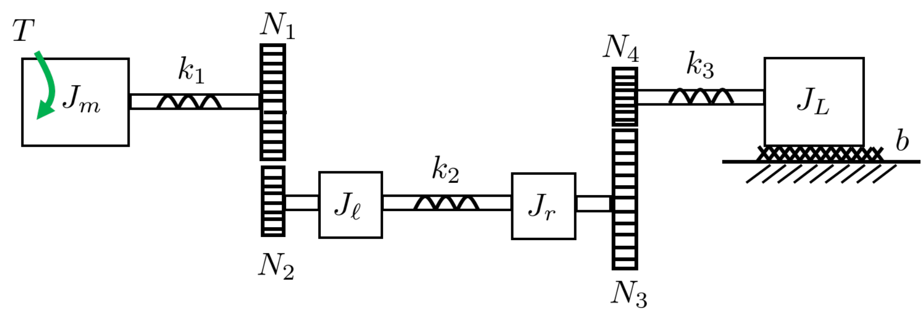
University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Homework 3

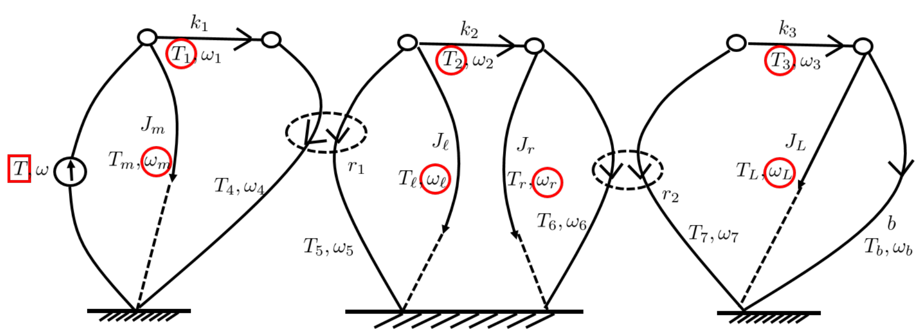
Due: October 7 (Monday), 2019, 3pm

For the gear-train system depicted below derive the state equation by using the linear graph (output equation is not necessary). The notations are explained in the following table. Masses of the flexible shafts are assumed to be negligible. The input is the motor torque T [Nm].

Notation	Unit	Meaning
J_m	[kg·m^2]	Moment of inertia of the motor
J_L	[kg·m^2]	Moment of inertia of the load
J_ℓ, J_r	[kg·m^2]	Lumped moment of inertia of the gears
k_i, $i = 1, 2, 3$	[Nm/rad]	Torsional spring constants
N_i, $i = 1, 2, 3, 4$	[–]	The number of gear teeth
b	[Nm/(rad/s)]	Rotational friction for the load



Solution: Linear graph is depicted as follows:



The states are selected as

$$x := \begin{bmatrix} \omega_m \\ \omega_\ell \\ \omega_r \\ \omega_L \\ T_1 \\ T_2 \\ T_3 \end{bmatrix}.$$

Constitutive equations are given by

$$
\begin{array}{ll}
\text{Inertia:} & \dot{\omega}_m = \dfrac{1}{J_m}T_m, \quad \dot{\omega}_\ell = \dfrac{1}{J_\ell}T_\ell, \quad \dot{\omega}_r = \dfrac{1}{J_r}T_r, \quad \dot{\omega}_L = \dfrac{1}{J_L}T_L, \\
\text{Torsional spring:} & \dot{T}_1 = k_1\omega_1, \quad \dot{T}_2 = k_2\omega_2, \quad \dot{T}_3 = k_3\omega_3, \\
\text{Rotational friction:} & T_b = b\omega_b, \\
\text{Transformer 1:} & \begin{bmatrix} \omega_5 \\ T_5 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & -1/r_1 \end{bmatrix} \begin{bmatrix} \omega_4 \\ T_4 \end{bmatrix}, \quad r_1 := \dfrac{N_1}{N_2}, \\
\text{Transformer 2:} & \begin{bmatrix} \omega_7 \\ T_7 \end{bmatrix} = \begin{bmatrix} r_2 & 0 \\ 0 & -1/r_2 \end{bmatrix} \begin{bmatrix} \omega_6 \\ T_6 \end{bmatrix}, \quad r_2 := \dfrac{N_3}{N_4}.
\end{array}
$$

Loop equations are:

$$
\begin{array}{l}
\omega = \omega_m, \quad \omega_m = \omega_1 + \omega_4, \quad \omega_5 = \omega_\ell = \omega_2 + \omega_r, \quad \omega_r = \omega_6, \\
\omega_7 = \omega_3 + \omega_L, \quad \omega_L = \omega_b
\end{array}
$$

Node equations are:

$$
\begin{array}{l}
T = T_1 + T_m, \quad T_1 = T_4, \\
T_2 + T_5 + T_\ell = 0, \quad T_2 = T_r + T_6, \\
T_7 + T_3 = 0, \quad T_3 = T_L + T_b.
\end{array}
$$

The derivations of the state equation are given below.

$$
\begin{array}{rcl}
\dot{\omega}_m &=& \dfrac{1}{J_m}T_m = \dfrac{1}{J_m}(T - T_1) \\
\dot{\omega}_\ell &=& \dfrac{1}{J_\ell}T_\ell = \dfrac{1}{J_\ell}(-T_5 - T_2) = \dfrac{1}{J_\ell}(\dfrac{T_4}{r_1} - T_2) = \dfrac{1}{J_\ell}(\dfrac{T_1}{r_1} - T_2) \\
\dot{\omega}_r &=& \dfrac{1}{J_r}T_r = \dfrac{1}{J_r}(T_2 - T_6) = \dfrac{1}{J_r}(T_2 + r_2 T_7) = \dfrac{1}{J_r}(T_2 - r_2 T_3) \\
\dot{\omega}_L &=& \dfrac{1}{J_L}T_L = \dfrac{1}{J_L}(T_3 - T_b) = \dfrac{1}{J_L}(T_3 - b\omega_b) = \dfrac{1}{J_L}(T_3 - b\omega_L) \\
\dot{T}_1 &=& k_1\omega_1 = k_1(\omega_m - \omega_4) = k_1(\omega_m - \dfrac{1}{r_1}\omega_5) = k_1(\omega_m - \dfrac{1}{r_1}\omega_\ell) \\
\dot{T}_2 &=& k_2\omega_2 = k_2(\omega_\ell - \omega_r) \\
\dot{T}_3 &=& k_3\omega_3 = k_3(\omega_7 - \omega_L) = k_3(r_2\omega_6 - \omega_L) = k_3(r_2\omega_r - \omega_L)
\end{array}
$$

Thus, the state equation becomes:

$$
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{1}{J_m} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{r_1 J_\ell} & -\frac{1}{J_\ell} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{J_r} & -\frac{r_2}{J_r} \\
0 & 0 & 0 & -\frac{b}{J_L} & 0 & 0 & \frac{1}{J_L} \\
k_1 & -\frac{k_1}{r_1} & 0 & 0 & 0 & 0 & 0 \\
0 & k_2 & -k_2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_3 r_2 & -k_3 & 0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix} \frac{1}{J_m} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} T
$$