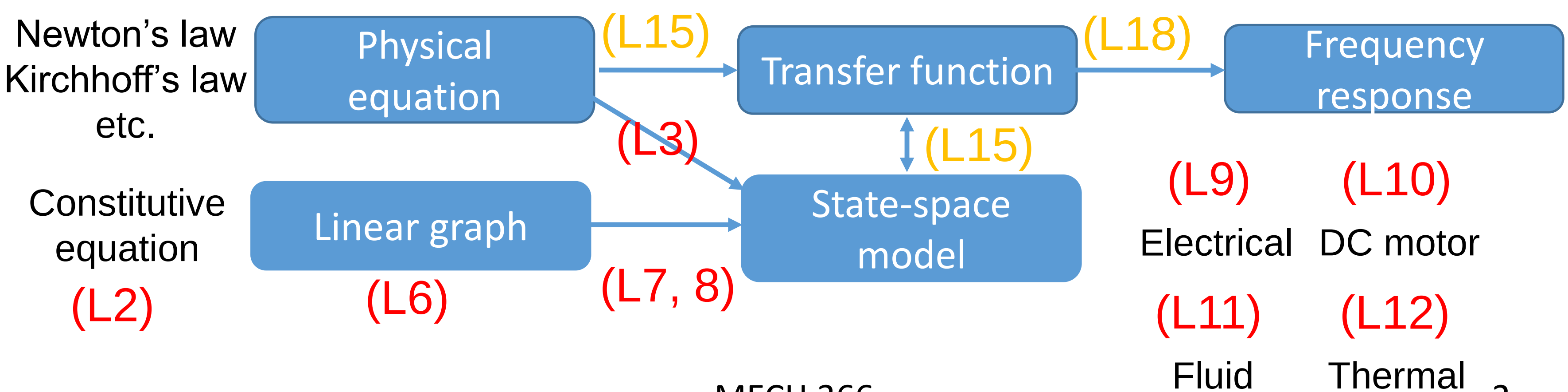
MECH366 : Modeling of Mechatronic Systems

L14 : ODE solution via Laplace transform

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Review and today's topic

- Up to now, we have studied state-space modeling based on linear graphs.
- From now on, we will learn another type of models, i.e. transfer functions, based on Laplace transform.
- Various models and their relations



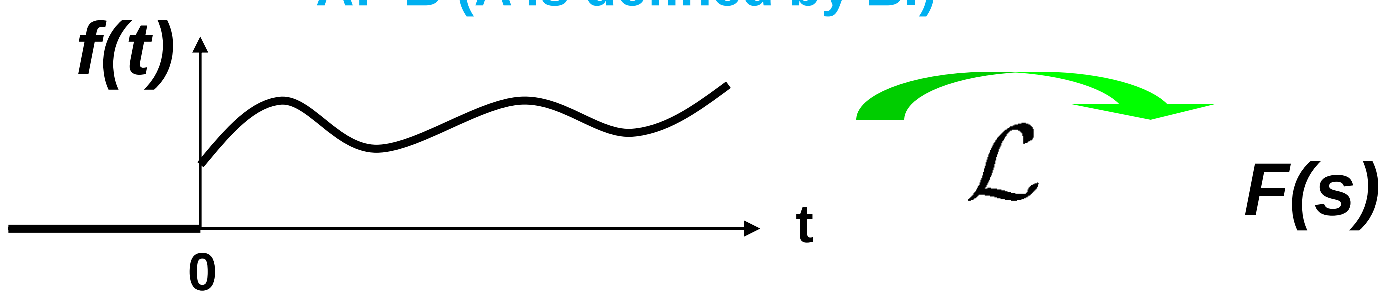
Laplace transform (review)

- Definition: For a function *f(t)* (*f(t)=0* for *t<0*),

$$\boxed{F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt}$$

A:=B (A is defined by B.)

(*s*: complex variable)



- We denote Laplace transform of *f(t)* by *F(s)*.

Laplace transform table (review)

$f(t)$		$F(s)$
$\delta(t)$		1
$u(t)$	$\mathcal{L}$ $\rightarrow$	$\frac{1}{s}$
$tu(t)$		$\frac{1}{s^2}$
$t^n u(t)$	$\mathcal{L}^{-1}$ $\leftarrow$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$		$\frac{1}{s+a}$
$\sin \omega t \cdot u(t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t \cdot u(t)$		$\frac{s}{s^2+\omega^2}$
$te^{-at}u(t)$		$\frac{1}{(s+a)^2}$

Inverse Laplace Transform

(u(t) is often omitted.)

