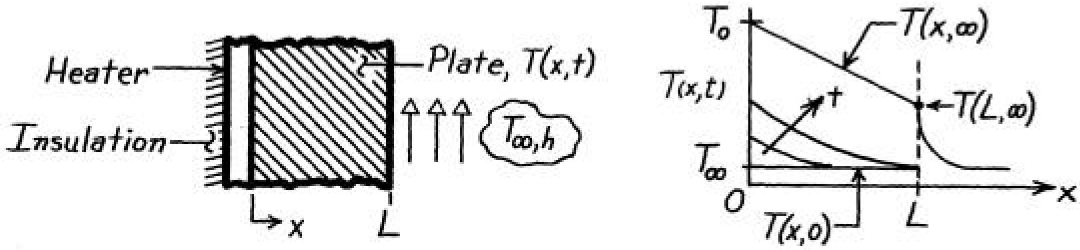
PROBLEM 5.1

KNOWN: Electrical heater attached to backside of plate while front surface is exposed to convection process (T_∞,h); initially plate is at a uniform temperature of the ambient air and suddenly heater power is switched on providing a constant q''_o.

FIND: (a) Sketch temperature distribution, T(x,t), (b) Sketch the heat flux at the outer surface, $q''_x(L,t)$ as a function of time.

SCHEMATIC:



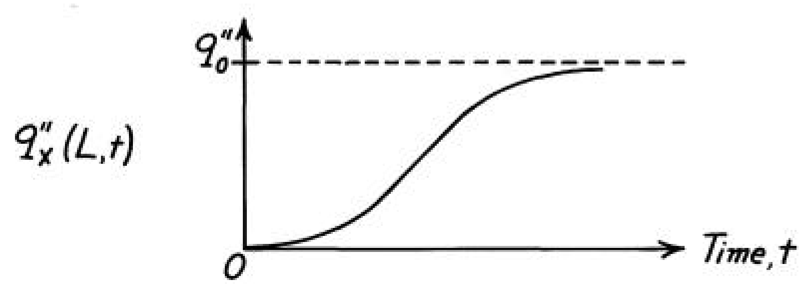
ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible heat loss from heater through insulation.

ANALYSIS: (a) The temperature distributions for four time conditions including the initial distribution, T(x,0), and the steady-state distribution, T(x,∞), are as shown above.

Note that the temperature gradient at x = 0, $-dT/dx)_{x=0}$, for t > 0 will be a constant since the flux, $q''_x(0)$, is a constant. Noting that $T_o = T(0,\infty)$, the steady-state temperature distribution will be linear such that

$$q''_o = k\frac{T_o - T(L,\infty)}{L} = h\left[T(L,\infty) - T_\infty\right].$$

(b) The heat flux at the front surface, x = L, is given by $q''_x(L,t) = -k(dT/dx)_{x=L}$. From the temperature distribution, we can construct the heat flux-time plot.



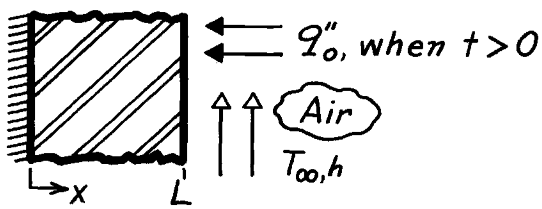
COMMENTS: At early times, the temperature and heat flux at x = L will not change from their initial values. Hence, we show a zero slope for $q''_x(L,t)$ at early times. Eventually, the value of $q''_x(L,t)$ will reach the steady-state value which is q''_o.

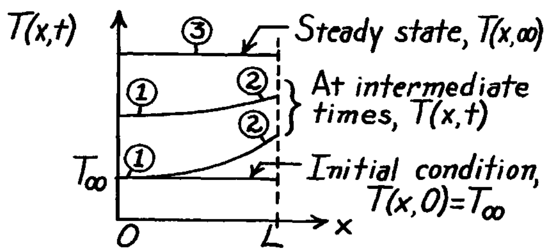
PROBLEM 5.2

KNOWN: Plane wall whose inner surface is insulated and outer surface is exposed to an airstream at T_∞. Initially, the wall is at a uniform temperature equal to that of the airstream. Suddenly, a radiant source is switched on applying a uniform flux, q''_o, to the outer surface.

FIND: (a) Sketch temperature distribution on T-x coordinates for initial, steady-state, and two intermediate times, (b) Sketch heat flux at the outer surface, $q''_x(L,t)$, as a function of time.

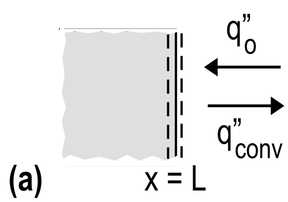
SCHEMATIC:

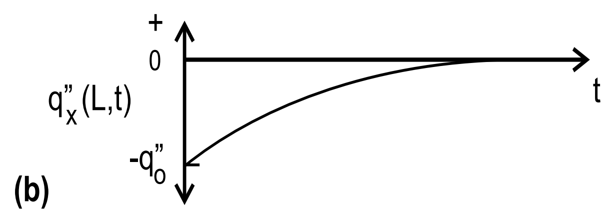




ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal generation, $\dot{E}_g = 0$, (4) Surface at x = 0 is perfectly insulated, (5) All incident radiant power is absorbed and negligible radiation exchange with surroundings.

ANALYSIS: (a) The temperature distributions are shown on the T-x coordinates and labeled accordingly. Note these special features: (1) Gradient at x = 0 is always zero, (2) gradient is more steep at early times and (3) for steady-state conditions, the radiant flux is equal to the convective heat flux (this follows from an energy balance on the CS at x = L),
$q''_o = q''_{conv} = h\left[T(L,\infty) - T_\infty\right]$.





(b) The heat flux at the outer surface, $q''_x(L,t)$, as a function of time appears as shown above.

COMMENTS: The sketches must reflect the initial and boundary conditions:

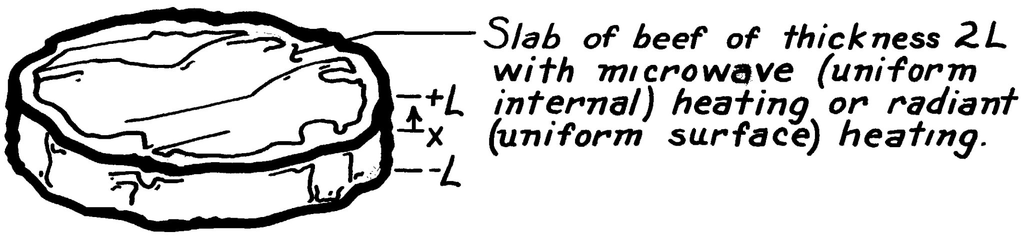
$T(x,0) = T_\infty$	uniform initial temperature.
$-k\dfrac{\partial T}{\partial x}\Big\vert_{x=0} = 0$	insulated at x = 0.
$-k\dfrac{\partial T}{\partial x}\Big\vert_{x=L} = h\left[T(L,t) - T_\infty\right] - q''_o$	surface energy balance at x = L.

PROBLEM 5.3

KNOWN: Microwave and radiant heating conditions for a slab of beef.

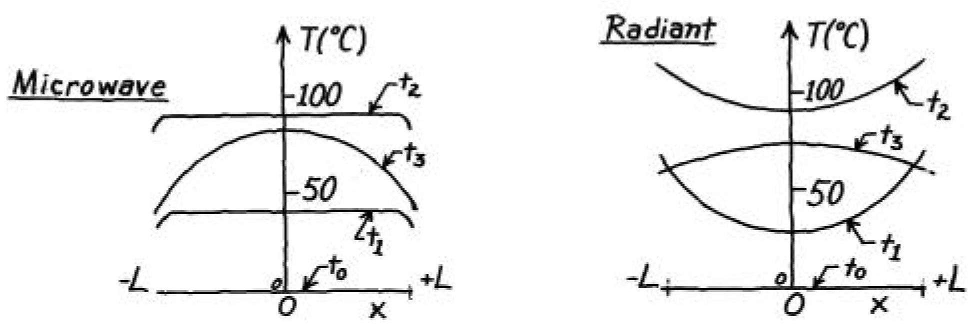
FIND: Sketch temperature distributions at specific times during heating and cooling.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Uniform internal heat generation for microwave, (3) Uniform surface heating for radiant oven, (4) Heat loss from surface of meat to surroundings is negligible during the heating process, (5) Symmetry about midplane.

ANALYSIS:



COMMENTS: (1) With uniform generation and negligible surface heat loss, the temperature distribution remains nearly uniform during *microwave heating*. During the subsequent surface cooling, the maximum temperature is at the midplane.

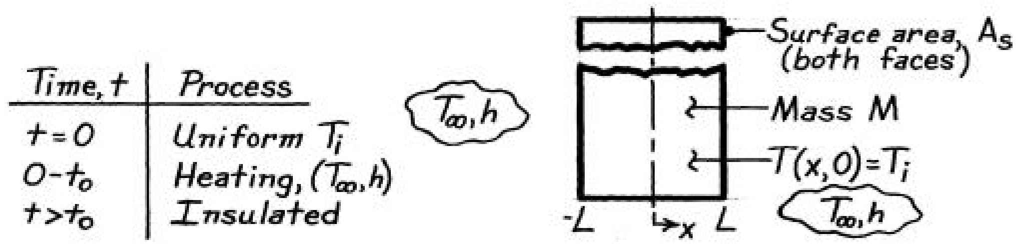
(2) The interior of the meat is heated by conduction from the hotter surfaces during *radiant heating*, and the lowest temperature is at the midplane. The situation is reversed shortly after cooling begins, and the maximum temperature is at the midplane.

PROBLEM 5.4

KNOWN: Plate initially at a uniform temperature T_i is suddenly subjected to convection process (T_∞,h) on both surfaces. After elapsed time t_o, plate is insulated on both surfaces.

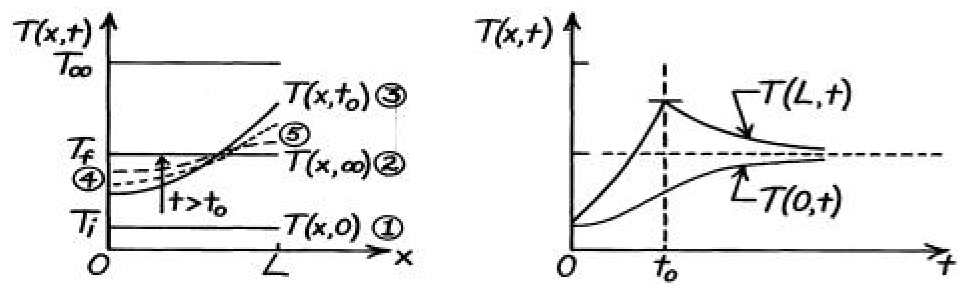
FIND: (a) Assuming Bi >> 1, sketch on T - x coordinates: initial and steady-state ($t \rightarrow \infty$) temperature distributions, $T(x,t_o)$ and distributions for two intermediate times $t_o < t < \infty$, (b) Sketch on T - t coordinates midplane and surface temperature histories, (c) Repeat parts (a) and (b) assuming Bi << 1, and (d) Obtain expression for $T(x,\infty) = T_f$ in terms of plate parameters (M,c_p), thermal conditions (T_i, T_∞, h), surface temperature T(L,t) and heating time t_o.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal generation, (4) Plate is perfectly insulated for $t > t_o$, (5) $T(0, t < t_o) < T_\infty$.

ANALYSIS: (a,b) With Bi >> 1, appreciable temperature gradients exist in the plate following exposure to the heating process.



On *T-x* coordinates: (1) initial, uniform temperature, (2) steady-state conditions when $t \rightarrow \infty$, (3) distribution at t_o just before plate is covered with insulation, (4) gradients are always zero (symmetry), and (5) when $t > t_o$ (dashed lines) gradients approach zero everywhere.

(c) If Bi << 1, plate is space-wise isothermal (no gradients). On *T-x* coordinates, the temperature distributions are flat; on T-t coordinates, T(L,t) = T(0,t).

(d) The conservation of energy requirement for the interval of time $\Delta t = t_o$ is

$$E_{in} - E_{out} = \Delta E = E_{final} - E_{initial} \qquad 2\int_0^{t_o} hA_s\left[T_\infty - T(L,t)\right]dt - 0 = Mc_p\left(T_f - T_i\right)$$

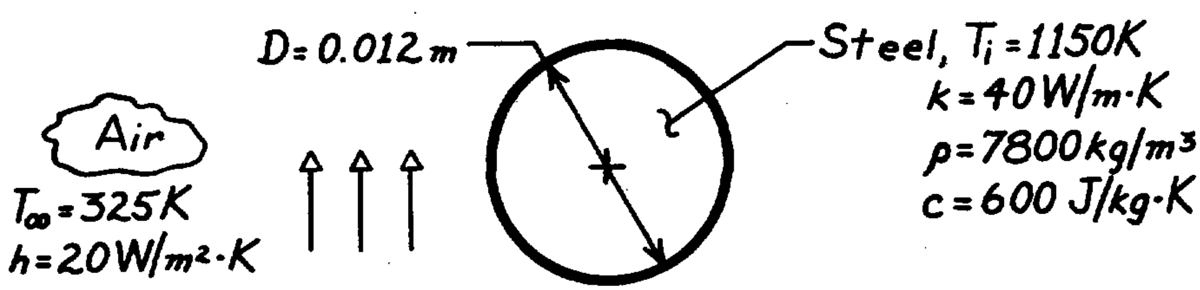
where E_{in} is due to convection heating over the period of time $t = 0 \rightarrow t_o$. With knowledge of T(L,t), this expression can be integrated and a value for T_f determined.

PROBLEM 5.5

KNOWN: Diameter and initial temperature of steel balls cooling in air.

FIND: Time required to cool to a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible radiation effects, (2) Constant properties.

ANALYSIS: Applying Eq. 5.10 to a sphere ($L_c = r_o/3$),

$$Bi = \frac{hL_c}{k} = \frac{h\left(r_o/3\right)}{k} = \frac{20\ W/m^2 \cdot K\ \left(0.002m\right)}{40\ W/m \cdot K} = 0.001.$$

Hence, the temperature of the steel remains approximately uniform during the cooling process, and the lumped capacitance method may be used. From Eqs. 5.4 and 5.5,

$$t = \frac{r V c_p}{hA_s} \ln\frac{T_i - T_\infty}{T - T_\infty} = \frac{r\left(p D^3/6\right)c_p}{hp\ D^2} \ln\frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{7800 kg/m^3 \left(0.012m\right) 600 J/kg \cdot K}{6 \times 20\ W/m^2 \cdot K} \ln\frac{1150 - 325}{400 - 325}$$

$$t = 1122\ s\ = 0.312h$$ <

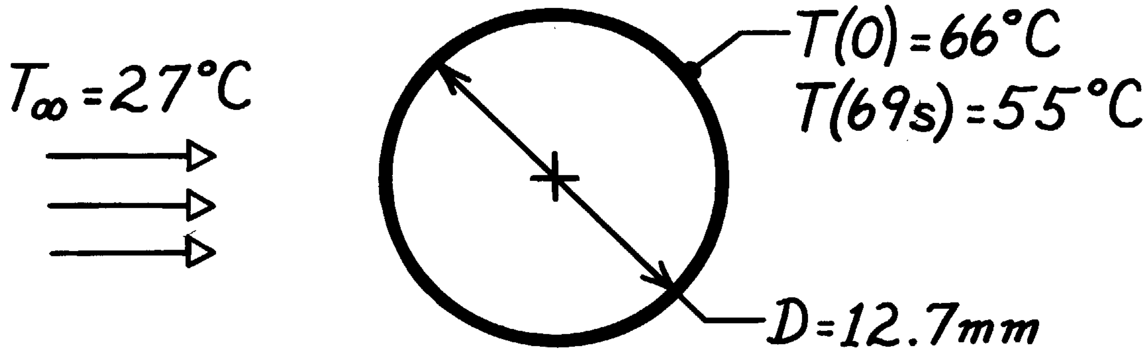
COMMENTS: Due to the large value of T_i, radiation effects are likely to be significant during the early portion of the transient. The effect is to shorten the cooling time.

PROBLEM 5.6

KNOWN: The temperature-time history of a pure copper sphere in an air stream.

FIND: The heat transfer coefficient between the sphere and the air stream.

SCHEMATIC:



ASSUMPTIONS: (1) Temperature of sphere is spatially uniform, (2) Negligible radiation exchange, (3) Constant properties.

PROPERTIES: *Table A-1*, Pure copper (333K): ρ = 8933 kg/m^3, c_p = 389 J/kg·K, k = 398 W/m·K.

ANALYSIS: The time-temperature history is given by Eq. 5.6 with Eq. 5.7.

$$\frac{q(\mathrm{t})}{q_\mathrm{i}} = \exp\left(-\frac{\mathrm{t}}{\mathrm{R_t C_t}}\right) \qquad \text{where} \qquad \mathrm{R_t} = \frac{1}{\mathrm{hA_s}} \qquad \mathrm{A_s} = p\ \mathrm{D}^2$$

$$\mathrm{C_t} = r\mathrm{Vc_p} \qquad \mathrm{V} = \frac{p\ \mathrm{D}^3}{6}$$

$$q = \mathrm{T} - \mathrm{T}_\infty.$$

Recognize that when t = 69s,

$$\frac{q(\mathrm{t})}{q_\mathrm{i}} = \frac{(55-27)^\circ\mathrm{C}}{(66-27)^\circ\mathrm{C}} = 0.718 = \exp\left(-\frac{\mathrm{t}}{t_\mathrm{t}}\right) = \exp\left(-\frac{69\mathrm{s}}{t_\mathrm{t}}\right)$$

and noting that $t_\mathrm{t} = \mathrm{R_t C_t}$ find

$$t_\mathrm{t} = 208\mathrm{s}.$$

Hence,

$$\mathrm{h} = \frac{r\mathrm{Vc_p}}{\mathrm{A_s}t_\mathrm{t}} = \frac{8933\ \mathrm{kg/m^3}\left(p\,0.0127^3\ \mathrm{m^3}/6\right)389\mathrm{J/kg\cdot K}}{p\,0.0127^2\mathrm{m}^2\times 208\mathrm{s}}$$

$$\mathrm{h} = 35.3\ \mathrm{W/m^2\cdot K}. \qquad <$$

COMMENTS: Note that with $L_c = D_o/6$,

$$\mathrm{Bi} = \frac{\mathrm{hL_c}}{\mathrm{k}} = 35.3\ \mathrm{W/m^2\cdot K}\times\frac{0.0127}{6}\mathrm{m}/398\ \mathrm{W/m\cdot K} = 1.88\times10^{-4}.$$

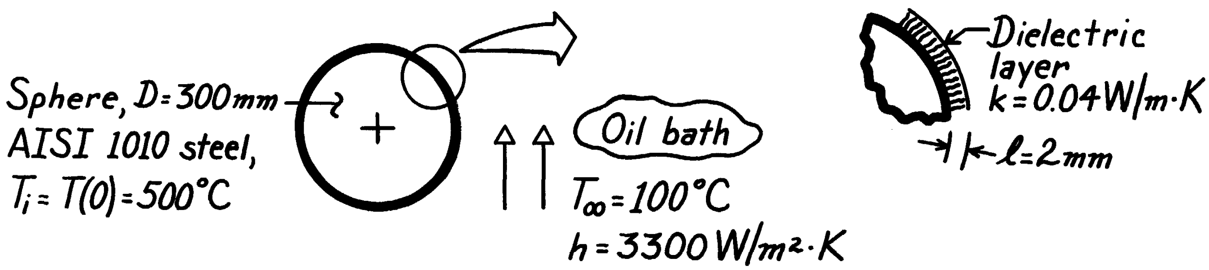
Hence, Bi < 0.1 and the spatially isothermal assumption is reasonable.

PROBLEM 5.7

KNOWN: Solid steel sphere (AISI 1010), coated with dielectric layer of prescribed thickness and thermal conductivity. Coated sphere, initially at uniform temperature, is suddenly quenched in an oil bath.

FIND: Time required for sphere to reach 140°C.

SCHEMATIC:



PROPERTIES: *Table A-1*, AISI 1010 Steel $\left(\bar{T} = [500 + 140]^\circ C/2 = 320^\circ C \approx 600K\right)$:

$r = 7832\ kg/m^3,\ c = 559\ J/kg\cdot K,\ k = 48.8\ W/m\cdot K.$

ASSUMPTIONS: (1) Steel sphere is space-wise isothermal, (2) Dielectric layer has negligible thermal capacitance compared to steel sphere, (3) Layer is thin compared to radius of sphere, (4) Constant properties.

ANALYSIS: The thermal resistance to heat transfer from the sphere is due to the dielectric layer and the convection coefficient. That is,

$$R'' = \frac{\ell}{k} + \frac{1}{h} = \frac{0.002m}{0.04\ W/m\cdot K} + \frac{1}{3300\ W/m^2\cdot K} = (0.050 + 0.0003) = 0.0503\frac{m^2\cdot K}{W},$$

or in terms of an overall coefficient, $U = 1/R'' = 19.88\ W/m^2\cdot K$. The effective Biot number is

$$Bi_e = \frac{UL_c}{k} = \frac{U(r_o/3)}{k} = \frac{19.88\ W/m^2\cdot K \times (0.300/6)m}{48.8\ W/m\cdot K} = 0.0204$$

where the characteristic length is $L_c = r_o/3$ for the sphere. Since $Bi_e < 0.1$, the lumped capacitance approach is applicable. Hence, Eq. 5.5 is appropriate with h replaced by U,

$$t = \frac{rc}{U}\left[\frac{V}{A_s}\right]\ln\frac{q_i}{q_o} = \frac{rc}{U}\left[\frac{V}{A_s}\right]\ln\frac{T(0) - T_\infty}{T(t) - T_\infty}.$$

Substituting numerical values with $(V/A_s) = r_o/3 = D/6$,

$$t = \frac{7832\ kg/m^3 \times 559\ J/kg\cdot K}{19.88\ W/m^2\cdot K}\left[\frac{0.300m}{6}\right]\ln\frac{(500 - 100)^\circ C}{(140 - 100)^\circ C}$$

$$t = 25{,}358s = 7.04h. \qquad <$$

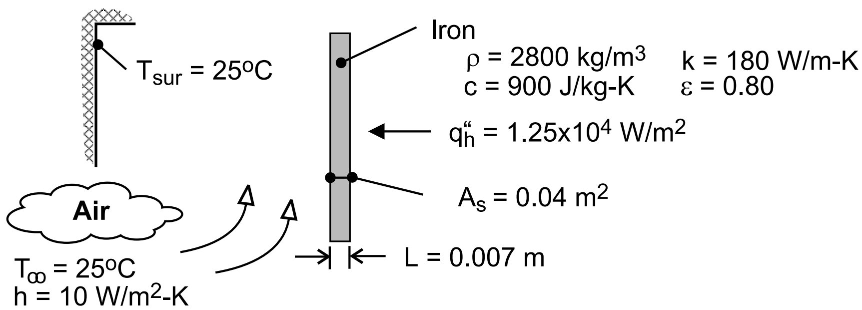
COMMENTS: (1) Note from calculation of R'' that the resistance of the dielectric layer dominates and therefore nearly all the temperature drop occurs across the layer.

PROBLEM 5.8

KNOWN: Thickness, surface area, and properties of iron base plate. Heat flux at inner surface. Temperature of surroundings. Temperature and convection coefficient of air at outer surface.

FIND: Time required for plate to reach a temperature of 135°C. Operating efficiency of iron.

SCHEMATIC:



ASSUMPTIONS: (1) Radiation exchange is between a small surface and large surroundings, (2) Convection coefficient is independent of time, (3) Constant properties, (4) Iron is initially at room temperature ($T_i = T_\infty$).

ANALYSIS: Biot numbers may be based on convection heat transfer and/or the maximum heat transfer by radiation, which would occur when the plate reaches the desired temperature (T = 135°C). From Eq. (1.9) the corresponding radiation transfer coefficient is $h_r = \varepsilon\sigma(T + T_{sur})\left(T^2 + T_{sur}^2\right) = 0.8 \times 5.67 \times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4\ (408 + 298)\ \text{K}\ (408^2 + 298^2)\ \text{K}^2 = 8.2\ \text{W/m}^2\cdot\text{K}$. Hence,

$$Bi = \frac{hL}{k} = \frac{10\,\text{W/m}^2\cdot\text{K}\,(0.007\text{m})}{180\,\text{W/m}\cdot\text{K}} = 3.9\times10^{-4}$$

$$Bi_r = \frac{h_r L}{k} = \frac{8.2\,\text{W/m}^2\cdot\text{K}\,(0.007\text{m})}{180\,\text{W/m}\cdot\text{K}} = 3.2\times10^{-4}$$

With convection and radiation considered independently or collectively, Bi, Bi_r, Bi + Bi_r << 1 and the lumped capacitance analysis may be used.

The energy balance, Eq. (5.15), associated with Figure 5.5 may be applied to this problem. With $\dot{E}g = 0$, the integral form of the equation is

$$T - T_i = \frac{A_s}{\rho Vc}\int_0^t \left[q''_h - h\left(T - T_\infty\right) - \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right]dt$$

Integrating numerically, we obtain, for T = 135°C,

$$t = 168\text{s}$$ <

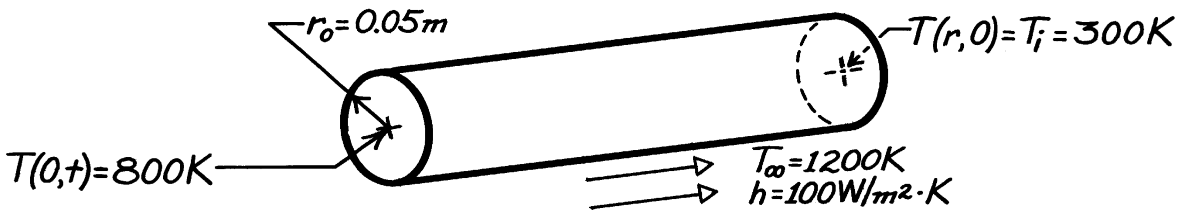
COMMENTS: Note that, if heat transfer is by natural convection, h, like h_r, will vary during the process from a value of 0 at t = 0 to a maximum at t = 168s.

PROBLEM 5.9

KNOWN: Diameter and radial temperature of AISI 1010 carbon steel shaft. Convection coefficient and temperature of furnace gases.

FIND: Time required for shaft centerline to reach a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction, (2) Constant properties.

PROPERTIES: AISI 1010 carbon steel, *Table A.1* $\left(\bar{T} = 550\ K\right)$: $r = 7832\ kg/m^3$, k = 51.2 W/m·K, c = 541 J/kg·K, $\alpha = 1.21 \times 10^{-5}\ m^2/s$.

ANALYSIS: The Biot number is

$$Bi = \frac{hr_o/2}{k} = \frac{100\ W/m^2 \cdot K\left(0.05\ m/2\right)}{51.2\ W/m \cdot K} = 0.0488.$$

Hence, the lumped capacitance method can be applied. From Equation 5.6,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hAs}{r\,Vc}\right)t\right] = \exp\left[-\frac{4h}{r\,cD}t\right]$$

$$\ln\left(\frac{800-1200}{300-1200}\right) = -0.811 = -\frac{4 \times 100\ W/m^2 \cdot K}{7832\ kg/m^3\left(541\ J/kg \cdot K\right)0.1\ m}t$$

$$t = 859\ s.$$

<

COMMENTS: To check the validity of the foregoing result, use the one-term approximation to the series solution. From Equation 5.49c,

$$\frac{T_o - T_\infty}{T_i - T_\infty} = \frac{-400}{-900} = 0.444 = C_1\ \exp\left(-V_1^2 Fo\right)$$

For Bi = hr_o/k = 0.0976, Table 5.1 yields ς_1 = 0.436 and C_1 = 1.024. Hence

$$\frac{-\left(0.436\right)^2\left(1.2 \times 10^{-5}\ m^2/s\right)}{\left(0.05\ m\right)^2}t = \ln\left(0.434\right) = -0.835$$

$$t = 915\ s.$$

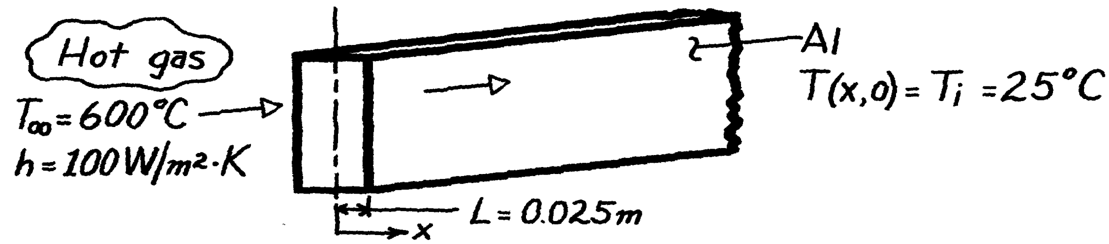
The results agree to within 6%. The lumped capacitance method underestimates the actual time, since the response at the centerline lags that at any other location in the shaft.

PROBLEM 5.10

KNOWN: Configuration, initial temperature and charging conditions of a thermal energy storage unit.

FIND: Time required to achieve 75% of maximum possible energy storage. Temperature of storage medium at this time.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible radiation exchange with surroundings.

PROPERTIES: *Table A-1*, Aluminum, pure $\left(\overline{T} \approx 600K = 327^\circ C\right)$: k = 231 W/m·K, c = 1033 J/kg·K, r = 2702 kg/m^3.

ANALYSIS: Recognizing the characteristic length is the half thickness, find

$$Bi = \frac{hL}{k} = \frac{100\ W/m^2 \cdot K \times 0.025m}{231\ W/m \cdot K} = 0.011.$$

Hence, the lumped capacitance method may be used. From Eq. 5.8,

$$Q = (rVc)q_i\left[1 - \exp(-t/t_t)\right] = -\Delta E_{st} \tag{1}$$

$$-\Delta E_{st,max} = (rVc)q_i. \tag{2}$$

Dividing Eq. (1) by (2),

$$\Delta E_{st}/\Delta E_{st,max} = 1 - \exp(-t/t_{th}) = 0.75.$$

Solving for $t_{th} = \frac{rVc}{hA_s} = \frac{rLc}{h} = \frac{2702\ kg/m^3 \times 0.025m \times 1033\ J/kg \cdot K}{100\ W/m^2 \cdot K} = 698s.$

Hence, the required time is

$$-\exp(-t/698s) = -0.25 \qquad \text{or} \qquad t = 968s.$$ <

From Eq. 5.6,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-t/t_{th})$$

$$T = T_\infty + (T_i - T_\infty)\exp(-t/t_{th}) = 600^\circ C - \left(575^\circ C\right)\exp(-968/698)$$

$$T = 456^\circ C.$$ <

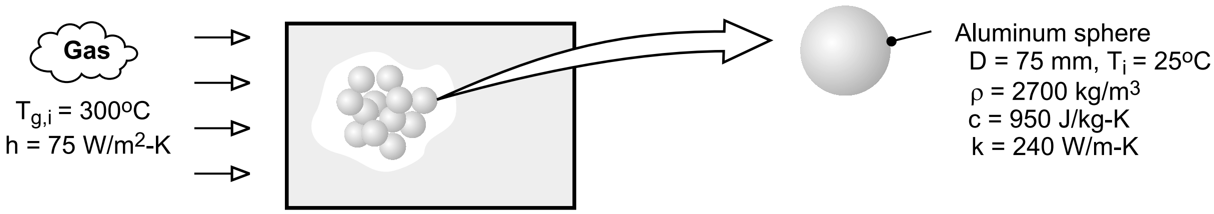
COMMENTS: For the prescribed temperatures, the property temperature dependence is significant and some error is incurred by assuming constant properties. However, selecting properties at 600K was reasonable for this estimate.

PROBLEM 5.11

KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere to acquire 90% of maximum possible thermal energy and the corresponding center temperature. Potential advantage of using copper in lieu of aluminum.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute Bi = $h(r_o/3)/k = 75\ W/m^2 \cdot K\ (0.025m)/150\ W/m \cdot K = 0.013 < 0.1$. Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time. From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$\frac{Q}{\rho c V \theta_i} = 0.90 = 1 - \exp(-t/\tau_t)$$

where $\tau_t = \rho V c / hA_s = \rho D c / 6h = 2700\,kg/m^3 \times 0.075m \times 950\,J/kg\cdot K / 6 \times 75\,W/m^2 \cdot K = 427s$. Hence,

$$t = -\tau_t \ln(0.1) = 427s \times 2.30 = 984s \qquad <$$

From Eq. (5.6), the corresponding temperature at any location in the sphere is

$$T(984s) = T_{g,i} + \left(T_i - T_{g,i}\right)\exp(-6ht/\rho D c)$$

$$T(984s) = 300°C - 275°C\ \exp\left(-6 \times 75\,W/m^2 \cdot K \times 984s / 2700\,kg/m^3 \times 0.075m \times 950\,J/kg \cdot K\right)$$

$$T(984)s = 272.5°C \qquad <$$

Obtaining the density and specific heat of copper from Table A-1, we see that $(\rho c)_{Cu} \approx 8900\ kg/m^3 \times 400\ J/kg \cdot K = 3.56 \times 10^6\ J/m^3 \cdot K > (\rho c)_{Al} = 2.57 \times 10^6\ J/m^3 \cdot K$. Hence, for an equivalent sphere diameter, the copper can store approximately 38% more thermal energy than the aluminum.

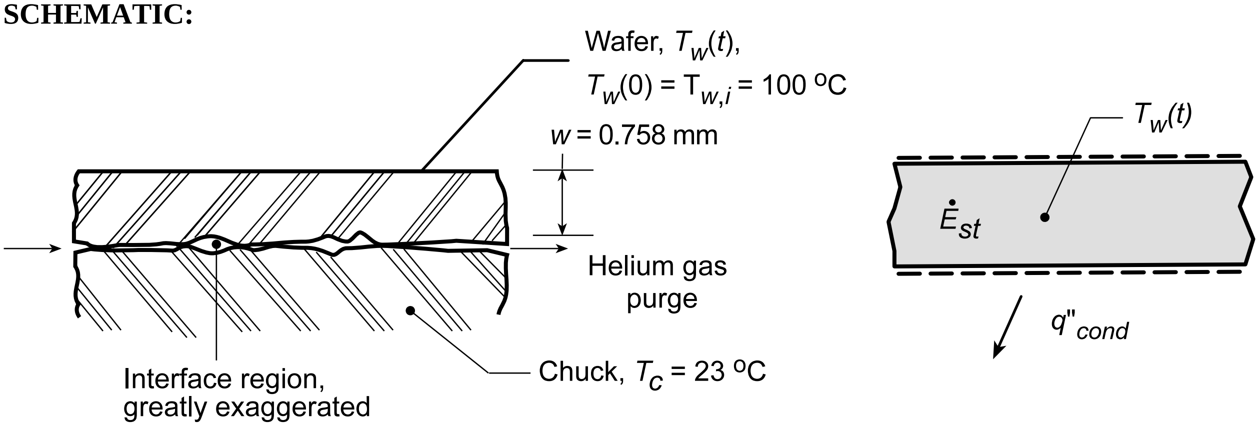
COMMENTS: Before the packed bed becomes fully charged, the temperature of the gas decreases as it passes through the bed. Hence, the time required for a sphere to reach a prescribed state of thermal energy storage increases with increasing distance from the bed inlet.

PROBLEM 5.12

KNOWN: Wafer, initially at 100°C, is suddenly placed on a chuck with uniform and constant temperature, 23°C. Wafer temperature after 15 seconds is observed as 33°C.

FIND: (a) Contact resistance, R''_{tc}, between interface of wafer and chuck through which helium slowly flows, and (b) Whether R''_{tc} will change if air, rather than helium, is the purge gas.

SCHEMATIC:



PROPERTIES: Wafer (silicon, typical values): ρ = 2700 kg/m³, c = 875 J/kg·K, k = 177 W/m·K.

ASSUMPTIONS: (1) Wafer behaves as a space-wise isothermal object, (2) Negligible heat transfer from wafer top surface, (3) Chuck remains at uniform temperature, (4) Thermal resistance across the interface is due to conduction effects, not convective, (5) Constant properties.

ANALYSIS: (a) Perform an energy balance on the wafer as shown in the Schematic.

$$\dot{E}''_{in} - \dot{E}''_{out} + \dot{E}_g = \dot{E}_{st} \tag{1}$$

$$-q''_{cond} = \dot{E}''_{st} \tag{2}$$

$$-\frac{T_w(t) - T_c}{R''_{tc}} = \rho wc \frac{dT_w}{dt} \tag{3}$$

Separate and integrate Eq. (3)

$$-\int_0^t \frac{dt}{\rho wc R''_{tc}} = \int_{T_{wi}}^{T_w} \frac{dT_w}{T_w - T_c} \tag{4}$$

$$\frac{T_w(t) - T_c}{T_{wi} - T_c} = \exp\left[-\frac{t}{\rho wc R''_{tc}}\right] \tag{5}$$

Substituting numerical values for $T_w(15s) = 33$°C,

$$\frac{(33-23)^\circ C}{(100-23)^\circ C} = \exp\left[\frac{15\,s}{2700\,kg/m^3 \times 0.758\times10^{-3}\,m \times 875\,J/kg\cdot K \times R''_{tc}}\right] \tag{6}$$

$$R''_{tc} = 0.0041\,m^2\cdot K/W$$ <

(b) R''_{tc} will increase since $k_{air} < k_{helium}$. See Table A.4.

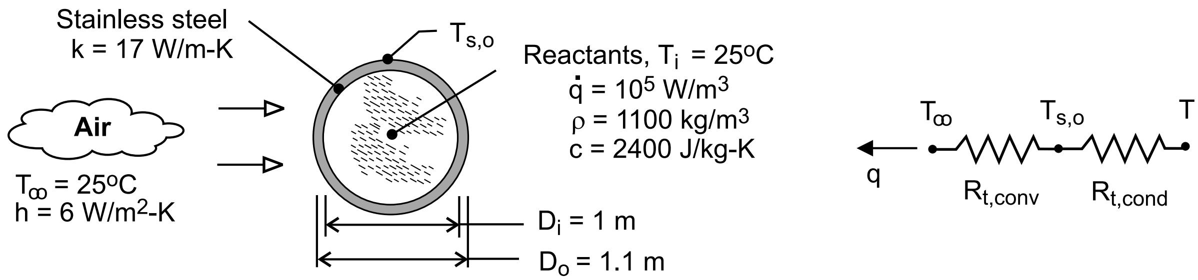
COMMENTS: Note that $Bi = R_{int}/R_{ext} = (w/k)/R''_{tc} = 0.001$. Hence the spacewise isothermal assumption is reasonable.

PROBLEM 5.13

KNOWN: Inner diameter and wall thickness of a spherical, stainless steel vessel. Initial temperature, density, specific heat and heat generation rate of reactants in vessel. Convection conditions at outer surface of vessel.

FIND: (a) Temperature of reactants after one hour of reaction time, (b) Effect of convection coefficient on thermal response of reactants.

SCHEMATIC:



ASSUMPTIONS: (1) Temperature of well stirred reactants is uniform at any time and is equal to inner surface temperature of vessel ($T = T_{s,i}$), (2) Thermal capacitance of vessel may be neglected, (3) Negligible radiation exchange with surroundings, (4) Constant properties.

ANALYSIS: (a) Transient thermal conditions within the reactor may be determined from Eq. (5.25), which reduces to the following form for $T_i - T_\infty = 0$.

$$T = T_\infty + (b/a)\left[1 - \exp(-at)\right]$$

where $a = UA/\rho Vc$ and $b = \dot{E}_g / \rho Vc = \dot{q}/\rho c$. From Eq. (3.19) the product of the overall heat transfer coefficient and the surface area is $UA = (R_{cond} + R_{conv})^{-1}$, where from Eqs. (3.36) and (3.9),

$$R_{t,cond} = \frac{1}{2\pi k}\left(\frac{1}{D_i} - \frac{1}{D_o}\right) = \frac{1}{2\pi(17\,W/m\cdot K)}\left(\frac{1}{1.0m} - \frac{1}{1.1m}\right) = 8.51\times10^{-4}\,K/W$$

$$R_{t,conv} = \frac{1}{hA_o} = \frac{1}{\left(6\,W/m^2\cdot K\right)\pi(1.1m)^2} = 0.0438\,K/W$$

Hence, UA = 24.4 W/K. It follows that, with $V = \pi D_i^3/6$,

$$a = \frac{UA}{\rho Vc} = \frac{6(22.4\,W/K)}{1100\,kg/m^3 \times \pi(1m)^3\,2400\,J/kg\cdot K} = 1.620\times10^{-5}s^{-1}$$

$$b = \frac{\dot{q}}{\rho c} = \frac{10^4\,W/m^3}{1100\,kg/m^3 \times 2400\,J/kg\cdot K} = 3.788\times10^{-3}K/s$$

With (b/a) = 233.8°C and t = 18,000s,

$$T = 25°C + 233.8°C\left[1 - \exp\left(-1.62\times10^{-5}s^{-1}\times18,000s\right)\right] = 84.1°C \qquad <$$

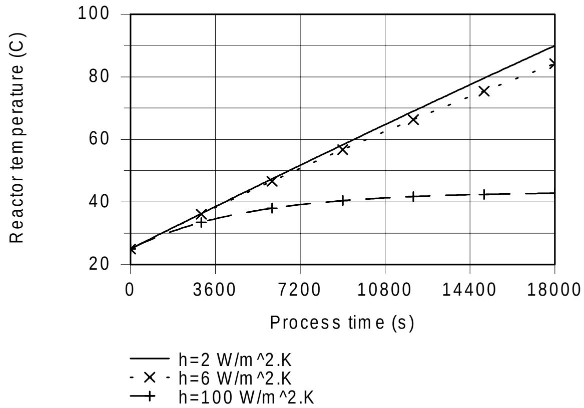
Neglecting the thermal capacitance of the vessel wall, the heat rate by conduction through the wall is equal to the heat transfer by convection from the outer surface, and from the thermal circuit, we know that

Continued …..

$$\frac{T - T_{s,o}}{T_{s,o} - T_\infty} = \frac{R_{t,cond}}{R_{t,conv}} = \frac{8.51 \times 10^{-4}\,K/W}{0.0438\,K/W} = 0.0194$$

$$T_{s,o} = \frac{T + 0.0194\,T_\infty}{1.0194} = \frac{84.1°C + 0.0194(25°C)}{1.0194} = 83.0°C$$ <

(b) Representative low and high values of h could correspond to 2 W/m^2·K and 100 W/m^2·K for free and forced convection, respectively. Calculations based on Eq. (5.25) yield the following temperature histories.



Forced convection is clearly an effective means of reducing the temperature of the reactants and accelerating the approach to steady-state conditions.

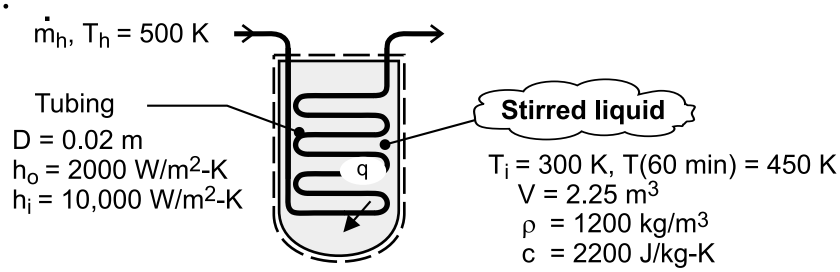
COMMENTS: The validity of neglecting thermal energy storage effects for the vessel may be assessed by contrasting its thermal capacitance with that of the reactants. Selecting values of ρ = 8000 kg/m^3 and c = 475 J/kg·K for stainless steel from Table A-1, the thermal capacitance of the vessel is $C_{t,v} = (\rho Vc)_{st} = 6.57 \times 10^5$ J/K, where $V = (\pi/6)\left(D_o^3 - D_i^3\right)$. With $C_{t,r} = (\rho Vc)_r = 2.64 \times 10^6$ J/K for the reactants, $C_{t,r}/C_{t,v} \approx 4$. Hence, the capacitance of the vessel is not negligible and should be considered in a more refined analysis of the problem.

PROBLEM 5.14

KNOWN: Volume, density and specific heat of chemical in a stirred reactor. Temperature and convection coefficient associated with saturated steam flowing through submerged coil. Tube diameter and outer convection coefficient of coil. Initial and final temperatures of chemical and time span of heating process.

FIND: Required length of submerged tubing. Minimum allowable steam flowrate.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible heat loss from vessel to surroundings, (3) Chemical is isothermal, (4) Negligible work due to stirring, (5) Negligible thermal energy generation (or absorption) due to chemical reactions associated with the batch process, (6) Negligible tube wall conduction resistance, (7) Negligible kinetic energy, potential energy, and flow work changes for steam.

ANALYSIS: Heating of the chemical can be treated as a transient, lumped capacitance problem, wherein heat transfer from the coil is balanced by the increase in thermal energy of the chemical. Hence, conservation of energy yields

$$\frac{dU}{dt} = \rho V c \frac{dT}{dt} = U A_s \left(T_h - T\right)$$

Integrating, $$\int_{T_i}^{T} \frac{dT}{T_h - T} = \frac{U A_s}{\rho V c} \int_{o}^{t} dt$$

$$-\ln \frac{T_h - T}{T_h - T_i} = \frac{U A_s t}{\rho V c}$$

$$A_s = -\frac{\rho V c}{U t} \ln \frac{T_h - T}{T_h - T_i} \qquad (1)$$

$$U = \left(h_i^{-1} + h_o^{-1}\right)^{-1} = \left[(1/10{,}000) + (1/2000)\right]^{-1} \mathrm{W/m^2 \cdot K}$$

$$U = 1670\,\mathrm{W/m^2 \cdot K}$$

$$A_s = -\frac{\left(1200\,\mathrm{kg/m^3}\right)\left(2.25\mathrm{m^3}\right)\left(2200\,\mathrm{J/kg \cdot K}\right)}{\left(1670\,\mathrm{W/m^2 \cdot K}\right)\left(3600\mathrm{s}\right)} \ln \frac{500 - 450}{500 - 300} = 1.37\mathrm{m^2}$$

$$L = \frac{A_s}{\pi D} = \frac{1.37\mathrm{m^2}}{\pi\left(0.02\mathrm{m}\right)} = 21.8\mathrm{m}$$ <

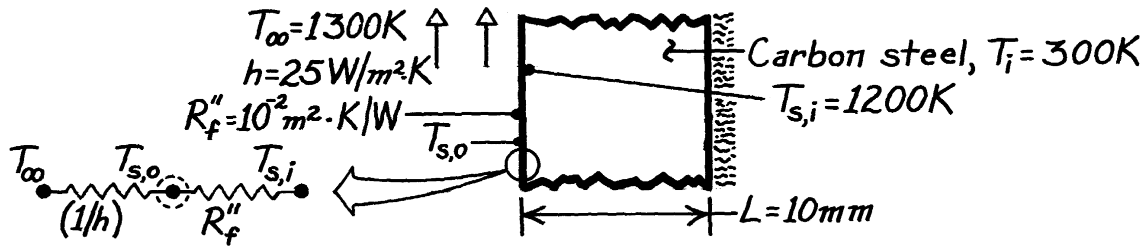
COMMENTS: Eq. (1) could also have been obtained by adapting Eq. (5.5) to the conditions of this problem, with T_∞ and h replaced by T_h and U, respectively.

PROBLEM 5.15

KNOWN: Thickness and properties of furnace wall. Thermal resistance of film on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of film surface temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible film thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel (given): $\rho = 7850\ \text{kg/m}^3$, $c = 430$ J/kg·K, $k = 60$ W/m·K.

ANALYSIS: The overall coefficient for heat transfer from the surface of the steel to the gas is

$$U = \left(R''_{tot}\right)^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \text{W/m}^2\cdot\text{K}} + 10^{-2}\,\text{m}^2\cdot\text{K/W}\right)^{-1} = 20\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$Bi = \frac{UL}{k} = \frac{20\ \text{W/m}^2\cdot\text{K}\times 0.01\ \text{m}}{60\ \text{W/m}\cdot\text{K}} = 0.0033$$

and the lumped capacitance method can be used.

(a) It follows that

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-t/\tau_t\right) = \exp\left(-t/RC\right) = \exp\left(-Ut/\rho Lc\right)$$

$$t = -\frac{\rho Lc}{U}\ln\frac{T - T_\infty}{T_i - T_\infty} = -\frac{7850\ \text{kg/m}^3\left(0.01\ \text{m}\right)430\ \text{J/kg}\cdot\text{K}}{20\ \text{W/m}^2\cdot\text{K}}\ln\frac{1200 - 1300}{300 - 1300}$$

$$t = 3886\text{s} = 1.08\text{h}.$$ <

(b) Performing an energy balance at the outer surface (s,o),

$$h\left(T_\infty - T_{s,o}\right) = \left(T_{s,o} - T_{s,i}\right)/R''_f$$

$$T_{s,o} = \frac{hT_\infty + T_{s,i}/R''_f}{h + \left(1/R''_f\right)} = \frac{25\ \text{W/m}^2\cdot\text{K}\times 1300\ \text{K} + 1200\ \text{K}/10^{-2}\,\text{m}^2\cdot\text{K/W}}{\left(25 + 100\right)\text{W/m}^2\cdot\text{K}}$$

$$T_{s,o} = 1220\ \text{K}.$$ <

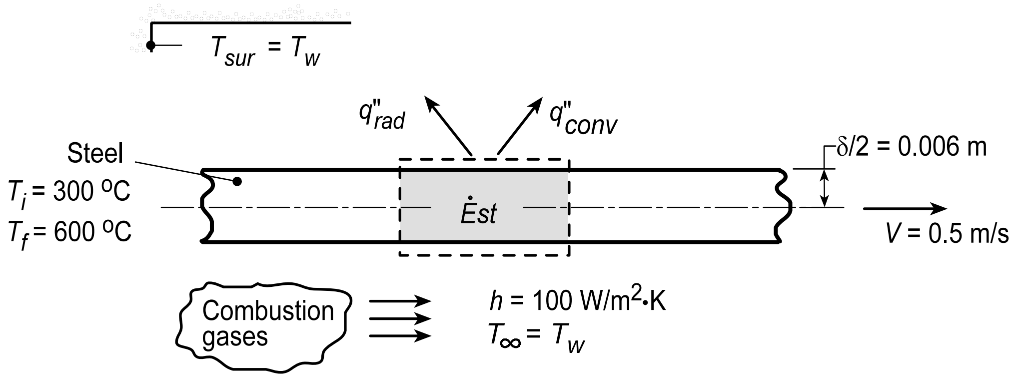
COMMENTS: The film increases τ_t by increasing R_t but not C_t.

PROBLEM 5.16

KNOWN: Thickness and properties of strip steel heated in an annealing process. Furnace operating conditions.

FIND: (a) Time required to heat the strip from 300 to 600°C. Required furnace length for prescribed strip velocity (V = 0.5 m/s), (b) Effect of wall temperature on strip speed, temperature history, and radiation coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible temperature gradients in transverse direction across strip, (c) Negligible effect of strip conduction in longitudinal direction.

