

Lectures Notes

BV functions and sets of finite perimeter

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Lecture by Giovanni Comi

Notes taken by Dennis Schmeckpeper

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Chapter 1

Introduction?

1.0 Motivation

Let's consider

$$\inf \left\{ \int_{\Omega} |\nabla u| dx : u \in W^{1,1}(\Omega), \|u\|_{L^1} = K > 0 \right\} =: m_K,$$

where $W^{1,1}(\Omega) := \{u \in L^1(\Omega) : Du \in L^1(\Omega, \mathbb{R}^n)\}$ is the Sobolev space. Then there exists a sequence $(u_j \in W^{1,1}(\Omega))_{j \in \mathbb{N}}$ such that $\|u_j\|_{L^1(\Omega)} = K$ and $\|\nabla u_j\|_{L^1(\Omega)} \rightarrow m_K$ for $j \rightarrow \infty$. Now, in general this does *not* imply that there is a subsequence $(u_{j_k})_{k \in \mathbb{N}}$ which will converge even only weakly to an $v \in W^{1,1}(\Omega)$ with $\|v\|_{L^1} = K$ and $\|\nabla v\|_{L^1} = m_K$.

The reason for this is essentially because L^1 is not a (topological) dual of any space, though it is contained in one.

Another example is the *Isoperimetric problem*:

$$\min \{ \sigma_{n-1}(\partial F) : F \text{ with some regularity, } |F| = K > 0 \} =: \gamma_K,$$

where $|F| =: \mathcal{L}^n(F)$ denotes the n -dimensional Lebesgue measure.

Lacks details

1.1 Measures

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ (or 2^X) the *power set*, that is, the collection of all subsets of X .

Definition ((outer) measure). A mapping $\mu : \mathcal{P}(X) \rightarrow [0, +\infty]$ satisfying

$$(1) \quad \mu(\emptyset) = 0$$

$$(2) \quad \mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k) \text{ if } A \subset \bigcup_{k=1}^{\infty} A_k \quad (\sigma\text{-subadditivity})$$

is called an (outer) measure.

Remark. The (outer) measure is not decreasing, that is, for $A \subset B$, where $A, B \in \mathcal{P}(X)$, we have $\mu(A) \leq \mu(B)$.

Definition (Restriction of a measure). If $Y \subset X$, the *restriction of μ to Y* , denoted by $\mu \llcorner Y$, is defined as $(\mu \llcorner Y)(A) := \mu(Y \cap A)$.

Definition (μ -measurable). We call a subset $A \subset X$ μ -measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \quad \text{for all } B \subseteq X.$$

Remark. This definition is meaningful since *Vitali* found that there exists a set $E \subset \mathbb{R}$ which is *not* \mathcal{L}^1 -measurable.

Definition (σ -algebra). A subset $\mathfrak{F} \subset \mathcal{P}(X)$ is called a σ -algebra of sets if holds

$$(1) \quad \emptyset, X \in \mathfrak{F},$$

- (2) for $A \in \mathfrak{F}$ also $X \setminus A \in \mathfrak{F}$,
(3) for a family $(A_i \in \mathfrak{F})_{i \in I}$ we have $\bigcup_{i \in I} A_i \in \mathfrak{F}$.

Theorem. Let μ be a (outer) measure on X , then the restriction to the σ -algebra of μ -measurable sets is σ -additive, that is, if $(A_j)_{j \in I}$ is a (at most) countable disjoint μ -measurable family of subsets of X , then

$$\mu \left(\bigcup_{j \in I} A_j \right) = \sum_{j \in I} \mu(A_j).$$

Definition. Here we collect some important definitions

- (1) Let $\mathfrak{C} \subset \mathcal{P}(X)$, we call the smallest σ -algebra containing \mathfrak{C} , the σ -algebra generated by \mathfrak{C} .[#]
(2) The *Borel-algebra* on \mathbb{R}^n , denoted by $\mathcal{B}(\mathbb{R}^n)$, is the σ -algebra generated by the family of open sets in \mathbb{R}^n (in the standard topology). The elements of the Borel-algebra are called *Borel sets*.
(3) A (outer) measure μ in \mathbb{R}^n is called a *Borel measure* if each Borel sets is μ -measurable.
(4) A (outer) measure μ in \mathbb{R}^n is called *Borel regular* if for all subsets $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.
(5) A Borel regular measure μ which is locally finite (e.g. $\mu(K) < \infty$ for all compact subsets $K \subset \mathbb{R}^n$), is called a *Radon measure*.

