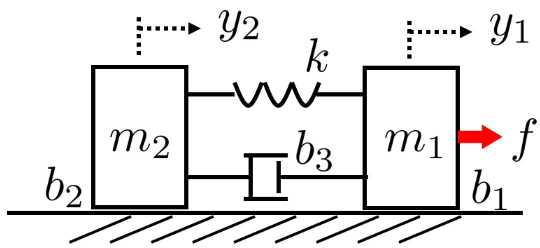
University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Homework 1: Solutions

1. Consider 2-DOF mass-spring-damper system in the figure below.



(a) By selecting the following states, obtain the state-space model.

i. $x_1 := y_1$, $x_2 := \dot{y}_1$, $x_3 := y_2$, $x_4 := \dot{y}_2$.

Solution: By Newton's second law, we have the equations of motion as

$$\begin{aligned} m_1\ddot{y}_1 &= f - k(y_1 - y_2) - b_3(\dot{y}_1 - \dot{y}_2) - b_1\dot{y}_1, \\ m_2\dot{y}_2 &= -k(y_2 - y_1) - b_3(\dot{y}_2 - \dot{y}_1) - b_2\dot{y}_2. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \ddot{y}_1 &= \frac{1}{m_1}\left(-ky_1 - (b_1 + b_3)\dot{y}_1 + ky_2 + b_3\dot{y}_2 + f\right), \\ \ddot{y}_2 &= \frac{1}{m_2}\left(ky_1 + b_3\dot{y}_1 - ky_2 - (b_2 + b_3)\dot{y}_2\right). \end{aligned}$$

Thus, the state-space model is given by

$$\dot{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -(b_1+b_3)/m_1 & k/m_1 & b_3/m_1 \\ 0 & 0 & 0 & 1 \\ k/m_2 & b_3/m_2 & -k/m_2 & -(b_2+b_3)/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} f$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

ii. $x_1 := \dot{y}_2$, $x_2 := y_2$, $x_3 := \dot{y}_1$, $x_4 := y_1$.

Solution: By exchanging the rows and columns of the state-space model obtained in (a) appropriately, we can get the state-space model for the new state selection as

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -(b_2+b_3)/m_2 & -k/m_2 & b_3/m_2 & k/m_2 \\ 1 & 0 & 0 & 0 \\ b_3/m_1 & k/m_1 & -(b_1+b_3)/m_1 & -k/m_1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} f$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) **Solution:** Due to the nonlinear spring, the equations of motion are modified as

$$\begin{aligned} m_1\ddot{y}_1 &= f - k_1(y_1-y_2) - k_2(y_1-y_2)^3 - b_3(\dot{y}_1-\dot{y}_2) - b_1\dot{y}_1, \\ m_2\dot{y}_2 &= -k_1(y_2-y_1) - k_2(y_2-y_1)^3 - b_3(\dot{y}_2-\dot{y}_1) - b_2\dot{y}_2. \end{aligned}$$

Thus, the nonlinear state equation is

$$\dot{x} = g(x, f),$$

where

$$g(x,f) := \begin{bmatrix} x_2 \\ \frac{1}{m_1}(f - k_1(x_1-x_3) - k_2(x_1-x_3)^3 - b_3(x_2-x_4) - b_1x_2) \\ x_4 \\ \frac{1}{m_2}(-k_1(x_3-x_1) - k_2(x_3-x_1)^3 - b_3(x_4-x_2) - b_2x_4) \end{bmatrix}$$

The Jacobian of g can be calculated as

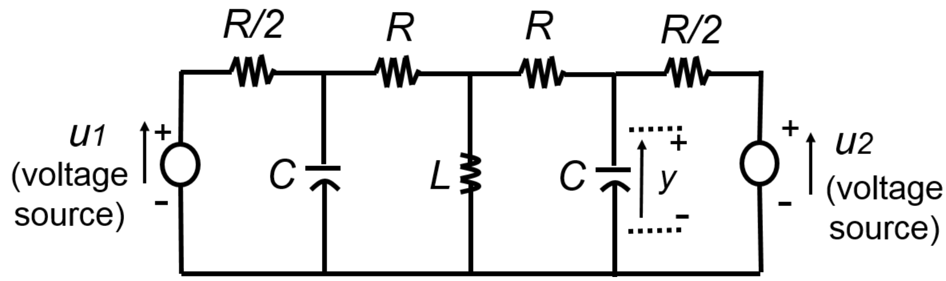
$$\frac{\partial g}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (-k_1 - 3k_2(x_1-x_3)^2)/m_1 & -(b_1+b_3)/m_1 & (k_1+3k_2(x_1-x_3)^2)/m_1 & b_3/m_1 \\ 0 & 0 & 0 & 1 \\ (k_1+3k_2(x_3-x_1)^2)/m_2 & b_3/m_2 & (-k_1-3k_2(x_3-x_1)^2)/m_2 & -(b_2+b_3)/m_2 \end{bmatrix}$$

