Lectures Notes

BV functions and sets of finite perimeter

Winter Term 2019/20

Lecture by Giovanni E. Comi

Written by Giovanni E. Comi and Dennis Schmeckpeper

Version: November 19, 2019

Contents

Introduction

♯ *Geometric Measure Theory* is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces generalizing techniques from differential geometry through measure theoretic arguments. The theory of *functions of bounded variations* and *sets of finite perimeter* is one of the core topics of Geometric Measure Theory, since it deals with extension of the classical notion of Sobolev functions and regular surfaces.

The 1-Laplace operator and BV as a natural extension of $W^{1,1}$

In the Calculus of Variation, the *Direct Method* is a general way of proving the existence of a minimizer for a given functional. More precisely, let X be a topological space and $F : X \to (-\infty, +\infty]$ be a functional. We are interested in finding a minimizer of F in X; that is, a $u \in X$ such that $F(u) \le F(v)$ for any $v \in X$. Assume that

$$m := \inf\{F(v) \,:\, v \in X\} > -\infty.$$

This ensure the existence of a minimizing sequence $\{v_j\}$; that is, a sequence of elements $v_j \in X$ such that $F(v_j) \to m$. Then, the Direct Method consists in the following steps:

(1) show that $\{v_j\}$ admits a converging subsequence $\{v_{j_k}\}$ and $u \in X$ such that $v_{j_k} \to u$, with respect to a the topology of X;

(2) show that F is (sequentially) lower semicontinuous with respect to the topology of X; that is, if $z_j \to z_0$ in X, then
$$F(z_0) \le \liminf_{j \to +\infty} F(z_j).$$

If these properties hold true, we can conclude that u is a minimizer of F. Indeed, we have

$$m = \lim_{k \to +\infty} F(v_{j_k}) \ge \liminf_{k \to +\infty} F(v_{j_k}) \ge F(u) \ge m,$$

from which we immediately conclude that $F(u) = \min\{F(v) \,:\, v \in X\}$.

This method is fundamental in proving the existence of solutions to minimization problems related to boundary value problems. Let us consider for instance the classical Dirichlet problem for the Laplace equation on an open set Ω with C^1-smooth boundary:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$. It is possible to see this system as the Euler-Lagrange equations for the functional
$$F(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx - \int_\Omega f u \, dx$$
defined on the space

$$X = W_0^{1,2}(\Omega) := \{u \in L^2(\Omega) \,:\, Du \in L^2(\Omega; \mathbb{R}^n), u = 0 \text{ on } \partial\Omega\};$$

♯These notes have been written for the course of *BV Functions and Sets of Finite Perimeter* held in the Department of Mathematics of the Hamburg Universität. The main references are the books [**?**, **?**, **?**]. Please write an email to giovanni.comi@uni-hamburg.de if you have corrections, comments, suggestions or questions.

that is, the space of 2-summable weakly differentiable Sobolev functions with zero trace on $\partial\Omega$[♯]. As customary, we denote by Du the weak gradient of u. Thanks to Poincaré inequality, we can prove that

$$\inf\{F(u) \,:\, u \in W_0^{1,2}(\Omega)\} > -\infty.$$

Hence, we can find the solution looking for minizers of F through the Direct Method: let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence. It is possible to show that $\{u_j\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, which is an Hilbert space, and in particular reflexive: as a consequence, there exists a subsequence $\{u_{j_k}\}$ converging to some $u \in W_0^{1,2}(\Omega)$ with respect to the weak topology. In addition, F is lower semicontinuous with respect to the weak topology, and so we infer the existence of a solution for the minimization problem

$$\min\left\{\int_\Omega \frac{1}{2}|Du|^2 - f u\, dx \,:\, u \in W_0^{1,2}(\Omega)\right\}.$$

It seems natural now to wonder if we could substitute the exponent 2 with any $p \in (1,\infty)$. Thanks to the Poincaré inequality and the reflexivity of the L^p-spaces for $p \in (1,\infty)$, it is indeed possible to show that, for any $f \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the problem

$$\min\left\{\int_\Omega \frac{1}{p}|Du|^p - f u\, dx \,:\, u \in W_0^{1,p}(\Omega)\right\}$$

admits a solution, where

$$W_0^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, Du \in L^p(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

The minimizers to this problem solves the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) =: \Delta_p u$ is the p-Laplace operator.

The next logical step is to consider also the case $p = 1$: for a given $f \in L^\infty(\Omega)$, we want to find a function u which realizes

$$\inf\left\{\int_\Omega |Du| - f u\, dx \,:\, u \in W_0^{1,1}(\Omega)\right\} =: m, \tag{0.0.1}$$

where

$$W_0^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

If we assume $\|f\|_{L^\infty(\Omega)}$ to be sufficiently small, we can again employ the Poincaré inequality to prove that $m \in (-\infty, +\infty]$. Hence, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $W_0^{1,1}(\Omega)$ such that

$$\lim_{j\to+\infty} \int_\Omega |Du_j| - f u_j\, dx = m.$$

However, in this case we cannot argue as above in the case $p > 1$, since, in general this does *not* imply that the existence of a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ weakly converging to some $u \in W_0^{1,1}(\Omega)$ such that

$$\int_\Omega |Du| - f u\, dx = m.$$

The reason for this lies in the fact that $L^1(\Omega)$ is not reflexive, and actually it is not the topological dual of any separable space. However, $L^1(\Omega)$ is contained in the space of finite Radon measures on Ω, $\mathcal{M}(\Omega)$, and this space can be see as the dual of the space of continuous functions vanishing on the boundary of Ω, $C_0(\Omega)$.

This fact suggests the definition of a space which contains the Sobolev space $W^{1,1}(\Omega)$ and which, although not reflexive, enjoys the property that bounded sets are weakly* compact: the space of *functions with bounded variation*,

$$BV(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n)\}.$$

[♯] We refer to [?, Chapter 5] and to [?, Chapter 4] for a detailed account on Sobolev spaces.

It is not difficult to prove that the total variation of the Radon measure Du over Ω is indeed lower semicontinuous with respect to the weak* converge of the gradient measures. This indicates that the correct space where to look solutions to (0.0.1) is the space of functions with bounded variation with zero trace$^\flat$,

$$BV_0(\Omega) := \{u \in L^1(\Omega) \,:\, Du \in \mathcal{M}(\Omega;\mathbb{R}^n), u = 0 \text{ on } \partial\Omega\}.$$

Finally, it is relevant to mention the fact that the minimizers to (0.0.1) solve the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) =: \Delta_1 u$ is the 1-Laplace operator, which is non trivially defined on nonsmooth functions because of the highly degenerate term $\frac{\nabla u}{|\nabla u|}$.

Minimal area problems and sets of finite perimeter

Other historically relevant problems from the Calculus of Variation are the minimal area problems, among which the most famous example is the *Euclidean isoperimetric problem*: find the possibly unique set with minimal surface area among those with fixed volume. In mathematical terms, if we denote by $|F|$ the n-dimensional volume of a set $F \subset \mathbb{R}^n$ (hence, its Lebesgue measure $\mathscr{L}^n(F)$) and by $\sigma_{n-1}(\partial F)$ its surface area (under the assumption the ∂F is regular enough), we are looking for the set which realizes

$$\inf\{\sigma_{n-1}(\partial F) \,:\, \partial F \in \mathcal{R}, |F| = k\} =: \gamma_k,$$

where $\mathcal{R}$ is a class of sufficiently smooth surfaces and $k > 0$. The Direct Method now consists in considering a minimizing sequence of sets F_j such that

$$\partial F_j \in \mathcal{R}, \quad |F_j| = k \quad \text{and} \quad \sigma_{n-1}(\partial F_j) \to \gamma_k, \tag{0.0.2}$$

and then in trying to prove the convergence (possibly up to subsequences) to some limit set E such that

$$\partial E \in \mathcal{R}, \quad |E| = k \quad \text{and} \quad \sigma_{n-1}(\partial E) = \gamma_k.$$

In order to achieve this result, some compactness property in the family of sets satisfying (0.0.2) is required. In addition, the surface measure σ_{n-1} need to be a lower semicontinuous with respect to the chosen convergence of sets, in the sense that

$$\sigma_{n-1}(\partial E) \le \liminf_{j\to+\infty} \sigma_{n-1}(\partial F_j)$$

if $F_j \to E$ in a suitable sense. However, these compactness and lower semicontinuity properties in general fail to be true in family of sets with regular topological boundary. In addition, we notice that the topological boundary is very unstable under modification of a set by Lebesgue negligible sets. For instance, let

$$E_1 = B(0,1) \quad \text{and} \quad E_2 = B(0,1) \cup (\partial B(0,2) \cap \mathbb{Q}^n).$$

It is plain to see that $|E_1 \Delta E_2| = 0$, so that these two sets are equivalent with respect to the Lebesgue measure, and so they have the same volume. However, their topological boundary, which are smooth surfaces, are very different:

$$\partial E_1 = \partial B(0,1) \quad \text{and} \quad \partial E_2 = \partial B(0,1) \cup \partial B(0,2).$$

The need of ruling out these problems and of recovering a notion of compactness and a lower semicontinuity property for the surface area is one of the main reasons for the birth of Geometric Measure Theory. This theory concerns methods to study the geometric properties of rough, irregular sets from a measure theoretic point of view. In this course we shall see how to exploit this

$^\flat$It can be proved that the trace of a function with bounded variation is well defined on any C^1-regular surface, as in the Sobolev case.

approach to give a meaningful notion of surface area for an irregular set and to define a suitable class of sets for which we can apply the Direct Method of the Calculus of Variation in order to deal with minimal area problems: the *sets of finite perimeter*. Broadly speaking, the notion of set of finite perimeter extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to minimal area problems and other similar geometric variational minimization problems. More precisely, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ is its characteristic function χ_E is a function with locally bounded variation.

Chapter 1

Notions of abstract Measure Theory

1.1 General measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*; that is, the collection of all subsets of X.

Definition 1.1.1 (Measures)**.** A mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ satisfying

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ if $A \subset \bigcup_{k=1}^{\infty} A_k$ (σ-subadditivity),

is called a measure.

It should be noticed that in the literature a mapping as the one in Definition 1.1.1 is also called an *outer measure*, while the name of measure is used to denote the restriction of the mapping to the family of measurable set (see Definition 1.1.4 below). We shall nevertheless follow the notation of [**?**], in order to be able to assign a measure even to nonmeasurable sets.

Remark 1.1.2. Thanks to σ-subadditivity, any measure is not decreasing; that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition 1.1.3 (Restriction of a measure)**.** If $Y \subset X$, the *restriction of μ to Y*, denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$ for any $A \subset X$.

Definition 1.1.4 (μ-measurable sets)**.** We call a subset $A \subset X$ μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \qquad \text{for all} \quad B \subseteq X.$$

Remark 1.1.5. This definition is meaningful, since the italian mathematician *Giuseppe Vitali* proved in 1905 that there exists a set $E \subset \mathbb{R}$ which is *not* $\mathscr{L}^1$-measurable [**?**]. For a modern presentation of his construction, we refer to [**?**, Section I.1.2].

Definition 1.1.6 (σ-algebra)**.** A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a *σ-algebra of sets* if the following conditions hold:

(1) $\emptyset, X \in \mathfrak{F}$,

(2) for any $A \in \mathfrak{F}$ we have $X \setminus A \in \mathfrak{F}$,

(3) for any countable family of sets $\{A_i\}_{i \in I}$ such that $A_i \in \mathfrak{F}$ for any $i \in I$ we have have

$$\bigcup_{i \in I} A_i \in \mathfrak{F}.$$

Theorem 1.1.7. *Given any measure μ on X, the family of μ-measurable sets forms a σ-algebra.*

Theorem 1.1.8. *Let μ be a measure on X, then the restriction to the σ-algebra of μ-measurable sets is σ-additive, that is, if $(A_j)_{j \in I}$ is a countable disjoint μ-measurable family of subsets of X, then*

$$\mu \left(\bigcup_{j \in I} A_j \right) = \sum_{j \in I} \mu (A_j) .$$

We list now some relevant definitions.

Definition 1.1.9.

(1) Given any $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ-algebra containing $\mathfrak{C}$, the *σ-algebra generated by* $\mathfrak{C}$.

(2) The *Borel σ-algebra* on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ-algebra generated by the family of open sets in $\mathbb{R}^n$ (in the standard topology). The elements of the Borel σ-algebra are called *Borel sets.*

(3) A measure μ in $\mathbb{R}^n$ is called a *Borel measure* if each Borel sets is μ-measurable.

(4) A measure μ in $\mathbb{R}^n$ is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

(5) A Borel regular measure μ which is locally finite (i.e. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure.*

Theorem 1.1.10. *Let μ be a Radon measure on $\mathbb{R}^n$. Then we have*

(1) $\mu(A) = \inf\{\mu(U) : U \supset A,\ U$ open$\}$ for all $A \subseteq \mathbb{R}^n$ (outer regularity),

(2) $\mu(B) = \sup\{\mu(K) : K \subset B,\ K$ compact$\}$ for all μ-measurable sets B (inner regularity).

Theorem 1.1.11 (Carathéodory's criterion)**.** *Let μ be a measure on $\mathbb{R}^n$. If for all $A, B \subset \mathbb{R}^n$ such that* $\operatorname{dist}(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

then μ is a Borel measure.

