Simulation: Analysis of results

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Outline

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Goodness of a model

During developement of the model, we face 2 problems

- Is the model correctly implemented?
  $\Rightarrow$ verification
- Does the model present the real system?
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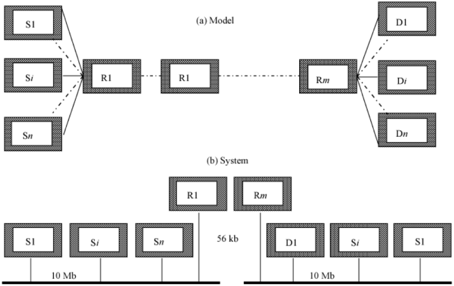
- Verification = "debugging"
  (person-in-charge: programming person)
- Validation = "representiveness of assumptions"
  (person-in-charge: modeling person)

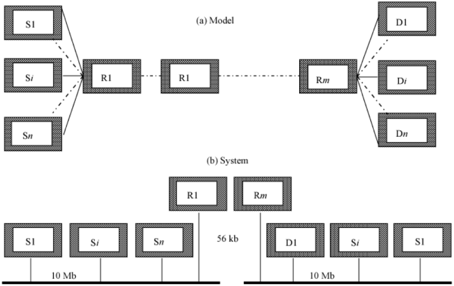
Verification techniques (software engineering)

- Top-Down Modular Design
- Anti-bugging
- Structured Walk-Through
- Deterministic Models
- Trace
- Run Simplified Cases
- On-Line Graphic Displays
- Continuity Test
- Degeneracy Tests
- Consistency Tests
- Seed Independence



Interconnected LAN

A system = two local-area networks (source LAN, destination LAN) connected through *m* intermediate nodes



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Topdown modular design =

- Modularity: modules + interfaces
- Top-down: "divide-and-conquer" (hierarchical) to small modules (easily debugged)

Verification: other techniques

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  - Total of probabilities is equal to 1. If not, print the error message
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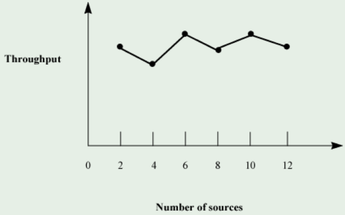
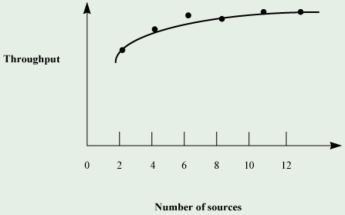
  For example,
  - Only one packet; only one source; only one intermediate node
- Trace: use a time-ordered list of events and variables.
  - Levels of details: events, procedures, variables

Verification: continuity test

Which could be error?





- Continuity test: run for different values of input parameters.
  - Slight change in input $\rightarrow$ slight change in output

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Validation = comparison

Compare the key aspects obtained from 3 possible sources

- Expert intuition
- Real system measurements
- Theoretical results (Analysis = Simulation)

Outline

Steady-state analysis

Only the steady-state performance is of interest.
$\Rightarrow$ Detecting transient state to be removed.

No exact definition of transient state $\Rightarrow$ heuristics:

- Long runs
- Proper initialization
- Truncation
- Initial data deletion
- Moving average of independent replications
- Batch means

Long runs and proper initialization

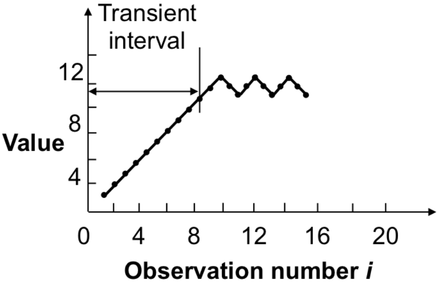
- Long runs: long enough to ensure that initial conditions will not affect the result
  - Wasting resources
  - Difficult to insure that it is long enough
- Proper initialization: start in a state close to expected steady state.
  - E.g., number of jobs in queue is initialized by previous simulations or by simple analysis

Truncation

Assumption

Variability is lower during steady state.

