Calculus of Variations

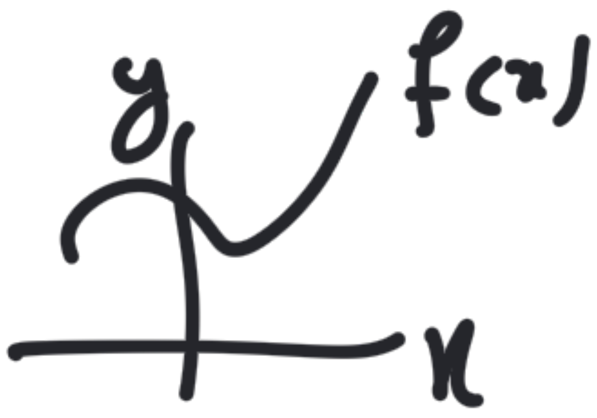
$f(x)$ $\quad$ $f(x,y)$ $\quad$ $f(\underline{x})$ $\quad x \in \mathbb{R}^d$

plug $\underline{x}$ get a real number

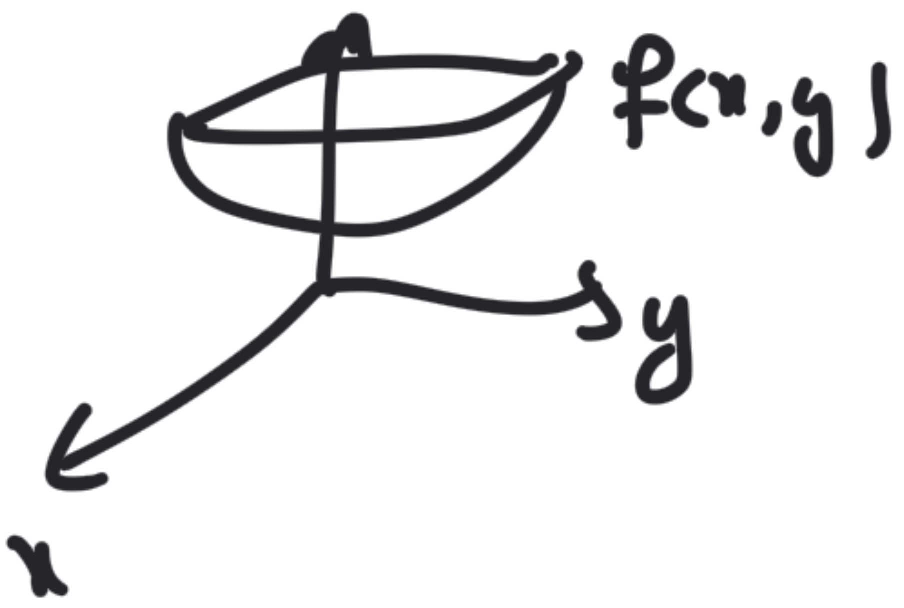
e.g. $f(5,7,2) = 25$

$$f: \mathbb{R}^d \longrightarrow \mathbb{R}$$

graph: $f(x)$



$f(x,y)$



limitation in graphing.

- $f: \mathbb{R}^d \longrightarrow \mathbb{R}$ $\quad f(x_1, x_2, \dots, x_d)$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_d} \right)$$

$$\frac{\partial f}{\partial \underline{n}} = \nabla f \cdot \underline{n} = \lim_{h \to 0} \frac{f(\underline{x} + h\underline{n}) - f(\underline{x})}{h}$$

directional derivative

Calculus on Functionals

Functional: function of a function

$E(u)$

fn that lives in some fn space
eg. $u \in$ Cont. fns on $\mathbb{R}$ $C(\mathbb{R})$
$\in$ diff. fns on $\mathbb{R}$ $C^1(\mathbb{R})$
$\in$ smooth fns on $\mathbb{R}$ $C^\infty(\mathbb{R})$

E : fn space $\longrightarrow \mathbb{R}$

Compare with a fn:

$f: \mathbb{R}^d \longrightarrow \mathbb{R}$

Example: • $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$f(x,y) = x^2 + y^2$ so $f(0,1) = 0^2 + 1^2 = 1$

• $E: C^1(\mathbb{R}) \longrightarrow \mathbb{R}$
$E(u) = \int_0^1 u'^2 + u \, dx$

Calculate $E(x^2)$

$E(x^2) = \int_0^1 (2x)^2 + x^2 \, dx = \int_0^1 4x^2 + x^2 \, dx$
$= \int_0^1 5x^2 \, dx = \left. \frac{5x^3}{3} \right|_0^1 = \frac{5}{3}$

Let's do Calculus on functionals:
Derivatives and finding critical pts
and finding $\underbrace{\text{minima and maxima}}_{\text{optimization}}$

This is called $\underbrace{\text{Calculus of Variations}}$

Calculus on fnl's istead of Calculus
on fns!!

Given a fn'l $E(u)$, we want its minimizer

$$\min_{u \in \text{some fn space}} E(u)$$

then we must find its critical pts!!
Critical pts happen when equivalent of
$\underbrace{\text{"derivative of } E(u)\text{" in the fn space} = 0}_{\text{Euler-Lagrange eqn}}$

This give us
ordinary differential eqns
and
partial differential eqns!!

Calculus $\rightsquigarrow$ PDEs
and ODEs

1) Fn $f: \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_d})$$

$$f = f(x_1, \ldots, x_d)$$

let x_i depend on time

$$f = f(x_1(t), x_2(t), \ldots, x_d(t))$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_d}\frac{dx_d}{dt}$$

$$= \nabla f \cdot \frac{d}{dt}(x_1(t), x_2(t), \ldots, x_d(t))$$

$$= \nabla f \cdot \dot{x}(t)$$

$$\dot{f} = \nabla f \cdot \dot{x}(t)$$

2) fn'l $\quad E$: fn space $\longrightarrow \mathbb{R}$

usually called an energy

Prb: find $\quad \min\limits_{\substack{u \in \text{fn. space} \\ \text{on some domain}}} E(u)$

If there is a minimizer, it will happen at the critical pt of $E(u)$ in the fn space.

