vitali covering property for $\mathscr{L}^n$)**.** *For all open U and for all $\delta > 0$ there exists a family of disjoint closed balls $\{\overline{B_k}\}_{k=1}^\infty$ such that $\operatorname{diam} B_k < \delta$ and $\mathscr{L}^n(U \setminus \bigcup_{k=1}^\infty \overline{B_k}) = 0$.*

Theorem 1.2.20 (isodiametric inequality)**.** *For all $\mathscr{L}^n$-measurable sets $E \subset \mathbb{R}^n$ we have*

$$|E| \le \omega_n\left(\frac{\operatorname{diam} E}{2}\right)^n.$$

Proof of theorem 1.2.18. We show the proof in three steps.

(Step 1) To show: $\mathscr{L}^n(A) \le \mathscr{H}^n(A)$ for all $A \subset \mathbb{R}^n$. @GC:A measurable?

Fix $\delta > 0$. Let $\{C_j\}_{j\in I}$: $A \subset \bigcup_{j\in I} C_j$, $\operatorname{diam} C_j \le \delta$. From this follows

$$\mathscr{L}^n(A) \le \sum_{j=1}^\infty \mathscr{L}^n(C_j) \le \sum_{j=1}^\infty \omega_n\left(\frac{\operatorname{diam} C_j}{2}\right)^n,$$

where in the last inequality we used the *isometric inequality* (e.g. theorem 1.2.20). Taking the infimum over all $\{C_j\}$ we arrive at claim

$$\mathscr{L}^n(A) \le \mathscr{H}^n_\delta(A) \qquad \text{for all} \quad \delta > 0.$$

(Step 2) To show: $\mathscr{H}^n_\delta \le C_n \mathscr{L}^n$ for some $C_n \ge 1$.
With the definition of the Lebesgue measure we get

$$\begin{aligned}
\mathscr{L}^n(A) &= \inf\left\{\sum_{j=1}^\infty \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j\right\} \\
&= \inf\left\{\sum_{j=1}^\infty \mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j,\, \operatorname{diam} Q_j < \delta\right\} \\
&= \frac{2^n}{\left(\sqrt{n}\right)^n \omega_n} \inf\left\{\sum_{j=1}^\infty \omega_n \left(\frac{\operatorname{diam} Q_j}{2}\right)^2 \mid A \subset \bigcup Q_j,\, \operatorname{diam} Q_j < \delta\right\} \\
&\ge \frac{1}{C_n} \mathscr{H}^n_\delta(A),
\end{aligned}$$

where for the second equality we used that

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}}, \qquad \mathscr{L}^1 = \mathscr{H}^1_\delta \quad \text{in } \mathbb{R} \quad \text{for all } \delta > 0, \quad \text{and} \quad \mathscr{L}^n(Q_j) = \left(\frac{\operatorname{diam} Q_j}{\sqrt{n}}\right)^n.$$

(Step 3) To show: $\mathscr{H}^n_\delta(A) \le \mathscr{L}^n(A) + \varepsilon$ for any $\varepsilon > 0$.
By definition of $\mathscr{L}^n$: For all $\delta, \varepsilon > 0$, there exists a family $\{Q_j\}_{j=1}^\infty$ such that $A \subset \bigcup_{j=1}^\infty Q_j$, $\operatorname{diam} Q_j \le \delta$ and $\sum_{j=1}^\infty \mathscr{L}^n(Q_j) \le \mathscr{L}^n(A) + \varepsilon$. Now, with lemma 1.2.19, there exists a family $(B^i_j)_{i=1}^\infty$ of disjoint closed balls such that $B^i_j \subset Q_j$ for all ($\operatorname{diam} B^i_j \le \delta$) and

$$\mathscr{L}^n\left(\mathring{Q}_j \setminus \bigcup_{i=1}^\infty \overline{B^i_j}\right) = 0 = \mathscr{L}^n\left(Q_j \setminus \bigcup_{i=1}^\infty \overline{B^i_j}\right).$$

So with step 2 we also have

$$\mathscr{H}^n_\delta\left(Q_j \setminus \bigcup_{i=1}^\infty \overline{B^i_j}\right) = 0,$$

from which we can infer that

$$\begin{aligned}
\mathscr{H}^n_\delta(A) &\le \sum_{j=1}^\infty \mathscr{H}^n_\delta(Q_j) = \sum_{j=1}^\infty \mathscr{H}^n_\delta\left(\bigcup_{i=1}^\infty \overline{B^i_j}\right) \\
&= \sum_{j=1}^\infty \sum_{i=1}^\infty \mathscr{H}^n_\delta\left(B^i_j\right) \le \sum_{j=1}^\infty \sum_{i=1}^\infty \underbrace{\omega_n \left(\frac{\operatorname{diam} B^i_j}{2}\right)^n}_{=\mathscr{L}^n\left(\overline{B^i_j}\right)} \\
&= \sum_{j=1}^\infty \mathscr{L}^n\left(\bigcup_{i=1}^\infty \overline{B^i_j}\right) = \sum_{j=1}^\infty \mathscr{L}^n(Q_j) \\
&\le \mathscr{L}^n(A) + \varepsilon.
\end{aligned}$$

And since the $\varepsilon > 0$ is arbitrary in (step 2) we arrive at the claim. □