Advantages of *s*-domain

- We can transform an ordinary differential equation into an algebraic equation which is easy to solve.

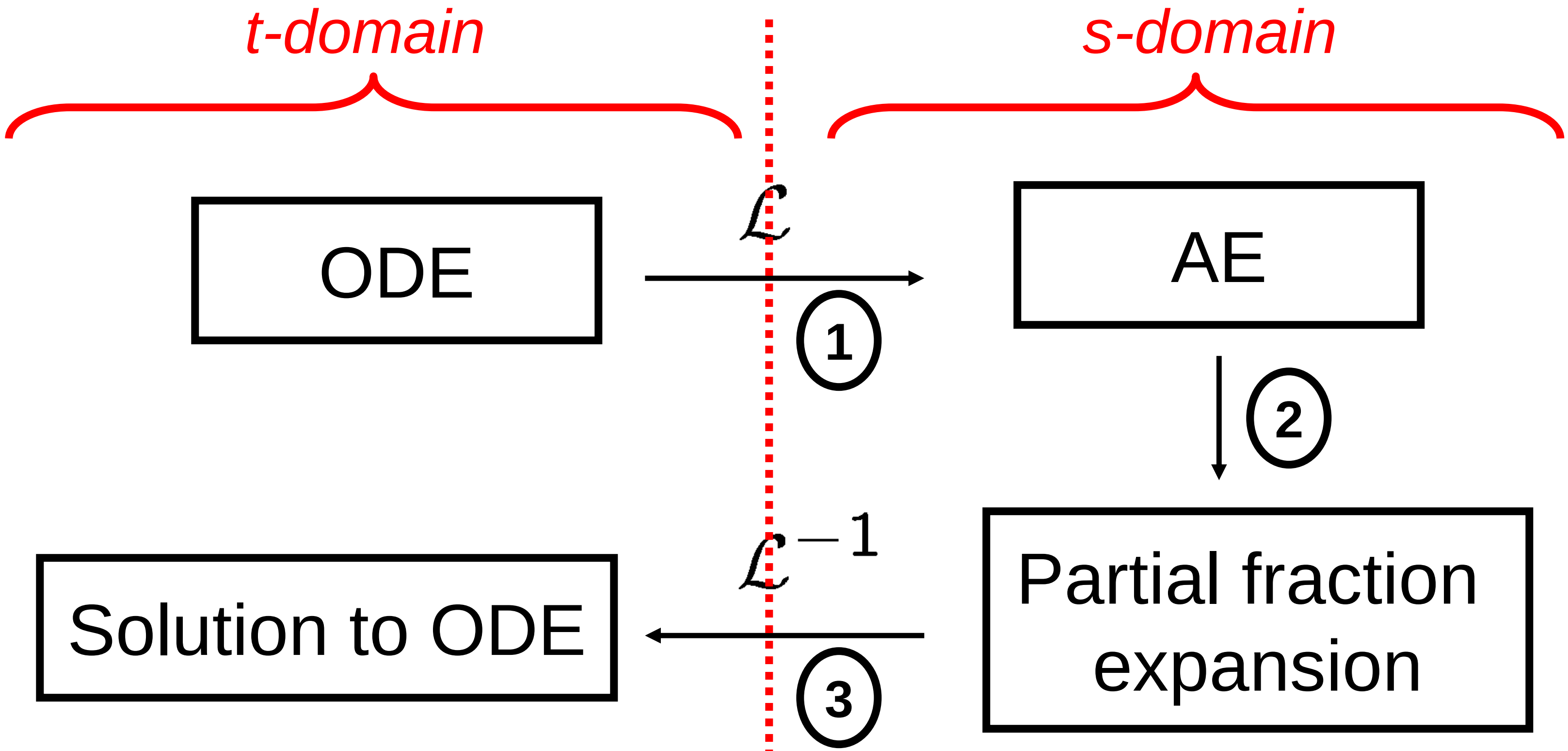
  (Today's class)
- It is easy to analyze and design interconnected (series, parallel, feedback etc.) systems.

  (In classical control such as MECH467, next slide)
- Frequency domain information of signals can be dealt with.