PROPERTIES: Steel: ρ = 7900 kg/m^3, c_p = 640 J/kg·K, k = 30 W/m·K, ε= 0.7.

ANALYSIS: (a) Considering a fixed (control) mass of the moving strip, its temperature variation with time may be obtained from an energy balance which equates the change in energy storage to heat transfer by convection and radiation. If the surface area associated with one side of the control mass is designated as A_s, $A_{s,c} = A_{s,r} = 2A_s$ and $V = \delta A_s$ in Equation 5.15, which reduces to

$$\rho c \delta \frac{dT}{dt} = -2\left[h\left(T - T_\infty\right) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right]$$

or, introducing the radiation coefficient from Equations 1.8 and 1.9 and integrating,

$$T_f - T_i = -\frac{1}{\rho c(\delta/2)}\int_o^{t_f}\left[h\left(T - T_\infty\right) + h_r\left(T - T_{sur}\right)\right]dt$$

Using the IHT *Lumped Capacitance Model* to integrate numerically with T_i = 573 K, we find that T_f = 873 K corresponds to

$$t_f \approx 209\text{s}$$ <

in which case, the required furnace length is

$$L = Vt_f \approx 0.5\,\text{m/s} \times 209\,\text{s} \approx 105\,\text{m}$$ <

(b) For T_w = 1123 K and 1273 K, the numerical integration yields $t_f \approx$ 102s and 62s respectively. Hence, for L = 105 m , $V = L/t_f$ yields

$$V\left(T_w = 1123\text{K}\right) = 1.03\,\text{m/s}$$

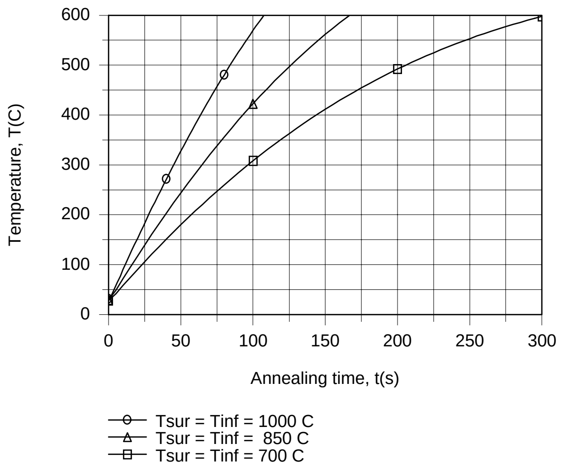
$$V\left(T_w = 1273\text{K}\right) = 1.69\,\text{m/s}$$ <

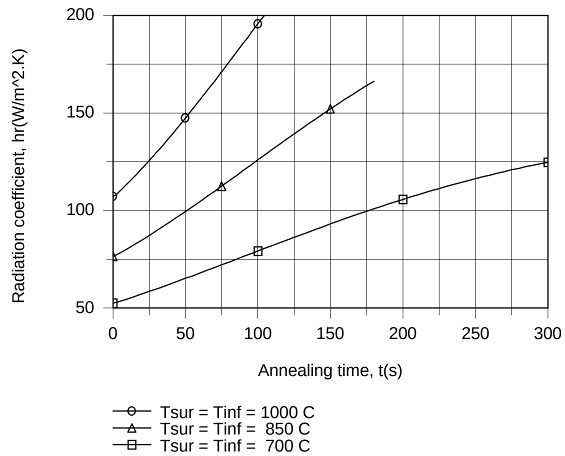
Continued...

PROBLEM 5.16 (Cont.)

which correspond to increased process rates of 106% and 238%, respectively. Clearly, productivity can be enhanced by increasing the furnace environmental temperature, albeit at the expense of increasing energy utilization and operating costs.

If the annealing process extends from 25°C (298 K) to 600°C (873 K), numerical integration yields the following results for the prescribed furnace temperatures.





As expected, the heating rate and time, respectively, increase and decrease significantly with increasing T_w. Although the radiation heat transfer rate decreases with increasing time, the coefficient h_r increases with t as the strip temperature approaches T_w.

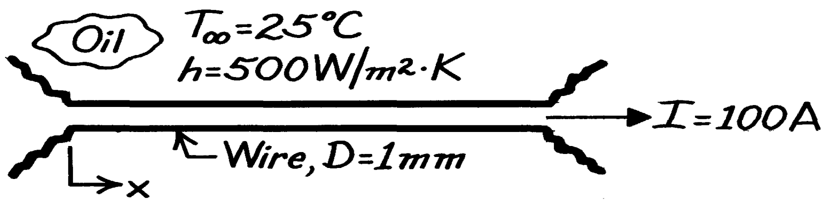
COMMENTS: To check the validity of the lumped capacitance approach, we calculate the Biot number based on a maximum cumulative coefficient of $(h + h_r) \approx 300\ \mathrm{W/m^2 \cdot K}$. It follows that $Bi = (h + h_r)(\delta/2)/k = 0.06$ and the assumption is valid.

PROBLEM 5.17

KNOWN: Diameter, resistance and current flow for a wire. Convection coefficient and temperature of surrounding oil.

FIND: Steady-state temperature of the wire. Time for the wire temperature to come within 1°C of its steady-state value.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Wire temperature is independent of x.

PROPERTIES: Wire (given): $\rho = 8000 \text{ kg/m}^3$, $c_p = 500 \text{ J/kg·K}$, $k = 20 \text{ W/m·K}$, $R'_e = 0.01\ \Omega/\text{m}$.

ANALYSIS: Since

$$\text{Bi} = \frac{h(r_o/2)}{k} = \frac{500 \text{ W/m}^2\cdot\text{K}\left(2.5\times10^{-4}\text{m}\right)}{20 \text{ W/m}\cdot\text{K}} = 0.006 < 0.1$$

the lumped capacitance method can be used. The problem has been analyzed in Example 1.3, and without radiation the steady-state temperature is given by

$$p\ Dh\left(T - T_\infty\right) = I^2 R'_e.$$

Hence

$$T = T_\infty + \frac{I^2 R'_e}{p\ Dh} = 25^\circ\text{C} + \frac{(100\text{A})^2\ 0.01\Omega/\text{m}}{p\left(0.001 \text{ m}\right)500 \text{ W/m}^2\cdot\text{K}} = 88.7^\circ\text{C}.$$ <

With no radiation, the transient thermal response of the wire is governed by the expression (Example 1.3)

$$\frac{dT}{dt} = \frac{I^2 R'_e}{r c_p\left(p D^2/4\right)} - \frac{4h}{r\ c_p D}\left(T - T_\infty\right).$$

With $T = T_i = 25°\text{C}$ at $t = 0$, the solution is

$$\frac{T - T_\infty - \left(I^2 R'_e/p\ Dh\right)}{T_i - T_\infty - \left(I^2 R'_e/p\ Dh\right)} = \exp\left(-\frac{4h}{r\ c_p D}t\right).$$

Substituting numerical values, find

$$\frac{87.7 - 25 - 63.7}{25 - 25 - 63.7} = \exp\left(-\frac{4\times500 \text{ W/m}^2\cdot\text{K}}{8000 \text{ kg/m}^3\times500 \text{ J/kg}\cdot\text{K}\times0.001 \text{ m}}t\right)$$

$$t = 8.31\text{s}.$$ <

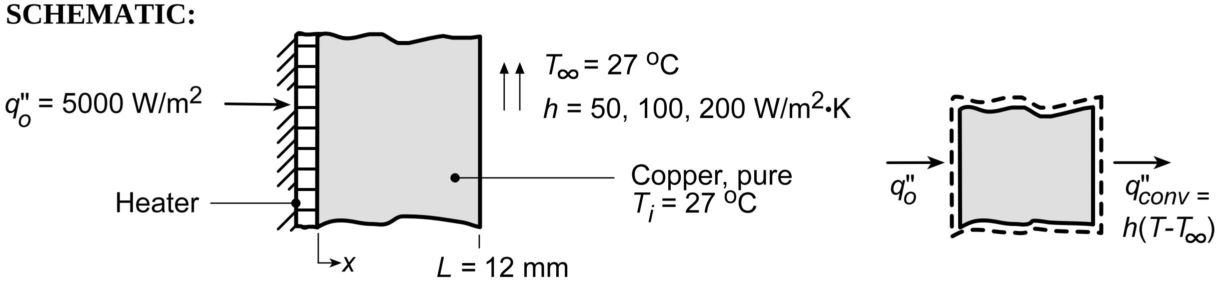
COMMENTS: The time to reach steady state increases with increasing ρ, c_p and D and with decreasing h.

PROBLEM 5.18

KNOWN: Electrical heater attached to backside of plate while front is exposed to a convection process (T_∞, h); initially plate is at uniform temperature T_∞ before heater power is switched on.

FIND: (a) Expression for temperature of plate as a function of time assuming plate is spacewise isothermal, (b) Approximate time to reach steady-state and $T(\infty)$ for prescribed T_∞, h and q''_o when wall material is pure copper, (c) Effect of h on thermal response.

SCHEMATIC:



ASSUMPTIONS: (1) Plate behaves as lumped capacitance, (2) Negligible loss out backside of heater, (3) Negligible radiation, (4) Constant properties.

PROPERTIES: *Table A-1*, Copper, pure (350 K): k = 397 W/m·K, c_p = 385 J/kg·K, ρ = 8933 kg/m^3.

ANALYSIS: (a) Following the analysis of Section 5.3, the energy conservation requirement for the system is $\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st}$ or $q''_o - h(T - T_\infty) = \rho L c_p \, dT/dt$. Rearranging, and with R''_t = 1/h and C''_t = ρLc$_p$,

$$T - T_\infty - q''_o/h = -R''_t \cdot C''_t \, dT/dt \tag{1}$$

Defining $\theta(t) \equiv T - T_\infty - q''_o/h$ with dθ = dT, the differential equation is

$$\theta = -R''_t C''_t \frac{d\theta}{dt}. \tag{2}$$

Separating variables and integrating,

$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t \frac{dt}{R''_t C''_t}$$

it follows that

$$\frac{\theta}{\theta_i} = \exp\left(-\frac{t}{R''_t C''_t}\right) \tag{3}$$ <

where $\theta_i = \theta(0) = T_i - T_\infty - (q''_o/h)$ (4)

(b) For h = 50 W/m^2 ·K, the steady-state temperature can be determined from Eq. (3) with $t \to \infty$; that is,

$$\theta(\infty) = 0 = T(\infty) - T_\infty - q''_o/h \qquad \text{or} \qquad T(\infty) = T_\infty + q''_o/h,$$

giving T(∞) = 27°C + 5000 W/m^2 /50 W/m^2·K = 127°C. To estimate the time to reach steady-state, first determine the thermal time constant of the system,

$$\tau_t = R''_t C''_t = \left(\frac{1}{h}\right)(\rho c_p L) = \left(\frac{1}{50\,W/m^2\cdot K}\right)\left(8933\,kg/m^3 \times 385\,J/kg\cdot K \times 12\times10^{-3}\,m\right) = 825\,s$$

Continued...

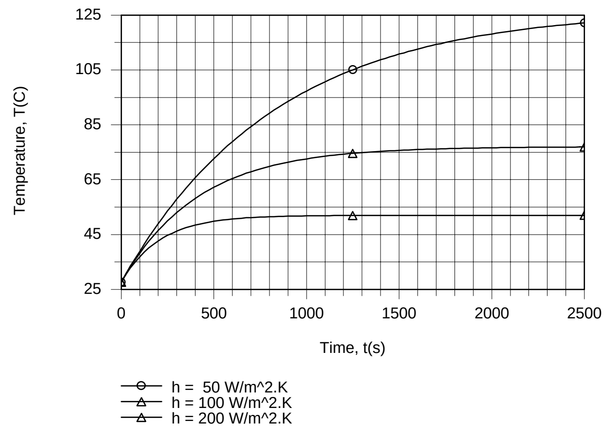
PROBLEM 5.18 (Cont.)

When $t = 3\tau_t = 3 \times 825\text{s} = 2475\text{s}$, Eqs. (3) and (4) yield

$$\theta(3\tau_t) = T(3\tau_t) - 27^\circ C - \frac{5000\,W/m^2}{50\,W/m^2 \cdot K} = e^{-3}\left[27^\circ C - 27^\circ C - \frac{5000\,W/m^2}{50\,W/m^2 \cdot K}\right]$$

$T(3\tau_t) = 122°C$ $\quad <$

(c) As shown by the following graphical results, which were generated using the IHT *Lumped Capacitance Model*, the steady-state temperature and the time to reach steady-state both decrease with increasing h.



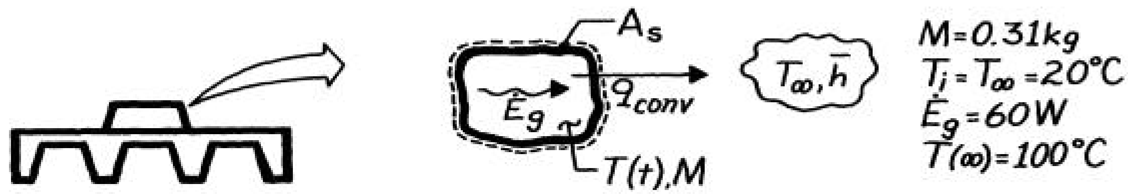
COMMENTS: Note that, even for $h = 200\ W/m^2 \cdot K$, $Bi = hL/k \ll 0.1$ and assumption (1) is reasonable.

PROBLEM 5.19

KNOWN: Electronic device on aluminum, finned heat sink modeled as spatially isothermal object with internal generation and convection from its surface.

FIND: (a) Temperature response after device is energized, (b) Temperature rise for prescribed conditions after 5 min.

SCHEMATIC:



ASSUMPTIONS: (1) Spatially isothermal object, (2) Object is primarily aluminum, (3) Initially, object is in equilibrium with surroundings at T_∞.

PROPERTIES: *Table A-1*, Aluminum, pure $\left(\bar{T} = (20+100)^\circ \text{C}/2 \approx 333\text{K}\right)$: c = 918 J/kg·K.

ANALYSIS: (a) Following the general analysis of Section 5.3, apply the conservation of energy requirement to the object,

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \qquad \dot{E}_g - \bar{h}A_s(T - T_\infty) = Mc\frac{dT}{dt} \tag{1}$$

where T = T(t). Consider now steady-state conditions, in which case the storage term of Eq. (1) is zero. The temperature of the object will be T(∞) such that

$$\dot{E}_g = \bar{h}A_s\left(T(\infty) - T_\infty\right). \tag{2}$$

Substituting for $\dot{E}_g$ using Eq. (2) into Eq. (1), the differential equation is

$$\left[T(\infty) - T_\infty\right] - \left[T - T_\infty\right] = \frac{Mc}{\bar{h}A_s}\frac{dT}{dt} \qquad \text{or} \qquad q = -\frac{Mc}{\bar{h}A_s}\frac{dq}{dt} \tag{3,4}$$

with θ ≡ T - T(∞) and noting that dθ = dT. Identifying $R_t = 1/\bar{h}A_s$ and $C_t = Mc$, the differential equation is integrated with proper limits,

$$\frac{1}{R_t C_t}\int_0^t dt = -\int_{q_i}^{q}\frac{dq}{q} \qquad \text{or} \qquad \frac{q}{q_i} = \exp\left[-\frac{t}{R_t C_t}\right] \tag{5}$$ <

where $\theta_i = \theta(0) = T_i - T(\infty)$ and T_i is the initial temperature of the object.

(b) Using the information about steady-state conditions and Eq. (2), find first the thermal resistance and capacitance of the system,

$$R_t = \frac{1}{\bar{h}A_s} = \frac{T(\infty) - T_\infty}{\dot{E}_g} = \frac{(100-20)^\circ \text{C}}{60 \text{ W}} = 1.33 \text{ K/W} \qquad C_t = Mc = 0.31 \text{ kg} \times 918 \text{ J/kg}\cdot\text{K} = 285 \text{ J/K}.$$

Using Eq. (5), the temperature of the system after 5 minutes is

$$\frac{q(5\text{min})}{q_i} = \frac{T(5\text{min}) - T(\infty)}{T_i - T(\infty)} = \frac{T(5\text{min}) - 100^\circ\text{C}}{(20-100)^\circ \text{C}} = \exp\left[-\frac{5\times 60\text{s}}{1.33 \text{ K/W} \times 285 \text{ J/K}}\right] = 0.453$$

$$T(5\text{min}) = 100^\circ\text{C} + (20-100)^\circ\text{C} \times 0.453 = 63.8^\circ\text{C}$$ <

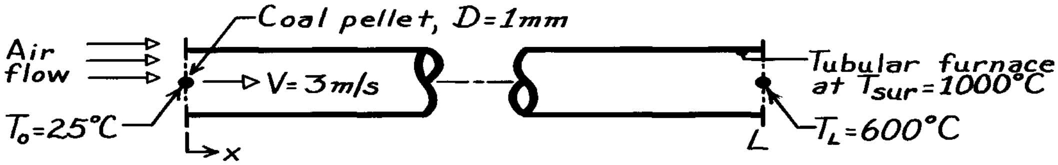
COMMENTS: Eq. 5.24 may be used directly for Part (b) with a = hA_s/Mc and $b = \dot{E}_g/Mc$.

PROBLEM 5.20

KNOWN: Spherical coal pellet at 25°C is heated by radiation while flowing through a furnace maintained at 1000°C.

FIND: Length of tube required to heat pellet to 600°C.

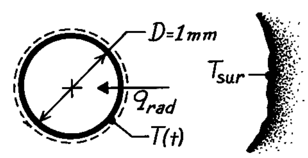
SCHEMATIC:



ASSUMPTIONS: (1) Pellet is suspended in air flow and subjected to only radiative exchange with furnace, (2) Pellet is small compared to furnace surface area, (3) Coal pellet has emissivity, ε = 1.

PROPERTIES: *Table A-3*, Coal $\left(\overline{T} = (600 + 25)^\circ C/2 = 585K,\ \text{however, only 300K data available}\right)$: ρ = 1350 kg/m^3, c_p = 1260 J/kg·K, k = 0.26 W/m·K.

ANALYSIS: Considering the pellet as spatially isothermal, use the lumped capacitance method of Section 5.3 to find the time required to heat the pellet from T_o = 25°C to T_L = 600°C. From an energy balance on the pellet $\dot{E}_{in} = \dot{E}_{st}$ where



$$\dot{E}_{in} = q_{rad} = \sigma A_s\left(T_{sur}^4 - T_s^4\right) \qquad \dot{E}_{st} = \rho \forall c_p \frac{dT}{dt}$$

giving
$$A_s\sigma\left(T_{sur}^4 - T_s^4\right) = \rho\forall c_p \frac{dT}{dt}.$$

Separating variables and integrating with limits shown, the temperature-time relation becomes

$$\frac{A_s\sigma}{\rho\forall c_p}\int_0^t dt = \int_{T_o}^{T_L}\frac{dT}{T_{sur}^4 - T^4}.$$

The integrals are evaluated in Eq. 5.18 giving

$$t = \frac{\rho\forall c_p}{4A_s\sigma T_{sur}^3}\left\{\ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left[\frac{T}{T_{sur}}\right] - \tan^{-1}\left[\frac{T_i}{T_{sur}}\right]\right]\right\}.$$

Recognizing that $A_s = \pi D^2$ and $\forall = \pi D^3/6$ or $A_s/\forall = 6/D$ and substituting values,

$$t = \frac{1350\ kg/m^3\,(0.001\ m)1260\ J/kg\cdot K}{24\times 5.67\times 10^{-8}\ W/m^2\cdot K^4\,(1273\ K)^3}\left\{\ln\frac{1273+873}{1273-873} - \ln\frac{1273+298}{1273-298}\right.$$

$$\left. +2\left[\tan^{-1}\left(\frac{873}{1273}\right) - \tan^{-1}\left(\frac{298}{1273}\right)\right]\right\} = 1.18s.$$

Hence, $\qquad L = V\cdot t = 3m/s\times 1.18s = 3.54m.$ <

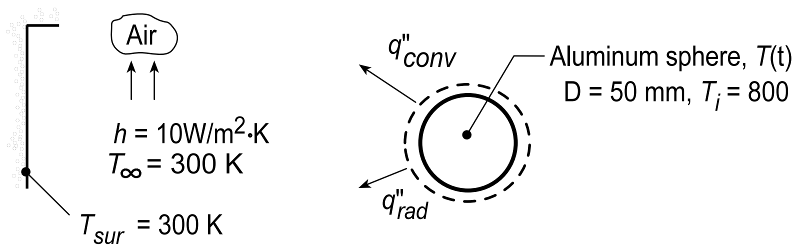
The validity of the lumped capacitance method requires $Bi = h(\forall/A_s)k < 0.1$. Using Eq. (1.9) for $h = h_r$ and $\forall/A_s = D/6$, find that when T = 600°C, Bi = 0.19; but when T = 25°C, Bi = 0.10. At early times, when the pellet is cooler, the assumption is reasonable but becomes less appropriate as the pellet heats.

PROBLEM 5.21

KNOWN: Metal sphere, initially at a uniform temperature T_i, is suddenly removed from a furnace and suspended in a large room and subjected to a convection process (T_∞, h) and to radiation exchange with surroundings, T_{sur}.

FIND: (a) Time it takes for sphere to cool to some temperature T, neglecting radiation exchange, (b) Time it takes for sphere to cool to some temperature t, neglecting convection, (c) Procedure to obtain time required if both convection and radiation are considered, (d) Time to cool an anodized aluminum sphere to 400 K using results of Parts (a), (b) and (c).

SCHEMATIC:



ASSUMPTIONS: (1) Sphere is spacewise isothermal, (2) Constant properties, (3) Constant heat transfer convection coefficient, (4) Sphere is small compared to surroundings.

PROPERTIES: *Table A-1*, Aluminum, pure ($\overline{T}$ = [800 + 400] K/2 = 600 K): ρ = 2702 kg/m^3, c = 1033 J/kg·K, k = 231 W/m·K, $\alpha = k/\rho c = 8.276 \times 10^{-5}$ m^2/s; Aluminum, anodized finish: ε = 0.75, polished surface: ε = 0.1.

ANALYSIS: (a) Neglecting radiation, the time to cool is predicted by Eq. 5.5,

$$t = \frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = \frac{\rho Dc}{6h}\ln\frac{T_i - T_\infty}{T - T_\infty} \qquad (1) <$$

where $V/A_s = (\pi D^3/6)/(\pi D^2) = D/6$ for the sphere.

(b) Neglecting convection, the time to cool is predicted by Eq. 5.18,

$$t = \frac{\rho Dc}{24\varepsilon\sigma T_{sur}^3}\left\{\ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{sur}}\right) - \tan^{-1}\left(\frac{T_i}{T_{sur}}\right)\right]\right\} \qquad (2)$$

where $V/A_{s,r} = D/6$ for the sphere.

(c) If convection and radiation exchange are considered, the energy balance requirement results in Eq. 5.15 (with $q_s'' = \dot{E}_g = 0$). Hence

$$\frac{dT}{dt} = \frac{6}{\rho Dc}\left[h\left(T - T_\infty\right) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right] \qquad (3) <$$

where $A_{s(c,r)} = A_s = \pi D^2$ and $V/A_{s(c,r)} = D/6$. This relation must be solved numerically in order to evaluate the time-to-cool.

(d) For the aluminum (pure) sphere with an anodized finish and the prescribed conditions, the times to cool from T_i = 800 K to T = 400 K are:

Continued...

PROBLEM 5.21 (Cont.)

Convection only, Eq. (1)

$$t = \frac{2702\,\mathrm{kg/m^3} \times 0.050\,\mathrm{m} \times 1033\,\mathrm{J/kg \cdot K}}{6 \times 10\,\mathrm{W/m^2 \cdot K}} \ln \frac{800-300}{400-300} = 3743\mathrm{s} = 1.04\mathrm{h}$$ <

Radiation only, Eq. (2)

$$t = \frac{2702\,\mathrm{kg/m^3} \times 0.050\,\mathrm{m} \times 1033\,\mathrm{J/kg \cdot K}}{24 \times 0.75 \times 5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4} \times (300\,\mathrm{K})^3} \cdot \left\{ \left(\ln \frac{400+300}{400-300} - \ln \frac{800+300}{800-300} \right) + 2\left[\tan^{-1}\frac{400}{300} - \tan^{-1}\frac{800}{300} \right] \right\}$$

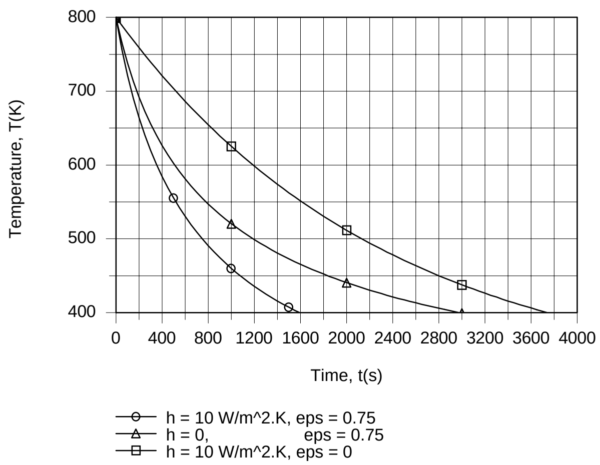
$$t = 5.065 \times 10^3 \{1.946 - 0.789 + 2(0.927 - 1.212)\} = 2973\mathrm{s} = 0.826\mathrm{h}$$ <

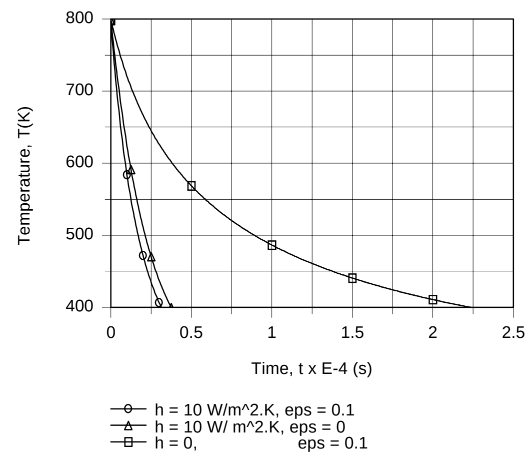
Radiation and convection, Eq. (3)

Using the IHT *Lumped Capacitance Model*, numerical integration yields

$$t \approx 1600\mathrm{s} = 0.444\mathrm{h}$$

In this case, heat loss by radiation exerts the stronger influence, although the effects of convection are by no means negligible. However, if the surface is polished (ε = 0.1), convection clearly dominates. For each surface finish and the three cases, the temperature histories are as follows.





COMMENTS: 1. A summary of the analyses shows the relative importance of the various modes of heat loss:

Active Modes	Time required to cool to 400 K (h)	
	ε = 0.75	ε = 0.1
Convection only	1.040	1.040
Radiation only	0.827	6.194
Both modes	0.444	0.889

2. Note that the spacewise isothermal assumption is justified since Be << 0.1. For the convection-only process,

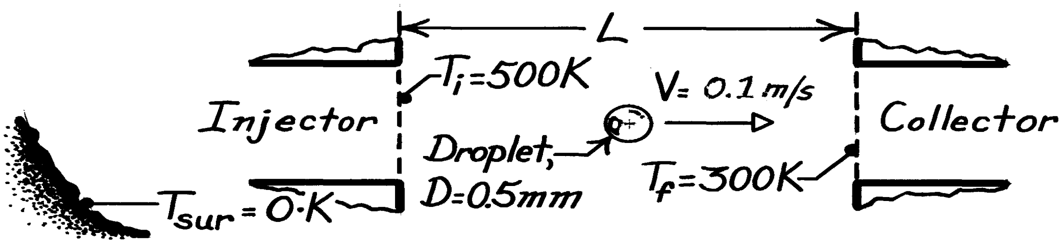
$$Bi = h(r_o/3)/k = 10\ \mathrm{W/m^2 \cdot K}\ (0.025\ \mathrm{m}/3)/231\ \mathrm{W/m \cdot K} = 3.6 \times 10^{-4}$$

PROBLEM 5.22

KNOWN: Droplet properties, diameter, velocity and initial and final temperatures.

FIND: Travel distance and rejected thermal energy.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible radiation from space.

PROPERTIES: Droplet (given): $\rho = 885\ \text{kg/m}^3$, $c = 1900$ J/kg·K, $k = 0.145$ W/m·K, $\varepsilon = 0.95$.

ANALYSIS: To assess the suitability of applying the lumped capacitance method, use Equation 1.9 to obtain the maximum radiation coefficient, which corresponds to $T = T_i$.

$$h_r = \varepsilon\sigma T_i^3 = 0.95 \times 5.67 \times 10^{-8}\,\text{W/m}^2\cdot\text{K}^4\,(500\ \text{K})^3 = 6.73\ \text{W/m}^2\cdot\text{K}.$$

Hence

$$Bi_r = \frac{h_r\,(r_o/3)}{k} = \frac{\left(6.73\ \text{W/m}^2\cdot\text{K}\right)\left(0.25\times10^{-3}\ \text{m}/3\right)}{0.145\ \text{W/m}\cdot\text{K}} = 0.0039$$

and the lumped capacitance method can be used. From Equation 5.19,

$$t = \frac{L}{V} = \frac{\rho c\left(\pi D^3/6\right)}{3\varepsilon\left(\pi D^2\right)\sigma}\left(\frac{1}{T_f^3} - \frac{1}{T_i^3}\right)$$

$$L = \frac{(0.1\ \text{m/s})\,885\ \text{kg/m}^3\,(1900\ \text{J/kg}\cdot\text{K})\,0.5\times10^{-3}\ \text{m}}{18\times0.95\times5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4}\left(\frac{1}{300^3} - \frac{1}{500^3}\right)\frac{1}{\text{K}^3}$$

$$L = 2.52\ \text{m}.$$ <

The amount of energy rejected by each droplet is equal to the change in its internal energy.

$$E_i - E_f = \rho Vc\,(T_i - T_f) = 885\ \text{kg/m}^3\,\pi\,\frac{\left(5\times10^{-4}\text{m}\right)^3}{6}\,1900\ \text{J/kg}\cdot\text{K}\,(200\ \text{K})$$

$$E_i - E_f = 0.022\ \text{J}.$$ <

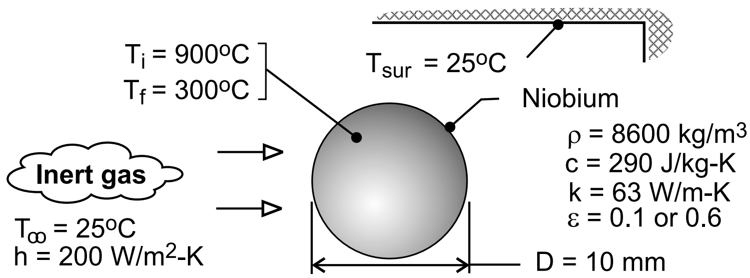
COMMENTS: Because some of the radiation emitted by a droplet will be intercepted by other droplets in the stream, the foregoing analysis overestimates the amount of heat dissipated by radiation to space.

PROBLEM 5.23

KNOWN: Initial and final temperatures of a niobium sphere. Diameter and properties of the sphere. Temperature of surroundings and/or gas flow, and convection coefficient associated with the flow.

FIND: (a) Time required to cool the sphere exclusively by radiation, (b) Time required to cool the sphere exclusively by convection, (c) Combined effects of radiation and convection.

SCHEMATIC:



ASSUMPTIONS: (1) Uniform temperature at any time, (2) Negligible effect of holding mechanism on heat transfer, (3) Constant properties, (4) Radiation exchange is between a small surface and large surroundings.

ANALYSIS: (a) If cooling is exclusively by radiation, the required time is determined from Eq. (5.18). With $V = \pi D^3/6$, $A_{s,r} = \pi D^2$, and $\varepsilon = 0.1$,

$$t = \frac{8600\ \text{kg/m}^3 (290\,\text{J/kg}\cdot\text{K})0.01\text{m}}{24(0.1)5.67\times10^{-8}\text{W/m}^2\cdot\text{K}^4(298\text{K})^3}\left\{\ln\left|\frac{298+573}{298-573}\right| - \ln\left|\frac{298+1173}{298-1173}\right|\right.$$

$$\left. +2\left[\tan^{-1}\left(\frac{573}{298}\right) - \tan^{-1}\left(\frac{1173}{298}\right)\right]\right\}$$

$$t = 6926\text{s}\{1.153 - 0.519 + 2(1.091 - 1.322)\} = 1190\text{s} \qquad (\varepsilon = 0.1)$$ <

If ε = 0.6, cooling is six times faster, in which case,

$$t = 199\text{s} \qquad (\varepsilon = 0.6)$$ <

(b) If cooling is exclusively by convection, Eq. (5.5) yields

$$t = \frac{\rho c D}{6h}\ln\left(\frac{T_i - T_\infty}{T_f - T_\infty}\right) = \frac{8600\ \text{kg/m}^3(290\,\text{J/kg}\cdot\text{K})0.010\text{m}}{1200\,\text{W/m}^2\cdot\text{K}}\ln\left(\frac{875}{275}\right)$$

$$t = 24.1\text{s}$$ <

(c) With both radiation and convection, the temperature history may be obtained from Eq. (5.15).

$$\rho\left(\pi D^3/6\right)c\frac{dT}{dt} = -\pi D^2\left[h(T - T_\infty) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right]$$

Integrating numerically from T_i = 1173 K at t = 0 to T = 573K, we obtain

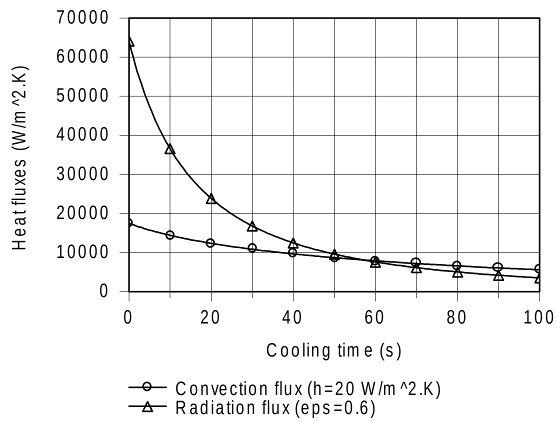
$$t = 21.0\text{s}$$ <

Continued …..

Cooling times corresponding to representative changes in ε and h are tabulated as follows

$h(W/m^2\cdot K)$	200	200	20	500
ε	0.6	1.0	0.6	0.6
t(s)	21.0	19.4	102.8	9.1

For values of h representative of forced convection, the influence of radiation is secondary, even for a maximum possible emissivity of 1.0. Hence, to accelerate cooling, it is necessary to increase h. However, if cooling is by natural convection, radiation is significant. For a representative natural convection coefficient of $h = 20\ W/m^2\cdot K$, the radiation flux exceeds the convection flux at the surface of the sphere during early to intermediate stages of the transient.



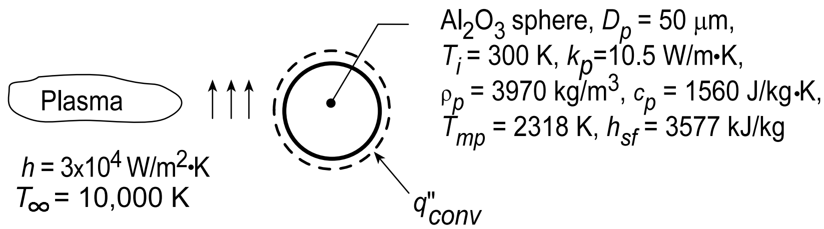
COMMENTS: (1) Even for h as large as $500\ W/m^2\cdot K$, $Bi = h\,(D/6)/k = 500\ W/m^2\cdot K\ (0.01m/6)/63\ W/m\cdot K = 0.013 < 0.1$ and the lumped capacitance model is appropriate. (2) The largest value of h_r corresponds to $T_i = 1173$ K, and for $\varepsilon = 0.6$ Eq. (1.9) yields $h_f = 0.6 \times 5.67 \times 10^{-8}\ W/m^2\cdot K^4\ (1173 + 298)K\ (1173^2 + 298^2)K^2 = 73.3\ W/m^2\cdot K$.

PROBLEM 5.24

KNOWN: Diameter and thermophysical properties of alumina particles. Convection conditions associated with a two-step heating process.

FIND: (a) Time-in-flight (t_{i-f}) required for complete melting, (b) Validity of assuming negligible radiation.

SCHEMATIC:



ASSUMPTIONS: (1) Particle behaves as a lumped capacitance, (2) Negligible radiation, (3) Constant properties.

ANALYSIS: (a) The two-step process involves (i) the time t_1 to heat the particle to its melting point and (ii) the time t_2 required to achieve complete melting. Hence, $t_{i-f} = t_1 + t_2$, where from Eq. (5.5),

$$t_1 = \frac{\rho_p V c_p}{h A_s} \ln \frac{\theta_i}{\theta} = \frac{\rho_p D_p c_p}{6h} \ln \frac{T_i - T_\infty}{T_{mp} - T_\infty}$$

$$t_1 = \frac{3970\,\text{kg/m}^3 \left(50\times10^{-6}\,\text{m}\right) 1560\,\text{J/kg}\cdot\text{K}}{6\left(30{,}000\,\text{W/m}^2\cdot\text{K}\right)} \ln \frac{(300-10{,}000)}{(2318-10{,}000)} = 4\times10^{-4}\,\text{s}$$

Performing an energy balance for the second step, we obtain

$$\int_{t_1}^{t_1+t_2} q_{conv}\,dt = \Delta E_{st}$$

where $q_{conv} = hA_s(T_\infty - T_{mp})$ and $\Delta E_{st} = \rho_p V h_{sf}$. Hence,

$$t_2 = \frac{\rho_p D_p}{6h} \frac{h_{sf}}{\left(T_\infty - T_{mp}\right)} = \frac{3970\,\text{kg/m}^3 \left(50\times10^{-6}\,\text{m}\right)}{6\left(30{,}000\,\text{W/m}^2\cdot\text{K}\right)} \times \frac{3.577\times10^6\,\text{J/kg}}{(10{,}000-2318)\text{K}} = 5\times10^{-4}\,\text{s}$$

Hence $t_{i-f} = 9\times10^{-4}\,\text{s} \approx 1\,\text{ms}$ <

(b) Contrasting the smallest value of the convection heat flux, $q''_{conv,min} = h\left(T_\infty - T_{mp}\right) = 2.3\times10^8\,\text{W/m}^2$ to the largest radiation flux, $q''_{rad,max} = \varepsilon\sigma\left(T_{mp}^4 - T_{sur}^4\right) = 6.5\times10^5\,\text{W/m}^2$, we conclude that radiation is, in fact, negligible.

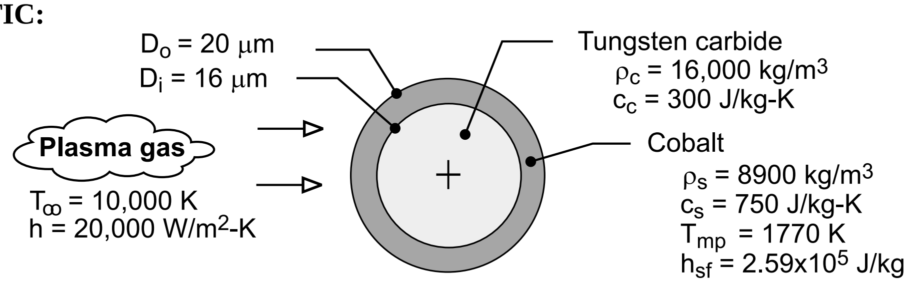
COMMENTS: (1) Since Bi = $(hr_p/3)/k \approx 0.05$, the lumped capacitance assumption is good. (2) In an actual application, the droplet should impact the substrate in a superheated condition ($T > T_{mp}$), which would require a slightly larger t_{i-f}.

PROBLEM 5.25

KNOWN: Diameters, initial temperature and thermophysical properties of WC and Co in composite particle. Convection coefficient and freestream temperature of plasma gas. Melting point and latent heat of fusion of Co.

FIND: Times required to reach melting and to achieve complete melting of Co.

SCHEMATIC:



ASSUMPTIONS: (1) Particle is isothermal at any instant, (2) Radiation exchange with surroundings is negligible, (3) Negligible contact resistance at interface between WC and Co, (4) Constant properties.

ANALYSIS: From Eq. (5.5), the time required to reach the melting point is

$$t_1 = \frac{(\rho Vc)_{tot}}{h\pi D_o^2} \ln \frac{T_i - T_\infty}{T_{mp} - T_\infty}$$

where the total heat capacity of the composite particle is

$$(\rho Vc)_{tot} = (\rho Vc)_c + (\rho Vc)_s = 16,000\,kg/m^3 \left[\pi\left(1.6\times10^{-5}m\right)^3/6\right]300\,J/kg\cdot K$$

$$+8900\,kg/m^3 \left\{\pi/6\left[\left(2.0\times10^{-5}m\right)^3 - \left(1.6\times10^{-5}m\right)^3\right]\right\}750\,J/kg\cdot K$$

$$= \left(1.03\times10^{-8} + 1.36\times10^{-8}\right)J/K = 2.39\times10^{-8}\,J/K$$

$$t_1 = \frac{2.39\times10^{-8}\,J/K}{\left(20,000\,W/m^2\cdot K\right)\pi\left(2.0\times10^{-5}m\right)^2} \ln \frac{(300-10,000)K}{(1770-10,000)K} = 1.56\times10^{-4}s \qquad <$$

The time required to melt the Co may be obtained by applying the first law, Eq. (1.11b) to a control surface about the particle. It follows that

$$E_{in} = h\pi D_o^2\left(T_\infty - T_{mp}\right)t_2 = \Delta E_{st} = \rho_s(\pi/6)\left(D_o^3 - D_i^3\right)h_{sf}$$

$$t_2 = \frac{8900\,kg/m^3(\pi/6)\left[\left(2\times10^{-5}m\right)^3 - \left(1.6\times10^{-5}m\right)^3\right]2.59\times10^5\,J/kg}{\left(20,000\,W/m^2\cdot K\right)\pi\left(2\times10^{-5}m\right)^2(10,000-1770)K} = 2.28\times10^{-5}s \qquad <$$

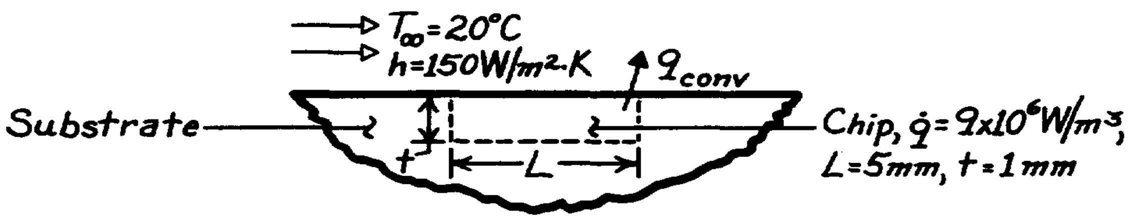
COMMENTS: (1) The largest value of the radiation coefficient corresponds to $h_r = \varepsilon\sigma\,(T_{mp} + T_{sur})\left(T_{mp}^2 + T_{sur}^2\right)$. For the maximum possible value of ε = 1 and T_{sur} = 300K, h_r = 378 W/m$^2\cdot$K << h = 20,000 W/m$^2\cdot$K. Hence, the assumption of negligible radiation exchange is excellent. (2) Despite the large value of h, the small values of D_o and D_i and the large thermal conductivities (~ 40 W/m·K and 70 W/m·K for WC and Co, respectively) render the lumped capacitance approximation a good one. (3) A detailed treatment of plasma heating of a composite powder particle is provided by Demetriou, Lavine and Ghoniem (Proc. 5th ASME/JSME Joint Thermal Engineering Conf., March, 1999).

PROBLEM 5.26

KNOWN: Dimensions and operating conditions of an integrated circuit.

FIND: Steady-state temperature and time to come within 1°C of steady-state.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible heat transfer from chip to substrate.

PROPERTIES: Chip material (given): ρ = 2000 kg/m^3, c = 700 J/kg·K.

ANALYSIS: At steady-state, conservation of energy yields

$$-\dot{E}_{out} + \dot{E}_g = 0$$

$$-h\left(L^2\right)\left(T_f - T_\infty\right) + \dot{q}\left(L^2 \cdot t\right) = 0$$

$$T_f = T_\infty + \frac{\dot{q}t}{h}$$

$$T_f = 20^\circ\text{C} + \frac{9\times10^6\ \text{W/m}^3 \times 0.001\ \text{m}}{150\ \text{W/m}^2\cdot\text{K}} = 80^\circ\text{C}. \qquad <$$

From the general lumped capacitance analysis, Equation 5.15 reduces to

$$r\left(L^2 \cdot t\right)c\frac{dT}{dt} = \dot{q}\left(L^2 \cdot t\right) - h\left(T - T_\infty\right)L^2.$$

With

$$a \equiv \frac{h}{r\,tc} = \frac{150\ \text{W/m}^2\cdot\text{K}}{\left(2000\ \text{kg/m}^3\right)\left(0.001\ \text{m}\right)\left(700\ \text{J/kg}\cdot\text{K}\right)} = 0.107\ \text{s}^{-1}$$

$$b \equiv \frac{\dot{q}}{r\,c} = \frac{9\times10^6\ \text{W/m}^3}{\left(2000\ \text{kg/m}^3\right)\left(700\ \text{J/kg}\cdot\text{K}\right)} = 6.429\ \text{K/s}.$$

From Equation 5.24,

$$\exp\left(-at\right) = \frac{T - T_\infty - b/a}{T_i - T_\infty - b/a} = \frac{\left(79 - 20 - 60\right)\text{K}}{\left(20 - 20 - 60\right)\text{K}} = 0.01667$$

$$t = -\frac{\ln\left(0.01667\right)}{0.107\ \text{s}^{-1}} = 38.3\ \text{s}. \qquad <$$

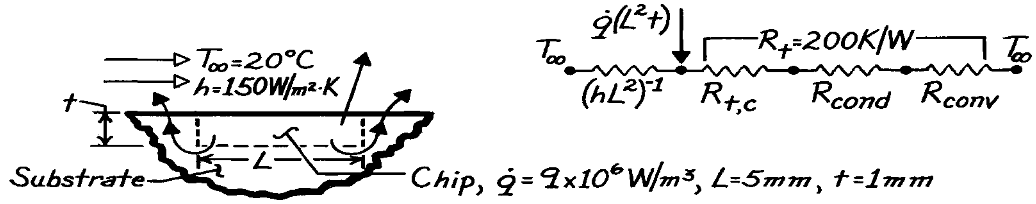
COMMENTS: Due to additional heat transfer from the chip to the substrate, the actual values of T_f and t are less than those which have been computed.

PROBLEM 5.27

KNOWN: Dimensions and operating conditions of an integrated circuit.

FIND: Steady-state temperature and time to come within 1°C of steady-state.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties.

PROPERTIES: Chip material (given): ρ = 2000 kg/m^3, c_p = 700 J/kg·K.

ANALYSIS: The direct and indirect paths for heat transfer from the chip to the coolant are in parallel, and the equivalent resistance is

$$R_{equiv} = \left[hL^2 + R_t^{-1}\right]^{-1} = \left[\left(3.75\times10^{-3} + 5\times10^{-3}\right)\text{W/K}\right]^{-1} = 114.3\ \text{K/W}.$$

The corresponding overall heat transfer coefficient is

$$U = \frac{\left(R_{equiv}\right)^{-1}}{L^2} = \frac{0.00875\ \text{W/K}}{(0.005\ \text{m})^2} = 350\ \text{W/m}^2\cdot\text{K}.$$

To obtain the steady-state temperature, apply conservation of energy to a control surface about the chip.

$$-\dot{E}_{out} + \dot{E}_g = 0 \qquad -UL^2\left(T_f - T_\infty\right) + \dot{q}\left(L^2 \cdot t\right) = 0$$

$$T_f = T_\infty + \frac{\dot{q}t}{U} = 20^\circ\text{C} + \frac{9\times10^6\ \text{W/m}^3 \times 0.001\ \text{m}}{350\ \text{W/m}^2\cdot\text{K}} = 45.7^\circ\text{C}. \qquad <$$

From the general lumped capacitance analysis, Equation 5.15 yields

$$\rho\left(L^2t\right)c\frac{dT}{dt} = \dot{q}\left(L^2t\right) - U\left(T - T_\infty\right)L^2.$$

With

$$a \equiv \frac{U}{\rho\ tc} = \frac{350\ \text{W/m}^2\cdot\text{K}}{\left(2000\ \text{kg/m}^3\right)(0.001\ \text{m})(700\ \text{J/kg}\cdot\text{K})} = 0.250\ \text{s}^{-1}$$

$$b = \frac{\dot{q}}{\rho\ c} = \frac{9\times10^6\ \text{W/m}^3}{\left(2000\ \text{kg/m}^3\right)(700\ \text{J/kg}\cdot\text{K})} = 6.429\ \text{K/s}$$

Equation 5.24 yields

$$\exp(-at) = \frac{T - T_\infty - b/a}{T_i - T_\infty - b/a} = \frac{(44.7 - 20 - 25.7)\text{K}}{(20 - 20 - 25.7)\text{K}} = 0.0389$$

$$t = -\ln(0.0389)/0.250\ \text{s}^{-1} = 13.0\ \text{s}. \qquad <$$

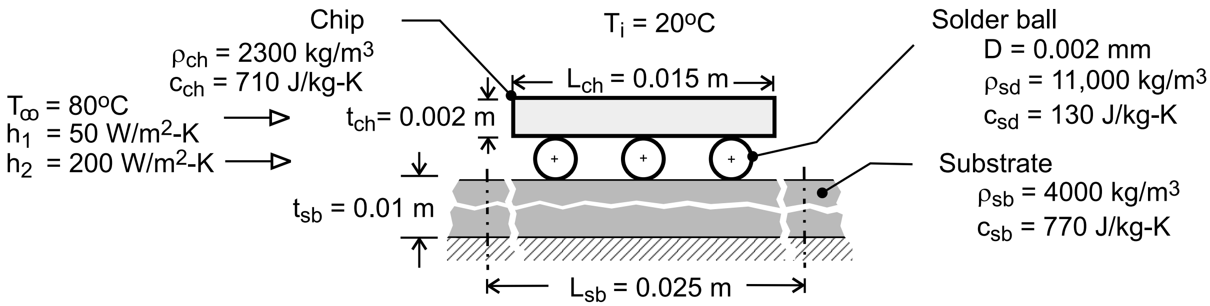
COMMENTS: Heat transfer through the substrate is comparable to that associated with direct convection to the coolant.

PROBLEM 5.28

KNOWN: Dimensions, initial temperature and thermophysical properties of chip, solder and substrate. Temperature and convection coefficient of heating agent.

FIND: (a) Time constants and temperature histories of chip, solder and substrate when heated by an air stream. Time corresponding to maximum stress on a solder ball. (b) Reduction in time associated with using a dielectric liquid to heat the components.

SCHEMATIC:



ASSUMPTIONS: (1) Lumped capacitance analysis is valid for each component, (2) Negligible heat transfer between components, (3) Negligible reduction in surface area due to contact between components, (4) Negligible radiation for heating by air stream, (5) Uniform convection coefficient among components, (6) Constant properties.

