# Single queues

Tran, Van Hoai (hoai@hcmut.edu.vn)
Le, Hong Trang (lhtrang@hcmut.edu.vn)

Faculty of Computer Science & Engineering
HCMC University of Technology

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# Outline


1. Basic structures and components
   - Kendall Notation
2. Performance metrics
3. Little's Law
4. Birth-death processes
5. Rules for All Queues
6. M/M/1
   - Exercise
7. M/M/n
   - Exercise

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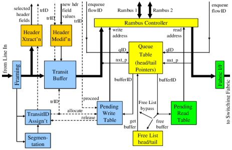
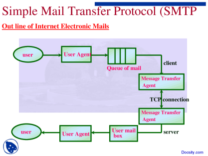
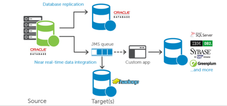
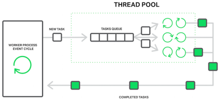
# Queues in real world



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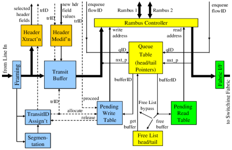
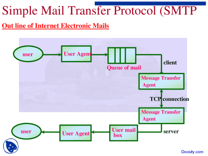
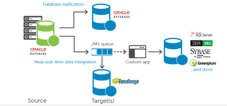
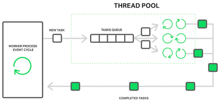


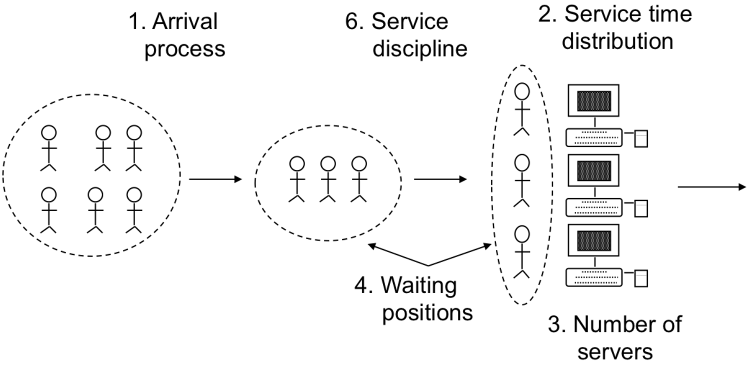




Are there any common structures of queues?

# Basic components of a queue




- Customers can be people, parts, vehicles, machines, jobs,...
- Queue might not be a physical line.

# Kendall Notation System

- $A/S/m/B/K/SD$
  - $A$: Arrival process,
  - $S$: Service time distribution,
  - $m$: Number of servers,
  - $B$: Number of buffers (system capacity),
  - $K$: Population size,
  - $SD$: Service discipline.

- Arrival times: $t_1, t_2, \ldots, t_j$.
- Interarrival times: $\tau_j = t_j - t_{j-1}$.
- $\tau_j$ form a sequence of Independent and Identically Distributed (IID) random variables.
- Exponential $+$ IID $\rightarrow$ Poisson.
- Notation:
  - $M$ $=$ Memoryless $=$ Poisson,
  - $E$ $=$ Erlang,
  - $H$ $=$ Hyper-exponential,
  - $G$ $=$ General $\rightarrow$ Results valid for all distributions.

- Time each student spends at the terminal.
- Service times are IID.
- Distribution: $M$, $E$, $H$, or $G$.
- Device $=$ Service center $=$ Queue.
- Buffer $=$ Waiting positions.

- First-Come-First-Served (FCFS);
- Last-Come-First-Served (LCFS);
- Last-Come-First-Served with Preempt and Resume (LCFS-PR);
- Round-Robin (RR) with a fixed quantum.
- Small Quantum $\rightarrow$ Processor Sharing (PS);
- Infinite Server: (IS) $=$ fixed delay;
- Shortest Processing Time first (SPT);
- Shortest Remaining Processing Time first (SRPT);
- Shortest Expected Processing Time first (SEPT);
- Shortest Expected Remaining Processing Time first (SERPT).
- Biggest-In-First-Served (BIFS);
- Loudest-Voice-First-Served (LVFS).