  (Frequency responses)

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Example 1 (distinct roots)

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t) = 5u(t),\ y(0) = -1,\ y'(0) = 2$$

1. Laplace transform

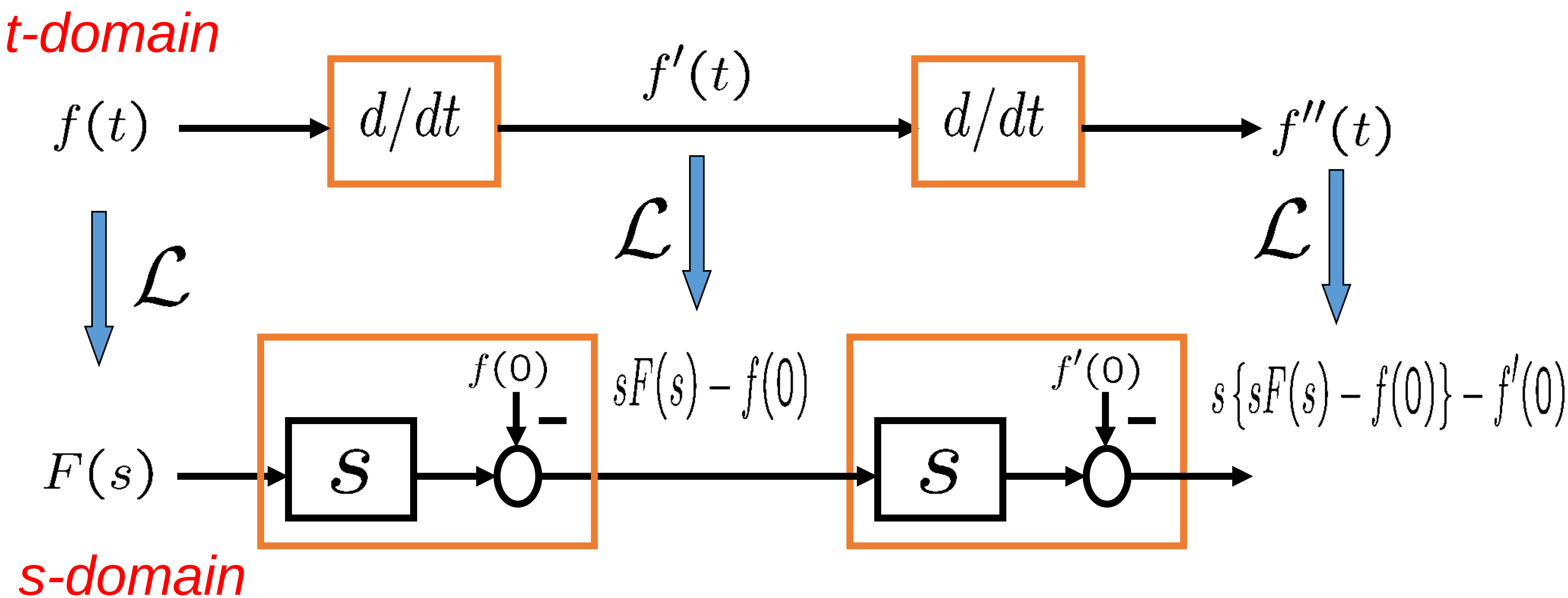
$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+3\underbrace{\{sY(s) - y(0)\}}_{\mathcal{L}\left\{y'(t)\right\}}+2Y(s) = \frac{5}{s}$$

$$\Rightarrow \quad Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

distinct roots

Properties of Laplace transform
Differentiation (review)

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)}$$



Example 1 (cont'd)

unknowns

2. Partial fraction expansion

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by *s(s+1)(s+2)* :

$$-s^2 - s + 5 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad -1 = A + B + C \\ s^1\text{-term} &: \quad -1 = 3A + 2B + C \\ s^0\text{-term} &: \quad 5 = 2A \end{aligned} \Rightarrow \begin{cases} A = \dfrac{5}{2} \\ B = -5 \\ C = \dfrac{3}{2} \end{cases}$$

Example 1 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

(You may omit u(t).)

$$\Rightarrow \quad y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t} \right) u(t)$$

If we are interested in only the final value of *y(t)*, apply the Final Value Theorem, without explicitly computing *y(t)*:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Example 2 (repeated roots)

ODE with zero initial conditions (ICs)

$$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+4y(t)=2\delta(t),\ y(0)=y'(0)=y''(0)=0$$

