University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Final exam: SOLUTIONS

Examiner: Dr. Ryozo Nagamune
December 10 (Monday), 2018, 3:30-6pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: one-page letter-size hand-written cheat-sheet (both sides).
- Not-allowed: laptop, calculator.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 50 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

If you finish early ...

- If you would like to leave the room **before 5:50pm**, **raise your hand with this booklet**, and **wait at your seat** until an invigilator comes to you and collects your exam booklet.

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		10
2		10
3		10
4		10
5		10
Total		50

1. Answer the following questions concisely. (2pt each)

(a) Explain what the "model validation" is.

Write your answer here.

Model validation is the process to check if your obtained model gives similar input-output relations to those of the real system to be modeled.

(b) Using the constitutive relation for the thermal capacitor, prove that the thermal energy is $C_t T$, where C_t is the thermal capacitance and T is the temperature.

Write your answer here.

The constitutive relation for the thermal capacitor is

$$C_t \frac{dT}{dt} = Q,$$

where Q is the heat transfer rate (power). The thermal energy can be obtained by integrating the power as

$$\int Q dt = \int C_t \frac{dT}{dt} dt = C_t \int dT = C_t T.$$

(c) Obtain the Laplace transform of the function $y(t) = e^{-t+1}u(t-2)$. Here, $u(t)$ is the unit step function. (Hint: No complicated calculations are necessary.)

Write your answer here.

$$\mathcal{L}\left\{e^{-t+1}u(t-2)\right\} = \mathcal{L}\left\{e^{-(t-2)-1}u(t-2)\right\} = e^{-1}\mathcal{L}\left\{e^{-(t-2)}u(t-2)\right\} = e^{-1} \cdot \frac{e^{-2s}}{s+1}$$

(d) Is the following statement true or false? Motivate your answer (i.e., 'true' or 'false') properly.

"For an s-domain function $Y(s) = \dfrac{-1}{s(s-1)}$, its corresponding time-domain function $y(t)$ will converge to 1 due to the Final Value Theorem."

Write your answer here.

Since $sY(s) = \dfrac{-1}{s-1}$ has a pole $s = 1$ which is not in the open left-half plane, the Final Value Theorem does not apply.
In fact, in this case,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = 1 - e^{t},$$

and the signal $y(t)$ will diverge.

(e) Suppose that a state-space model is represented as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

where

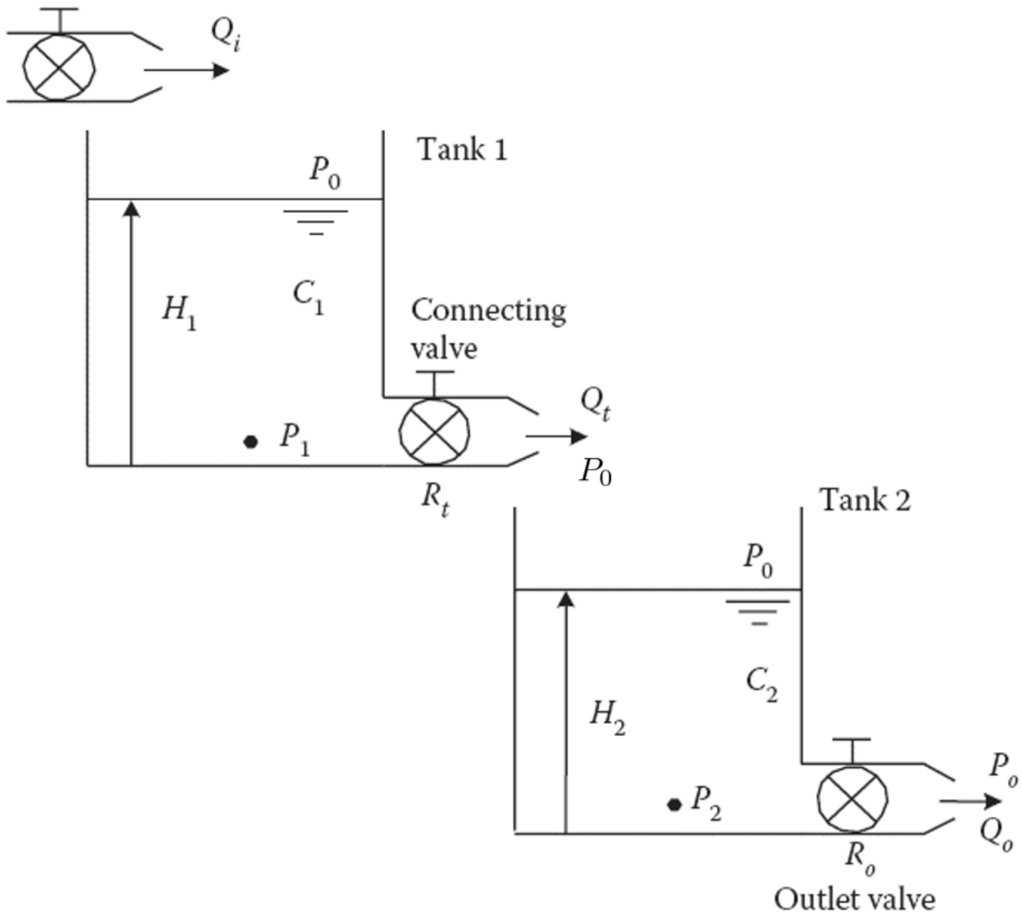
- the number of states x is four (4),
- the number of inputs u is three (3), and
- the number of outputs y is two (2).