Critical pts happen at pts where

$$\nabla_{\text{fn space}} E = 0$$

have to define $\nabla_{\text{fn space}}$

new: gradient in a fn space

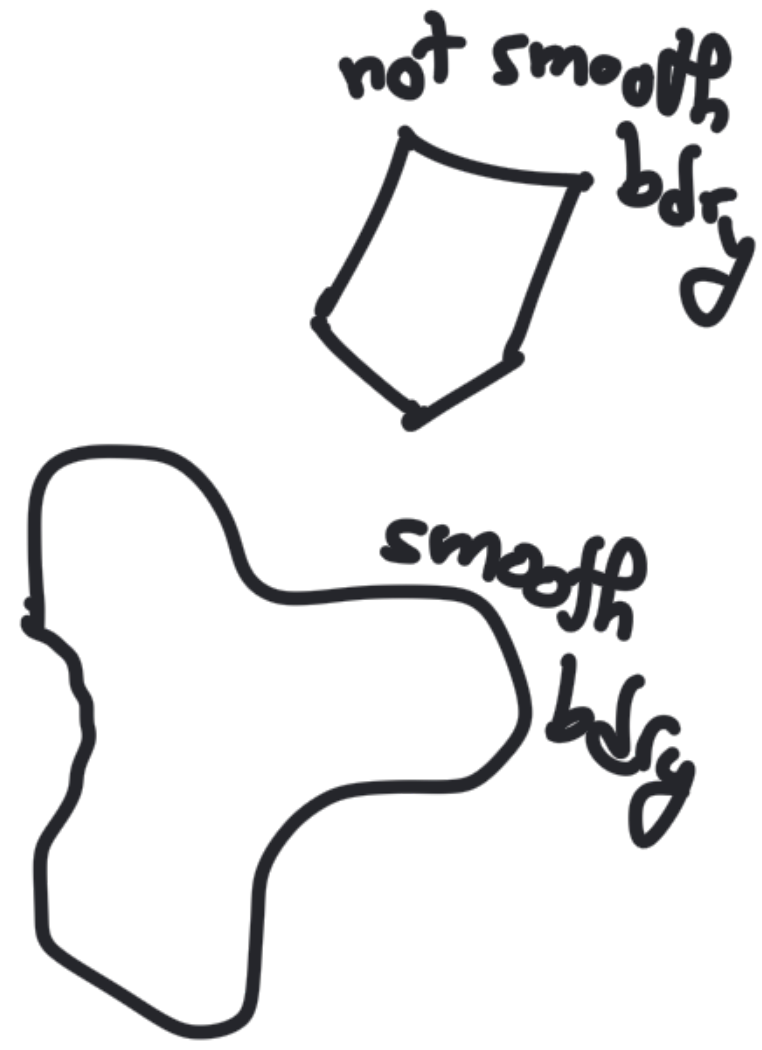
We know ∇f when $f : \mathbb{R}^d \to \mathbb{R}$

We also know $\frac{\partial f}{\partial n} = \nabla f \cdot \underline{n}$

directional derivative

<u>Example:</u> Consider (famous) the Dirichlet energy fn'l:

$$E(u) = \int_D \frac{1}{2} |\nabla u|^2 \, dx$$



D is a domain in $\mathbb{R}^d$

∂D is the bdry of D

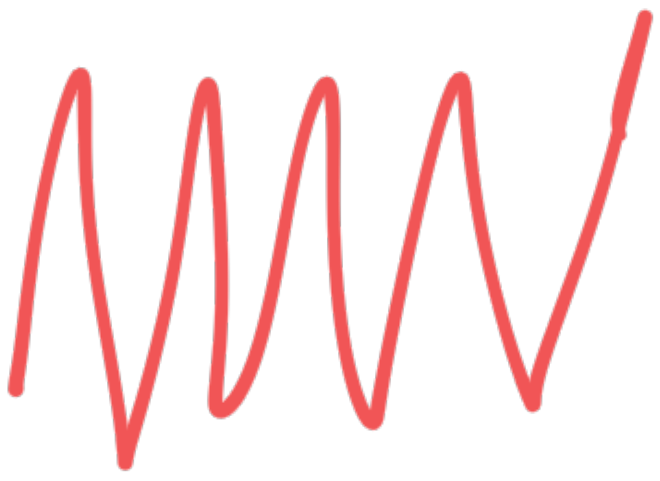
$u \in H^1(D)$ which means that u has finite $\int_D |u|^2 + |\nabla u|^2 \, dx$

$H^1(D)$ is called a Sobolov Space.

Sobolov spaces allow a wide range of fns that are not so nice

nice fn

lives in cont. and continuously diff. fn spaces



not nice fn

not nice fn

to be solns of important PDE that represent real life situations.

minimizer of the Dirchlet energy satisfies

$$\underbrace{\nabla_{\text{fn space}} E(u) = 0}$$

Euler-Lagrange eqn associated with the energy fn'l $E(u)$
also called variational principle.

Soln: $E(u) = \int_D \frac{1}{2} |\nabla u|^2 \, dx$

$$\dot{E}(u) = \int_D \frac{1}{2} \left(|\nabla u|^2\right)^{\bullet} dx$$

$$= \int_D \frac{1}{2} (\nabla u . \nabla u)^{\bullet} dx$$

$|\nabla u|^2 = (u_{x_1}^2 + \cdots + u_{x_d}^2)$
$= \nabla u . \nabla u$

$$= \int_D \frac{1}{2} \left((\nabla u)^{\bullet} . \nabla u + (\nabla u) . (\nabla u)^{\bullet} \right) dx$$

$$= \int_D \nabla u \cdot (\nabla u)^{\bullet} \, dx$$

$$= \int_D \nabla u \cdot \nabla \dot{u} \, dx$$

goal:
$\dot{E}(u) = \nabla E(u) \cdot \dot{u}$
fn space

most imp. trick in Calculus
integration by parts!

Calc I: $\displaystyle\int_a^b u \, v_x \, dx = -\int_a^b u_x v \, dx + uv\Big|_a^b$

Calc III: Int. by parts Green's thm.

follows from div. thm

$$\int_D u \nabla v \, dx = -\int_D \nabla u \, v \, dx + \int_{\partial D} u \, v \, \underline{n} \, ds$$

divergence thm: $\displaystyle\int_D \nabla \cdot \underline{F} \, dx = \int_{\partial D} \underline{F} \cdot \underline{n} \, ds$

$$\Rightarrow \dot{E}(u) = -\int_D \nabla \cdot (\nabla u) \, \dot{u} \, dx + \int_{\partial D} \nabla u \, \dot{u} \cdot \underline{n} \, ds$$

$$= -\int_D \Delta u \, \dot{u} \, dx + \int_{\partial D} \dot{u} \, \nabla u \cdot \underline{n} \, ds$$

$$\dot{E}(u) = -\int_D \Delta u \, \dot{u} \, dx + \int_{\partial D} \dot{u} \frac{\partial u}{\partial n} \, ds$$

true for any $\dot{u}$ in my allowed fn space.