- Plot max-min of $n - l$ remaining observations for $l = 1, 2, \ldots$
- When $(l+1)^{\text{th}}$ observation is neither the minimum nor maximum $\rightarrow$ transient state ended.
- At $l = 9$, Range $= (9, 11)$, next observation $= 10$



Sometimes it gives incorrect results.

Initial data deletion

- Purpose: delete some initial observation (as truncation).
- Ideas:
  - Compute average
    Use several replications to smoothen the average
    m replications of size n each: $x_{ij} = j^{\text{th}}$ observation in the i^{th} replication.
  - No change $\rightarrow$ Steady state.

Initial data deletion: steps (1)

1. Get a mean trajectory by averaging across replications

$$\bar{x}_j = \frac{1}{m}\sum_{i=1}^{m} x_{ij} \quad j = 1, 2, \ldots, n.$$

2. Get the overall mean

$$\bar{\bar{x}} = \frac{1}{n}\sum_{j=1}^{n} \bar{x}_j.$$

   Set $l := 1$ and proceed to the next step.

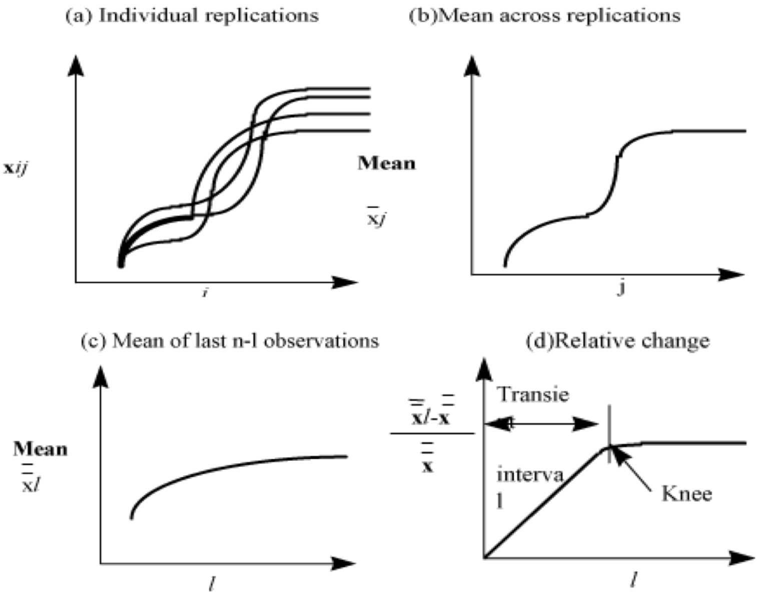
3. Delete the first l observations and get an overall mean from the remaining $n - l$ values

$$\bar{\bar{\bar{x}}}_l = \frac{1}{n-l}\sum_{j=l+1}^{n} \bar{x}_j.$$

4. Compute the relative change: $\frac{\bar{\bar{x}}_l - \bar{\bar{x}}}{\bar{\bar{x}}}$.

5. Repeat steps 3 and 4 by varying l from 1 to $n - 1$.
   Plot the overall mean and the relative change.
   l at knee is the length of the transient interval.

Initial Data Deletion: steps (2)



Moving average of independent replications (1)

Mean over a moving time interval window.

1. Get a mean trajectory by averaging across replications:

$$\bar{x}_j = \frac{1}{m}\sum_{i=1}^{m} x_{ij} \quad j = 1, 2, \ldots, n.$$

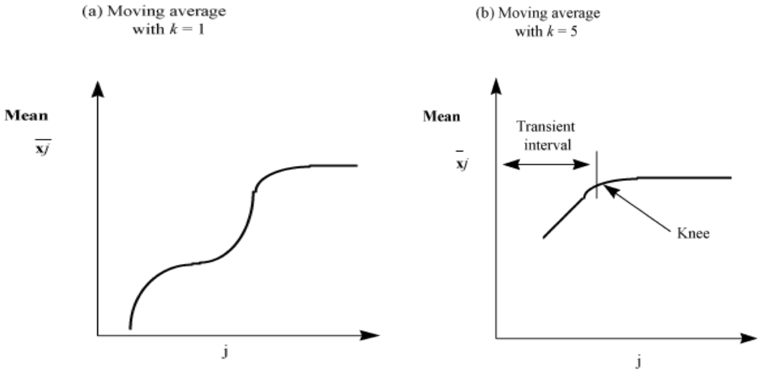
   Set $k := 1$ and proceed to the next step.