What are the sizes of the matrices A, B, C and D?

Write your answer here.

$$\begin{aligned} A &: 4 \times 4 \text{ (4-rows and 4-columns)} \\ B &: 4 \times 3 \text{ (4-rows and 3-columns)} \\ C &: 2 \times 4 \text{ (2-rows and 4-columns)} \\ D &: 2 \times 3 \text{ (2-rows and 3-columns)} \end{aligned}$$

2. Consider the following non-interacting two-tank fluid system.



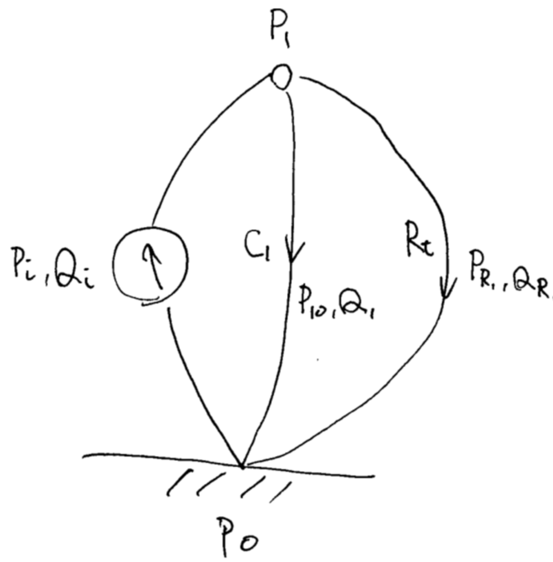
Notations in the figure are given in the table below. Fluid inertances are assumed to be negligible.

Symbol	Meaning
C_1 and C_2	fluid capacitances of Tank 1 and Tank 2
R_t and R_o	fluid resistances at the outlets of Tank 1 and Tank 2
$P_0 = P_o$	ambient pressure
Q_i and Q_o	input and output volume flow rates
Q_t	volume flow rate into Tank 2
P_1 and P_2	pressures at the bottom of Tank 1 and Tank 2
ρ	mass density of the fluid
g	acceleration due to the gravity
H_1 and H_2	fluid heights of Tank 1 and Tank 2

(a) For Tank 1 (upper tank), with the volume flow rate Q_i as an input, derive the state equation **by using the linear graph**. Here, take the state variable as $P_{10} := P_1 - P_0$. (4pt)

Write your answer here.