- $M$: Exponential,
- $E_k$: Erlang with parameter $k$,
- $H_k$: Hyper-exponential with parameter $k$,
- $D$: Deterministic $\rightarrow$ constant,
- $G$: General $\rightarrow$ All.
- Memoryless:
  - Expected time to the next arrival is always $1/\lambda$ regardless of the time since the last arrival,
  - Remembering the past history does not help.

# Example: $M/M/3/20/1500/FCFS$

- Time between successive arrivals is exponentially distributed.
- Service times are exponentially distributed.
- Three servers,
- 20 Buffers = 3 service + 17 waiting,
  After 20, all arriving jobs are lost,
- Total of 1500 jobs that can be serviced.
- Service discipline is FCFS.
- Defaults:
  - Infinite buffer capacity,
  - Infinite population size,
  - FCFS service discipline.
- $G/G/1 = G/G/1/1/1/FCFS$.

- Bulk arrivals/service.
- $M^{[x]}$: $x$ represents the group size.
- $G^{[x]}$: a bulk arrival or service process with general inter-group times.
- Example:
  - $M^{[x]}/M/1$: Single server queue with bulk Poisson arrivals and exponential service times;
  - $M/G^{[x]}/m$: Poisson arrival process, bulk service with general service time distribution, and $m$ servers.

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# Typical performance questions

What is the ...

- average number of customers in the system?
- average time a customer spends in the system?
- probability a customer is rejected?
- fraction of time a server is idle?

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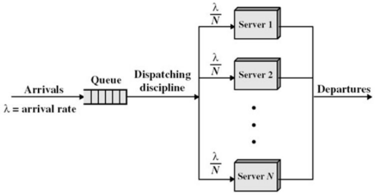


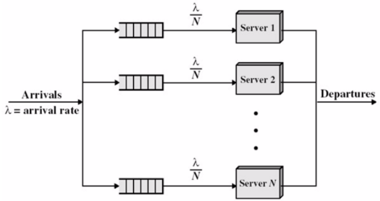


- What is average time waiting in the queue?
- What is variability of time in the queue?

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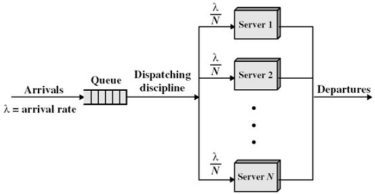
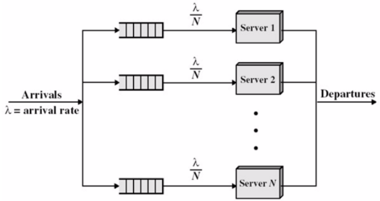
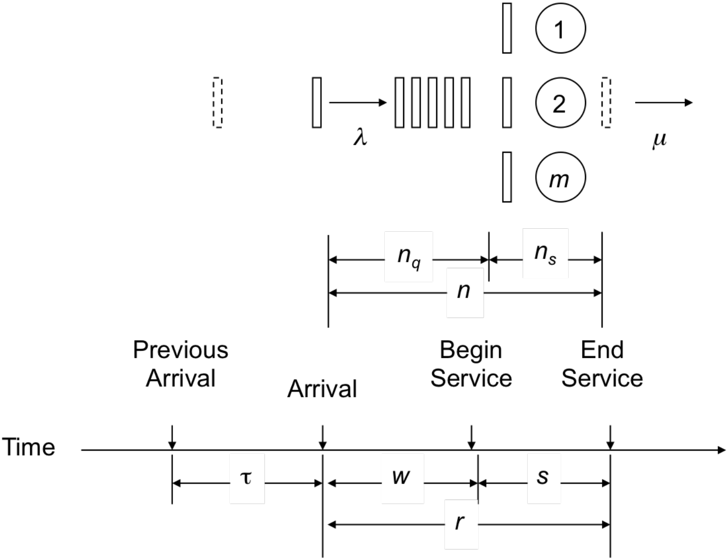



- What is average time waiting in the queue?
- What is variability of time in the queue?

None of done incorrectly outcomes are investigated to produce metrics.

# Key variables (1)



# Key variables (2)

- $\tau$: Inter-arrival time = time between two successive arrivals.
- $\lambda$: Average arrival rate $= 1/E[\tau]$
  - May be a function of the state of the system
    E.g., number of jobs already in the system.
- $s$: Service time per job.
- $\mu$: Average service rate per server $= 1/E[s]$.
- Total service rate for $m$ servers is $m\mu$.
- $n$: Number of jobs in the system.
  Note: $n$ includes jobs currently receiving service as well as those waiting in the queue.