1. Laplace transform

$$\begin{aligned}&s^3Y(s)-s^2y(0)-sy'(0)-y''(0) && \longleftarrow \mathcal{L}\left\{y'''(t)\right\}\\ &+5\left\{s^2Y(s)-sy(0)-y'(0)\right\} && \longleftarrow 5\mathcal{L}\left\{y''(t)\right\}\\ &+8\left\{sY(s)-y(0)\right\}+4Y(s)\\ &=2\end{aligned}$$

$$\Rightarrow \quad Y(s)=\frac{2}{(s+1)(s+2)^2}$$

Repeated roots

Example 2 (cont'd)

unknowns

2. Partial fraction expansion

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiply both sides by $(s+1)(s+2)^2$

$$2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &\quad : \quad 0 = A + B \\ s^1\text{-term} &\quad : \quad 0 = 4A + 3B + C \\ s^0\text{-term} &\quad : \quad 2 = 4A + 2B + C \end{aligned} \quad \Rightarrow \quad \begin{cases} A = 2 \\ B = -2 \\ C = -2 \end{cases}$$

Example 2 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

(u(t) omitted.)

$$\Rightarrow \quad y(t) = \underbrace{2}_{A} e^{-t} + \underbrace{(-2)}_{B} e^{-2t} + \underbrace{(-2)}_{C} te^{-2t}$$

If we are interested in only the final value of *y(t)*, apply the Final Value Theorem, without explicitly computing *y(t)*:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{2s}{(s+1)(s+2)^2} = 0$$

Properties of Laplace transform
Frequency shift theorem (review)

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$

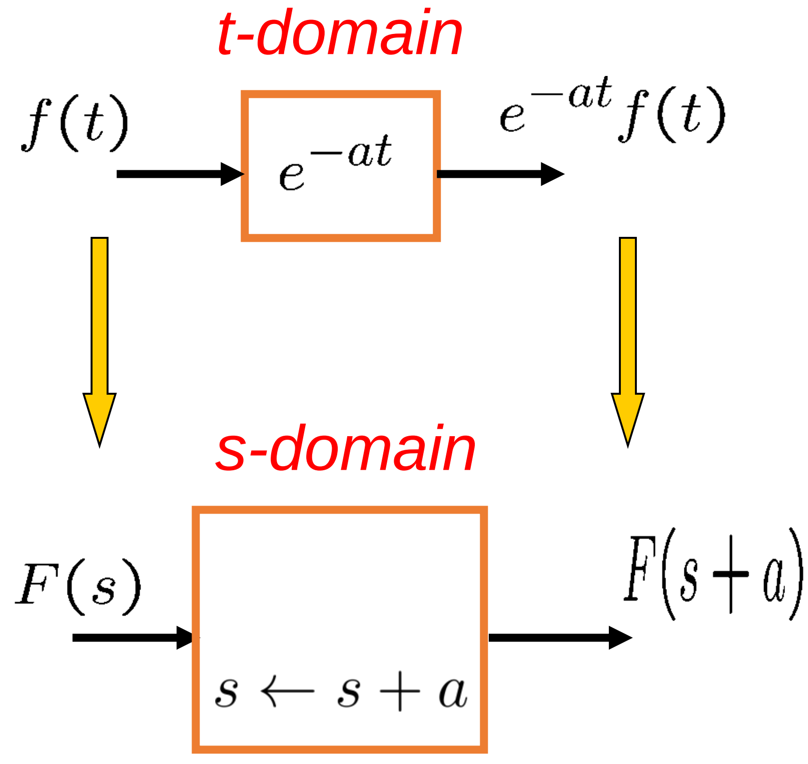
Proof.

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^\infty e^{-at}f(t)e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

Ex.

$$\mathcal{L}\left\{te^{-2t}\right\} = \frac{1}{(s+2)^2}$$



Example 3 (complex roots)

ODE with zero initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t)=3u(t),\ \ y(0)=0,\ \ y'(0)=0$$

1. Laplace transform

$$s^2Y(s)+2sY(s)+5Y(s)=\frac{3}{s}$$

$$\Rightarrow\ Y(s)=\frac{3}{s(s^2+2s+5)}$$

Complex roots

Example 3 (cont'd)

unknowns

2. Partial fraction expansion

$$Y(s) = \frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Multiply both sides by $s(s^2+2s+5)$

$$3 = A(s^2+2s+5) + s(Bs+C)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &\quad : \quad 0 = A + B \\ s^1\text{-term} &\quad : \quad 0 = 2A + C \\ s^0\text{-term} &\quad : \quad 3 = 5A \end{aligned} \Rightarrow \begin{cases} A = \frac{3}{5} \\ B = -\frac{3}{5} \\ C = -\frac{6}{5} \end{cases}$$