Theorem. Let μ be a Radon measure on \mathbb{R}^n . We have

- (1) for all $A \subseteq \mathbb{R}^n$ holds $\mu(A) = \inf \{ \mu(U) : U \supset A, U \text{ open} \}$ (outer regularity),
(2) for all μ -measurable sets B holds $\mu(B) = \sup \{ \mu(K) : K \subset B, K \text{ compact} \}$ (inner regularity).

Theorem (Carathéodory's criteria). Let μ be a (outer) measure on \mathbb{R}^n . If for all $A, B \subset \mathbb{R}^n$ that satisfy $\text{dist}(A, B) > 0$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$, then μ is a Borel measure.

Examples.

- (1) For $x \in \mathbb{R}^n$ we can define that *dirac measure* by

$$\delta_x(A) := \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

This is in fact a Radon measure.

- (2) We define the *counting measure* by

$$\#(E) = \begin{cases} \text{card}(E) & \text{if } E \text{ is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

This measure is Borel regular, but *not* a Radon measure (since it is clearly not locally finite).

- (3) The also have the well-known *Lebesgue measure* defined by

$$\mathcal{L}^n(A) := \inf \left\{ \sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i, Q_i \text{ cubes} \right\},$$

where $\mathcal{L}^n(Q_i)$ is equal to the side length of the cubes Q_i to the n -th power. In particular, we have

$$\mathcal{L}^1(A) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam } C_j \mid A \subset \bigcup_{i=1}^{\infty} C_j, C_j \subset \mathbb{R} \right\}$$

and so we can characterize

$$\mathcal{L}^n = \underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n\text{-times}} = \mathcal{L}^{n-1} \times \mathcal{L}^1.$$

[#]Here $\mathfrak{C} = \mathcal{P}(X)$

- (4) (**Hausdorff measure**) Consider $A \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $\delta \in (0, +\infty]$, we define the *Hausdorff α -dimensional content* of A as

$$\mathcal{H}_\delta^\alpha(A) := \inf \left\{ \sum_{j \in I} \omega_\alpha \left(\frac{\text{diam } C_j}{2} \right)^\alpha \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j, \text{diam } C_j \leq \delta, C_j \subseteq \mathbb{R}^n \right\},$$

where the infimum is taking over all the (at most countable) coverings $(C_j \subset \mathbb{R}^n)_{j \in I}$ of A , and set

$$\omega_\alpha := \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2} + 1\right)}.$$

Since $\mathcal{H}_\delta^\alpha$ is a not-increasing function in δ the following limit

$$\mathcal{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathcal{H}_\delta^\alpha(A) = \sup_{\delta > 0} \mathcal{H}_\delta^\alpha(A)$$

always exists in the extended real numbers. This limit is defined to be the *Hausdorff measure*.

Theorem (Hausdorff measure is Borel regular). \mathcal{H}^α is a Borel regular measure on \mathbb{R}^n for all $\alpha \geq 0$.

Theorem (Basic properties of the Hausdorff measure).

- (1) $\mathcal{H}^0 = \#$
- (2) $\mathcal{H}^1 = \mathcal{L}^1$ on \mathbb{R}
- (3) $\mathcal{H}^\alpha \equiv 0$ for all $\alpha > n$ in \mathbb{R}^n .
- (4) $\mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A)$ for all $A \subseteq \mathbb{R}^n$ and $\lambda > 0$
- (5) $\mathcal{H}^\alpha(L(A)) = \mathcal{H}^\alpha(A)$ for all affine isometry $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Proof.

- (1) Since $\omega_0 = 1$ we have

$$\begin{aligned} \mathcal{H}^0(A) &= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j \in I} \left(\frac{\text{diam}(C_j)}{2} \right)^0 \mid A \subset \bigcup_{j \in I \subset \mathbb{N}} C_j, \text{diam } C_j \leq \delta \right\} \\ &= \lim_{\delta \searrow 0} \inf \left\{ \sum_{j \in I} 1 \mid A \subset \bigcup_{j \in I} C_j, \text{diam } C_j \leq \delta \right\} \\ &= \begin{cases} \text{card}(A) & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- (2) We estimate the Lebesgue measure \mathcal{L}^1 from both sides by the Hausdorff measure: Since $\omega_1 = 2 = |(-1, 1)|$ we first get

$$\begin{aligned} \mathcal{L}^1(A) &= \inf \left\{ \sum_{j \in I} \text{diam } C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \\ &\leq \inf \left\{ \sum_{j \in I} \text{diam } C_j \mid A \subset \bigcup_{j \in I} C_j, \text{diam } C_j \leq \delta \right\} = \mathcal{H}_\delta^1(A), \end{aligned}$$

which is true for all $\delta > 0$ so we obtained $\mathcal{L}^1(A) \leq \mathcal{H}^1(A)$.