# Key variables (3)

- $n_q$: Number of jobs waiting.
- $n_s$: Number of jobs receiving service.
- $r$: Response time or the time in the system (system time)
  - time waiting + time receiving service.
- $w$: Waiting time
  - Time between arrival and beginning of service.

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# Little's Law

Named after Little (1961).

**Little's Law**

For any queuing system that has a steady state and has an average rate of $\lambda$,

$$E[n] = \lambda E[r]$$

If the average system time is 2 hours, and customers arrive at a rate of 3 per hour then on average, there are 6 customers in the system.

# Discussion on Little's Law

Based on a black-box view of the system.


Little's law can be used for a system or any part of the system.

- Average number in queue = arrival rate × average waiting time
- Average number in service = arrival rate × average service time

Little's law requires no assumptions about arrival or service time distribution, the size of population, or limits on the system.

# Example on Little's Law

- Consider problem
  - A monitor on a disk server showed that the average time to satisfy an I/O request was 100 milliseconds.
  - The I/O rate was about 100 requests per second.
  - What was the average number of requests at the disk server?
- Using Little's law, average number in the disk server

$$
\begin{aligned}
&= \text{Arrival rate} \times \text{system time} \\
&= 100 \text{ (requests/second)} \times \text{(0.1 seconds)} \\
&= 10 \text{ requests.}
\end{aligned}
$$

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# (Recall) Stochastic processes

- Process: sequence (family) of random variables that are functions of time.
  E.g., $n(t)$: number of jobs waiting for CPU of a computer system at time $t$.
- Each $n(t)$ is random variable which can be defined by a probability distribution. $\Rightarrow$ Stochastic process.

# (Recall) Stochastic processes

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  E.g., $n(t)$: number of jobs waiting for CPU of a computer system at time $t$.
- Each $n(t)$ is random variable which can be defined by a probability distribution. $\Rightarrow$ Stochastic process.
- Types of stochastic processes
  - Discrete or Continuous State Processes
    - $n(t)$ is a discrete-state process
    - $w(t)$ is a continuous-state process
  - Markov Processes
  - Birth-death Processes
  - Poisson Processes

# Markov processes

- A **Markov process** = A stochastic process in which future states are independent of the past and depend only on the present. ⇒ easier to analyze, not have to remember past trajectory.
- A **Markov chain** = A discrete-state Markov process
- Some remarks:
  - Knowing current (present) state is sufficient
  - Not necessary to know how long the process has been in the current state. ⇒State time has a memoryless distribution.

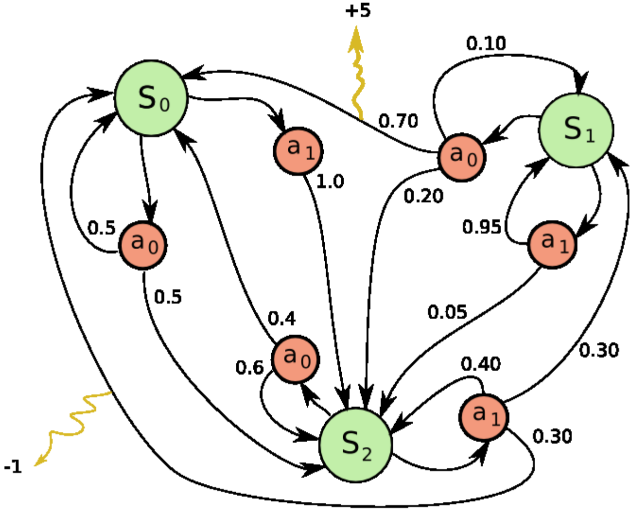
E.g., exponential distribution

$$\begin{aligned} Pr\{X > s+t | X > t\} &= \frac{Pr\{X>s+t \text{ and } X>t\}}{Pr\{X>t\}} \\ &= \frac{Pr\{X>s+t\}}{Pr\{X>s\}} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} \end{aligned}$$

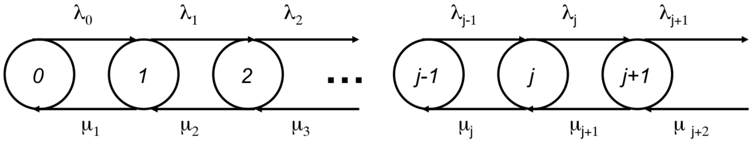
The time spent by a job in such a queue is a Markov process and the number of jobs in the queue is a Markov chain.