In particular, let $\dot{u} = 0$ on ∂D

$$\Rightarrow \int_{\partial D} \dot{u} \frac{\partial u}{\partial n} \, ds = 0$$

and $\dot{E}(u) = -\int_D \Delta u \, \dot{u} \, dx$

$$= \langle -\Delta u, \dot{u} \rangle_{L^2(D)}$$

$$= \nabla_{L^2(D)} E(u) \cdot \dot{u}$$

$$\Rightarrow \nabla_{L^2(D)} E(u) = -\Delta u$$

minizer satisfies E-L eqn: $\nabla_{L^2(D)} E(u) = 0$

$\Rightarrow -\Delta u = 0$ or $\Delta u = 0$ Laplace eqn PDE

I defined inner product in a fn space called L^2:

$$\langle f, g \rangle_{L^2(D)} = \int_D f(x) g(x) \, dx$$

Therefore fns that minimize the Dirichlet energy

$$E(u) = \int_D \frac{1}{2} |\nabla u|^2 \, dx$$

are harmonic fns (fns that satisfy Laplace eqn on D)

$$\Delta u = 0 \quad \text{on } D$$

E-L eqn for E(u) is a PDE

say in 2D: $u_{xx} + u_{yy} = 0$

Look up examples of Harmonic fns.

Recall: Calc III

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}\right)$$: vector

div $\underline{v}$ = $\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \dots + \frac{\partial v_d}{\partial x_d}$: scalar

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}\right)$$

as an operator

$v = (v_1, v_2, \dots, v_n)$: random vector

$\nabla \cdot \underline{v}$ = dot product between operator ∇ and components of v

$$\nabla \cdot (\nabla u) = \underbrace{\frac{\partial^2}{\partial x_1^2} u_1 + \frac{\partial^2}{\partial x_2^2} u_2 + \dots + \frac{\partial^2}{\partial x_d^2} u_d}_{= \Delta u}$$

in 2D:

$$\Delta u = u_{xx} + u_{yy}$$

scalars

in 3D $\Delta u = u_{xx} + u_{yy} + u_{zz}$

We proved that if

$$E(u) = \int_D \frac{1}{2} |\nabla u|^2 \, dx$$

then $\nabla_{L^2(D)} E(u) = -\Delta u$

How: allow u to depend on t as well

$$u = u(x,t)$$

$$\dot{E}(u) = \langle \nabla_{L^2} E(u), \dot{u} \rangle_{L^2} = \langle -\Delta u, \dot{u} \rangle_{L^2}$$

Recall: Similar to: $f: \mathbb{R}^d \to \mathbb{R}$

$f(x)$, allow x to depend on t

$f(x(t))$

$$\dot{f}(x(t)) = \nabla f \, \dot{x}(t)$$

didn't have to take $\lim_{h \to 0} \frac{E(u+h\eta) - E(u)}{h}$

Goal: find u in appropriate fn space with bdry cond. $u = h(x)$ on ∂D

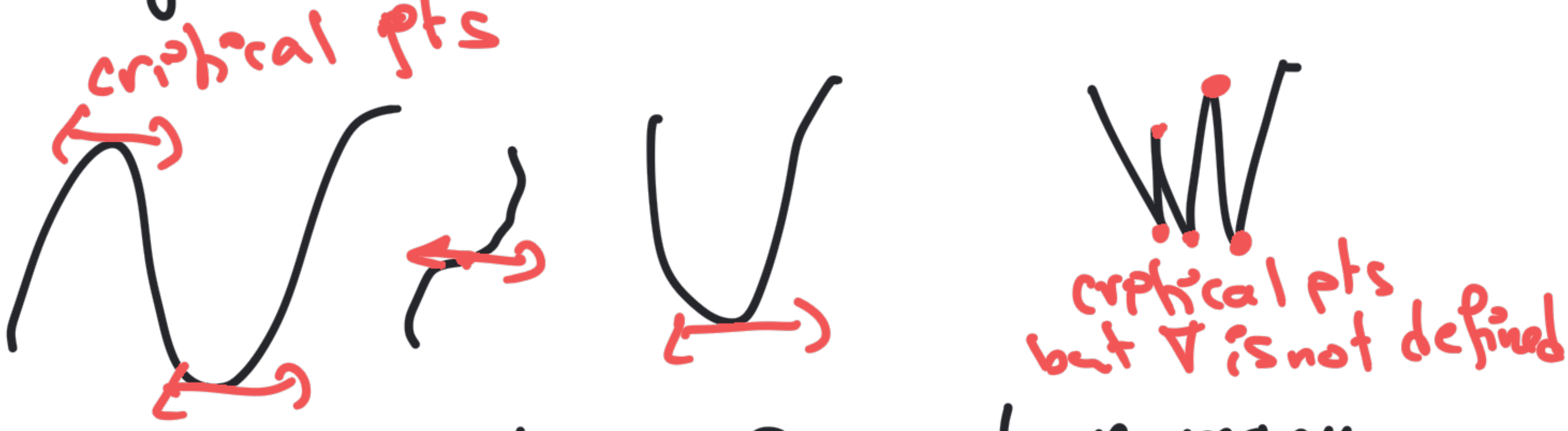
that minimizes

$$E(u) = \int_D \frac{1}{2} |\nabla u(x)|^2 \, dx$$

We found that a critical pt of $E(u)$ must satisfy $\begin{cases} \Delta u = 0 & \text{in } D \\ u = h(x) & \text{on } \partial D \end{cases}$ $(\nabla_{L^2} E(u) = 0)$

Harmonic fn with $h(x)$ on ∂D.

. In general: $E(u)$ could look like



general landscapes can have many peaks and valleys, mountain passes (saddles) don't have to be smooth etc.

How do I know that the harmonic fn on D with $u = h(x)$ on ∂D is a minimizer of $E(u)$ not only a critical pt ?

Answer: Shape of $E(u)$ forces that:

$E(u)$ is convex and $E(u)$ is bounded below by zero

critical pt must be a minimizer

Recall: $f(x) = x^2$

Convex, bd below by zero

Critical pt:

$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$(0,0)$ is definitely a minizer.

How does the heat eqn come into play??