ANALYSIS: (a) From Eq. (5.7), $\tau_t = (\rho V c)/hA$

Chip: $V = \left(L_{ch}^2\right)t_{ch} = (0.015\text{m})^2(0.002\text{m}) = 4.50\times10^{-7}\,\text{m}^3,\ A_s = \left(2L_{ch}^2 + 4L_{ch}t_{ch}\right)$

$$= 2(0.015\text{m})^2 + 4(0.015\text{m})0.002\text{m} = 5.70\times10^{-4}\,\text{m}^2$$

$$\tau_t = \frac{2300\,\text{kg/m}^3\times4.50\times10^{-7}\,\text{m}^3\times710\,\text{J/kg}\cdot\text{K}}{50\,\text{W/m}^2\cdot\text{K}\times5.70\times10^{-4}\,\text{m}^2} = 25.8\text{s} \qquad <$$

Solder: $V = \pi D^3/6 = \pi(0.002\text{m})^3/6 = 4.19\times10^{-9}\,\text{m}^3,\ A_s = \pi D^2 = \pi(0.002\text{m})^2 = 1.26\times10^{-5}\,\text{m}^2$

$$\tau_t = \frac{11,000\,\text{kg/m}^3\times4.19\times10^{-9}\,\text{m}^3\times130\,\text{J/kg}\cdot\text{K}}{50\,\text{W/m}^2\cdot\text{K}\times1.26\times10^{-5}\,\text{m}^2} = 9.5\text{s} \qquad <$$

Substrate: $V = \left(L_{sb}^2\,t_{sb}\right) = (0.025\text{m})^2(0.01\text{m}) = 6.25\times10^{-6}\,\text{m}^3,\ A_s = L_{sb}^2 = (0.025\text{m})^2 = 6.25\times10^{-4}\,\text{m}^2$

$$\tau_t = \frac{4000\,\text{kg/m}^3\times6.25\times10^{-6}\,\text{m}^3\times770\,\text{J/kg}\cdot\text{K}}{50\,\text{W/m}^2\cdot\text{K}\times6.25\times10^{-4}\,\text{m}^2} = 616.0\text{s} \qquad <$$

Substituting Eq. (5.7) into (5.5) and recognizing that $(T - T_i)/(T_\infty - T_i) = 1 - (\theta/\theta_i)$, in which case $(T - T_i)/(T_\infty - T_i) = 0.99$ yields $\theta/\theta_i = 0.01$, it follows that the time required for a component to experience 99% of its maximum possible temperature rise is

$$t_{0.99} = \tau\ln(\theta_i/\theta) = \tau\ln(100) = 4.61\tau$$

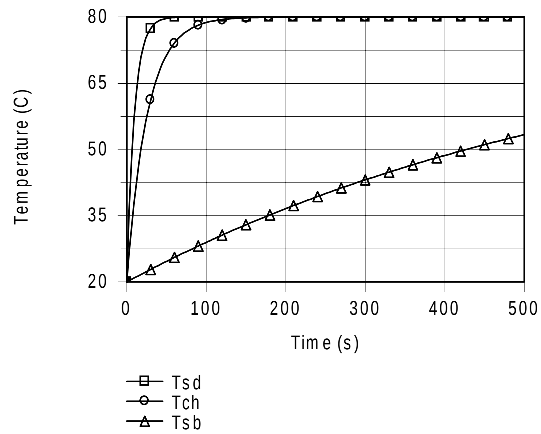
Hence,

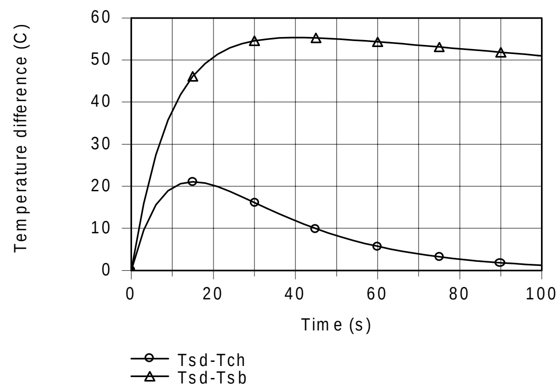
Chip: $t = 118.9\text{s}$, *Solder*: $t = 43.8\text{s}$, *Substrate*: $t = 2840$ <

Continued …..

PROBLEM 5.28 (Cont.)

Histories of the three components and temperature differences between a solder ball and its adjoining components are shown below.





Commensurate with their time constants, the fastest and slowest responses to heating are associated with the solder and substrate, respectively. Accordingly, the largest temperature difference is between these two components, and it achieves a maximum value of 55°C at

$$t(\text{maximum stress}) \approx 40\text{s} \qquad <$$

(b) With the 4-fold increase in h associated with use of a dielectric liquid to heat the components, the time constants are each reduced by a factor of 4, and the times required to achieve 99% of the maximum temperature rise are

Chip: $t = 29.5\text{s}$, *Solder*: $t = 11.0\text{s}$, *Substrate*: $t = 708\text{s}$ $<$

The time savings is approximately 75%.

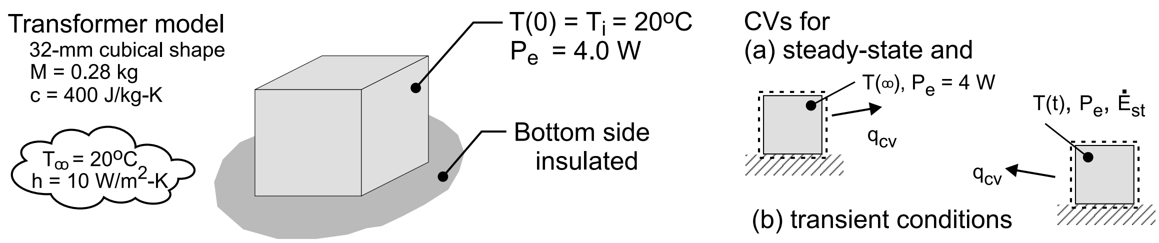
COMMENTS: The foregoing analysis provides only a first, albeit useful, approximation to the heating problem. Several of the assumptions are highly approximate, particularly that of a uniform convection coefficient. The coefficient will vary between components, as well as on the surfaces of the components. Also, because the solder balls are flattened, there will be a reduction in surface area exposed to the fluid for each component, as well as heat transfer between components, which reduces differences between time constants for the components.

PROBLEM 5.29

KNOWN: Electrical transformer of approximate cubical shape, 32 mm to a side, dissipates 4.0 W when operating in ambient air at 20°C with a convection coefficient of 10 $W/m^2 \cdot K$.

FIND: (a) Develop a model for estimating the steady-state temperature of the transformer, $T(\infty)$, and evaluate $T(\infty)$, for the operating conditions, and (b) Develop a model for estimating the temperature-time history of the transformer if initially the temperature is $T_i = T_\infty$ and suddenly power is applied. Determine the time required to reach within 5°C of its steady-state operating temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Transformer is spatially isothermal object, (2) Initially object is in equilibrium with its surroundings, (3) Bottom surface is adiabatic.

ANALYSIS: (a) Under steady-state conditions, for the control volume shown in the schematic above, the energy balance is

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = 0 \qquad 0 - q_{cv} + P_e = -h\,A_s\left[T(\infty) - T_\infty\right] + P_e = 0 \qquad (1)$$

where $A_s = 5 \times L^2 = 5 \times 0.032\text{m} \times 0.032\text{m} = 5.12 \times 10^{-3}\ \text{m}^2$, find

$$T(\infty) = T_\infty + P_e / h\,A_s = 20°\text{C} + 4\,\text{W}/\left(10\ \text{W/m}^2 \cdot \text{K} \times 5.12 \times 10^{-3}\text{m}^2\right) = 98.1°\text{C} \qquad <$$

(b) Under transient conditions, for the control volume shown above, the energy balance is

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st} \qquad 0 - q_{cv} + P_e = Mc\frac{dT}{dt} \qquad (2)$$

Substitute from Eq. (1) for P_e, separate variables, and define the limits of integration.

$$-h\left[T(t) - T_\infty\right] + h\left[T(\infty) - T_\infty\right] = Mc\frac{dT}{dt}$$

$$-h\left[T(t) - T(\infty)\right] = Mc\frac{d}{dt}\left(T - T(\infty)\right) \qquad \frac{h}{Mc}\int_0^{t_o} dt = -\int_{\theta_i}^{\theta_o} \frac{d\theta}{\theta}$$

where $\theta = T(t) - T(\infty)$; $\theta_i = T_i - T(\infty) = T_\infty - T(\infty)$; and $\theta_o = T(t_o) - T(\infty)$ with t_o as the time when $\theta_o = -5°\text{C}$. Integrating and rearranging find (see Eq. 5.5),

$$t_o = \frac{Mc}{h\,A_s}\ell n\frac{\theta_i}{\theta_o}$$

$$t_o = \frac{0.28\ \text{kg} \times 400\ \text{J/kg} \cdot \text{K}}{10\ \text{W/m}^2 \cdot \text{K} \times 5.12 \times 10^{-3}\text{m}^2}\ell n\frac{(20 - 98.1)°\text{C}}{-5°\text{C}} = 1.67\ \text{hour} \qquad <$$

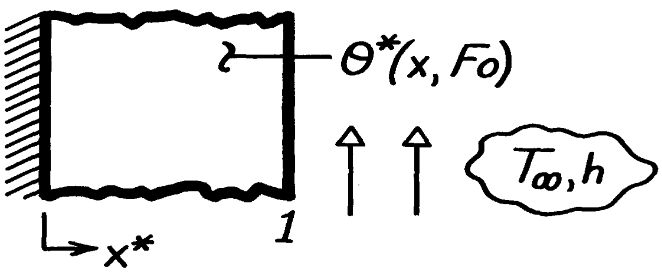
COMMENTS: The spacewise isothermal assumption may not be a gross over simplification since most of the material is copper and iron, and the external resistance by free convection is high. However, by ignoring internal resistance, our estimate for t_o is optimistic.

PROBLEM 5.30

KNOWN: Series solution, Eq. 5.39, for transient conduction in a plane wall with convection.

FIND: Midplane (x*=0) and surface (x*=1) temperatures θ* for Fo=0.1 and 1, using Bi=0.1, 1 and 10 with only the first four eigenvalues. Based upon these results, discuss the validity of the approximate solutions, Eqs. 5.40 and 5.41.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Constant properties.

ANALYSIS: The series solution, Eq. 5.39a, is of the form,

$$q^* = \sum_{n=1}^{\infty} C_n \ \exp\left(-z_n^2 Fo\right)\cos\left(z_n x^*\right)$$

where the eigenvalues, z_n, and the constants, C_n, are from Eqs. 5.39b and 5.39c.

$$z_n \tan z_n = Bi \qquad C_n = 4\sin z_n / \left(2z_n + \sin\left(2z_n\right)\right).$$

The eigenvalues are tabulated in Appendix B.3; note, however, that z_1 and C_1 are available from Table 5.1. The values of z_n and C_n used to evaluate θ* are as follows:

Bi	z_1	C_1	z_2	C_2	z_3	C_3	z_4	C_4
0.1	0.3111	1.0160	3.1731	-0.0197	6.2991	0.0050	9.4354	-0.0022
1	0.8603	1.1191	3.4256	-0.1517	6.4373	0.0466	9.5293	-0.0217
10	1.4289	1.2620	4.3058	-0.3934	7.2281	0.2104	10.2003	-0.1309

Using z_n and C_n values, the terms of q^*, designated as q_1^*, q_2^*, q_3^* and q_4^*, are as follows:

Fo=0.1

	Bi=0.1		Bi=1.0		Bi=10	
x*	0	1	0	1	0	1
q_1^*	1.0062	0.9579	1.0393	0.6778	1.0289	0.1455
q_2^*	-0.0072	0.0072	-0.0469	0.0450	-0.0616	0.0244
q_3^*	0.0001	0.0001	0.0007	0.0007	0.0011	0.0006
q_4^*	-2.99×10^{-7}	3.00×10^{-7}	2.47×10^{-6}	2.46×10^{-7}	-3.96×10^{-6}	2.83×10^{-6}
q^*	0.9991	0.9652	0.9931	0.7235	0.9684	0.1705

Continued …..

PROBLEM 5.30(Cont.)

Fo=1						
	Bi=0.1		Bi=1.0		Bi=10	
x*	0	1	0	1	0	1
q_1^*	0.9223	0.8780	0.5339	0.3482	0.1638	0.0232
q_2^*	8.35×10^{-7}	8.35×10^{-7}	-1.22×10^{-5}	1.17×10^{-6}	3.49×10^{-9}	1.38×10^{-9}
q_3^*	7.04×10^{-20}	-	4.70×10^{-20}	-	4.30×10^{-24}	-
q_4^*	4.77×10^{-42}	-	7.93×10^{-42}	-	8.52×10^{-47}	-
q^*	0.9223	0.8780	0.5339	0.3482	0.1638	0.0232

The tabulated results for $q^* = q^*\left(x^*, \text{Bi}, \text{Fo}\right)$ demonstrate that for Fo=1, the first eigenvalue is sufficient to accurately represent the series. However, for Fo=0.1, three eigenvalues are required for accurate representation.

A more detailed analysis would show that a practical criterion for representation of the series solution by one eigenvalue is Fo>0.2. For these situations the approximate solutions, Eqs. 5.40 and 5.41, are appropriate. For the midplane, x^*=0, the first two eigenvalues for Fo=0.2 are:

	Fo=0.2	x*=0	
Bi	0.1	1.0	10
q_1^*	0.9965	0.9651	0.8389
q_2^*	-0.00226	-0.0145	-0.0096
q^*	0.9939	0.9506	0.8293
Error,%	+0.26	+1.53	+1.16

The percentage error shown in the last row of the above table is due to the effect of the second term. For Bi=0.1, neglecting the second term provides an error of 0.26%. For Bi=1, the error is 1.53%.

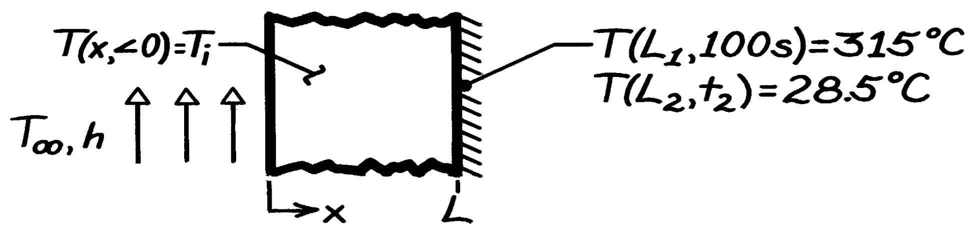
Hence we conclude that the approximate series solutions (with only one eigenvalue) provides systematically high results, but by less than 1.5%, for the Biot number range from 0.1 to 10.

PROBLEM 5.31

KNOWN: One-dimensional wall, initially at a uniform temperature, T_i, is suddenly exposed to a convection process (T_∞, h). For wall #1, the time (t_1 = 100s) required to reach a specified temperature at x = L is prescribed, $T(L_1, t_1)$ = 315°C.

FIND: For wall #2 of different thickness and thermal conditions, the time, t_2, required for $T(L_2, t_2)$ = 28°C.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties.

ANALYSIS: The properties, thickness and thermal conditions for the two walls are:

Wall	L(m)	α(m^2/s)	k(W/m·K)	T_i(°C)	T_∞(°C)	h(W/m^2·K)
1	0.10	15×10^{-6}	50	300	400	200
2	0.40	25×10^{-6}	100	30	20	100

The dimensionless functional dependence for the one-dimensional, transient temperature distribution, Eq. 5.38, is

$$\theta^* = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = f\left(x^*, \text{Bi}, \text{Fo}\right)$$

where

$$x^* = x/L \qquad \text{Bi} = hL/k \qquad \text{Fo} = \alpha t/L^2.$$

If the parameters x*, Bi, and Fo are the same for both walls, then $\theta_1^* = \theta_2^*$. Evaluate these parameters:

Wall	x*	Bi	Fo	θ*
1	1	0.40	0.150	0.85
2	1	0.40	$1.563\times10^{-4}\ t_2$	0.85

where

$$\theta_1^* = \frac{315-400}{300-400} = 0.85 \qquad \theta_2^* = \frac{28.5-20}{30-20} = 0.85.$$

It follows that

$$\text{Fo}_2 = \text{Fo}_1 \qquad 1.563\times10^{-4} t_2 = 0.150$$

$$t_2 = 960\text{s}.$$

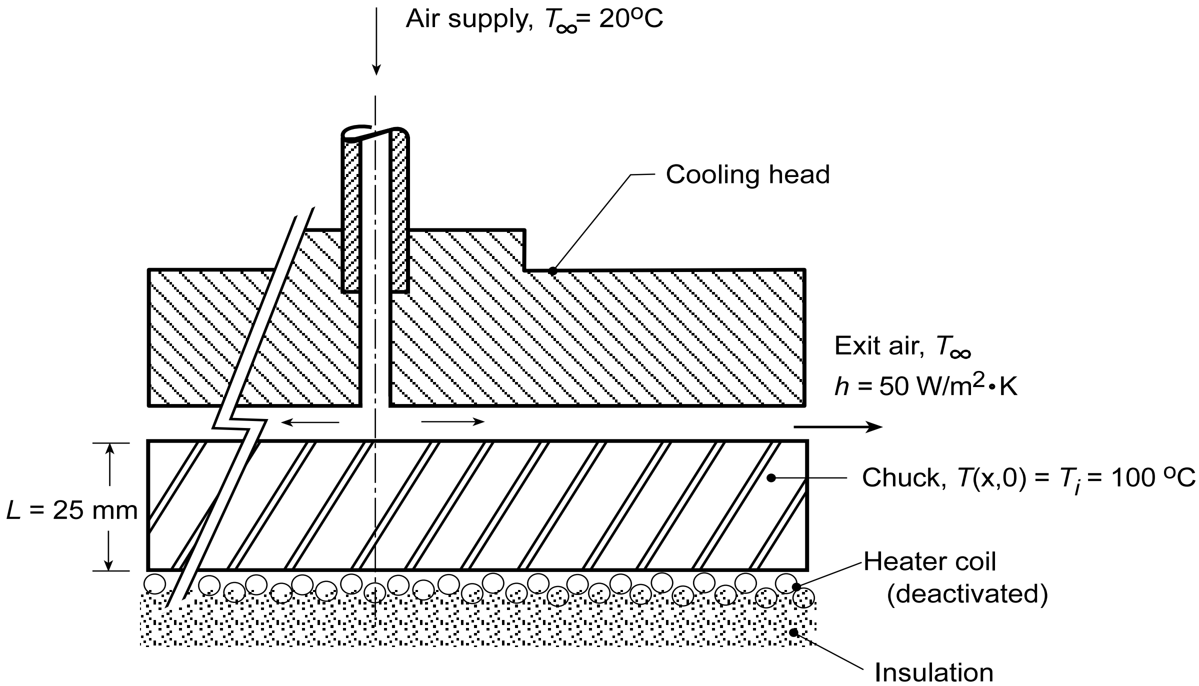
<

PROBLEM 5.32

KNOWN: The chuck of a semiconductor processing tool, initially at a uniform temperature of T_i = 100°C, is cooled on its top surface by supply air at 20°C with a convection coefficient of 50 W/m^2·K.

FIND: (a) Time required for the lower surface to reach 25°C, and (b) Compute and plot the time-to-cool as a function of the convection coefficient for the range 10 ≤ h ≤ 2000 W/m^2·K; comment on the effectiveness of the head design as a method for cooling the chuck.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction in the chuck, (2) Lower surface is perfectly insulated, (3) Uniform convection coefficient and air temperature over the upper surface of the chuck, and (4) Constant properties.

PROPERTIES: *Table A.1*, Aluminum alloy 2024 ((25 + 100)°C / 2 = 335 K): ρ = 2770 kg/m^3, c_p = 880 J/kg· K, k = 179 W/m·K.

ANALYSIS: (a) The Biot number for the chuck with h = 50 W/m^2·K is

$$\mathrm{Bi} = \frac{hL}{k} = \frac{50\,\mathrm{W/m^2\cdot K}\times 0.025\,\mathrm{m}}{179\,\mathrm{W/m\cdot K}} = 0.007 \le 0.1 \qquad (1)$$

so that the lumped capacitance method is appropriate. Using Eq. 5.5, with $V/A_s = L$,

$$t = \frac{\rho V c}{h A_s}\ln\frac{\theta_i}{\theta} \qquad \theta = T - T_\infty \qquad \theta_i = T_i - T_\infty$$

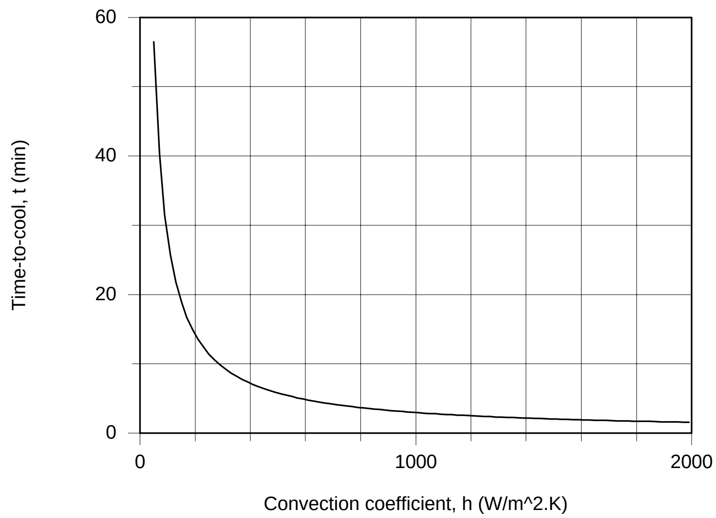
$$t = \left(2770\,\mathrm{kg/m^3}\times 0.025\,\mathrm{m}\times 880\,\mathrm{J/kg\cdot K}\Big/50\,\mathrm{W/m^2\cdot K}\right)\ln\frac{(100-20)^\circ\,\mathrm{C}}{(25-20)^\circ\,\mathrm{C}}$$

$$t = 3379\,\mathrm{s} = 56.3\,\mathrm{min}$$ <

Continued...

PROBLEM 5.32 (Cont.)

(b) When h = 2000 W/m^2·K, using Eq. (1), find Bi = 0.28 > 0.1 so that the series solution, Section 5.51, for the plane wall with convection must be used. Using the *IHT Transient Conduction*, *Plane Wall Model*, the time-to-cool was calculated as a function of the convection coefficient. Free convection cooling conduction corresponds to h ≈ 10 W/m^2·K and the time-to-cool is 282 minutes. With the cooling head design, the time-to-cool can be substantially decreased if the convection coefficient can be increased as shown below.

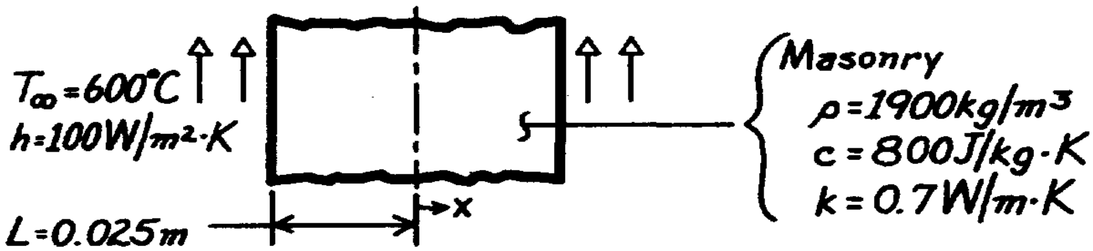


PROBLEM 5.33

KNOWN: Configuration, initial temperature and charging conditions of a thermal energy storage unit.

FIND: Time required to achieve 75% of maximum possible energy storage and corresponding minimum and maximum temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible radiation exchange with surroundings.

ANALYSIS: For the system, find first

$$\mathrm{Bi} = \frac{hL}{k} = \frac{100\ \mathrm{W/m^2 \cdot K} \times 0.025\mathrm{m}}{0.7\ \mathrm{W/m \cdot K}} = 3.57$$

indicating that the lumped capacitance method cannot be used.

Groeber chart, Fig. D.3: $Q/Q_o = 0.75$

$$a = \frac{k}{r\ c} = \frac{0.7\ \mathrm{W/m \cdot K}}{1900\ \mathrm{kg/m^3} \times 800\ \mathrm{J/kg \cdot K}} = 4.605 \times 10^{-7}\ \mathrm{m^2/s}$$

$$\mathrm{Bi}^2\mathrm{Fo} = \frac{h^2 a\ t}{k^2} = \frac{\left(100\ \mathrm{W/m^2K}\right)^2 \times \left(4.605 \times 10^{-7} \mathrm{m^2/s}\right) \times t(s)}{\left(0.7\ \mathrm{W/m \cdot K}\right)^2} = 9.4 \times 10^{-3} t$$

Find $\mathrm{Bi}^2\mathrm{Fo} \approx 11$, and substituting numerical values

$$t = 11/9.4 \times 10^{-3} = 1170\mathrm{s}. \qquad <$$

Heisler chart, Fig. D.1: T_{min} is at x = 0 and T_{max} at x = L, with

$$\mathrm{Fo} = \frac{a\ t}{L^2} = \frac{4.605 \times 10^{-7} \mathrm{m^2/s} \times 1170\ \mathrm{s}}{(0.025\mathrm{m})^2} = 0.86 \qquad \mathrm{Bi}^{-1} = 0.28.$$

From Fig. D.1, $q_o^* \approx 0.33$. Hence,

$$T_o \approx T_\infty + 0.33\left(T_i - T_\infty\right) = 600^\circ\mathrm{C} + 0.33\left(-575^\circ\mathrm{C}\right) = 410^\circ\mathrm{C} = T_{min}. \qquad <$$

From Fig. D.2, $\theta/\theta_o \approx 0.33$ at x = L, for which

$$T_{x=L} \approx T_\infty + 0.33\left(T_o - T_\infty\right) = 600^\circ\mathrm{C} + 0.33(-190)^\circ\mathrm{C} = 537^\circ\mathrm{C} = T_{max}. \qquad <$$

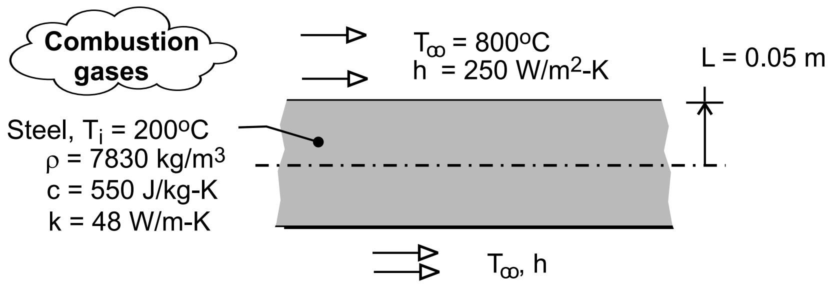
COMMENTS: Comparing masonry (m) with aluminum (Al), see Problem 5.10, $(\rho c)_{Al} > (\rho c)_m$ and $k_{Al} > k_m$. Hence, the aluminum can store more energy and can be charged (or discharged) more quickly.

PROBLEM 5.34

KNOWN: Thickness, properties and initial temperature of steel slab. Convection conditions.

FIND: Heating time required to achieve a minimum temperature of 550°C in the slab.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible radiation effects, (3) Constant properties.

ANALYSIS: With a Biot number of hL/k = (250 W/m^2·K × 0.05m)/48 W/m·K = 0.260, a lumped capacitance analysis should not be performed. At any time during heating, the lowest temperature in the slab is at the midplane, and from the one-term approximation to the transient thermal response of a plane wall, Eq. (5.41), we obtain

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{(550-800)°C}{(200-800)°C} = 0.417 = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

With $\zeta_1 \approx 0.488$ rad and $C_1 \approx 1.0396$ from Table 5.1 and $\alpha = k/\rho c = 1.115\times10^{-5} m^2/s$,

$$-\zeta_1^2\left(\alpha t/L^2\right) = \ln(0.401) = -0.914$$

$$t = \frac{0.914 L^2}{\zeta_1^2 \alpha} = \frac{0.841(0.05m)^2}{(0.488)^2 1.115\times10^{-5} m^2/s} = 861s$$ <

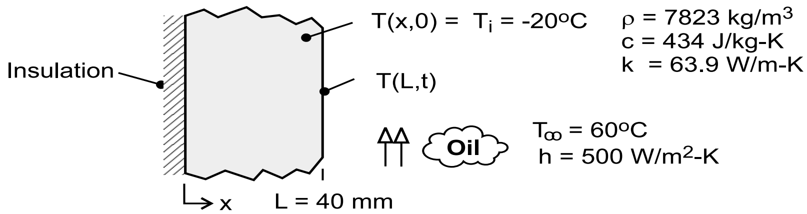
COMMENTS: The surface temperature at t = 861s may be obtained from Eq. (5.40b), where $\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right) = 0.417\cos(0.488 \text{ rad}) = 0.368$. Hence, $T(L, 792s) \equiv T_s = T_\infty + 0.368(T_i - T_\infty)$ $= 800°C - 221°C = 579°C$. Assuming a surface emissivity of ε = 1 and surroundings that are at $T_{sur} = T_\infty = 800°C$, the radiation heat transfer coefficient corresponding to this surface temperature is $h_r = \varepsilon\sigma(T_s + T_{sur})\left(T_s^2 + T_{sur}^2\right) = 205 \text{ W/m}^2\cdot\text{K}$. Since this value is comparable to the convection coefficient, radiation is not negligible and the desired heating will occur well before t = 861s.

PROBLEM 5.35

KNOWN: Pipe wall subjected to sudden change in convective surface condition. See Example 5.4.

FIND: (a) Temperature of the inner and outer surface of the pipe, heat flux at the inner surface, and energy transferred to the wall after 8 min; compare results to the hand calculations performed for the Text Example; (b) Time at which the outer surface temperature of the pipe, T(0,t), will reach 25°C; (c) Calculate and plot on a single graph the temperature distributions, T(x,t) vs. x, for the initial condition, the final condition and the intermediate times of 4 and 8 min; explain key features; (d) Calculate and plot the temperature-time history, T(x,t) vs. t, for the locations at the inner and outer pipe surfaces, x = 0 and L, and for the range $0 \le t \le 16$ min. Use the *IHT | Models | Transient Conduction | Plane Wall* model as the solution tool.

SCHEMATIC:



ASSUMPTIONS: (1) Pipe wall can be approximated as a plane wall, (2) Constant properties, (3) Outer surface of pipe is adiabatic.

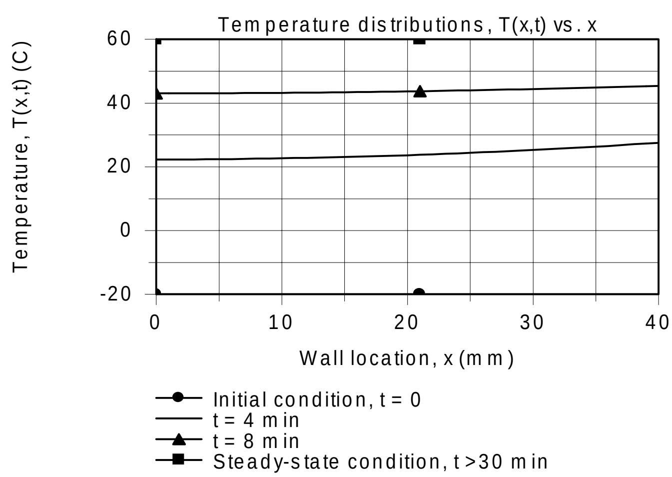
ANALYSIS: The IHT model represents the series solution for the plane wall providing temperatures and heat fluxes evaluated at (x,t) and the total energy transferred at the inner wall at (t). Selected portions of the IHT code used to obtain the results tabulated below are shown in the Comments.

(a) The code is used to evaluate the tabulated parameters at t = 8 min for locations x = 0 and L. The agreement is very good between the one-term approximation of the Example and the multiple-term series solution provided by the IHT model.

	Text Ex 5.4	IHT Model
T(L, 8min), °C	45.2	45.4
T(0, 8 min), °C	42.9	43.1
$Q'(8\ \text{min})\times10^{-7}$, J/m	-2.73	-2.72
$q''_x(L,\ 8\ \text{min})$, W/m^2	-7400	-7305

(b) To determine the time t_o for which T(0,t) = 25°C, the IHT model is solved for t_o after setting x = 0 and T_xt = 25°C. Find, t_o = 4.4 min. <

(c) The temperature distributions, T(x,t) vs x, for the initial condition (t = 0), final condition ($t \to \infty$) and intermediate times of 4 and 8 min. are shown on the graph below.

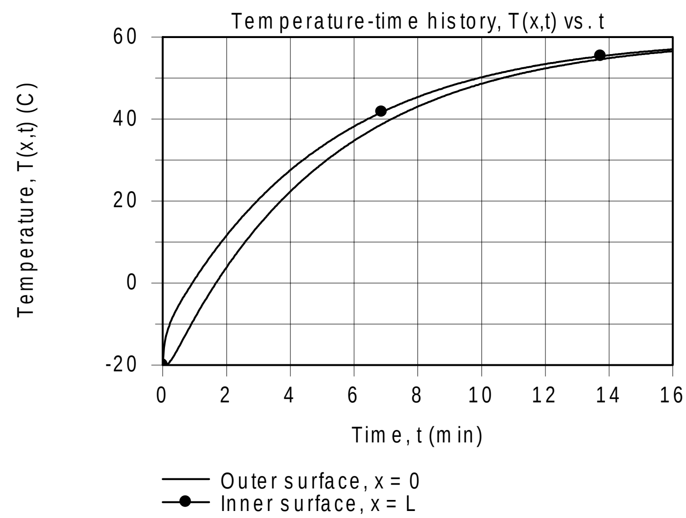


Continued …..

PROBLEM 5.35 (Cont.)

The final condition corresponds to the steady-state temperature, $T(x,\infty) = T_\infty$. For the intermediate times, the gradient is zero at the insulated boundary (x = 0, the pipe exterior). As expected, the temperature at x = 0 will be less than at the boundary experiencing the convection process with the hot oil, x = L. Note, however, that the difference is not very significant. The gradient at the inner wall, x = L, decreases with increasing time.

(d) The temperature history T(x,t) for the locations at the inner and outer pipe surfaces are shown in the graph below. Note that the temperature difference between the two locations is greatest at the start of the transient process and decreases with increasing time. After a 16 min. duration, the pipe temperature is almost uniform, but yet 3 or 4°C from the steady-state condition.



COMMENTS: (1) Selected portions of the IHT code for the plane wall model are shown below. Note the relation for the pipe volume, vol, used in calculating the total heat transferred per unit length over the time interval t.

```
// Models | Transient Conduction | Plane Wall
// The temperature distribution is
T_xt = T_xt_trans("Plane Wall",xstar,Fo,Bi,Ti,Tinf)    // Eq 5.39
//T_xt = 25                       // Part (b) surface temperature, x = 0
// The heat flux in the x direction is
q''_xt = qdprime_xt_trans("Plane Wall",x,L,Fo,Bi,k,Ti,Tinf)   // Eq 2.6

// The total heat transfer from the wall over the time interval t is
QoverQo = Q_over_Qo_trans("Plane Wall",Fo,Bi)     // Eq 5.45
Qo = rho * cp * vol * (Ti - Tinf)     // Eq 5.44
//vol = 2 * As * L                // Appropriate for wall of 2L thickness
vol = pi * D * L                  // Pipe wall of diameter D, thickness L and unit length
Q = QoverQo * Qo                  // Total energy transfered per unit length
```

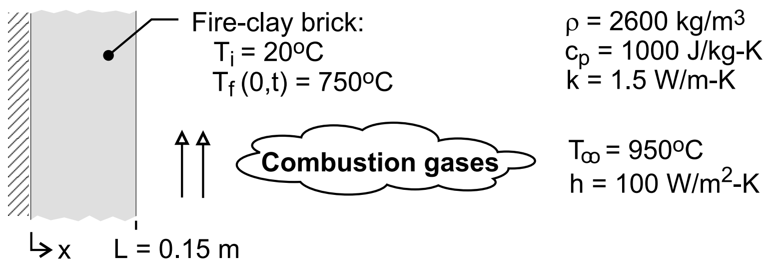
(2) Can you give an explanation for why the inner and outer surface temperatures are not very different? What parameter provides a measure of the temperature non-uniformity in a system during a transient conduction process?

PROBLEM 5.36

KNOWN: Thickness, initial temperature and properties of furnace wall. Convection conditions at inner surface.

FIND: Time required for outer surface to reach a prescribed temperature. Corresponding temperature distribution in wall and at intermediate times.

SCHEMATIC:



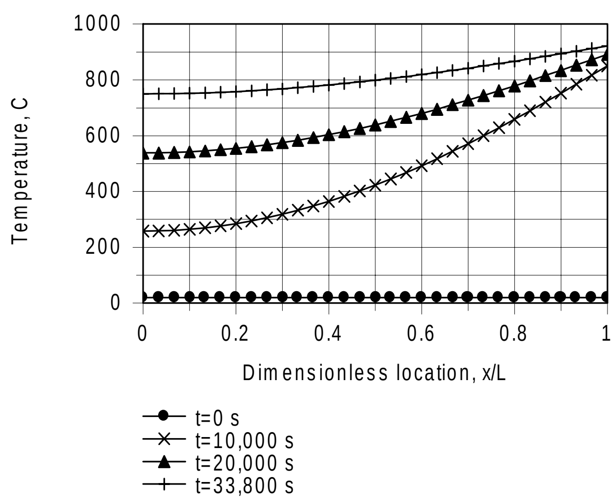
ASSUMPTIONS: (1) One-dimensional conduction in a plane wall, (2) Constant properties, (3) Adiabatic outer surface, (4) Fo > 0.2, (5) Negligible radiation from combustion gases.

ANALYSIS: The wall is equivalent to one-half of a wall of thickness 2L with symmetric convection conditions at its two surfaces. With Bi = hL/k = 100 $W/m^2 \cdot K \times 0.15m/1.5\ W/m \cdot K$ = 10 and Fo > 0.2, the one-term approximation, Eq. 5.41 may be used to compute the desired time, where $\theta_o^* = (T_o - T_\infty)/(T_i - T_\infty) = 0.215$. From Table 5.1, C_1 = 1.262 and ζ_1 = 1.4289. Hence,

$$Fo = -\frac{\ln\left(\theta_o^* / C_1\right)}{\zeta_1^2} = -\frac{\ln(0.215/1.262)}{(1.4289)^2} = 0.867$$

$$t = \frac{Fo\ L^2}{\alpha} = \frac{0.867(0.15m)^2}{\left(1.5\,W/m \cdot K / 2600\ kg/m^3 \times 1000\ J/kg \cdot K\right)} = 33{,}800\,s$$ <

The corresponding temperature distribution, as well as distributions at t = 0, 10,000, and 20,000 s are plotted below



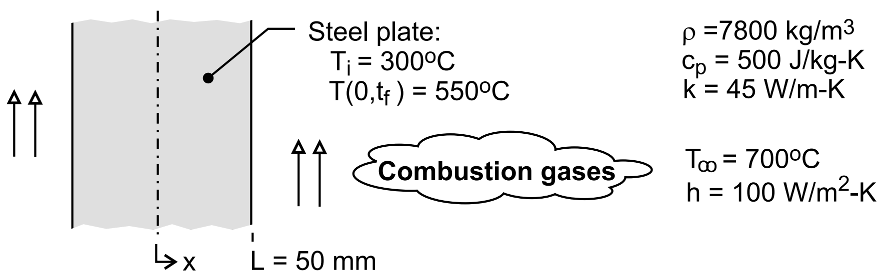
COMMENTS: Because Bi >>1, the temperature at the inner surface of the wall increases much more rapidly than at locations within the wall, where temperature gradients are large. The temperature gradients decrease as the wall approaches a steady-state for which there is a uniform temperature of 950°C.

PROBLEM 5.37

KNOWN: Thickness, initial temperature and properties of steel plate. Convection conditions at both surfaces.

FIND: Time required to achieve a minimum temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in plate, (2) Symmetric heating on both sides, (3) Constant properties, (4) Negligible radiation from gases, (5) Fo > 0.2.

ANALYSIS: The smallest temperature exists at the midplane and, with Bi = hL/k = 500 $W/m^2 \cdot K \times$ 0.050m/45 W/m·K = 0.556 and Fo > 0.2, may be determined from the one-term approximation of Eq. 5.41. From Table 5.1, C_1 = 1.076 and ζ_1 = 0.682. Hence, with $\theta_o^* = (T_o - T_\infty)/(T_i - T_\infty) = 0.375$,

$$\mathrm{Fo} = -\frac{\ln\left(\theta_o^* / C_1\right)}{\zeta_1^2} = -\frac{\ln(0.375/1.076)}{(0.682)^2} = 2.266$$

$$t = \frac{\mathrm{Fo}\ L^2}{\alpha} = \frac{2.266(0.05\mathrm{m})^2}{\left(45\,\mathrm{W/m\cdot K}/7800\ \mathrm{kg/m^3} \times 500\ \mathrm{J/kg\cdot K}\right)} = 491\mathrm{s} \qquad <$$

COMMENTS: From Eq. 5.40b, the corresponding surface temperature is

$$T_s = T_\infty + (T_i - T_\infty)\theta_o^* \cos(\zeta_1) = 700°\mathrm{C} - 400°\mathrm{C} \times 0.375 \times 0.776 = 584°\mathrm{C}$$

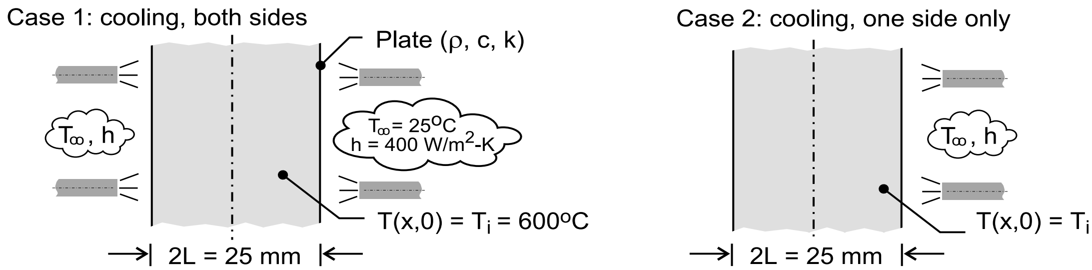
Because Bi is not much larger than 0.1, temperature gradients in the steel are moderate.

PROBLEM 5.38

KNOWN: Plate of thickness 2L = 25 mm at a uniform temperature of 600°C is removed from a hot pressing operation. Case 1, cooled on both sides; case 2, cooled on one side only.

FIND: (a) Calculate and plot on one graph the temperature histories for cases 1 and 2 for a 500-second cooling period; use the *IHT* software; Compare times required for the maximum temperature in the plate to reach 100°C; and (b) For both cases, calculate and plot on one graph, the variation with time of the maximum temperature difference in the plate; Comment on the relative magnitudes of the temperature gradients within the plate as a function of time.

SCHEMATIC:



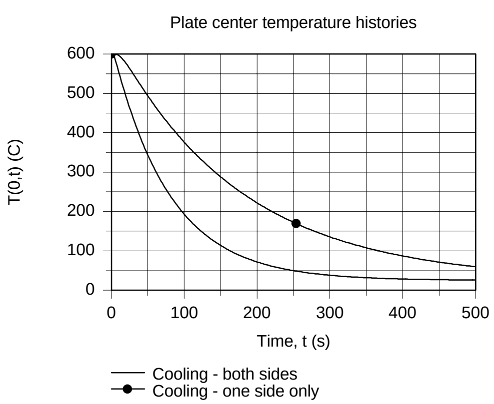
ASSUMPTIONS: (1) One-dimensional conduction in the plate, (2) Constant properties, and (3) For case 2, with cooling on one side only, the other side is adiabatic.

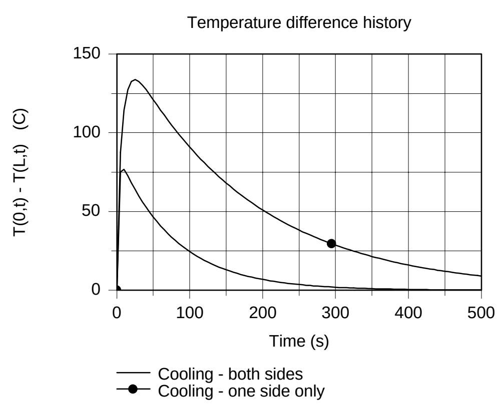
PROPERTIES: Plate (*given*): $\rho = 3000\ \text{kg/m}^3$, c = 750 J/kg·K, k = 15 W/m·K.

ANALYSIS: (a) From *IHT*, call up *Plane Wall, Transient Conduction* from the *Models* menu. For case 1, the plate thickness is 25 mm; for case 2, the plate thickness is 50 mm. The plate center (x = 0) temperature histories are shown in the graph below. The times required for the center temperatures to reach 100°C are

$t_1 = 164$ s $\qquad\qquad t_2 = 367$ s $\qquad\qquad <$

(b) The plot of T(0, t) – T(1, t), which represents the maximum temperature difference in the plate during the cooling process, is shown below.





COMMENTS: (1) From the plate center-temperature history graph, note that it takes more than twice as long for the maximum temperature to reach 100°C with cooling on only one side.

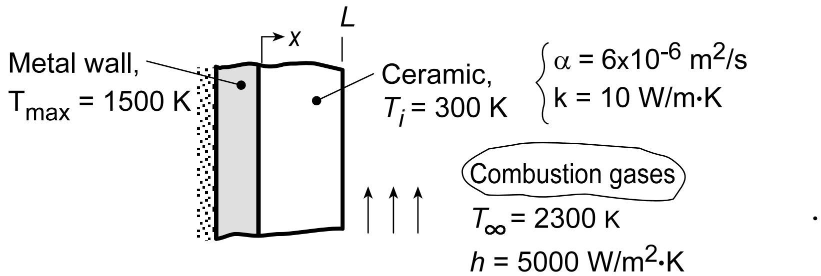
(2) From the maximum temperature-difference graph, as expected, cooling from one side creates a larger maximum temperature difference during the cooling process. The effect could cause microstructure differences, which could adversely affect the mechanical properties within the plate.

PROBLEM 5.39

KNOWN: Properties and thickness L of ceramic coating on rocket nozzle wall. Convection conditions. Initial temperature and maximum allowable wall temperature.

FIND: (a) Maximum allowable engine operating time, t_{max}, for L = 10 mm, (b) Coating inner and outer surface temperature histories for L = 10 and 40 mm.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in a plane wall, (2) Constant properties, (3) Negligible thermal capacitance of metal wall and heat loss through back surface, (4) Negligible contact resistance at wall/ceramic interface, (5) Negligible radiation.

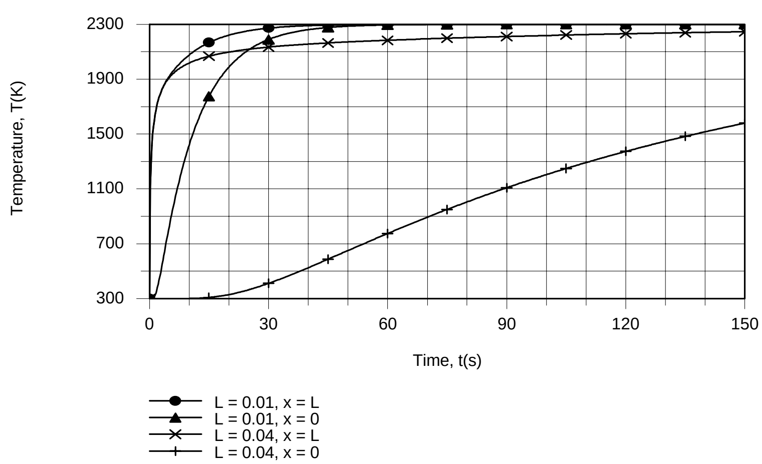
ANALYSIS: (a) Subject to assumptions (3) and (4), the maximum wall temperature corresponds to the ceramic temperature at x = 0. Hence, for the ceramic, we wish to determine the time t_{max} at which T(0,t) = $T_o(t)$ = 1500 K. With Bi = hL/k = 5000 W/m^2·K(0.01 m)/10 W/m·K = 5, the lumped capacitance method cannot be used. Assuming Fo > 0.2, obtaining ζ_1 = 1.3138 and C_1 = 1.2402 from Table 5.1, and evaluating $\theta_o^* = (T_o - T_\infty)/(T_i - T_\infty)$ = 0.4, Equation 5.41 yields

$$\mathrm{Fo} = -\frac{\ln\left(\theta_o^*/C_1\right)}{\zeta_1^2} = -\frac{\ln(0.4/1.2402)}{(1.3138)^2} = 0.656$$

confirming the assumption of Fo > 0.2. Hence,

$$t_{max} = \frac{\mathrm{Fo}\left(L^2\right)}{\alpha} = \frac{0.656(0.01\,\mathrm{m})^2}{6\times10^{-6}\,\mathrm{m^2/s}} = 10.9\,\mathrm{s}$$ <

(b) Using the IHT *Lumped Capacitance Model* for a *Plane Wall*, the inner and outer surface temperature histories were computed and are as follows:



Continued...

PROBLEM 5.39 (Cont.)

The increase in the inner ($x = 0$) surface temperature lags that of the outer surface, but within $t \approx 45s$ both temperatures are within a few degrees of the gas temperature for $L = 0.01$ m. For $L = 0.04$ m, the increased thermal capacitance of the ceramic slows the approach to steady-state conditions. The thermal response of the inner surface significantly lags that of the outer surface, and it is not until $t \approx 137s$ that the inner surface reaches 1500 K. At this time there is still a significant temperature difference across the ceramic, with $T(L,t_{max}) = 2240$ K.

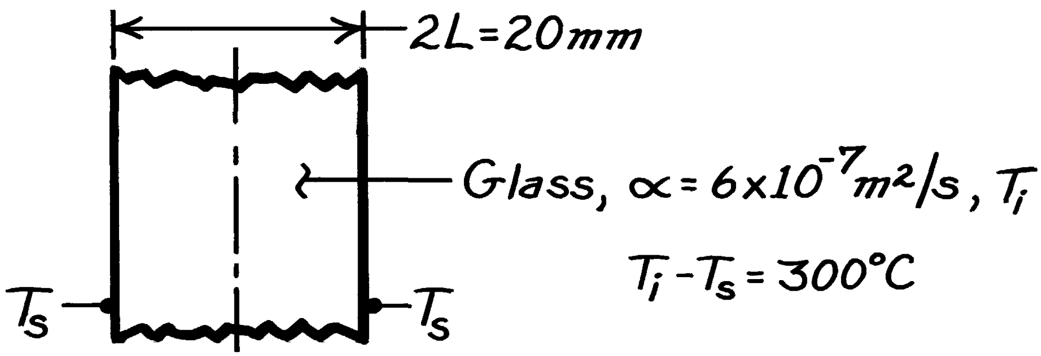
COMMENTS: The allowable engine operating time increases with increasing thermal capacitance of the ceramic and hence with increasing L.