# Markov decision process in learning

## Vision



# Birth-death processes




- A **Birth-death process** = A discrete-state Markov process in which the transitions are restricted to neighboring states.
  $\Rightarrow$ Process in state $n$ can change only to state $n+1$ or $n-1$.

Number of jobs in a queue with a single server and individual arrivals (not bulk arrivals).

- An arrival (birth): state changed by $+1$.
- A departure (death): state changed by -1.

**Theorem**

The steady-state probability $p_n$ of a birth-death process being in state $n$ is given by

$$p_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} p_0, n = 1, 2, \ldots, \infty$$

Here, $p_0$ is the probability of being in the zero state.

# Poisson processes (1)

- Inter-arrival time $\tau =$ IID (identically and independently distributed) and exponential
  - number of arrivals $n$ over a given interval $(t, t + x)$ has a Poisson distribution $P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}$
  - Arrival process is a **Poisson process** or **Poisson stream**.

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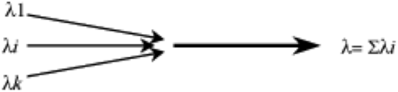
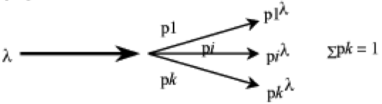
- Properties

(1) Merging: $\lambda = \sum_{i=1}^{k} \lambda_i$.

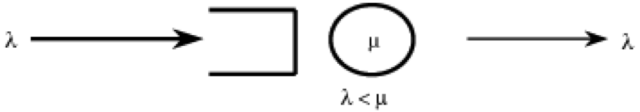



(2) Splitting: If the probability of a job going to $i^{\text{th}}$ substream is $p_i$, each substream is also Poisson with a mean rate of $p_i\lambda$.

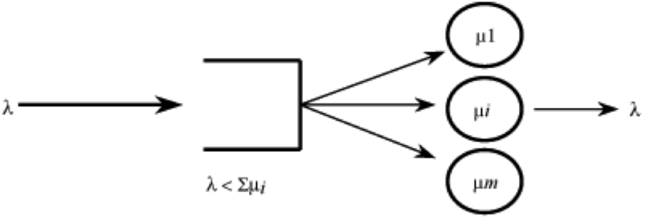


# Poisson processes (2)

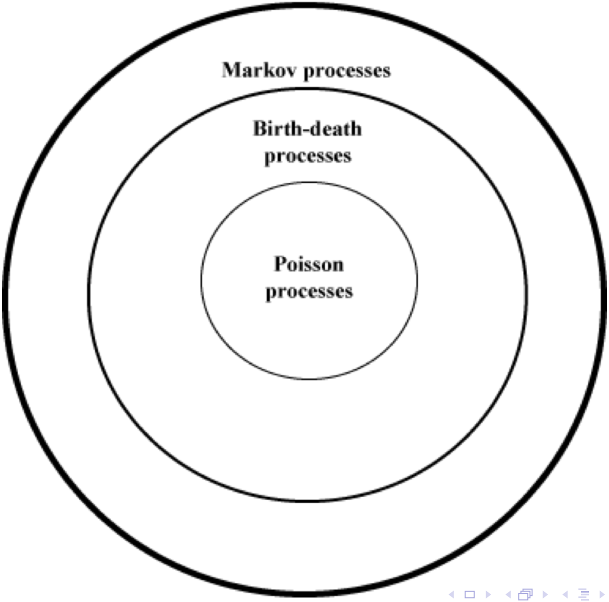
- Properties (cont.)
  (3) If the arrivals to a single server with exponential service time are Poisson with mean rate $\lambda$, the departures are also Poisson with the same rate $\lambda$ provided $\lambda < \mu$.



- Properties (cont.)
  (4) If the arrivals to a service facility with $m$ service centers are Poisson with a mean rate $\lambda$, the departures also constitute a Poisson stream with the same rate $\lambda$, provided $\lambda < \sum_i \mu_i$.
    - Here, the servers are assumed to have exponentially distributed service times.