Not any Borel regular measure is a Radon measure. However, it is possible to obtain a Radon measure as a restriction of a Borel regular one, as stated in the followin theorem.

Theorem 1.1.12. *If μ is a Borel regular measure in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.*

Example 1.1.13 (Dirac delta)**.** For $x \in \mathbb{R}^n$ we define the *Dirac*$^\sharp$ *measure centered in x* by setting

$$\delta_x(A) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

It is easy to check that this is indeed a Radon measure. In addition, any set in $\mathbb{R}^n$ is δ_x-measurable.

Example 1.1.14 (The counting measure)**.** We define the *counting measure* by setting

$$\#(E) = \begin{cases} \operatorname{card}(E) & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure, since it is clearly not locally finite.

$^\sharp$Named after Paul Adrien Maurice Dirac (1902-1984), English theoretical physicist who shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory". He actually introduced the so-called *Dirac delta function* as a "convenient notation" in his influential 1930 book *The Principles of Quantum Mechanics.* The name "delta function" was chosen since it works like a continuous analogue of the discrete Kronecker delta

$$\delta_{i\,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Indeed, for any sequence $\{a_j\}_{j \in \mathbb{Z}}$, we have

$$\sum_{j=-\infty}^{\infty} a_j \delta_{i\,j} = a_i,$$

and, analogously, for any $x \in \mathbb{R}^n$ and any continuous function $f : \mathbb{R} \to \mathbb{R}$, the Dirac delta satisfies the property

$$\int_{-\infty}^{+\infty} f(y)\delta(x - y)\, dy = \int_{-\infty}^{\infty} f(y)\, d\delta_x(y) = f(x).$$

Example 1.1.15 (The Lebesgue measure)**.** The well-known *Lebesgue measure* is defined by

$$\mathscr{L}^n(A) := \inf\left\{\sum_{i=1}^{\infty}\mathscr{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i,\, Q_i \text{ cubes}\right\},$$

where $\mathscr{L}^n(Q_i) = (l(Q_i)^n$ and $l(Q_i)$ is the side length of the cube Q_i. It is actually possible to show that in one dimension we have

$$\mathscr{L}^1(A) = \inf\left\{\sum_{ij=1}^{\infty}\operatorname{diam} C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j,\, C_j \subset \mathbb{R}\right\}$$

and that we can characterize $\mathscr{L}^n$ in an alternative way as

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}} = \mathscr{L}^{n-1} \times \mathscr{L}^1.$$

1.2 The Hausdorff measure

Definition 1.2.1 (Hausdorff content)**.** Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *α-dimensional Hausdorff content* of A as

$$\mathscr{H}^\alpha_\delta(A) := \inf\left\{\sum_{j\in I}\omega_\alpha\left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha \mid A \subset \bigcup_{j\in I\subset\mathbb{N}} C_j,\, \operatorname{diam} C_j \leq \delta, C_j \subseteq \mathbb{R}^n\right\},$$

where the infimum is taken over all the (at most countable) *δ-coverings* $\{C_j\}_{j\in I}$ of A, and we set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

We notice that $\mathscr{H}^\alpha_\delta(A)$ is a non-decreasing function in δ, so that we can take the limit as $\delta \searrow 0$ and it always exists in the extended real numbers. This justifies the following definition.

Definition 1.2.2 (Hausdorff measure)**.** For any $A \subset \mathbb{R}^n$ and $\alpha \geq 0$, we define the *α-dimensional Hausdorff measure* of A as

$$\mathscr{H}^\alpha(A) := \lim_{\delta\searrow 0}\mathscr{H}^\alpha_\delta(A) = \sup_{\delta>0}\mathscr{H}^\alpha_\delta(A).$$

Roughly speaking, we take the limit as $\delta \searrow 0$ since it forces the coverings to follow the local geometry of the set A. Indeed, the key idea behind the definition of the Hausdorff measure is that it should be able to capture the properties of thin sets in $\mathbb{R}^n$ (in particular, Lebesgue negligible sets). As we shall see in the following, if $\alpha = k \in \{1, \ldots, n-1\}$, then $\mathscr{H}^k$ agrees with the k-dimensional surface area on sufficiently regular sets, as for instance k-dimensional planes.

It is not too difficult to prove that, as a consequence of Carathéodory's criterion, Theorem 1.1.11, any Borel set is $\mathscr{H}^\alpha$-measurable, for any $\alpha \geq 0$.

Theorem 1.2.3 (Hausdorff measures are Borel regular)**.** *$\mathscr{H}^\alpha$ is a Borel regular measure on $\mathbb{R}^n$ for all $\alpha \geq 0$.*

Theorem 1.2.4 (Basic properties of the Hausdorff measure)**.** *The following statements hold true:*

(1) $\mathscr{H}^0 = \#$;

(2) $\mathscr{H}^1 = \mathscr{H}^1_\delta = \mathscr{L}^1$ on $\mathbb{R}$, for any $\delta > 0$;

(3) $\mathscr{H}^\alpha \equiv 0$ for all $\alpha > n$ in $\mathbb{R}^n$;

(4) $\mathscr{H}^\alpha(\lambda A) = \lambda^\alpha\mathscr{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$;

(5) $\mathscr{H}^\alpha(L(A)) = \mathscr{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$.

Proof.

(1) Since $\omega_0 = 1$, we have $\mathscr{H}^0_\delta(\{x\}) = 1$ for every $x \in \mathbb{R}^n$ and $\delta > 0$. Indeed,

$$\omega_0 \left(\frac{\operatorname{diam}(C_j)}{2} \right)^0 = 1,$$

which implies $\mathscr{H}^0_\delta(\{x\}) \geq 1$, and, on the other hand, we can clearly cover the singleton only with itself. Hence, $\mathscr{H}^0(\{x\}) = 1$ for every $x \in \mathbb{R}^n$. Since $\mathscr{H}^0$ is a Borel measure, it is σ-additive on Borel sets, so that

$$\mathscr{H}^0(A) = \sum_{x \in A} \mathscr{H}^0(\{x\}) = \#A,$$

for any finite or countable set A. Finally, if A is incountable, then A contains a countable set B, and so $\mathscr{H}^0(A) \geq \mathscr{H}^0(B) = +\infty$.

(2) We estimate the Lebesgue measure $\mathscr{L}^1$ from both sides by the Hausdorff measure. Since $\omega_1 = 2 = |(-1,1)|$, for any $\delta > 0$ we first get

$$\begin{aligned}
\mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\
&\leq \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j,\ \operatorname{diam} C_j \leq \delta \right\} = \mathscr{H}^1_\delta(A),
\end{aligned}$$

Now, we define a partition of $\mathbb{R}$ by setting $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$. These are intervals of diameter δ, so that, for every $j \in I$, we get

$$\operatorname{diam}(C_j \cap J_{k,\delta}) \leq \delta. \tag{1.2.1}$$

In addition, we have

$$\sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \leq \operatorname{diam} C_j, \tag{1.2.2}$$

since $\{J_{k,\delta}\}_{k \in \mathbb{Z}}$ is a partition $\mathbb{R}$ of essentially disjoint intervals, because $\#(J_{k,\delta} \cap J_{m,\delta}) \leq 1$ for any $k \neq m$. Therefore, by (1.2.2) we get

$$\begin{aligned}
\mathscr{L}^1(A) &= \inf \left\{ \sum_{j \in I} \operatorname{diam} C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\
&\geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \operatorname{diam}(C_j \cap J_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap J_{k,\delta} \right\}.
\end{aligned}$$

We set now $C_j \cap J_{k,\delta} =: \widetilde{C}_{i_{j,k}}$, by relabeling the indexes sets I and $\mathbb{Z}$ to an index set $\widetilde{I}$. Thanks to (1.2.1), we have $\operatorname{diam}(\widetilde{C}_i) \leq \delta$ and so we get

$$\mathscr{L}^1(A) \geq \inf \left\{ \sum_{j \in \widetilde{I}} \widetilde{C}_i \mid A \subset \bigcup_{i \in \widetilde{I}} \widetilde{C}_i,\ \operatorname{diam} \widetilde{C}_i \leq \delta \right\} \geq \mathscr{H}^1_\delta(A).$$

All in all, we get $\mathscr{L}^1 = \mathscr{H}^1_\delta$ for any $\delta > 0$, from which it easily follows $\mathscr{L}^1 = \mathscr{H}^1$ on $\mathbb{R}$.

(3) Let $\alpha > n$ and Q be a unit cube in $\mathbb{R}^n$. It is easy to see that, for any fixed $m \in \mathbb{N}$, Q can be covered by m^n smaller cubes Q_i with side length $\frac{1}{m}$. Clearly, we have $\operatorname{diam} Q_i = \frac{\sqrt{n}}{m}$. Therefore, we obtain

$$\mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\operatorname{diam} Q_i}{2} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha},$$

from which we deduce that, since $n < \alpha$,

$$\mathscr{H}^\alpha(Q) = \lim_{m\to\infty} \mathscr{H}^\alpha_{\frac{\sqrt{n}}{m}}(Q) \le \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} \lim_{m\to\infty} m^{n-\alpha} = 0.$$

Thus, the claim easily follows, since $\mathbb{R}^n$ can be covered by a countable collection of unit cubes and $\mathscr{H}^n$ is σ-subadditive.

The proofs of (4) and (5) are left as an exercise. □

Lemma 1.2.5. *Let $A \subset \mathbb{R}^n$ and $\delta_0 > 0$ such that $\mathscr{H}^\alpha_{\delta_0}(A) = 0$, then we have $\mathscr{H}^\alpha(A) = 0$.*

Proof. Since the Hausdorff content is non-increasing in δ, we have $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_\delta(A)$ for any $\delta > 0$. In particular, this means that $\mathscr{H}^\alpha_\infty(A) \le \mathscr{H}^\alpha_{\delta_0}(A) = 0$, so that, for every $\varepsilon > 0$, there exists a countable family of sets $\{C_j\}_{j\in I}$ such that

$$A \subseteq \bigcup_{j\in I} C_j \quad \text{and} \quad \sum_{j\in I} \omega_\alpha \left(\frac{\operatorname{diam} C_j}{2}\right)^\alpha < \varepsilon.$$

In particular, the second condition immediately implies

$$\operatorname{diam} C_j \le 2\left(\frac{\varepsilon}{\omega_\alpha}\right)^{\frac{1}{\alpha}} =: \delta_\varepsilon.$$

Hence, we have $\mathscr{H}^\alpha_{\delta_\varepsilon} \le \varepsilon$, and $\delta_\varepsilon \searrow 0$ if and only if $\varepsilon \searrow 0$. This implies the claim $\mathscr{H}^\alpha(A) = 0$. □

Proposition 1.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \le s < t < \infty$.*

(1) If $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$.

(2) If $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a countable family of subsets $\{C_j\}_{j\in I}$ such that

$$\operatorname{diam} C_j \le \delta \quad \text{and} \quad \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le \mathscr{H}^s_\delta(A) + 1 \le \mathscr{H}^s(A) + 1.$$

From this, it follows that

$$\begin{aligned}\mathscr{H}^t_\delta(A) \le \sum_{j\in I} \omega_t \left(\frac{\operatorname{diam} C_j}{2}\right)^t &= \frac{\omega_t}{\omega_s} 2^{s-t} \sum_{j\in I} \omega_s \left(\frac{\operatorname{diam} C_j}{2}\right)^s (\operatorname{diam} C_j)^{t-s} \\ &\le C_{s,t}\delta^{t-s}\left(\mathscr{H}^s(A) + 1\right) \longrightarrow 0 \quad \text{as} \quad \delta \to 0,\end{aligned}$$

which implies the claim $\mathscr{H}^t(A) = 0$.

(2) If by contradiction $\mathscr{H}^s(A) < \infty$, then by (1) if follows that $\mathscr{H}^r(A) = 0$ for all $r > s$ and in particular for $r = t$, which is clearly absurd.

□

Definition 1.2.7. We call the *Hausdorff dimension*[♯] of a set $A \subset \mathbb{R}^n$ the number

$$\dim_{\mathscr{H}}(A) := \inf\{\alpha \ge 0 : \mathscr{H}^\alpha(A) = 0\}.$$

Remark 1.2.8. Let $\alpha = \dim_{\mathscr{H}}(A)$. Then one has

$$\mathscr{H}^s(A) = 0 \quad \text{for all} \quad s > \alpha \qquad \text{and} \qquad \mathscr{H}^t(A) = +\infty \quad \text{for all} \quad t < \alpha. \tag{1.2.3}$$

The first part of (1.2.3) follows clearly from the definition of the Hausdorff dimension. The second, instead, can be proved by contradiction. Suppose by contradiction that $\mathscr{H}^t(A) < \infty$ for some $t < \alpha$, then, by the Proposition 1.2.6, we have $\mathscr{H}^r(A) = 0$ for all $r > t$. This implies

$$\alpha = \inf\left\{\beta \ge 0 : \mathscr{H}^\beta(A) = 0\right\} \le t < \alpha,$$

which is clearly absurd.

It should be noticed that, in general, $\mathscr{H}^\alpha(A)$ can be any number in $[0, +\infty]$.

[♯]The interested reader may find a detailed exposition on Hausdorff's and other related concepts of dimension in the monograph [?].

We state now an important result on the equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure, whose proof we postpone to the end of the section.