PROBLEM 5.40

KNOWN: Initial temperature, thickness and thermal diffusivity of glass plate. Prescribed surface temperature.

FIND: (a) Time to achieve 50% reduction in midplane temperature, (b) Maximum temperature gradient at that time.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties.

ANALYSIS: Prescribed surface temperature is analogous to h → ∞ and $T_\infty = T_s$. Hence, Bi = ∞. Assume validity of one-term approximation to series solution for T (x,t).

(a) At the midplane,

$$q_o^* = \frac{T_o - T_s}{T_i - T_s} = 0.50 = C_1 \exp\left(-z_1^2 Fo\right)$$

$$z_1 \tan z_1 = Bi = \infty \rightarrow z_1 = p/2.$$

Hence

$$C_1 = \frac{4 \sin z_1}{2z_1 + \sin(2z_1)} = \frac{4}{p} = 1.273$$

$$Fo = -\frac{\ln\left(q_o^*/C_1\right)}{z_1^2} = 0.379$$

$$t = \frac{Fo L^2}{a} = \frac{0.379(0.01\text{ m})^2}{6\times10^{-7}\text{ m}^2/\text{s}} = 63\text{ s.}$$ <

(b) With $q^* = C_1 \exp\left(-z_1^2 Fo\right)\cos z_1 x^*$

$$\frac{\P\ T}{\P\ x} = \frac{(T_i - T_s)}{L}\frac{\P q^*}{\P\ x^*} = -\frac{(T_i - T_s)}{L} z_1 C_1 \exp\left(-z_1^2 Fo\right)\sin z_1 x^*$$

$$\P\ T/\P\ x\Big|_{max} = \P\ T/\P\ x\Big|_{x^*=1} = -\frac{300^\circ C}{0.01\text{ m}}\frac{p}{2}0.5 = -2.36\times10^4\ ^\circ C/m.$$ <

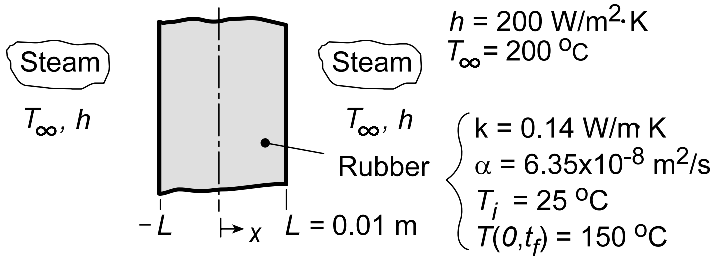
COMMENTS: Validity of one-term approximation is confirmed by Fo > 0.2.

PROBLEM 5.41

KNOWN: Thickness and properties of rubber tire. Convection heating conditions. Initial and final midplane temperature.

FIND: (a) Time to reach final midplane temperature. (b) Effect of accelerated heating.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in a plane wall, (2) Constant properties, (3) Negligible radiation.

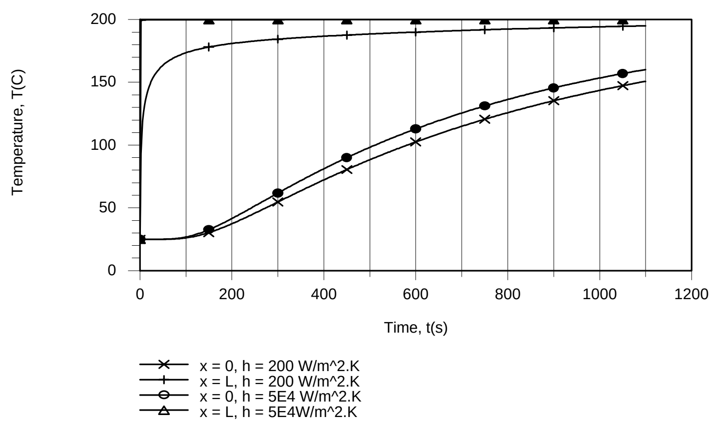
ANALYSIS: (a) With Bi = hL/k = 200 W/m^2·K(0.01 m)/0.14 W/m·K = 14.3, the lumped capacitance method is clearly inappropriate. Assuming Fo > 0.2, Eq. (5.41) may be used with C_1 = 1.265 and $\zeta_1 \approx$ 1.458 rad from Table 5.1 to obtain

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = C_1 \exp\left(-\zeta_1^2 \mathrm{Fo}\right) = 1.265 \exp\left(-2.126\,\mathrm{Fo}\right)$$

With $\theta_o^* = \left(T_o - T_\infty\right)/\left(T_i - T_\infty\right)$ = (-50)/(-175) = 0.286, $\mathrm{Fo} = -\ln\left(0.286/1.265\right)/2.126 = 0.70 = \alpha t_f / L^2$

$$t_f = \frac{0.7\left(0.01\mathrm{m}\right)^2}{6.35\times10^{-8}\,\mathrm{m}^2/\mathrm{s}} = 1100\,\mathrm{s}$$ <

(b) The desired temperature histories were generated using the IHT *Transient Conduction Model* for a *Plane Wall*, with h = 5×10^4 W/m^2·K used to approximate imposition of a surface temperature of 200°C.



The fact that imposition of a constant surface temperature (h → ∞) does not significantly accelerate the heating process should not be surprising. For h = 200 W/m^2·K, the Biot number is already quite large (Bi = 14.3), and limits to the heating rate are principally due to conduction in the rubber and not to convection at the surface. Any increase in h only serves to reduce what is already a small component of the total thermal resistance.

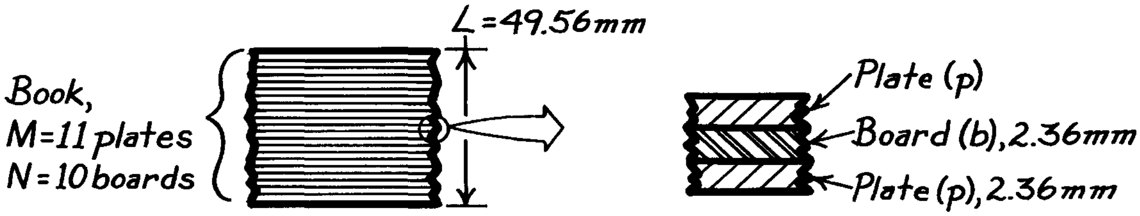
COMMENTS: The heating rate could be accelerated by increasing the steam temperature, but an upper limit would be associated with avoiding thermal damage to the rubber.

PROBLEM 5.42

KNOWN: Stack or book comprised of 11 metal plates (p) and 10 boards (b) each of 2.36 mm thickness and prescribed thermophysical properties.

FIND: Effective thermal conductivity, k, and effective thermal capacitance, (ρc_p).

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible contact resistance between plates and boards.

PROPERTIES: Metal plate (p, given): ρ_p = 8000 kg/m^3, $c_{p,p}$ = 480 J/kg·K, k_p = 12 W/m·K; Circuit boards (b, given): ρ_b = 1000 kg/m^3, $c_{p,b}$ = 1500 J/kg·K, k_b = 0.30 W/m·K.

ANALYSIS: The thermal resistance of the book is determined as the sum of the resistance of the boards and plates,

$$R''_{tot} = NR''_b + MR''_p$$

where M,N are the number of plates and boards in the book, respectively, and $R''_i = L_i / k_i$ where L_i and k_i are the thickness and thermal conductivities, respectively.

$$R''_{tot} = M\left(L_p / k_p\right) + N\left(L_b / k_b\right)$$
$$R''_{tot} = 11\left(0.00236\text{ m}/12\text{ W/m}\cdot\text{K}\right) + 10\left(0.00236\text{ m}/0.30\text{ W/m}\cdot\text{K}\right)$$
$$R''_{tot} = 2.163\times10^{-3} + 7.867\times10^{-2} = 8.083\times10^{-2}\text{ K/W.}$$

The effective thermal conductivity of the book of thickness (10 + 11) 2.36 mm is

$$k = L/R''_{tot} = \frac{0.04956\text{ m}}{8.083\times10^{-2}\text{ K/W}} = 0.613\text{ W/m}\cdot\text{K.} \qquad <$$

The thermal capacitance of the stack is

$$C''_{tot} = M\left(\rho_p L_p c_p\right) + N\left(\rho_b L_b c_b\right)$$
$$C''_{tot} = 11\left(8000\text{ kg/m}^3\times0.00236\text{ m}\times480\text{ J/kg}\cdot\text{K}\right) + 10\left(1000\text{ kg/m}^3\times0.00236\text{ m}\times1500\text{ J/kg}\cdot\text{K}\right)$$
$$C''_{tot} = 9.969\times10^4 + 3.540\times10^4 = 1.35\times10^5\text{ J/m}^2\cdot\text{K.}$$

The effective thermal capacitance of the book is

$$\left(\rho c_p\right) = C''_{tot} / L = 1.351\times10^5\text{ J/m}^2\cdot\text{K}/0.04956\text{ m} = 2.726\times10^6\text{ J/m}^3\cdot\text{K.} \qquad <$$

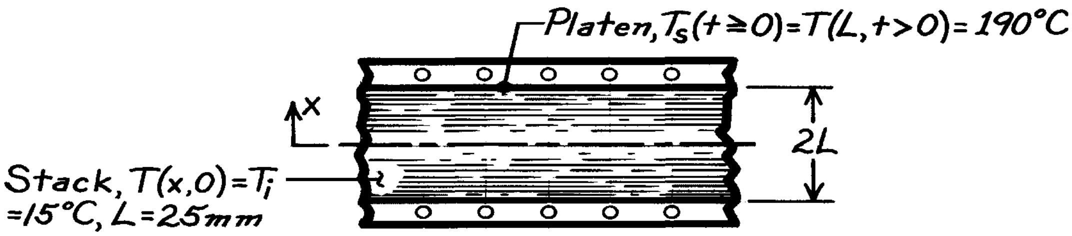
COMMENTS: The results of the analysis allow for representing the stack as a homogeneous medium with *effective* properties: k = 0.613 W/m·K and $\alpha = (k/\rho c_p) = 2.249\times10^{-7}$ m^2/s. See for example, Problem 5.38.

PROBLEM 5.43

KNOWN: Stack of circuit board-pressing plates, initially at a uniform temperature, is subjected by upper/lower platens to a higher temperature.

FIND: (a) Elapsed time, t_e, required for the mid-plane to reach cure temperature when platens are suddenly changed to $T_s = 190°C$, (b) Energy removal from the stack needed to return its temperature to T_i.

SCHEMATIC:



PROPERTIES: Stack (given): $k = 0.613$ W/m·K, $\rho c_p = 2.73\times10^6$ J/m^3·K; $\alpha = k/\rho c_p = 2.245\times10^{-7}$ m^2/s.

ANALYSIS: (a) Recognize that sudden application of surface temperature corresponds to $h \to \infty$, or $Bi^{-1} = 0$ (Heisler chart) or $Bi \to \infty$ (100, Table 5.1). With $T_s = T_\infty$,

$$\theta_o^* = \frac{T(0,t) - T_s}{T_i - T_s} = \frac{(170-190)^\circ C}{(15-190)^\circ C} = 0.114.$$

Using Eq. 5.41 with values of $\zeta_1 = 1.552$ and $C_1 = 1.2731$ at $Bi = 100$ (Table 5.1), find Fo

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

$$Fo = -\frac{1}{\zeta_1^2}\ln\left(\theta_o^*/C_1\right) = -\frac{1}{(1.552)^2}\ln(0.114/1.2731) = 1.002$$

where $Fo = \alpha t/L^2$,

$$t = \frac{FoL^2}{\alpha} = \frac{1.002\left(25\times10^{-3}\text{ m}\right)^2}{2.245\times10^{-7}\text{ m}^2/\text{s}} = 2.789\times10^3 \text{s} = 46.5 \text{ min.}$$ <

The Heisler chart, Figure D.1, could also be used to find Fo from values of θ_o^* and $Bi^{-1} = 0$.

(b) The energy removal is equivalent to the energy gained by the stack per unit area for the time interval $0 \to t_e$. With Q_o'' corresponding to the maximum amount of energy that could be transferred,

$$Q_o'' = \rho c(2L)(T_i - T_\infty) = 2.73\times10^6 \text{ J/m}^3\cdot\text{K}\left(2\times25\times10^{-3}\text{ m}\right)(15-190)\text{K} = -2.389\times10^7 \text{ J/m}^2.$$

Q'' may be determined from Eq. 5.46,

$$\frac{Q''}{Q_o''} = 1 - \frac{\sin\zeta_1}{\zeta_1}\theta_o^* = 1 - \frac{\sin(1.552\text{rad})}{1.552\text{rad}}\times0.114 = 0.795$$

We conclude that the energy to be removed from the stack per unit area to return it to T_i is

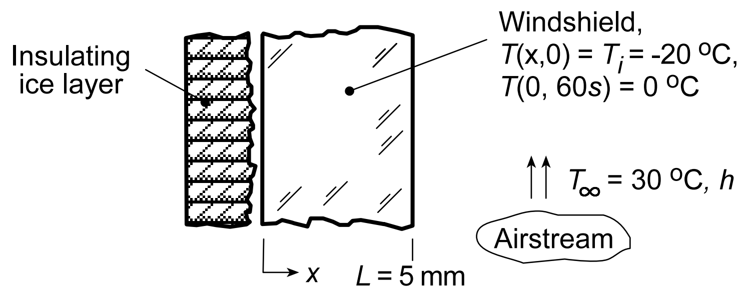
$$Q'' = 0.795Q_o'' = 0.795\times2.389\times10^7 \text{ J/m}^2 = 1.90\times10^7 \text{ J/m}^2.$$ <

PROBLEM 5.44

KNOWN: Car windshield, initially at a uniform temperature of -20°C, is suddenly exposed on its interior surface to the defrost system airstream at 30°C. The ice layer on the exterior surface acts as an insulating layer.

FIND: What airstream convection coefficient would allow the exterior surface to reach 0°C in 60 s?

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction in the windshield, (2) Constant properties, (3) Exterior surface is perfectly insulated.

PROPERTIES: Windshield (Given): ρ = 2200 kg/m^3, c_p = 830 J/kg·K and k = 1.2 W/m·K.

ANALYSIS: For the prescribed conditions, from Equations 5.31 and 5.33,

$$\frac{\theta(0,60\text{s})}{\theta_i} = \frac{\theta_o}{\theta_i} = \frac{T(0,60\text{s}) - T_\infty}{T_i - T_\infty} = \frac{(0-30)^\circ\ \text{C}}{(-20-30)^\circ\ \text{C}} = 0.6$$

$$\text{Fo} = \frac{kt}{\rho c L^2} = \frac{1.2\,\text{W/m}\cdot\text{K}\times 60}{2200\,\text{kg}\big/\text{m}^3 \times 830\,\text{J}\big/\text{kg}\cdot\text{K}\times(0.005\,\text{m})^2} = 1.58$$

The single-term series approximation, Eq. 5.41, along with Table 5.1, requires an iterative solution to find an appropriate Biot number. Alternatively, the Heisler charts, Appendix D, Figure D.1, for the midplane temperature could be used to find

$$\text{Bi}^{-1} = k/hL = 2.5$$

$$h = 1.2\,\text{W/m}\cdot\text{K}/2.5\times 0.005\,\text{m} = 96\,\text{W}\big/\text{m}^2\cdot\text{K}$$ <

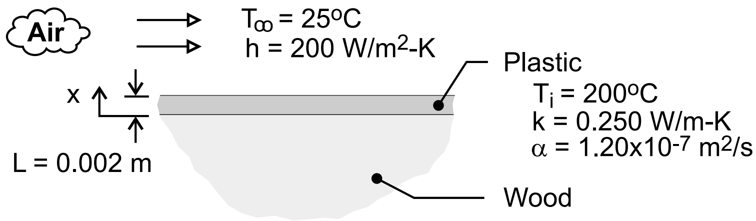
COMMENTS: Using the *IHT*, *Transient Conduction*, *Plane Wall Model*, the convection coefficient can be determined by solving the model with an assumed h and then sweeping over a range of h until the T(0,60s) condition is satisfied. Since the model is based upon multiple terms of the series, the result of h = 99 W/m^2·K is more precise than that found using the chart.

PROBLEM 5.45

KNOWN: Thickness, initial temperature and properties of plastic coating. Safe-to-touch temperature. Convection coefficient and air temperature.

FIND: Time for surface to reach safe-to-touch temperature. Corresponding temperature at plastic/wood interface.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in coating, (2) Negligible radiation, (3) Constant properties, (4) Negligible heat of reaction, (5) Negligible heat transfer across plastic/wood interface.

ANALYSIS: With Bi = hL/k = 200 $W/m^2\cdot K \times 0.002m/0.25\ W/m\cdot K = 1.6 > 0.1$, the lumped capacitance method may not be used. Applying the approximate solution of Eq. 5.40a, with C_1 = 1.155 and ζ_1 = 0.990 from Table 5.1,

$$\theta_s^* = \frac{T_s - T_\infty}{T_i - T_\infty} = \frac{(42-25)^\circ C}{(200-25)^\circ C} = 0.0971 = C_1 \exp\left(-\zeta_1^2 Fo\right)\cos\left(\zeta_1 x^*\right) = 1.155\exp(-0.980\ Fo)\cos(0.99)$$

Hence, for $x^* = 1$,

$$Fo = -\ln\left(\frac{0.0971}{1.155\cos(0.99)}\right)/(0.99)^2 = 1.914$$

$$t = \frac{Fo\,L^2}{\alpha} = \frac{1.914(0.002m)^2}{1.20\times10^{-7}\,m^2/s} = 63.8\,s$$ <

From Eq. 5.41, the corresponding interface temperature is

$$T_o = T_\infty + (T_i - T_\infty)\exp\left(-\zeta_1^2 Fo\right) = 25^\circ C + 175^\circ C\exp(-0.98\times1.914) = 51.8^\circ C$$ <

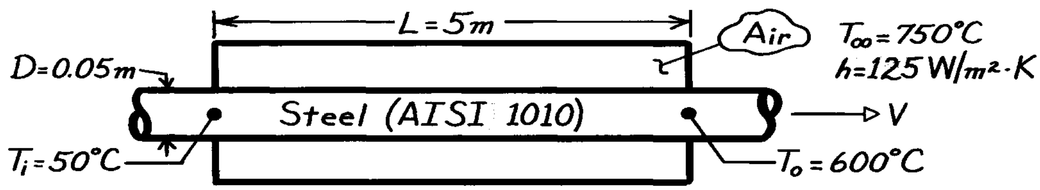
COMMENTS: By neglecting conduction into the wood and radiation from the surface, the cooling time is overpredicted and is therefore a conservative estimate. However, if energy generation due to solidification of polymer were significant, the cooling time would be longer.

PROBLEM 5.46

KNOWN: Inlet and outlet temperatures of steel rods heat treated by passage through an oven.

FIND: Rod speed, V.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction (axial conduction is negligible), (2) Constant properties, (3) Negligible radiation.

PROPERTIES: *Table A-1,* AISI 1010 Steel $\left(\bar{T} \approx 600\text{K}\right)$: k = 48.8 W/m·K, ρ = 7832 kg/m^3, c_p = 559 J/kg·K, α = (k/ρc_p) = 1.11×10^{-5}m^2/s.

ANALYSIS: The time needed to traverse the rod through the oven may be found from Fig. D.4.

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{600-750}{50-750} = 0.214$$

$$\text{Bi}^{-1} \equiv \frac{k}{hr_o} = \frac{48.8\ \text{W/m}\cdot\text{K}}{125\ \text{W/m}^2\cdot\text{K}\left(0.025\text{m}\right)} = 15.6.$$

Hence,

$$\text{Fo} = \alpha\ t/r_o^2 \approx 12.2$$

$$t = 12.2\left(0.025\text{m}\right)^2 / 1.11\times10^{-5}\text{m}^2/\text{s} = 687\ \text{s}.$$

The rod velocity is

$$V = \frac{L}{t} = \frac{5\text{m}}{687\text{s}} = 0.0073\ \text{m/s}.$$

COMMENTS: (1) Since (h r_o/2)/k = 0.032, the lumped capacitance method could have been used. From Eq. 5.5 it follows that t = 675 s.

(2) Radiation effects decrease t and hence increase V, assuming there is net radiant transfer from the oven walls to the rod.

(3) Since Fo > 0.2, the approximate analytical solution may be used. With Bi = hr_o/k =0.0641, Table 5.1 yields $\zeta_1 = 0.3549$ rad and C_1 = 1.0158. Hence from Eq. 5.49c

$$\text{Fo} = -\left(\zeta_1^2\right)^{-1}\ \ln\left[\frac{\theta_o^*}{C_1}\right] = 12.4,$$

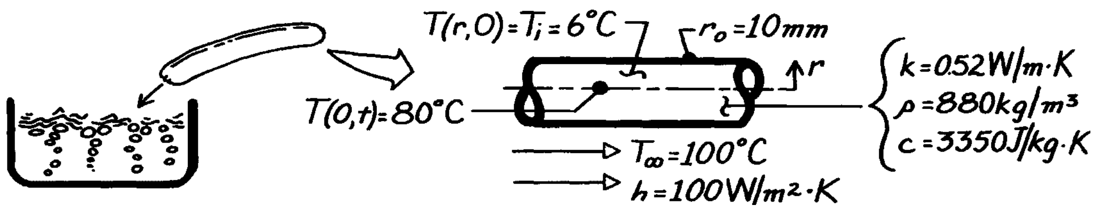
which is in good agreement with the graphical result.

PROBLEM 5.47

KNOWN: Hot dog with prescribed thermophysical properties, initially at 6°C, is immersed in boiling water.

FIND: Time required to bring centerline temperature to 80°C.

SCHEMATIC:



ASSUMPTIONS: (1) Hot dog can be treated as infinite cylinder, (2) Constant properties.

ANALYSIS: The Biot number, based upon Eq. 5.10, is

$$\mathrm{Bi} \equiv \frac{h\,L_c}{k} = \frac{h\,r_o/2}{k} = \frac{100\ \mathrm{W/m^2\cdot K}\left(10\times10^{-3}\mathrm{m}/2\right)}{0.52\ \mathrm{W/m\cdot K}} = 0.96$$

Since Bi > 0.1, a lumped capacitance analysis is not appropriate. Using the Heisler chart, Figure D.4 with

$$\mathrm{Bi} \equiv \frac{hr_o}{k} = \frac{100\mathrm{W/m^2\cdot K} \times 10\times10^{-3}\mathrm{m}}{0.52\ \mathrm{W/m\cdot K}} = 1.92 \quad \text{or} \quad \mathrm{Bi}^{-1} = 0.52$$

and $$\theta_o^* = \frac{\theta_o}{\theta_i} = \frac{T(0,t)-T_\infty}{T_i - T_\infty} = \frac{(80-100)^\circ\mathrm{C}}{(6\text{-}100)^\circ\mathrm{C}} = 0.21 \qquad (1)$$

find $$\mathrm{Fo} = t^* = \frac{\alpha\,t}{r_o^2} = 0.8 \qquad t = \frac{r_o^2}{\alpha}\cdot\mathrm{Fo} = \frac{\left(10\times10^{-3}\mathrm{m}\right)^2}{1.764\times10^{-7}\mathrm{m^2/s}}\times0.8 = 453.5\mathrm{s} = 7.6\ \mathrm{min} \quad <$$

where $$\alpha = k/\rho\,c = 0.52\ \mathrm{W/m\cdot K}/880\ \mathrm{kg/m^3}\times3350\ \mathrm{J/kg\cdot K} = 1.764\times10^{-7}\mathrm{m^2/s}.$$

COMMENTS: (1) Note that $L_c = r_o/2$ when evaluating the Biot number for the lumped capacitance analysis; however, in the Heisler charts, $\mathrm{Bi} \equiv hr_o/k$.

(2) The surface temperature of the hot dog follows from use of Figure D.5 with $r/r_o = 1$ and $\mathrm{Bi}^{-1} = 0.52$; find $\theta(1,t)/\theta_o \approx 0.45$. From Eq. (1), note that $\theta_o = 0.21\,\theta_i$ giving

$$\theta(1,t) = T(r_o,t) - T_\infty = 0.45\theta_o = 0.45\left(0.21\left[T_i - T_\infty\right]\right) = 0.45\times0.21\left[6-100\right]^\circ\mathrm{C} = -8.9^\circ\mathrm{C}$$

$$T(r_o,t) = T_\infty - 8.9^\circ\mathrm{C} = (100-8.9)^\circ\mathrm{C} = 91.1^\circ\mathrm{C}$$

(3) Since Fo ≥ 0.2, the approximate solution for θ*, Eq. 5.49, is valid. From Table 5.1 with Bi = 1.92, find that $\zeta_1 = 1.3245$ rad and $C_1 = 1.2334$. Rearranging Eq. 5.49 and substituting values,

$$\mathrm{Fo} = -\frac{1}{\zeta_1^2}\ln\left(\theta_o^*/C_1\right) = \frac{1}{(1.3245\ \mathrm{rad})^2}\ln\left[\frac{0.213}{1.2334}\right] = 1.00$$

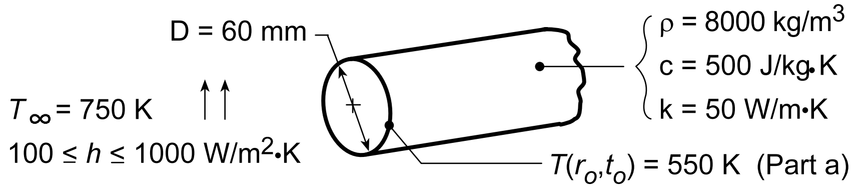
This result leads to a value of t = 9.5 min or 20% higher than that of the graphical method.

PROBLEM 5.48

KNOWN: Long rod with prescribed diameter and properties, initially at a uniform temperature, is heated in a forced convection furnace maintained at 750 K with a convection coefficient of h = 1000 $W/m^2 \cdot K$.

FIND: (a) The corresponding center temperature of the rod, $T(0, t_o)$, when the surface temperature $T(r_o, t_o)$ is measured as 550 K, (b) Effect of h on centerline temperature history.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction in rod, (2) Constant properties, (3) Rod, when initially placed in furnace, had a uniform (but unknown) temperature, (4) Fo ≥ 0.2.

ANALYSIS: (a) Since the rod was initially at a uniform temperature and Fo ≥ 0.2, the approximate solution for the infinite cylinder is appropriate. From Eq. 5.49b,

$$\theta^*\left(r^*, Fo\right) = \theta_o^*(Fo) J_0\left(\zeta_1 r^*\right) \qquad (1)$$

where, for $r^* = 1$, the dimensionless temperatures are, from Eq. 5.31,

$$\theta^*(1, Fo) = \frac{T(r_o, t_o) - T_\infty}{T_i - T_\infty} \qquad \theta_o^*(Fo) = \frac{T(0, t_o) - T_\infty}{T_i - T_\infty} \qquad (2,3)$$

Combining Eqs. (2) and (3) with Eq. (1) and rearranging,

$$\frac{T(r_o, t_o) - T_\infty}{T_i - T_\infty} = \frac{T(0, t_o) - T_\infty}{T_i - T_\infty} J_0(\zeta_1 \cdot 1)$$

$$T(0, t_o) = T_\infty + \frac{1}{J_0(\zeta_1)}\left[T(r_o, t_o) - T_\infty\right] \qquad (4)$$

The eigenvalue, ζ_1 = 1.0185 rad, follows from Table 5.1 for the Biot number

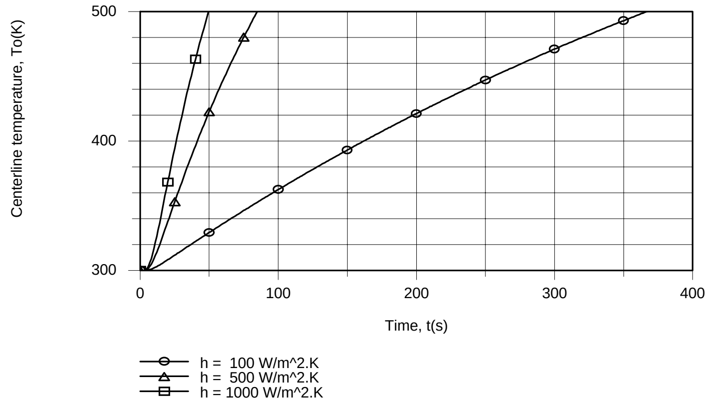
$$Bi = \frac{hr_o}{k} = \frac{1000\, W/m^2 \cdot K\,(0.060\, m/2)}{50\, W/m \cdot K} = 0.60.$$

From Table B-4, with ζ_1 = 1.0185 rad, $J_0(1.0185) = 0.7568$. Hence, from Eq. (4)

$$T(0, t_o) = 750\,K + \frac{1}{0.7568}[550 - 750]\,K = 486\,K \qquad <$$

(b) Using the IHT *Transient Conduction Model* for a *Cylinder*, the following temperature histories were generated.

Continued...



The times required to reach a centerline temperature of 500 K are 367, 85 and 51s, respectively, for h = 100, 500 and 1000 $W/m^2 \cdot K$. The corresponding values of the Biot number are 0.06, 0.30 and 0.60. Hence, even for h = 1000 $W/m^2 \cdot K$, the convection resistance is not negligible relative to the conduction resistance and significant reductions in the heating time could still be effected by increasing h to values considerably in excess of 1000 $W/m^2 \cdot K$.

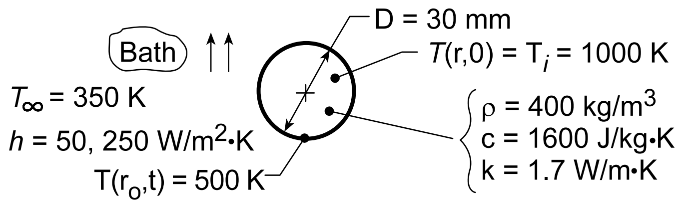
COMMENTS: For Part (a), recognize why it is not necessary to know T_i or the time t_o. We require that $Fo \geq 0.2$, which for this sphere corresponds to $t \geq 14s$. For this situation, the time dependence of the surface and center are the same.

PROBLEM 5.49

KNOWN: A long cylinder, initially at a uniform temperature, is suddenly quenched in a large oil bath.

FIND: (a) Time required for the surface to reach 500 K, (b) Effect of convection coefficient on surface temperature history.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties, (3) Fo > 0.2.

ANALYSIS: (a) Check first whether lumped capacitance method is applicable. For h = 50 W/m^2·K,

$$Bi_c = \frac{hL_c}{k} = \frac{h(r_o/2)}{k} = \frac{50\,W/m^2 \cdot K\,(0.015\,m/2)}{1.7\,W/m \cdot K} = 0.221.$$

Since $Bi_c > 0.1$, method is not suited. Using the approximate series solution for the infinite cylinder,

$$\theta^*\left(r^*, Fo\right) = C_1 \exp\left(-\zeta_1^2 Fo\right) \times J_0\left(\zeta_1 r^*\right) \tag{1}$$

Solving for Fo and setting $r^* = 1$, find

$$Fo = -\frac{1}{\zeta_1^2} \ln\left[\frac{\theta^*}{C_1 J_0(\zeta_1)}\right]$$

where $\theta^* = (1, Fo) = \dfrac{T(r_o, t_o) - T_\infty}{T_i - T_\infty} = \dfrac{(500 - 350)K}{(1000 - 350)K} = 0.231$.

From Table 5.1, with Bi = 0.441, find $\zeta_1 = 0.8882$ rad and $C_1 = 1.1019$. From Table B.4, find $J_0(\zeta_1) = 0.8121$. Substituting numerical values into Eq. (2),

$$Fo = -\frac{1}{(0.8882)^2} \ln[0.231/1.1019 \times 0.8121] = 1.72.$$

From the definition of the Fourier number, Fo = $\alpha t / r_o^2$, and $\alpha = k/\rho c$,

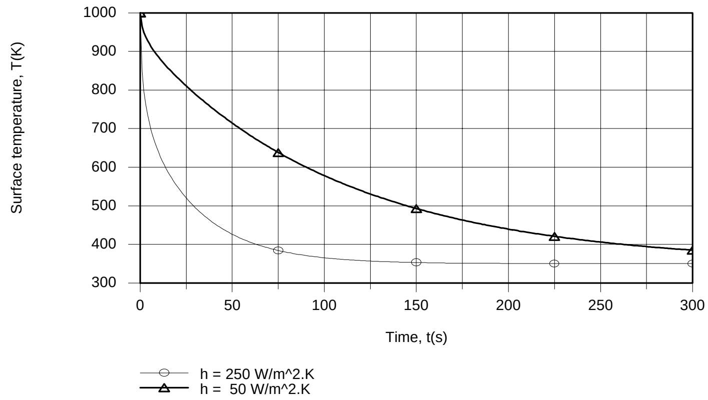
$$t = Fo\frac{r_o^2}{\alpha} = Fo \cdot r_o^2 \frac{\rho c}{k}$$

$$t = 1.72(0.015\,m)^2 \times 400\,kg/m^3 \times 1600\,J/kg \cdot K / 1.7\,W/m \cdot K = 145s.$$ <

(b) Using the IHT *Transient Conduction Model* for a *Cylinder*, the following surface temperature histories were obtained.

Continued...

PROBLEM 5.49 (Cont.)



Increasing the convection coefficient by a factor of 5 has a significant effect on the surface temperature, greatly accelerating its approach to the oil temperature. However, even with h = 250 $W/m^2 \cdot K$, Bi = 1.1 and the convection resistance remains significant. Hence, in the interest of accelerated cooling, additional benefit could be achieved by further increasing the value of h.

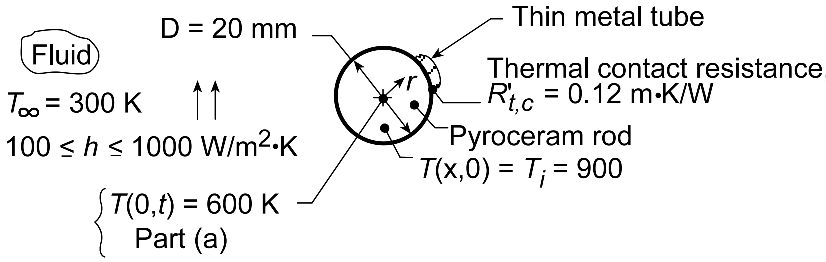
COMMENTS: For Part (a), note that, since Fo = 1.72 > 0.2, the approximate series solution is appropriate.

PROBLEM 5.50

KNOWN: Long pyroceram rod, initially at a uniform temperature of 900 K, and clad with a thin metallic tube giving rise to a thermal contact resistance, is suddenly cooled by convection.

FIND: (a) Time required for rod centerline to reach 600 K, (b) Effect of convection coefficient on cooling rate.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Thermal resistance and capacitance of metal tube are negligible, (3) Constant properties, (4) Fo ≥ 0.2.

PROPERTIES: *Table A-2*, Pyroceram ($\bar{T}$ = (600 + 900)K/2 = 750 K): ρ = 2600 kg/m^3, c = 1100 J/kg·K, k = 3.13 W/m·K.

ANALYSIS: (a) The thermal contact and convection resistances can be combined to give an overall heat transfer coefficient. Note that $R'_{t,c}$ [m·K/W] is expressed per unit length for the outer surface. Hence, for h = 100 W/m^2·K,

$$U = \frac{1}{1/h + R'_{t,c}(\pi D)} = \frac{1}{1/100\,\mathrm{W/m^2\cdot K} + 0.12\,\mathrm{m\cdot K/W}\,(\pi\times 0.020\,\mathrm{m})} = 57.0\,\mathrm{W/m^2\cdot K}.$$

Using the approximate series solution, Eq. 5.50c, the Fourier number can be expressed as

$$\mathrm{Fo} = -\left(1/\zeta_1^2\right)\ln\left(\theta_o^*/C_1\right).$$

From Table 5.1, find ζ_1 = 0.5884 rad and C_1 = 1.0441 for

$$\mathrm{Bi} = Ur_o/k = 57.0\,\mathrm{W/m^2\cdot K}\,(0.020\,\mathrm{m}/2)/3.13\,\mathrm{W/m\cdot K} = 0.182.$$

The dimensionless temperature is

$$\theta_o^*(0,\mathrm{Fo}) = \frac{T(0,t)-T_\infty}{T_i - T_\infty} = \frac{(600-300)\,\mathrm{K}}{(900-300)\,\mathrm{K}} = 0.5.$$

Substituting numerical values to find Fo and then the time t,

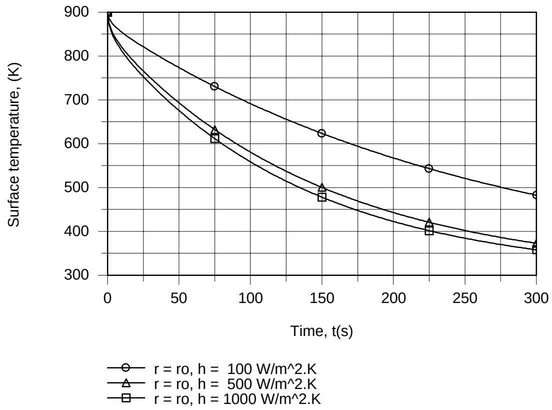
$$\mathrm{Fo} = \frac{-1}{(0.5884)^2}\ln\frac{0.5}{1.0441} = 2.127$$

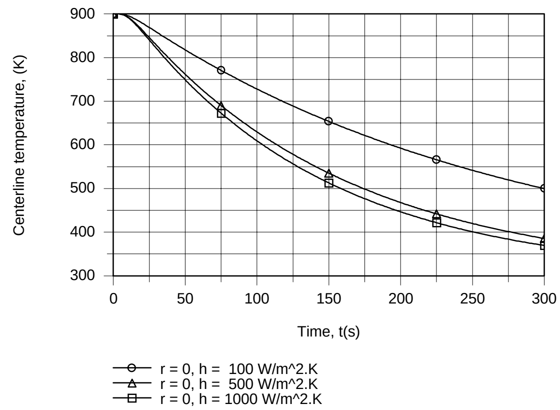
$$t = \mathrm{Fo}\frac{r_o^2}{\alpha} = \mathrm{Fo}\cdot r_o^2\,\frac{\rho c}{k}$$

$$t = 2.127\,(0.020\,\mathrm{m}/2)^2\,2600\,\mathrm{kg/m^3}\times 1100\,\mathrm{J/kg\cdot K}/3.13\,\mathrm{W/m\cdot K} = 194\,\mathrm{s}. \qquad <$$

(b) The following temperature histories were generated using the IHT *Transient conduction Model* for a *Cylinder*.

Continued...





While enhanced cooling is achieved by increasing h from 100 to 500 $W/m^2 \cdot K$, there is little benefit associated with increasing h from 500 to 1000 $W/m^2 \cdot K$. The reason is that for h much above 500 $W/m^2 \cdot K$, the contact resistance becomes the dominant contribution to the total resistance between the fluid and the rod, rendering the effect of further reductions in the convection resistance negligible. Note that, for h = 100, 500 and 1000 $W/m^2 \cdot K$, the corresponding values of U are 57.0, 104.8 and 117.1 $W/m^2 \cdot K$, respectively.

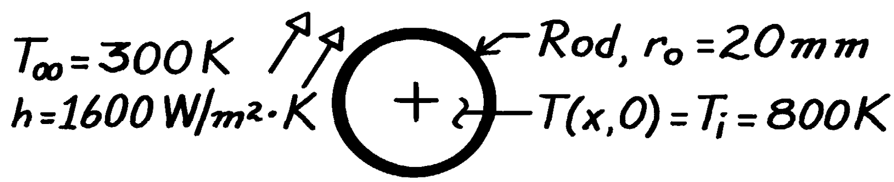
COMMENTS: For Part (a), note that, since Fo = 2.127 > 0.2, Assumption (4) is satisfied.

PROBLEM 5.51

KNOWN: Sapphire rod, initially at a uniform temperature of 800K is suddenly cooled by a convection process; after 35s, the rod is wrapped in insulation.

FIND: Temperature rod reaches after a long time following the insulation wrap.

SCHEMATIC:



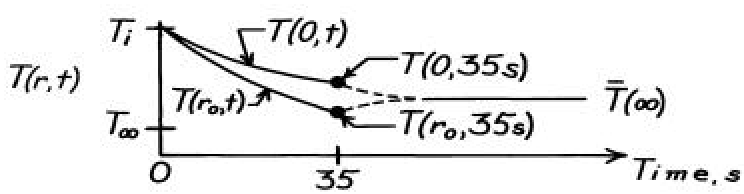
ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties, (3) No heat losses from the rod when insulation is applied.

PROPERTIES: *Table A-2*, Aluminum oxide, sapphire (550K): $\rho = 3970\ kg/m^3$, $c = 1068\ J/kg\cdot K$, $k = 22.3\ W/m\cdot K$, $\alpha = 5.259\times10^{-5}\ m^2/s$.

ANALYSIS: First calculate the Biot number with $L_c = r_o/2$,

$$Bi = \frac{h\,L_c}{k} = \frac{h\,(r_o/2)}{k} = \frac{1600\ W/m^2\cdot K\,(0.020\ m/2)}{22.3\ W/m\cdot K} = 0.72.$$

Since Bi > 0.1, the rod cannot be approximated as a lumped capacitance system. The temperature distribution during the cooling process, $0 \le t \le 35s$, and for the time following the application of insulation, $t > 35s$, will appear as



Eventually ($t \to \infty$), the temperature of the rod will be uniform at $\bar{T}(\infty)$. To find $\bar{T}(\infty)$, write the conservation of energy requirement for the rod on a *time interval* basis, $E_{in} - E_{out} = \Delta E \equiv E_{final} - E_{initial}$.

Using the nomenclature of Section 5.5.3 and basing energy relative to T_∞, the energy balance becomes

$$-Q = r\ cV\left(\bar{T}(\infty) - T_\infty\right) - Q_o$$

where $Q_o = \rho cV(T_i - T_\infty)$. Dividing through by Q_o and solving for $\bar{T}(\infty)$, find

$$\bar{T}(\infty) = T_\infty + (T_i - T_\infty)(1 - Q/Q_o).$$

From the Groeber chart, Figure D.6, with

$$Bi = \frac{hr_o}{k} = \frac{1600\ W/m^2\cdot K \times 0.020m}{22.3\ W/m\cdot K} = 1.43$$

$$Bi^2Fo = Bi^2\left(a\ t/r_o^2\right) = (1.43)^2\ \left(5.259\times10^{-6}\ m^2/s\times35s/(0.020m)^2\right) = 0.95.$$

find $Q/Q_o \approx 0.57$. Hence,

$$\bar{T}(\infty) = 300K + (800-300)K\ (1\text{-}0.57) = 515\ K.$$ <

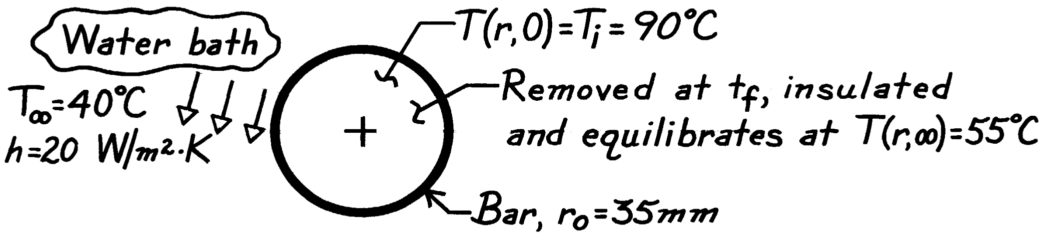
COMMENTS: From use of Figures D.4 and D.5, find $T(0,35s) = 525K$ and $T(r_o,35s) = 423K$.

PROBLEM 5.52

KNOWN: Long bar of 70 mm diameter, initially at 90°C, is suddenly immersed in a water bath (T_∞ = 40°C, h = 20 W/m^2·K).

FIND: (a) Time, t_f, that bar should remain in bath in order that, when removed and allowed to equilibrate while isolated from surroundings, it will have a uniform temperature T(r, ∞) = 55°C.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties.

PROPERTIES: Bar (given): ρ = 2600 kg/m^3, c = 1030 J/kg·K, k = 3.50 W/m·K, α = k/ρc = 1.31×10^{-6} m^2/s.

ANALYSIS: Determine first whether conditions are space-wise isothermal

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{h\left(r_o/2\right)}{k} = \frac{20\ \mathrm{W/m^2\cdot K}\left(0.035\ \mathrm{m}/2\right)}{3.50\ \mathrm{W/m\cdot K}} = 0.10$$

and since Bi ≥ 0.1, a Heisler solution is appropriate.

(a) Consider an overall energy balance on the bar during the time interval Δt = t_f (the time the bar is in the bath).

$$E_{in} - E_{out} = \Delta E$$
$$0 - Q = E_{final} - E_{initial} = Mc\left(T_f - T_\infty\right) - Mc\left(T_i - T_\infty\right)$$
$$-Q = Mc\left(T_f - T_\infty\right) - Q_o$$
$$\frac{Q}{Q_o} = 1 - \frac{T_f - T_\infty}{T_i - T_\infty} = 1 - \frac{\left(55-40\right)^\circ \mathrm{C}}{\left(90-40\right)^\circ \mathrm{C}} = 0.70$$

where Q_o is the initial energy in the bar (relative to T_∞; Eq. 5.44). With Bi = hr_o/k = 0.20 and Q/Q_o = 0.70, use Figure D.6 to find Bi^2Fo = 0.15; hence Fo = $0.15/Bi^2$ = 3.75 and

$$t_f = Fo\cdot r_o^2/a = 3.75\left(0.035\ \mathrm{m}\right)^2/1.31\times10^{-6}\ \mathrm{m^2/s} = 3507\ \mathrm{s}. \qquad <$$

(b) To determine T(r_o, t_f), use Figures D.4 and D.5 for $\theta(r_o,t)/\theta_i$ (Fo = 3.75, Bi^{-1} = 5.0) and θ_o/θ_i (Bi^{-1} = 5.0, r/r_o = 1, respectively, to find

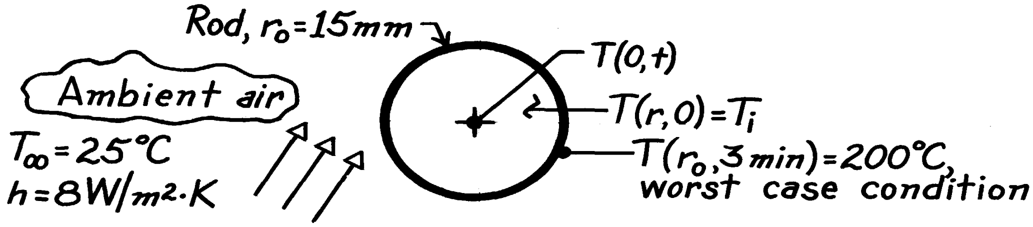
$$T\left(r_o, t_f\right) = T_\infty + \frac{q\left(r_o, t\right)}{q_o}\cdot\frac{q_o}{q_i}\cdot q_i = 40^\circ\mathrm{C} + 0.25\times0.90\left(90-50\right)^\circ\mathrm{C} = 49^\circ\mathrm{C}. \qquad <$$

PROBLEM 5.53

KNOWN: Long plastic rod of diameter D heated uniformly in an oven to T_i and then allowed to convectively cool in ambient air (T_∞, h) for a 3 minute period. Minimum temperature of rod should not be less than 200°C and the maximum-minimum temperature within the rod should not exceed 10°C.

FIND: Initial uniform temperature T_i to which rod should be heated. Whether the 10°C internal temperature difference is exceeded.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties, (3) Uniform and constant convection coefficients.

PROPERTIES: Plastic rod (given): k = 0.3 W/m·K, ρc_p = 1040 kJ/m^3·K.

ANALYSIS: For the worst case condition, the rod cools for 3 minutes and its outer surface is at least 200°C in order that the subsequent pressing operation will be satisfactory. Hence,

$$\mathrm{Bi} = \frac{hr_o}{k} = \frac{8\ \mathrm{W/m^2 \cdot K} \times 0.015\ \mathrm{m}}{0.3\ \mathrm{W/m \cdot K}} = 0.40$$

$$\mathrm{Fo} = \frac{\alpha t}{r_o^2} = \frac{k}{\rho c_p} \cdot \frac{t}{r_o^2} = \frac{0.3\ \mathrm{W/m \cdot K}}{1040 \times 10^3\ \mathrm{J/m^3 \cdot K}} \times \frac{3 \times 60\mathrm{s}}{(0.015\ \mathrm{m})^2} = 0.2308.$$

Using Eq. 5.49a and $\zeta_1 = 0.8516$ rad and C_1 = 1.0932 from Table 5.1,

$$\theta^* = \frac{T(r_o,t) - T_\infty}{T_i - T_\infty} = C_1 J_0(\zeta_1 r_o^*) \exp(-\zeta_1^2 \mathrm{Fo}).$$

With $r_o^* = 1$, from Table B.4, $J_0(\zeta_1 \times 1) = J_0(0.8516) = 0.8263$, giving

$$\frac{200 - 25}{T_i - 25} = 1.0932 \times 0.8263 \exp(-0.8516^2 \times 0.2308) \qquad T_i = 254°\mathrm{C}.$$ <

At this time (3 minutes) what is the difference between the center and surface temperatures of the rod? From Eq. 5.49b,

$$\frac{\theta^*}{\theta_o} = \frac{T(r_o,t) - T_\infty}{T(0,t) - T_\infty} = \frac{200 - 25}{T(0,t) - 25} = J_0(\zeta_1 r_o^*) = 0.8263$$

which gives T(0,t) = 237°C. Hence,

$$\Delta T = T(0,180\mathrm{s}) - T(r_o,180\mathrm{s}) = (237 - 200)°\mathrm{C} = 37°\mathrm{C}.$$ <

Hence, the desired max-min temperature difference sought (10°C) is not achieved.

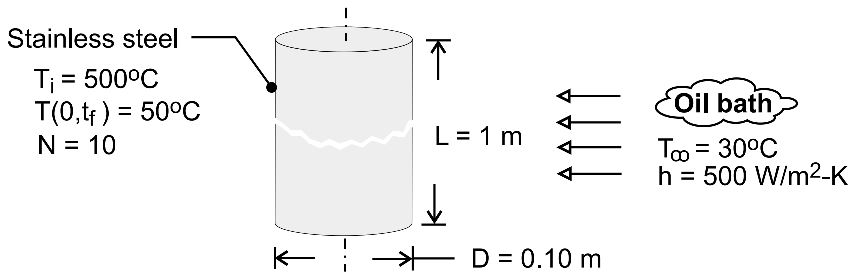
COMMENTS: ΔT could be reduced by decreasing the cooling rate; however, h can not be made much smaller. Two solutions are (a) increase ambient air temperature and (b) non-uniformly heat rod in oven by controlling its residence time.

PROBLEM 5.54

KNOWN: Diameter and initial temperature of roller bearings. Temperature of oil bath and convection coefficient. Final centerline temperature. Number of bearings processed per hour.

FIND: Time required to reach centerline temperature. Cooling load.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction in rod, (2) Constant properties.