Heat eqn: $u(x,t)$: usually temp. or conc. of a solute in a liquid, or conc. of gas in a room

or any quantity that diffuses

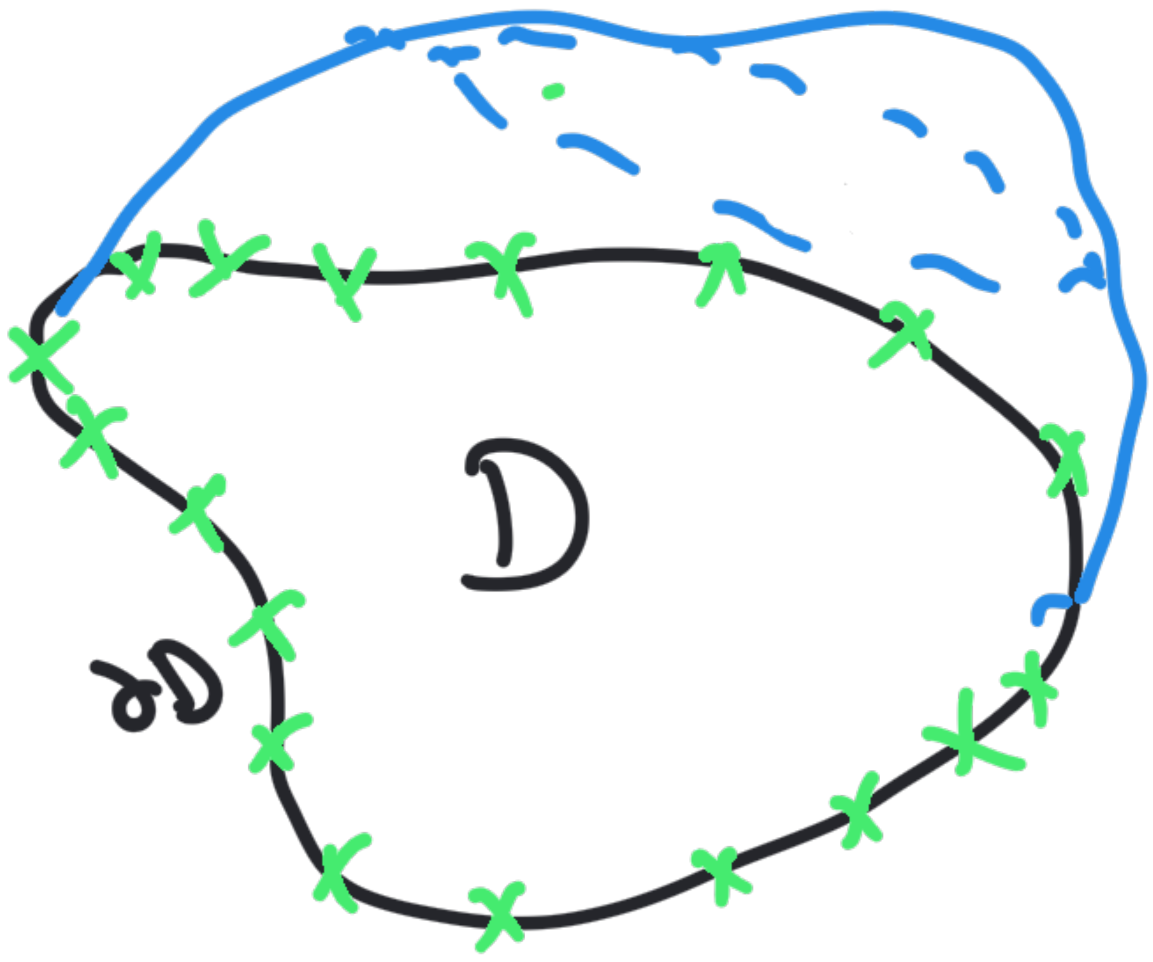
chemicals in a city (chemical weapons)

$$\begin{cases} u_t = \Delta u(x,t) & x\in D \ t>0 \\ u(x,0) = g(x) & x\in D \\ u(x,t) = 0 & x\in \partial D \end{cases}$$

zero bdry cond.

Recall:

$\Delta u(x,t) = u_{xx}(x,t) + u_{yy}(x,t)$ in 2D.



Note: So far from PDEs we have seen

1) Laplace eqn

$$\begin{cases} \Delta u = 0 & x\in D \\ u = g & x\in\partial D \end{cases}$$

2) Heat eqn

$$\begin{cases} u_t = \Delta u & x\in D, t>0 \\ u(x,0) = \ldots & \text{Initial Cond.} \\ u(x,t) = \ldots & \text{B.C.} \end{cases}$$

Put things in perspective:

A whole PDE course is

linear $\begin{cases} \text{Heat eqn} \\ \text{Wave eqn} \\ \text{Laplace eqn} \end{cases}$ Burger's eqn (nonlinear) $\Big\}$ undergrad.

Numerics

<u>Recall:</u> $\Delta u = -\nabla_{L^2} E(u)$

So heat eqn:

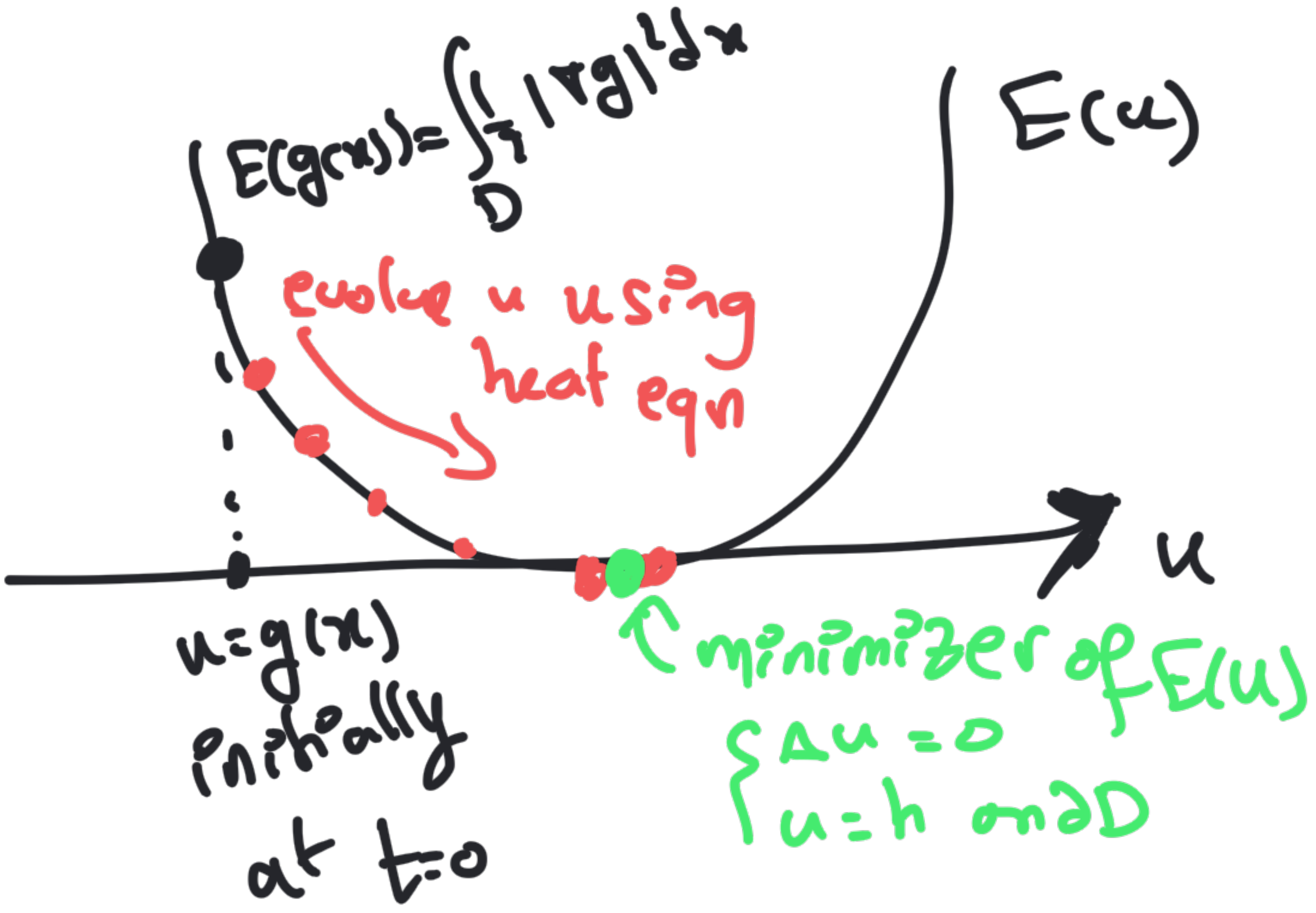
$$u_t = \Delta u = -\nabla_{L^2} E(u)$$

$$\Rightarrow u_t = -\nabla_{L^2} E(u)$$

So how does u evolve in time according to the heat eqn?

<u>answer:</u> $u(x,t)$ starts at $g(x)$ and then evolves in a way that decreases the Dir. Energy $E(u)$ the fastest: steepest descent direction

Concl: The heat eqn does L^2-steepest descent for the Dir. Energy.



<u>Happens in nature:</u> Many systems evolve (you think evolution is random) but in reality they are decreasing some higher energy in the most efficient way!!

The heat eqn provides a minimizing scheme for the Dirichlet energy.

Start somewhere: a random fn $g(x)$, then evolve the heat eqn (numerically if you don't have an explicit analytical soln)

So next time step you're at a new fn $g_1(x)$, keep going:

You know:

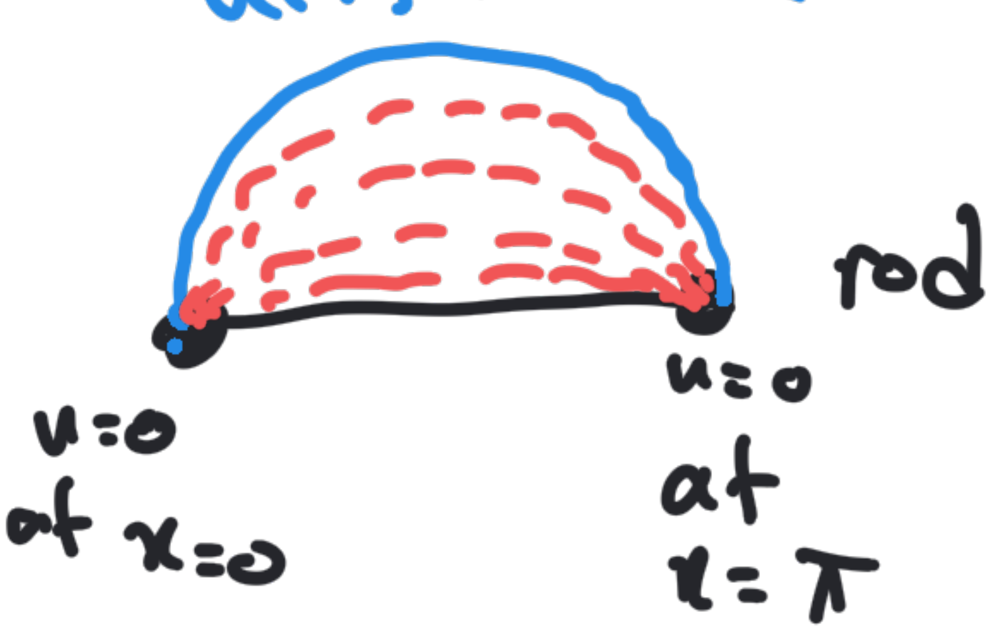
→ $E(g_1(x)) \leq E(g(x))$ & so on

→ Fastest way to decrease this energy

→ When you settle down (stationary soln), then you arrived at the minizer of the Dir. energy in the fastest possible way.

