MECH366 : Modeling of Mechatronic Systems

L17 : Performance measures in time domain Step response for second-order systems

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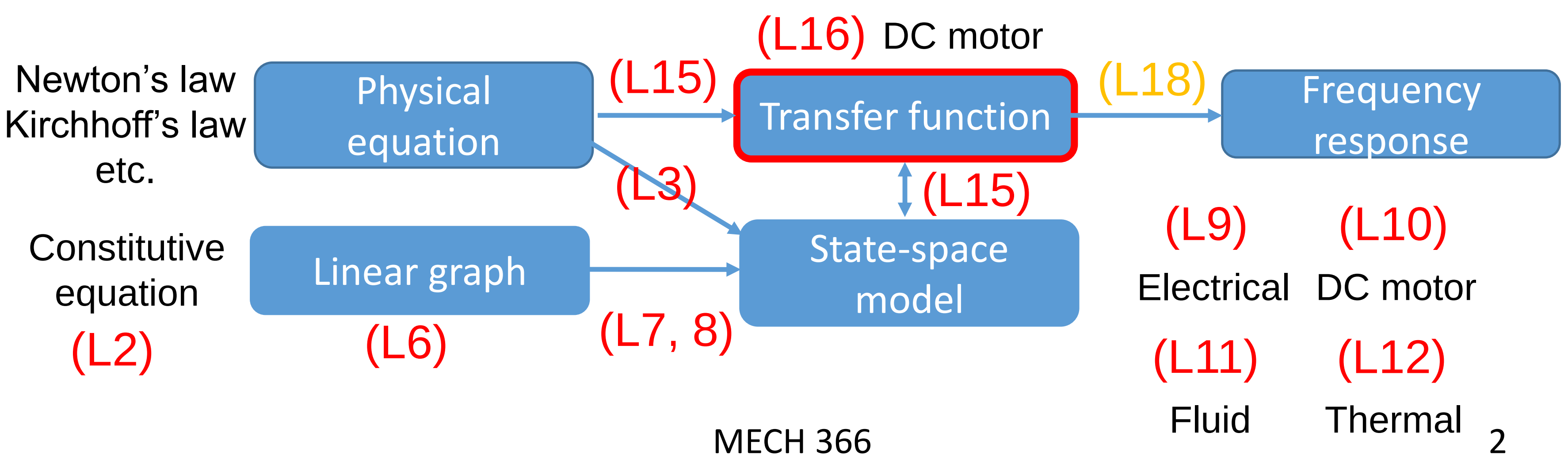
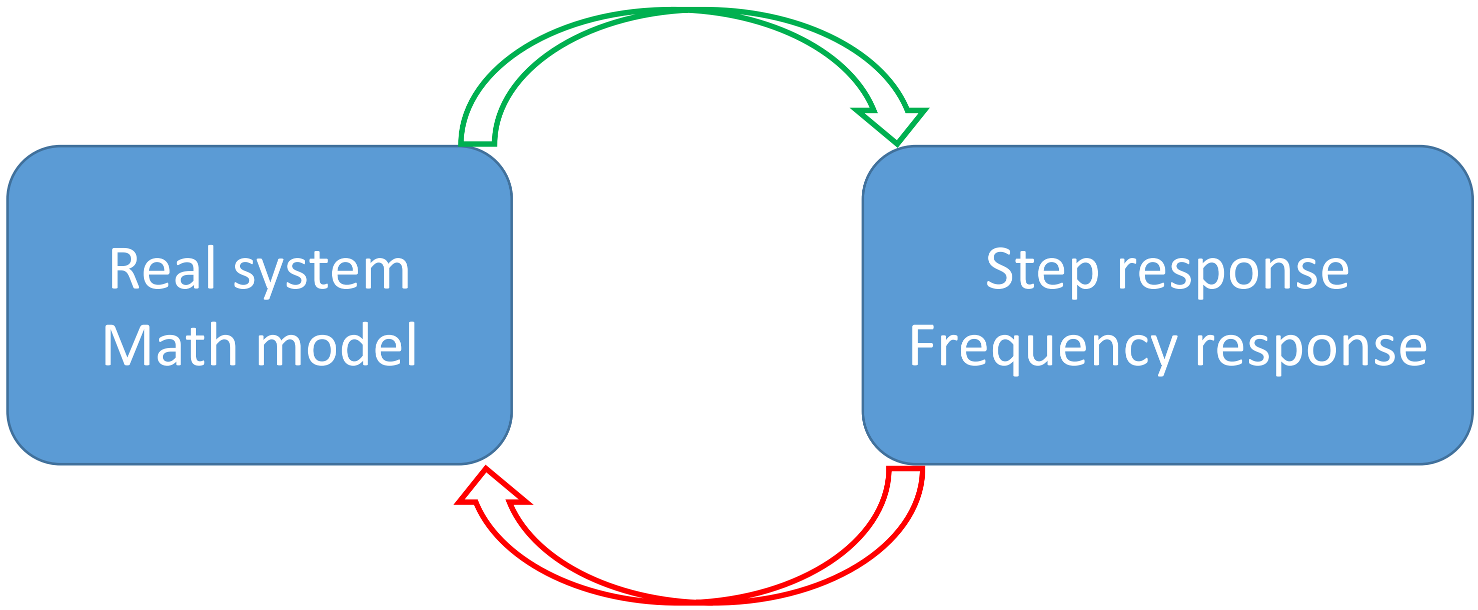
Review and today's topic

- Up to now, we have studied state-space modeling, transfer func., step response for 1st-order systems.
- Today, we will learn performance measures and step response of 2nd-order systems.
- Various models and their relations