PROPERTIES: *Table A.1*, St. St. 304 $\left(\overline{T} = 548\,K\right)$: ρ=7900 kg/m^3, k = 19.0 W/m·K, c_p = 546 J/kg·K, $\alpha = 4.40 \times 10^{-6}$ m^2/s.

ANALYSIS: With Bi = h $(r_o/2)/k$ = 0.658, the lumped capacitance method can not be used. From the one-term approximation of Eq. 5.49 c for the centerline temperature,

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{50-30}{500-30} = 0.0426 = C_1 \exp\left(-\zeta_1^2 Fo\right) = 1.1382 \exp\left[-\left(0.9287\right)^2 Fo\right]$$

where, for Bi = hr_o/k = 1.316, C_1 = 1.1382 and ζ_1 = 0.9287 from Table 5.1.

$$Fo = -\ell n\left(0.0374\right)/0.863 = 3.81$$

$$t_f = Fo\, r_o^2/\alpha = 3.81\left(0.05\,m\right)^2/4.40\times10^{-6} = 2162\,s = 36\,min$$ <

From Eqs. 5.44 and 5.51, the energy extracted from a single rod is

$$Q = \rho c V\left(T_i - T_\infty\right)\left[1 - \frac{2\theta_o^*}{\zeta_1} J_1\left(\zeta_1\right)\right]$$

With J_1 (0.9287) = 0.416 from Table B.4,

$$Q = 7900\,kg/m^3 \times 546\,J/kg\cdot K\left[\pi\left(0.05\,m\right)^2 1m\right]470\,K\left[1 - \frac{0.0852\times0.416}{0.9287}\right] = 1.53\times10^7\ J$$

The nominal cooling load is

$$\overline{q} = \frac{NQ}{t_f} = \frac{10\times1.53\times10^7\ J}{2162\,s} = 70,800\,W = 7.08\ kW$$ <

COMMENTS: For a centerline temperature of 50°C, Eq. 5.49b yields a surface temperature of

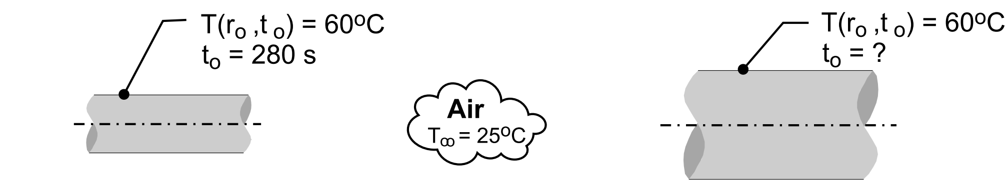
$$T\left(r_o, t\right) = T_\infty + \left(T_i - T_\infty\right)\theta_o^*\ J_o\left(\zeta_1\right) = 30°C + 470°C\times0.0426\times0.795 = 45.9°C$$

PROBLEM 5.55

KNOWN: Long rods of 40 mm- and 80-mm diameter at a uniform temperature of 400°C in a curing oven, are removed and cooled by forced convection with air at 25°C. The 40-mm diameter rod takes 280 s to reach a *safe-to-handle* temperature of 60°C.

FIND: Time it takes for a 80-mm diameter rod to cool to the same safe-to-handle temperature. Comment on the result? Did you anticipate this outcome?

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial (cylindrical) conduction in the rods, (2) Constant properties, and (3) Convection coefficient same value for both rods.

PROPERTIES: Rod (*given*): $\rho = 2500\ kg/m^3$, $c = 900\ J/kg\cdot K$, $k = 15\ W/m\cdot K$.

ANALYSIS: Not knowing the convection coefficient, the Biot number cannot be calculated to determine whether the rods behave as spacewise isothermal objects. Using the relations from Section 5.6, Radial Systems with Convection, for the infinite cylinder, Eq. 5.50, evaluate $Fo = \alpha t / r_o^2$, and knowing $T(r_o, t_o)$, a trial-and-error solution is required to find $Bi = h\, r_o/k$ and hence, h. Using the *IHT Transient Conduction* model for the *Cylinder*, the following results are readily calculated for the 40-mm rod. With $t_o = 280$ s,

$$Fo = 4.667 \qquad Bi = 0.264 \qquad h = 197.7\ W/m^2\cdot K$$

For the 80-mm rod, with the foregoing value for h, with $T(r_o, t_o) = 60°C$, find

$$Bi = 0.528 \qquad Fo = 2.413 \qquad t_o = 579\ s$$ <

COMMENTS: (1) The time-to-cool, t_o, for the 80-mm rod is slightly more than twice that for the 40-mm rod. Did you anticipate this result? Did you believe the times would be proportional to the diameter squared?

(2) The simplest approach to explaining the relationship between t_o and the diameter follows from the lumped capacitance analysis, Eq. 5.13, where for the same θ/θ_i, we expect $Bi\cdot Fo_o$ to be a constant. That is,

$$\frac{h\cdot r_o}{k}\times\frac{\alpha\, t_o}{r_o^2} = C$$

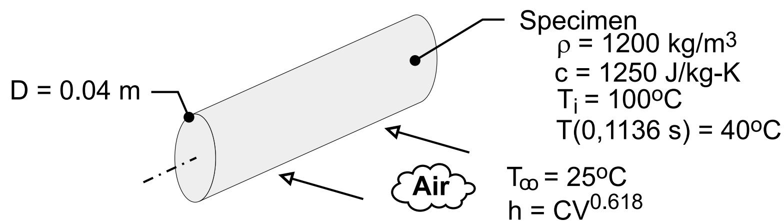
yielding $t_o \sim r_o$ (not r_o^2).

PROBLEM 5.56

KNOWN: Initial temperature, density, specific heat and diameter of cylindrical rod. Convection coefficient and temperature of air flow. Time for centerline to reach a prescribed temperature. Dependence of convection coefficient on flow velocity.

FIND: (a) Thermal conductivity of material, (b) Effect of velocity and centerline temperature and temperature histories for selected velocities.

SCHEMATIC:



ASSUMPTIONS: (1) Lumped capacitance analysis can not be used but one-term approximation for an infinite cylinder is appropriate, (2) One-dimensional conduction in r, (3) Constant properties, (4) Negligible radiation, (5) Negligible effect of thermocouple hole on conduction.

ANALYSIS: (a) With $\theta_o^* = [T_o(0,1136\text{s}) - T_\infty]/(T_i - T_\infty) = (40 - 25)/(100 - 25) = 0.20$, Eq. 5.49c yields

$$\text{Fo} = \frac{\alpha t}{r_o^2} = \frac{k\ t}{\rho c_p r_o^2} = \frac{k(1136\,\text{s})}{1200\ \text{kg/m}^3 \times 1250\,\text{J/kg}\cdot\text{K} \times (0.02\,\text{m})^2} = -\ln(0.2/C_1)/\zeta_1^2 \qquad (1)$$

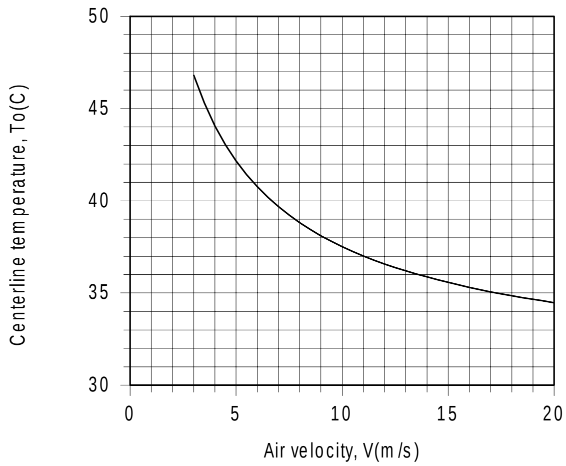
Because C_1 and ζ_1 depend on Bi = hr_o/k, a trial-and-error procedure must be used. For example, a value of k may be assumed and used to calculate Bi, which may then be used to obtain C_1 and ζ_1 from Table 5.1. Substituting C_1 and ζ_1 into Eq. (1), k may be computed and compared with the assumed value. Iteration continues until satisfactory convergence is obtained, with

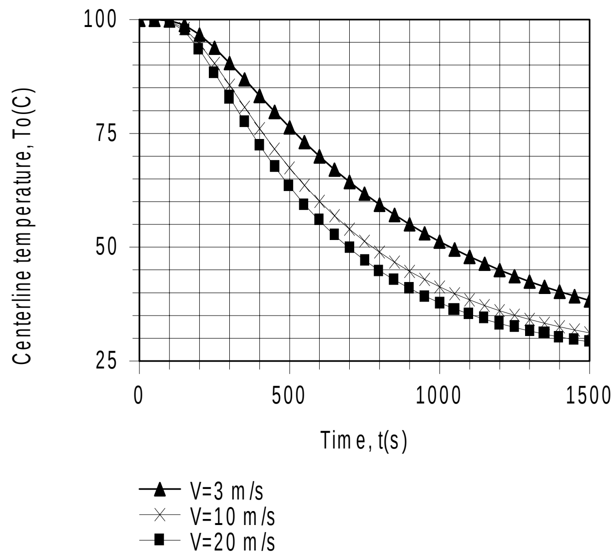
$$k \approx 0.30\ \text{W/m}\cdot\text{K}$$ <

and, hence, Bi = 3.67, C_1 = 1.45, ζ_1 = 1.87 and Fo = 0.568. For the above value of k, $-\ln(0.2/C_1)/\zeta_1^2 = 0.567$, which equals the Fourier number, as prescribed by Eq. (1).

(b) With h = 55 $\text{W/m}^2\cdot\text{K}$ for V = 6.8 m/s, $h = CV^{0.618}$ yields a value of C = 16.8 $\text{W}\cdot\text{s}^{0.618}/\text{m}^{2.618}\cdot\text{K}$. The desired variations of the centerline temperature with velocity (for t = 1136 s) and time (for V = 3, 10 and 20 m/s) are as follows:

Continued …..





With increasing V from 3 to 20 m/s, h increases from 33 to 107 $W/m^2 \cdot K$, and the enhanced cooling reduces the centerline temperature at the prescribed time. The accelerated cooling associated with increasing V is also revealed by the temperature histories, and the time required to achieve thermal equilibrium between the air and the cylinder decreases with increasing V.

COMMENTS: (1) For the smallest value of h = 33 $W/m^2 \cdot K$, $Bi \equiv h\,(r_o/2)/k = 1.1 >> 0.1$, and use of the lumped capacitance method is clearly inappropriate.

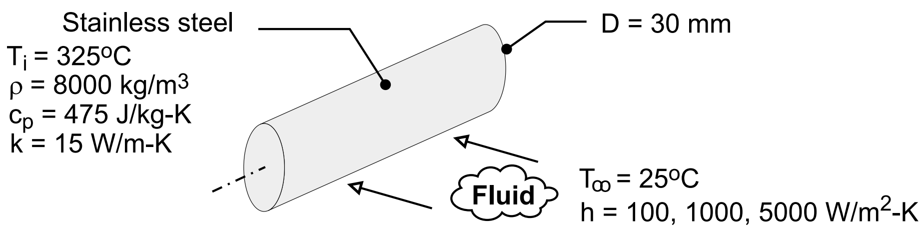
(2) The *IHT* Transient Conduction Model for a cylinder was used to perform the calculations of Part (b). Because the model is based on the exact solution, Eq. 5.47a, it is accurate for values of Fo < 0.2, as well as Fo > 0.2. Although in principle, the model may be used to calculate the thermal conductivity for the conditions of Part (a), convergence is elusive and may only be achieved if the initial guesses are close to the correct results.

PROBLEM 5.57

KNOWN: Diameter, initial temperature and properties of stainless steel rod. Temperature and convection coefficient of coolant.

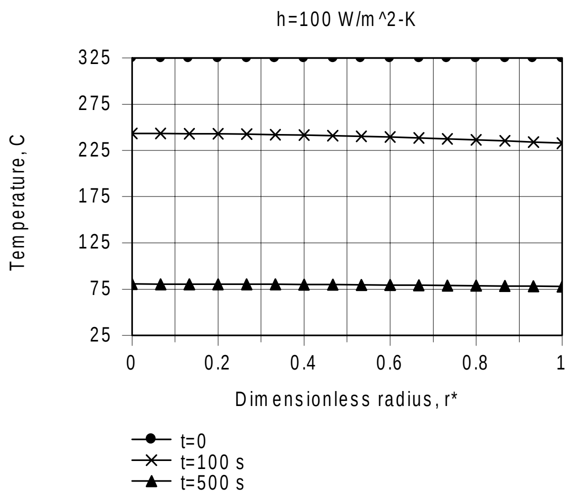
FIND: Temperature distributions for prescribed convection coefficients and times.

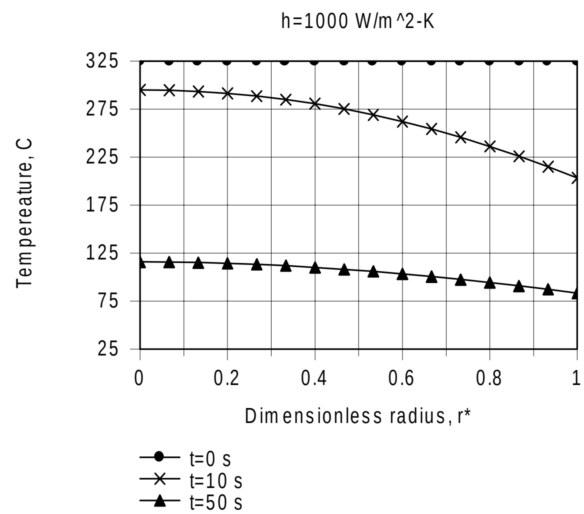
SCHEMATIC:



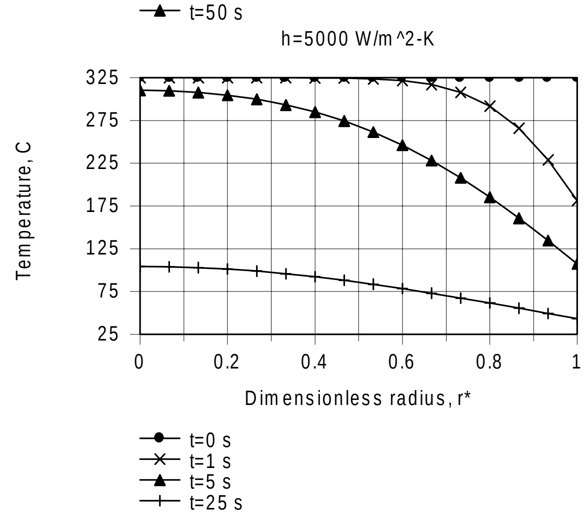
ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties.

ANALYSIS: The *IHT* model is based on the exact solution to the heat equation, Eq. 5.47. The results are plotted as follows





For h = 100 W/m^2·K, Bi = hr_o/k = 0.1, and as expected, the temperature distribution is nearly uniform throughout the rod. For h = 1000 W/m^2·K (Bi = 1), temperature variations within the rod are not negligible. In this case the centerline-to-surface temperature difference is comparable to the surface-to-fluid temperature difference. For h = 5000 W/m^2·K (Bi = 5), temperature variations within the rod are large and [T (0,t) – T (r_o,t)] is substantially larger than [T (r_o,t) - T_∞].



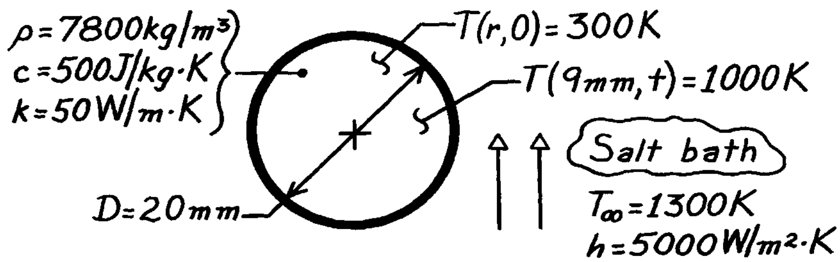
COMMENTS: With increasing Bi, conduction within the rod, and not convection from the surface, becomes the limiting process for heat loss.

PROBLEM 5.58

KNOWN: A ball bearing is suddenly immersed in a molten salt bath; heat treatment to harden occurs at locations with T > 1000K.

FIND: Time required to harden outer layer of 1mm.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties, (3) Fo ≥ 0.2.

ANALYSIS: Since any location within the ball whose temperature exceeds 1000K will be hardened, the problem is to find the time when the location r = 9mm reaches 1000K. Then a 1mm outer layer will be hardened. Begin by finding the Biot number.

$$\mathrm{Bi} = \frac{h\,r_o}{k} = \frac{5000\ \mathrm{W/m^2 \cdot K}\left(0.020\mathrm{m}/2\right)}{50\ \mathrm{W/m \cdot K}} = 1.00.$$

Using the one-term approximate solution for a sphere, find

$$\mathrm{Fo} = -\frac{1}{\zeta_1^2}\ln\left[\theta^* / C_1 \frac{1}{\zeta_1 r^*}\sin\left(\zeta_1 r^*\right)\right].$$

From Table 5.1 with Bi = 1.00, for the sphere find $\zeta_1 = 1.5708$ rad and C_1 = 1.2732. With r* = r/r_o = (9mm/10mm) = 0.9, substitute numerical values.

$$\mathrm{Fo} = \frac{-1}{(1.5708)^2}\ln\left[\frac{(1000-1300)\mathrm{K}}{(300-1300)\mathrm{K}}/1.2732\frac{1}{1.5708\times 0.9}\sin(1.5708\times 0.9\ \mathrm{rad})\right] = 0.441.$$

From the definition of the Fourier number with α = k/ρc,

$$t = \mathrm{Fo}\frac{r_o^2}{\alpha} = \mathrm{Fo}\cdot r_o^2\frac{\rho\ c}{k} = 0.441\times\left[\frac{0.020\mathrm{m}}{2}\right]^2 7800\frac{\mathrm{kg}}{\mathrm{m}^3}\times 500\frac{\mathrm{J}}{\mathrm{kg \cdot K}}/50\ \mathrm{W/m \cdot K} = 3.4\mathrm{s}. \quad <$$

COMMENTS: (1) Note the very short time required to harden the ball. At this time it can be easily shown the center temperature is T(0,3.4s) = 871 K.

(2) The Heisler charts can also be used. From Fig. D.8, with Bi^{-1} = 1.0 and r/r_o = 0.9, read θ/θ_o = 0.69(±0.03). Since

$$\theta = T - T_\infty = 1000 - 1300 = -300\mathrm{K} \qquad \theta_i = T_i - T_\infty = -1000\mathrm{K}$$

it follows that

$$\frac{\theta}{\theta_i} = 0.30. \quad \text{Since} \quad \frac{\theta}{\theta_i} = \frac{\theta}{\theta_o}\cdot\frac{\theta_o}{\theta_i}, \quad \text{then} \quad \frac{\theta}{\theta_i} = 0.69\ \frac{\theta_o}{\theta_i}$$

and $\theta_o/\theta_i = 0.30/0.69 = 0.43\ (\pm 0.02)$.

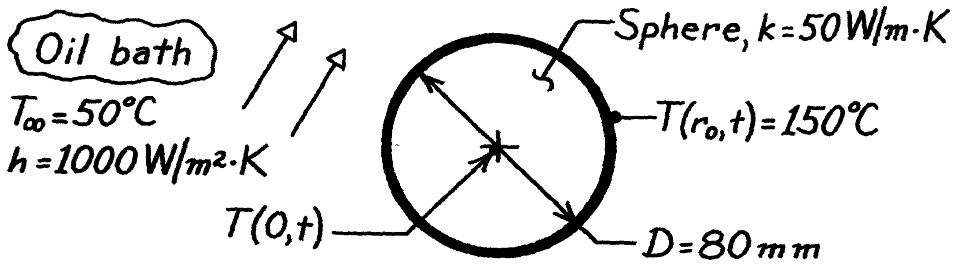
From Fig. D.7 at θ_o/θ_i=0.43, Bi^{-1}=1.0, read Fo = 0.45 (±0.03) and t = 3.5 (±0.2)s. Note the use of tolerances associated with reading the charts to ±5%.

PROBLEM 5.59

KNOWN: An 80mm sphere, initially at a uniform elevated temperature, is quenched in an oil bath with prescribed T_∞, h.

FIND: The center temperature of the sphere, T(0,t) at a certain time when the surface temperature is $T(r_o,t) = 150°C$.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Initial uniform temperature within sphere, (3) Constant properties, (4) Fo ≥ 0.2.

ANALYSIS: Check first to see if the sphere is spacewise isothermal.

$$Bi_c = \frac{hL_c}{k} = \frac{h(r_o/3)}{k} = \frac{1000\ W/m^2 \cdot K \times 0.040m/3}{50\ W/m \cdot K} = 0.26.$$

Since $Bi_c > 0.1$, lumped capacitance method is not appropriate. Recognize that when Fo ≥ 0.2, the time dependence of the temperature at any point within the sphere will be the same as the center. Using the Heisler chart method, Fig. D.8 provides the relation between $T(r_o,t)$ and T(0,t). Find first the Biot number,

$$Bi = \frac{hr_o}{k} = \frac{1000\ W/m^2 \cdot K \times 0.040m}{50\ W/m \cdot K} = 0.80.$$

With $Bi^{-1} = 1/0.80 = 1.25$ and $r/r_o = 1$, read from Fig. D.8,

$$\frac{q}{q_o} = \frac{T(r_o,t) - T_\infty}{T(0,t) - T_\infty} = 0.67.$$

It follows that

$$T(0,t) = T_\infty + \frac{1}{0.67}\left[T(r_o,t) - T_\infty\right] = 50^\circ C + \frac{1}{0.67}[150 - 50]^\circ C = 199^\circ C.$$ <

COMMENTS: (1) There is sufficient information to evaluate Fo; hence, we require that the time be sufficiently long after the start of quenching for this solution to be appropriate.

(2)The approximate series solution could also be used to obtain T(0,t). For Bi = 0.80 from Table 5.1, $z_1 = 1.5044$ rad. Substituting numerical values, r* = 1,

$$\frac{q^*}{q_o^*} = \frac{T(r_o,t) - T_\infty}{T(0,t) - T_\infty} = \frac{1}{z_1 r^*}\sin\left(z_1 r^*\right) = \frac{1}{1.5044}\sin(1.5044\ rad) = 0.663.$$

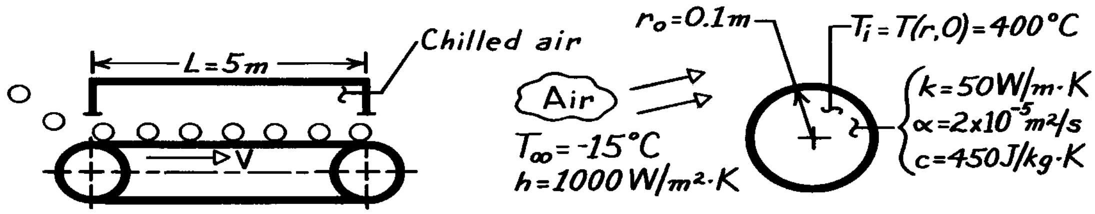
It follows that T(0,t) = 201°C.

PROBLEM 5.60

KNOWN: Steel ball bearings at an initial, uniform temperature are to be cooled by convection while passing through a refrigerated chamber; bearings are to be cooled to a temperature such that 70% of the thermal energy is removed.

FIND: Residence time of the balls in the 5m-long chamber and recommended drive velocity for the conveyor.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible conduction between ball and conveyor surface, (2) Negligible radiation exchange with surroundings, (3) Constant properties, (4) Uniform convection coefficient over ball's surface.

ANALYSIS: The Biot number for the lumped capacitance analysis is

$$\mathrm{Bi} \equiv \frac{hL_c}{k} = \frac{h\left(r_o/3\right)}{k} = \frac{1000\ \mathrm{W/m^2\cdot K}\ \left(0.1\mathrm{m}/3\right)}{50\ \mathrm{W/m\cdot K}} = 0.67.$$

Since Bi > 0.1, lumped capacitance analysis is not appropriate. In Figure D.9, the internal energy change is shown as a function of Bi and Fo. For

$$\frac{Q}{Q_o} = 0.70 \qquad \text{and} \qquad \mathrm{Bi} = \frac{hr_o}{k} = \frac{1000\ \mathrm{W/m^2\cdot K}\times 0.1\mathrm{m}}{50\ \mathrm{W/m\cdot K}} = 2.0,$$

find $\mathrm{Bi}^2\,\mathrm{Fo} \approx 1.2$. The Fourier number is

$$\mathrm{Fo} = \frac{\alpha\ t}{r_o^2} = \frac{2\times10^{-5}\mathrm{m^2/s}\ \times\ t}{\left(0.1\ \mathrm{m}\right)^2} = 2.0\times10^{-3}t$$

giving

$$t = \frac{\mathrm{Fo}}{2.0\times10^{-3}} = \frac{1.2/\mathrm{Bi}^2}{2.0\times10^{-3}} = \frac{1.2/\left(2.0\right)^2}{2.0\times10^{-3}} = 150\mathrm{s}.$$

The velocity of the conveyor is expressed in terms of the length L and residence time t. Hence

$$V = \frac{L}{t} = \frac{5\mathrm{m}}{150\mathrm{s}} = 0.033\mathrm{m/s} = 33\mathrm{mm/s}.$$ <

COMMENTS: Referring to Eq. 5.10, note that for a sphere, the characteristic length is

$$L_c = V/A_s = \frac{4}{3}\pi\ r_o^3/4\pi\ r_o^2 = \frac{r_o}{3}.$$

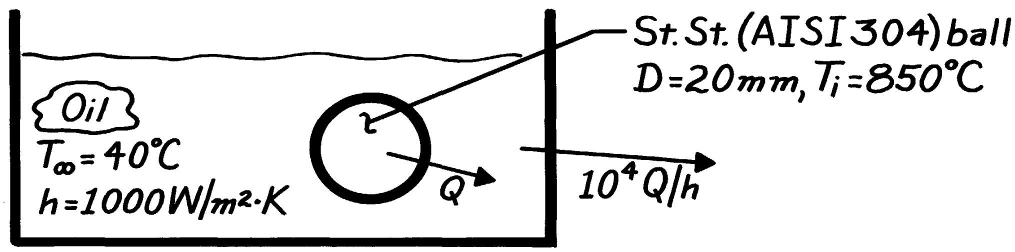
However, when using the Heisler charts, note that $\mathrm{Bi} \equiv h\ r_o/k$.

PROBLEM 5.61

KNOWN: Diameter and initial temperature of ball bearings to be quenched in an oil bath.

FIND: (a) Time required for surface to cool to 100°C and the corresponding center temperature, (b) Oil bath cooling requirements.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction in ball bearings, (2) Constant properties.

PROPERTIES: *Table A-1*, St. St., AISI 304, ($T \approx 500°C$): $k = 22.2$ W/m·K, $c_p = 579$ J/kg·K, $\rho = 7900$ kg/m^3, $\alpha = 4.85\times10^{-6}$ m^2/s.

ANALYSIS: (a) To determine whether use of the lumped capacitance method is suitable, first compute

$$\mathrm{Bi} = \frac{h(r_o/3)}{k} = \frac{1000\ \mathrm{W/m^2\cdot K}(0.010\mathrm{m}/3)}{22.2\ \mathrm{W/m\cdot K}} = 0.15.$$

We conclude that, although the lumped capacitance method could be used as a first approximation, the Heisler charts should be used in the interest of improving accuracy. Hence, with

$$\mathrm{Bi}^{-1} = \frac{k}{hr_o} = \frac{22.2\ \mathrm{W/m\cdot K}}{1000\ \mathrm{W/m^2\cdot K}(0.01\mathrm{m})} = 2.22 \qquad \text{and} \qquad \frac{r}{r_o} = 1,$$

Fig. D.8 gives

$$\frac{\theta(r_o,t)}{\theta_o(t)} \approx 0.80.$$

Hence, with

$$\frac{\theta(r_o,t)}{\theta_i} = \frac{T(r_o,t) - T_\infty}{T_i - T_\infty} = \frac{100-40}{850-40} = 0.074,$$

Continued …..

PROBLEM 5.61 (Cont.)

it follows that

$$\frac{q_o}{q_i} = \frac{q(r_o,t)/q_i}{q(r_o,t)/q_o} = \frac{0.074}{0.80} = 0.093.$$

From Fig. D.7, with $q_o/q_i = 0.093$ and $Bi^{-1} = k/hr_o = 2.22$, find

$$t^* = Fo \approx 2.0$$

$$t = \frac{r_o^2 Fo}{a} = \frac{(0.01m)^2(2.0)}{4.85\times10^{-6}m^2/s} = 41s.$$ <

Also,

$$q_o = T_o - T_\infty = 0.093(T_i - T_\infty) = 0.093(850 - 40) = 75^\circ C$$

$$T_o = 115^\circ C$$ <

(b) With $Bi^2 Fo = (1/2.2)^2 \times 2.0 = 0.41$, where $Bi \equiv (hr_o/k) = 0.45$, it follows from Fig. D.9 that for a single ball

$$\frac{Q}{Q_o} \approx 0.93.$$

Hence, from Eq. 5.44,

$$Q = 0.93\ r c_p V (T_i - T_\infty)$$

$$Q = 0.93 \times 7900\ kg/m^3 \times 579\ J/kg \cdot K \times \frac{p}{6}(0.02m)^3 \times 810^\circ C$$

$$Q = 1.44\times10^4 J$$

is the amount of energy transferred from a single ball during the cooling process. Hence, the oil bath cooling rate must be

$$q = 10^4 Q/3600s$$

$$q = 4\times10^4 W = 40\ kW.$$ <

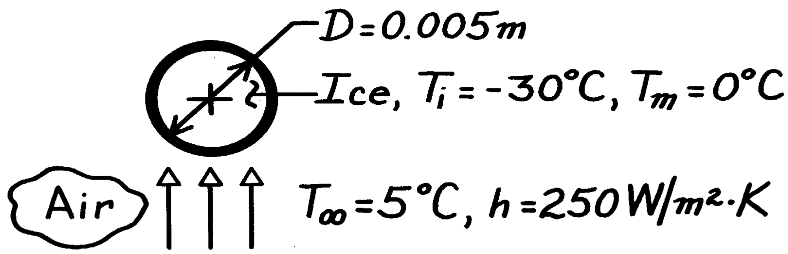
COMMENTS: If the lumped capacitance method is used, the cooling time, obtained from Eq. 5.5, would be t = 39.7s, where the ball is assumed to be uniformly cooled to 100°C. This result, and the fact that $T_o - T(r_o) = 15^\circ C$ at the conclusion, suggests that use of the lumped capacitance method would have been reasonable. Note that, when using the Heisler charts, accuracy to better than 5% is seldom possible.

PROBLEM 5.62

KNOWN: Diameter and initial temperature of hailstone falling through warm air.

FIND: (a) Time, t_m, required for outer surface to reach melting point, $T(r_o,t_m) = T_m = 0°C$, (b) Centerpoint temperature at that time, (c) Energy transferred to the stone.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties.

PROPERTIES: *Table A-3*, Ice (253K): $\rho = 920\ kg/m^3$, $k = 2.03\ W/m \cdot K$, $c_p = 1945\ J/kg \cdot K$; $\alpha = k/\rho c_p = 1.13 \times 10^{-6}\ m^2/s$.

ANALYSIS: (a) Calculate the lumped capacitance Biot number,

$$Bi = \frac{h(r_o/3)}{k} = \frac{250\ W/m^2 \cdot K(0.0025m/3)}{2.03\ W/m \cdot K} = 0.103.$$

Since Bi > 0.1, use the Heisler charts for which

$$\frac{q(r_o,t_m)}{q_i} = \frac{T(r_o,t_m) - T_\infty}{T_i - T_\infty} = \frac{0-5}{-30-5} = 0.143$$

$$Bi^{-1} = \frac{k}{hr_o} = \frac{2.03\ W/m \cdot K}{250\ W/m^2 \cdot K \times 0.0025m} = 3.25.$$

From Fig. D.8, find $\dfrac{q(r_o,t_m)}{q_o(t_m)} \approx 0.86.$

It follows that $\dfrac{q_o(t_m)}{q_i} = \dfrac{q(r_o,t_m)/q_i}{q(r_o,t_m)/q_o(t_m)} \approx \dfrac{0.143}{0.86} \approx 0.17.$

From Fig. D.7 find Fo ≈ 2.1. Hence,

$$t_m \approx \frac{Fo\ r_o^2}{a} = \frac{2.1(0.0025)^2}{1.13 \times 10^{-6} m^2/s} = 12s.$$ <

(b) Since $(\theta_o/\theta_i) \approx 0.17$, find

$$T_o - T_\infty \approx 0.17(T_i - T_\infty) \approx 0.17\ (-30-5) \approx -6.0°C$$

$$T_o(t_m) \approx -1.0°C.$$ <

(c) With $Bi^2 Fo = (1/3.25)^2 \times 2.1 = 0.2$, from Fig. D.9, find $Q/Q_o \approx 0.82$. From Eq. 5.44,

$$Q_o = r V c_p q_i = \left(920\ kg/m^3\right)\ (p/6)(0.005m)^3 1945(J/kg \cdot K)(-35K) = -4.10\ J$$

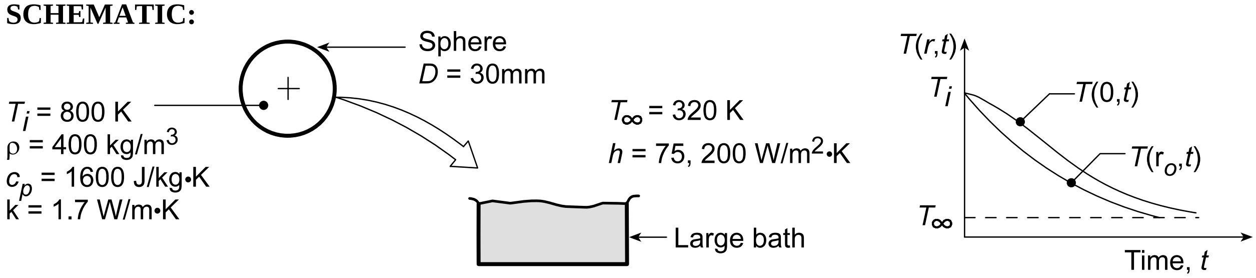
$$Q = 0.82\ Q_o = 0.82(-4.10\ J) = -3.4\ J.$$ <

PROBLEM 5.63

KNOWN: Sphere quenching in a constant temperature bath.

FIND: (a) Plot T(0,t) and T(r_o,t) as function of time, (b) Time required for surface to reach 415 K, t', (c) Heat flux when T(r_o, t') = 415 K, (d) Energy lost by sphere in cooling to T(r_o, t') = 415 K, (e) Steady-state temperature reached after sphere is insulated at t = t', (f) Effect of h on center and surface temperature histories.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Constant properties, (3) Uniform initial temperature.

ANALYSIS: (a) Calculate Biot number to determine if sphere behaves as spatially isothermal object,

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{h(r_o/3)}{k} = \frac{75\,\mathrm{W/m^2\cdot K}\,(0.015\,\mathrm{m}/3)}{1.7\,\mathrm{W/m\cdot K}} = 0.22\,.$$

Hence, temperature gradients exist in the sphere and T(r,t) vs. t appears as shown above.

(b) The Heisler charts may be used to find t' when T(r_o, t') = 415 K. Using Fig. D.8 with r/r_o = 1 and Bi^{-1} = k/hr_o = 1.7 W/m·K/(75 W/m^2·K × 0.015 m) = 1.51, $\theta(1,t')/\theta_o \approx 0.72$. In order to enter Fig. D.7, we need to determine θ_o/θ_i, which is

$$\frac{\theta_o}{\theta_i} = \frac{\theta(1,t')}{\theta_i}\bigg/\frac{\theta(1,t')}{\theta_o} \approx \frac{(415-320)\,\mathrm{K}}{(800-320)\,\mathrm{K}}\bigg/0.72 = 0.275$$

Hence, for Bi^{-1} = 1.51, Fo ≡ $\alpha t'/r_o^2$ ≈ 0.87 and

$$t' = \mathrm{Fo}\frac{r_o^2}{\alpha} = \mathrm{Fo}\cdot\frac{\rho c_p}{k}\cdot r_o^2 \approx 0.87\,\frac{400\,\mathrm{kg/m^3}\times 1600\,\mathrm{J/kg\cdot K}}{1.7\,\mathrm{W/m\cdot K}}\times(0.015\,\mathrm{m})^2 = 74\mathrm{s} \qquad <$$

(c) The heat flux at the outer surface at time t' is given by Newton's law of cooling

$$q'' = h\left[T(r_o,t') - T_\infty\right] = 75\,\mathrm{W/m^2\cdot K}\,[415-320]\,\mathrm{K} = 7125\,\mathrm{W/m^2}.. \qquad <$$

The manner in which q'' is calculated indicates that energy is leaving the sphere.

(d) The energy lost by the sphere during the cooling process from t = 0 to t' can be determined from the Groeber chart, Fig. D.9. With Bi = 1/1.51 = 0.67 and $\mathrm{Bi}^2\mathrm{Fo} = (1/1.51)^2 \times 0.87 \approx 0.4$, the chart yields $Q/Q_o \approx 0.75$. The energy loss by the sphere with V = (πD^3)/6 is therefore

$$Q \approx 0.85 Q_o = 0.85\rho\left(\pi D^3/6\right)c_p\left(T_i - T_\infty\right)$$

$$Q \approx 0.85\times 400\,\mathrm{kg/m^3}\left(\pi\,[0.030\,\mathrm{m}]^3/6\right)1600\,\mathrm{J/kg\cdot K}\,(800-320)\,\mathrm{K} = 3691\mathrm{J} \qquad <$$

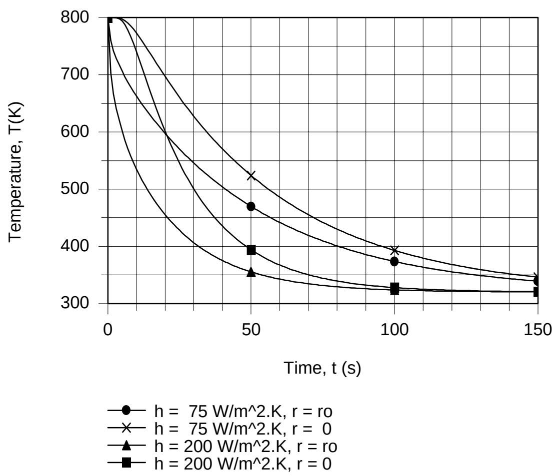
Continued...

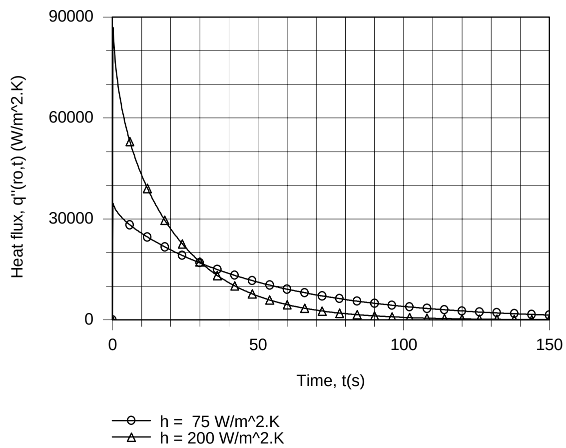
PROBLEM 5.63 (Cont.)

(e) If at time t' the surface of the sphere is perfectly insulated, eventually the temperature of the sphere will be uniform at $T(\infty)$. Applying conservation of energy to the sphere over a *time interval*, $E_{in} - E_{out} = \Delta E \equiv E_{final} - E_{initial}$. Hence, $-Q = \rho cV[T(\infty) - T_\infty] - Q_o$, where $Q_o \equiv \rho cV[T_i - T_\infty]$. Dividing by Q_o and regrouping, we obtain

$$T(\infty) = T_\infty + (1 - Q/Q_o)(T_i - T_\infty) \approx 320\,\text{K} + (1 - 0.75)(800 - 320)\,\text{K} = 440\,\text{K} \qquad <$$

(f) Using the IHT *Transient Conduction Model* for a *Sphere*, the following graphical results were generated.





The quenching process is clearly accelerated by increasing h from 75 to 200 $W/m^2 \cdot K$ and is virtually completed by $t \approx 100$s for the larger value of h. Note that, for both values of h, the temperature difference $[T(0,t) - T(r_o,t)]$ decreases with increasing t. Although the surface heat flux for h = 200 $W/m^2 \cdot K$ is initially larger than that for h = 75 $W/m^2 \cdot K$, the more rapid decline in $T(r_o,t)$ causes it to become smaller at $t \approx 30$s.

COMMENTS: 1. There is considerable uncertainty associated with reading Q/Q_o from the Groeber chart, Fig. D.9, and it would be better to use the one-term approximation solutions of Section 5.6.2. With Bi = 0.662, from Table 5.1, find $\zeta_1 = 1.319$ rad and $C_1 = 1.188$. Using Eq. 5.50, find Fo = 0.852 and $t' = 72.2$ s. Using Eq. 5.52, find $Q/Q_o = 0.775$ and $T(\infty) = 428$ K.

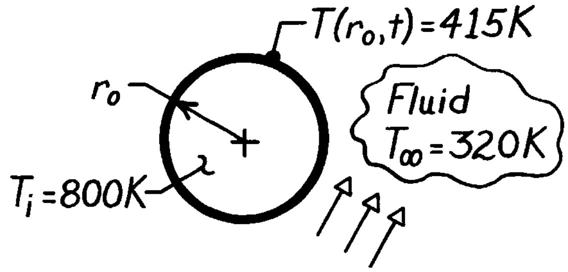
2. Using the *Transient Conduction/Sphere* model in *IHT* based upon multiple-term series solution, the following results were obtained: $t' = 72.1$ s; $Q/Q_o = 0.7745$, and $T(\infty) = 428$ K.

PROBLEM 5.64

KNOWN: Two spheres, A and B, initially at uniform temperatures of 800K and simultaneously quenched in large, constant temperature baths each maintained at 320K; properties of the spheres and convection coefficients.

FIND: (a) Show in a qualitative manner, on T-t coordinates, temperatures at the center and the outer surface for each sphere; explain features of the curves; (b) Time required for the outer surface of each sphere to reach 415K, (c) Energy gained by each bath during process of cooling spheres to a surface temperature of 415K.

SCHEMATIC:



	Sphere A	Sphere B
r_o (mm)	150	15
ρ (kg/m^3)	1600	400
c (J/kg·K)	400	1600
k (W/m·K)	170	1.7
h (W/m^2·K)	5	50

ASSUMPTIONS: (1) One-dimensional radial conduction, (2) Uniform properties, (3) Constant convection coefficient.

ANALYSIS: (a) From knowledge of the Biot number and the thermal time constant, it is possible to qualitatively represent the temperature distributions. From Eq. 5.10, with $L_c = r_o/3$, find

$$\mathrm{Bi}_A = \frac{5\ \mathrm{W/m^2\cdot K}\left(0.150\mathrm{m}/3\right)}{170\ \mathrm{W/m\cdot K}} = 1.47\times10^{-3} \tag{1}$$

$$\mathrm{Bi} = \frac{h\left(r_o/3\right)}{k}$$

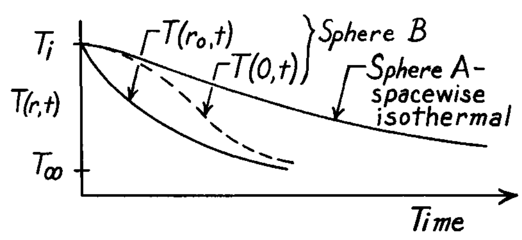
$$\mathrm{Bi}_B = \frac{50\ \mathrm{W/m^2\cdot K}\left(0.015\mathrm{m}/3\right)}{1.7\ \mathrm{W/m\cdot K}} = 0.147 \tag{2}$$

The thermal time constant for a lumped capacitance system from Eq. 5.7 is

$$\tau = \left[\frac{1}{hA_s}\right](\rho Vc) \qquad \tau_A = \frac{1600\ \mathrm{kg/m^3}\times(0.150\mathrm{m})400\ \mathrm{J/kg\cdot K}}{3\times5\ \mathrm{W/m^2\cdot K}} = 6400\mathrm{s} \tag{3}$$

$$\tau = \frac{\rho\, r_o\, c}{3h} \qquad \tau_B = \frac{400\ \mathrm{kg/m^3}\times(0.015\mathrm{m})1600\ \mathrm{J/kg\cdot K}}{3\times50\ \mathrm{W/m^2\cdot K}} = 64\mathrm{s} \tag{4}$$

When Bi << 0.1, the sphere will cool in a spacewise isothermal manner (Sphere A). For sphere B, Bi > 0.1, hence gradients will be important. Note that the thermal time constant of A is much larger than for B; hence, A will cool much slower. See sketch for these features.



(b) Recognizing that $\mathrm{Bi}_A < 0.1$, Sphere A can be treated as spacewise isothermal and analyzed using the lumped capacitance method. From Eq. 5.6 and 5.7, with T = 415 K

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-t/\tau) \tag{5}$$

Continued …..

PROBLEM 5.64 (Cont.)

$$t_A = -\tau_A\left[\ln\frac{T-T_\infty}{T_i-T_\infty}\right] = -6400s\left[\ln\frac{415-320}{800-320}\right] = 10,367s = 2.88h. \qquad <$$

Note that since the sphere is nearly isothermal, the surface and inner temperatures are approximately the same.

Since $Bi_B > 0.1$, *Sphere B* must be treated by the Heisler chart method of solution beginning with Figure D.8. Using

$$Bi_B \equiv \frac{hr_o}{k} = \frac{50\ W/m^2\cdot K\times(0.015m)}{1.7\ W/m\cdot K} = 0.44 \qquad or \qquad Bi_B^{-1} = 2.27,$$

find that for $r/r_o = 1$,

$$\frac{\theta(1,t)}{\theta_o} = \frac{T(r_o,t)-T_\infty}{\theta_o} = \frac{(415-320)}{\theta_o} = 0.8. \qquad (6)$$

Using Eq. (6) and Figure D.7, find the Fourier number,

$$\frac{\theta_o}{\theta_i} = \frac{\left(T(r_o,t)-T_\infty\right)/0.8}{T_i-T_\infty} = \frac{(415-320)K/0.8}{(800-320)K} = 0.25 \qquad Fo = \frac{\alpha\ t}{r_o^2} = 1.3.$$

$$t_B = \frac{Fo\ r_o^2}{\alpha} = \frac{1.3\ (0.015m)^2}{2.656\times10^{-6}m^2/s} = 110s = 1.8\ min \qquad <$$

where $\alpha = k/\rho c = 1.7\ W/m\cdot K/400\ kg/m^3 \times 1600\ J/kg\cdot K = 2.656\times10^{-6} m^2/s$.

(c) To determine the energy change by the spheres during the cooling process, apply the conservation of energy requirement on a time interval basis.

Sphere A:

$$E_{in} - E_{out} = \Delta E \qquad -Q_A = \Delta E = E(t) - E(0).$$

$$Q_A = \rho\ cV\left[T(t)-T_i\right] = 1600kg/m^3\times400J/kg\cdot K\times(4/3)\pi(0.150m)^3\left[415-800\right]K$$

$$Q_A = 3.483\times10^6 J. \qquad <$$

Note that this simple expression is a consequence of the spacewise isothermal behavior.

Sphere B: $\quad E_{in} - E_{out} = \Delta E \qquad -Q_B = E(t) - E(0).$

For the nonisothermal sphere, the Groeber chart, Figure D.9, can be used to evaluate Q_B. With Bi = 0.44 and $Bi^2Fo = (0.44)^2\times1.3 = 2.52$, find $Q/Q_o = 0.74$. The energy transfer from the sphere during the cooling process, using Eq. 5.44, is

$$Q_B = 0.74\ Q_o = 0.74\left[\rho\ cV\left(T_i - T_\infty\right)\right]$$

$$Q_B = 0.75\times400kg/m^3\times1600J/kg\cdot K(4/3)\pi(0.015m)^3(800-320)K = 3257\ J. \qquad <$$

COMMENTS: (1) In summary:

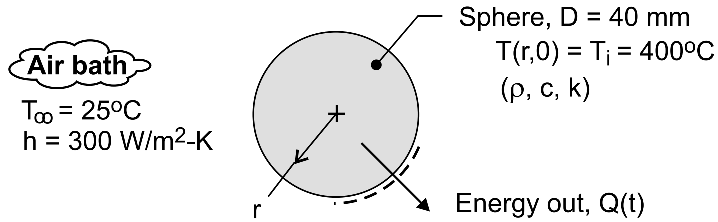
Sphere	$Bi = hr_o/k$	τ(s)	t(s)	Q(J)
A	4.41×10^{-3}	6400	10,370	3.48×10^6
B	0.44	64	110	3257

PROBLEM 5.65

KNOWN: Spheres of 40-mm diameter heated to a uniform temperature of 400°C are suddenly removed from an oven and placed in a forced-air bath operating at 25°C with a convection coefficient of 300 $W/m^2 \cdot K$.

FIND: (a) Time the spheres must remain in the bath for 80% of the thermal energy to be removed, and (b) Uniform temperature the spheres will reach when removed from the bath at this condition and placed in a carton that prevents further heat loss.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction in the spheres, (2) Constant properties, and (3) No heat loss from sphere after removed from the bath and placed into the packing carton.

PROPERTIES: Sphere (*given*): $\rho = 3000\ kg/m^3$, $c = 850\ J/kg \cdot K$, $k = 15\ W/m \cdot K$.

ANALYSIS: (a) From Eq. 5.52, the fraction of thermal energy removed during the time interval $\Delta t = t_o$ is

$$\frac{Q}{Q_o} = 1 - 3\theta_o^* / \zeta_1^3 \left[\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)\right] \quad (1)$$

where $Q/Q_o = 0.8$. The Biot number is

$$Bi = hr_o / k = 300\ W/m^2 \cdot K \times 0.020\,m / 15\ W/m \cdot K = 0.40$$

and for the one-term series approximation, from Table 5.1,

$$\zeta_1 = 1.0528\ rad \qquad C_1 = 1.1164 \quad (2)$$

The dimensionless temperature θ_o^*, Eq. 5.31, follows from Eq. 5.50.