Example Showing an analytical soln of heat eqn on a simple domain: (1 dim.)

$$\begin{cases} u_t = u_{xx} & x \in [0,\pi],\ t>0 \\ u(x,0) = \sin x & \text{I.C} \\ u(0)=0,\ u(\pi)=0 & \text{B.C} \end{cases}$$



soln: (from any PDE course using sep. of vars)

rod of length π
zero bdry cond.'s

$$u(x,t) = e^{-t} \sin x$$

: initial sin x is scaled by e^{-t}
• as $t \to \infty$ soln $\to 0$

(let's verify this is the soln:

$$u_t = -e^{-t} \sin x$$

$$u_x = e^{-t} \cos x \qquad u_{xx} = -e^{-t} \sin x$$

$$\Rightarrow u_t = u_{xx} \quad \checkmark$$

I.C. : $u(x,0) = e^{-0} \sin x = \sin x$ ✓

B.C. : $u(0,t) = e^{-t} \sin 0 = 0$ ✓
$u(\pi,t) = e^{-t} \sin \pi = 0$ ✓)

Note that as $t \to \infty$,

$u(x,t) = e^{-t} \sin x \longrightarrow 0$ (as expected)

minimizer of Dir. energy

satisfies $\begin{cases} \Delta u = u_{xx} = 0 \\ \text{and } u(0) = 0 = u(\pi) \end{cases}$

soln of Laplace eqn with zero bdry cond's

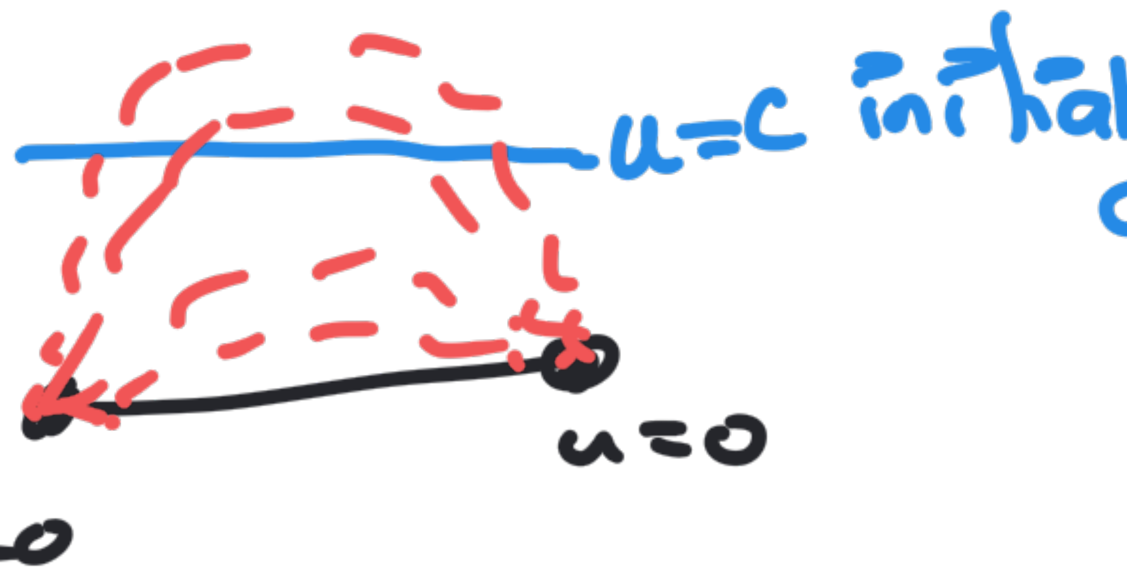
is the zero fn

So All consistent.

Soln of heat eqn indeed converges to the minimizer of Dir. energy as $t \to \infty$.

What happens if we start with a const. temp. througout the rod and not the $\sin x$ temp.

From PDE class, again from sep. of variables, the soln:



exp decay of the temp.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx$$

where the coef's B_n

$$B_n = \frac{2}{\pi}\int_0^{\pi} c \sin nx \, dx$$

$$= \frac{2}{\pi} c \left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi}$$

$$= \frac{2c}{n\pi}\left(1-(-1)^n\right)$$

Note: In general it's not easy to find energy fn'l(s) that decrease with the evolution of a certain PDE (system) and when/if you find these energy fn'l(s) it is very helpful for the community working on that PDE

(Helps understand the evolution of the system "from above". Also provides bounded quantities that are needed for analysis).

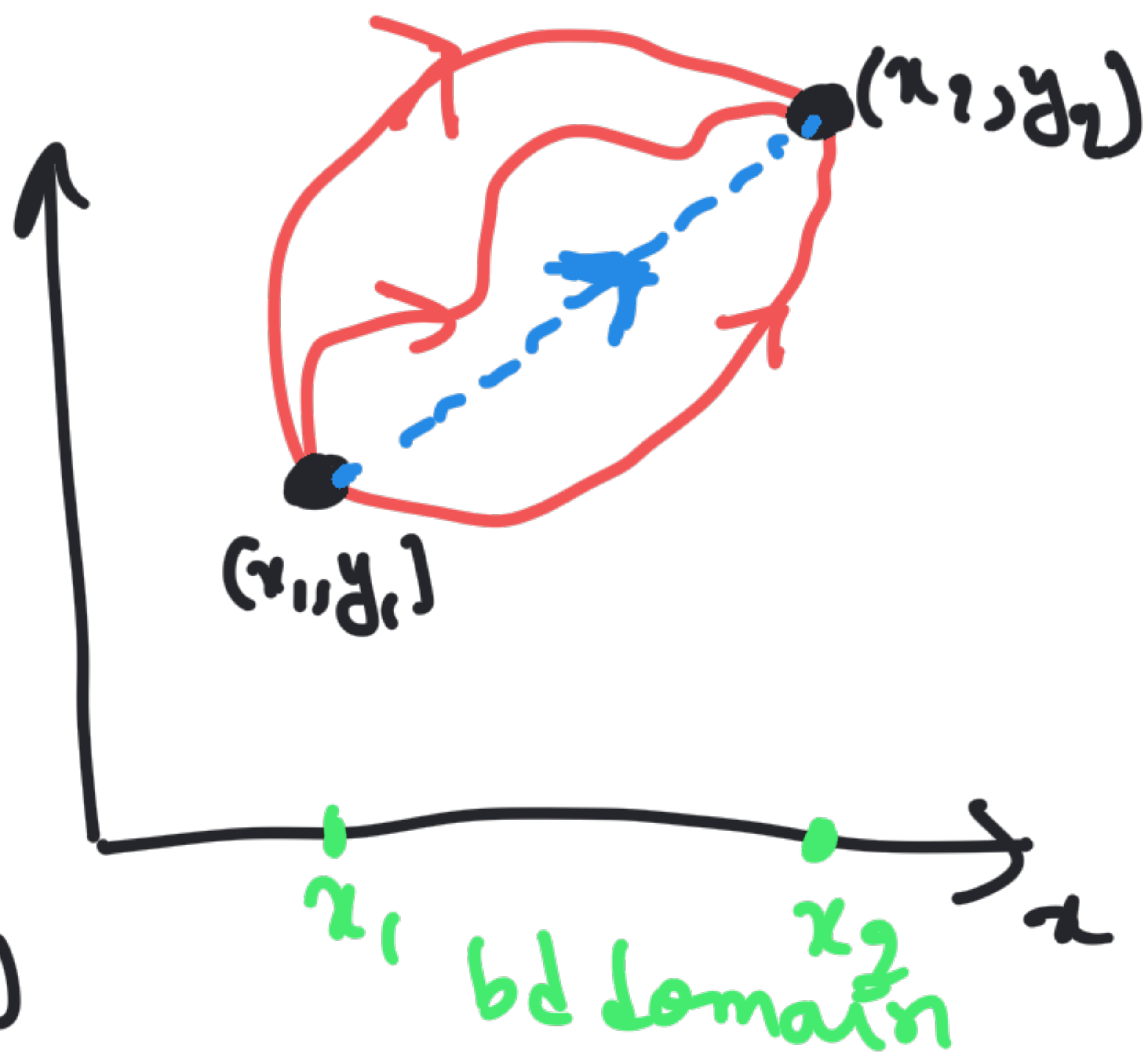
Note: Integration by parts helps find many energy fnl's !