Proof of isodiametric inequality (lemma 1.2.20). If $E \subset B(x, \frac{\operatorname{diam} E}{2})$ for some x then it's trivial. w.l.o.g, E is compact. $\operatorname{diam} A = \operatorname{diam} \overline{A}$. Steiner symmetrization (1838). Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q : \mathbb{R}^n \to \mathbb{R}$ so that $x = (px, qx)$, $q(x) = x_n$

$$\forall z \in \mathbb{R}^{n-1} \quad E_z := \{t \in \mathbb{R} : (z,t) \in E\} \quad \text{vertical section}$$

define

$$E^s := \left\{x \in \mathbb{R}^n : |q(x)| \le \frac{\mathscr{L}^1(E_{p(x)})}{2}\right\}.$$

By Fubini's theorem, E_z is $\mathscr{L}^1$-measurable for $\mathscr{L}^{n-1}$-a.e. z, $z \mapsto \mathscr{L}^1(E_z)$ is Lebesgue measurable and so we get

$$|E| = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z^s)dz = |E^s|,$$

where the first equality follows with Fubini, and the second equal sign is due to

$$(E^s)_z = \{t \in \mathbb{R} : (z,t) \in E^s\} = \left\{t \in \mathbb{R} : |t| \leq \frac{\mathscr{L}^1(E_z)}{2}\right\} = \left[-\frac{\mathscr{L}^1(E_z)}{2}, \frac{\mathscr{L}^1(E_z)}{2}\right].$$

Now we claim

$$\operatorname{diam} E^s \leq \operatorname{diam} E.$$

To proof this, let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned} &p(m(x)) = p(M(x)) = px \\ &q(m(x)) = q(z) \leq q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x). \end{aligned}$$

Let $x, y \in E^s$,

$$\begin{aligned} |q(x) - q(y)| &\leq \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\} \overset{w.l.o.g.}{=} |q(M(x)) - q(m(y))| \\ |x-y|^2 &= |p(x-y)|^2 + |q(x-y)|^2 \leq \max\{|M(x) - m(y)|, |M(y) - m(x)|\}^2 \leq (\operatorname{diam} E)^2. \end{aligned}$$

From this follows $|x - y| \leq \operatorname{diam} E$ for all $x, y \in E^s$.

Given a $\mathscr{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. $E_0 := E$, $E_i := (E_{i=1})^i$ with $i \in \{1, 2, \dots, n\}$. Then $|E_n| = |E|$, $\operatorname{diam} E_n \leq \operatorname{diam} E$ and, if $x \in E_n$, then $-x \in E_n$. From this follows $E_n \subset B\left(0, \frac{\operatorname{diam} E_n}{2}\right)$. And with this we are done! □

1.3 Integration

Let $X \neq \emptyset$, μ be a measure on X.

Definition 1.3.1.

(1) A function $u : X \to [-\infty, \infty] =: \overline{\mathbb{R}}$ is *μ-measurable* if $\{u > t\} = \{x \in X : u(x) > t\}$ is μ-measurable for all $t \in \mathbb{R}$.

(2) u is a *μ-simple function* if it is μ-measurable and $u(X)$ is countable (that is $u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x)$)

(3) If u is a non-negative μ-simple function, we define

$$\int_X u d\mu := \sum_{t \in u(X)} t\mu(\{u = t\}) = \sum_{k=1}^{\infty} u_k \mu(E_k) \in [0, \infty]$$

where $0 \cdot \infty = 0$.

(4) Set $u^{\pm} := \max\{\pm u, 0\}$, $u = u^+ - u^-$. If u is μ-simple and $\int_X u^+ d\mu$ or $\int_X u^- d\mu < \infty$, then

$$\int_X u d\mu := \int_X u^+ d\mu - \int_X \mu^- d\mu \in [-\infty, \infty]$$

If v satisfies (4), is called μ-integrable simple function.

(5) If u is μ-measurable, we define the upper and lower integrals of u as

$$\int_X^* u d\mu := \inf\left\{\int_X v d\mu \mid v \geq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

or

$${}_*\!\int_X u d\mu := \sup\left\{\int_X v d\mu \mid v \leq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

respectively. If

$${}_*\!\int_X u d\mu = \int_X^* u d\mu,$$

then u is μ-integrable.

(6) A measurable function u is *μ-summable* if $|u|$ is μ-integrable and

$$\int_X |u| d\mu < \infty$$

(7) Now we define the following sets

$L^1(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mu\text{-summable}\}$
$L^1_{\text{loc}}(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid u\chi_K \text{ is } \mu\text{-summable for all } K \subset X \text{ compact}\}$
$L^p(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid |u|^p \text{ is } \mu\text{-summable}\}$
$L^1_{\text{loc}}(X,\mu) := \{u : X \to \overline{\mathbb{R}} \mid |u|^p\chi_K \text{ is } \mu\text{-summable for all } K \subset X \text{ compact}\}$

Definition 1.3.2. If $u : X \to [0,\infty]$ μ-measurable, then we define $\nu = u\mu$ (or $\mu \llcorner u$) as

$$\nu(A) = \int_A u d\mu = \int_X u\chi_A d\mu \qquad \text{for all} \quad \mu\text{-measurable } A$$

Definition 1.3.3 (Radon measure)**.** Given a Radon measure μ on an open subset $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to [-\infty,\infty]$ locally μ-summable. Then $\nu := f\mu$ defined by

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A\, \mu\text{-measurable}$$

is said to be a *signed Radon measure* on Ω.