Now, we define a partition of \mathbb{R} by $J_{k,\delta} := [k\delta, (k+1)\delta]$ for $k \in \mathbb{Z}$ and first fixed $\delta > 0$. These are intervals of diameter δ so for every $j \in I$ we get $\text{diam}(C_j \cap I_{k,\delta}) \leq \delta$. Also we have

$\sum_{k \in \mathbb{Z}} \text{diam}(C_j \cap I_{k,\delta}) \leq \text{diam } C_j$, since $I_{k,\delta}$ are a partition \mathbb{R} of disjoint intervals in k . So we get

$$\mathcal{L}^1(A) = \inf \left\{ \sum_{j \in I} \text{diam } C_j \mid A \subset \bigcup_{j \in I} C_j \right\} \geq \inf \left\{ \sum_{j \in I} \sum_{k \in \mathbb{Z}} \text{diam}(C_j \cap I_{k,\delta}) \mid A \subset \bigcup_{j \in I} \bigcup_{k \in \mathbb{Z}} C_j \cap I_{k,\delta} \right\}$$

since $\text{diam}(C_j \cap I_{k,\delta}) \leq \delta$ and after relabeling the index sets I and \mathbb{Z} to an index set $I^{(k,\delta)}$ this last expressions reads

$$\dots = \inf \left\{ \sum_{j \in I^{(k,\delta)}} C_j^{(k)} \mid A \subset \bigcup_{j \in I^{(k,\delta)}} C_j^{(k)}, \text{diam } C_j^{(k)} \leq \delta \right\} \geq \mathcal{H}_\delta^1.$$

And since this is true for every $\delta > 0$ we arrive at the claim.

- (3) Let $\alpha > n$ and Q be a unit cube in \mathbb{R}^n . If we consider cubes Q_i with side length $\frac{1}{m}$ for any fixed $m \in \mathbb{N}$, we get $\text{diam } Q_i = \frac{\sqrt{n}}{m}$ and $Q \subset \bigcup_{i=1}^{m^n} Q_i$. From this we can infer

$$\mathcal{H}_{\frac{\sqrt{n}}{m}}^\alpha(Q) \leq \sum_{j=1}^{m^n} \omega_\alpha \left(\frac{\text{diam } Q_i}{2} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} \sum_{j=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = \frac{\omega_\alpha}{2^\alpha} n^{\frac{\alpha}{2}} m^{n-\alpha}$$

and read off with the crucial assumptions $n < \alpha$

$$\mathcal{H}^\alpha(Q) = \lim_{m \rightarrow \infty} \mathcal{H}_{\frac{\sqrt{n}}{m}}^\alpha(Q) = \frac{\omega_\alpha}{2^\alpha} \lim_{m \rightarrow \infty} n^{\frac{\alpha}{2}} m^{n-\alpha} = 0.$$

This is the claim, since \mathbb{R}^n can be covered by a countable collection of unit cubes. □

Lemma. Let $A \subset \mathbb{R}^n$ and $\delta > 0$ such that $\mathcal{H}_\delta^\alpha = 0$, then follows that $\mathcal{H}^\alpha(A) = 0$.

Proof. Since the Hausdorff content is non-increasing in δ , we have $0 = \mathcal{H}_\delta^\alpha(A) \geq \mathcal{H}_\infty^\alpha(A)$. That convergence means that for every $\varepsilon > 0$ there exists a (at most countable) family of subsets $(C_j)_{j \in I}$ such that

$$A \subseteq \bigcup_{j \in I} C_j \quad \text{and} \quad \sum_{j \in I} \omega_\alpha \left(\frac{\text{diam } C_j}{2} \right)^\alpha < \varepsilon.$$

For this to be true the diameter of every C_j must be controlled by $\text{diam } C_j \leq 2 \left(\frac{\varepsilon}{\omega_\alpha} \right)^{\frac{1}{\alpha}} =: \delta_\varepsilon$. So we have $\mathcal{H}_{\delta_\varepsilon}^\alpha \leq \varepsilon$ and as $\delta_\varepsilon \searrow 0$ so will $\varepsilon \searrow 0$. But this is the claim $\mathcal{H}^\alpha(A) = 0$. □

Proposition. Let $A \subseteq \mathbb{R}^n$, $0 \leq s < t < \infty$.