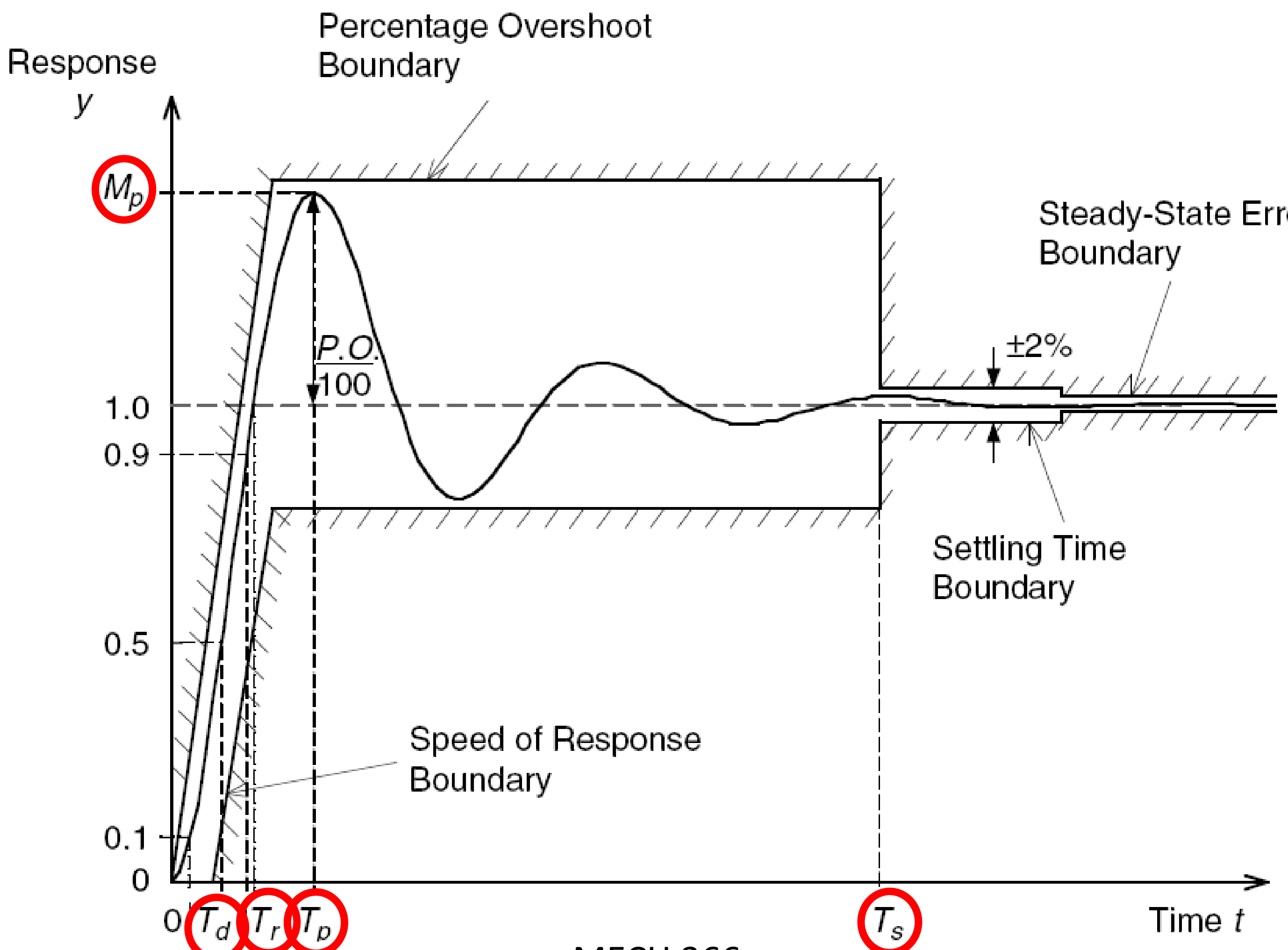
Analysis and modeling are relevant!

Analysis (in time/frequency domain)



Modeling ← (Course title)
System identification
Parameter estimation

Performance measures in time domain (for unit step input)



Performance measures

Transient measures

T_r: *Rise time*: time between 10% and 90% of SS value

T_d: *Delay time*: time to reach 50% of SS value

★ T_s: *Settling time*: time to settle down within 2% of SS value

★ T_p: *Peak time*: time to reach the maximum value

M_p: *Peak value*: the maximum value

★ PO: *Percent overshoot*: $\mathrm{PO} = 100\,(Mp - 1)\%$

Steady-state measure

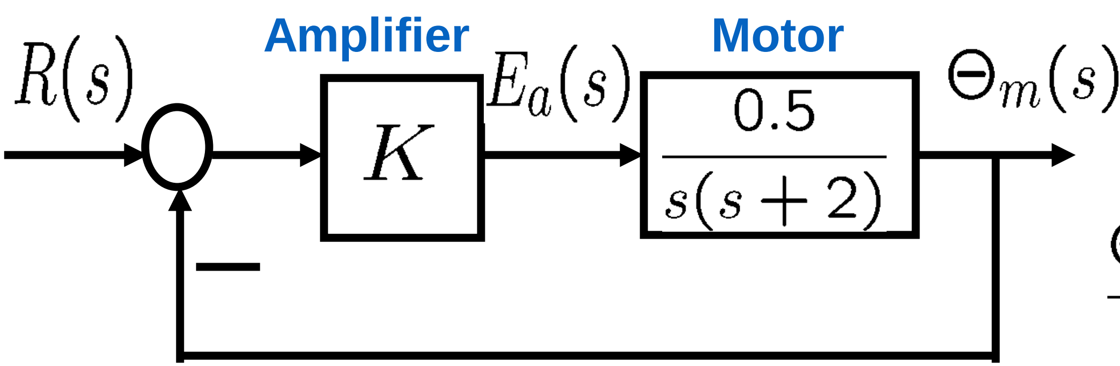
★ *Steady-state error*: deviation of SS value from desired value

Second-order systems

- A standard form of the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \begin{cases} \zeta : & \text{damping ratio} \\ \omega_n : & \text{undamped natural frequency} \end{cases}$$

- DC motor position control example



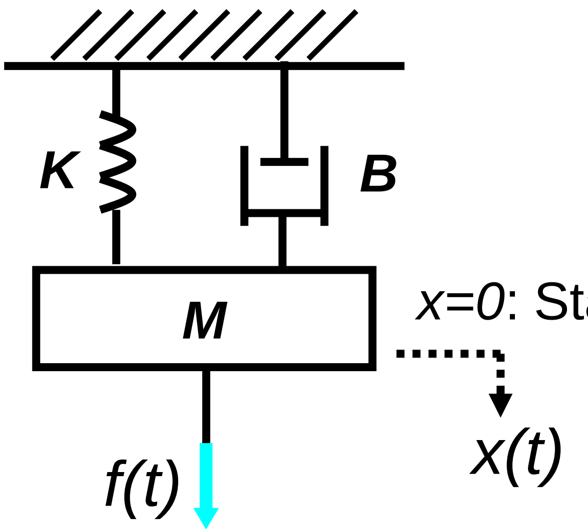
Closed-loop TF

$$\frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2 + 2s + 0.5K}$$

Second-order system

- Mass spring damper system

$$\begin{aligned}\frac{X(s)}{F(s)} &= \frac{1}{Ms^2+Bs+K} \\ &= \frac{1}{K}\cdot\frac{1}{(M/K)s^2+(B/K)s+1} \\ &= \frac{1}{K}\cdot\frac{(K/M)}{s^2+(B/M)s+(K/M)} \\ &=: \frac{1}{K}\cdot\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\end{aligned}$$



