a place of mind

MECH366 : Modeling of Mechatronic Systems

L4 : Linearization

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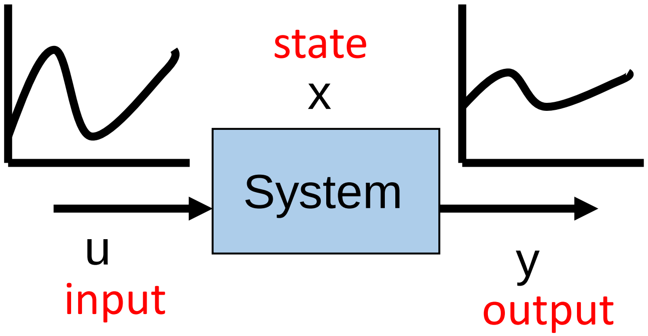
Review and today's topic

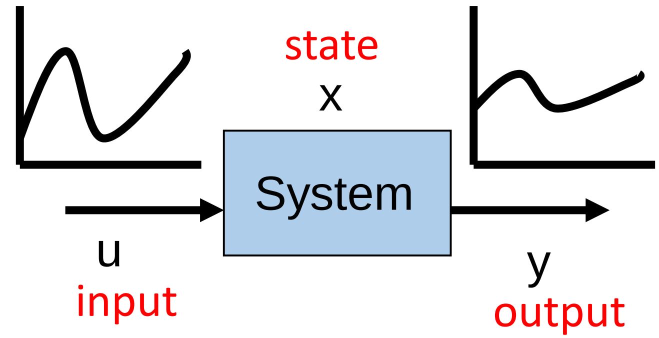
- Last lecture was about:
  - Linear state-space model

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$



  - Mechanical and electrical examples
- Today, we will study the linearization of nonlinear state-space models.

Linear system



- A system having *Principle of Superposition*

For zero initial state $x(0) = 0$:

$$\left.\begin{array}{l} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{array}\right\} \Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

$\forall \alpha_1, \alpha_2 \in \mathbb{R}$ ('$\forall$' means 'for all')

- A nonlinear system is a system which does not satisfy the principle of superposition.

Linear and nonlinear SS models

- Linear state-space model: Right-hand sides of the state-space model is linear with respect to *x* and *u*.

$$\begin{cases} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{cases}$$

- Nonlinear state-space model: Right-hand sides of the state-space model has nonlinear terms with respect to *x* and *u*.

$$\begin{cases} \dot{x} & = & f(x, u) \\ y & = & h(x, u) \end{cases}$$

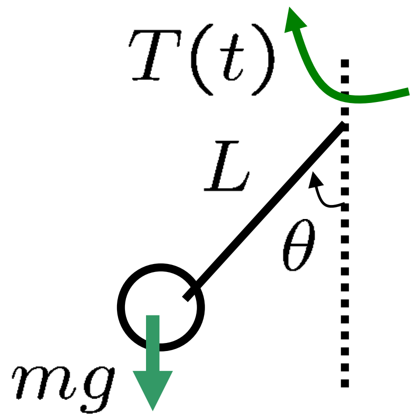
Examples of nonlinear terms

$$x_1^2, \; x_1 x_2, \; x_1 u, \; \sin(x_1), \; \sqrt{x_1}$$

Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!) *Ex.* $f(t)=Kx(t)$, $v(t)=Ri(t)$
- Nonlinear systems are difficult to deal with mathematically.
- Many mechatronics system analysis/design techniques are available for linear systems.
- Linear approximation is often good enough for system analysis and design purposes.
- How to linearize nonlinear systems?

Example 1: A pendulum



- Motion of the pendulum

$$mL^2\ddot{\theta}(t) = T(t) - mgL\sin\theta(t)$$

- Define state variables

$$x_1(t) := \theta(t),\ x_2(t) := \dot{\theta}(t)$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{g}{L}\underline{\sin x_1(t)} + \frac{1}{mL^2}T(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Nonlinear!