Theorem 1.2.9. *$\mathscr{H}^n_\delta = \mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$, for any $\delta > 0$.*

Remark 1.2.10. As a consequence of Theorem 1.2.9, we see that $\mathscr{H}^\alpha$ is *not* a Radon measure for all $\alpha \in [0, n)$. Indeed, it is not bounded on some compact sets. Take for example the closed unit ball $\overline{B(0,1)}$ in $\mathbb{R}^n$. We know that

$$\mathscr{H}^n(\overline{B(0,1)}) = \mathscr{L}^n(\overline{B(0,1)}) = \omega_n \in (0, \infty)$$

and so, by Proposition 1.2.6, $\mathscr{H}^\alpha(\overline{B(0,1)}) = +\infty$ for all $\alpha < n$.

Even though $\mathscr{H}^\alpha$ is not a Radon measure for $\alpha \in [0, n)$, it is possible to show that its restriction to some suitable Borel set is indeed a Radon measure.

Proposition 1.2.11. *If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathscr{H}^\alpha(E) \in (0, \infty)$, then $\mathscr{H}^\alpha \llcorner E$ is a Radon measure.*

Proof. It is a simple consequence of Theorem 1.1.12. $\square$

We investigate now the behaviour of the Hausdorff measure under the action of Lipschitz and Hölder functions. We recall first the definition of such family of functions.

Definition 1.2.12 (Lipschitz and Hölder functions)**.** Let $E \subset \mathbb{R}^n$.

(1) We say that $f : E \to \mathbb{R}^m$ is *Lipschitz continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y| \quad \text{for any } x, y \in E. \tag{1.2.4}$$
The smallest constant for which (1.2.4) holds is called the *Lipschitz constant* of f on E and it is denoted by $\mathrm{Lip}(f, E)$.

(2) We say that $f : E \to \mathbb{R}^m$ is *locally Lipschitz continuous* on E if, for all compact sets $K \subset E$, f is Lipschitz continuous on K.

(3) Let $\gamma \in (0, 1)$. We say that $f : E \to \mathbb{R}^m$ is *γ-Hölder continuous* on E if there exists a constant $C > 0$ such that
$$|f(x) - f(y)| \le C|x - y|^\gamma \quad \text{for any } x, y \in E. \tag{1.2.5}$$

(4) We say that $f : E \to \mathbb{R}^m$ is *locally γ-Hölder continuous* on E if, for all compact sets $K \subset E$, f is γ-Hölder continuous on K.

From this point on, we shall refer to Lipschitz continuous and Hölder continuous functions simply as Lipschitz and Hölder functions.

Exercise 1.2.13. Show that any Lipschitz or γ-Hölder function (for some $\gamma \in (0, 1)$) is indeed continuous.

Exercise 1.2.14. Show that the Lipschitz constant of f on E satisfies

$$\mathrm{Lip}(f, E) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in E, x \ne y \right\}. \tag{1.2.6}$$

This equality can indeed be used as an alternative definition.

Remark 1.2.15. It is easy to notice that Lipschitz functions can be seen as 1-Hölder functions. In addition, for any open set $\Omega \subset \mathbb{R}^n$ and any $\gamma \in [0, 1]$, we can define the space $C^{0,\gamma}(\overline{\Omega}; \mathbb{R}^m)$ of bounded γ-Hölder functions as the set of continuous bounded functions $f : \overline{\Omega} \to \mathbb{R}^m$ for which there exists a constant $C > 0$ such that (1.2.5) holds. If $\gamma = 0$, we have $C^{0,0}(\overline{\Omega}; \mathbb{R}^m) = C^0(\overline{\Omega}; \mathbb{R}^m)$. Such spaces may be equipped with the following norms:

$$\|f\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} + [f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)},$$

where

$$\|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)} := \sup_{x\in\overline{\Omega}} |f(x)|,$$

$$[f]_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)} := \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|^\gamma} : x,y\in\overline{\Omega}, x\neq y \right\},$$

for $\gamma \in (0,1]$, while we set $\|f\|_{C^{0,0}(\overline{\Omega};\mathbb{R}^m)} := \|f\|_{C^0(\overline{\Omega};\mathbb{R}^m)}$. It is not difficult to see that, for all $\gamma\in[0,1]$, $C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)$ equipped with the norm $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega};\mathbb{R}^m)}$ is a Banach space.

Exercise 1.2.16. Let $\gamma > 1$ and $f:\Omega\to\mathbb{R}^m$ be such that there exists a constant $C>0$ such that (1.2.5) holds. Show that f is constant.

Proposition 1.2.17. *Let $\alpha\geq 0$, $A\subset\mathbb{R}^n$.*

(1) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is Lipschitz, then $\mathscr{H}^\alpha(f(A)) \leq (\mathrm{Lip}(f))^\alpha \mathscr{H}^\alpha(A)$.

(2) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is γ-Hölder, for some $\gamma\in(0,1)$, then $\mathscr{H}^\alpha(f(A)) \leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}(A)$.

Proof. Thanks to Remark 1.2.15, it is enough to prove (2) for any $\gamma\in(0,1]$. Fix $\delta>0$, and take a countable family of sets $\{C_j\}_{j\in I}$ such that $A\subset\bigcup_{j\in I} C_j$ and $\operatorname{diam} C_j\leq\delta$. It is clear that

$$f(A)\subseteq\bigcup_{j\in I} f(C_j).$$

Thanks to (1.2.5), we see that $f(C_j)$ satisfies

$$\operatorname{diam} f(C_j) \leq C\,(\operatorname{diam} C_j)^\gamma \leq C\delta^\gamma,$$

where $C = \mathrm{Lip}(f)$ is $\gamma = 1$. Hence, we obtain

$$\mathscr{H}^\alpha_{C\delta^\gamma}(f(A)) \leq \sum_{j\in I}\omega_\alpha\left(\frac{\operatorname{diam} f(C_j)}{2}\right)^\alpha \leq \underbrace{\frac{\omega_\alpha}{2^\alpha}\frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=:C_{\alpha,\gamma}}\sum_{j\in I}\omega_{\alpha\gamma}\left(\frac{\operatorname{diam} C_j}{2}\right)^{\alpha\gamma}$$

and by taking the infimum over all δ-coverings $\{C_j\}_{j\in I}$ we get

$$\mathscr{H}^\alpha_{C\delta^\gamma}(g(A)) \leq C_{\alpha,\gamma}\mathscr{H}^{\alpha\gamma}_\delta(A),$$

where $C_{\alpha,\gamma} = \mathrm{Lip}(f)^\alpha$, if $\gamma = 1$. By sending $\delta\searrow 0$ we conclude the proof. □

Example 1.2.18 (Sierpinski triangle[♭]). We provide an example on the estimation of the Hausdorff measure for a set with non integer Hausdorff dimension. Let us construct a self-similar fractal in $\mathbb{R}^2$ in the following way:

1. Take S_0 to be an equilateral triangle with side length 1.
2. Divide each side in half, then connect the three middle points, so that S_0 becomes the union of four congruent equilateral triangles. Then, remove the open triangle in the center and denote by S_1 the union of the three remaining closed triangles with side length 1/2.
3. Now repeat the step in 2. in each one of the three equilateral triangles in S_1 in order to generate nine triangles of side length 1/4 which form S_2.

By iterating this procedure k times, we construct the set S_k as the union of 3^k equilateral triangles with side length 2^{-k}. Notice that $S_{k+1}\subset S_k$ and each one of the S_k's is compact and nonempty. Hence, we define the *Sierpinski triangle* as the set

$$S := \bigcup_{k=0}^{\infty} S_k,$$

[♭]Fractal described by Waclaw Sierpinski in 1915, [?], and appearing in Italian art from the 11th century [?].

which is therefore compact and nonempty. Since the area of an equilateral triangle of side length l is $\frac{\sqrt{3}}{4}l^2$, so that we have

$$\mathscr{L}^2(S) \leq \mathscr{L}^2(S_k) = 3^k \frac{\sqrt{3}}{4} 4^{-k}$$

for any $k \geq 0$, so that, by taking the limit as $k \to +\infty$, we conclude that $\mathscr{L}^2(S) = 0$. We proceed now to estimate the Hausdorff measure of S. We notice that

$$S_k = \bigcup_{j=1}^{3^k} S_{k,j},$$

if we denote by $S_{k,j}$ the j-th equilateral triangle of the k-th iteration step. It is not difficult to see that $\operatorname{diam}(S_{k,j}) = 2^{-k}$. Therefore, since clearly $S \subset S_k$, for any $k \geq 0$, we see that, by choosing $\delta = 2^{-k}$, we obtain the following estimate

$$\mathscr{H}^{\alpha}_{\frac{1}{2^k}}(S) \leq \sum_{j=1}^{3^k} \frac{\omega_\alpha}{2^\alpha} (\operatorname{diam} S_{k,j})^\alpha = \frac{\omega_\alpha}{2^\alpha} 3^k 2^{-k\alpha},$$

which goes to zero for $k \to \infty$ if $\alpha > \frac{\log^3}{\log^2}$. Thus, we can conclude that, for all $\alpha > \frac{\log 3}{\log 2}$, we have $\mathscr{H}^\alpha(S) = 0$, and this yields an upper bound on the Hausdorff dimension of S:

$$\dim_{\mathcal{H}}(S) \leq \frac{\log 3}{\log 2}.$$

We come now to the proof of Theorem 1.2.9, which is crucially based on the two following statements.

Lemma 1.2.19 (Vitali covering property for $\mathscr{L}^n$)**.** *For all open U and for all $\delta > 0$ there exists a family of disjoint closed balls $\{\overline{B_k}\}_{k=1}^\infty$ such that* $\operatorname{diam} B_k < \delta$ *and* $\mathscr{L}^n(U \setminus \bigcup_{k=1}^\infty \overline{B_k}) = 0$.

Theorem 1.2.20 (Isodiametric inequality)**.** *For all $\mathscr{L}^n$-measurable sets $E \subset \mathbb{R}^n$ we have*

$$|E| \leq \omega_n \left(\frac{\operatorname{diam} E}{2} \right)^n.$$

Proof of theorem 1.2.9. The proof consists of three steps.

(Step 1) Claim: $\mathscr{L}^n(A) \leq \mathscr{H}^n_\delta(A)$ for all $A \subset \mathbb{R}^n$ and for all $\delta > 0$.
Fix $\delta > 0$. Let $\{C_j\}_{j \in I}$: $A \subset \bigcup_{j \in I} C_j$, $\operatorname{diam} C_j \leq \delta$. By the σ-subadditivity of the Lebesgue measure, we have

$$\mathscr{L}^n(A) \leq \sum_{j=1}^\infty \mathscr{L}^n(C_j) \leq \sum_{j=1}^\infty \omega_n \left(\frac{\operatorname{diam} C_j}{2} \right)^n,$$

where in the last inequality we used the *isometric inequality*, Theorem 1.2.20. Taking the infimum over all such δ-coverings $\{C_j\}_{j \in J}$, we obtain the claim

$$\mathscr{L}^n(A) \leq \mathscr{H}^n_\delta(A) \qquad \text{for all} \quad \delta > 0.$$

(Step 2) Claim: for all $\delta > 0$, there exists $C_n \geq 1$ such that $\mathscr{H}^n_\delta \leq C_n \mathscr{L}^n$.
Notice that for any cube Q we have

$$\mathscr{L}^n(Q) = \left(\frac{\operatorname{diam} Q}{\sqrt{n}} \right)^n.$$

By the definition of the Lebesgue measure we get

$$\begin{aligned}
\mathscr{L}^n(A) &= \inf\left\{\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j\right\}\\
&= \inf\left\{\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j) \mid A \subset \bigcup Q_j,\ \operatorname{diam} Q_j \leq \delta\right\}\\
&= \frac{2^n}{\left(\sqrt{n}\right)^n\omega_n}\inf\left\{\sum_{j=1}^{\infty}\omega_n\left(\frac{\operatorname{diam} Q_j}{2}\right)^n \mid A \subset \bigcup Q_j,\ \operatorname{diam} Q_j < \delta\right\}\\
&\geq \frac{1}{C_n}\mathscr{H}^n_\delta(A),
\end{aligned}$$

where in the second equality we used the fact that

$$\mathscr{L}^n = \underbrace{\mathscr{L}^1 \times \cdots \times \mathscr{L}^1}_{n-\text{times}}, \quad \text{and} \quad \mathscr{L}^1 = \mathscr{H}^1_\delta \quad \text{in } \mathbb{R} \quad \text{for all } \delta > 0.$$

(Step 3) Claim: $\mathscr{H}^n_\delta(A) \leq \mathscr{L}^n(A) + \varepsilon$ for any $\varepsilon > 0$.
By the definition of $\mathscr{L}^n$ we see that, for all fixed $\delta, \varepsilon > 0$, there exists a family $\{Q_j\}_{j=1}^{\infty}$ such that $A \subset \bigcup_{j=1}^{\infty} Q_j$, $\operatorname{diam} Q_j \leq \delta$ and $\sum_{j=1}^{\infty}\mathscr{L}^n(Q_j) \leq \mathscr{L}^n(A) + \varepsilon$. Now, by Lemma 1.2.19, there exists a family $(\overline{B^i_j})_{i=1}^{\infty}$ of disjoint closed balls such that $\overline{B^i_j} \subset Q_j$ for all ($\operatorname{diam} B^i_j \leq \delta$) and

$$\mathscr{L}^n\left(Q_j \setminus \bigcup_{i=1}^{\infty}\overline{B^i_j}\right) = \mathscr{L}^n\left(\mathring{Q}_j \setminus \bigcup_{i=1}^{\infty}\overline{B^i_j}\right) = 0.$$

Therefore, by Step 2 we also have

$$\mathscr{H}^n_\delta\left(Q_j \setminus \bigcup_{i=1}^{\infty}\overline{B^i_j}\right) = 0,$$

from which we deduce that

$$\begin{aligned}
\mathscr{H}^n_\delta(A) &\leq \sum_{j=1}^{\infty}\mathscr{H}^n_\delta(Q_j) = \sum_{j=1}^{\infty}\mathscr{H}^n_\delta\left(\bigcup_{i=1}^{\infty}\overline{B^i_j}\right)\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\mathscr{H}^n_\delta\left(B^i_j\right) \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\underbrace{\omega_n\left(\frac{\operatorname{diam} B^i_j}{2}\right)^n}_{=\mathscr{L}^n\left(\overline{B^i_j}\right)}\\
&= \sum_{j=1}^{\infty}\mathscr{L}^n\left(\bigcup_{i=1}^{\infty}\overline{B^i_j}\right) = \sum_{j=1}^{\infty}\mathscr{L}^n(Q_j)\\
&\leq \mathscr{L}^n(A) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this inequality ends the proof. □

Proof of the isodiametric inequality (Theorem 1.2.20). Without loss of generality, we may assume E to be compact. Indeed, notice that $\operatorname{diam} A = \operatorname{diam} \overline{A}$, and, if $\operatorname{diam} E = +\infty$, the inequality is trivially true.