2. Plot a trajectory of the moving average of successive $2k+1$ values:

$$\bar{\bar{x}}_j = \frac{1}{2k+1}\sum_{l=-k}^{k} \bar{x}_{j+l} \quad j = k+1, \ldots, n-k.$$

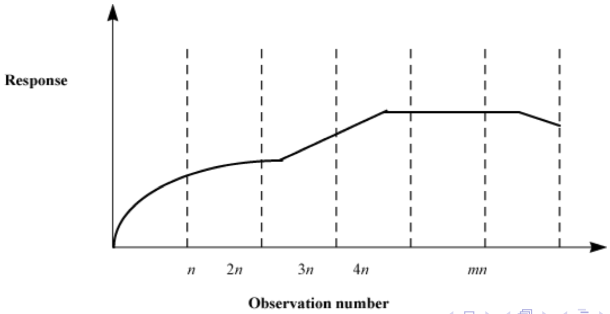
3. Repeat step 2, with $k = 2, 3$, and so on until the plot is smooth.
4. Value of j at the knee gives the length of the transient phase.

Moving average of independent replications (2)



Batch means

- Run a long simulation and divide into equal duration part.
- Part = Batch = Sub-sample.
- Study variance of batch means as a function of the batch size.
    - m: number of batches
    - n: batch size



Batch means: steps

1. For each batch, compute a batch mean:

$$\bar{x}_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij} \quad i = 1, 2, \ldots, m.$$

2. Compute overall mean:

$$\bar{\bar{x}} = \frac{1}{m} \sum_{i=1}^{m} \bar{x}_i.$$

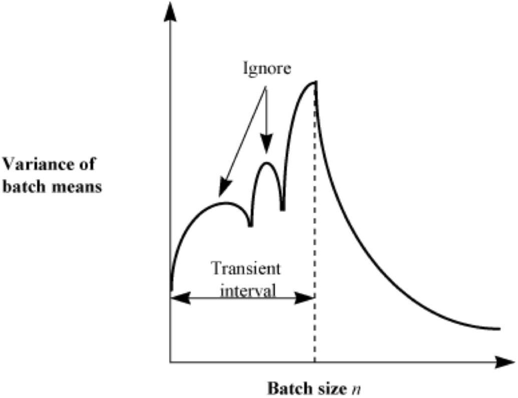
3. Compute the variance of the batch means:

$$\mathrm{Var}(\bar{x}) = \frac{1}{m-1} \sum_{i=1}^{m} (\bar{x}_i - \bar{\bar{x}})^2.$$

4. Repeat steps 1 and 3, for $n = 3, 4, 5, \ldots$
   Plot the variance as a function of batch size n.
   Value of n at which the variance definitely starts decreasing gives transient interval.

Batch means

- Ignore peaks followed by an upswing



Outline

Terminating simulations

- Transient performance is of interest, however, some systems never reach a steady state.
  E.g, network traffic of small file transfering, but simulations with large files are not of interest.
  E.g., System shutdowns at a given point of time.
- They are called $\Rightarrow$ terminating simulation
- Final conditions:
  - May need to exclude the final portion from results.
  - Techniques similar to transient removal.

Treatment of leftover entities

- Mean service time

$$\frac{\text{total service time}}{\text{Num. of jobs that completed service}}.$$

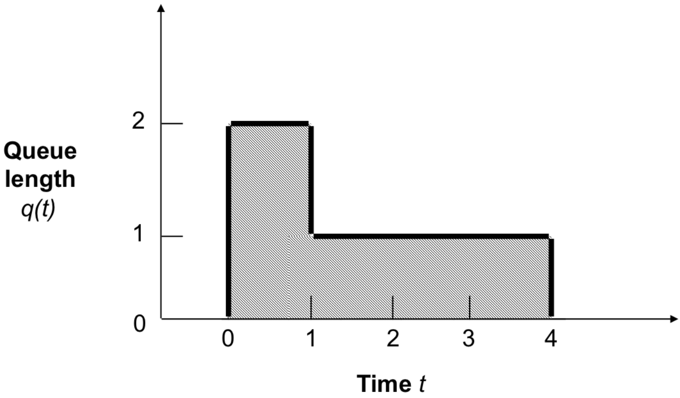
- Mean waiting time

$$\frac{\text{sum of waiting time}}{\text{Num. of jobs that completed service}}.$$