Example 2: Water level in a tank

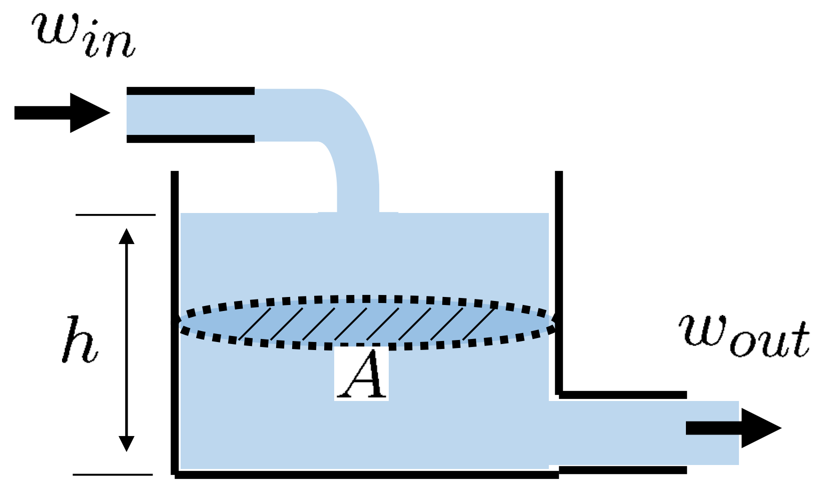
- Mass flow equation

$$
\begin{aligned}
\rho A\dot{h}(t) &= -w_{out}(t) + w_{in}(t) \\
&= -\frac{(\rho g)^{1/\alpha}}{R} h(t)^{1/\alpha} + w_{in}(t)
\end{aligned}
$$

$$
\Rightarrow \begin{cases} \dot{h}(t) = -\dfrac{(\rho g)^{1/\alpha}}{\rho A R} h(t)^{1/\alpha} + \dfrac{1}{\rho A} w_{in}(t) \\ y(t) = h(t) \end{cases}
$$

$\alpha = 1 \;\Rightarrow\;$ linear (Laminar flow)

$\alpha \neq 1 \;\Rightarrow\;$ nonlinear (Turbulent flow)



w_{in}, w_{out} : mass flow rate

h : water height

A : tank area

ρ : liquid density

R, α : constant depending on restriction. $1 \leq \alpha \leq 2$

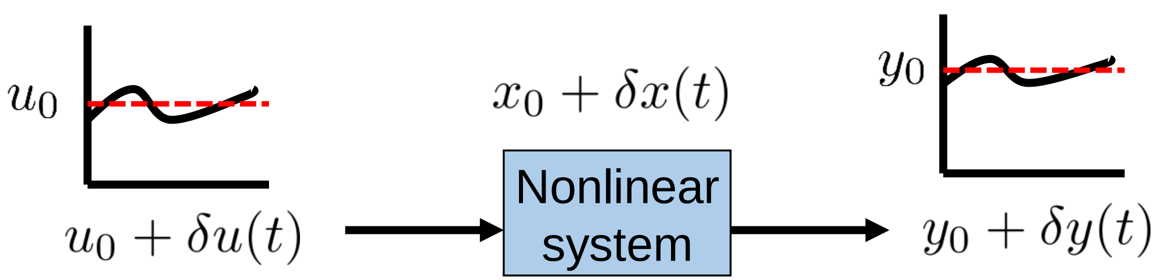
Linearization of nonlinear system

$$\begin{cases} \dot{x} & = & f(x,u) \\ y & = & h(x,u) \end{cases}$$

- Suppose constants (x_0,u_0,y_0) satisfies $\begin{cases} 0 = f(x_0,u_0) \\ y_0 = h(x_0,u_0) \end{cases}$

  (x_0,u_0,y_0) are called equilibrium (operating) points.

- If *u(t)* perturbs from u_0, then *x(t)* and *y(t)* also perturb from (x_0,y_0).



Linearization of state equation

$$\frac{d}{dt}(\cancel{x_0} + \delta x(t)) = f(x_0 + \delta x(t), u_0 + \delta u(t))$$

$$= \cancel{f(x_0, u_0)} + \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u + \underline{H.O.T.}$$

Negligible for small δx(t) & δu(t)

$$\Rightarrow \quad \frac{d}{dt}(\delta x(t)) = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)}}_{=:A} \delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)}}_{=:B} \delta u$$

(Often, we remove "δ". Then, $\dot{x} = Ax + Bu$)

Linearization of output equation

$$\begin{aligned}\cancel{y_0} + \delta y(t) &= h(x_0 + \delta x(t), u_0 + \delta u(t)) \\ &= \cancel{h(x_0, u_0)} + \left.\frac{\partial h}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial h}{\partial u}\right|_{(x_0,u_0)} \delta u + \underline{H.O.T.}\end{aligned}$$