Linear graph



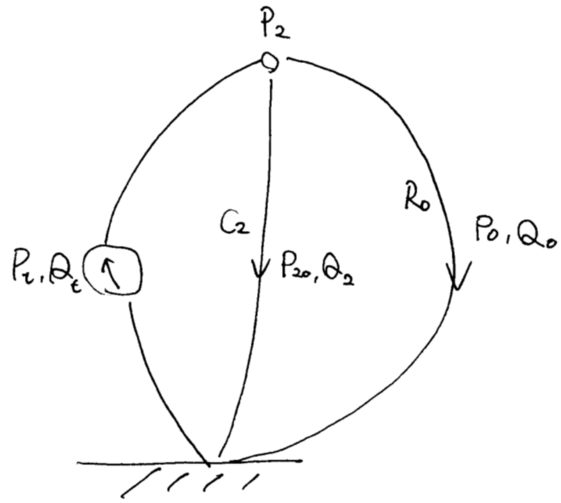
Derivation of the state equation

$$\dot{P}_{10} = \frac{1}{C_1}Q_1 = \frac{1}{C_1}(Q_i - Q_{R_1}) = \frac{1}{C_1}\left(Q_i - \frac{1}{R_t}P_{10}\right)$$

(b) For Tank 2 (lower tank), with the volume flow rate Q_t as an input, derive the state equation **by using the linear graph**. Here, take the state variable as $P_{20} := P_2 - P_0$. (4pt)

Write your answer here.

Linear graph



Derivation of the state equation

$$\dot{P}_{20} = \frac{1}{C_2} Q_2 = \frac{1}{C_2}(Q_t - Q_o) = \frac{1}{C_2}\left(Q_t - \frac{1}{R_o} P_{20}\right)$$

(c) Obtain the state-space model of the two-tank system with:

- **one input**: the volume flow rate Q_i, and
- **three outputs**: the heights H_1 and H_2, and volume flow rate Q_o.

(In this question, you do not need to draw the linear graph.) (2pt)

Write your answer here.

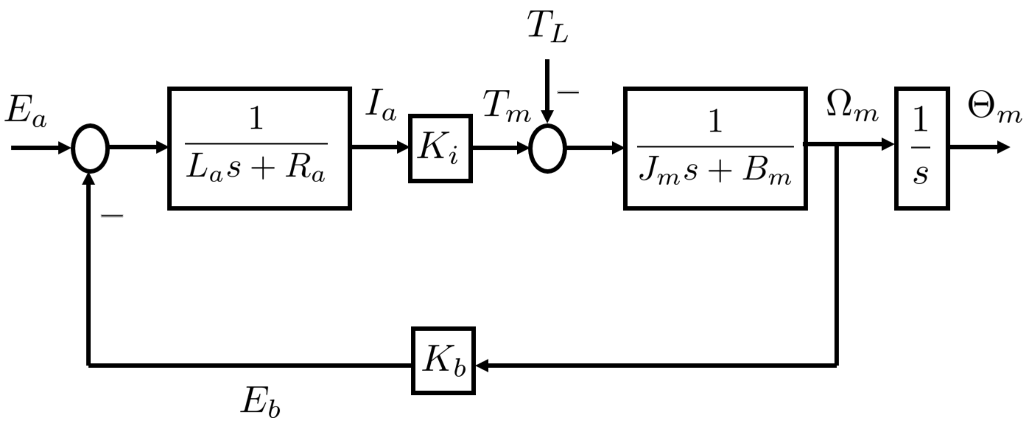
From Questions (a) and (b), we have

$$\dot{P}_{10} = \frac{1}{C_1}\left(Q_i - \frac{1}{R_t}P_{10}\right)$$
$$\dot{P}_{20} = \frac{1}{C_2}\left(\underbrace{\frac{1}{R_t}P_{10}}_{Q_t} - \frac{1}{R_o}P_{20}\right)$$

The relation between height and pressure is given by $P_{i0} = \rho g H_i$, $i = 1, 2$. Therefore, the state-space model is given as follows.

$$\begin{bmatrix} \dot{P}_{10} \\ \dot{P}_{20} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1 R_t} & 0 \\ \frac{1}{C_2 R_t} & -\frac{1}{C_2 R_o} \end{bmatrix} \begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} Q_i$$
$$\begin{bmatrix} H_1 \\ H_2 \\ Q_o \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho g} & 0 \\ 0 & \frac{1}{\rho g} \\ 0 & \frac{1}{R_o} \end{bmatrix} \begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix}$$

3. Consider the DC motor block diagram below, where all the constants which appear in the blocks (L_a, R_a, K_i, J_m, B_m and K_b) are positive.



(a) Obtain the following transfer functions:

i. from the motor voltage E_a to the rotor speed Ω_m. (2pt)

ii. from the motor current I_a to the rotor speed Ω_m. (2pt)

Write your answer here.

i.