# Relationship among stochastics processes



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# Rules for All Queues: apply to G/G/m queues (1)

(1) Stability Condition: $\lambda < m\mu$

- Finite-population and infinite-buffer systems are always stable.

(2) Number in System versus Number in Queue:

- $n = n_q + n_s$, where $n, n_q$, and $n_s$ are random variables;
- $E[n] = E[n_q] + E[n_s]$;
- If the service rate is independent of the number in the queue, $Cov(n_q, n_s) = 0$

$$\mathrm{Var}[n] = \mathrm{Var}[n_q] + \mathrm{Var}[n_s].$$

# Rules for All Queues: apply to G/G/m queues (2)

(3) Number versus Time: if jobs are not lost due to insufficient buffers,
   - Mean number of jobs in the system = Arrival rate $\times$ Mean response time.

(4) Similarly,
   - Mean number of jobs in the queue = Arrival rate $\times$ Mean waiting time.
   - This is Little's law as mentioned later.

(5) Time in System versus Time in Queue $r = w + s$, where $r$, $w$, and $s$ are random variables.
   - $E[r] = E[w] + E[s]$.

(6) If the service rate is independent of the number of jobs in the queue, $Cov(w, s) = 0$

$$\mathrm{Var}[r] = \mathrm{Var}[w] + \mathrm{Var}[s].$$

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# M/M/1

**Definition**

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- One server.
- No limitation on buffer and population.
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# M/M/1

**Definition**

- Interarrival times, service times are exponentially distributed.
- One server.
- No limitation on buffer and population.
- FCFS service discipline.

- It is the most commonly used type of queue.
- Need to know only arrival rate $\lambda$ and service rate $\mu$.

- A birth-death process with

$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots, \infty$$
$$\mu_n = \mu, \quad n = 1, 2, \ldots, \infty$$

- Probability of $n$ jobs in the system

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, n = 1, 2, \ldots, \infty$$

- Traffic intensity $\rho = \lambda/\mu$.

$$p_n = \rho^n p_0$$

We have

$$\sum_{i=0}^{\infty} p_i = 1$$
$$p_0(1 + \rho^1 + \rho^2 + \ldots) = 1$$
$$p_0 = \frac{1}{1+\rho^1+\rho^2+\ldots} = 1 - \rho$$

$$\Rightarrow p_n = \rho^n(1 - \rho)$$

- Utilization of the server = Probability of having one or more jobs in the system

$$U = 1 - p_0 = \rho$$

- Mean number of jobs in the system

$$E[n] = \sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} n \rho^n (1-\rho) = \frac{\rho}{1-\rho}$$

- Variance of the number of jobs in the system

$$\begin{aligned} \mathrm{Var}[n] &= E[n^2] - (E[n])^2 \\ &= \sum_{n=1}^{\infty} n^2 (1-\rho)\rho^n - (E[n])^2 \\ &= \frac{\rho}{(1-\rho)^2} \end{aligned}$$

- Probability of $n$ or more jobs in the system

$$P(\geq n \text{ jobs in the system}) = \sum_{j=n}^{\infty} p_j = \rho^n$$

- By Little's Law

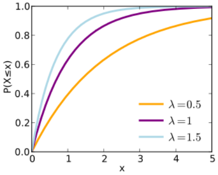
$$E[n] = \lambda E[r]$$

Then,

$$E[r] = \frac{E[n]}{\lambda} = \frac{1/\mu}{1-\rho}$$

- Cumulative distribution function of the response time

$$F(r) = 1 - e^{-r\mu(1-\rho)}.$$




$\Rightarrow$ The response time is exponentially distributed.

- $q$-percentile of the response time (i.e., $F(r) = q/100$)

$$1 - e^{-r_q\mu(1-\rho)} = \frac{q}{100}.$$

Hence,

$$r_q = \frac{1}{\mu(1-\rho)\ln\left(\frac{100}{100-q}\right)}.$$

- Cumulative distribution function of the waiting time

$$F(w) = 1 - \rho e^{-w\mu(1-\rho)}.$$

- Mean waiting time

$$E[w] = \rho\frac{1/\mu}{1-\rho} = E[r] - \frac{1}{\mu}$$

- This is a truncated exponential distribution. Its q-percentile is given by

$$w_q = \frac{1}{\mu(1-\rho)\ln\left(\frac{100\rho}{100-q}\right)}.$$

- The above formula applies only if $q$ is greater than $100(1-\rho)$. All lower percentiles are zero.

$$w_q = \max\left\{0, \frac{E[w]}{\rho}\ln\left(\frac{100\rho}{100-q}\right)\right\}.$$

- Mean number of jobs in the queue:

$$E[n_q] = \sum_{n=1}^{\infty}(n-1)p_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}.$$
$$E[n_q] = E[n] - \rho$$

**Note**

All results for $M/M/1$ queues including some for the busy period are summarized in Box 31.1 in the book of R. Jain.