Simplest examples from Calculus of Variations:

1) Shortest path prb:

prb:

min length of a fn $y(x)$ between two pts (x_1, y_1) and (x_2, y_2)



Let $y(x)$ be a fn that passes through (x_1, y_1) and (x_2, y_2). Formula for arc length of a fn

$$\min_{\substack{y(x_1)=y_1,\, y(x_2)=y_2 \\ y(x) \\ \text{cont., one time cont. diff.}}} \underbrace{\int_{x_1}^{x_2} \sqrt{1+y'(x)^2}\, dx}_{\text{arc length}}$$

o o o why is this the formula for an arc length?

$$E(y) = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$

Want $\nabla_{\text{some space}} E(y) = 0$ at a minimizer candidate.

Euler-Lagrange eqn

Assume also dependence on time

$$\dot{E}(y) = \langle \nabla_{\text{fnspace}} E(y), \dot{y} \rangle_{\text{fnspace}}$$

$$E(y) = \int_{x_1}^{x_2} (1 + y'^2)^{1/2}\, dx$$

$$\dot{E}(y) = \int_{x_1}^{x_2} \left((1 + y'^2)^{1/2} \right)^{\bullet} dx$$

$$= \int_{x_1}^{x_2} \frac{1}{2} (1 + y'^2)^{\frac{1}{2}-1} (1 + y'^2)^{\bullet}\, dx$$

$$= \int_{x_1}^{x_2} \frac{1}{2\sqrt{1+y'^2}} (0 + 2y'\dot{y}')\, dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sqrt{1+y'^2}}\, y' \dot{y}'\, dx$$

Recall:
$f = f(x(t))$
$\frac{df}{dt} = \dot{f} = \nabla f \cdot \dot{x}$
chain rule

integration by parts so I can get $\dot{y}$ alone inside the integral

$$= -\int_{x_1}^{x_2} \left(\frac{y'}{\sqrt{1+y'^2}}\right)' \dot{y}\, dx + \left(\frac{y'}{\sqrt{1+y'^2}}\right)\dot{y}\Big|_{x_1}^{x_2}$$

Valid for all $\dot{y}$. Let $\dot{y}=0$ on x_1 and x_2

this kills the boundry term.

Left with:

$$\dot{E}(y) = \int_{x_1}^{x_2} -\left(\frac{y'}{\sqrt{1+y'^2}}\right)' \dot{y}\, dx$$

$$= \left\langle -\left(\frac{y'}{\sqrt{1+y'^2}}\right)', \dot{y} \right\rangle_{L^2([x_1,x_2])}$$

$$\Rightarrow \nabla_{L^2([x_1,x_2])} E(y) = -\left(\frac{y'}{\sqrt{1+y'^2}}\right)'$$

Now at a minimizer, $\nabla_{L^2([x_1,x_2])} E(y) = 0$

$$\Rightarrow \left(\frac{y'}{\sqrt{1+y'^2}}\right)' = 0 \quad \text{E-L eqn}$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c$$ (since its derivative is zero)

eventually $y(x) = ax+b$ connecting (x_1, y_1) and (x_2, y_2)

$$y' = c\sqrt{1+y'^2}$$

square both sides

$$y'^2 = c^2(1+y'^2)$$

$$y'^2 = c^2 + c^2 y'^2$$

$$(1-c^2)y'^2 = c^2$$

$$y'^2 = \frac{c^2}{1-c^2}$$

$$y'(x) = \pm\sqrt{\frac{c^2}{1-c^2}}$$ constant

$$y'(x) = a$$

$$\Rightarrow y(x) = ax+b$$ st. line.

satisfying b.c. $y(x_1) = y_1$ and $y(x_2) = y_2$

and we just proved that shortest path is indeed a st. line. What's the exact eqn of this minimizer?

$$y - y_1 = \text{slope}(x - x_1)$$

$$y = \text{slope}(x-x_1) + y_1 \Rightarrow \boxed{y = \frac{y_2-y_1}{x_2-x_1}(x-x_1) + y_1}$$

Note: $E(y) = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$

is actually convex and bd below by zero.

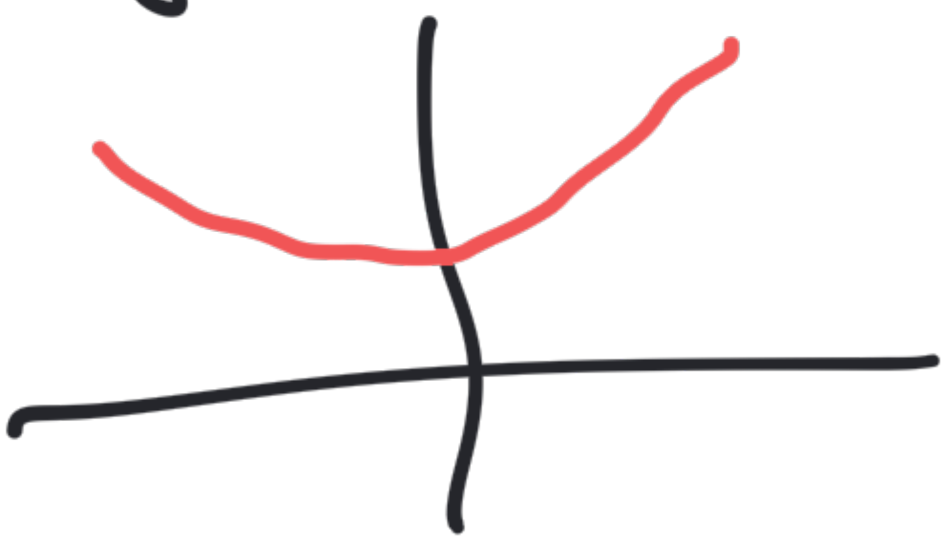
This means that the critical pt that we found: fn $y(x)$ satisfying

$$\nabla_{L^2} E(y) = 0 \qquad \text{(E-L eqn)}$$

is indeed a minimizer.