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \quad (3)$$

where $Fo = \alpha t_o / r_o^2$. Substituting Eq. (3) into Eq. (1), solve for Fo and t_o.

$$\frac{Q}{Q_o} = 1 - 3\ C_1 \exp\left(-\zeta_1^2 Fo\right) / \zeta_1^3 \left[\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)\right] \quad (4)$$

$$Fo = 1.45 \qquad t_o = 98.6\ s \qquad <$$

(b) Performing an overall energy balance on the sphere during the interval of time $t_o \le t \le \infty$,

$$E_{in} - E_{out} = \Delta E = E_f - E_i = 0 \quad (5)$$

where E_i represents the thermal energy in the sphere at t_o,

$$E_i = (1 - 0.8) Q_o = (1 - 0.8) \rho c V (T_i - T_\infty) \quad (6)$$

and E_f represents the thermal energy in the sphere at $t = \infty$,

$$E_f = \rho c V \left(T_{avg} - T_\infty\right) \quad (7)$$

Combining the relations, find the average temperature

$$\rho c V \left[\left(T_{avg} - T_\infty\right) - (1 - 0.8)(T_i - T_\infty)\right] = 0$$

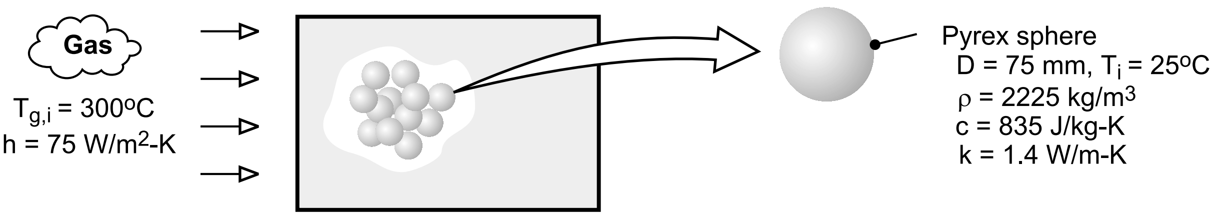
$$T_{avg} = 100°C \qquad <$$

PROBLEM 5.66

KNOWN: Diameter, density, specific heat and thermal conductivity of Pyrex spheres in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial conduction in sphere, (2) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with adjoining spheres, (3) Constant properties.

ANALYSIS: With $Bi \equiv h(r_o/3)/k = 75\ W/m^2 \cdot K\ (0.0125m)/1.4\ W/m \cdot K = 0.67$, the approximate solution for one-dimensional transient conduction in a sphere is used to obtain the desired results. We first use Eq. (5.52) to obtain θ_o^*.

$$\theta_o^* = \frac{\zeta_1^3}{3\left[\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)\right]}\left(1 - \frac{Q}{Q_o}\right)$$

With $Bi \equiv hr_o/k = 2.01$, $\zeta_1 \approx 2.03$ and $C_1 \approx 1.48$ from Table 5.1. Hence,

$$\theta_o^* = \frac{0.1(2.03)^3}{3\left[0.896 - 2.03(-0.443)\right]} = \frac{0.837}{5.386} = 0.155$$

The center temperature is therefore

$$T_o = T_{g,i} + 0.155\left(T_i - T_{g,i}\right) = 300°C - 42.7°C = 257.3°C$$ <

From Eq. (5.50c), the corresponding time is

$$t = -\frac{r_o^2}{\alpha \zeta_1^2} \ln\left(\frac{\theta_o^*}{C_1}\right)$$

where $\alpha = k/\rho c = 1.4\ W/m \cdot K / \left(2225\ kg/m^3 \times 835\ J/kg \cdot K\right) = 7.54 \times 10^{-7}\ m^2/s$.

$$t = -\frac{(0.0375m)^2 \ln(0.155/1.48)}{7.54 \times 10^{-7}\ m^2/s\ (2.03)^2} = 1{,}020s$$ <

COMMENTS: The surface temperature at the time of interest may be obtained from Eq. (5.50b). With $r^* = 1$,

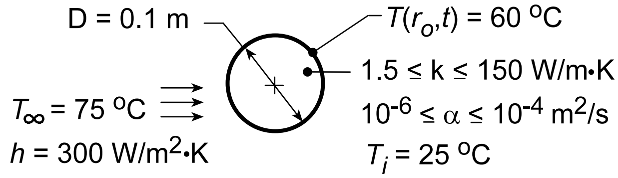
$$T_s = T_{g,i} + \left(T_i - T_{g,i}\right)\frac{\theta_o^* \sin(\zeta_1)}{\zeta_1} = 300°C - 275°C\left(\frac{0.155 \times 0.896}{2.03}\right) = 280.9°C$$ <

PROBLEM 5.67

KNOWN: Initial temperature and properties of a solid sphere. Surface temperature after immersion in a fluid of prescribed temperature and convection coefficient.

FIND: (a) Time to reach surface temperature, (b) Effect of thermal diffusivity and conductivity on thermal response.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction, (2) Constant properties.

ANALYSIS: (a) For k = 15 W/m·K, the Biot number is

$$Bi = \frac{h(r_o/3)}{k} = \frac{300\,W/m^2\cdot K\,(0.05\,m/3)}{15\,W/m\cdot K} = 0.333.$$

Hence, the lumped capacitance method cannot be used. From Equation 5.50a,

$$\frac{T - T_\infty}{T_i - T_\infty} = C_1 \exp\left(-\zeta_1^2 Fo\right)\frac{\sin\left(\zeta_1 r^*\right)}{\zeta_1 r^*}.$$

At the surface, $r^* = 1$. From Table 5.1, for Bi = 1.0, $\zeta_1 = 1.5708$ rad and $C_1 = 1.2732$. Hence,

$$\frac{60-75}{25-75} = 0.30 = 1.2732\exp\left(-1.5708^2\,Fo\right)\frac{\sin 90^\circ}{1.5708}$$

$$\exp(-2.467Fo) = 0.370$$

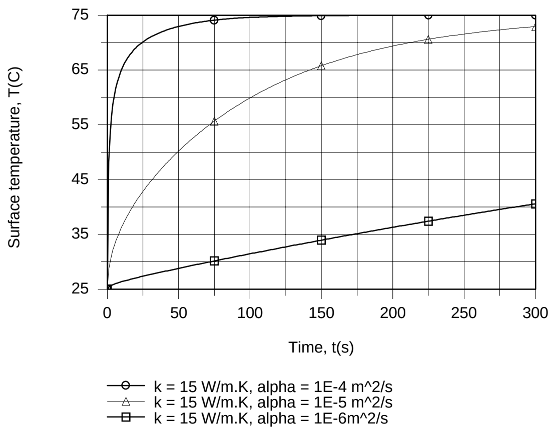
$$Fo = \frac{\alpha t}{r_o^2} = 0.403$$

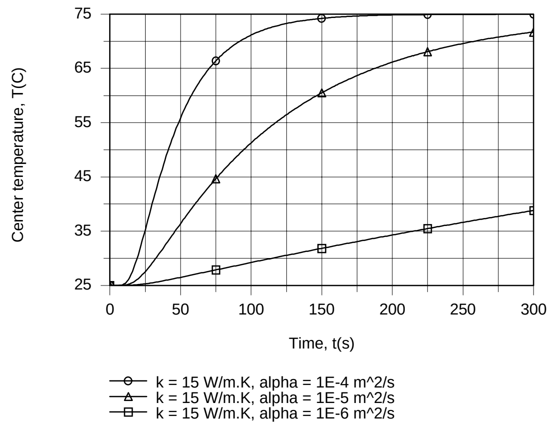
$$t = 0.403\frac{r_o^2}{\alpha} = 0.403\frac{(0.05\,m)^2}{10^{-5}\,m^2/s} = 100s \qquad <$$

(b) Using the IHT *Transient Conduction Model* for a *Sphere* to perform the parametric calculations, the effect of α is plotted for k = 15 W/m·K.

Continued...

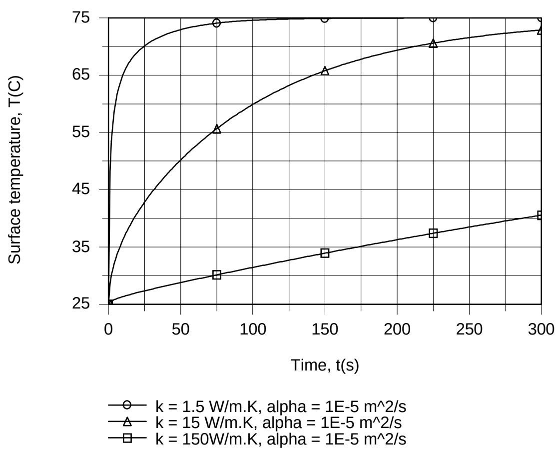
PROBLEM 5.67 (Cont.)

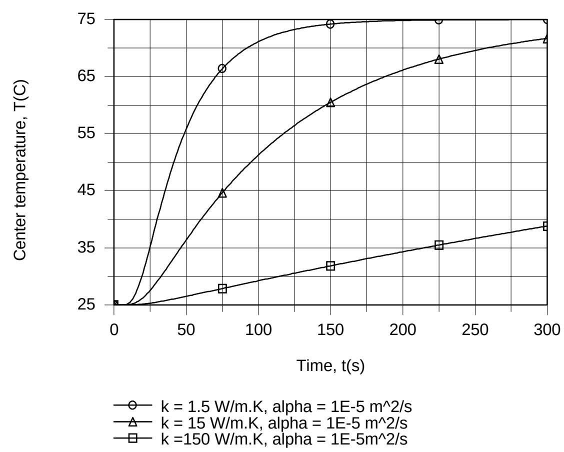




For fixed k and increasing α, there is a reduction in the thermal capacity (ρc_p) of the material, and hence the amount of thermal energy which must be added to increase the temperature. With increasing α, the material therefore responds more quickly to a change in the thermal environment, with the response at the center lagging that of the surface.

The effect of k is plotted for $\alpha = 10^{-5}$ m^2/s.





With increasing k for fixed alpha, there is a corresponding increase in ρc_p, and the material therefore responds more slowly to a thermal change in its surroundings. The thermal response of the center lags that of the surface, with temperature differences, $T(r_o,t) - T(0,t)$, during early stages of solidification increasing with decreasing k.

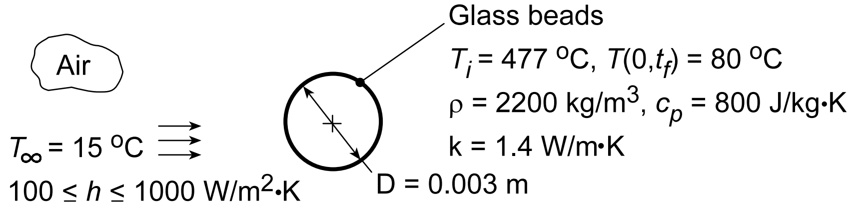
COMMENTS: Use of this technique to determine h from measurement of $T(r_o)$ at a prescribed t requires an interative solution of the governing equations.

PROBLEM 5.68

KNOWN: Properties, initial temperature, and convection conditions associated with cooling of glass beads.

FIND: (a) Time required to achieve a prescribed center temperature, (b) Effect of convection coefficient on center and surface temperature histories.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in r, (2) Constant properties, (3) Negligible radiation, (4) Fo ≥ 0.2.

ANALYSIS: (a) With h = 400 W/m^2·K, Bi ≡ h(r$_o$/3)/k = 400 W/m^2·K(0.0005 m)/1.4 W/m·K = 0.143 and the lumped capacitance method should not be used. From the one-term approximation for the center temperature, Eq. 5.50c,

$$\theta_o^* \equiv \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{80-15}{477-15} = 0.141 = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

For Bi ≡ hr$_o$/k = 0.429, Table 5.1 yields ζ_1 = 1.101 rad and C_1 = 1.128. Hence,

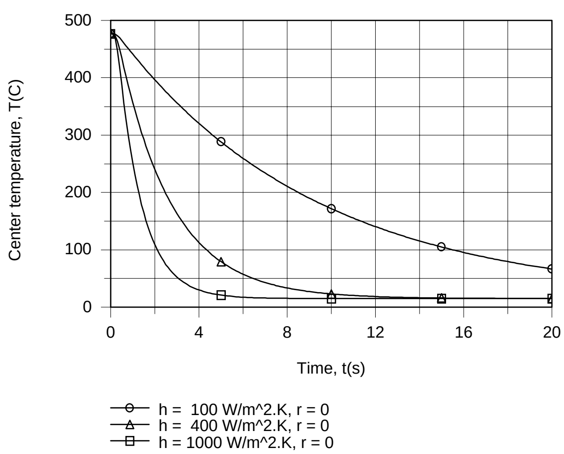
$$Fo = -\frac{1}{\zeta_1^2}\ln\left(\frac{\theta_o^*}{C_1}\right) = -\frac{1}{(1.101)^2}\ln\left(\frac{0.141}{1.128}\right) = 1.715$$

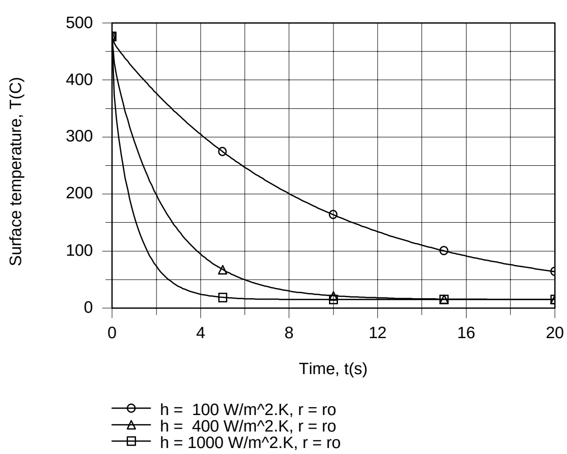
$$t = 1.715 r_o^2 \frac{\rho c_p}{k} = 1.715(0.0015\,m)^2 \frac{2200\,kg/m^3 \times 800\,J/kg\cdot K}{1.4\,W/m\cdot K} = 4.85s \qquad <$$

From Eq. 5.50b, the corresponding surface (r^* = 1) temperature is

$$T(r_o, t) = T_\infty + (T_i - T_\infty)\theta_o^* \frac{\sin\zeta_1}{\zeta_1} = 15^\circ C + \left(462^\circ C\right)0.141\frac{0.892}{1.101} = 67.8^\circ C \qquad <$$

(b) The effect of h on the surface and center temperatures was determined using the IHT *Transient Conduction Model* for a *Sphere*.





Continued...

PROBLEM 5.68 (Cont.)

The cooling rate increases with increasing h, particularly from 100 to 400 $W/m^2 \cdot K$. The temperature difference between the center and surface decreases with increasing t and, during the early stages of solidification, with decreasing h.

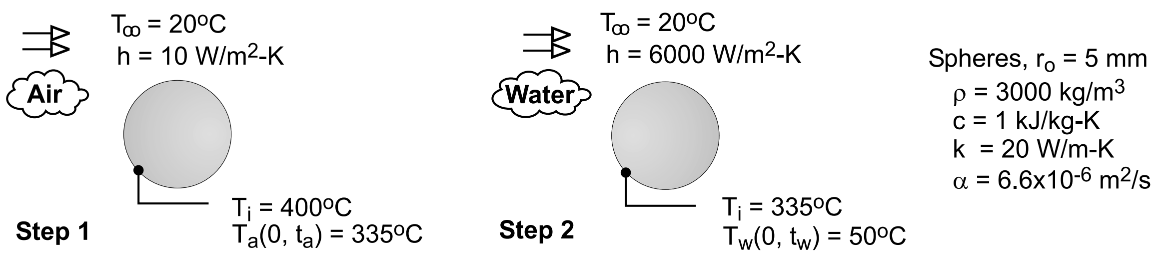
COMMENTS: Temperature gradients in the glass are largest during the early stages of solidification and increase with increasing h. Since thermal stresses increase with increasing temperature gradients, the propensity to induce defects due to crack formation in the glass increases with increasing h. Hence, there is a value of h above which product quality would suffer and the process should not be operated.

PROBLEM 5.69

KNOWN: Temperature requirements for cooling the spherical material of Ex. 5.4 in air and in a water bath.

FIND: (a) For step 1, the time required for the center temperature to reach T(0,t) = 335°C while cooling in air at 20°C with h = 10 W/m^2·K; find the Biot number; do you expect radial gradients to be appreciable?; compare results with hand calculations in Ex. 5.4; (b) For step 2, time required for the center temperature to reach T(0,t) = 50°C while cooling in water bath at 20°C with h = 6000 W/m^2·K; and (c) For step 2, calculate and plot the temperature history, T(x,t) vs. t, for the center and surface of the sphere; explain features; when do you expect the temperature gradients in the sphere to the largest? Use the IHT *Models* | *Transient Conduction* | *Sphere* model as your solution tool.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the radial direction, (2) Constant properties.

ANALYSIS: The IHT model represents the series solution for the sphere providing the temperatures evaluated at (r,t). A selected portion of the IHT code used to obtain results is shown in the Comments.

(a) Using the IHT model with step 1 conditions, the time required for $T(0,t_a)$ = T_xt = 335°C with r = 0 and the Biot number are:

$$t_a = 94.2 \text{ s} \qquad\qquad Bi = 0.0025$$ <

Radial temperature gradients will not be appreciable since Bi = 0.0025 << 0.1. The sphere behaves as space-wise isothermal object for the air-cooling process. The result is identical to the lumped-capacitance analysis result of the Text example.

(b) Using the IHT model with step 2 conditions, the time required for $T(0,t_w)$ = T_xt = 50°C with r = 0 and T_i = 335°C is

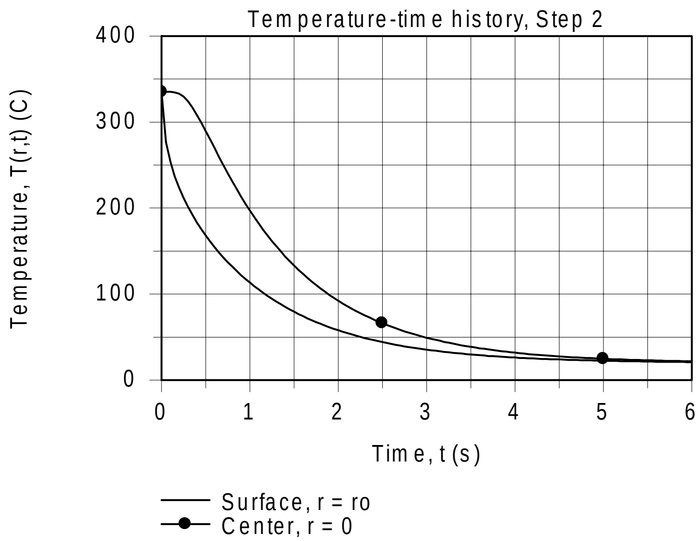
$$t_w = 3.0 \text{ s}$$ <

Radial temperature gradients will be appreciable, since Bi = 1.5 >> 0.1. The sphere does not behave as a space-wise isothermal object for the water-cooling process.

(c) For the step 2 cooling process, the temperature histories for the center and surface of the sphere are calculated using the IHT model.

Continued …..

PROBLEM 5.69 (Cont.)



At early times, the difference between the center and surface temperature is appreciable. It is in this time region that thermal stresses will be a maximum, and if large enough, can cause fracture. Within 6 seconds, the sphere has a uniform temperature equal to that of the water bath.

COMMENTS: Selected portions of the IHT sphere model codes for steps 1 and 2 are shown below.

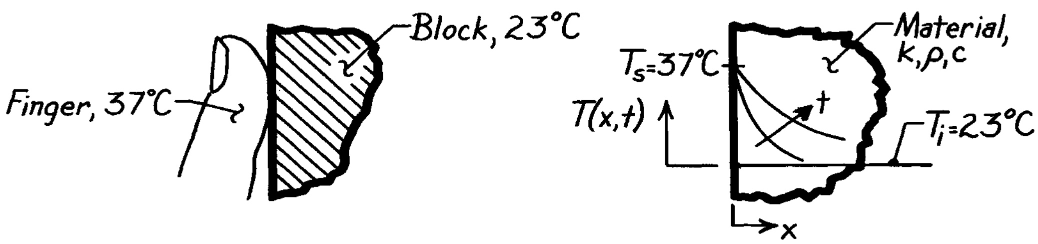
```
/* Results, for part (a), step 1,  air cooling; clearly negligible gradient
Bi              Fo       t        T_xt     Ti      r        ro
0.0025          25.13    94.22    335      400     0        0.005  */

// Models | Transient Conduction | Sphere - Step 1, Air cooling
// The temperature distribution T(r,t) is
T_xt = T_xt_trans("Sphere",rstar,Fo,Bi,Ti,Tinf)    // Eq 5.47
T_xt = 335              // Surface temperature


/* Results, for part (b), step 2, water cooling; Ti = 335 C
Bi      Fo       t        T_xt     Ti      r        ro
1.5     0.7936   2.976    50       335     0        0.005  */

// Models | Transient Conduction | Sphere - Step 2, Water cooling
// The temperature distribution T(r,t) is
T_xt = T_xt_trans("Sphere",rstar,Fo,Bi,Ti,Tinf)    // Eq 5.47
//T_xt = 335              // Surface temperature from Step 1; initial temperature for Step 2
T_xt = 50                 // Center temperature, end of Step 2
```

PROBLEM 5.70

KNOWN: Two large blocks of different materials – like copper and concrete – at room temperature, 23°C.

FIND: Which block will feel cooler to the touch?

SCHEMATIC:



ASSUMPTIONS: (1) Blocks can be treated as semi-infinite solid, (2) Hand or finger temperature is 37°C.

PROPERTIES: *Table A-1*, Copper (300K): ρ = 8933 kg/m^3, c = 385 J/kg·K, k = 401 W/m·K; *Table A-3*, Concrete, stone mix (300K): ρ = 2300 kg/m^3, c = 880 J/kg·K, k = 1.4 W/m·K.

ANALYSIS: Considering the block as a semi-infinite solid, the heat transfer situation corresponds to a sudden change in surface temperature, Case 1, Figure 5.7. The sensation of coolness is related to the heat flow from the hand or finger to the block. From Eq. 5.58, the surface heat flux is

$$q_s''(t) = k(T_s - T_i)/(\pi\alpha\, t)^{1/2} \tag{1}$$

or

$$q_s''(t) \sim (k\rho\, c)^{1/2} \qquad \text{since} \quad \alpha = k/\rho\, c. \tag{2}$$

Hence for the same temperature difference, $T_s - T_i$, and elapsed time, it follows that the heat fluxes for the two materials are related as

$$\frac{q''_{s,copper}}{q''_{s,concrete}} = \frac{(k\rho\, c)^{1/2}_{copper}}{(k\rho\, c)^{1/2}_{concrete}} = \frac{\left[401\dfrac{W}{m\cdot K}\times 8933\dfrac{kg}{m^3}\times 385\dfrac{J}{kg\cdot K}\right]^{1/2}}{\left[1.4\dfrac{W}{m\cdot K}\times 2300\dfrac{kg}{m^3}\times 880\dfrac{J}{kg\cdot K}\right]^{1/2}} = 22.1$$

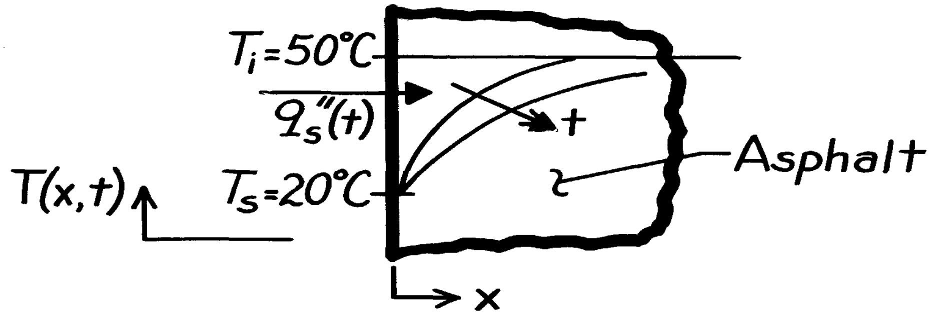
Hence, the heat flux to the copper block is more than 20 times larger than to the concrete block. The *copper* block will therefore feel noticeably cooler than the concrete one.

PROBLEM 5.71

KNOWN: Asphalt pavement, initially at 50°C, is suddenly exposed to a rainstorm reducing the surface temperature to 20°C.

FIND: Total amount of energy removed (J/m^2) from the pavement for a 30 minute period.

SCHEMATIC:



ASSUMPTIONS: (1) Asphalt pavement can be treated as a semi-infinite solid, (2) Effect of rainstorm is to suddenly reduce the surface temperature to 20°C and is maintained at that level for the period of interest.

PROPERTIES: *Table A-3*, Asphalt (300K): ρ = 2115 kg/m^3, c = 920 J/kg·K, k = 0.062 W/m·K.

ANALYSIS: This solution corresponds to Case 1, Figure 5.7, and the surface heat flux is given by Eq. 5.58 as

$$q_s''(t) = k(T_s - T_i)/(\pi\alpha\ t)^{1/2} \tag{1}$$

The energy into the pavement over a period of time is the integral of the surface heat flux expressed as

$$Q'' = \int_0^t q_s''(t)\,dt. \tag{2}$$

Note that $q_s''(t)$ is into the solid and, hence, Q represents energy into the solid. Substituting Eq. (1) for $q_s''(t)$ into Eq. (2) and integrating find

$$Q'' = k(T_s - T_i)/(\pi\alpha)^{1/2} \int_0^t t^{-1/2} dt = \frac{k(T_s - T_i)}{(\pi\alpha)^{1/2}} \times 2\, t^{1/2}. \tag{3}$$

Substituting numerical values into Eq. (3) with

$$\alpha = \frac{k}{\rho c} = \frac{0.062 \text{ W/m}\cdot\text{K}}{2115 \text{ kg/m}^3 \times 920 \text{ J/kg}\cdot\text{K}} = 3.18\times10^{-8} \text{m}^2/\text{s}$$

find that for the 30 minute period,

$$Q'' = \frac{0.062 \text{ W/m}\cdot\text{K}\ (20-50)\text{K}}{\left(\pi\times3.18\times10^{-8}\text{m}^2/\text{s}\right)^{1/2}} \times 2(30\times60\text{s})^{1/2} = -4.99\times10^5 \text{ J/m}^2.$$ <

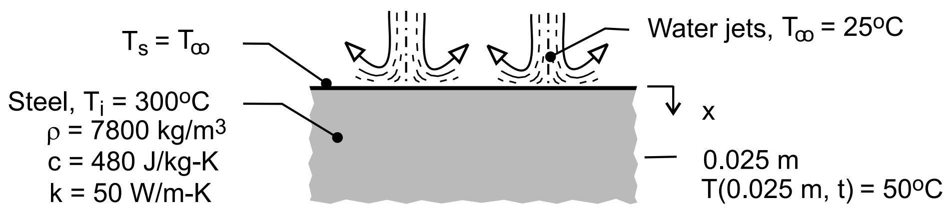
COMMENTS: Note that the sign for Q'' is negative implying that energy is removed from the solid.

PROBLEM 5.72

KNOWN: Thermophysical properties and initial temperature of thick steel plate. Temperature of water jets used for convection cooling at one surface.

FIND: Time required to cool prescribed interior location to a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in slab, (2) Validity of semi-infinite medium approximation, (3) Negligible thermal resistance between water jets and slab surface ($T_s = T_\infty$), (4) Constant properties.

ANALYSIS: The desired cooling time may be obtained from Eq. (5.57). With T(0.025m, t) = 50°C,

$$\frac{T(x,t)-T_s}{T_i-T_s} = \frac{(50-25)°C}{(300-25)°C} = 0.0909 = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\frac{x}{2\sqrt{\alpha t}} = 0.0807$$

$$t = \frac{x^2}{(0.0807)^2\,4\alpha} = \frac{(0.025\text{m})^2}{0.0261\left(1.34\times10^{-5}\text{m}^2/\text{s}\right)} = 1793\text{s}$$ <

where $\alpha = k/\rho c$ = 50 W/m·K/(7800 kg/m^3 × 480 J/kg·K) = 1.34 × 10^{-5} m^2/s.

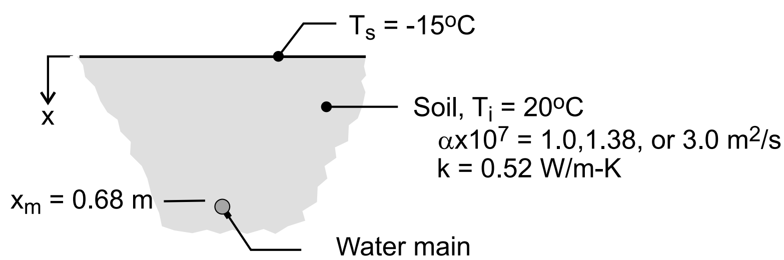
COMMENTS: (1) Large values of the convection coefficient ($h \sim 10^4$ W/m^2·K) are associated with water jet impingement, and it is reasonable to assume that the surface is immediately quenched to the temperature of the water. (2) The surface heat flux may be determined from Eq. (5.58). In principle, the flux is infinite at t = 0 and decays as $t^{1/2}$.

PROBLEM 5.73

KNOWN: Temperature imposed at the surface of soil initially at 20°C. See Example 5.5.

FIND: (a) Calculate and plot the temperature history at the burial depth of 0.68 m for selected soil thermal diffusivity values, $\alpha \times 10^7$ = 1.0, 1.38, and 3.0 m^2/s, (b) Plot the temperature distribution over the depth $0 \le x \le 1.0$ m for times of 1, 5, 10, 30, and 60 days with $\alpha = 1.38 \times 10^{-7}$ m^2/s, (c) Plot the surface heat flux, $q''_x(0,t)$, and the heat flux at the depth of the buried main, $q''_x(0.68\text{m}, t)$, as a function of time for a 60 day period with $\alpha = 1.38 \times 10^{-7}$ m^2/s. Compare your results with those in the Comments section of the example. Use the IHT *Models* | *Transient Conduction* | *Semi-infinite Medium* model as the solution tool.

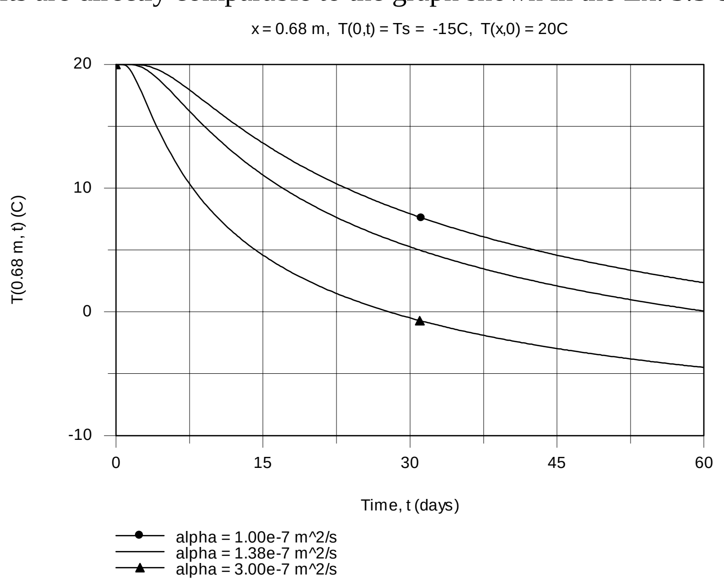
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the x-direction, (2) Soil is a semi-infinite medium, and (3) Constant properties.

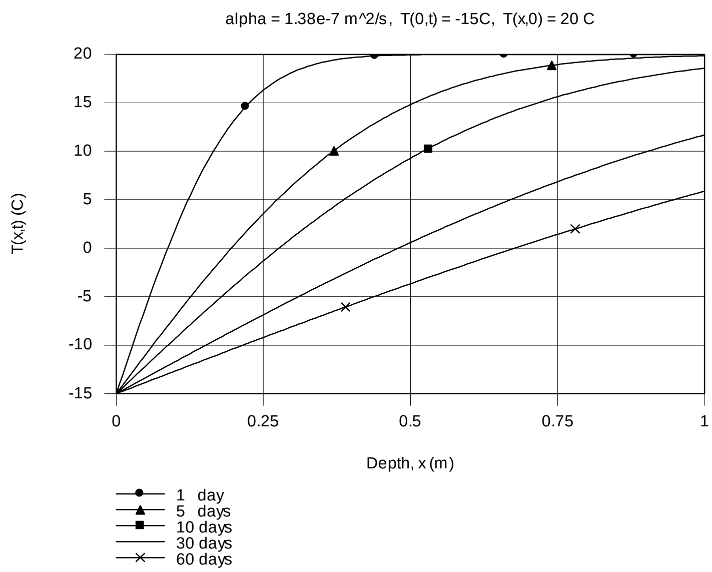
ANALYSIS: The IHT model corresponds to the case 1, constant surface temperature sudden boundary condition, Eqs. 5.57 and 5.58. Selected portions of the IHT code used to obtain the graphical results below are shown in the Comments.

(a) The temperature history T(x,t) for x = 0.68 m with selected soil thermal diffusivities is shown below. The results are directly comparable to the graph shown in the Ex. 5.5 comments.

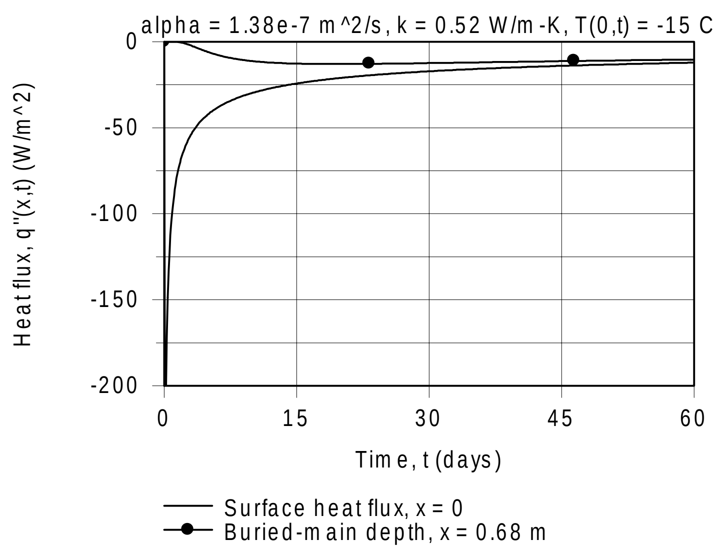


Continued …..

(b) The temperature distribution T(x,t) for selected times is shown below. The results are directly comparable to the graph shown in the Ex. 5.5 comments.



(c) The heat flux from the soil, $q''_x(0,t)$, and the heat flux at the depth of the buried main, $q''_x(0.68\text{m},t)$, are calculated and plotted for the time period $0 \le t \le 60$ days.



Both the surface and buried-main heat fluxes have a negative sign since heat is flowing in the negative x-direction. The surface heat flux is initially very large and, in the limit, approaches that of the buried-main heat flux. The latter is initially zero, and since the effect of the sudden change in surface temperature is delayed for a time period, the heat flux begins to slowly increase.

Continued …..

PROBLEM 5.73 (Cont.)

COMMENTS: (1) Can you explain why the surface and buried-main heat fluxes are nearly the same at t = 60 days? Are these results consistent with the temperature distributions? What happens to the heat flux values for times much greater than 60 days? Use your IHT model to confirm your explanation.

(2) Selected portions of the IHT code for the semi-infinite medium model are shown below.

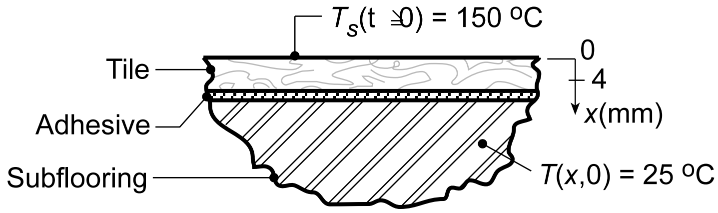
```
//  Models | Transient Conduction | Semi-infinite Solid | Constant temperature Ts
/* Model: Semi-infinite solid, initially with a uniform temperature T(x,0) = Ti, suddenly subjected to
prescribed surface boundary conditions.  */
// The temperature distribution (Tx,t) is
T_xt = T_xt_semi_CST(x,alpha,t,Ts,Ti)   // Eq 5.55
// The heat flux in the x direction is
q''_xt = qdprime_xt_semi_CST(x,alpha,t,Ts,Ti,k)    //Eq 5.56

// Input parameters
/* The independent variables for this system and their assigned numerical values are */
Ti =  20                    // initial temperature, C
k = 0.52                    // thermal conductivity, W/m.K; base case condition
alpha = 1.38e-7        // thermal diffusivity, m^2/s; base case
//alpha = 1.0e-7
//alpha = 3.0e-7

// Calculating at x-location and time t,
x = 0                                          // m, surface
// x = 0.68                                    // m, burial depth
t = t_day * 24 * 3600                          // seconds to days time covnersion
//t_day = 60
//t_day = 1
//t_day = 5
//t_day = 10
//t_day = 30
t_day = 20

// Surface condition: constant surface temperature
Ts = -15              // surface temperature, K
```

PROBLEM 5.74

KNOWN: Tile-iron, 254 mm to a side, at 150°C is suddenly brought into contact with tile over a subflooring material initially at T_i = 25°C with prescribed thermophysical properties. Tile adhesive softens in 2 minutes at 50°C, but deteriorates above 120°C.

FIND: (a) Time required to lift a tile after being heated by the tile-iron and whether adhesive temperature exceeds 120°C, (2) How much energy has been removed from the tile-iron during the time it has taken to lift the tile.

SCHEMATIC:



ASSUMPTIONS: (1) Tile and subflooring have same thermophysical properties, (2) Thickness of adhesive is negligible compared to that of tile, (3) Tile-subflooring behaves as semi-infinite solid experiencing one-dimensional transient conduction.

PROPERTIES: Tile-subflooring (given): k = 0.15 W/m·K, $\rho c_p = 1.5 \times 10^6$ J/m^3·K, $\alpha = k/\rho c_p = 1.00 \times 10^{-7}$ m^2/s.

ANALYSIS: (a) The tile-subflooring can be approximated as a semi-infinite solid, initially at a uniform temperature T_i = 25°C, experiencing a sudden change in surface temperature T_s = T(0,t) = 150°C. This corresponds to Case 1, Figure 5.7. The time required to heat the adhesive (x_o = 4 mm) to 50°C follows from Eq. 5.57

$$\frac{T(x_o,t_o)-T_s}{T_i-T_s}=\mathrm{erf}\left(\frac{x_o}{2(\alpha t_o)^{1/2}}\right)$$

$$\frac{50-150}{25-150}=\mathrm{erf}\left(\frac{0.004\,\mathrm{m}}{2\left(1.00\times10^{-7}\,\mathrm{m}^2/\mathrm{s}\times t_o\right)^{1/2}}\right)$$

$$0.80=\mathrm{erf}\left(6.325 t_o^{-1/2}\right)$$

t_o = 48.7s = 0.81 min

using error function values from Table B.2. Since the softening time, Δt_s, for the adhesive is 2 minutes, the time to lift the tile is

$$t_\ell = t_o + \Delta t_s = (0.81+2.0)\,\mathrm{min} = 2.81\,\mathrm{min}.$$ <

To determine whether the adhesive temperature has exceeded 120°C, calculate its temperature at t_ℓ = 2.81 min; that is, find $T(x_o, t_\ell)$

$$\frac{T(x_o,t_\ell)-150}{25-150}=\mathrm{erf}\left(\frac{0.004\,\mathrm{m}}{2\left(1.0\times10^{-7}\,\mathrm{m}^2/\mathrm{s}\times2.81\times60\mathrm{s}\right)^{1/2}}\right)$$

Continued...

PROBLEM 5.74 (Cont.)

$$T\left(x_o, t_\ell\right) - 150 = -125\,\mathrm{erf}\left(0.4880\right) = 125 \times 0.5098$$

$$T\left(x_o, t_\ell\right) = 86^\circ C \qquad <$$

Since $T(x_o,\ t_\ell) < 120°C$, the adhesive will not deteriorate.

(b) The energy required to heat a tile to the lift-off condition is

$$Q = \int_0^{t_\ell} q''_x\left(0, t\right) \cdot A_s dt\ .$$

Using Eq. 5.58 for the surface heat flux $q''_s(t) = q''_x(0,t)$, find

$$Q = \int_0^{t_\ell} \frac{k\left(T_s - T_i\right)}{\left(\pi\alpha\right)^{1/2}} A_s \frac{dt}{t^{1/2}} = \frac{2k\left(T_s - T_i\right)}{\left(\pi\alpha\right)^{1/2}} A_s t_\ell^{1/2}$$

$$Q = \frac{2 \times 0.15\,W/m \cdot K\left(150 - 25\right)^\circ C}{\left(\pi \times 1.00 \times 10^{-7}\,m^2/s\right)^{1/2}} \times \left(0.254\,m\right)^2 \times \left(2.81 \times 60s\right)^{1/2} = 56\,kJ \qquad <$$

COMMENTS: (1) Increasing the tile-iron temperature would decrease the time required to soften the adhesive, but the risk of burning the adhesive increases.

(2) From the energy calculation of part (b) we can estimate the size of an electrical heater, if operating continuously during the 2.81 min period, to maintain the tile-iron at a near constant temperature. The power required is

$$P = Q/t_\ell = 56\,kJ/2.81 \times 60s = 330\,W\ .$$

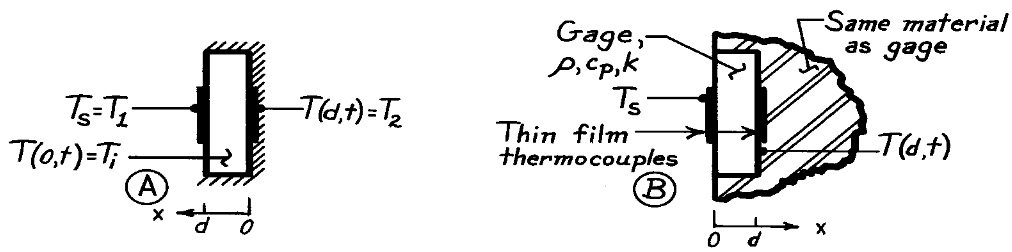
Of course a much larger electrical heater would be required to initially heat the tile-iron up to the operating temperature in a reasonable period of time.

PROBLEM 5.75

KNOWN: Heat flux gage of prescribed thickness and thermophysical properties (ρ, c_p, k) initially at a uniform temperature, T_i, is exposed to a sudden change in surface temperature $T(0,t) = T_s$.

FIND: Relationships for time constant of gage when (a) backside of gage is insulated and (b) gage is imbedded in semi-infinite solid having the same thermophysical properties. Compare with equation given by manufacturer, $\tau = \left(4d^2 \rho \; c_p\right)/\pi^2 k$.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties.

ANALYSIS: The time constant τ is defined as the time required for the gage to indicate, following a sudden step change, a signal which is 63.2% that of the steady-state value. The manufacturer's relationship for the time constant

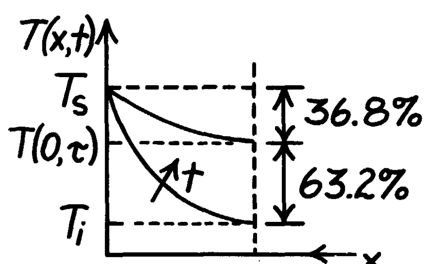
$$\tau = \left(4d^2 \rho \; c_p\right)/\pi^2 k$$

can be written in terms of the Fourier number as

$$Fo = \frac{\alpha\tau}{d^2} = \frac{k}{\rho \; c_p} \cdot \frac{\tau}{d^2} = \frac{4}{\pi^2} = 0.4053.$$

The Fourier number can be determined for the two different installations.

(a) For the gage having its backside insulated, the surface and backside temperatures are T_s and $T(0,t)$, respectively. From the sketch it follows that



$$\theta_o^* = \frac{T(0,\tau) - T_s}{T_i - T_s} = 0.368.$$

From Eq. 5.41,

$$\theta_o^* = 0.368 = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

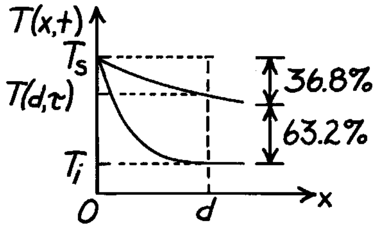
Using Table 5.1 with Bi = 100 (as the best approximation for Bi = hd/k $\to \infty$, corresponding to sudden surface temperature change with h $\to \infty$), ζ_1 = 1.5552 rad and C_1 = 1.2731. Hence,

$$0.368 = 1.2731 \exp(-1.5552^2 \times Fo_a)$$

$$Fo_a = 0.513.$$ <

Continued …..

(b) For the gage imbedded in a semi-infinite medium having the same thermophysical properties, Table 5.7 (case 1) and Eq. 5.57 yield



$$\frac{T(x,\tau)-T_s}{T_i-T_s}=0.368=\operatorname{erf}\left[d/2(\alpha\tau)^{1/2}\right]$$

$$d/2(\alpha\tau)^{1/2}=0.3972$$

$$Fo_b=\frac{\alpha\tau}{d^2}=\frac{1}{(2\times0.3972)^2}=1.585$$

$<$

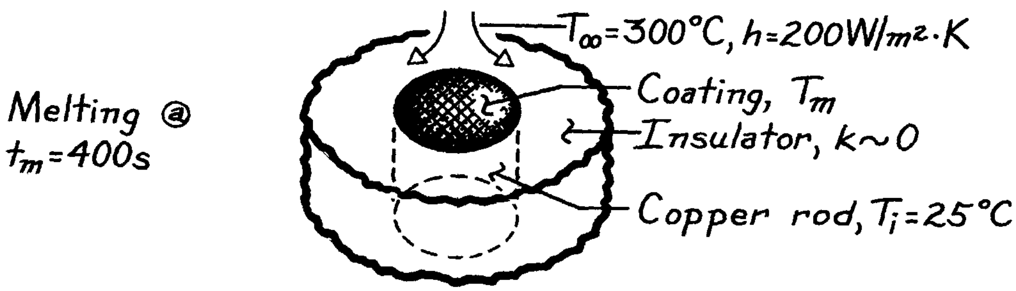
COMMENTS: Both models predict higher values of Fo than that suggested by the manufacturer. It is understandable why $Fo_b > Fo_a$ since for (b) the gage is thermally connected to an infinite medium, while for (a) it is isolated. From this analysis we conclude that the gage's transient response will depend upon the manner in which it is applied to the surface or object.

PROBLEM 5.76

KNOWN: Procedure for measuring convection heat transfer coefficient, which involves melting of a surface coating.

FIND: Melting point of coating for prescribed conditions.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in solid rod (negligible losses to insulation), (2) Rod approximated as semi-infinite medium, (3) Negligible surface radiation, (4) Constant properties, (5) Negligible thermal resistance of coating.

PROPERTIES: Copper rod (Given): k = 400 W/m·K, $\alpha = 10^{-4}$ m^2/s.

ANALYSIS: Problem corresponds to transient conduction in a semi-infinite solid. Themal response is given by

$$\frac{T(x,t)-T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2(\alpha t)^{1/2}}\right) - \left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2(\alpha t)^{1/2}}+\frac{h(\alpha t)^{1/2}}{k}\right)\right].$$

For x = 0, erfc(0) = 1 and T(x,t) = T(0,t) = T_s. Hence

$$\frac{T_s - T_i}{T_\infty - T_i} = 1 - \exp\left(\frac{h^2\alpha t}{k^2}\right)\operatorname{erfc}\left(\frac{h(\alpha t)^{1/2}}{k}\right)$$

with

$$\frac{h(\alpha t_m)^{1/2}}{k} = \frac{200\ \mathrm{W/m^2 \cdot K}\left(10^{-4}\mathrm{m^2/s} \times 400\ \mathrm{s}\right)^{1/2}}{400\ \mathrm{W/m \cdot K}} = 0.1$$

$$T_s = T_m = T_i + (T_\infty - T_i)\left[1 - \exp(0.01)\operatorname{erfc}(0.1)\right]$$

$$T_s = 25^\circ\mathrm{C} + 275^\circ\mathrm{C}[1\text{-}1.01 \times 0.888] = 53.5^\circ\mathrm{C}. \qquad <$$

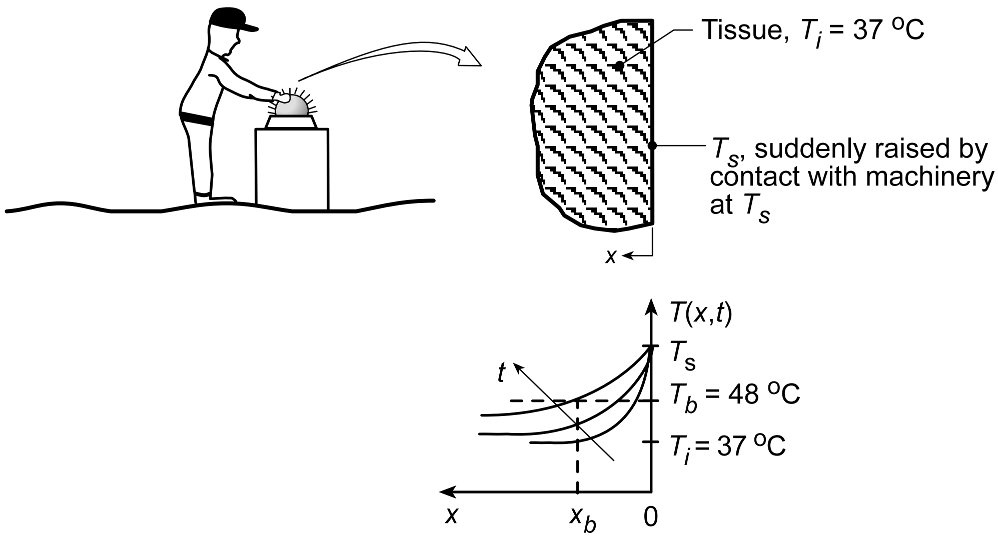
COMMENTS: Use of the procedure to evaluate h from measurement of t_m necessitates iterative calculations.

PROBLEM 5.77

KNOWN: Irreversible thermal injury (cell damage) occurs in living tissue maintained at T ≥ 48°C for a duration Δt ≥ 10s.

FIND: (a) Extent of damage for 10 seconds of contact with machinery in the temperature range 50 to 100°C, (b) Temperature histories at selected locations in tissue (x = 0.5, 1, 5 mm) for a machinery temperature of 100°C.

SCHEMATIC:



ASSUMPTIONS: (1) Portion of worker's body modeled as semi-infinite medium, initially at a uniform temperature, 37°C, (2) Tissue properties are constant and equivalent to those of water at 37°C, (3) Negligible contact resistance.

PROPERTIES: *Table A-6*, Water, liquid (T = 37°C = 310 K): $\rho = 1/v_f = 993.1\ \text{kg/m}^3$, c = 4178 J/kg·K, k = 0.628 W/m·K, $\alpha = k/\rho c = 1.513 \times 10^{-7}\ \text{m}^2/\text{s}$.

ANALYSIS: (a) For a given surface temperature -- suddenly applied -- the analysis is directed toward finding the skin depth x_b for which the tissue will be at $T_b \geq 48°C$ for more than 10s? From Eq. 5.57,

$$\frac{T(x_b, t) - T_s}{T_i - T_s} = \text{erf}\left[x_b/2(\alpha t)^{1/2}\right] = \text{erf}\,[w].$$

For the two values of T_s, the left-hand side of the equation is

$$T_s = 100^\circ C: \quad \frac{(48-100)^\circ C}{(37-100)^\circ C} = 0.825 \qquad\qquad T_s = 50^\circ C: \quad \frac{(48-50)^\circ C}{(37-50)^\circ C} = 0.154$$

The burn depth is

$$x_b = [w]2(\alpha t)^{1/2} = [w]2\left(1.513\times10^{-7}\ \text{m}^2/\text{s}\times t\right)^{1/2} = 7.779\times10^{-4}\,[w]\,t^{1/2}.$$

Continued...