x=0: Static equilibrium

$$\zeta = \frac{B}{2\sqrt{KM}}, \quad \omega_n = \sqrt{\frac{K}{M}}$$

Main messages from now on

- To second-order transfer functions,

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

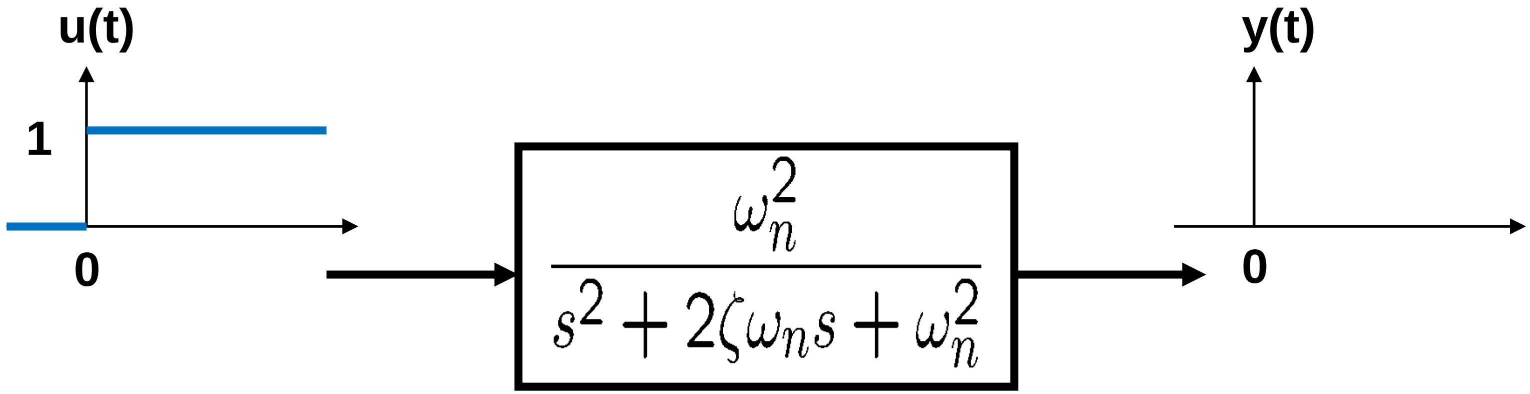
  when we apply the unit step input,
  - Steady state value is **DC gain** $G(0)$
  - Transient behavior is characterized by **poles** of $G(s)$
    - **Real part** determines settling time **Ts=4/|Re|**.
    - **Imaginary part** determines peak time **Tp=π/|Im|**.
    - **Angle of pole location** determines overshoot.

(Slide 15)

- This is true for first-order transfer functions too.

Step response of 2^{nd}-order system

- Input a unit step function to a 2nd-order system. What is the output?