- Mean queue length

$$\frac{1}{T}\int_0^T \text{QueueLength}(t)dt.$$

Treatment of leftover entities: example



- Three events: Arrival at $t = 0$, departures at $t = 1$ and $t = 4$.
- $Q = 2, 1, 0$ at these events. Avg $Q \neq (2 + 1 + 0)/3 = 1$.
- Avg $Q = \text{Area}/4 = 5/4$.

Confidence interval: review (1)

- Population characteristics $\neq$ sample characteristics
  Population mean $\mu \neq$ sample mean $\bar{x}$
  Impossible to get perfect estimate of population mean
- Two bounds c_1, c_2, such that population mean in the interval (c_1, c_2) with a high probability $1 - \alpha$.

$$\text{Probability}\{c_1 \leq \mu \leq c_2\} = 1 - \alpha$$

  - (c_1, c_2): confidence interval for population mean
  - α: significance level
  - $100(1 - \alpha)$: confidence level

Confidence interval: review (2)

Determinining 90% confidence interval

- Use 5-percentile and 95-percentile of sample means
- Take k samples, find sample means, sort them in increasing order
- Take $[1 + 0.05(k-1)]^{\text{th}}$ and $[1 + 0.95(k-1)]^{\text{th}}$ elements of the sorted list

Determinining 90% confidence interval

A $100(1-\alpha)\%$ confidence interval for the population mean

$$(\bar{x} - z_{1-\alpha/2} s/\sqrt{n} + z_{1-\alpha/2} s/\sqrt{(}n))$$

in which,

- $\bar{x}$: sample mean
- s: sample standard deviation
- n: sample size
- $z_{1-\alpha/2}$: $(1-\alpha/2)$-quantile of $\mathcal{N}(0,1)$

Stopping criteria: Variance Estimation

- Run until confidence interval is narrow enough

$$\bar{x} \pm z_{1-\alpha/2} \, \mathrm{Var}(\bar{x})$$

- For independent observations:

$$\mathrm{Var}(\bar{x}) = \frac{\mathrm{Var}(x)}{n}$$

- Independence not applicable to most simulations.
  - Large waiting time for i^{th} job $\rightarrow$ Large waiting time for $(i+1)^{\text{th}}$ job.
  - For correlated observations:

$$\text{Actual variance} \gg \frac{\mathrm{Var}(x)}{n}.$$

Variance estimation methods

1. Independent replications
2. Batch means
3. Method of regeneration

Independent replications

- Assumes: mean of independent replications are independent.
- Conduct m replications of size $n + n_0$ each (n_0 is transient length). First, remove n_0 observations of each replication
  1. Compute a mean for each replication
  $$\bar{x}_i = \frac{1}{n} \sum_{j=n_0+1}^{n_0+n} x_{ij} \quad i = 1, 2, \ldots, m.$$
  2. Compute an overall mean for all replications
  $$\bar{\bar{x}} = \frac{1}{m} \sum_{i=1}^{m} \bar{x}_i$$
  3. Calculate the variance of replicate means
  $$\mathrm{Var}(\bar{x}) = \frac{1}{m-1} \sum_{i=1}^{m} (\bar{x}_i - \bar{\bar{x}})^2$$
  4. Confidence interval for the mean is
  $$\left[\bar{\bar{x}} \pm z_{1-\alpha/2} \mathrm{Var}(\bar{x})\right]$$

Batch means

- Also called method of sub-samples.
- Run a long simulation run
- Discard initial transient interval, and Divide the remaining observations run into several batches or sub-samples.
  1. Compute means for each batch:
  $$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \quad i = 1, 2, \ldots, m.$$
  2. Compute an overall mean:
  $$\bar{\bar{x}} = \frac{1}{m}\sum_{i=1}^{m} x_i.$$
  3. Calculate the variance of batch means:
  $$\mathrm{Var}(\bar{x}) = \frac{1}{m-1}\sum_{i=1}^{m} (\bar{x}_i - \bar{\bar{x}})^2.$$
  4. Confidence interval for the mean response is
  $$\left[\bar{\bar{x}} \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}(\bar{x})/m}\right].$$