Next, observe that, if $E \subset B(x, \frac{\operatorname{diam} E}{2})$ for some $x \in \mathbb{R}^n$, then there is nothing to prove. We employ Steiner symmetrization[♭] in order to reduce ourselves to such a case.

Decompose $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ and let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q : \mathbb{R}^n \to \mathbb{R}$ be the orthogonal projections,

$$p(x) = (x_1, \ldots, x_{n-1}), \quad q(x) = x_n,$$

[♭] Introduced in 1838 by Jakob Steiner [?].

so that

$$x = (p(x), q(x)) \quad \text{and} \quad |x|^2 = |p(x)|^2 + |q(x)|^2.$$

T hen, for any $z \in \mathbb{R}^{n-1}$ we define the *verital section*

$$E_z := \{t \in \mathbb{R} : (z,t) \in E\},$$

and, as a consequence, we introduce the *symmetrization* of E with respect n-th coordinate axis:

$$E^s := \left\{ x \in \mathbb{R}^n : |q(x)| \leq \frac{\mathscr{L}^1(E_{p(x)})}{2} \right\}.$$

By Fubini's theorem, E_z is $\mathscr{L}^1$-measurable for $\mathscr{L}^{n-1}$-a.e. z, $z \mapsto \mathscr{L}^1(E_z)$ is Lebesgue measurable and so we get

$$|E| = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \mathscr{L}^1(E_z^s)dz = |E^s|, \tag{1.2.7}$$

where the first equality follows by Fubini's theorem, and the second one is a consequence of the fact that

$$(E^s)_z = \{t \in \mathbb{R} : (z,t) \in E^s\} = \left\{ t \in \mathbb{R} : |t| \leq \frac{\mathscr{L}^1(E_z)}{2} \right\} = \left[-\frac{\mathscr{L}^1(E_z)}{2}, \frac{\mathscr{L}^1(E_z)}{2} \right].$$

Now we claim that

$$\operatorname{diam} E^s \leq \operatorname{diam} E. \tag{1.2.8}$$

In order to prove this, let $x \in E^s$ and define $M(x), m(x) \in E$ to be the points for which

$$\begin{aligned} &p(m(x)) = p(M(x)) = p(x) \\ &q(m(x)) \leq q(z) \leq q(M(x)) \qquad \text{for all} \quad z \in E \quad \text{with} \quad p(z) = p(x). \end{aligned}$$

Hence, for all $x, y \in E^s$, we have

$$|q(x) - q(y)| \leq \max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\},$$

in particular, without loss of generality, we can assume that

$$\max\{|q(M(x)) - q(m(y))|, |q(M(y)) - q(m(x))|\} = |q(M(x)) - q(m(y))|,$$

so that

$$|q(x) - q(y)| \leq |q(M(x)) - q(m(y))|.$$

As a consequence, we see that

$$\begin{aligned} |x-y|^2 &= |p(x-y)|^2 + |q(x-y)|^2 \leq |p(M(x)) - p(m(y))|^2 + |q(M(x)) - q(m(y))|^2 \\ &= |M(x) - m(y)|^2 = \max\{|M(x) - m(y)|, |M(y) - m(x)|\}^2 \leq (\operatorname{diam} E)^2. \end{aligned}$$

This means that $|x - y| \leq \operatorname{diam} E$ for all $x, y \in E^s$, which immediately implies (1.2.8).

Given a $\mathscr{L}^n$ measurable set F, we define F^i to be the Steiner symmetrization with respect to the i-th coordinate axis. Hence, if we set $E_0 := E$, $E_i := (E_{i=1})^i$ with $i \in \{1, 2, \dots, n\}$, then, by (1.2.7) we have $|E_n| = |E|$ and $\operatorname{diam} E_n \leq \operatorname{diam} E$ by (1.2.8). In addition, we notice that, if $x \in E_n$, then $-x \in E_n$, which implies $E_n \subset B\left(0, \frac{\operatorname{diam} E_n}{2}\right)$. Thus, we conclude that

$$|E| = |E_n| \leq \omega_n \left(\frac{\operatorname{diam} E_n}{2} \right)^n \leq \omega_n \left(\frac{\operatorname{diam} E}{2} \right)^n.$$

And so we are done! $\square$

1.3 Integration and Radon measures

In this section, let $X \neq \emptyset$, and μ be a measure on X. Recall the definition of the *extended real line*

$$\overline{\mathbb{R}} := [-\infty, \infty].$$

Definition 1.3.1.

(1) A function $u : X \to \overline{\mathbb{R}}$ is *μ-measurable* if the *superlevel set*

$$\{u > t\} := \{x \in X : u(x) > t\}$$

is μ-measurable for all $t \in \overline{\mathbb{R}}$.

(2) u is a *μ-simple function* if it is μ-measurable and $u(X)$ is countable; that is,

$$u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x),$$

for some sequences of real numbers $\{u_k\}$ and of μ-measurable disjoint sets $\{E_k\}$.

(3) If u is a non-negative μ-simple function, we define

$$\int_X u\, d\mu := \sum_{t \in u(X)} t\mu(\{u = t\}) = \sum_{k=1}^{\infty} u_k \mu(E_k) \in [0, \infty]$$

with the convention that $0 \cdot \infty = 0$.

(4) We set $u^{\pm} := \max\{\pm u, 0\}$, so that $u = u^+ - u^-$ and $|u| = u^+ + u^-$. If u is μ-simple and $\int_X u^+ d\mu$ or $\int_X u^- d\mu < \infty$, then we define

$$\int_X u\, d\mu := \int_X u^+ d\mu - \int_X u^- d\mu \in [-\infty, \infty] \tag{1.3.1}$$

If u satisfies (1.3.1), then it is called *μ-integrable simple function.*

(5) If u is μ-measurable, we define the *upper and lower integrals* of u as

$$\int_X^* u\, d\mu := \inf\left\{\int_X v\, d\mu \mid v \geq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

or

$$\int_{*X} u\, d\mu := \sup\left\{\int_X v\, d\mu \mid v \leq u \ \mu\text{-a.e, } v \ \mu\text{-integrable simple function}\right\}$$

respectively. If

$$\int_{*X} u\, d\mu = \int_X^* u\, d\mu,$$

then u is *μ-integrable.*

(6) A measurable function u is *μ-summable* if $|u|$ is μ-integrable and

$$\int_X |u|\, d\mu < \infty.$$

Example 1.3.2 (Integral with respect to the Dirac measure)**.** Let $x_0 \in X$ and $\mu = \delta_{x_0}$. Notice that any subset in X is δ_{x_0}-measurable, so that any function $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-measurable. Then, any $u : X \to \overline{\mathbb{R}}$ simple function is μ-integrable. Indeed, assuming at first $u : X \to [0, \infty]$, for some sequence of nonnegative real numbers $\{u_k\}$ and a partition $\{E_k\}$ of X, we have

$$u(x) = \sum_{k=1}^{\infty} u_k \chi_{E_k}(x),$$

so that
$$\int_X u\,d\delta_{x_0} = \sum_{k=1}^{\infty} u_k \delta_{x_0}(E_k) = u_{k_0},$$
where k_0 satisfies $E_{k_0} \ni x_0$, which implies $u(x_0) = u_{k_0}$. Therefore, we can easily see that
$$\int_X u\,d\delta_{x_0} = u^+(x_0) - u^-(x_0) = u(x_0)$$
for any simple function u. As a consequence, any $u : X \to \overline{\mathbb{R}}$ is δ_{x_0}-integrable. Indeed, for any simple function $v \geq u$ and any simple function $w \leq u$, we have
$$w(x_0) \leq \quad {}_*\!\int_X u\,d\delta_{x_0} \leq \int_X^* u\,d\delta_{x_0} \leq v(x_0),$$
so that we get
$${}_*\!\int_X u\,d\delta_{x_0} = \int_X^* u\,d\delta_{x_0} = u(x_0),$$
by choosing
$$v(x) = \begin{cases} u(x_0) & x = x_0, \\ +\infty & x \neq x_0, \end{cases}$$
and
$$w(x) = \begin{cases} u(x_0) & x = x_0, \\ -\infty & x \neq x_0. \end{cases}$$
Thus, we conclude that, for any $u : X \to \overline{\mathbb{R}}$, we have
$$\int_X u\,d\delta_{x_0} = u(x_0),$$
and that u is δ_{x_0}-summable if and only if $|u(x_0)| < \infty$.

We define now general versions of the familiar L^p-function spaces.

Definition 1.3.3. Let $p \in (1, \infty)$.

$L^1(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mu\text{-summable} \right\}$ and we set
$$\|u\|_{L^1(X,\mu)} := \int_X |u|\,d\mu.$$

$L^1_{\mathrm{loc}}(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^1(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact} \right\}.$

$L^p(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} \mid |u|^p \text{ is } \mu\text{-summable} \right\}$ and we set
$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u|^p\,d\mu \right)^{\frac{1}{p}}.$$

$L^p_{\mathrm{loc}}(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^p(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact} \right\}.$

Let $u : X \to \overline{R}$ be μ-measurable. We set
$$\|u\|_{L^\infty(X,\mu)} := \inf\{\lambda > 0 : \mu(\{|u| > \lambda\}) = 0\}.$$

As a consequence, we define $L^\infty(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} : \|u\|_{L^\infty(X,\mu)} < \infty \right\}$.

$L^\infty_{\mathrm{loc}}(X, \mu) := \left\{ u : X \to \overline{\mathbb{R}} \mid \|u\chi_K\|_{L^\infty(X,\mu)} < \infty \text{ for all } K \subset X \text{ compact} \right\}.$

In the case $X = \mathbb{R}^n$ and $\mu = \mathscr{L}^n$, we shall set for brevity $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \mathscr{L}^n)$, for any $p \in [1, \infty]$.

Definition 1.3.4 (Integral measures)**.** If $u : X \to [0, \infty]$ μ-measurable, then we define the *integral measure* $\nu = u\mu$ (or $\mu \llcorner u$) as
$$\nu(A) = \int_A u\,d\mu = \int_X u\chi_A\,d\mu \qquad \text{for all } \mu\text{-measurable } A.$$

1.4 Real and vector valued Radon measures

Through this section, let $\Omega \subset \mathbb{R}^n$ be an open set. We exploit now the concept of integral measure introduced in Definition 1.3.4 to define signed and vector valued Radon measures. In order to avoid ambiguity, from this point on we shall refer to the Radon measure introduced in Definition 1.1.9 as nonnegative Radon measures.

Definition 1.4.1 (Signed Radon measures)**.** Given a non-negative Radon measure μ on Ω and $f : \Omega \to [-\infty, \infty]$ locally μ-summable. Then we set $\nu := f\mu$ to be the integral measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact.}$$

ν is said to be a *signed Radon measure* on Ω.

Definition 1.4.2 (Vector valued Radon measures)**.** Given a non-negative Radon measure μ on Ω and $f : \Omega \to \mathbb{R}^m$ is locally μ-summable. Then we set $\nu := f\mu$ to be the vector valued Radon measure satisfying

$$\nu(K) = \int_K f \, d\mu \qquad \text{for all } K \text{ compact.}$$

ν is said to be a *vector valued Radon measure* on Ω.

Definition 1.4.3 (Alternative approach)**.**

- A non-negative Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ which is σ-additive and finite on compact sets. We denote the space of such measures by $\mathcal{M}^+_{\text{loc}}(\Omega)$.
- A vector valued (real or signed if $m = 1$) Radon measure is a mapping $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ which is σ-additive and its total variation $|\mu|$ is finite on compact sets; that is

$$|\mu|(K) := \sup \left\{ \sum_{j=1}^{\infty} |\mu(B_j)| \mid K = \bigcup_j B_j, B_i \cap B_i = \emptyset \text{ if } i \neq j \right\} < \infty \text{ for all } K \text{ compact in } \Omega.$$

  The space of such measures is denote by $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$; and by $\mathcal{M}_{\text{loc}}(\Omega)$ if $m = 1$.
- We say that a non-negative Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty)$ is finite if $\mu(\Omega) < \infty$; and we denote by $\mathcal{M}^+(\Omega)$ the space of such measures.
- We say that a non-negative vector-valued Radon measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is finite if $|\mu|(\Omega) < \infty$; and we denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$, and $\mathcal{M}(\Omega)$ if $m = 1$, the space of such measures.