Convexity of $E(y)$ follows from the convexity of the Calculus fn that's in the integrand:

$$g(s) = \sqrt{1+s^2}$$



Good news

Review from Calc III:

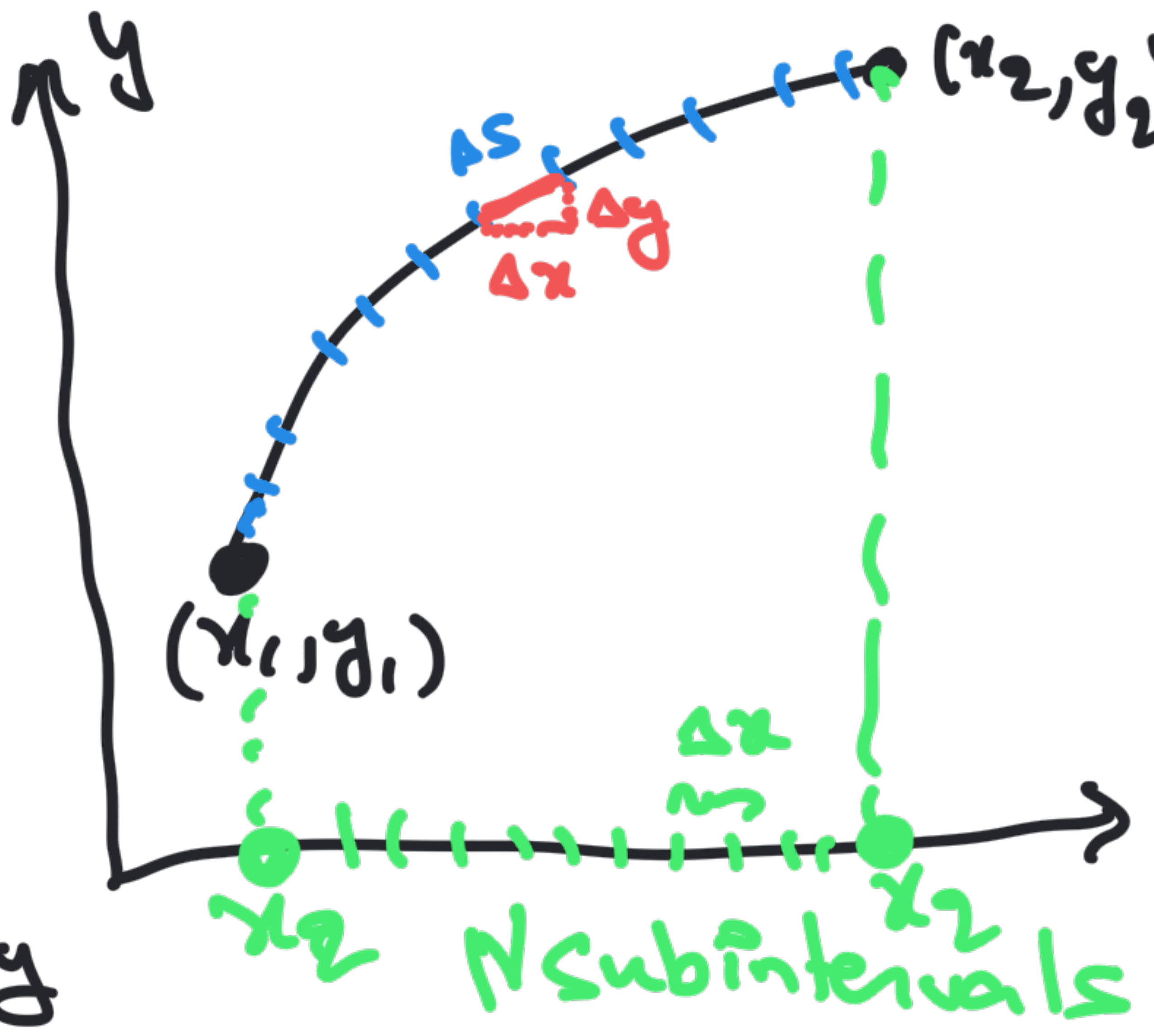
Why is the arc length of a fn between (x_1, y_1) and (x_2, y_2) equal to

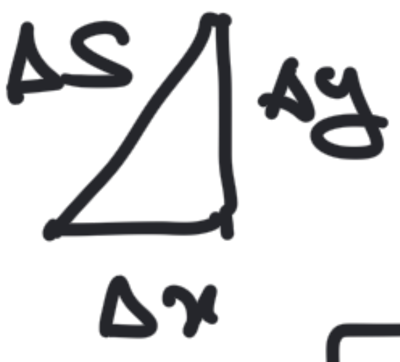
$$\int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}\, dx \quad ??$$

Answer:

At a small enough scale:

Δs as linear not curved





length of $\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 \left(1 + \frac{\Delta y^2}{\Delta x^2}\right)}$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Now length of the arc of $y(x) = \lim_{N \to \infty} \sum_{i=1}^{N} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$

$$\Rightarrow \text{Arc length} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

• Second simplest prb in Calculus of Variations

minimal surface prb

D

think of a soap bubbe maker. Pull out the tip get flat soap film.

Intuitively

fn with $f=0$ on D

with minimal

surface area is is

the flat surface (minimal curvature)

Write a minimization prb (formula for surface area of a fn)

then E-L. eqn : $\underset{\substack{\text{some}\\\text{fn space}}}{\nabla E(u)} = 0$

End up getting

$$\begin{cases} \nabla \cdot \dfrac{\nabla u}{(1+|\nabla u|^2)^{1/2}} = 0 & \text{in } D \\ u = \alpha & \text{on } \partial D \end{cases}$$

Donot solve this PDE. Just set the min. pr[b] up (var'l principle and get to this E-L [eqn])

- Third prb in Calculus of var'ns.
  Isoperimetric prb. Look it up.
  This will be on the midterm.

NEXT WEEK NEW TOPIC