DC gain

$$\lim_{t\to\infty} y(t) = G(0) = 1 \text{ if } G \text{ is stable}$$

All poles are in open LHP

Step response of 2nd-order system for various damping ratios

- Undamped

$$\zeta = 0$$

- Underdamped

Today's topic

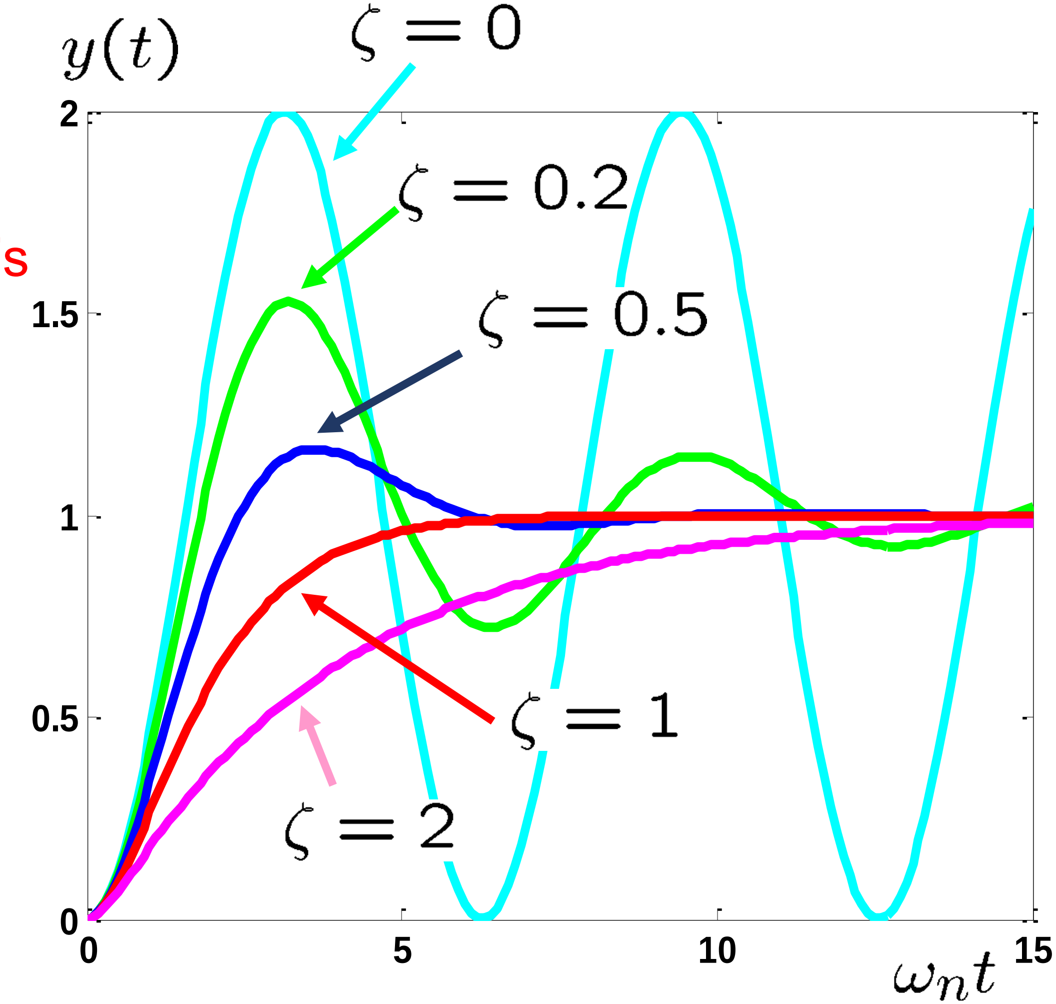
$$0 < \zeta < 1$$

- Critically damped

$$\zeta = 1$$

- Overdamped

$$\zeta > 1$$



Step response of 2nd-order system

Underdamped case $0 < \zeta < 1$

- Math expression of *y(t)* for underdamped case

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

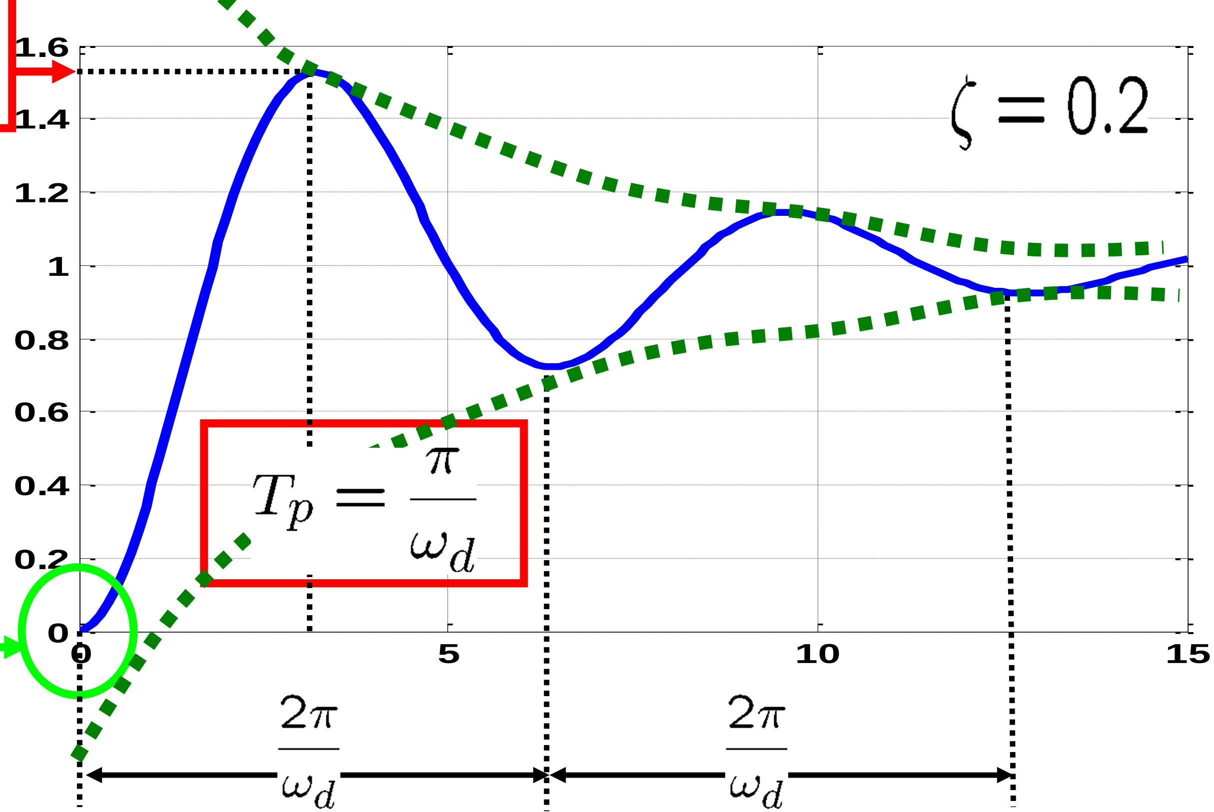
$$\xrightarrow{\mathcal{L}^{-1}} \quad \boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)}$$