Remarks (Basic facts)**.**

- If $\mu \in \mathcal{M}_{loc}(\Omega, \mathbb{R}^m)$, then $|\mu| \in \mathcal{M}^+_{loc}(\Omega)$, where

$$|\mu|(B) := \sup \left\{ \sum_{j=1}^{\infty} |\mu(B_j)| \mid B = \bigcup B_j, B_j \cap B_i = \emptyset \; i \neq j, B_j \in \mathcal{B}(\Omega) \right\}$$

  for any $B \Subset \Omega$. In particular, $\sum \mu(B_j)$ is absolutely convergent for all $\{B_j\}$ partition of a some set $B \Subset \Omega$.
- The total variation is the smallest non-negative Radon measure ν such that $\nu(B) \geq |\mu(B)|$ for all $B \in \mathcal{B}(\Omega)$.
- If $\mu \in \mathcal{M}(\Omega)$, we define the *positive and negative parts* of μ

$$\mu^+ := \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu^- := \frac{|\mu| - \mu}{2}.$$

  It is easy to notice that $\mu^\pm \geq 0$ and that $\mu = \mu^+ - \mu^-$, which is the *Jordan decomposition*, and it is unique. In addition, $|\mu| = \mu^+ + \mu^-$.

Lemma 1.4.4. *If $\mu \in \mathcal{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu; \mathbb{R}^m)$, then $f\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $|f\mu| = |f|\mu$.*

Proof. Let $B \in \mathcal{B}(\Omega)$.

- It is easy to notice that
$$|(f\mu)(B)| := |\int_B f d\mu| \le \int_B |f| d\mu.$$
From this it follows immediately that $|f\mu| \le |f|\mu$, which implies $f\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$.

- Let $\varepsilon > 0$ and $D = \{z_h\}_{h\in\mathbb{N}}$ countable dense set in $\mathbb{S}^{m-1}$, let $B \in \mathcal{B}(\Omega)$. We define
$$\sigma(x) := \min\{h \in \mathbb{N} : f(x)z_h \ge (1-\varepsilon)|f(x)|\}$$
it is Borel measurable. Then, we set
$$B_h := \sigma^{-1}(\{h\}) \cap B,$$
and we notice that
$$B_h \in \mathcal{B}(\Omega),\ B = \bigcup_{h\in\mathbb{N}} B_h \text{ and } B_h \cap B_k = \text{ if } h \ne k.$$
This implies that
$$\int_B |f| d\mu = \sum_{k\in\mathbb{N}} \int_{B_h} |f| d\mu \le \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} \int_{B_h} f z_h d\mu \le \frac{1}{1-\varepsilon} \sum_{h\in\mathbb{N}} |(f\mu)(B_h)| \le \frac{1}{1-\varepsilon} |f\mu|(B),$$
since
$$\int_{B_h} f z_h d\mu = z_h \int_{B_h} f d\mu \le \left| \int_{B_h} f d\mu \right| = |(f\mu)(B_h)|$$

□

Definition 1.4.5. Let μ be a non-negative measure on Ω.

- We say that μ is concentrated on a set $E \subset \Omega$ if
$$\mu(\Omega \setminus E) = 0.$$

- We call the support of μ, $\operatorname{supp}\mu$, the smallest closed set on which μ is concentrated :
$$\operatorname{supp}(\mu) := \bigcap_{C \text{ closed}, \mu(\Omega\setminus C)=0} C.$$

Exercise 1.4.6. Equivalently, we may characterize the support of a non-negative Radon measure μ in terms of its behaviour on balls:
$$\operatorname{supp}(\mu) = \{x \in \Omega \mid \mu(B(x,r)) > 0, \forall r > 0 \text{ such that } B(x,r) \subset \Omega\}$$

Remark 1.4.7. Notice that a non-negative Radon measure may be concentrated on a set strictly smaller than its support. Indeed, let $\Omega = \mathbb{R}$ and
$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{\frac{1}{k}}.$$
It is clear that μ is concentrated on the set $E = \left\{\frac{1}{k}\right\}_{k\ge1}$, but $\mu((-r,r)) > 0$ for any $r > 0$, so that $0 \in \operatorname{supp}(\mu)$. In fact, it is not difficult to check that $\operatorname{supp}(\mu) = \{0\} \cup \left\{\frac{1}{k}\right\}_{k\ge1} = \overline{E}$.

Definition 1.4.8. 1. Let $\mu \in \mathcal{M}^+(\Omega)$, $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$. We say that μ is absolutely continuous with respect to μ, and we write $\nu \ll \mu$, if for all $B \in \mathcal{B}(\Omega)$ such that $\mu(B) = 0$, then $|\nu|(B) = 0$.

2. If $\mu, \nu \in \mathcal{M}^+(\Omega)$, we say that they are mutually singular if there exists $E, F \in \mathcal{B}(\Omega)$ such that $\mu(F) = 0$, $\mu(E) = 0$ and
$$\mu(B) = \mu(B \cap E) \qquad \text{and} \qquad \nu(B) = \nu(B \cap F)$$
for all $B \in \mathcal{B}(\Omega)$ and we write $\mu \perp \nu$. If $\mu, \nu \in \mathcal{M}(\Omega, \mathbb{R}^m)$,
$$\mu \perp \nu \iff |\mu| \perp |\mu|$$

Theorem 1.4.9 (Radon-Nikodym)**.** *Let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$. Then there exist unique measures $\nu^{ac}, \nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m)$ such that $\nu^{ac} \ll \mu$, $\nu^s \perp \mu$ and*

$$\nu = \nu^{ac} + \nu^s. \tag{1.4.1}$$

In addition, there exists a unique measure $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that $\nu^{ac} = f\mu$.
In particular, if $\mu = \mathscr{L}^n$, every $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ can be uniquely decomposed in

$$\mu = f\mathscr{L}^n + \nu^s,$$

for some $f \in L^1(\Omega; \mathbb{R}^n)$ and $\nu^s \in \mathcal{M}(\Omega; \mathbb{R}^m), \nu^s \perp \mathscr{L}^n$.

The decomposition in (1.4.1) is called *Lebesgue decomposition* of the measure ν with respect to μ.

Definition 1.4.10. We say that a property holds $|\mu|$-almost everywhere or for $|\mu|$-almost every x if the set where the property does not hold is $|\mu|$-negligible; that is, it has zero $|\mu|$-measure.

Corollary 1.4.11 (Polar decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Then there exists a unique $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $|f(x)| = 1$ $|\mu|$-a.e and $\mu = f|\mu|$.*

Proof of corollary. Apply Radon-Nikodym theorem (Theorem 1.4.9) to μ and $|\mu|$. We know that $|\mu(B)| \leq |\mu|(B)$ for all $B \in \mathcal{B}(\Omega)$. From this follows $\mu \ll |\mu|$, and so there exists $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ such that $\mu = f|\mu|$.
We proved that $|f|\mu|| = |f||\mu|$, hence we obtain

$$|\mu| = |f|\mu|| = |f||\mu| \quad \text{and so} \quad (|f| - 1)|\mu| = 0.$$

This means that we have

$$\int_\Omega (|f| - 1) d|\mu| = 0,$$

which yields $|f(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$. □

Corollary 1.4.12 (Hahn decomposition)**.** *Let $\mu \in \mathcal{M}(\Omega)$, there exists a unique $A \in \mathcal{B}(\Omega)$ (up to $|\mu|$-negligible sets) such that*

$$\mu^+ = \mu \llcorner A \qquad \mu^- = -\mu \llcorner (\Omega \setminus A).$$

Proof. By the polar decomposition, there exists a unique $f \in L^1(\Omega, |\mu|)$ such that $\mu = f|\mu|$ and $f(x) \in \{\pm 1\}$ for $|\mu|$-a.e. $x \in \Omega$. This means that, if we set

$$A := \{f = 1\},$$

we have

$$f(x) = \chi_A - \chi_{\Omega \setminus A}.$$

Thus, we obtain

$$\mu^+ := \frac{|\mu| + \mu}{2} = \frac{1 + \chi_A - \chi_{\Omega\setminus A}}{2}|\mu| = \chi_A |\mu|,$$
$$\mu^- := \frac{|\mu| - \mu}{2} = \frac{1 - \chi_A + \chi_{\Omega\setminus A}}{2}|\mu| = \chi_{\Omega\setminus A}|\mu|.$$

□

1.5 Duality for Radon measures

Another characterization is given via the duality with continuous functions.

Definition 1.5.1. We say that $B \Subset \Omega$ if $\overline{B} \subset \Omega$ and it is compact in Ω.

$$C^0_C(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \operatorname{supp} u \Subset \Omega\}$$

$$C^0_0(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \forall \varepsilon > 0 \ \exists K \subset \Omega : |u(x)| < \varepsilon \quad \forall x \notin K\}$$

$$\|u\|_\infty := \sup_{x \in \Omega} |u(x)|$$

Remark 1.5.2. $C_0^0(\Omega;\mathbb{R}^m) = \overline{C_c^0(\Omega;\mathbb{R}^n)}^{\|\cdot\|_\infty}$, $(C_0^0(\Omega;\mathbb{R}^n), \|\cdot\|_\infty)$ is Banach. C_c^0 is separable, locally convex, topological vector space with the following topology:

$$\varphi_k \longrightarrow \varphi \quad \text{in} \quad C_c^0 \qquad \Longleftrightarrow \qquad \|\varphi_k - \varphi\|_\infty \to 0 \quad \text{and there exists } K \subset \Omega : \operatorname{supp}\varphi \cup \bigcup_{k\in\mathbb{N}} \operatorname{supp}\varphi_k \subset K$$

Theorem 1.5.3 (Lusin). *Let μ Borel on Ω and $u : \Omega \to \mathbb{R}$ is μ-measurable, $u \equiv 0$ in $\Omega \setminus E$ with $\mu(E) < \infty$. Then for all $\varepsilon > 0$ there exists $v \in C^0(\Omega)$ such that $\|v\|_\infty \le \|u\|_\infty$*

$$\mu(\{x \in \Omega : v(x) \neq u(x)\}) < \varepsilon.$$

Remark 1.5.4. An equivalent formulation states that, under the additional assumption $\mu(\Omega) < \infty$, then there exists a sequence of compact sets $\{K_h\}$ such that

$$\mu\left(\Omega \setminus \bigcup_{h=1}^\infty K_h\right) = 0 \qquad \text{and} \qquad u\big|_{K_h} \text{ is continuous.}$$

In other terms, this means that there exists a sequence of functions $\{u_h\} \in C^0(\Omega)$ such that $u = u_h$ on K_h and $\|u_h\|_\infty \le \|u\|_\infty$.

Proposition 1.5.5. *Let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$. Then for all $A \subset \Omega$ open we have*

$$|\mu|(A) = \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c^0(A;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\}, \tag{1.5.1}$$

with the convention that

$$\int_\Omega \varphi \cdot d\mu := \sum_{j=1}^m \int_\Omega \varphi_j d\mu_j.$$

Proof. Polar decomposition implies that $\mu = f|\mu|$, $|f| = 1$ μ-a.e. So we get

$$\int_\Omega \varphi \cdot d\mu = \int_A \varphi \cdot f d|\mu| \le |\mu|(A).$$

By Lusin theorem, for all $\varepsilon > 0$ there exists $\varphi \in C^0(A;\mathbb{R}^m)$ such that $\|\varphi\|_\infty \le 1$ and

$$|\mu|\left(\{x \in A : \varphi(x) \neq f(x)\}\right) < \varepsilon.$$

Take $K \subset A$ compact such that $|\mu|(A \setminus K) < \varepsilon$. Construct $\eta \in C_c^\infty(A)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on K, $\tilde{\varphi} = \varphi\eta \in C_c^0(A;\mathbb{R}^m)$ and

$$|\mu|\left(\{x : \tilde{\varphi}(x) \neq f(x)\}\right) \le |\mu|(A \setminus K) + |\mu|\left(\{x : \varphi(x) \neq f(x)\}\right) \le 2$$

to get

$$\int_A \tilde{\varphi} \cdot d\mu \ge |\mu|(K) - 2\varepsilon$$

and by sending K to A and $\varepsilon \searrow 0$ we arrive at the claim. $\square$

Proposition 1.5.5 shows that, given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$, we can define a linear continuous functional $L_\mu : C_0^0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ as

$$L_\mu(\varphi) := \int_\Omega \varphi \cdot d\mu,$$

for any $\varphi \in C_0^0(\Omega;\mathbb{R}^m)$. In addition, the operatorial norm of L_μ is equal to $|\mu|(\Omega)$, since, by the density of C_c^0 in C_0^0 with respect to the supremum norm and by (1.5.1), we have

$$\begin{aligned}\|L_\mu\| &:= \sup\{L_\mu(\varphi) : \varphi \in C_0^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\} \\ &= \sup\left\{\int_\Omega \varphi \cdot d\mu \mid \varphi \in C_c^0(\Omega;\mathbb{R}^m), \|\varphi\|_\infty \le 1\right\} = |\mu|(\Omega).\end{aligned}$$

This suggests that it is possible to characterize $\mathcal{M}(\Omega;\mathbb{R}^m)$ as a dual space. In such a way, we gain yields a weaker topology on the space of vector valued Radon measure, and therefore weak* compactness of bounded sequences.