- (1) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.
(2) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

Proof. (1) Fix $\delta > 0$ and a (at most countable) family of subsets $(C_j)_{j \in I}$ such that

$$\text{diam } C_j \leq \delta \quad \text{and} \quad \sum_{j \in I} \omega_s \left(\frac{\text{diam } C_j}{2} \right)^s \leq \mathcal{H}_\delta^s(A) + 1 \leq \mathcal{H}^s(A) + 1.$$

From this follows

$$\begin{aligned} \mathcal{H}_\delta^t(A) &\leq \sum_{j \in I} \omega_t (\text{diam } C_j)^t = \frac{\omega_t}{\omega_s} s^{s-t} \sum_{j \in I} \omega_s \left(\frac{\text{diam } C_j}{2} \right)^s (\text{diam } C_j)^{t-s} \\ &\leq C_{s,t} \delta^{t-s} (\mathcal{H}^s(A) + 1) \longrightarrow 0 \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

which is the claim $\mathcal{H}^t(A) = 0$.

- (2) If by contradiction $\mathcal{H}^s(A) < \infty$, then by (1) it follows that $\mathcal{H}^r(A) = 0$ for all $r > s$ and such in particular $r = t$. □

Definition. We call the *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$

$$\dim_{\mathcal{H}}(A) := \inf \{ \alpha \geq 0 : \mathcal{H}^\alpha(A) = 0 \}.$$

Remark. Let $\alpha = \dim_{\mathcal{H}}(A)$. Then one has

$$\mathcal{H}^s(A) = 0 \quad \text{for all } s > \alpha \quad \text{and also} \quad \mathcal{H}^t(A) = +\infty \quad \text{for all } t < \alpha,$$

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where the first equality is clear from the definition of the Hausdorff dimension and the second fact follows like this: Suppose by contradiction $\mathcal{H}^t(A) < \infty$ for some $t < \alpha$, then by the above proposition we have $\mathcal{H}^r(A) = 0$ for all $r > t$. But this means it even holds for all $r \in (t, \alpha)$ and this is a contradiction to the fact that α is indeed the infimum.

In particular \mathcal{H}^α is *not* a Radon measure for all $\alpha \in [0, n)$. Take for example the closed unit Ball $\overline{B}(0, 1)$ in \mathbb{R}^n . We know that $0 < \mathcal{H}^n(\overline{B}(0, 1)) < \infty$ and so $\mathcal{H}^\alpha(\overline{B}(0, 1)) = +\infty$ for all $\alpha < n$.

@GC Is this the right argument?

If a Borel set $E \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^\alpha(E) \in (0, \infty)$, then $\mathcal{H}^\alpha \llcorner E$ is a Radon measure. Indeed in general we have the following

Theorem. If μ is a Borel regular measure in \mathbb{R}^n and $A \subset \mathbb{R}^n$ is μ -measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is a Radon measure.

Theorem. $\mathcal{H}^n = \mathcal{L}^n$

Proposition. Let $\alpha \geq 0$, $A \subset \mathbb{R}^n$.

- (1) If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, then $\mathcal{H}^\alpha(f(A)) \leq (\text{Lip}(f))^\alpha \mathcal{H}^\alpha(A)$.
- (2) If $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is γ -Hölder[‡], then $\mathcal{H}^\alpha(g(A)) \leq C_{\alpha, \gamma} \mathcal{H}^\alpha(A)$.

Proof of (2). Fix $\delta > 0$, and take a (at most countable) family of subsets $(C_j)_{j \in I}$ such that $A \subset \bigcup_{j \in I} C_j$ and $\text{diam } C_j \leq \delta$. For $g(C_j)$ we get for all $j \in I$

$$\text{diam } g(C_j) \leq C (\text{diam } C_j)^\gamma \leq C \delta^\gamma \quad \text{and} \quad g(A) \subseteq \bigcup_{j \in I} g(C_j).$$

Using this, we can infer

$$\mathcal{H}_{C\delta^\gamma}^\alpha(g(A)) \leq \sum_{j \in I} \omega_\alpha \left(\frac{\text{diam } g(C_j)}{2} \right)^\alpha \leq \underbrace{\frac{\omega_\alpha}{2^\alpha} \frac{C^\alpha 2^{\alpha\gamma}}{\omega_{\alpha\gamma}}}_{=: C_{\alpha, \gamma}} \sum_{j \in I} \omega_{\alpha\gamma} \left(\frac{\text{diam } C_j}{2} \right)^{\alpha\gamma}$$

and by taking the infimum over all coverings $(C_j)_{j \in I}$ we get

$$\mathcal{H}_{C\delta^\gamma}^\alpha(g(A)) \leq C_{\alpha, \gamma} \mathcal{H}_\delta^{\alpha\gamma}(A)$$

and by sending $\delta \searrow 0$ we arrive at the claim. □

[‡]E.g. $|g(x) - g(y)| \leq C_\gamma |x - y|^\gamma$ for a $\gamma \in (0, 1)$.