Damped natural frequency → $\omega_d := \omega_n\sqrt{1-\zeta^2}$

Peak value/time
Underdamped case

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

$$\boxed{M_p = 1 + e^{-\frac{\zeta\pi}{1-\zeta^2}}}$$

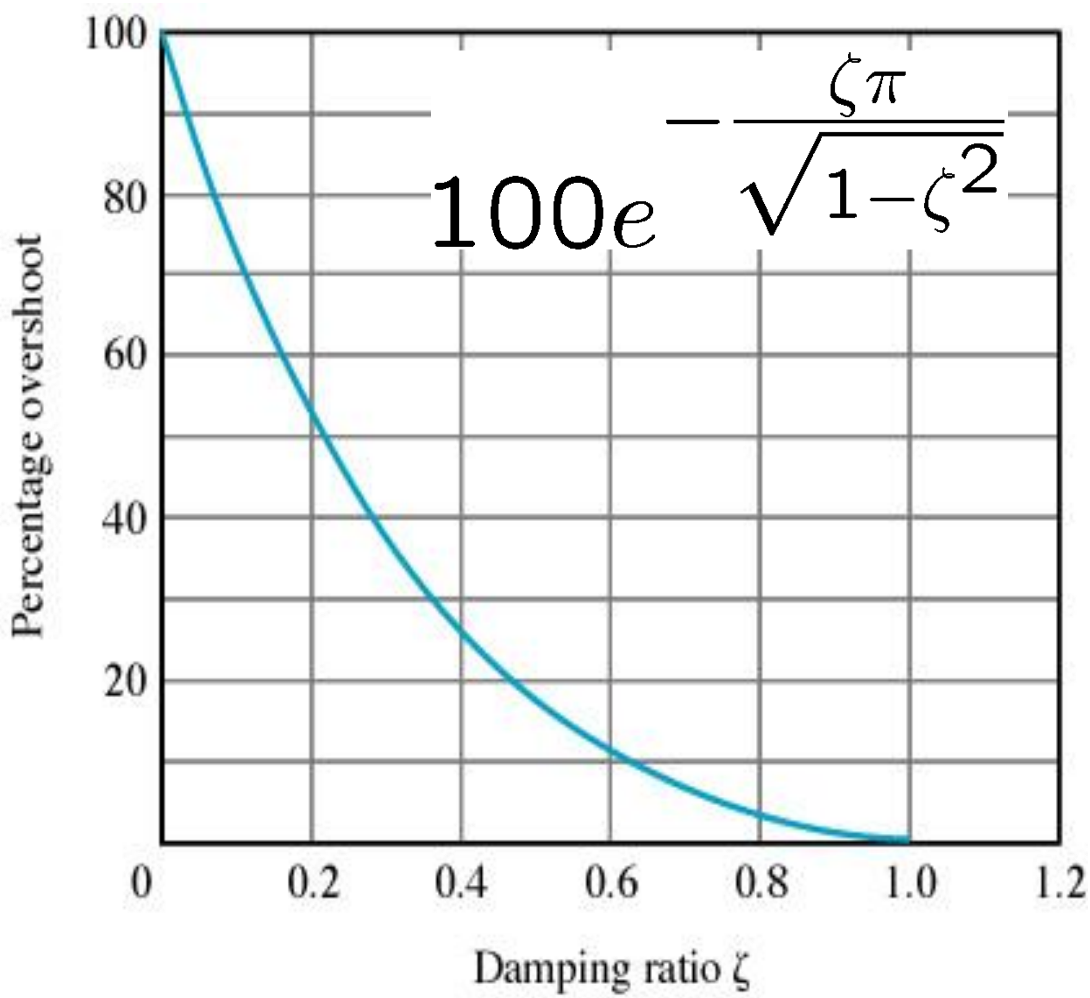


$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}}$$

Step response properties of underdamped 2nd-order system in terms of ζ and ω_n

Performance measure	Formula
Rise time	$T_r = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$
Peak time	$T_p = \frac{\pi}{\omega_d}$
Peak value	$M_p = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$
Percent overshoot	$PO = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$
Time constant	$\tau = \frac{1}{\zeta\omega_n}$
Settling time	$T_s \approx 4\tau = \frac{4}{\zeta\omega_n}$

P.O. vs. damping ratio

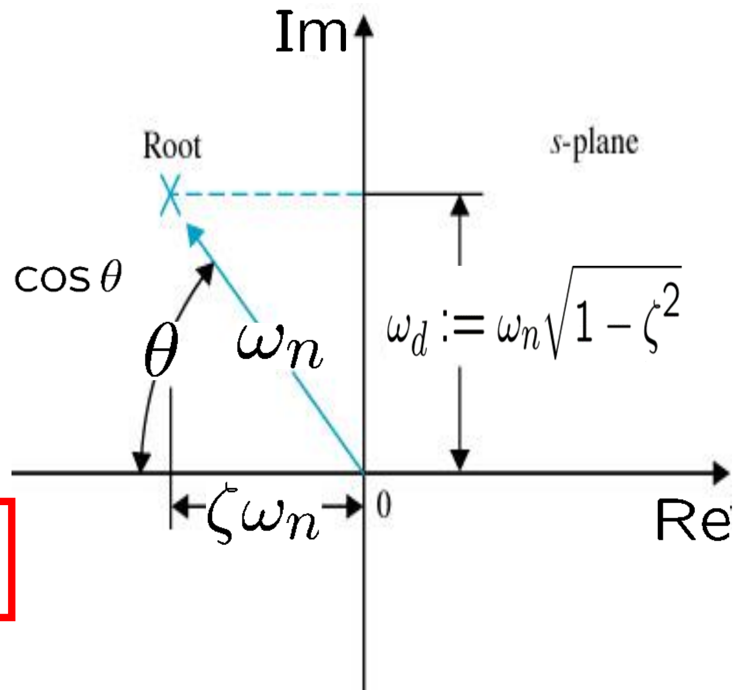


$$100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Step response properties of underdamped 2^{nd}-order system in terms of pole locations

Poles ($0<\zeta<1$) $s = -\zeta\omega_n \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$

$\zeta = \cos\theta$

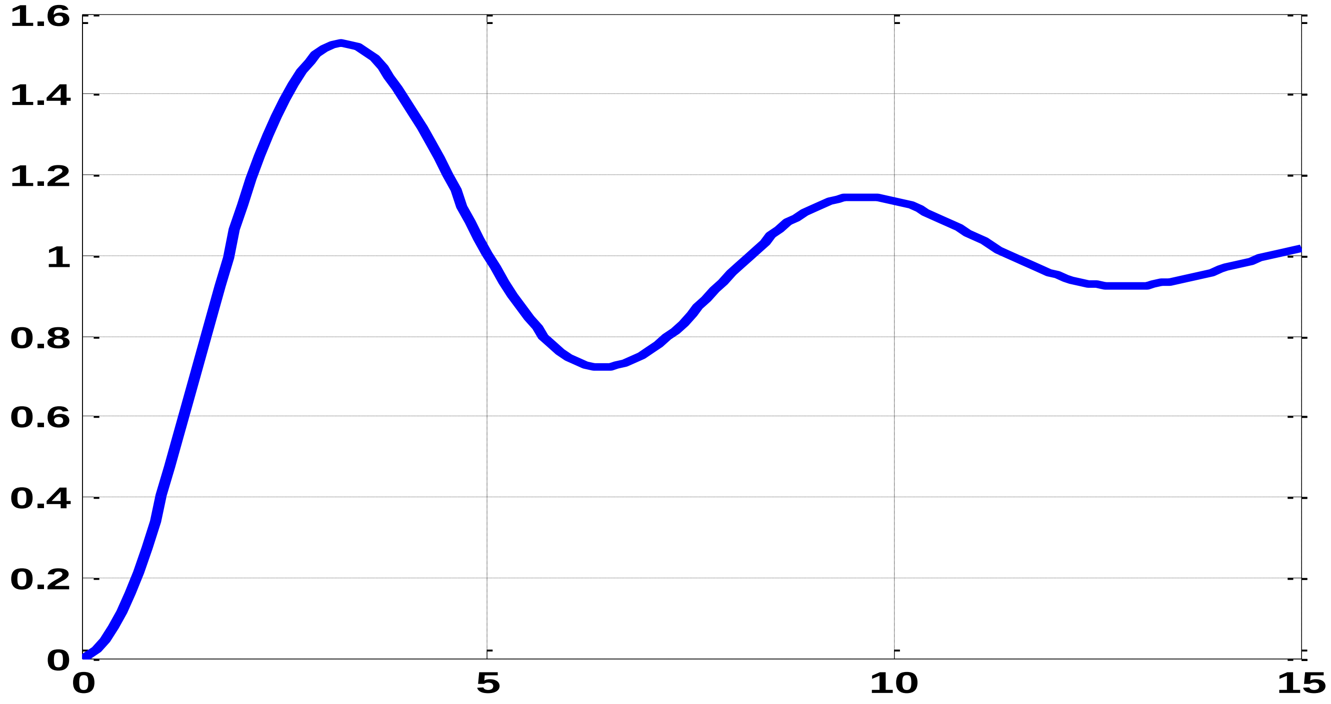


Performance measure	Formula	
Rise time	$T_r = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$	
Peak time	$T_p = \frac{\pi}{\omega_d}$	$\boxed{\frac{\pi}{\lvert\mathrm{Im}\rvert}}$
Peak value	$M_p = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	
Percent overshoot	$PO = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$	
Time constant	$\tau = \frac{1}{\zeta\omega_n}$	
Settling time	$T_s \approx 4\tau = \frac{4}{\zeta\omega_n}$	$\boxed{\frac{4}{\lvert\mathrm{Re}\rvert}}$