Negligible for small δx(t) & δu(t)

$$\Rightarrow \quad \delta y(t) = \underbrace{\left.\frac{\partial h}{\partial x}\right|_{(x_0,u_0)}}_{=:C} \delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{(x_0,u_0)}}_{=:D} \delta u$$

(Often, we remove "δ". Then, $y = Cx + Du$)

Ex 1: A pendulum revisited

- Nonlinear model $\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} x_2(t) \\ -\frac{g}{L}\sin x_1(t) + \frac{1}{mL^2}u(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$

- Linearization around $(x_{10}, x_{20}, u_0) = (0, 0, 0)$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 + \frac{1}{mL^2}u \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & 0 \end{bmatrix} \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

$$\Rightarrow \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u(t)$$

$$(\text{In this case, } x = \delta x,\ u = \delta u)$$

Ex 2: Water level in a tank revisited

- Nonlinear model
$$\begin{cases} \dot{x}(t) &= -\underbrace{\dfrac{(\rho g)^{1/\alpha}}{\rho A R}}_{\tilde{A}} x(t)^{1/\alpha} + \underbrace{\dfrac{1}{\rho A}}_{\tilde{B}} u(t) \\ y(t) &= x(t) \end{cases}$$

- Linearization around $(x_0, u_0) = (h_0, \dfrac{\tilde{A}}{\tilde{B}} h_0^{1/\alpha})$

$$f(x,u) = -\tilde{A} x^{1/\alpha} + \tilde{B} u \quad \Rightarrow \quad \frac{\partial f}{\partial x} = -\frac{\tilde{A}}{\alpha} x^{(1-\alpha)/\alpha}, \quad \frac{\partial f}{\partial u} = \tilde{B}$$

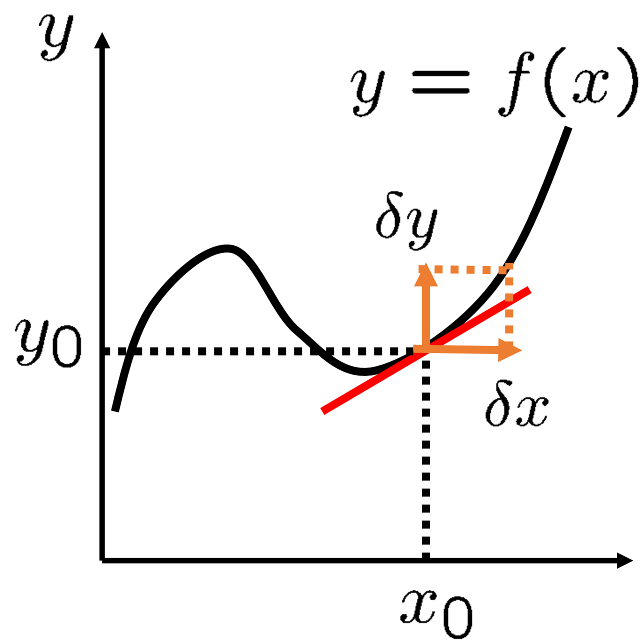
$$\Rightarrow \quad \dot{\delta x}(t) = -\frac{\tilde{A}}{\alpha} h_0^{(1-\alpha)/\alpha} \delta x(t) + \tilde{B} \delta u(t)$$

Summary

- Today's topics
  - Linearization of nonlinear systems
  - Examples
    - A pendulum
    - Water level in a tank
- Next, other linearization examples
- Homework 1: Due Sep 23 (Monday), 3pm
- Lab 1 starts on Sep 20 (Friday).
  - Check the schedule. Go to Kaiser 1160 in time.
  - Read the manual before going to the lab.

Linearization: 1-dim case

- Linearize a function *y=f(x)* around $x=x_0$ (scalar)
  - Consider a solution (x_0,y_0)
    $$y_0 = f(x_0)$$
  - If *x* perturbs from x_0, then *y* also perturbs from y_0.



$$\begin{aligned} \cancel{y_0} + \delta y &= f(x_0 + \delta x) \\ &= \cancel{f(x_0)} + \left.\frac{df}{dx}\right|_{x=x_0} \delta x + \underline{H.O.T.} \end{aligned}$$

(Taylor expansion)