Example 3 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{B(s+1)+C-B}{(s+1)^2+4}\right\} \\
&= B\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \frac{C-B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\
&= Be^{-t}\cos 2t + \frac{C-B}{2}e^{-t}\sin 2t \\
\mathcal{L}^{-1}\{Y(s)\} &= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)
\end{aligned}$$

Laplace transform table

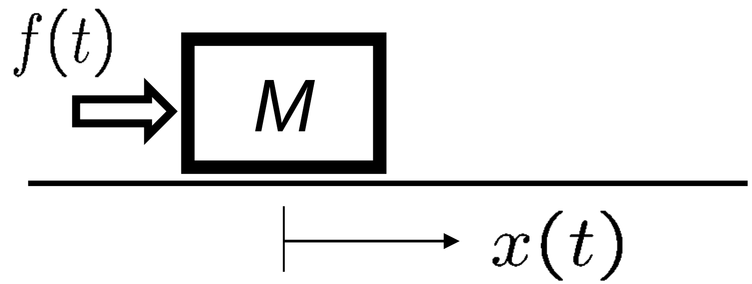
$f(t)$	$F(s)$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$e^{-\alpha t}\sin\omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-\alpha t}\cos\omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Frequency shift theorem

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$

Example: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



Want to know position *x(t)* when force *f(t)* is applied.

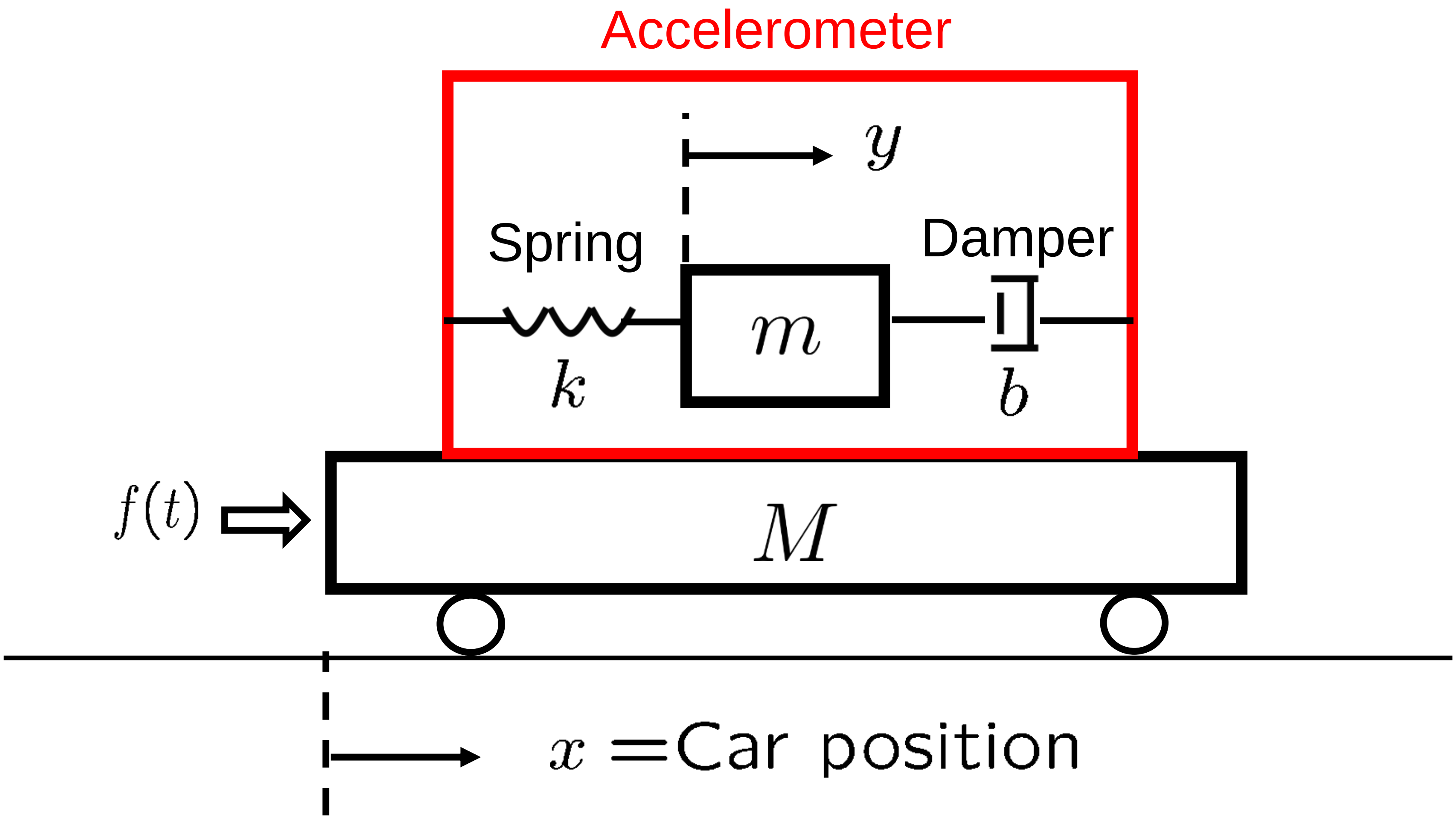
$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