$\boxed{\theta \quad \text{determines} \quad PO}$

System identification

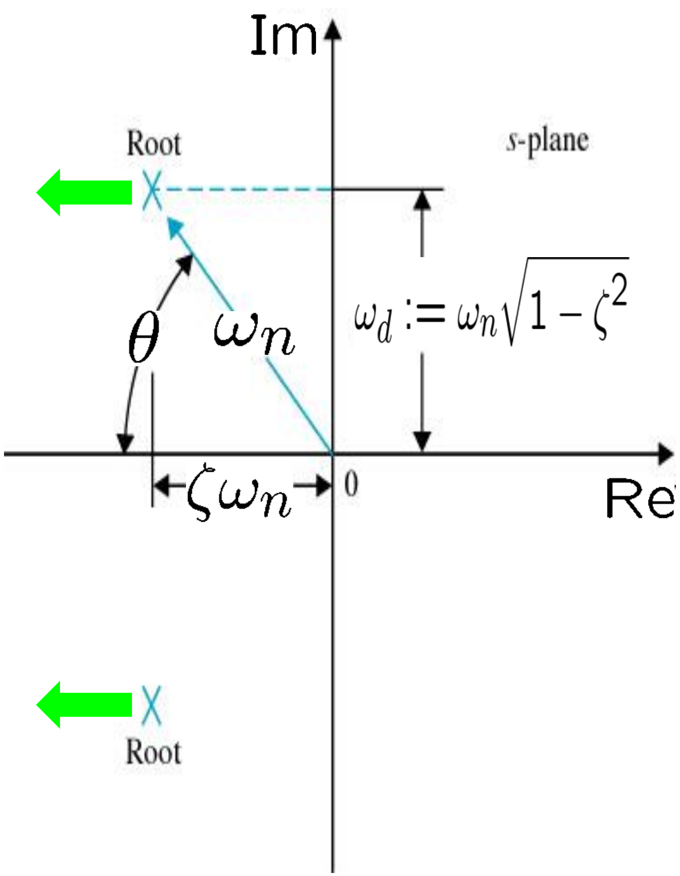
- Suppose we got the following unit step response for a system. How to get a transfer function?

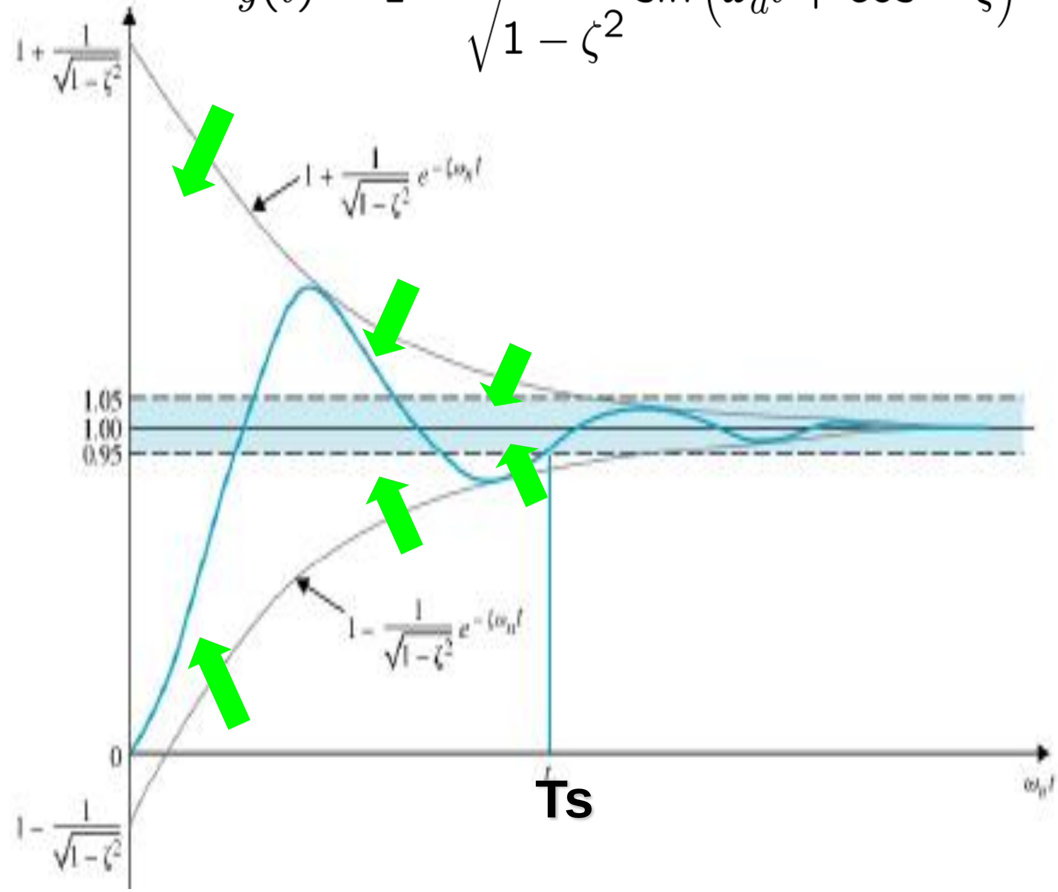


Influence of real part of poles

Settling time *Ts* decreases.