Negligible for small δx

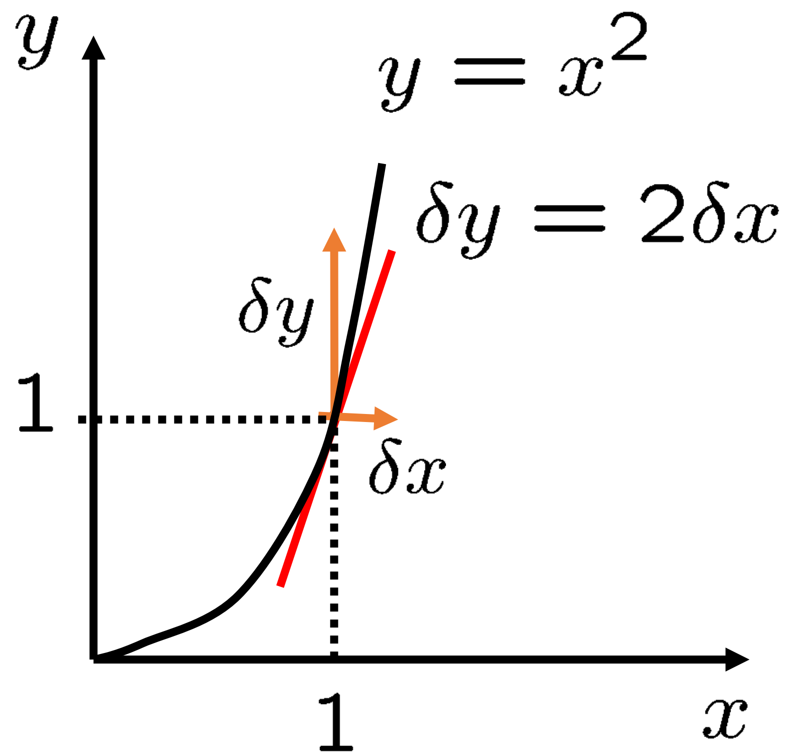
$$\Rightarrow \quad \delta y = \left.\frac{df}{dx}\right|_{x=x_0} \delta x$$

Example: 1-dim case

- Linearization of a function $y = x^2$ around *x=1*.

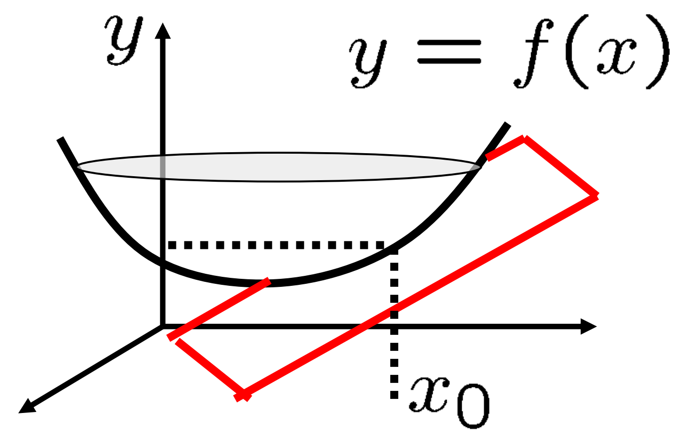
$$\begin{cases} \delta y = \left.\dfrac{df}{dx}\right|_{x=x_0} \delta x \\ \left.\dfrac{df}{dx}\right|_{x=1} = 2x|_{x=1} = 2 \end{cases}$$

$\Rightarrow$ $\delta y = 2\delta x$



Linearization: 2-dim case

- Linearize a function *y=f(x)* around $x=x_0 \in \mathbb{R}^2$
  - Consider a solution
    $$y_0 = f(x_0)$$
  - If *x* perturbs from x_0, then *y* also perturbs from y_0.



$$\cancel{y_0} + \delta y = f(x_0 + \delta x)$$
$$= \cancel{f(x_0)} + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} \delta x_1 + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_0} \delta x_2 + \underline{H.O.T.}$$

Negligible for small δx

$$\Rightarrow \delta y = \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} & \left.\frac{\partial f}{\partial x_2}\right|_{x=x_0} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \left.\underline{\frac{\partial f}{\partial x}}\right|_{x=x_0} \delta x$$

Jacobian

Example: 2-dim case

- Linearize a function below around $x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$y = x_1^2 + \sin(x_1 x_2^2)$$

- Linearized equation $\delta y = \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \delta x$
  - Jacobian computation

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \left.\begin{bmatrix} 2x_1 + x_2^2 \cos(x_1 x_2^2) & 2x_1 x_2 \cos(x_1 x_2^2) \end{bmatrix}\right|_{x=x_0}$$

$$= \begin{bmatrix} 4 + \cos 2 & 4\cos 2 \end{bmatrix}$$