$$\Rightarrow \quad X(s) = \underbrace{\frac{1}{Ms^2}F(s)} + \underbrace{\frac{x(0)}{s} + \frac{x'(0)}{s^2}}$$

(Total response) = (Forced response) + (IC response)

$$\Rightarrow \quad x(t) = \overbrace{\mathcal{L}^{-1}\left\{\frac{1}{Ms^2}F(s)\right\}} + \overbrace{x(0)u(t) + x'(0)tu(t)}$$

Ex: Mechanical accelerometer



Ex: Accelerometer (cont'd)

- Want to know how *y(t)* moves when unit step *f(t)* is applied with zero ICs.
- By Newton's 2nd law

$$\begin{cases} m\dfrac{d^2}{dt^2}(x(t)+y(t)) = -b\dfrac{dy(t)}{dt} - ky(t) \\ M\dfrac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

$$\Rightarrow \quad m\ddot{y}(t) + b\dot{y}(t) + ky(t) = -\frac{m}{M}f(t)$$

$$\xrightarrow[\mathcal{L}]{} \quad Y(s) = -\frac{1}{M}\cdot\frac{1}{s^2+(b/m)s+(k/m)}\cdot\frac{1}{s}$$

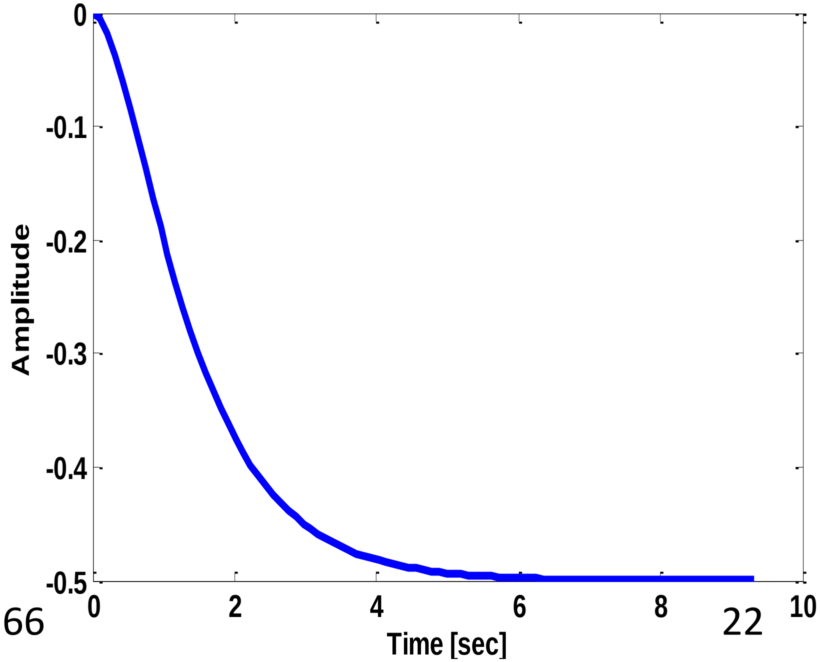
Ex: Accelerometer (cont'd)

- Suppose that *b/m=3*, *k/m=2* and *M=1*.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} = -\frac{1}{2s}+\frac{1}{s+1}-\frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$y(t) = \left(-\frac{1}{2}+e^{-t}-\frac{1}{2}e^{-2t}\right)u(t)$$



Summary

- Solution to ODE via Laplace transform
  1. Laplace transform
  2. Partial fraction expansion
  3. Inverse Laplace transform
- Next,
  - Transfer function
- Homework 5: Due Nov 4 (Monday), 3pm
- Lab 3 report: Due Nov 1 (Friday), 6pm
- Lab 4: Nov 1 (Friday)