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{\frac{K_i}{(L_a s+R_a)(J_m s+B_m)}}{1 + \frac{K_i K_b}{(L_a s+R_a)(J_m s+B_m)}} = \frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_i K_b}$$

ii.

$$\frac{\Omega_m(s)}{I_a(s)} = \frac{K_i}{J_m s + B_m}$$

(b) **By finding the poles of the transfer functions**, discuss the stability of the following transfer functions:

i. from the motor voltage E_a to the rotor speed Ω_m. (2pt)

ii. from the motor voltage E_a to the rotor position Θ_m. (2pt)

Write your answer here.

i. Transfer function

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i}{L_a J_m s^2 + (L_a B_m + R_a J_m)s + R_a B_m + K_i K_b}$$

Poles

$$s = \frac{-(L_a B_m + R_a J_m) \pm \sqrt{(L_a B_m + R_a J_m)^2 - 4L_a J_m (R_a B_m + K_i K_b)}}{2L_a J_m}$$

- In the case when $(L_a B_m + R_a J_m)^2 \leq 4L_a J_m (R_a B_m + K_i K_b)$, the real part of the poles is

$$\mathrm{Re}(s) = -\frac{L_a B_m + R_a J_m}{2L_a L_m} < 0.$$

This means that all the poles are in the open left-half plane.

- In the case when $(L_a B_m + R_a J_m)^2 > 4L_a J_m (R_a B_m + K_i K_b)$, the poles are negative real values, because

$$L_a B_m + R_a J_m > \sqrt{(L_a B_m + R_a J_m)^2 - 4L_a J_m (R_a B_m + K_i K_b)}$$

Therefore, the transfer function $\frac{\Omega_m(s)}{E_a(s)}$ is stable (asymptotically stable, BIBO stable).

ii.

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{1}{s} \cdot \frac{K_i}{L_a J_m s^2 + (L_a B_m + R_a J_m)s + R_a B_m + K_i K_b}$$

Since all poles are in the left half-plane and there is one simple pole at the origin, the transfer function $\frac{\Theta_m(s)}{E_a(s)}$ is marginally stable.

(c) When we apply the unit step voltage E_a and the unit step load torque T_L simultaneously, what value does the rotor speed converge? In other words, what is $\lim_{t\to\infty} \omega_m(t)$? (Here, $\omega_m(t) = \mathcal{L}^{-1}\{\Omega_m(s)\}$.) (2pt)

Write your answer here.

Due to the final value theorem, we have

$$\lim_{t\to\infty} \omega_m(t) = G_{E_a\to\omega_m}(0) + G_{T_L\to\omega_m}(0),$$

where

$$\begin{aligned} G_{E_a\to\omega_m}(s) &:= \frac{K_i}{L_a J_m s^2 + (L_a B_m + R_a J_m)s + R_a B_m + K_i K_b} \\ G_{T_L\to\omega_m}(s) &:= -\frac{L_a s + R_a}{L_a J_m s^2 + (L_a B_m + R_a J_m)s + R_a B_m + K_i K_b} \end{aligned}$$