Theorem 1.5.6. (Riesz Representation Theorem) *Let $L : C_0(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a continuous linear functional; that is, L is linear and satisfies*

$$\sup\{L(\phi) : \phi \in C_0(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

Moreover,

$$|\mu|(\Omega) = \sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1\} = \|L\|.$$

For the proof we refer to [**?**, Theorem 1.54].

The following corollary is a direct consequence of the global version of the Riesz Representation Theorem.

Corollary 1.5.7. *Let $L : C_c(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), \|\phi\|_\infty \leq 1, \mathrm{supp}(\phi) \subset K\} < \infty,$$

for any compact set $K \subset \Omega$. Then there exists a unique $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Thus we can identify any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with a continuous linear functional on $C_0(\Omega;\mathbb{R}^m)$, written as

$$L_\mu(\phi) := \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m),$$

and analogously $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega;\mathbb{R}^m)$. These facts lead us to a notion of weak* convergence for Radon measure.

1.6 Weak* convergence for Radon measures

Definition 1.6.1. Given a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega)$, we say that μ_k *weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

If $\{\mu_k\}$ and μ are in $\mathcal{M}_{\mathrm{loc}}(\Omega)$, we say that μ_k *locally weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Lemma 1.6.2. *Let $\{\mu_k\} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ be a weak-star convergent sequence, and let μ be its limit. Then we have*

$$\limsup_{k\to+\infty} |\mu_k|(\Omega) < \infty$$

and

$$|\mu|(\Omega) \leq \liminf_{k\to+\infty} |\mu_k|(\Omega).$$

Proof. The first assertion follows from Uniform Boundedness Principle (Banach-Steinhaus Theorem), since $L_{\mu_k}(\phi) \to L_\mu(\phi)$ for each $\phi \in C_0(\Omega;\mathbb{R}^m)$ and therefore $\{L_{\mu_k}(\phi)\}$ is a bounded sequence in $\mathbb{R}$.

The second inequality comes from:

$$|L_{\mu_k}(\phi)| \leq \|\phi\|_\infty |\mu_k|(\Omega)$$

then, passing to the limit we have $|L_\mu(\phi)| \leq \liminf_{k\to+\infty} \|\phi\|_\infty |\mu_k|(\Omega)$ and taking supremum in ϕ yields the result. $\square$

Remark 1.6.3. Weak-star convergence of finite Radon measures is equivalent to local weak-star convergence with the condition that $\sup |\mu_k|(\Omega) = C < \infty$. We observe that, by Lemma 1.6.2, this condition implies $|\mu|(\Omega) \leq C$.
Clearly weak-star convergence always implies local weak-star convergence.
On the other hand, if we suppose that μ_k locally weak-star converges to μ, then, given $\psi \in C_0(\Omega; \mathbb{R}^m)$, for any $\epsilon > 0$ there exists $\phi \in C_c(\Omega; \mathbb{R}^m)$ such that $\|\psi - \phi\|_\infty < \epsilon$ and so

$$\begin{aligned}\left|\int_\Omega \psi \cdot d\mu_k - \int_\Omega \psi \cdot d\mu\right| &\leq \left|\int_\Omega (\psi - \phi) \cdot d\mu_k\right| + \left|\int_\Omega (\psi - \phi) \cdot d\mu\right| \\ &\quad + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right| \\ &\leq 2C\epsilon + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right|.\end{aligned}$$

Now, $\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu$ and so, since ϵ is arbitrary, we obtain weak-star convergence.
Therefore, in what follows, we will always write $\mu_k \overset{*}{\rightharpoonup} \mu$ to denote local weak-star convergence, and, in the case of finite Radon measures, we will also check the condition $\sup |\mu_k|(\Omega) < \infty$.

Remark 1.6.4. Let μ be a positive Radon measure. If $\{A_t\}_{t \in \mathcal{I}}$, where $\mathcal{I}$ is uncountable, is a family of μ-measurable sets in Ω such that their boundaries are disjoint, $\bigcup_{t \in \mathcal{I}} \partial A_t = \Omega$ and for every compact K there exists an uncountable set of indices $\mathcal{J} \subset \mathcal{I}$ such that $K \cap \partial A_t \neq \emptyset$, $\forall t \in \mathcal{J}$, then there exists a countable set $\mathcal{N}$ such that

$$\mu(K \cap \partial A_t) = 0 \quad \forall t \notin \mathcal{N}.$$

We claim that, if such a set $\mathcal{N}$ did not exist, then there would be an uncountable set $\mathcal{Y}$ such that $\mu(K \cap \partial A_t) > \epsilon > 0$, $\forall t \in \mathcal{Y}$. Suppose to the contrary that for each $\epsilon > 0$ the set of t's which satisfy $\mu(K \cap \partial A_t) > \epsilon$ is countable.
We set $\epsilon_j = \frac{1}{j}$ and we have

$$\{t \in \mathcal{I} : \mu(K \cap \partial A_t) \neq 0\} = \bigcup_{j=1}^{+\infty} \left\{t \in \mathcal{I} : \mu(K \cap \partial A_t) > \frac{1}{j}\right\},$$

so this set, being countable union of countable sets, is itself countable, contradicting our assumption. We extract now from $\mathcal{Y}$ a sequence $\{t_j\}$.
By the monotonicity and the σ-additivity, we have

$$\mu(K) \geq \sum_{j=1}^{+\infty} \mu(K \cap \partial A_{t_j}) = +\infty,$$

which is absurd, since μ is a Radon measure. Therefore, such a $\mathcal{Y}$ cannot exist and so $\mathcal{N}$ exists.
In the applications, the sets $\{A_t\}$ will usually be balls $B(x, r)$.

Finally, we state a characterization of nonnegative linear functionals on $C_c^\infty(\Omega)$.

Lemma 1.6.5. *Let $L : C_c^\infty(\Omega) \to \mathbb{R}$ be linear and nonnegative; that is,*

$$L(\phi) \geq 0, \quad \forall \phi \in C_c^\infty(\Omega) \text{ with } \phi \geq 0.$$

Then there exists a positive Radon measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c^\infty(\Omega).$$

Proof. We choose a compact set $K \subset \Omega$ and we select a smooth function $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on K and $0 \leq \zeta \leq 1$. Then, for any $\phi \in C_c^\infty(\Omega)$ with $\mathrm{supp}(\phi) \subset K$, we set $\psi = \|\phi\|_\infty \zeta - \phi \geq 0$. Therefore, since L is nonnegative, we have $0 \leq L(\psi) = \|\phi\|_\infty L(\zeta) - L(\phi)$ and so $L(\phi) \leq C\|\phi\|_\infty$, with $C := L(\zeta)$.
L thus may be extended to a linear mapping $\hat{L} : C_c(\Omega) \to \mathbb{R}$ such that, for any compact $K \subset \Omega$,

$$\sup\{L(\phi) : \phi \in C_c(\Omega; \mathbb{R}^m), \|\phi\|_\infty \leq 1, \mathrm{supp}(\phi) \subset K\} < \infty.$$

Hence, Corollary 1.5.7 yields the existence of a real Radon measure μ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c(\Omega).$$

By the polar decomposition of measures, $\mu = h|\mu|$, where $|h| = 1$ $|\mu|$-a.e. The fact that L is nonnegative implies that $h = 1$ $|\mu|$-a.e.; that is, μ is a positive Radon measure. □

Theorem 1.6.6 (Criterions for weak$*$ convergence)**.** *Let $\mu_h, \mu \in \mathcal{M}^+_{\mathrm{loc}}(\Omega)$. The following are equivalent*

(1) $\mu_h \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

(2) For all $U \subset \Omega$ open and for all $K \subset \Omega$ compact we have

$$\liminf_{h\to\infty} \mu_h(U) \geq \mu(U), \tag{1.6.1}$$

$$\limsup_{h\to\infty} \mu_h(K) \leq \mu(K). \tag{1.6.2}$$

(3) For all Borel sets $B \Subset \Omega$ such that $\mu(\partial B) = 0$, we have

$$\lim_{h\to\infty} \mu_h(B) = \mu(B). \tag{1.6.3}$$

Proof. (1) $\Rightarrow$ (2) Let $K \subset U \subset \Omega$ where K is compact and U is open, and choose $\varphi \in C_0(\Omega)$, $\operatorname{supp}\varphi \subset U$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K. By our assumption, we have

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu.$$

Hence, we get

$$\mu(K) \leq \int_\Omega \varphi \, d\mu = \lim_{h\to\infty} \int_\Omega \varphi \, d\mu_h \leq \liminf_{h\to\infty} \mu_h(U),$$

and we deduce (1.6.1) by taking the supremum in $K \subset U$ and using the innner regularity of μ. On the other hand, it also clear that we have

$$\mu(U) \geq \int_\Omega \varphi \, d\mu = \lim_{h\to\infty} \int_\Omega \varphi \, d\mu_h \geq \limsup_{h\to\infty} \mu_h(K),$$

from which we deduce (1.6.2) by taking the infimum in $U \supset K$ and using the outer regularity.

(2) $\Rightarrow$ (3) Notice that $B = \mathring{B} \cup (\partial B \cap B)$. Therefore, using $\mu(\partial B) = 0$, we have

$$\begin{aligned} \mu(B) = \mu(\mathring{B}) + \mu(\partial B \cap B) = \mu(\mathring{B}) &\leq \liminf_{h\to\infty} \mu_h(\mathring{B}) \\ &\leq \limsup_{h\to\infty} \mu_h(\mathring{B}) \leq \limsup_{h\to\infty} \mu_h(\overline{B}) \leq \mu(\overline{B}) = \mu(B). \end{aligned}$$

(3) $\Rightarrow$ (1) Let $\varepsilon > 0$ and $\varphi \in C^0_c(\Omega)$. We need to prove that

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu.$$

Let us at first assume $\varphi \geq 0$. Choose $0 = t_0 < t_1 < \cdots < t_N := 2\|\varphi\|_\infty$, such that $0 < t_i - t_{i-1} < \varepsilon$ and $\mu(\varphi^{-1}\{t_i\}) = 0$. By Remark 1.6.4, it is always possible to choose such good t_i's. Let $B_i = \varphi^{-1}((t_{i-1}, t_i))$, then $\mu(\partial B_i) = 0$. Hence, by (1.6.3) we have

$$\mu_h(B_i) \to \mu(B_i).$$

In addition, it is easy to notice that

$$\sum_{i=2}^N t_{i-1}\mu_h(B_i) \leq \int_\Omega \varphi \, d\mu_h \leq \sum_{i=2}^N t_i \mu_h(B_i) + t_1 \mu_h(B_0),$$

$$\sum_{i=2}^N t_{i-1}\mu(B_i) \leq \int_\Omega \varphi \, d\mu \leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B_0).$$

Therefore, by the triangle inequality and the subadditivity of the limsup, we have

$$\begin{aligned}
\limsup_{h\to+\infty}\left|\int_\Omega \varphi\,d\mu_h - \int_\Omega \varphi\,d\mu\right| &\le \limsup_{h\to+\infty}\left|\int_\Omega \varphi\,d\mu_h - \sum_{i=2}^N t_{i-1}\mu_h(B_i)\right| + \\
&+\left|\sum_{i=2}^N t_{i-1}\mu_h(B_i) - \sum_{i=2}^N t_{i-1}\mu(B_i)\right| + \left|\int_\Omega \varphi\,d\mu - \sum_{i=2}^N t_{i-1}\mu(B_i)\right| \\
&\le \limsup_{h\to+\infty} t_1\mu_h(B_0) + t_1\mu(B_0) = 2t_1\mu(B_0) \\
&< \varepsilon\mu(\operatorname{supp}\varphi),
\end{aligned}$$

from which we conclude, since ε is arbitrary. Let us now consider the general case of $\varphi : \Omega \to \mathbb{R}$, and consider

$$\psi := \varphi + \|\varphi\|_\infty \eta,$$

for some $\eta \in C_c(\Omega)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\operatorname{supp}(\varphi)$. It is plain to see that $\psi \ge 0$ and $\psi \in C_c(\Omega)$, so that we have

$$\begin{aligned}
\int_\Omega \varphi\,d\mu_h &= \int_\Omega \psi\,d\mu_h - \int_\Omega \|\varphi\|_\infty \eta\,d\mu_h \\
&\to \int_\Omega \psi\,d\mu - \int_\Omega \|\varphi\|_\infty \eta\,d\mu = \int_\Omega \varphi\,d\mu
\end{aligned}$$

and this ends the proof.

$\square$

We quote now a useful result about weak* convergence of vector valued Radon measures.