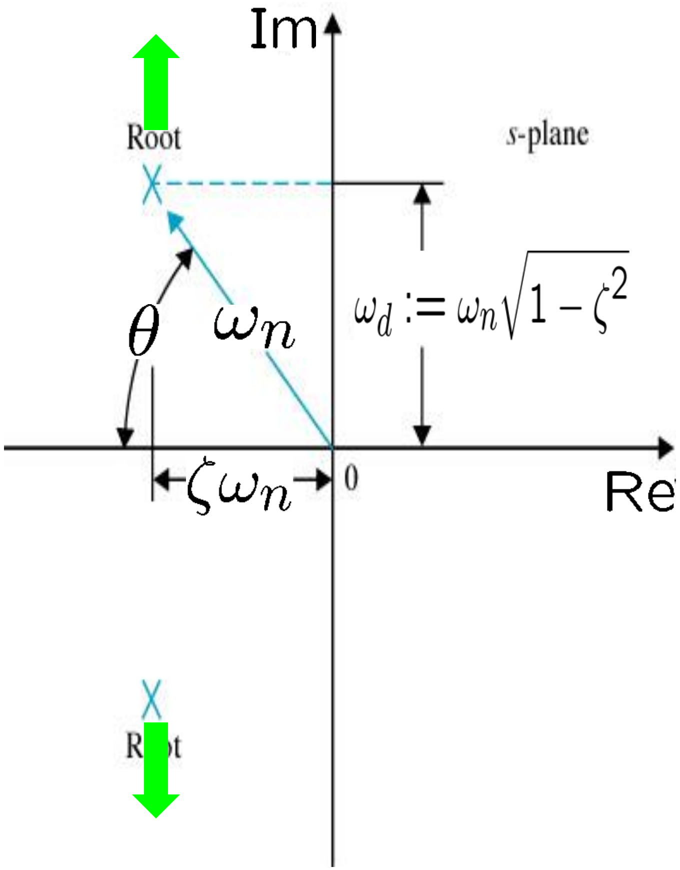
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