By plugging the equilibrium point to the Jacobian, we can get the same A matrix as in (a)-i.
The linearized state-space model is

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta f \\ \delta y &= C\delta x, \end{aligned}$$

where B and C are the same matrices as in (a)-i, $\delta x = x$, $\delta f = f$ and $\delta y = y$.
The linearization around $y_{10} = 1$ [m], $y_{20} = 0$ [m] and $\dot{y}_{10} = \dot{y}_{20} = 0$ [m/s] is impossible because that is not an equilibrium point.

2. Obtain a state-space model for the RLC circuit depicted below.



Solution: Using Kirchhoff voltage law, we can obtain

$$u_1 = i\frac{R}{2} + \frac{1}{C}\int i_1$$

$$\frac{1}{C}\int i_1 = (i - i_1)R + L\frac{di_2}{dt}$$

$$L\frac{di_2}{dt} = (i - i_1 - i_2)R + \frac{1}{C}\int i_3$$

$$\frac{1}{C}\int i_3 = (i - i_1 - i_2 - i_3)\frac{R}{2} + u_2$$

Define the states for across variables (i.e., voltages) for capacitors and for through variable (i.e., current) for inductor as

$$x_1 := \frac{1}{C}\int i_1, \quad x_2 := i_2, \quad x_3 := \frac{1}{C}\int i_3.$$

Then,

$$u_1 = i\frac{R}{2} + x_1 \;\Rightarrow\; iR = 2(u_1 - x_1) \tag{1}$$

$$x_1 = (i - i_1)R + L\dot{x}_2 \;\Rightarrow\; (i - i_1)R = x_1 - L\dot{x}_2 \tag{2}$$

$$L\dot{x}_2 = (i - i_1 - x_2)R + x_3 \;\Rightarrow\; (i - i_1 - x_2)R = L\dot{x}_2 - x_3 \tag{3}$$

$$x_3 = (i - i_1 - i_2 - i_3)\frac{R}{2} + u_2 \;\Rightarrow\; (i - i_1 - x_2 - i_3)R = 2(x_3 - u_2) \tag{4}$$

Now,

$$Rx_2 = x_1 - 2L\dot{x}_2 + x_3 \quad \text{by taking (2)-(3)} \;\Rightarrow\; \dot{x}_2 = \frac{1}{2L}(x_1 - Rx_2 + x_3) \tag{5}$$

$$\dot{x}_1 = \frac{1}{C}i_1 = \frac{1}{CR}(2u_1 - 3x_1 + L\dot{x}_2) \quad \text{by taking (1)-(2)} \tag{6}$$

$$\dot{x}_1 = \frac{1}{CR}\left(2u_1 - 3x_1 + \frac{1}{2}(x_1 - Rx_2 + x_3)\right) \quad \text{by plugging (5) in (6)} \tag{7}$$

$$\dot{x}_3 = \frac{1}{C} i_3 = \frac{1}{CR}(L\dot{x}_2 - 3x_3 + 2u_2) \quad \text{by taking (3)-(4)} \tag{8}$$

$$\dot{x}_3 = \frac{1}{CR}\left(\frac{1}{2}(x_1 - Rx_2 + x_3) - 3x_3 + 2u_2\right) \quad \text{by plugging (5) in (7)} \tag{9}$$

Therefore, we can obtain the state-space model as

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -5/(2CR) & -1/(2C) & 1/(2CR) \\ 1/(2L) & -R/(2L) & 1/(2L) \\ 1/(2CR) & -1/(2C) & -5/(2CR) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{CR} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}$$