Lemma 1.6.7. *If μ_k and μ are $\mathbb{R}^m$-vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then $|\mu| \le \nu$. Moreover, if a μ-measurable set $E \subset\subset \Omega$ satisfies $\nu(\partial E) = 0$, then*

$$\mu(E) = \lim_{k\to+\infty} \mu_k(E).$$

More generally, if $f : \Omega \to \mathbb{R}^m$ is a bounded Borel function with compact support such that the set of its discontinuity points is ν-neglegible, then

$$\lim_{k\to+\infty} \int_\Omega f \cdot d\mu_k = \int_\Omega f \cdot d\mu.$$

Remark 1.6.8. By Remark 1.6.4 and Lemma 1.6.7, we can assert that, if μ_k and μ are positive Radon measures in Ω, for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\mu(\partial B(x, r)) = 0$ and so, if $\mu_k \overset{*}{\rightharpoonup} \mu$, $\mu_k(B(x, r)) \to \mu(B(x, r))$.
Moreover, if μ_k and μ are vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$, then for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\nu(\partial B(x, r)) = 0$ and $\mu_k(B(x, r)) \to \mu(B(x, r))$.

Chapter 2

Basic results from Geometric Measure Theory

2.1 Covering theorems and differentiation of measures

2.1.1 Mollification of Radon measure

Let $\mu \in \mathcal{M}_{loc}(\Omega;\mathbb{R}^m)$ and $g \in C_c(\mathbb{R}^n)$. We denote by $(\mu * g)$ the convolution :

$$(\mu * g)(x) := \int_\Omega g(x-y)d\mu(y) \quad \forall x : \quad \text{the map} \quad y \mapsto g(x-y) \text{ is } C_c$$

In particular, let $\rho \in C_c^\infty(B(0,1))$, $\rho(-x) = \rho(x)$, $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho dx = 1$ (ρ is a mollifier), set $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, then

$$(\mu * \rho_\varepsilon)(x) = \int_\Omega \rho_\varepsilon(x-y)d\mu(y)$$

is well defined for all $x \in \Omega^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Theorem 2.1.1 (Properties of mollifications)**.** *Let $\mu \in \mathcal{M}_{loc}(\Omega;\mathbb{R}^m)$, ρ a mollifier. Then $\mu * \rho_\varepsilon \in C^\infty(\Omega^\varepsilon;\mathbb{R}^m)$, $D^\alpha(\mu * \rho_\varepsilon) = \mu * D^\alpha \rho_\varepsilon$ (for all $\alpha \in \mathbb{N}_0^n$). In addition*

$$\mu * \rho_\varepsilon \overset{*}{\rightharpoonup} \mu$$

for all E Lebesgue measure, $\int_E |\mu * \rho_\varepsilon| dx \leq |\mu|(E_\varepsilon)$ for all $\varepsilon > 0$.

Proof. Regularity and differentiability is an exercise. Let $\varphi \in C_c(\Omega;\mathbb{R}^m)$

$$\int_\Omega \varphi \cdot (\mu * \rho_\varepsilon)dx = \int_\Omega \int_\Omega \varphi(x)\rho_\varepsilon(x-y) \cdot d\mu(y)dx = \int_\Omega (\varphi * \rho_\varepsilon)(y) \cdot d\mu(y) \to \int \varphi \cdot d\mu(y),$$

where we used $\rho_\varepsilon(x-y) = \rho_\varepsilon(y-x)$ and $(\varphi * \rho_\varepsilon)(y) \to \varphi(y)$ uniformly. By Fubini's Theorem, TODO ☐

2.1.2 Differentiation of Radon and Hausdorff measures

We say that a family $\mathcal{F}$ is disjoint if $F \cap F' = \emptyset$ for all $F, F' \in \mathcal{F}$, $F \neq F'$.

Theorem 2.1.2 (Besicovitch covering theorem)**.** *There exists a $\xi_n \in \mathbb{N}$ such that for all families of closed balls $\mathcal{F}$ such that the set $A := \{x \in \mathbb{R}^n \mid \exists \varrho > 0 : \overline{B(x,\varrho)} \in \mathcal{F}\}$ is bounded, there exists at mos ξ_n disjoint subfamilies $\mathcal{F}_i \subset \mathcal{F}$ such that*

$$A \subset \bigcup_{i=1}^{\xi_n} \bigcup_{\overline{B} \in \mathcal{F}_i} \overline{B}.$$

Remark 2.1.3. It works also with open balls.

Theorem 2.1.4 (consequence of Besicovitch)**.** *Let A be a bounded set and $\varrho : A \to (0, \infty)$. Then there exists $S \subset A$ at most countable such that $A \subset \bigcup_{x \in S} B(x, \varrho(x))$ and for all $y \in \mathbb{R}^n$*

$$\sum_{x \in S} \chi_{B(x,\varrho(x))}(y) \leq \xi_n.$$

Proof. Let $\mathcal{F} := \{(B(x, \varrho(x)) \mid x \in A\}$ TODO □

Definition 2.1.5. Let $\mathcal{F}$ be a family of closed balls and $A \subset \mathbb{R}^n$. $\mathcal{F}$ is called a fine covering of A if

$$\inf \left\{ \varrho > 0 \mid \overline{B(x, \varrho)} \in \mathcal{F} \right\} = 0 \quad \text{for all } x \in A.$$

Theorem 2.1.6 (Vitali)**.** *Let A be a bounded Borel set and $\mathcal{F}$ be a fine covering of A. In any case $\mu \in \mathcal{M}^+_{loc}(\mathbb{R}^n)$ there exists $\mathcal{F}' \subset \mathcal{F}$ disjoint such that*

$$\mu \left(A \setminus \bigcup_{\overline{B} \in \mathcal{F}'} \overline{B} \right) = 0$$

Proof. TODO □

Theorem 2.1.7. *let $\mu \in \mathcal{M}^+_{loc}(\Omega)$, $\lambda \in \mathcal{M}_{loc}(\Omega; \mathbb{R}^m)$, $|\lambda| << \mu$. Then, for μ-a.e. $x \in \operatorname{supp} u$, the limit*

$$\lim_{\varrho \to 0} \frac{\lambda(B(x, \varrho))}{\mu(B(x, \varrho))}$$

exists in $\mathbb{R}^n$.

By Radon-Nikodyn, $\lambda = f\mu$ for some $f \in \mathscr{L}^1_{\text{loc}}(\Omega; \mu)$. Thus

$$f(x) = \lim_{\varrho \to 0} \frac{\lambda(B(x, \varrho))}{\mu(B(x, \varrho))}$$

for μ-a.e. $x \in \operatorname{supp} \mu$.

2.2 Fine properties of Lipschitz functions

We devote this section to the discussion of some properties of Lipschitz functions, which proved to be very useful in the framework of Geometric Measure Theory. The choice of working with Lipschitz functions is due to the fact that such functions have a less rigid structure than C^1-differentiable functions (for instance, extension theorems are much easier to prove, see McShane's lemma), while they enjoy differentiability properties almost everywhere (see Rademacher's theorem).

Lemma 2.2.1 (McShane's lemma)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be a Lipschitz function. Then the function $f^+ : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$f^+(x) := \inf\{f(y) + \mathrm{Lip}(f, E)|x - y| : y \in E\}$$

is Lipschitz and it satisfies $f^+(x) = f(x)$ for any $x \in E$ and $\mathrm{Lip}(f, E) = \mathrm{Lip}(f^+, \mathbb{R}^n)$.

Proof. For any $x, z \in \mathbb{R}^n$, by the triangle inequality, we have

$$f^+(x) \leq \inf\{f(y) + \mathrm{Lip}(f, E)(|x - z| + |z - y|) : y \in E\} = f^+(z) + \mathrm{Lip}(f, E)|x - z|.$$

Then, interchanging the role of x and z, we immediately get

$$|f^+(x) - f^+(z)| \leq \mathrm{Lip}(f, E)|x - z|.$$

Finally, let $x \in E$. It is easy to see that $f^+(x) \leq f(x)$. In order to obtain the reverse inequality, notice that

$$f(x) \leq f(y) + \mathrm{Lip}(f, E)|x - y|$$

for any $y \in E$, since f is Lipschitz on E. □

Remark 2.2.2. The extension given in McShane's lemma is the largest extension of f, while, arguing analogously, one can show that the smaller extension is given by

$$f^-(x) := \sup\{f(y) - \mathrm{Lip}(f, E)|x - y| : y \in E\}.$$

It is not difficult to see that McShane's lemma can be extended to vector valued Lipschitz functions by hands; however, in such a way we loose the equality between the Lipschitz constants.

Corollary 2.2.3. *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\widetilde{f} = f$ on E and $\mathrm{Lip}(\widetilde{f}, \mathbb{R}^n) \leq \sqrt{m}\,\mathrm{Lip}(f, E)$.*

Proof. Apply McShane's lemma (Lemma 2.2.1) to each component of f, thus defining

$$\widetilde{f} := (f_1^+, \dots, f_m^+).$$

Then it is easy to see that $\widetilde{f} = f$ on E. As for the Lipschitz constant, notice that

$$|\widetilde{f}(x) - \widetilde{f}(y)|^2 = \sum_{i=1}^{m} |f_i^+(x) - f_i^+(y)|^2 \leq m(\mathrm{Lip}(f, E))^2|x - y|^2.$$

This ends the proof. ☐

A more refined result was found by Kirszbraun.

Theorem 2.2.4 (Kirszbraun theorem)**.** *Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g = f$ on E and $\mathrm{Lip}(g, \mathbb{R}^n) = \mathrm{Lip}(f, E)$.*

A practical consequence of these extension results for Lipschitz functions is that we may always assume, without loss of generality, that our Lipschitz maps are defined on the whole space $\mathbb{R}^n$.

We shall now see that, quite surprisingly, the Lipschitz continuity property is enough to ensure differentiability outside of a Lebesgue negligible set. We start by recalling the notion of differentiability.

Definition 2.2.5. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $x \in \mathbb{R}^n$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0.$$

This linear mapping is denoted by $\nabla f(x)$ or $df(x)$.

Theorem 2.2.6 (Rademacher's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable at $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. In particular, $\nabla f(x)$ is well defined for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$ and belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m \times \mathbb{R}^n)$, with*

$$\|\nabla f\|_{L^\infty(K; \mathbb{R}^m \times \mathbb{R}^n)} \leq \mathrm{Lip}(f, K)$$

for any compact set K.

An interesting consequence of this result is that the differential of a Lipschitz function vanishes on the level sets of the function.

Theorem 2.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz and $t \in \mathbb{R}$. Then $\nabla f(x) = 0$ for $\mathscr{L}^n$-a.e. $x \in \{f = t\} := \{y \in \mathbb{R}^n : f(y) = t\}$.*

2.3 The area and coarea formulas

2.3.1 Linear maps and Jacobians

We recall here some standard definitions and facts from linear algebra.

Definition 2.3.1.

i) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if

$$(Ox) \cdot (Oy) = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

ii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^m$ is *symmetric* if

$$x \cdot (Sy) = (Sx) \cdot y$$

for all $x, y \in \mathbb{R}^n$.

iii) Let $A : \mathbb{R}^n \to \mathbb{R}^m$. The *adjoint* of A is the linear map $A^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$x \cdot (A^* y) = (Ax) \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Proposition 2.3.2.

i) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps. Then we have $A^{**} = A$ and $(A \circ B)^* = B^* \circ A^*$.

ii) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear map. Then $S^* = S$.

iii) If $O : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal linear map, then $n \leq m$ and

$$O^* \circ O = I \text{ on } \mathbb{R}^n,$$
$$O \circ O^* = I \text{ on } \mathbb{R}^m.$$

Theorem 2.3.3 (Polar decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.*

i) If $n \leq m$, there exists a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$L = O \circ S.$$

ii) If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$L = S \circ O^*.$$

Definition 2.3.4 (Jacobian)**.** Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

i) If $n \leq m$, we write $L = O \circ S$ as above, and we define the *Jacobian* of L as

$$\mathbf{J}L := |\det S|.$$

ii) If $n \leq m$, we write $L = S \circ O^*$ as above, and we define the *Jacobian* of L as

$$\mathbf{J}L := |\det S|.$$

In the literature, these two different definitions of Jacobian are also called *n-dimensional Jacobian* (or *area factor*), and m-dimensional *coarea factor*, respectively, and are denoted by $\mathbf{J}_n$ and $\mathbf{C}_m$.

Theorem 2.3.5 (Representation of Jacobian)**.**

i) If $n \leq m$,

$$\mathbf{J}L = \sqrt{\det (L^* \circ L)}.$$

ii) If $n \geq m$,

$$\mathbf{J}L = \sqrt{\det (L \circ L^*)}.$$

Proof. Let $n \leq m$ and $L = O \circ S$, by Theorem 2.3.3. Then we have $L^* = S \circ O^*$, so that

$$L^* \circ L = S \circ O^* \circ O \circ S = S^2,$$

since O is orthogonal and so $O^* \circ O$ is the identity mapping on $\mathbb{R}^n$ (by Proposition 2.3.2). Hence

$$\det(L^* \circ L) = \det S^2 = (\mathbf{J}L)^2.$$

The proof of (ii) is similar. $\square$

Remark 2.3.6. The definition of the Jacobian of L is independent of the choices of O and S, and we have $\mathbf{J}L = \mathbf{J}L^*$.

Proposition 2.3.7 (Cauchy-Binet formula)**.** *If $n \leq m$ and $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then*

$$\mathbf{J}L = \sqrt{\sum_B \overset{2}{\det}(B)}$$

where the sum is taken over all $n \times n$ minor of any matrix representation of L.

Let now $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, be a Lipschitz map. By Rademacher's theorem, f is differentiable $\mathscr{L}^n$-a.e. and therefore the gradient matrix

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}$$

is well defined and can be considered a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$.