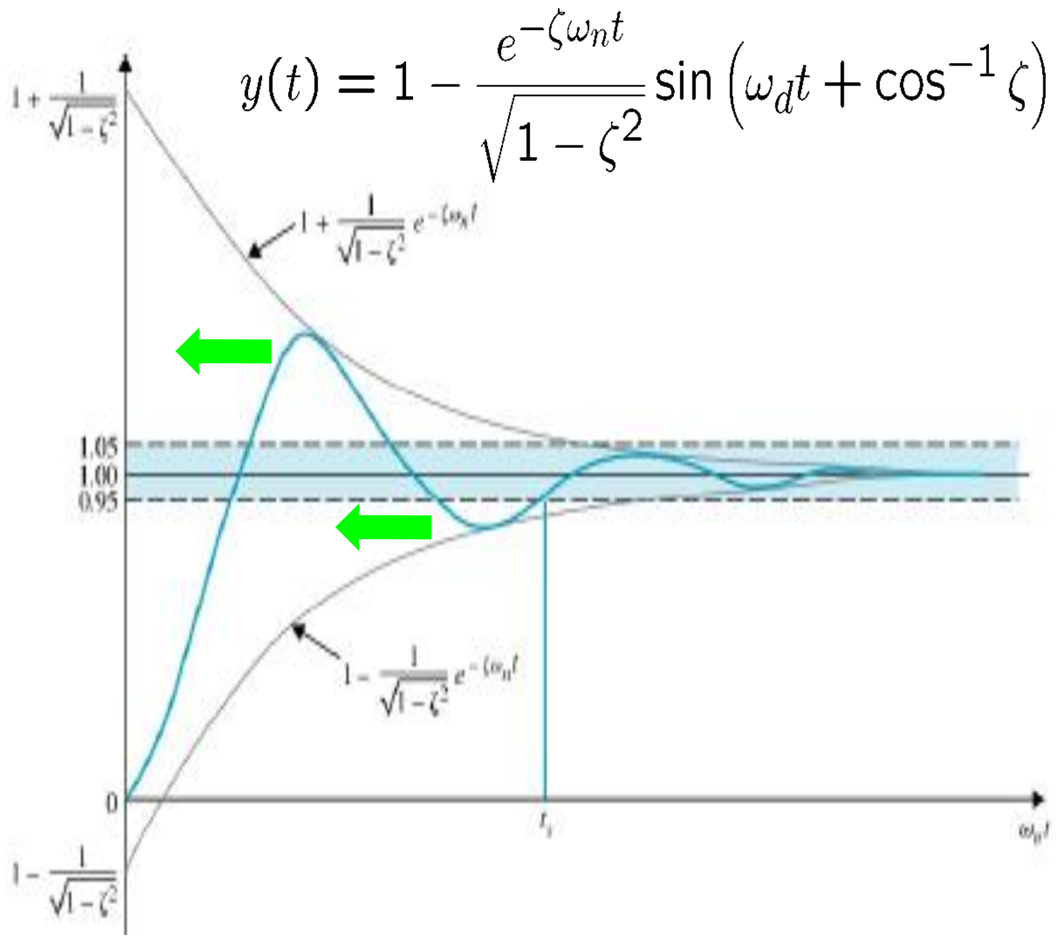




Influence of imag. part of poles

Oscillation frequency ω_d increases

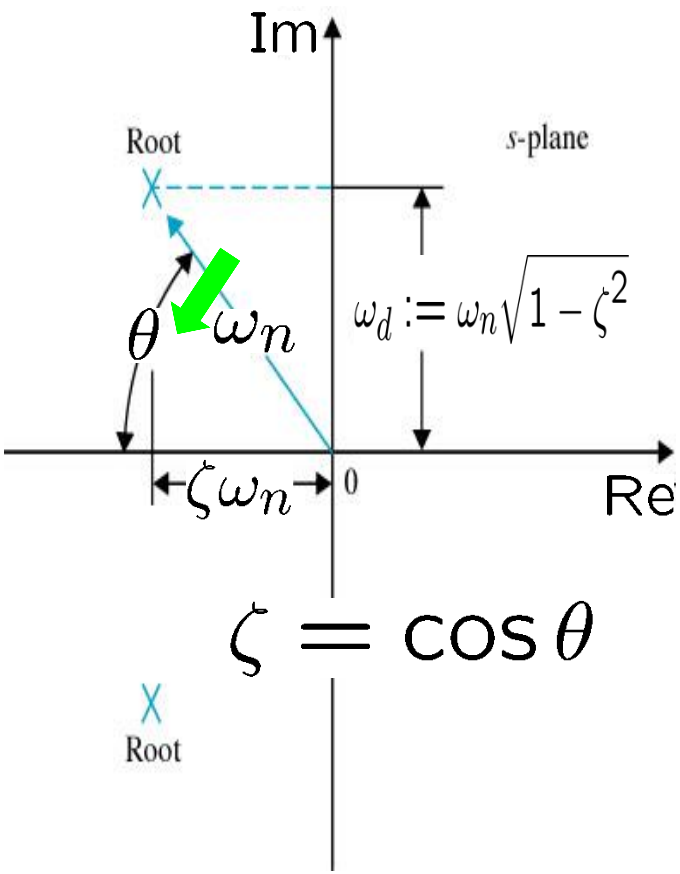




Peak time T_p decreases

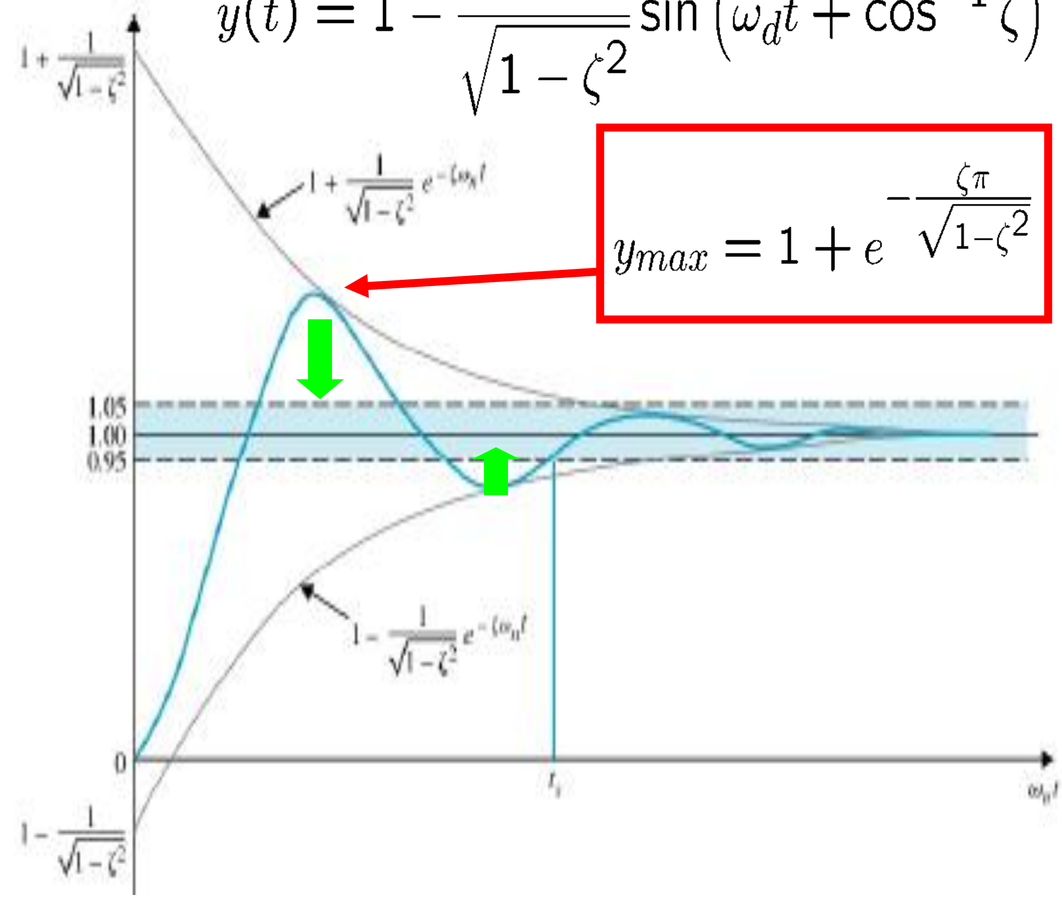
Influence of angle of poles

Over/under-shoot decreases.



$$\zeta = \cos\theta$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

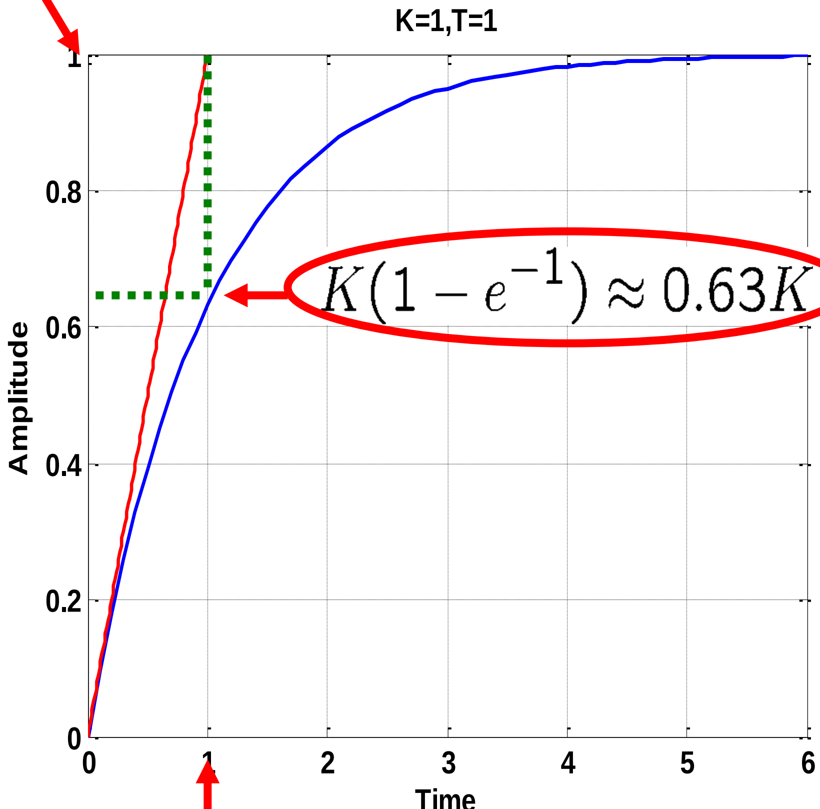


$$\boxed{y_{max} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

Step response of first-order systems (review)

$$G(s) = \frac{K}{Ts+1}$$

K *DC gain* $\lim_{t\to\infty} y(t) = G(0) = K$



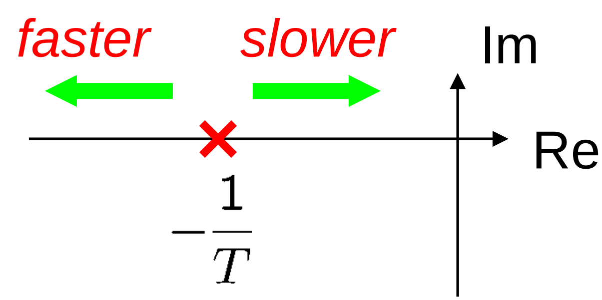
$$K(1-e^{-1}) \approx 0.63K$$

T *Time constant*

2% settling time

$$T_s = \frac{4}{|\mathrm{Re}|} = 4T$$

× Pole



Summary

- Performance measures in time domain
- Step response of second order systems: Underdamped case (Overdamped case: later)
- Response characterization by pole locations
- Next, frequency response
- Project: Fridays Nov 15, 22, 29 (presentation)
- Homework 6: Due Nov 12 (Tuesday), 6pm
- Lab 4 report: Due Nov 25 (Monday), 6pm