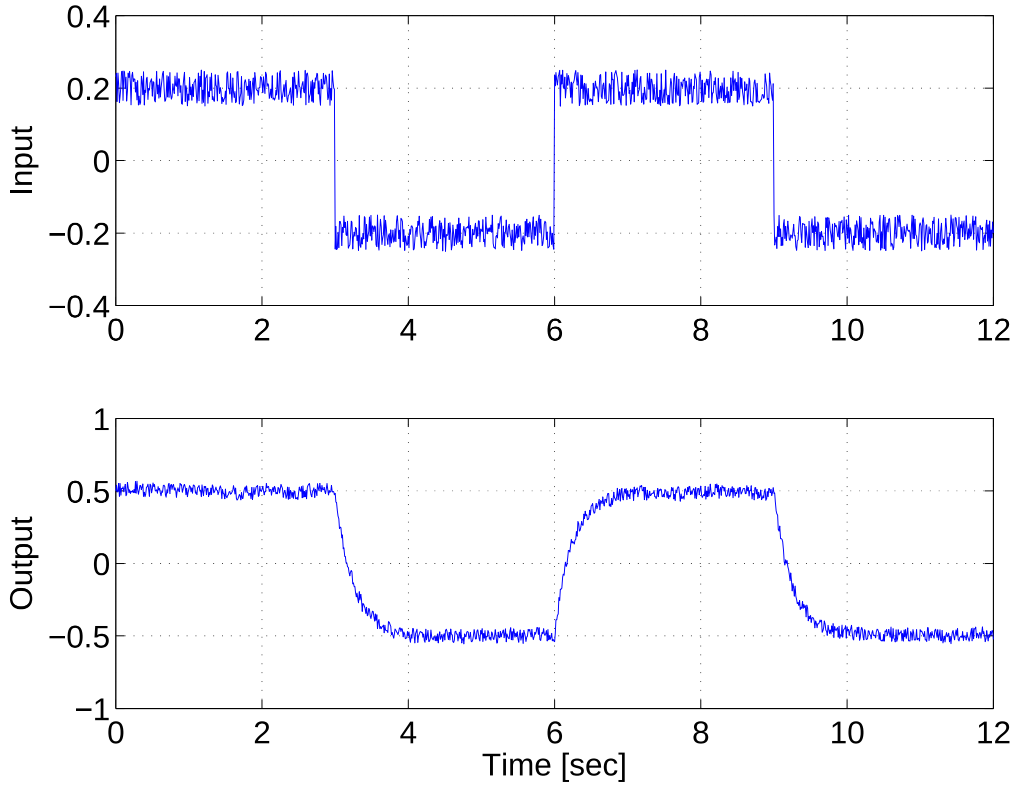
Therefore,

$$\lim_{t\to\infty} \omega_m(t) = \frac{K_i - R_a}{R_a B_m + K_i K_b}$$

Write your answer here.

4. Answer the following modeling questions based on experimental data.

   (a) For an unknown system, we applied a noisy square-wave input signal with the period 6 seconds, and obtained a noisy output signal, as shown below. Estimate the transfer function of the system. (5pt)

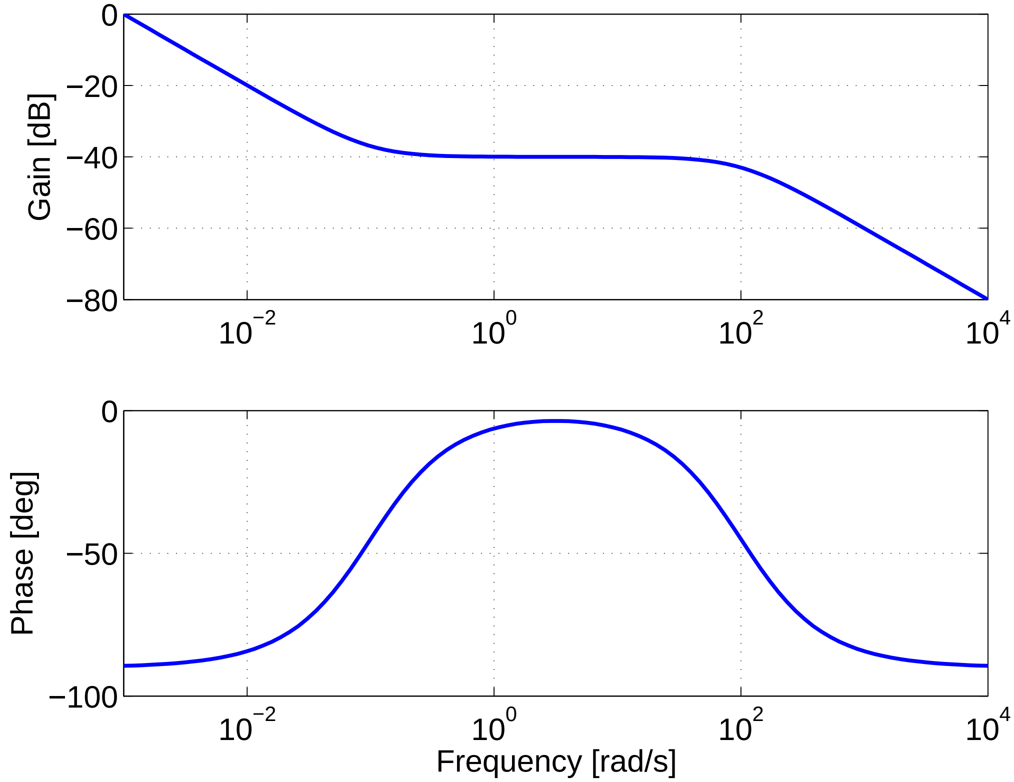


Write your answer here.