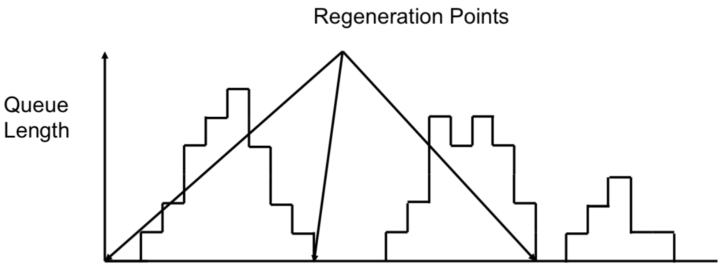
Batch means

- Less waste than independent replications.
- Keep batches long to avoid correlation.
- Check: Compute the auto-covariance of successive batch means:

$$\mathrm{Cov}(\bar{x}_i, \bar{x}_{i+1}) = \frac{1}{m-2} \sum_{i=1}^{m-1} (\bar{x}_i - \bar{\bar{x}})(\bar{x}_{i+1} - \bar{\bar{x}}).$$

- Double n until auto covariance is small.

Method of regeneration (1)



- Behavior after idle period does not depend upon the past history
  - System takes a new birth,
  - Regeneration point.

Note

The regeneration point are the beginning of the idle interval.

Method of regeneration (2)

- Regeneration cycle: Between two successive regeneration points.
- Use means of regeneration cycles
- Problems:
  - Not all systems are regenerative
    Different lengths $\rightarrow$ Computation complex.
- Overall mean $\neq$ Average of cycle means.
- Cycle means are given by:

$$\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_i.$$

- Overall mean:

$$\bar{\bar{x}} \neq \frac{1}{m} \sum_{i=1}^{m} \bar{x}_i.$$

Method of regeneration (3)

1. Compute cycle sums: $y_i = \sum_{j=1}^{n_i} x_{ij}$.
2. Compute overall mean: $\bar{\bar{x}} = \frac{\sum_{i=1}^{m} y_i}{\sum_{i=1}^{m} n_i}$.
3. Calculate the difference between expected and observed cycle sums: $w_i = y_i - n_i \bar{\bar{x}}, \quad i = 1, 2, \ldots, m$.
4. Calculate the variance of the differences:
$$\mathrm{Var}(w) = s_w^2 = \frac{1}{m-1} \sum_{i=1}^{m} w_i^2.$$
5. Compute mean cycle length:
$$\bar{n} = \frac{1}{m} \sum_{i=1}^{m} n_i.$$
6. Confidence interval for the mean response is given by
$$\bar{\bar{x}} \mp z_{1-\alpha/2} \frac{s_w}{\bar{n}\sqrt{m}}.$$
7. No need to remove transient observations.

Outline

Exercises

Problem 1

Imagine that you have been called as an expert to review a simulation study. Which of the following simulation results would you consider non-intuitive and would want it carefully validated:

1. The throughput of a system increases as its load increases.
2. The throughput of a system decreases as its load increases.
3. The response time increases as the load increases.
4. The response time of a system decreases as its load increases.
5. The loss rate of a system decreases as the load increases.

Problem 2

Find the duration of the transient interval for the following sample: $11, 4, 2, 6, 5, 7, 10, 9, 10, 9, 10, 9, 10, \ldots$, Does the method of truncation give the correct result in this case?

Exercises

Problem 3

The observed queue lengths at time $t = 0, 1, 2, \ldots, 32$ in a simulation are:
$0, 1, 2, 4, 5, 6, 7, 7, 5, 3, 3, 2, 1, 0, 0, 0, 1, 1, 3, 5, 4, 5,$
$4, 4, 2, 0, 0, 0, 1, 2, 3, 2, 0$. A plot of this data is shown below. Apply method of regeneration to compute the confidence interval for the mean queue length.