# $M/M/1$: exercise

## Problem

On a network gateway, measurements show that the packets arrive at a mean rate of 125 packets per second (pps) and the gateway takes about two milliseconds to forward them.

- Using an $M/M/1$ model, analyze the gateway.
- What is the probability of buffer overflow if the gateway had only 13 buffers?
- How many buffers do we need to keep packet loss below one packet per million?

# $M/M/1$: exercise(1)

- Arrival rate $\lambda = 125$ pps.
- Service rate $\mu = 1/.002 = 500$ pps.
- Gateway Utilization $\rho = \lambda/\mu = 0.25$.
- Probability of $n$ packets in the gateway:
  $(1-\rho)\rho^n = 0.75 \times 0.25\rho^n$.
- Mean Number of packets in the gateway:
  $\rho/(1-\rho) = 0.25/0.75$.
- Mean time spent in the gateway:
  $(1/\mu)/(1-\rho) = (1/500)/(1-0.25) = 2.66$ milliseconds.
- Probability of buffer overflow:
  $P$(more than 13 packets in gateway)
  $$= \rho^{13} = 0.25^{13} = 14.9 \times 10^{-9}$$
  $$\approx 15 \text{ packets per billion packets.}$$

# $M/M/1$: exercise(2)

- An airport runway for arrivals only
- Arriving aircraft join a single queue for the runway
- Exponentially distributed service time with a rate $\mu = 27$ arrivals/hour.
- Poisson arrivals with a rate $\lambda = 20$ arrivals/hour.
- Compute
  - Time in the airport runway system
  - Number of aircrafts in the runway system
  - Waiting time for the runway
  - Number of aircrafts waiting for the runway

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- Compute
  - Time in the airport runway system
    $E[r] = \frac{1}{\mu - \lambda} = \frac{1}{27-20} = \frac{1}{7}$hour.
  - Number of aircrafts in the runway system
    $E[n] = \lambda E[r] = \frac{20}{27-20} = 2.9$ aircrafts.
  - Waiting time for the runway
    $E[w] = E[r] - 1/\mu = \frac{1}{7} - \frac{1}{27} = 6.4$ min
  - Number of aircrafts waiting for the runway
    $E[n_q] = \ldots$

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## Definition

- Interarrival times, service times are exponentially distributed.
- $n$ **servers.**
- No limitation on buffer and population.
- FCFS service discipline.

# $M/M/m$

## Some results

- A birth-death process with

$$\lambda_n = \lambda$$
$$\mu_n = \begin{cases} n\mu & n = 1, 2, \ldots, m-1 \\ m\mu & n = m, m+1, \ldots, +\infty \end{cases}$$

- Traffic intensity $\rho = \lambda/(m\mu)$
- Probability of zero job in the system

$$p_0 = \left[ 1 + \frac{(m\rho)^m}{m!(1-\rho} + \sum_{n=1}^{m-1} \frac{(m\rho)^n}{n!} \right]^{-1}$$

- Probability of $n$ jobs in the system

$$\mu_n = \begin{cases} \frac{\lambda^n}{n!\mu^n} p_0 & n = 1, 2, \ldots, m-1 \\ \frac{\lambda^n}{m!m^{n-m}\mu^n} p_0 & n = m, m+1, \ldots, +\infty \end{cases}$$

### Computer center

Students arrive at a computer center in Poisson manner of rate 10 students/hour. Each student spends an average of 20 minutes at a terminal in exponential distribution. The center has 5 terminals. Let analyze the center usage.

- Traffic intensity $\rho = \lambda/(5\mu) = 0.167/(5 \times 0.05) = 0.67$
- Probability of all terminals being idle is $p_0 = ... = 0.0318$
- Probability of all terminals being busy is $\frac{(m\rho)^m}{m!(1-\rho)} p_0 = 0.33$.