PROBLEM 5.77 (Cont.)

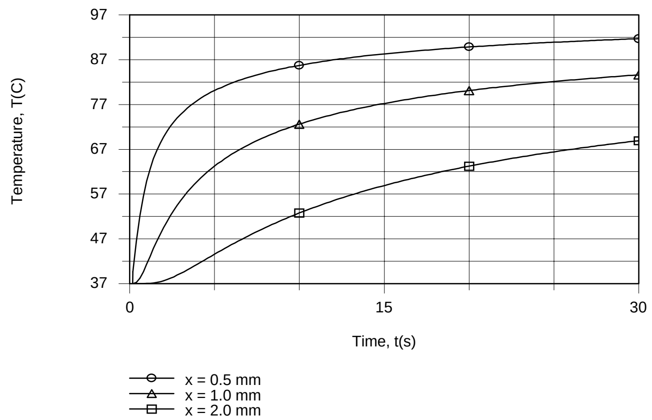
Using Table B.2 to evaluate the error function and letting t = 10s, find x_b as

$T_s = 100°C$: $x_b = 7.779 \times 10^{-4}\,[0.96](10s)^{1/2} = 2.362 \times 10^{3}$ m = 2.36 mm <

$T_s = 50°C$: $x_b = 7.779 \times 10^{-4}\,[0.137](10s)^{1/2} = 3.37 \times 10^{3}$ m = 0.34 mm <

Recognize that tissue at this depth, x_b, has not been damaged, but will become so if T_s is maintained for the next 10s. We conclude that, for $T_s = 50°C$, only superficial damage will occur for a contact period of 20s.

(b) Temperature histories at the prescribed locations are as follows.



The critical temperature of 48°C is reached within approximately 1s at x = 0.5 mm and within 7s at x = 2 mm.

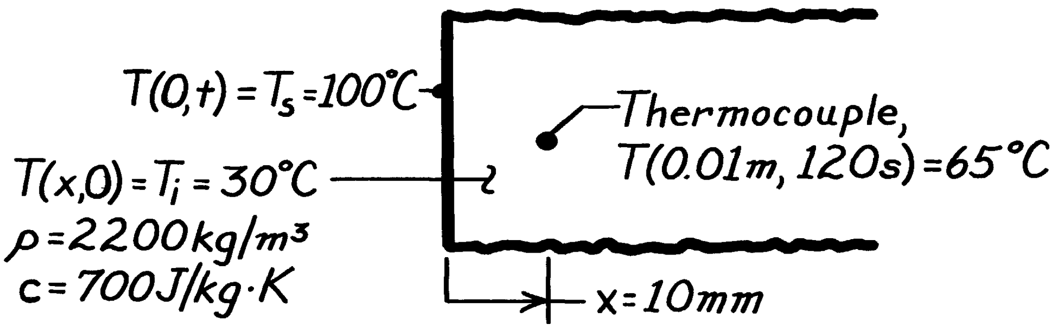
COMMENTS: Note that the burn depth x_b increases as $t^{1/2}$.

PROBLEM 5.78

KNOWN: Thermocouple location in thick slab. Initial temperature. Thermocouple measurement two minutes after one surface is brought to temperature of boiling water.

FIND: Thermal conductivity of slab material.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Slab is semi-infinite medium, (3) Constant properties.

PROPERTIES: Slab material (given): ρ = 2200 kg/m^3, c = 700 J/kg·K.

ANALYSIS: For the semi-infinite medium from Eq. 5.57,

$$\frac{T(x,t)-T_s}{T_i-T_s} = \operatorname{erf}\left[\frac{x}{2(\alpha\ t)^{1/2}}\right]$$

$$\frac{65-100}{30-100} = \operatorname{erf}\left[\frac{0.01\text{m}}{2(\alpha\times 120\text{s})^{1/2}}\right]$$

$$\operatorname{erf}\left[\frac{0.01\text{m}}{2(\alpha\times 120\text{s})^{1/2}}\right] = 0.5.$$

From Appendix B, find for erf w = 0.5 that w = 0.477; hence,

$$\frac{0.01\text{m}}{2(\alpha\times 120\text{s})^{1/2}} = 0.477$$

$$(\alpha\times 120)^{1/2} = 0.0105$$

$$\alpha = 9.156\times 10^{-7}\ \text{m}^2/\text{s}.$$

It follows that since $\alpha = k/\rho c$,

$$k = \alpha\rho\ c$$

$$k = 9.156\times 10^{-7}\ \text{m}^2/\text{s}\times 2200\ \text{kg/m}^3\times 700\ \text{J/kg}\cdot\text{K}$$

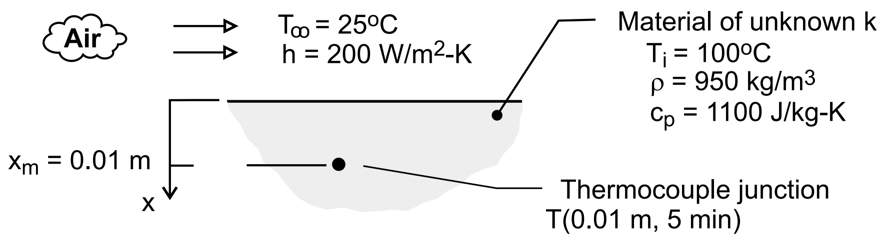
$$k = 1.41\ \text{W/m·K}.$$ <

PROBLEM 5.79

KNOWN: Initial temperature, density and specific heat of a material. Convection coefficient and temperature of air flow. Time for embedded thermocouple to reach a prescribed temperature.

FIND: Thermal conductivity of material.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Sample behaves as a semi-infinite modium, (3) Constant properties.

ANALYSIS: The thermal response of the sample is given by Case 3, Eq. 5.60,

$$\frac{T(x,t)-T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)\right]$$

where, for x = 0.01m at t = 300 s, $[T(x,t) - T_i]/(T_\infty - T_i) = 0.533$. The foregoing equation must be solved iteratively for k, with $\alpha = k/\rho c_p$. The result is

$$k = 0.45\ W/m \cdot K \qquad <$$

with $\alpha = 4.30 \times 10^{-7}\ m^2/s$.

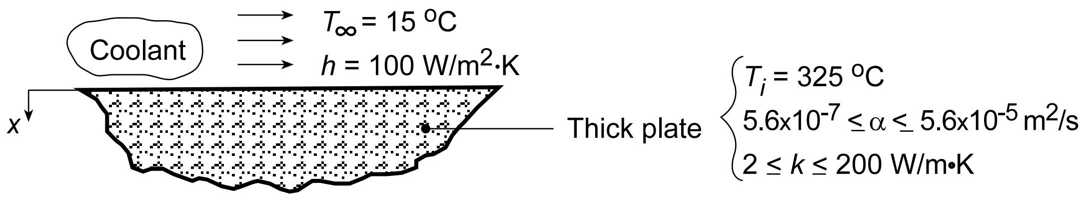
COMMENTS: The solution may be effected by inserting the *Transient Conduction*/*Semi-infinite Solid*/*Surface Conduction Model* of *IHT* into the work space and applying the *IHT Solver*. However, the ability to obtain a converged solution depends strongly on the initial guesses for k and α.

PROBLEM 5.80

KNOWN: Very thick plate, initially at a uniform temperature, T_i, is suddenly exposed to a surface convection cooling process (T_∞,h).

FIND: (a) Temperatures at the surface and 45 mm depth after 3 minutes, (b) Effect of thermal diffusivity and conductivity on temperature histories at x = 0, 0.045 m.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Plate approximates semi-infinite medium, (3) Constant properties, (4) Negligible radiation.

ANALYSIS: (a) The temperature distribution for a semi-infinite solid with surface convection is given by Eq. 5.60.

$$\frac{T(x,t)-T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2(\alpha t)^{1/2}}\right) - \left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2(\alpha t)^{1/2}}+\frac{h(\alpha t)^{1/2}}{k}\right)\right].$$

At the surface, x = 0, and for t = 3 min = 180s,

$$\frac{T(0,180s)-325^\circ C}{(15-325)^\circ C} = \operatorname{erfc}(0) - \left[\exp\left(0+\frac{100^2\ W^2/m^4 K^2 \times 5.6\times10^{-6}\ m^2/s \times 180s}{(20\ W/m\cdot K)^2}\right)\right]$$

$$\times\left[\operatorname{erfc}\left(0+\frac{100\ W/m^2\cdot K\left(5.6\times10^{-6}\ m^2/s\times180s\right)^{1/2}}{20\ W/m\cdot K}\right)\right]$$

$$= 1-\left[\exp(0.02520)\right]\times\left[\operatorname{erfc}(0.159)\right] = 1-1.02552\times(1-0.178)$$

$$T(0,180s) = 325^\circ C - (15-325)^\circ C\cdot(1-1.0255\times0.822)$$

$$T(0,180s) = 325^\circ C - 49.3^\circ C = 276^\circ C .$$ <

At the depth x = 0.045 m, with t = 180s,

$$\frac{T(0.045m,180s)-325^\circ C}{(15-325)^\circ C} = \operatorname{erfc}\left(\frac{0.045\ m}{2\left(5.6\times10^{-6}\ m^2/s\times180s\right)^{1/2}}\right) - \left[\exp\left(\frac{100\ W/m^2\cdot K\times0.045\ m}{20\ W/m\cdot K}+0.02520\right)\right]$$

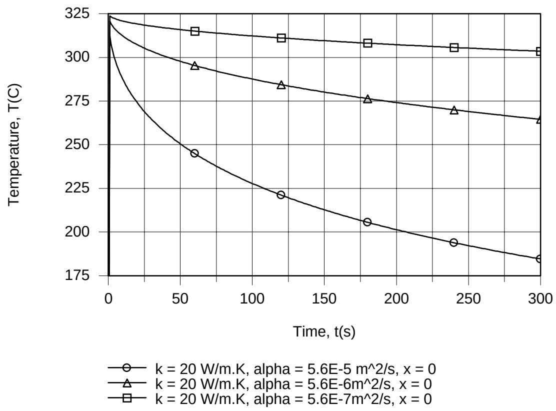
$$\times\left[\operatorname{erfc}\left(\frac{0.045\ m}{2\left(5.6\times10^{-6}\ m^2/s\times180s\right)^{1/2}}+0.159\right)\right]$$

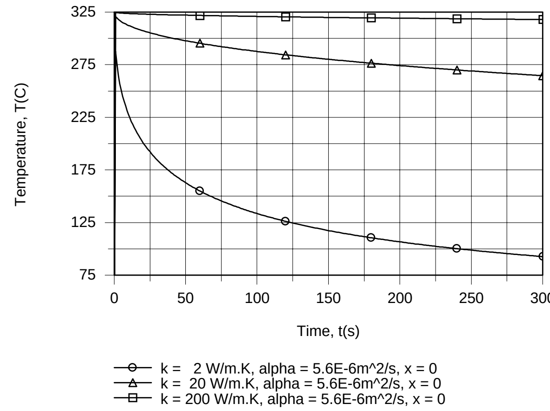
$$= \operatorname{erfc}(0.7087) + \left[\exp(0.225+0.0252)\right]\times\left[\operatorname{erfc}(0.7087+0.159)\right].$$

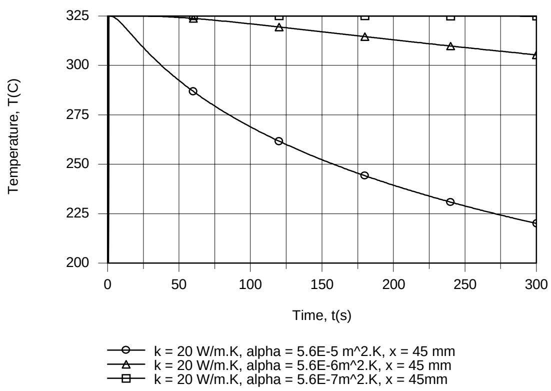
$$T(0.045m,180s) = 325^\circ C + (15-325)^\circ C\left[(1-0.684)-1.284(1-0.780)\right] = 315^\circ C$$ <

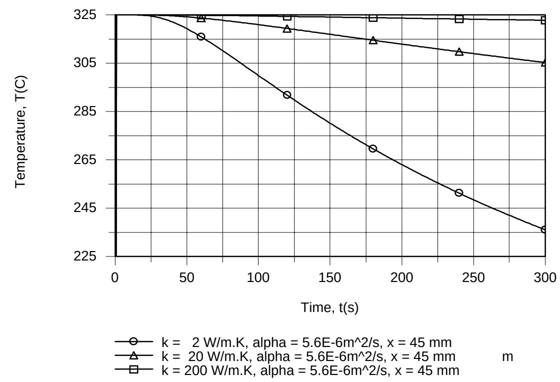
Continued...

(b) The IHT *Transient Conduction Model* for a *Semi-Infinite Solid* was used to generate temperature histories, and for the two locations the effects of varying α and k are as follows.









For fixed k, increasing alpha corresponds to a reduction in the thermal capacitance per unit volume (ρc_p) of the material and hence to a more pronounced reduction in temperature at both surface and interior locations. Similarly, for fixed α, decreasing k corresponds to a reduction in ρc_p and hence to a more pronounced decay in temperature.

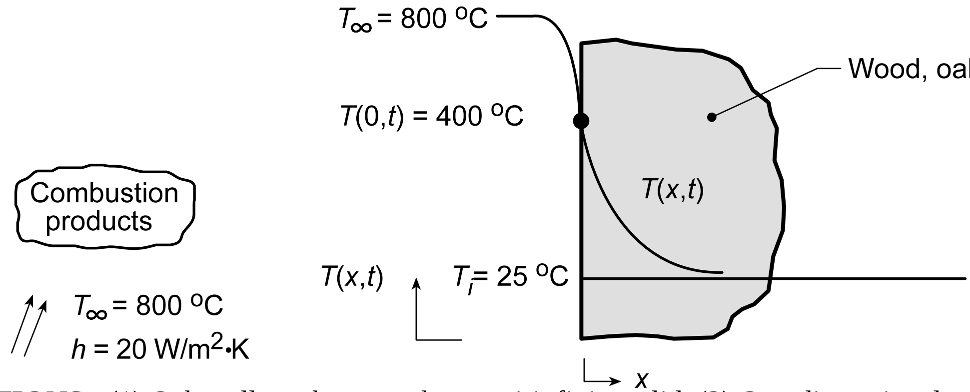
COMMENTS: In part (a) recognize that Fig. 5.8 could also be used to determine the required temperatures.

PROBLEM 5.81

KNOWN: Thick oak wall, initially at a uniform temperature of 25°C, is suddenly exposed to combustion products at 800°C with a convection coefficient of 20 W/m^2·K.

FIND: (a) Time of exposure required for the surface to reach an ignition temperature of 400°C, (b) Temperature distribution at time t = 325s.

SCHEMATIC:



ASSUMPTIONS: (1) Oak wall can be treated as semi-infinite solid, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible radiation.

PROPERTIES: *Table A-3*, Oak, cross grain (300 K): ρ = 545 kg/m^3, c = 2385 J/kg·K, k = 0.17 W/m·K, α = k/ρc = 0.17 W/m·K/545 kg/m^3 × 2385 J/kg·K = 1.31 × 10^{-7} m^2/s.

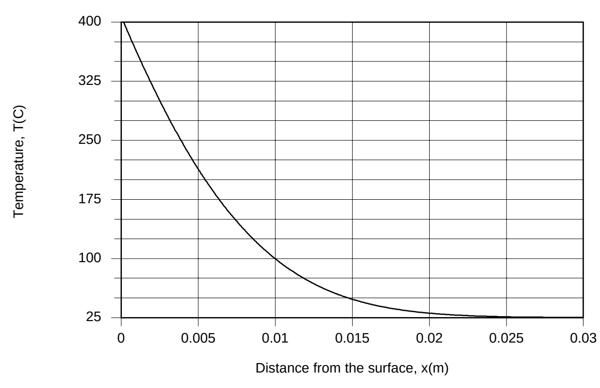
ANALYSIS: (a) This situation corresponds to Case 3 of Figure 5.7. The temperature distribution is given by Eq. 5.60 or by Figure 5.8. Using the figure with

$$\frac{T(0,t)-T_i}{T_\infty - T_i} = \frac{400-25}{800-25} = 0.48 \qquad \text{and} \qquad \frac{x}{2(\alpha t)^{1/2}} = 0$$

we obtain $h(\alpha t)^{1/2}/k \approx 0.75$, in which case $t \approx (0.75k/h\alpha^{1/2})^2$. Hence,

$$t \approx \left(0.75 \times 0.17\,\mathrm{W/m\cdot K} \Big/ 20\,\mathrm{W/m^2\cdot K}\left(1.31\times10^{-7}\,\mathrm{m^2/s}\right)^{1/2}\right)^2 = 310\mathrm{s} \qquad <$$

(b) Using the IHT *Transient Conduction Model* for a *Semi-infinite Solid*, the following temperature distribution was generated for t = 325s.



The temperature decay would become more pronounced with decreasing α (decreasing k, increasing ρc_p) and in this case the penetration depth of the heating process corresponds to x ≈ 0.025 m at 325s.

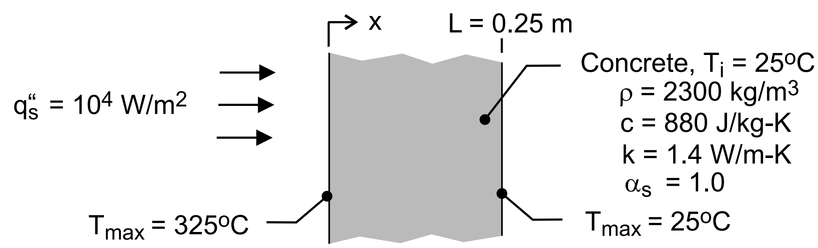
COMMENTS: The result of part (a) indicates that, after approximately 5 minutes, the surface of the wall will ignite and combustion will ensue. Once combustion has started, the present model is no longer appropriate.

PROBLEM 5.82

KNOWN: Thickness, initial temperature and thermophysical properties of concrete firewall. Incident radiant flux and duration of radiant heating. Maximum allowable surface temperatures at the end of heating.

FIND: If maximum allowable temperatures are exceeded.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in wall, (2) Validity of semi-infinite medium approximation, (3) Negligible convection and radiative exchange with the surroundings at the irradiated surface, (4) Negligible heat transfer from the back surface, (5) Constant properties.

ANALYSIS: The thermal response of the wall is described by Eq. (5.60)

$$T(x,t) = T_i + \frac{2\,q''_o(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where, $\alpha = k/\rho c_p = 6.92\times10^{-7}\,m^2/s$ and for $t = 30\ min = 1800s$, $2q''_o(\alpha t/\pi)^{1/2}/k = 284.5\,K$. Hence, at x = 0,

$$T(0, 30\ min) = 25°C + 284.5°C = 309.5°C < 325°C \qquad <$$

At $x = 0.25m$, $\left(-x^2/4\alpha t\right) = -12.54$, $q''_o x/k = 1,786K$, and $x/2(\alpha t)^{1/2} = 3.54$. Hence,

$$T(0.25m,\ 30\,min) = 25°C + 284.5°C\left(3.58\times10^{-6}\right) - 1786°C\times(\sim 0) \approx 25°C \qquad <$$

Both requirements are met.

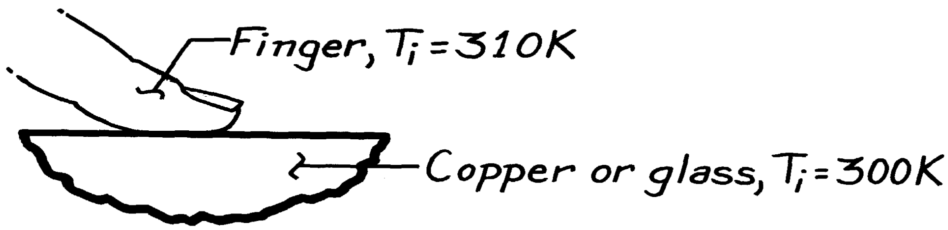
COMMENTS: The foregoing analysis is conservative since heat transfer at the irradiated surface due to convection and net radiation exchange with the environment have been neglected. If the emissivity of the surface and the temperature of the surroundings are assumed to be ε = 1 and T_{sur} = 298K, radiation exchange at T_s = 309.5°C would be $q''_{rad} = \varepsilon\sigma\left(T_s^4 - T_{sur}^4\right) = 6,080\ W/m^2\cdot K$, which is significant (~ 60% of the prescribed radiation).

PROBLEM 5.83

KNOWN: Initial temperature of copper and glass plates. Initial temperature and properties of finger.

FIND: Whether copper or glass feels cooler to touch.

SCHEMATIC:



ASSUMPTIONS: (1) The finger and the plate behave as semi-infinite solids, (2) Constant properties, (3) Negligible contact resistance.

PROPERTIES: Skin (given): $\rho = 1000\ kg/m^3$, c = 4180 J/kg·K, k = 0.625 W/m·K; *Table A-1* (T = 300K), Copper: $\rho = 8933\ kg/m^3$, c = 385 J/kg·K, k = 401 W/m·K; *Table A-3* (T = 300K), Glass: $\rho = 2500\ kg/m^3$, c = 750 J/kg·K, k = 1.4 W/m·K.

ANALYSIS: Which material feels cooler depends upon the contact temperature T_s given by Equation 5.63. For the three materials of interest,

$$(k\rho c)^{1/2}_{skin} = (0.625 \times 1000 \times 4180)^{1/2} = 1{,}616\ J/m^2 \cdot K \cdot s^{1/2}$$
$$(k\rho c)^{1/2}_{cu} = (401 \times 8933 \times 385)^{1/2} = 37{,}137\ J/m^2 \cdot K \cdot s^{1/2}$$
$$(k\rho c)^{1/2}_{glass} = (1.4 \times 2500 \times 750)^{1/2} = 1{,}620\ J/m^2 \cdot K \cdot s^{1/2}.$$

Since $(k\rho c)^{1/2}_{cu} >> (k\rho c)^{1/2}_{glass}$, the copper will feel much cooler to the touch. From Equation 5.63,

$$T_s = \frac{(k\rho c)^{1/2}_A T_{A,i} + (k\rho c)^{1/2}_B T_{B,i}}{(k\rho c)^{1/2}_A + (k\rho c)^{1/2}_B}$$

$$T_{s(cu)} = \frac{1{,}616(310) + 37{,}137(300)}{1{,}616 + 37{,}137} = 300.4\ K \qquad <$$

$$T_{s(glass)} = \frac{1{,}616(310) + 1{,}620(300)}{1{,}616 + 1{,}620} = 305.0\ K. \qquad <$$

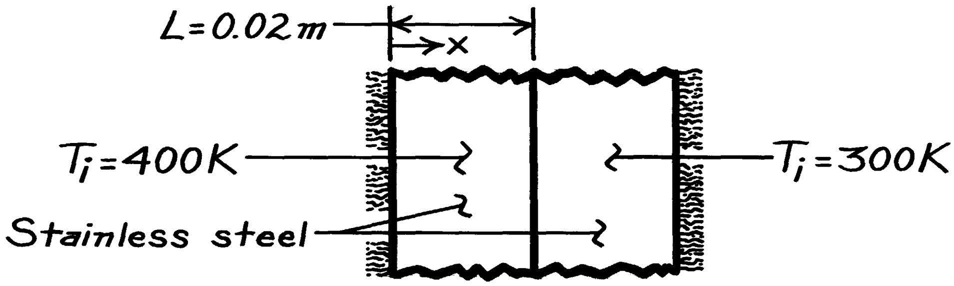
COMMENTS: The extent to which a material's temperature is affected by a change in its thermal environment is inversely proportional to $(k\rho c)^{1/2}$. Large k implies an ability to *spread* the effect by conduction; large ρc implies a large capacity for thermal energy *storage*.

PROBLEM 5.84

KNOWN: Initial temperatures, properties, and thickness of two plates, each insulated on one surface.

FIND: Temperature on insulated surface of one plate at a prescribed time after they are pressed together.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible contact resistance.

PROPERTIES: Stainless steel (given): $\rho = 8000\ kg/m^3$, c = 500 J/kg·K, k = 15 W/m·K.

ANALYSIS: At the instant that contact is made, the plates behave as semi-infinite slabs and, since the (ρkc) product is the same for the two plates, Equation 5.63 yields a surface temperature of

$$T_s = 350K.$$

The interface will remain at this temperature, even after thermal effects penetrate to the insulated surfaces. The transient response of the hot wall may therefore be calculated from Equations 5.40 and 5.41. At the insulated surface (x* = 0), Equation 5.41 yields

$$\frac{T_o - T_s}{T_i - T_s} = C_1 \exp\left(-z_1^2 Fo\right)$$

where, in principle, $h \to \infty$ and $T_\infty \to T_s$. From Equation 5.39c, $Bi \longrightarrow \infty$ yields z_1 = 1.5707, and from Equation 5.39b

$$C_1 = \frac{4 \sin z_1}{2z_1 + \sin(2z_1)} = 1.273$$

Also, $$Fo = \frac{\alpha t}{L^2} = \frac{3.75\times10^{-6} m^2/s(60s)}{(0.02\ m)^2} = 0.563.$$

Hence, $$\frac{T_o - 350}{400 - 350} = 1.273 \exp\left(-1.5707^2 \times 0.563\right) = 0.318$$

$$T_o = 365.9\ K.$$ <

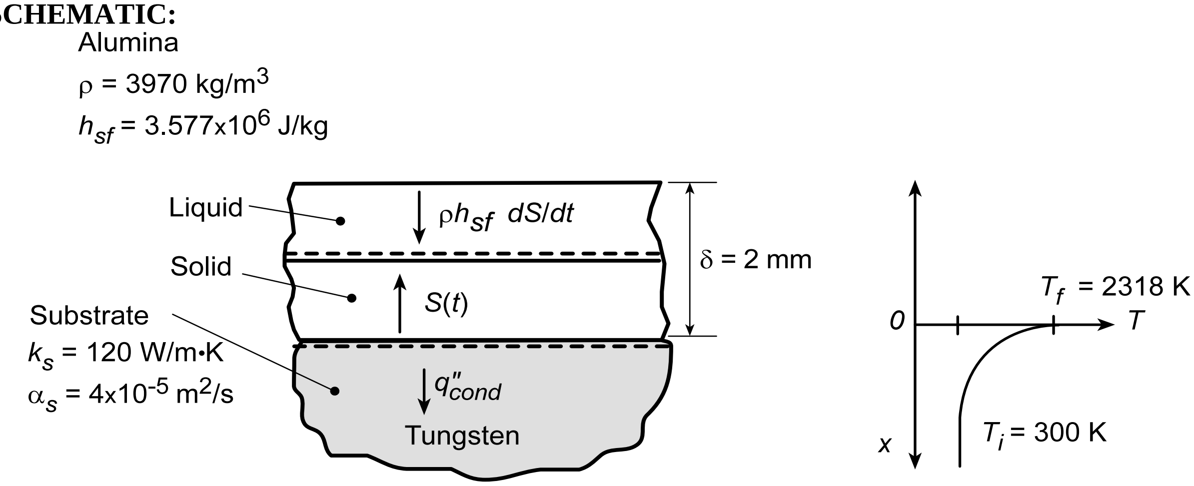
COMMENTS: Since Fo > 0.2, the one-term approximation is appropriate.

PROBLEM 5.85

KNOWN: Thickness and properties of liquid coating deposited on a metal substrate. Initial temperature and properties of substrate.

FIND: (a) Expression for time required to completely solidify the liquid, (b) Time required to solidify an alumina coating.

SCHEMATIC:



ASSUMPTIONS: (1) Substrate may be approximated as a semi-infinite medium in which there is one-dimensional conduction, (2) Solid and liquid alumina layers remain at fusion temperature throughout solidification (negligible resistance to heat transfer by conduction through solid), (3) Negligible contact resistance at the coating/substrate interface, (4) Negligible solidification contraction, (5) Constant properties.

ANALYSIS: (a) Performing an energy balance on the solid layer, whose thickness S increases with t, the latent heat released at the solid/liquid interface must be balanced by the rate of heat conduction into the solid. Hence, per unit surface area,

$$\rho h_{sf}\frac{dS}{dt} = q''_{cond}$$

where, from Eq. 5.58, $q''_{cond} = k\left(T_f - T_i\right)/\left(\pi\alpha t\right)^{1/2}$. It follows that

$$\rho h_{sf}\frac{dS}{dt} = \frac{k_s\left(T_f - T_i\right)}{\left(\pi\alpha_s t\right)^{1/2}}$$

$$\int_o^{\delta} dS = \frac{k_s\left(T_f - T_i\right)}{\rho h_{sf}\left(\pi\alpha_s\right)^{1/2}}\int_o^t \frac{dt}{t^{1/2}}$$

$$\delta = \frac{2k_s}{\left(\pi\alpha_s\right)^{1/2}}\left(\frac{T_f - T_i}{\rho h_{sf}}\right)t^{1/2}$$

$$t = \frac{\pi\alpha_s}{4k_s^2}\left(\frac{\delta\rho h_{sf}}{T_f - T_i}\right)^2$$ <

(b) For the prescribed conditions,

$$t = \frac{\pi\left(4\times10^{-5}\,m^2/s\right)}{4\left(120\,W/m\cdot K\right)^2}\left(\frac{0.002\,m\times3970\,kg/m^3\times3.577\times10^6\,J/kg}{2018\,K}\right)^2 = 0.43s$$ <

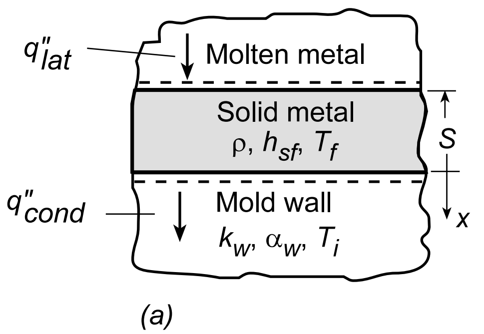
COMMENTS: Such solidification processes occur over short time spans and are typically termed *rapid solidification*.

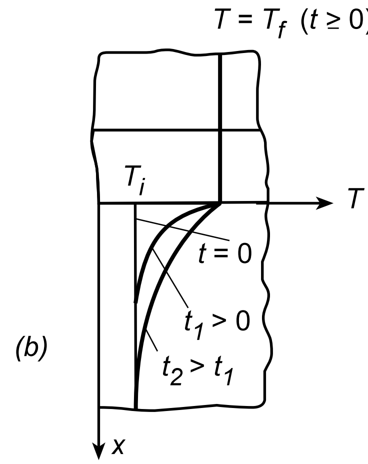
PROBLEM 5.86

KNOWN: Properties of mold wall and a solidifying metal.

FIND: (a) Temperature distribution in mold wall at selected times, (b) Expression for variation of solid layer thickness.

SCHEMATIC:





ASSUMPTIONS: (1) Mold wall may be approximated as a semi-infinite medium in which there is one-dimensional conduction, (2) Solid and liquid metal layers remain at fusion temperature throughout solidification (negligible resistance to heat transfer by conduction through solid), (3) Negligible contact resistance at mold/metal interface, (4) Constant properties.

ANALYSIS: (a) As shown in schematic (b), the temperature remains nearly uniform in the metal (at T_f) throughout the process, while both the temperature and temperature penetration increase with time in the mold wall.

(b) Performing an energy balance for a control surface about the solid layer, the latent energy released due to solidification at the solid/liquid interface is balanced by heat conduction into the solid, $q''_{lat} = q''_{cond}$, where $q''_{lat} = \rho h_{sf}\, dS/dt$ and q''_{cond} is given by Eq. 5.58. Hence,

$$\rho h_{sf} \frac{dS}{dt} = \frac{k_w (T_f - T_i)}{(\pi \alpha_w t)^{1/2}}$$

$$\int_o^S dS = \frac{k_w (T_f - T_i)}{\rho h_{sf} (\pi \alpha_w)^{1/2}} \int_o^t \frac{dt}{t^{1/2}}$$

$$S = \frac{2k_w (T_f - T_i)}{\rho h_{sf} (\pi \alpha_w)^{1/2}} t^{1/2}$$ <

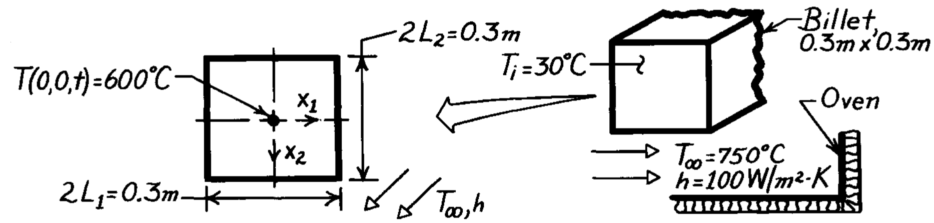
COMMENTS: The analysis of part (b) would only apply until the temperature field penetrates to the exterior surface of the mold wall, at which point, it may no longer be approximated as a semi-infinite medium.

PROBLEM 5.87

KNOWN: Steel (plain carbon) billet of square cross-section initially at a uniform temperature of 30°C is placed in a soaking oven and subjected to a convection heating process with prescribed temperature and convection coefficient.

FIND: Time required for billet center temperature to reach 600°C.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction in x_1 and x_2 directions, (2) Constant properties, (3) Heat transfer to billet is by convection only.

PROPERTIES: *Table A-1*, Steel, plain carbon (T = (30+600)°C/2 = 588K = ≈ 600K): ρ = 7854 kg/m^3, c_p = 559 J/kg·K, k = 48.0 W/m·K, $\alpha = k/\rho c_p = 1.093 \times 10^{-5}$ m^2/s.

ANALYSIS: The billet corresponds to Case (e), Figure 5.11 (infinite rectangular bar). Hence, the temperature distribution is of the form

$$\theta^*\left(x_1, x_2, t\right) = P\left(x_1, t\right) \times P\left(x_2, t\right)$$

where P(x,t) denotes the distribution corresponding to the plane wall. Because of symmetry in the x_1 and x_2 directions, the P functions are identical. Hence,

$$\frac{\theta(0,0,t)}{\theta_i} = \left[\frac{\theta_o(0,t)}{\theta_i}\right]^2_{\text{Plane wall}} \qquad \text{where} \begin{cases} \theta = T - T_\infty \\ \theta_i = T_i - T_\infty \\ \theta_o = T(0,t) - T_\infty \end{cases} \qquad \text{and } L = 0.15\text{m}.$$

Substituting numerical values, find

$$\frac{\theta_o(0,t)}{\theta_i} = \left[\frac{T(0,0,t) - T_\infty}{T_i - T_\infty}\right]^{1/2} = \left[\frac{(600-750)^\circ\,C}{(30-750)^\circ\,C}\right]^{1/2} = 0.46.$$

Consider now the Heisler chart for the plane wall, Figure D.1. For the values

$$\theta_o^* = \frac{\theta_o}{\theta_i} \approx 0.46 \qquad Bi^{-1} = \frac{k}{hL} = \frac{48.0 \text{ W/m}\cdot\text{K}}{100 \text{ W/m}^2\cdot\text{K} \times 0.15\text{m}} = 3.2$$

find

$$t^* = Fo = \frac{\alpha\, t}{L^2} \approx 3.2.$$

Hence,

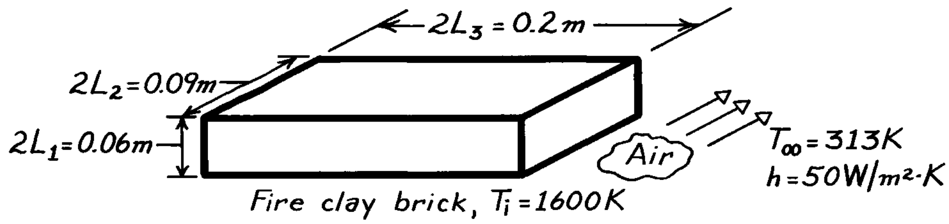
$$t = \frac{3.2\, L^2}{\alpha} = \frac{3.2\ (0.15\text{m})^2}{1.093\times 10^{-5}\text{m}^2/\text{s}} = 6587\text{s} = 1.83\text{h}.$$ <

PROBLEM 5.88

KNOWN: Initial temperature of fire clay brick which is cooled by convection.

FIND: Center and corner temperatures after 50 minutes of cooling.

SCHEMATIC:



ASSUMPTIONS: (1) Homogeneous medium with constant properties, (2) Negligible radiation effects.

PROPERTIES: *Table A-3*, Fire clay brick (900K): $\rho = 2050\ kg/m^3$, $k = 1.0\ W/m\cdot K$, $c_p = 960\ J/kg\cdot K$. $\alpha = 0.51 \times 10^{-6} m^2/s$.

ANALYSIS: From Fig. 5.11(h), the center temperature is given by

$$\frac{T(0,0,0,t) - T_\infty}{T_i - T_\infty} = P_1(0,t) \times P_2(0,t) \times P_3(0,t)$$

where P_1, P_2 and P_3 must be obtained from Fig. D.1.

$$L_1 = 0.03m: \qquad Bi_1 = \frac{h\,L_1}{k} = 1.50 \qquad Fo_1 = \frac{\alpha\,t}{L_1^2} = 1.70$$

$$L_2 = 0.045m: \qquad Bi_2 = \frac{h\,L_2}{k} = 2.25 \qquad Fo_2 = \frac{\alpha\,t}{L_2^2} = 0.756$$

$$L_3 = 0.10m: \qquad Bi_3 = \frac{h\,L_3}{k} = 5.0 \qquad Fo_3 = \frac{\alpha\,t}{L_3^2} = 0.153$$

Hence from Fig. D.1,

$$P_1(0,t) \approx 0.22 \qquad P_2(0,t) \approx 0.50 \qquad P_3(0,t) \approx 0.85.$$

Hence,
$$\frac{T(0,0,0,t) - T_\infty}{T_i - T_\infty} \approx 0.22 \times 0.50 \times 0.85 = 0.094$$

and the center temperature is

$$T(0,0,0,t) \approx 0.094(1600 - 313)K + 313K = 434K. \qquad <$$

Continued …..

PROBLEM 5.88 (Cont.)

The corner temperature is given by

$$\frac{T(L_1, L_2, L_3, t) - T_\infty}{T_i - T_\infty} = P(L_1, t) \times P(L_2, t) \times P(L_3, t)$$

where

$$P(L_1, t) = \frac{\theta(L_1, t)}{\theta_o} \cdot P_1(0, t), \text{ etc.}$$

and similar forms can be written for L_2 and L_3. From Fig. D.2,

$$\frac{\theta(L_1, t)}{\theta_o} \approx 0.55 \qquad \frac{\theta(L_2, t)}{\theta_o} \approx 0.43 \qquad \frac{\theta(L_3, t)}{\theta_o} \approx 0.25.$$

Hence,

$$P(L_1, t) \approx 0.55 \times 0.22 = 0.12$$
$$P(L_2, t) \approx 0.43 \times 0.50 = 0.22$$
$$P(L_3, t) \approx 0.85 \times 0.25 = 0.21$$

and

$$\frac{T(L_1, L_2, L_3, t) - T_\infty}{T_i - T_\infty} \approx 0.12 \times 0.22 \times 0.21 = 0.0056$$

or

$$T(L_1, L_2, L_3, t) \approx 0.0056(1600 - 313)\text{K} + 313\text{K}.$$

The corner temperature is then

$$T(L_1, L_2, L_3, t) \approx 320\text{K}.$$ <

COMMENTS: (1) The foregoing temperatures are overpredicted by ignoring radiation, which is significant during the early portion of the transient.

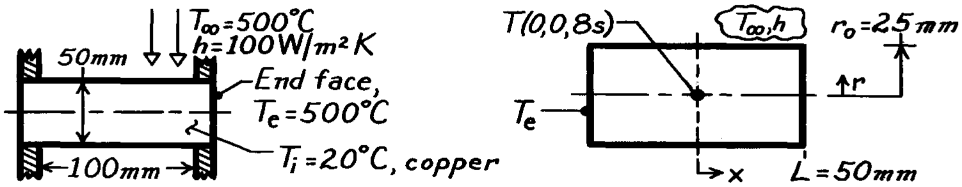
(2) Note that, if the time required to reach a certain temperature were to be determined, an iterative approach would have to be used. The foregoing procedure would be used to compute the temperature for an assumed value of the time, and the calculation would be repeated until the specified temperature were obtained.

PROBLEM 5.89

KNOWN: Cylindrical copper pin, 100mm long × 50mm diameter, initially at 20°C; end faces are subjected to intense heating, suddenly raising them to 500°C; at the same time, the cylindrical surface is subjected to a convective heating process (T_∞,h).

FIND: (a) Temperature at center point of cylinder after a time of 8 seconds from sudden application of heat, (b) Consider parameters governing transient diffusion and justify simplifying assumptions that could be applied to this problem.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Constant properties and convection heat transfer coefficient.

PROPERTIES: *Table A-1*, Copper, pure $\left(\bar{T} \approx (500+20)^\circ \text{C}/2 \approx 500\text{K}\right)$: ρ = 8933 kg/m³, c = 407 J/kg·K, k = 386 W/m·K, α = k/ρc = 386 W/m·K/8933 kg/m³ × 407 J/kg·K = 1.064 × 10⁻⁴ m²/s.

ANALYSIS: (1) The pin can be treated as a two-dimensional system comprised of an infinite cylinder whose surface is exposed to a convection process (T_∞,h) and of a plane wall whose surfaces are maintained at a constant temperature (T_e). This configuration corresponds to the short cylinder, Case (i) of Fig. 5.11,

$$\frac{\theta(\text{r,x,t})}{\theta_i} = C(\text{r,t}) \times P(\text{x,t}). \tag{1}$$

For the infinite cylinder, using Fig. D.4, with

$$\text{Bi} = \frac{hr_o}{k} = \frac{100\ \text{W/m}^2\cdot\text{K}\left(25\times10^{-3}\text{m}\right)}{385\ \text{W/m}\cdot\text{K}} = 6.47\times10^{-3} \quad \text{and} \quad \text{Fo} = \frac{\alpha\ t}{r_o^2} = \frac{1.064\times10^{-4}\ \frac{\text{m}^2}{\text{s}}\times 8\text{s}}{\left(25\times10^{-3}\text{m}\right)^2} = 1.36,$$

find $$C(0,8\text{s}) = \left.\frac{\theta(0,8\text{s})}{\theta_i}\right]_{\text{cyl}} \approx 1. \tag{2}$$

For the infinite plane wall, using Fig. D.1, with

$$\text{Bi} = \frac{hL}{k} \to \infty \quad \text{or} \quad \text{Bi}^{-1} \to 0 \quad \text{and} \quad \text{Fo} = \frac{\alpha\ t}{L^2} = \frac{1.064\times10^{-4}\ \text{m}^2/\text{s}\times 8\text{s}}{\left(50\times10^{-3}\text{m}\right)^2} = 0.34,$$

find $$P(0,8\text{s}) = \left.\frac{\theta(0,8\text{s})}{\theta_i}\right]_{\text{wall}} \approx 0.5. \tag{3}$$

Combining Eqs. (2) and (3) with Eq. (1), find $\dfrac{\theta(0,0,8\text{s})}{\theta_i} = \dfrac{T(0,0,8\text{s}) - T_\infty}{T_i - T_\infty} \approx 1\times0.5 = 0.5$

$$T(0,0,8\text{s}) = T_\infty + 0.5\left(T_i - T_\infty\right) = 500 + 0.5(20-500) = 260^\circ\text{C}. \qquad <$$

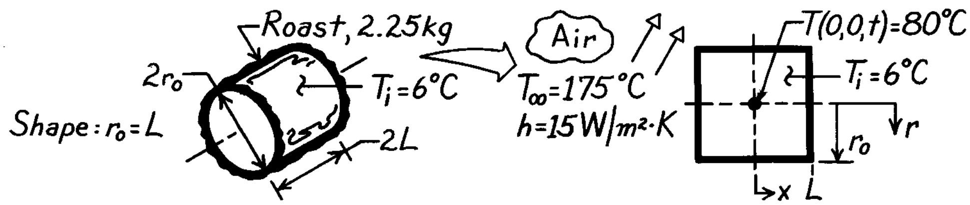
(b) The parameters controlling transient conduction with convective boundary conditions are the Biot and Fourier numbers. Since Bi << 0.1 for the cylindrical shape, we can assume radial gradients are negligible. That is, we need only consider conduction in the x-direction.

PROBLEM 5.90

KNOWN: Cylindrical-shaped meat roast weighing 2.25 kg, initially at 6°C, is placed in an oven and subjected to convection heating with prescribed (T_∞,h).

FIND: Time required for the center to reach a done temperature of 80°C.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction in x and r directions, (2) Uniform and constant properties, (3) Properties approximated as those of water.

PROPERTIES: *Table A-6*, Water, liquid $\left(\bar{T} = (80+6)^\circ \text{C}/2 \approx 315\text{K}\right)$:◁ $\rho = 1/v_f = 1/1.009 \times 10^{-3} \text{m}^3/\text{kg} = 991.1 \text{ kg/m}^3$, $c_{p,f} = 4179$ J/kg·K, k = 0.634 W/m·K, $\alpha = k/\rho c = 1.531 \times 10^{-7} \text{m}^2/\text{s}$.

ANALYSIS: The dimensions of the roast are determined from the requirement $r_o = L$ and knowledge of its weight and density,

$$M = \rho V = \rho \cdot 2L \cdot \pi r_o^2 \quad \text{or} \quad r_o = L = \left[\frac{M}{2\pi\rho}\right]^{1/3} = \left[\frac{2.25 \text{ kg}}{2\pi 991.1 \text{ kg/m}^3}\right]^{1/3} = 0.0712\text{m}. \qquad (1)$$

The roast corresponds to Case (i), Figure 5.11, and the temperature distribution may be expressed as the product of one-dimensional solutions, $\dfrac{T(x,r,t) - T_\infty}{T_i - T_\infty} = P(x,t) \times C(r,t)$, where P(x,t) and C(r,t) are defined by Eqs. 5.65 and 5.66, respectively. For the center of the cylinder,

$$\frac{T(0,0,t) - T_\infty}{T_i - T_\infty} = \frac{(80-175)^\circ \text{C}}{(6-175)^\circ \text{C}} = 0.56. \qquad (2)$$

In terms of the product solutions,

$$\frac{T(0,0,t) - T_\infty}{T_i - T_\infty} = 0.56 = \left.\frac{T(0,t) - T_\infty}{T_i - T_\infty}\right]_{\text{wall}} \times \left.\frac{T(0,t) - T_\infty}{T_i - T_\infty}\right]_{\text{cylinder}} \qquad (3)$$

For each of these shapes, we need to find values of θ_o/θ_i such that their product satisfies Eq. (3). For both shapes,

$$\text{Bi} = \frac{h\, r_o}{k} = \frac{hL}{k} = \frac{15 \text{ W/m}^2 \cdot \text{K} \times 0.0712\text{m}}{0.634 \text{ W/m} \cdot \text{K}} = 1.68 \qquad \text{or} \qquad \text{Bi}^{-1} \approx 0.6$$

$$\text{Fo} = \alpha\, t/r_o^2 = \alpha\, t/L^2 = 1.53 \times 10^{-7} \text{m}^2/\text{s} \times t/(0.0712\text{m})^2 = 3.020 \times 10^{-5} t.$$

Continued …..

PROBLEM 5.90 (Cont.)

A trial-and-error solution is necessary. Begin by assuming a value of Fo; obtain the respective θ_o/θ_i values from Figs. D.1 and D.4; test whether their product satisfies Eq. (3). Two trials are shown as follows:

Trial	Fo	t(hrs)	$\theta_o/\theta_i)_{wall}$	$\theta_o/\theta_i)_{cyl}$	$\left.\frac{\theta_o}{\theta_i}\right]_w \times \left.\frac{\theta_o}{\theta_i}\right]_{cyl}$
1	0.4	3.68	0.72	0.50	0.36
2	0.3	2.75	0.78	0.68	0.53

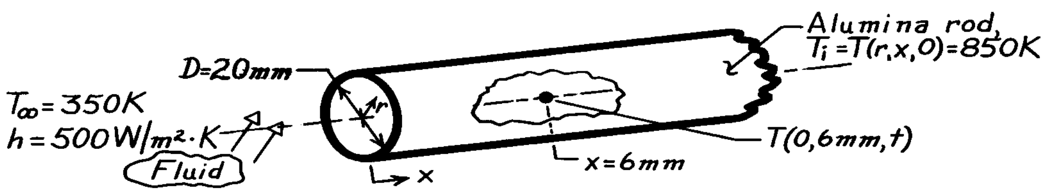
For Trial 2, the product of 0.53 agrees closely with the value of 0.56 from Eq. (2). Hence, it will take approximately 2 ¾ hours to roast the meat.

PROBLEM 5.91

KNOWN: A long alumina rod, initially at a uniform temperature of 850K, is suddenly exposed to a cooler fluid.

FIND: Temperature of the rod after 30s, at an exposed end, T(0,0,t), and at an axial distance 6mm from the end, T(0, 6mm, t).

SCHEMATIC:



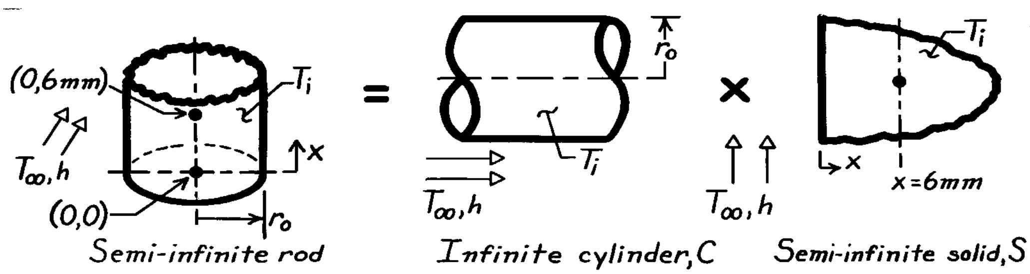
ASSUMPTIONS: (1) Two-dimensional conduction in (r,x) directions, (2) Constant properties, (3) Convection coefficient is same on end and cylindrical surfaces.

PROPERTIES: *Table A-2*, Alumina, polycrystalline aluminum oxide (assume $\bar{T} \approx (850+600)\text{K}/2 = 725\text{K}$): $\rho = 3970\ \text{kg/m}^3$, c = 1154 J/kg·K, k = 12.4 W/m·K.

ANALYSIS: First, check if system behaves as a lumped capacitance. Find

$$\text{Bi} = \frac{hL_c}{k} = \frac{h(r_o/2)}{k} = \frac{500\ \text{W/m}\cdot\text{K}(0.010\text{m}/2)}{12.4\ \text{W/m}\cdot\text{K}} = 0.202.$$

Since Bi > 0.1, rod does not behave as spacewise isothermal object. Hence, treat rod as a semi-infinite cylinder, the multi-dimensional system Case (f), Fig. 5.11.