Since the settling time of the response is about 1 second, the time constant is about $1/4 = 0.25$ second. Also, by comparing input and output amplitude, the DC gain is $0.5/0.2 = 2.5$. Thus, the model can be

$$G(s) = \frac{10}{s+4} = \frac{2.5}{0.25s+1}.$$

(b) For an unknown system, we took the experimental frequency response and plotted the Bode plot as below. Estimate the transfer function of the system. (5pt)

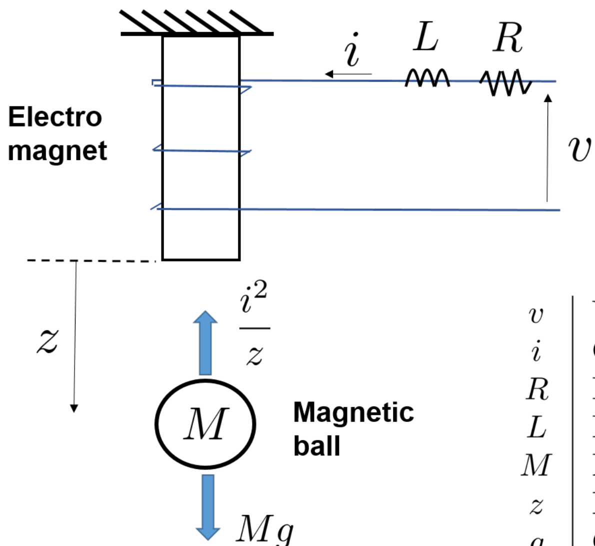


Write your answer here.

Since the low frequency gain slope is -20 dB/decade and the low frequency phase is -90 degree, the system contains $1/s$. Also, by looking at corner frequencies, we figure out that it has a zero at $s = -0.1$ and a pole at $s = -100$.

$$G(s) = \frac{s + 0.1}{s(s + 100)} = \frac{(10s + 1)}{1000s(0.01s + 1)}$$

5. Consider the magnetic-ball suspension system in the figure below. Here, the **input** is the applied voltage v, and the **output** is the displacement z of the ball, as indicated in the figure.



v	Voltage (Input)
i	Current
R	Lumped resistance
L	Lumped inductance
M	Mass of ball
z	Displacement of ball (Output)
g	Gravitational acceleration

Forces applied to the magnetic ball are indicated in the figure, where Mg is the gravitational force (downward force) and i^2/z is the electromagnetic force (upward force).

(a) Obtain the state-space model. (6pt)

(b) Around the equilibrium point $z = z_0$ (positive constant displacement), obtain the linearized state-space model. (4pt)

— End of Exam Questions —

Write your answer here.

Write your answer here.

(a) Electrical equation

$$v = Ri + L\frac{di}{dt}$$

Mechanical equation

$$M\ddot{z} = Mg - \frac{i^2}{z}$$

Selecting the states as $x_1 := z$, $x_2 := \dot{z}$, $x_3 := i$, and introducing the notations $u := i$ and $y := z$, we have the state-space model:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{x_3^2}{Mx_1} \\
\dot{x}_3 &= -\frac{R}{L}x_3 + \frac{1}{L}u \\
y &= x_1
\end{aligned}$$

(b) The linearization point is obtained by setting $x_{10} := z_0$ and $\dot{x} = 0$.

$$x_0 := \begin{bmatrix} z_0 \\ 0 \\ \pm\sqrt{Mgz_0} \end{bmatrix}, \; u_0 = \pm R\sqrt{Mgz_0}$$

The linearized model is

$$\begin{aligned}
\delta\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{z_0} & 0 & \mp\frac{2\sqrt{Mgz_0}}{Mz_0} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}\delta u \\
\delta y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\delta x
\end{aligned}$$

Extra page. Write the problem number before writing your answer.

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