Definition 2.3.8. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous and x is a differentiability point, we define the *Jacobian* of f as

$$\mathbf{J}f(x) := \mathbf{J}\nabla f(x).$$

Remark 2.3.9. Notice that $\mathbf{J}f \leq c_n \operatorname{Lip}(f)^n$.

2.3.2 The area formula

Through this subsection we assume $n \leq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ to be Lipschitz continuous.

Lemma 2.3.10. *Let $A \subset \mathbb{R}^n$ be $\mathscr{L}^n$-measurable. Then*

i) $f(A)$ is $\mathscr{H}^n$-measurable,

ii) the mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is $\mathscr{H}^n$-measurable on $\mathbb{R}^m$ and

$$\int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y) \leq (\operatorname{Lip}(f))^n \mathscr{L}^n(A)$$

Definition 2.3.11. The mapping $y \to \mathscr{H}^0(A \cap f^{-1}(y))$ is the *multiplicity function* of f in A.

Remark 2.3.12. It is easy to notice that $\mathscr{H}^0(A \cap f^{-1}(y))$ is equal to the cardinality of the set of

$$\{x \in A : f(x) = y\},$$

so that $f^{-1}(y)$ is finite for $\mathscr{H}^n$-a.e. $y \in \mathbb{R}^m$. In particular, if f is injective, then

$$\mathscr{H}^0(A \cap f^{-1}(y)) = \begin{cases} 1 & y \in f(A), \\ 0 & y \notin f(A). \end{cases}$$

Theorem 2.3.13 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \leq m$. Then, for all $\mathscr{L}^n$-measurable sets $A \subset \mathbb{R}^n$, we have*

$$\int_A \mathbf{J}f(x)\, dx = \int_{\mathbb{R}^m} \mathscr{H}^0(A \cap f^{-1}(y))\, d\mathscr{H}^n(y). \tag{2.3.1}$$

This means that the $\mathscr{H}^n$-measure of $f(A)$, counting multiplicity, is equal to the integral of the Jacobian of f over A. As an immediate consequence, we deduce a generalization of the classical change of variables formula.

Theorem 2.3.14 (General change of variables)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \le m$. Then, for all $\mathscr{L}^n$-summable functions $g : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int_{\mathbb{R}^n} g(x)\,\mathbf{J}f(x)\,dx = \int_{\mathbb{R}^m} \left(\sum_{x \in f^{-1}(y)} g(x) \right) d\mathscr{H}^n(y). \tag{2.3.2}$$

Corollary 2.3.15 (Injective maps)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous and $n \le m$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{L}^n$-summable function, and assume that f is injective on the support of g. Then, we have*

$$\int_{\mathbb{R}^n} g(x)\,\mathbf{J}f(x)\,dx = \int_{f(\mathbb{R}^n)} g(f^{-1}(y))\,d\mathscr{H}^n(y). \tag{2.3.3}$$

Equivalently, if $h : \mathbb{R}^m \to \mathbb{R}$ is such that $h \circ f$ is $\mathscr{L}^n$-summable and f is injective on the support of h, then we have

$$\int_{\mathbb{R}^n} h(f(x))\,\mathbf{J}f(x)\,dx = \int_{f(\mathbb{R}^n)} h(y)\,d\mathscr{H}^n(y). \tag{2.3.4}$$

If $g = \chi_A$ for some $\mathscr{L}^n$-measurable set A, then

$$\mathscr{H}^n(f(A)) = \int_A \mathbf{J}f(x)\,dx. \tag{2.3.5}$$

Remark 2.3.16. Theorem 2.3.14 and Corollary 2.3.15 hold also in the case $g : \mathbb{R}^n \to [0, +\infty]$ is $\mathscr{L}^n$-measurable; however, the left hand sides of (2.3.2) and (2.3.3) may be equal to $+\infty$. In addition, since any Borel function is Lebesgue measurable, Theorem 2.3.14 and Corollary 2.3.15 are valid for all Borel functions $g : \mathbb{R}^n \to \mathbb{R}$ either nonnegative or $\mathscr{L}^n$-summable.

We list here some remarkable applications of the area formula.

Example 2.3.17 (Length of a curve)**.** Let $n = 1, m \ge 1$. Assume $f : \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and injective. It is clear that, for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,

$$\mathbf{J}_1 f(t) = |\dot{f}(t)|.$$

Therefore, for any $a, b \in \mathbb{R}, a < b$, the length of a curve $C := f([a,b])$ is given by

$$\mathscr{H}^1(C) = \int_a^b |\dot{f}|\,dt,$$

thanks to (2.3.5).

Example 2.3.18 (Surface area of a graph)**.** Let $n \ge 1$ and $m = n+1$. Assume $g : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ as

$$f(x) := (x, g(x)).$$

Then

$$\nabla f(x) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix},$$

and so, by Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J}f = \sqrt{1 + |\nabla g|^2}.$$

For any open set $\Omega \subset \mathbb{R}^n$, we define the graph of g over Ω as

$$\Gamma(g, \Omega) := \{(x, g(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}.$$

Therefore, (2.3.5) yields

$$\mathscr{H}^n(\Gamma(g, \Omega)) = \int_\Omega \sqrt{1 + |\nabla g|^2}\,dx.$$

Example 2.3.19 (Surface area of a parametric hypersurface)**.** Let $n \geq 1$ and $m = n+1$. Assume $f : \mathbb{R}^n \to \mathbb{R}^{n+1}, f = (f^1, \dots, f^{n+1})$, is Lipschitz and injective. For any $k \in \{1, \dots, n+1\}$, we define

$$\hat{f}_k := (f^1, \dots, f^{k-1}, f^{k+1}, \dots, f^{n+1});$$

that is, the vector valued map f without its k-th component. Then, it is not difficult to see that

$$\mathbf{J}\hat{f}_k = |\det \nabla \hat{f}_k|,$$

and so, as a consequence of Cauchy-Binet formula (Proposition 2.3.7), we have

$$\mathbf{J}f = \sqrt{\sum_{k=1}^{n+1} (\mathbf{J}\hat{f}_k)^2}.$$

Thus, if we define $\Sigma(f, \Omega) := f(\Omega)$, for any open set $\Omega \subset \mathbb{R}^n$ to be a portiong of the parametric hypersurface, (2.3.5) yields

$$\mathscr{H}^n(\Sigma(f, \Omega)) = \int_\Omega \sqrt{\sum_{k=1}^{n+1} (\mathbf{J}\hat{f}_k)^2}\, dx.$$

2.3.3 The coarea formula

In this subsection we assume $n \geq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ to be Lipschitz.

Lemma 2.3.20. *Let A be $\mathscr{L}^n$-measurable. Then*

i) $A \cap f^{-1}(y)$ is $\mathscr{H}^{n-m}$ measurable for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$.

ii) the mapping $y \to \mathscr{H}^{n-m}(A \cap f^{-1}(y))$ is $\mathscr{L}^m$-measurable, and

$$\int_{\mathbb{R}^m} \mathscr{H}^{n-m}(A \cap f^{-1}(y))\, dy \leq c_{n,m} (\mathrm{Lip}(f))^m \mathscr{L}^n(A).$$

Theorem 2.3.21 (Coarea formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $n \geq m$. Then, for all $\mathscr{L}^n$-measurable sets $A \subset \mathbb{R}^n$, we have*

$$\int_A \mathbf{J}f\, dx = \int_{\mathbb{R}^m} \mathscr{H}^{n-m}(A \cap f^{-1}(y))\, dy. \tag{2.3.6}$$

Notice that the coarea formula can be seen as a generalized version of Fubini's theorem.

Remark 2.3.22 (Morse-Sard theorem)**.** If we apply the coarea formula to $A = \{\mathbf{J}f = 0\}$, it is immediate to see that

$$\mathscr{H}^{n-m}(\{\mathbf{J}f = 0\} \cap f^{-1}(y)) = 0$$

for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$. This is a weak variant of Morse-Sard theorem, which states that, if $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1 + n - m$, then

$$\{\mathbf{J}f = 0\} \cap f^{-1}(y) = \emptyset$$

for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$.

Theorem 2.3.23. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $n \geq m$. Then, for all $\mathscr{L}^n$-summable functions $g : \mathbb{R}^n \to \mathbb{R}$, we have $g|_{f^{-1}(y)}$ is $\mathscr{H}^{n-m}$-summable for $\mathscr{L}^m$-a.e. $y \in \mathbb{R}^m$, and*

$$\int_{\mathbb{R}^n} g\, \mathbf{J}f\, dx = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} g\, d\mathscr{H}^{n-m}\, dy. \tag{2.3.7}$$

Remark 2.3.24. Notice that $f^{-1}(y)$ is closed for all $y \in \mathbb{R}^m$, so that it is immediately $\mathscr{H}^{n-m}$-measurable.

We list now some relevant applications of the coarea formula.

Theorem 2.3.25 (Polar coordinates)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathscr{L}^n$-summable. Then*

$$\int_{\mathbb{R}^n} g\,dx = \int_0^{+\infty} \int_{\partial B(0,\rho)} g\,d\mathscr{H}^{n-1}\,d\rho$$

In particular, for any $r > 0$ and g such that $g\chi_{B(0,r)}$ is $\mathscr{L}^n$-summable, we have

$$\int_{B(0,r)} g\,dx = \int_0^r \int_{\partial B(0,\rho)} g\,d\mathscr{H}^{n-1}\,d\rho,$$

so that, for $\mathscr{L}^1$-a.e. $r > 0$,

$$\frac{d}{dr} \int_{B(0,r)} g\,dx = \int_{\partial B(0,r)} g\,d\mathscr{H}^{n-1}.$$

Proof. Apply Theorem 2.3.23 to $f(x) = |x|$, in the case $m = 1$. Then, for all $x \neq 0$, we have

$$\nabla f(x) = \frac{x}{|x|},\ \mathbf{J}f(x) = 1.$$

Finally, the second equality is a consequence of the first, as the third can be derived from the second. □

In general, in the case $m = 1$ it is easy to notice that $\mathbf{J}f = |\nabla f|$, so that we have the following result.

Theorem 2.3.26 (Integration over level sets)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then we have*

$$\int_{\mathbb{R}^n} |\nabla f|\,dx = \int_{-\infty}^{+\infty} \mathscr{H}^{n-1}(\{f = t\})\,dt. \tag{2.3.8}$$

If we assume also that $\operatorname{ess\,inf} |\nabla f| > 0$ *and we let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathscr{L}^n$-summable, then for all $t \in \mathbb{R}$ we obtain*

$$\int_{\{f>t\}} g\,dx = \int_t^{+\infty} \int_{\{f=s\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds. \tag{2.3.9}$$

In particular,

$$\frac{d}{dt} \int_{\{f>t\}} g\,dx = -\int_{\{f=t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$.

Proof. It is easy to see that (2.3.8) is a simple consequence of (2.3.6) when $m = 1$. Then, by Theorem 2.3.23, we have

$$\begin{aligned}
\int_{\{f>t\}} g\,dx &= \int_{\mathbb{R}^n} \chi_{\{f>t\}} \frac{g}{|\nabla f|} \mathbf{J}f\,dx \\
&= \int_{-\infty}^{+\infty} \int_{\{f=s\}} \chi_{\{f>t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds \\
&= \int_t^{+\infty} \int_{\{f=s\}} \chi_{\{f>t\}} \frac{g}{|\nabla f|}\,d\mathscr{H}^{n-1}\,ds.
\end{aligned}$$

Then, the final equality follow easily by (2.3.9). □

Finally, we conclude this section with the examination of the case in which f is the distance function from a compact set.

Theorem 2.3.27 (Integration over the level set of the distance function)**.** *Let $K \subset \mathbb{R}^n$ be a nonempty compact set and set*

$$d(x) := \operatorname{dist}(x, K).$$

Then, for all $0 < a < b$ and all $g : \mathbb{R}^n \to \mathbb{R}$ $\mathscr{L}^n$-summable we have

$$\int_a^b \int_{\{d=t\}} g\,d\mathscr{H}^{n-1}\,dt = \int_{\{a<d\leq b\}} g\,dx.$$

In particular,

$$\int_a^b \mathscr{H}^{n-1}(\{d = t\})\,dt = \mathscr{L}^n(\{a < d \leq b\}).$$

Proof. It is enough to prove that $|\nabla d(x)| = 1$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n \setminus K$, and then we need just to apply Theorem 2.3.26. We start by showing the d is Lipschitz. Let $x \in \mathbb{R}^n$: there exists a $c \in K$ such that $|x - c| = d(x)$. By the triangle inequality, we have

$$d(y) - d(x) \leq |y - c| - |x - c| \leq |x - y|.$$

If we interchange now the roles of x and y, we see that we get

$$|d(y) - d(x)| \leq |x - y|,$$

which shows that d is Lipschitz with $\mathrm{Lip}(d) \leq 1$. Hence, Rademacher's theorem implies that the function d is differentiable in $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. Let now $x \in \mathbb{R}^n \setminus K$ be such that $\nabla d(x)$ exists. Then we have $|\nabla d(x)| \leq 1$. In addition, if we select $c \in K$ as above, we also have

$$d(tx + (1 - t)c) = t|x - c|$$

for all $t \in [0, 1]$ (since the segment is the shortest path). Therefore, by taking a derivative in t, we get

$$|x - c| = \nabla d(x) \cdot (x - c) \leq |\nabla d(x)||x - c|,$$

which immediately implies $|\nabla d(x)| \geq 1$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n \setminus K$. Thus, the proof is completed. $\square$