The product solution can be written as

$$\theta^*(r,x,t) = \frac{\theta(r,x,t)}{\theta_i} = \frac{\theta(r,t)}{\theta_i} \times \frac{\theta(x,t)}{\theta_i} = C\left(r^*,t^*\right) \times S\left(x^*,t^*\right)$$

Infinite cylinder, C(r*,t*). Using the Heisler charts with r* = r = 0 and

$$\text{Bi}^{-1} = \left[\frac{h\,r_o}{k}\right]^{-1} = \left[\frac{500\ \text{W/m}^2\cdot\text{K}\times 0.01\text{m}}{12.4\ \text{W/m}\cdot\text{K}}\right]^{-1} = 2.48.$$

Evaluate $\alpha = k/\rho c = 2.71 \times 10^{-6}\text{m}^2/\text{s}$, find $\text{Fo} = \alpha\, t/r_o^2 = 2.71\times10^{-6}\text{m}^2/\text{s} \times 30\text{s}/(0.01\text{m})^2$ = 0.812. From the Heisler chart, Fig. D.4, with Bi^{-1} = 2.48 and Fo = 0.812, read C(0,t*) = $\theta(0,t)/\theta_i$ = 0.61.

Continued …..

Semi-infinite medium, S(x*,t*). Recognize this as Case (3), Fig. 5.7. From Eq. 5.60, note that the LHS needs to be transformed as follows,

$$\frac{T-T_i}{T_\infty - T_i} = 1 - \frac{T-T_\infty}{T_i - T_\infty} \qquad S(x,t) = \frac{T-T_\infty}{T_i - T_\infty}.$$

Thus,

$$S(x,t) = 1 - \left\{ \operatorname{erfc}\left[\frac{x}{2(\alpha t)^{1/2}}\right] - \left[\exp\left[\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right]\right]\left[\operatorname{erfc}\left[\frac{x}{2(\alpha t)^{1/2}} + \frac{h(\alpha t)^{1/2}}{k}\right]\right]\right\}.$$

Evaluating this expression at the surface (x = 0) and 6mm from the exposed end, find

$$S(0,30s) = 1 - \left\{ \operatorname{erfc}(0) - \left[\exp\left[0 + \frac{\left(500\ W/m^2 \cdot K\right)^2 2.71\times10^{-6} m^2/s \times 30s}{(12.4\ W/m \cdot K)^2}\right]\right]\right.$$

$$\left.\left[\operatorname{erfc}\left[0 + \frac{500\ W/m^2 \cdot K\left(2.71\times10^{-6} m^2/s \times 30s\right)^{1/2}}{12.4\ W/m \cdot K}\right]\right]\right\}$$

$$S(0,30s) = 1 - \left\{1 - \left[\exp(0.1322)\right]\left[\operatorname{erfc}(0.3636)\right]\right\} = 0.693.$$

Note that Table B.2 was used to evaluate the complementary error function, erfc(w).

$$S(6mm,30s) = 1 - \left\{ \operatorname{erfc}\left[\frac{0.006m}{2\left(2.71\times10^{-6} m^2/s \times 30s\right)^{1/2}}\right] - \right.$$

$$\left.\left[\exp\left[\frac{500\ W/m^2 \cdot K \times 0.006m}{12.4\ W/m \cdot K} + 0.1322\right]\right]\left[\operatorname{erfc}(0.3327 + 0.3636)\right]\right\} = 0.835.$$

The product solution can now be evaluated for each location. At (0,0),

$$\theta^*(0,0,t) = \frac{T(0,0,30s) - T_\infty}{T_i - T_\infty} = C\left(0,t^*\right) \times S\left(0,t^*\right) = 0.61 \times 0.693 = 0.423.$$

Hence, $\quad T(0,0,30s) = T_\infty + 0.423(T_i - T_\infty) = 350K + 0.423(850 - 350)K = 561K.$ <

At (0,6mm),

$$\theta^*(0,6mm,t) = C\left(0,t^*\right) \times S\left(6mm,t^*\right) = 0.61 \times 0.835 = 0.509$$

$$T(0,6mm,30s) = 604K.$$ <

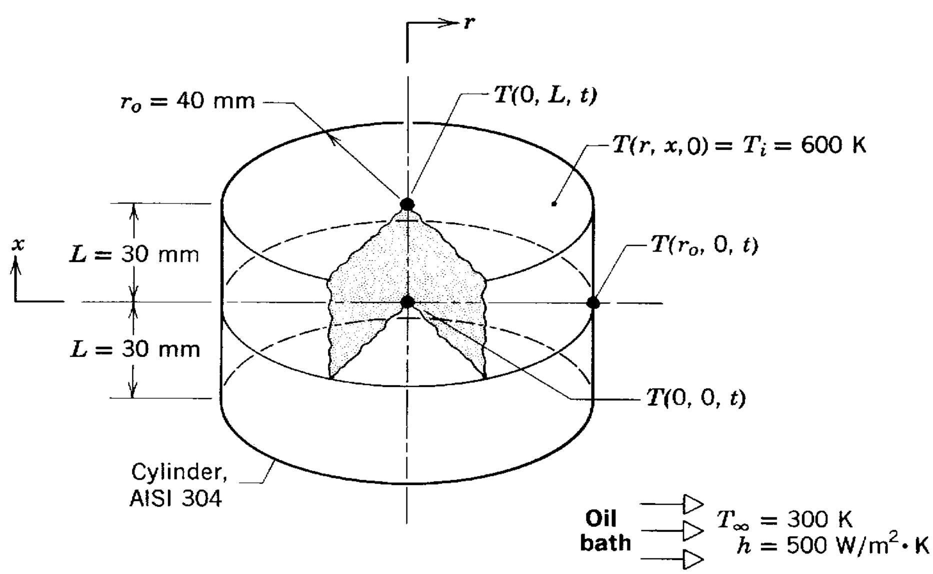
COMMENTS: Note that the temperature at which the properties were evaluated was a good estimate.

PROBLEM 5.92

KNOWN: Stainless steel cylinder of Ex. 5.7, 80-mm diameter by 60-mm length, initially at 600 K, suddenly quenched in an oil bath at 300 K with h = 500 W/m^2·K. Use the *Transient Conduction, Plane Wall* and *Cylinder* models of *IHT* to obtain the following solutions.

FIND: (a) Calculate the temperatures T(r,x,t) after 3 min: at the cylinder center, T(0, 0, 3 min), at the center of a circular face, T(0, L, 3 min), and at the midheight of the side, T(r_o, 0, 3 min); compare your results with those in the example; (b) Calculate and plot temperature histories at the cylinder center, T(0, 0, t), the mid-height of the side, T(r_o, 0, t), for $0 \le t \le 10$ min; comment on the gradients and what effect they might have on phase transformations and thermal stresses; and (c) For $0 \le t \le 10$ min, calculate and plot the temperature histories at the cylinder center, T(0, 0, t), for convection coefficients of 500 and 1000 W/m^2·K.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction in r- and x-coordinates, (2) Constant properties.

PROPERTIES: Stainless steel (*Example 5.7*): ρ = 7900 kg/m^3, c = 526 J/kg·K, k = 17.4 W/m·K.

ANALYSIS: The following results were obtained using the *Transient Conduction* models for the *Plane Wall* and *Cylinder* of *IHT*. Salient portions of the code are provided in the Comments.

(a) Following the methodology for a product solution outlined in Example 5.7, the following results were obtained at t = t_o = 3 min

(r, x, t)	P(x, t)	C(r, t)	T(r, x, t)-IHT (K)	T(r, x, t)-Ex (K)
0, 0, t_o	0.6357	0.5388	402.7	405
0, L, t_o	0.4365	0.5388	370.5	372
r_o, 0, t_o	0.6357	0.3273	362.4	365

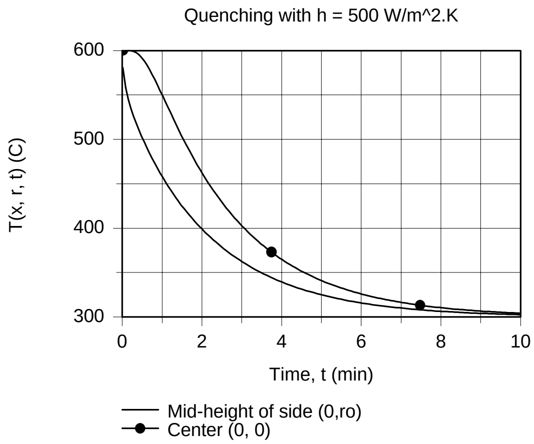
Continued …..

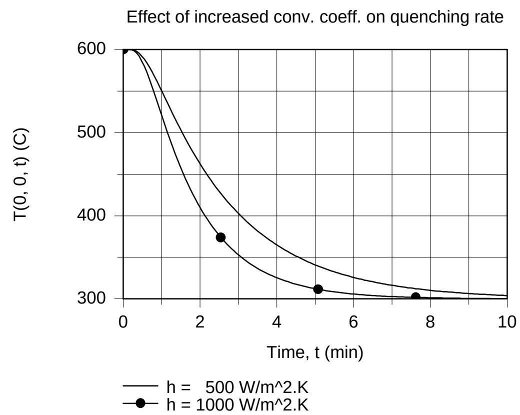
PROBLEM 5.92 (Cont.)

The temperatures from the one-term series calculations of the Example 5.7 are systematically higher than those resulting from the *IHT* multiple-term series model, which is the more accurate method.

(b) The temperature histories for the center and mid-height of the side locations are shown in the graph below. Note that at early times, the temperature difference between these locations, and hence the gradient, is large. Large differences could cause variations in microstructure and hence, mechanical properties, as well as induce residual thermal stresses.

(c) Effect of doubling the convection coefficient is to increase the quenching rate, but much less than by a factor of two as can be seen in the graph below.





COMMENTS: From *IHT* menu for *Transient Conduction* | *Plane Wall* and *Cylinder*, the models were combined to solve the product solution. Key portions of the code, less the input variables, are copied below.

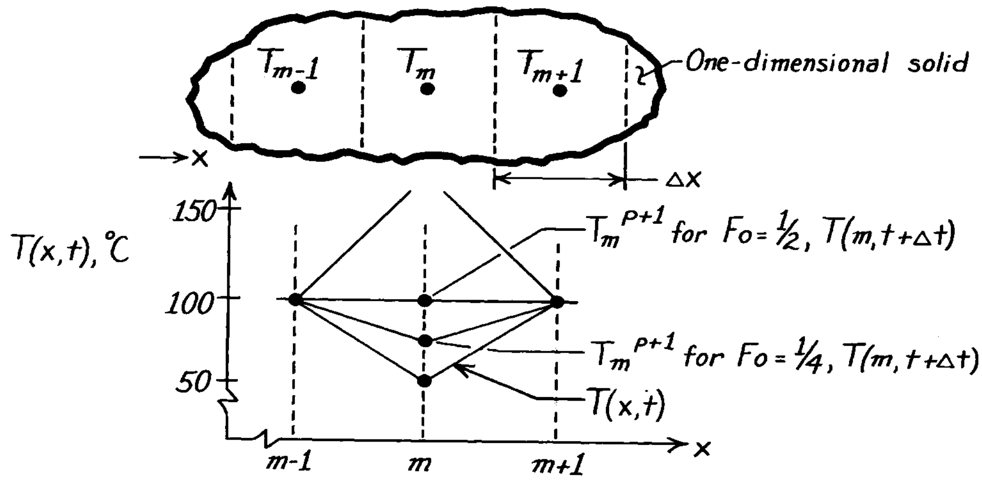
```
// Plane wall temperature distribution
// The temperature distribution is
T_xtP = T_xt_trans("Plane Wall",xstar,FoP,BiP,Ti,Tinf)    // Eq 5.39
// The dimensionless parameters are
xstar = x / L
BiP = h * L / k                  // Eq 5.9
FoP= alpha * t / L^2       // Eq 5.33
alpha = k/ (rho * cp)
// Dimensionless representation, P(x,t)
P_xt = (T_xtP - Tinf ) / (Ti - Tinf)

// Cylinder temperature distribution
// The temperature distribution T(r,t) is
T_rtC = T_xt_trans("Cylinder",rstar,FoC,BiC,Ti,Tinf)    // Eq 5.47
// The dimensionless parameters are
rstar = r / ro
BiC = h * ro / k
FoC= alpha * t / ro^2
// Dimensionless representation, C(r,t)
C_rt= (T_rtC - Tinf ) / (Ti - Tinf)

// Product solution temperature distribution
(T_xrt - Tinf) / (Ti - Tinf) = P_xt * C_rt
```

PROBLEM 5.93

KNOWN: Stability criterion for the explicit method requires that the coefficient of the T_m^p term of the one-dimensional, finite-difference equation be zero or positive.

FIND: For Fo > 1/2, the finite-difference equation will predict values of T_m^{p+1} which violate the Second law of thermodynamics. Consider the prescribed numerical values.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Constant properties, (3) No internal heat generation.

ANALYSIS: The explicit form of the finite-difference equation, Eq. 5.73, for an interior node is

$$T_m^{p+1} = Fo\left(T_{m+1}^p + T_{m-1}^p\right) + (1 - 2\ Fo)T_m^p.$$

The stability criterion requires that the coefficient of T_m^p be zero or greater. That is,

$$(1 - 2\ Fo) \geq 0 \qquad \text{or} \qquad Fo \leq \frac{1}{2}.$$

For the prescribed temperatures, consider situations for which Fo = 1, ½ and ¼ and calculate T_m^{p+1}.

$$Fo = 1 \qquad T_m^{p+1} = 1(100 + 100)^\circ C + (1 - 2 \times 1)50^\circ C = 250^\circ C$$
$$Fo = 1/2 \qquad T_m^{p+1} = 1/2(100 + 100)^\circ C + (1 - 2 \times 1/2)50^\circ C = 100^\circ C$$
$$Fo = 1/4 \qquad T_m^{p+1} = 1/4(100 + 100)^\circ C + (1 - 2 \times 1/4)50^\circ C = 75^\circ C.$$

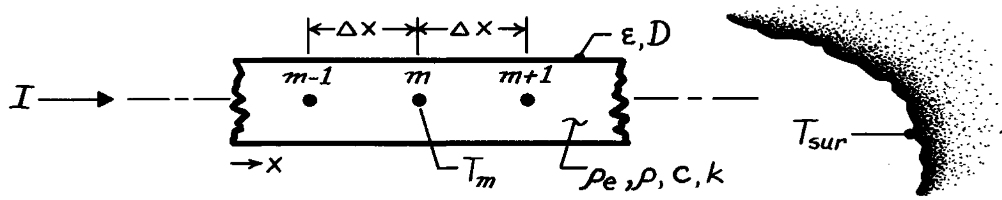
Plotting these distributions above, note that when Fo = 1, T_m^{p+1} is greater than 100°C, while for Fo = ½ and ¼ , $T_m^{p+1} \leq 100$°C. The distribution for Fo = 1 is thermodynamically impossible: heat is flowing into the node during the time period Δt, causing its temperature to rise; yet heat is flowing in the direction of increasing temperature. This is a violation of the Second law. When Fo = ½ or ¼, the node temperature increases during Δt, but the temperature gradients for heat flow are proper. This will be the case when Fo ≤ ½, verifying the stability criterion.

PROBLEM 5.94

KNOWN: Thin rod of diameter D, initially in equilibrium with its surroundings, T_{sur}, suddenly passes a current I; rod is in vacuum enclosure and has prescribed electrical resistivity, ρ_e, and other thermophysical properties.

FIND: Transient, finite-difference equation for node m.

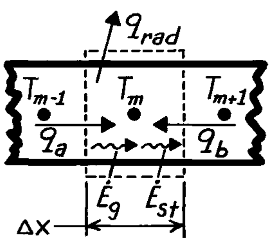
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction in rod, (2) Surroundings are much larger than rod, (3) Properties are constant and evaluated at an average temperature, (4) No convection within vacuum enclosure.

ANALYSIS: The finite-difference equation is derived from the energy conservation requirement on the control volume, $A_c\Delta x$, where $A_c = \pi D^2/4$ and $P = \pi D$.

The energy balance has the form



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad q_a + q_b - q_{rad} + I^2 R_e = \rho\, cV \frac{T_m^{p+1} - T_m^p}{\Delta t}.$$

where $\dot{E}_g = I^2 R_e$ and $R_e = \rho_e \Delta x / A_c$. Using Fourier's law to express the conduction terms, q_a and q_b, and Eq. 1.7 for the radiation exchange term, q_{rad}, find

$$kA_c \frac{T_{m-1}^p - T_m^p}{\Delta x} + kA_c \frac{T_{m+1}^p - T_m^p}{\Delta x} - \varepsilon P \Delta x \sigma \left(T_m^{4,p} - T_{sur}^4\right) + I^2 \frac{\rho_e \Delta x}{A_c} = \rho\, cA_c \Delta x \frac{T_m^{p+1} - T_m^p}{\Delta t}.$$

Divide each term by $\rho c A_c \Delta x/\Delta t$, solve for T_m^{p+1} and regroup to obtain

$$T_m^{p+1} = \frac{k}{\rho\, c} \cdot \frac{\Delta t}{\Delta x^2}\left(T_{m-1}^p + T_{m+1}^p\right) - \left[2 \cdot \frac{k}{\rho\, c} \cdot \frac{\Delta t}{\Delta x^2} - 1\right] T_m^p$$

$$- \frac{\varepsilon P \sigma}{A_c} \cdot \frac{\Delta t}{\rho\, c}\left(T_m^{4,p} - T_{sur}^4\right) + \frac{I^2 \rho_e}{A_c^2} \cdot \frac{\Delta t}{\rho\, c}.$$

Recognizing that $Fo = \alpha \Delta t / \Delta x^2$, regroup to obtain

$$T_m^{p+1} = Fo\left(T_{m-1}^p + T_{m+1}^p\right) + (1 - 2\, Fo) T_m^p - \frac{\varepsilon P \sigma \Delta x^2}{kA_c} \cdot Fo\left(T_m^{4,p} - T_{sur}^4\right) + \frac{I^2 \rho_e \Delta x^2}{kA_c^2} \cdot Fo.$$

The stability criterion is based upon the coefficient of the T_m^p term written as

$$Fo \le 1/2. \qquad <$$

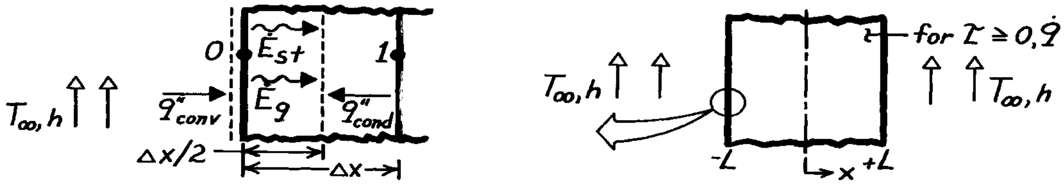
COMMENTS: Note that we have used the forward-difference representation for the time derivative; see Section 5.9.1. This permits convenient treatment of the non-linear radiation exchange term.

PROBLEM 5.95

KNOWN: One-dimensional wall suddenly subjected to uniform volumetric heating and convective surface conditions.

FIND: Finite-difference equation for node at the surface, x = -L.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Constant properties, (3) Uniform $\dot{q}$.

ANALYSIS: There are two types of finite-difference equations for the *explicit* and *implicit* methods of solution. Using the energy balance approach, both types will be derived.

Explicit Method. Perform an energy balance on the surface node shown above,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad q_{conv} + q_{cond} + \dot{q}V = \rho\, cV \frac{T_o^{p+1} - T_o^p}{\Delta t} \tag{1}$$

$$h(1 \cdot 1)\left(T_\infty - T_o^p\right) + k(1 \cdot 1)\frac{T_1^p - T_o^p}{\Delta x} + \dot{q}\left[1 \cdot 1 \cdot \frac{\Delta x}{2}\right] = \rho\, c\left[1 \cdot 1 \cdot \frac{\Delta x}{2}\right]\frac{T_o^{p+1} - T_o^p}{\Delta t}. \tag{2}$$

For the explicit method, the temperatures on the LHS are evaluated at the *previous* time (p). The RHS provides a *forward*-difference approximation to the time derivative. Divide Eq. (2) by $\rho c\Delta x/2\Delta t$ and solve for T_o^{p+1}.

$$T_o^{p+1} = 2\frac{h\Delta t}{\rho\, c\Delta x}\left(T_\infty - T_o^p\right) + 2\frac{k\Delta t}{\rho\, c\Delta x^2}\left(T_1^p - T_o^p\right) + \dot{q}\frac{\Delta t}{\rho\, c} + T_o^p.$$

Introducing the Fourier and Biot numbers,

$$Fo \equiv (k/\rho\, c)\Delta t/\Delta x^2 \qquad Bi \equiv h\Delta x/k$$

$$T_o^{p+1} = 2\, Fo\left[T_1^p + Bi \cdot T_\infty + \frac{\dot{q}\Delta x^2}{2k}\right] + (1 - 2\, Fo - 2\, Fo \cdot Bi)T_o^p. \tag{3}$$

The stability criterion requires that the coefficient of T_o^p be positive. That is,

$$(1 - 2\, Fo - 2\, Fo \cdot Bi) \geq 0 \qquad \text{or} \qquad Fo \leq 1/2(1 + Bi). \tag{4}$$ <

Implicit Method. Begin as above with an energy balance. In Eq. (2), however, the temperatures on the LHS are evaluated at the *new* (p+1) time. The RHS provides a *backward*-difference approximation to the time derivative.

$$h\left(T_\infty - T_o^{p+1}\right) + k\frac{T_1^{p+1} - T_o^{p+1}}{\Delta x} + \dot{q}\left[\frac{\Delta x}{2}\right] = \rho\, c\left[\frac{\Delta x}{2}\right]\frac{T_o^{p+1} - T_o^p}{\Delta t} \tag{5}$$

$$\left(1 + 2\, Fo(Bi + 1)\right)T_o^{p+1} - 2\, Fo \cdot T_1^{p+1} = T_o^p + 2Bi \cdot Fo \cdot T_\infty + Fo\frac{\dot{q}\Delta x^2}{k}. \tag{6}$$ <

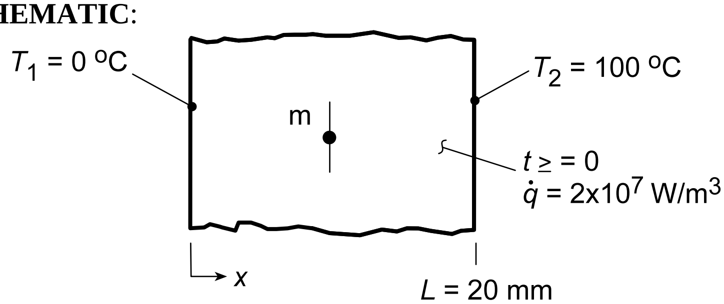
COMMENTS: Compare these results (Eqs. 3, 4 and 6) with the appropriate expression in Table 5.2.

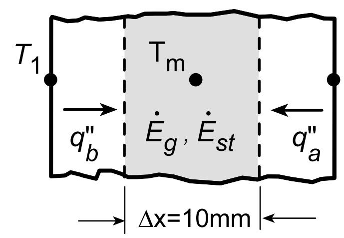
PROBLEM 5.96

KNOWN: Plane wall, initially having a linear, steady-state temperature distribution with boundaries maintained at $T(0,t) = T_1$ and $T(L,t) = T_2$, suddenly experiences a uniform volumetric heat generation due to the electrical current. Boundary conditions T_1 and T_2 remain fixed with time.

FIND: (a) On T-x coordinates, sketch the temperature distributions for the following cases: initial conditions ($t \leq 0$), steady-state conditions ($t \rightarrow \infty$) assuming the maximum temperature exceeds T_2, and two intermediate times; label important features; (b) For the three-nodal network shown, derive the finite-difference equation using either the implicit or explicit method; (c) With a time increment of Δt = 5 s, obtain values of T_m for the first 45s of elapsed time; determine the corresponding heat fluxes at the boundaries; and (d) Determine the effect of mesh size by repeating the foregoing analysis using grids of 5 and 11 nodal points.

SCHEMATIC:

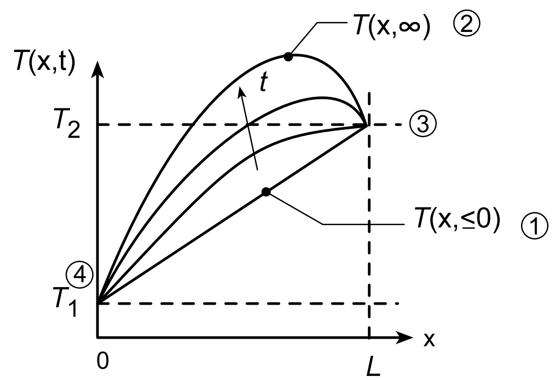




ASSUMPTIONS: (1) Two-dimensional, transient conduction, (2) Uniform volumetric heat generation for $t \geq 0$, (3) Constant properties.

PROPERTIES: Wall (Given): ρ = 4000 kg/m^3, c = 500 J/kg·K, k = 10 W/m·K.

ANALYSIS: (a) The temperature distribution on T-x coordinates for the requested cases are shown below. Note the following key features: (1) linear initial temperature distribution, (2) non-symmetrical parabolic steady-state temperature distribution, (3) gradient at x = L is first positive, then zero and becomes negative, and (4) gradient at x = 0 is always positive.



(b) Performing an energy balance on the control volume about node m above, for unit area, find

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$k(1)\frac{T_2 - T_m}{\Delta x} + k(1)\frac{T_1 - T_m}{\Delta x} + \dot{q}(1)\Delta x = \rho(1)c\Delta x\frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$Fo\left[T_1 + T_2 - 2T_m\right] + \frac{\dot{q}\Delta t}{\rho c_p} = T_m^{p+1} - T_m^p$$

For the T_m term in brackets, use "p" for explicit and "p+1" for implicit form,

Explicit: $$T_m^{p+1} = Fo\left(T_1^p + T_2^p\right) + (1 - 2Fo)T_m^p + \dot{q}\Delta t/\rho c_p \qquad (1)$$ <

Implicit: $$T_m^{p+1} = \left[Fo\left(T_1^{p+1} + T_2^{p+1}\right) + \dot{q}\,\Delta t/\rho c_p + T_m^p\right]\Big/(1 + 2Fo) \qquad (2)$$ <

Continued...

PROBLEM 5.96 (Cont.)

(c) With a time increment $\Delta t = 5s$, the FDEs, Eqs. (1) and (2) become

Explicit: $$T_m^{p+1} = 0.5T_m^p + 75 \qquad (3)$$

Implicit: $$T_m^{p+1} = \left(T_m^p + 75\right)\Big/1.5 \qquad (4)$$

where

$$Fo = \frac{k\Delta t}{\rho c \Delta x^2} = \frac{10\,W/m\cdot K \times 5\,s}{4000\,kg\big/m^3 \times 500\,J/kg\cdot K\left(0.010\,m\right)^2} = 0.25$$

$$\frac{\dot{q}\Delta t}{\rho c} = \frac{2\times 10^7\ W\big/m^3 \times 5\,s}{4000\,kg\big/m^3 \times 500\,J\big/kg\cdot K} = 50\,K$$

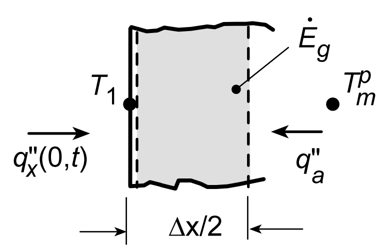
Performing the calculations, the results are tabulated as a function of time,

p	t (s)	T_1 (°C)	T_m (°C) Explicit	T_m (°C) Implicit	T_2 (°C)
0	0	0	50	50	100
1	5	0	100.00	83.33	100
2	10	0	125.00	105.55	100
3	15	0	137.50	120.37	100
4	20	0	143.75	130.25	100
5	25	0	146.88	136.83	100
6	30	0	148.44	141.22	100
7	35	0	149.22	144.15	100
8	40	0	149.61	146.10	100
9	45	0	149.80	147.40	100

<

The heat flux at the boundaries at t = 45s follows from the energy balances on control volumes about the boundary nodes, using the explicit results for T_m^p,

Node 1: $$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$



$$q''_x\left(0,t\right) + k\frac{T_m^p - T_1}{\Delta x} + \dot{q}\left(\Delta x/2\right) = 0$$

$$q''_x\left(0,t\right) = -k\left(T_m^p - T_1\right)\Big/\Delta x - \dot{q}\Delta x/2 \qquad (5)$$

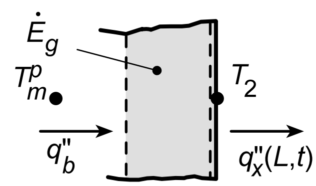
$$q''_x\left(0,45s\right) = -10\,W/m\cdot K\left(149.8-0\right)K\big/0.010\,m - 2\times 10^7\ W\big/m^3 \times 0.010\,m/2$$

$$q''_x\left(0,45s\right) = -149,800\,W\big/m^2 - 100,000\,W\big/m^2 = -249,800\,W\big/m^2$$

<

Node 2: $$k\frac{T_m^p - T_2}{\Delta x} - q''_x\left(L,t\right) + \dot{q}\left(\Delta x/2\right) = 0$$

$$q''_x\left(L,t\right) = k\left(T_m^p - T_2\right)\Big/\Delta x + \dot{q}\Delta x/2 = 0 \qquad (6)$$



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$$q''_x(L,t) = 10\,W/m\cdot K(149.80 - 100)C/0.010\,m + 2\times10^7\,W/m^3 \times 0.010\,m/2$$

$$q''_x(L,t) = 49,800\,W/m^2 + 100,000\,W/m^2 = +149,800\,W/m^2 \quad <$$

(d) To determine the effect of mesh size, the above analysis was repeated using grids of 5 and 11 nodal points, Δx = 5 and 2 mm, respectively. Using the *IHT Finite-Difference Equation Tool*, the finite-difference equations were obtained and solved for the temperature-time history. Eqs. (5) and (6) were used for the heat flux calculations. The results are tabulated below for t = 45s, where T_m^p (45s) is the center node,

Mesh Size Δx (mm)	T_m^p (45s) (°C)	q''_x (0,45s) kW/m^2	q''_x (L,45s) kW/m^2
10	149.8	-249.8	+149.8
5	149.3	-249.0	+149.0
2	149.4	-249.1	+149.0

COMMENTS: (1) The center temperature and boundary heat fluxes are quite insensitive to mesh size for the condition.

(2) The copy of the IHT workspace for the 5 node grid is shown below.

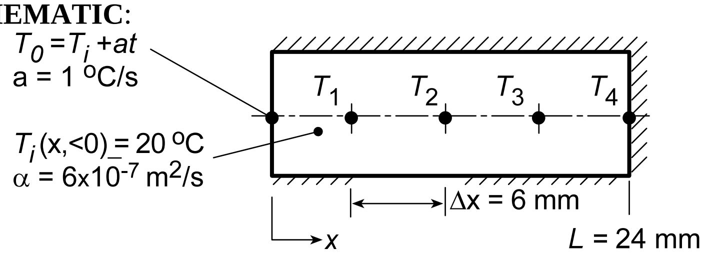
```
// Mesh size - 5 nodes, deltax = 5 mm
// Nodes a, b(m), and c are interior nodes

// Finite-Difference Equations Tool - nodal
equations
/* Node a: interior node; e and w labeled b and
1. */
rho*cp*der(Ta,t) =
fd_1d_int(Ta,Tb,T1,k,qdot,deltax)
/* Node b: interior node; e and w labeled c and
a. */
rho*cp*der(Tb,t) =
fd_1d_int(Tb,Tc,Ta,k,qdot,deltax)
/* Node c: interior node; e and w labeled 2 and
b. */
rho*cp*der(Tc,t) =
fd_1d_int(Tc,T2,Tb,k,qdot,deltax)

// Assigned Variables:
deltax = 0.005
k = 10
rho = 4000
cp = 500
qdot = 2e7
T1 = 0
T2 = 100

/* Initial Conditions:
Tai = 25
Tbi = 50
Tci = 75   */

/* Data Browser Results - Nodal
temperatures at 45s
Ta          Tb            Tc       t
99.5        149.3         149.5    45  */

// Boundary Heat Fluxes - at t = 45s
q''x0 = - k * (Taa - T1 ) / deltax - qdot
* deltax / 2
q''xL = k * (Tcc - T2 ) / deltax + qdot *
deltax /2
//where Taa = Ta (45s), Tcc =
Tc(45s)
Taa = 99.5
Tcc = 149.5
/* Data Browser results
q''x0             q''xL
-2.49E5   1.49E5     */
```

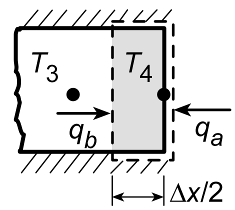
PROBLEM 5.97

KNOWN: Solid cylinder of plastic material ($\alpha = 6 \times 10^{-7}$ m^2/s), initially at uniform temperature of T_i = 20°C, insulated at one end (T_4), while other end experiences heating causing its temperature T_0 to increase linearly with time at a rate of a = 1°C/s.

FIND: (a) Finite-difference equations for the 4 nodes using the explicit method with Fo = 1/2 and (b) Surface temperature T_0 when T_4 = 35°C.

SCHEMATIC:





ASSUMPTIONS: (1) One-dimensional, transient conduction in cylinder, (2) Constant properties, and (3) Lateral and end surfaces perfectly insulated.

ANALYSIS: (a) The finite-difference equations using the *explicit* method for the interior nodes (m = 1, 2, 3) follow from Eq. 5.73 with Fo = 1/2,

$$T_m^{p+1} = Fo\left(T_{m+1}^p + T_{m-1}^p\right) + (1 - 2Fo)T_m^p = 0.5\left(T_{m+1}^p + T_{m-1}^p\right) \tag{1}$$

From an energy balance on the control volume node 4 as shown above yields with Fo = 1/2

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad q_a + q_b + 0 = \rho c V\left(T_4^{p+1} - T_4^p\right)\Big/\Delta t$$

$$0 + k\left(T_3^p - T_4^p\right)\Big/\Delta x = \rho c(\Delta x/2)\left(T_4^{p+1} - T_4^p\right)\Big/\Delta t$$

$$T_4^{p+1} = 2FoT_3^p + (1 - 2Fo)T_4^p = T_3^p \tag{2}$$

(b) Performing the calculations, the temperature-time history is tabulated below, where $T_0 = T_i + a \cdot t$ where a = 1°C/s and $t = p \cdot \Delta t$ with,

$$Fo = \alpha\Delta t\big/\Delta x^2 = 0.5 \qquad \Delta t = 0.5(0.006\,m)^2\Big/6\times10^{-7}\ m^2\big/s = 30s$$

p	t (s)	T_0 (°C)	T_1 (°C)	T_2 (°C)	T_3 (°C)	T_4 (°C)
0	0	20	20	20	20	20
1	30	50	20	20	20	20
2	60	80	35	20	20	20
3	90	110	50	27.5	20	20
4	120	140	68.75	35	23.75	20
5	150	170	87.5	46.25	27.5	23.75
6	180	200	108.1	57.5	35	27.5
7	210	230	-	-	-	35

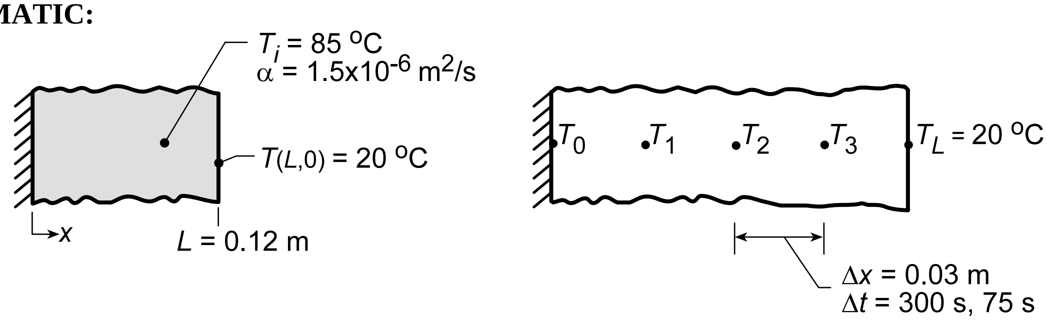
When T_4(210s, p = 7) = 35°C, find T_0(210s) = 230°C. <

PROBLEM 5.98

KNOWN: A 0.12 m thick wall, with thermal diffusivity 1.5×10^{-6} m^2/s, initially at a uniform temperature of 85°C, has one face suddenly lowered to 20°C while the other face is perfectly insulated.

FIND: (a) Using the explicit finite-difference method with space and time increments of Δx = 30 mm and Δt = 300s, determine the temperature distribution within the wall 45 min after the change in surface temperature; (b) Effect of Δt on temperature histories of the surfaces and midplane.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Constant properties.

ANALYSIS: (a) The finite-difference equations for the interior points, nodes 0, 1, 2, and 3, can be determined from Eq. 5.73,

$$T_m^{p+1} = Fo\left(T_{m-1}^p + T_{m+1}^p\right) + (1 - 2\,Fo)T_m^p \tag{1}$$

with

$$Fo = \alpha\Delta t/\Delta x^2 = 1.5\times10^{-6}\,m^2/s \times 300s\big/(0.03\,m)^2 = 1/2. \tag{2}$$

Note that the stability criterion, Eq. 5.74, Fo ≤ 1/2, is satisfied. Hence, combining Eqs. (1) and (2), $T_m^{p+1} = 1/2\left(T_{m-1}^p + T_{m+1}^p\right)$ for m = 0, 1, 2, 3. Since the adiabatic plane at x = 0 can be treated as a symmetry plane, $T_{m-1} = T_{m+1}$ for node 0 (m = 0). The finite-difference solution is generated in the table below using t = p·Δt = 300 p (s) = 5 p (min).

p	t(min)	T_0	T_1	T_2	T_3	T_L(°C)
0	0	85	85	85	85	20
1		85	85	85	52.5	20
2	10	85	85	68.8	52.5	20
3		85	76.9	68.8	44.4	20
4	20	76.9	76.9	60.7	44.4	20
5		76.9	68.8	60.7	40.4	20
6	30	68.8	68.8	54.6	40.4	20
7		68.8	61.7	54.6	37.3	20
8	40	61.7	61.7	49.5	37.3	20
9	45	61.7	55.6	49.5	34.8	20

<

The temperature distribution can also be determined from the Heisler charts. For the wall,

$$Fo = \frac{\alpha t}{L^2} = \frac{1.5\times10^{-6}\,m^2/s \times (45\times60)s}{(0.12\,m)^2} = 0.28 \qquad \text{and} \qquad Bi^{-1} = \frac{k}{hL} = 0.$$

Continued...

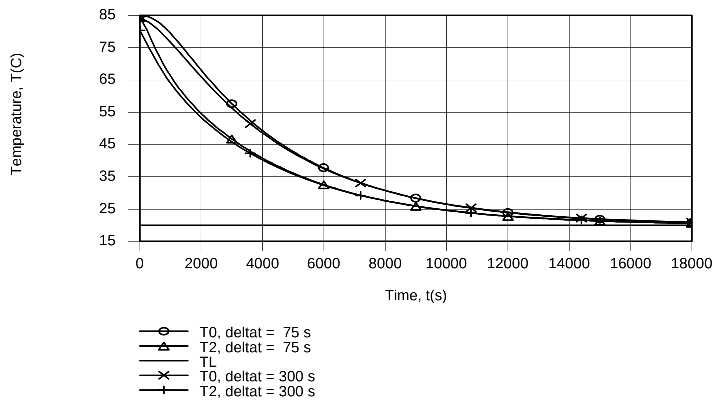
PROBLEM 5.98 (Cont.)

From Figure D.1, for $Bi^{-1} = 0$ and $Fo = 0.28$, find $\theta_o/\theta_i \approx 0.55$. Hence, for x = 0

$$\frac{T_o - T_\infty}{T_i - T_\infty} = \frac{\theta_o}{\theta_i} \quad \text{or} \quad T_o = T(0,t) = T_\infty + \frac{\theta_o}{\theta_i}(T_i - T_\infty) = 20^\circ C + 0.55(85-20)^\circ C = 55.8^\circ C .$$

This value is to be compared with 61.7°C for the finite-difference method.

(b) Using the IHT *Finite-Difference Equation* Tool Pad for *One-Dimensional Transient Conduction*, temperature histories were computed and results are shown for the insulated surface (T0) and the midplane, as well as for the chilled surface (TL).



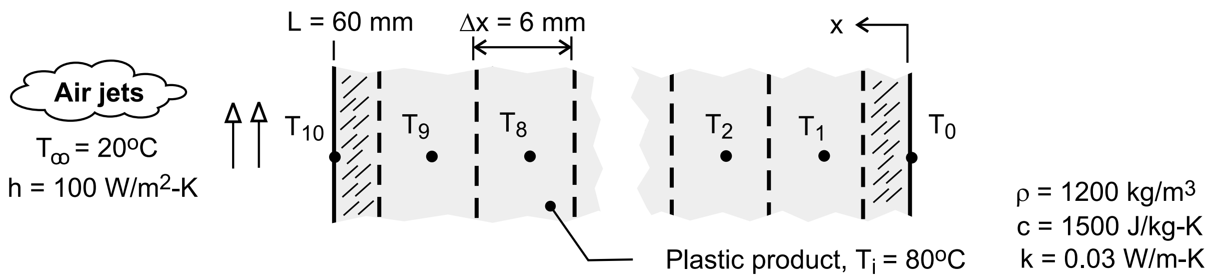
Apart from small differences during early stages of the transient, there is excellent agreement between results obtained for the two time steps. The temperature decay at the insulated surface must, of course, lag that of the midplane.

PROBLEM 5.99

KNOWN: Thickness, initial temperature and thermophysical properties of molded plastic part. Convection conditions at one surface. Other surface insulated.

FIND: Surface temperatures after one hour of cooling.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in product, (2) Negligible radiation, at cooled surface, (3) Negligible heat transfer at insulated surface, (4) Constant properties.

ANALYSIS: Adopting the implicit scheme, the finite-difference equation for the cooled surface node is given by Eq. (5.88), from which it follows that

$$(1+2\,\mathrm{Fo}+2\,\mathrm{FoBi})T_{10}^{p+1}-2\mathrm{Fo}\,T_9^{p+1}=2\,\mathrm{FoBi}\,T_\infty+T_{10}^{p}$$

The general form of the finite-difference equation for any interior node (1 to 9) is given by Eq. (5.89),

$$(1+2\,\mathrm{Fo})T_m^{p+1}-\mathrm{Fo}\left(T_{m-1}^{p+1}+T_{m+1}^{p+1}\right)=T_m^{p}$$

The finite-difference equation for the insulated surface node may be obtained by applying the symmetry requirement to Eq. (5.89); that is, $T_{m+1}^{p}=T_{m-1}^{p}$. Hence,

$$(1+2\,\mathrm{Fo})T_o^{p+1}-2\,\mathrm{Fo}\,T_1^{p+1}=T_o^{p}$$

For the prescribed conditions, Bi = hΔx/k = 100 W/m^2·K (0.006m)/0.30 W/m·K = 2. If the explicit method were used, the most restrictive stability requirement would be given by Eq. (5.79). Hence, for Fo (1+Bi) ≤ 0.5, Fo ≤ 0.167. With Fo = αΔt/Δx^2 and α = k/ρc = 1.67 ×10^{-7} m^2/s, the corresponding restriction on the time increment would be Δt ≤ 36s. Although no such restriction applies for the implicit method, a value of Δt = 30s is chosen, and the set of 11 finite-difference equations is solved using the *Tools* option designated as *Finite-Difference Equations*, *One-Dimensional*, and *Transient* from the IHT Toolpad. At t = 3600s, the solution yields:

$$T_{10}(3600\mathrm{s})=24.1°\mathrm{C} \qquad T_0(3600\mathrm{s})=71.5°\mathrm{C}$$ <

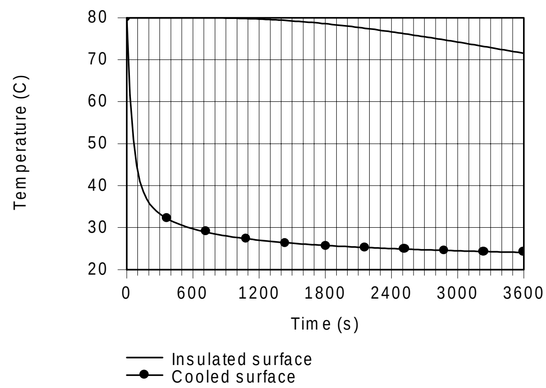
COMMENTS: (1) More accurate results may be obtained from the one-term approximation to the exact solution for one-dimensional, transient conduction in a plane wall. With Bi = hL/k = 20, Table 5.1 yields ζ_1 = 1.496 rad and C_1 = 1.2699. With Fo = αt/L^2 = 0.167, Eq. (5.41) then yields $T_o = T_\infty$ + (T_i - T_∞) C_1 exp $\left(-\zeta_1^2\mathrm{Fo}\right)$ = 72.4°C, and from Eq. (5.40b), $T_s = T_\infty$ + (T_i - T_∞) cos $\left(\zeta_1\right)$ = 24.5°C.

Since the finite-difference results do not change with a reduction in the time step (Δt < 30s), the difference between the numerical and analytical results is attributed to the use of a coarse grid. To improve the accuracy of the numerical results, a smaller value of Δx should be used.

Continued …..

PROBLEM 5.99 (Cont.)

(2) Temperature histories for the front and back surface nodes are as shown.



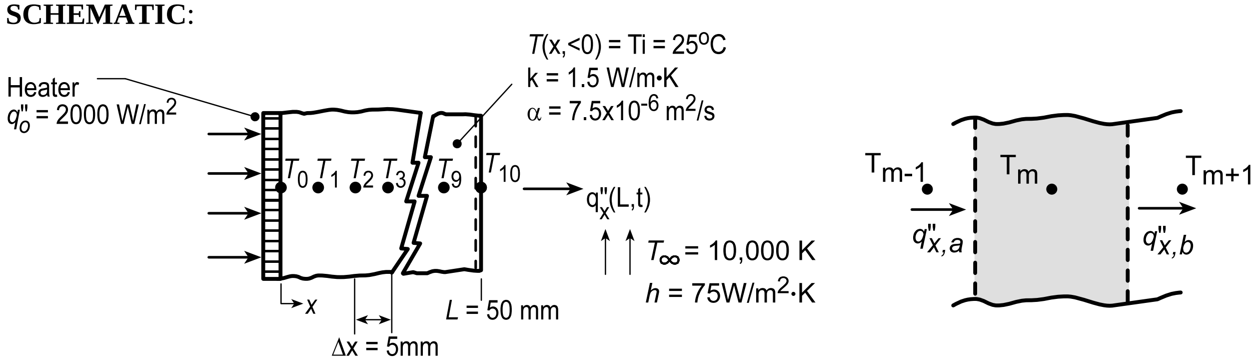
Although the surface temperatures rapidly approaches that of the coolant, there is a significant lag in the thermal response of the back surface. The different responses are attributable to the small value of α and the large value of Bi.

PROBLEM 5.100

KNOWN: Plane wall, initially at a uniform temperature T_i = 25°C, is suddenly exposed to convection with a fluid at T_∞ = 50°C with a convection coefficient h = 75 W/m^2·K at one surface, while the other is exposed to a constant heat flux q''_o = 2000 W/m^2. See also Problem 2.43.

FIND: (a) Using spatial and time increments of Δx = 5 mm and Δt = 20s, compute and plot the temperature distributions in the wall for the initial condition, the steady-state condition, and two intermediate times, (b) On q''_x-x coordinates, plot the heat flux distributions corresponding to the four temperature distributions represented in part (a), and (c) On q''_x-t coordinates, plot the heat flux at x = 0 and x = L.

SCHEMATIC:



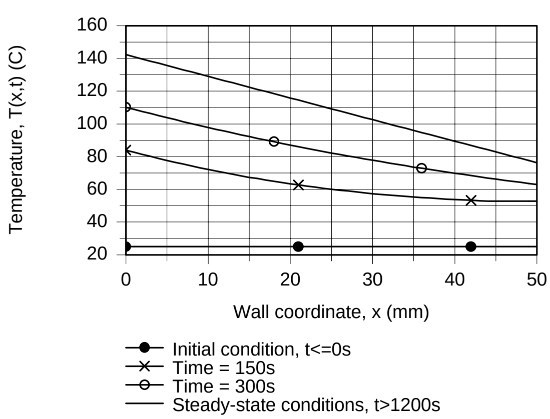
ASSUMPTIONS: (1) One-dimensional, transient conduction and (2) Constant properties.

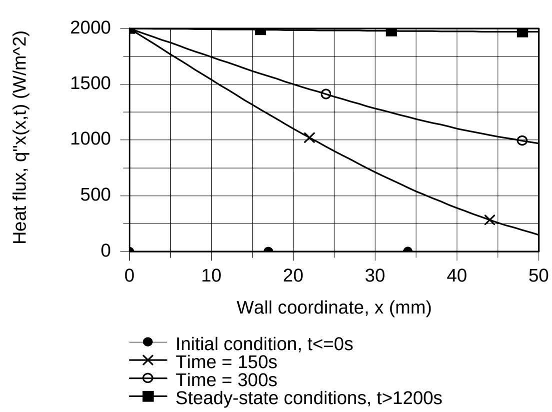
ANALYSIS: (a) Using the *IHT Finite-Difference Equations*, *One-Dimensional*, *Transient Tool*, the equations for determining the temperature distribution were obtained and solved with a spatial increment of Δx = 5 mm. Using the *Lookup Table* functions, the temperature distributions were plotted as shown below.

(b) The heat flux, q''_x (x,t), at each node can be evaluated considering the control volume shown with the schematic above

$$q''_x(m,p) = \left(q''_{x,a} + q''_{x,b}\right)/2 = \left[k(1)\frac{T^p_{m-1} - T^p_m}{\Delta x} + k(1)\frac{T^p_m - T^p_{m+1}}{\Delta x}\right]\Big/2 = k\left(T^p_{m-1} - T^p_{m+1}\right)\Big/2\Delta x$$

From knowledge of the temperature distribution, the heat flux at each node for the selected times is computed and plotted below.

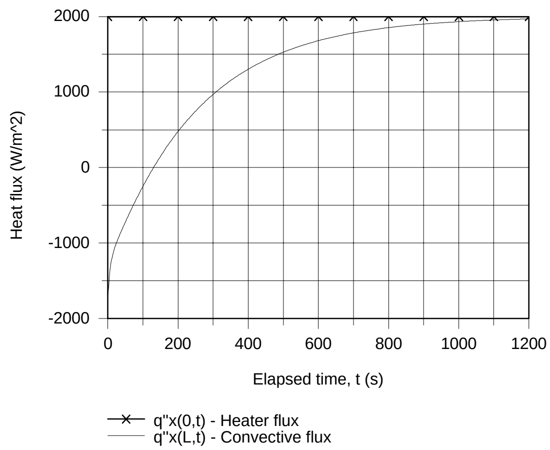




(c) The heat fluxes for the locations x = 0 and x = L, are plotted as a function of time. At the x = 0 surface, the heat flux is constant, q''_o = 2000 W/m^2. At the x = L surface, the heat flux is given by Newton's law of cooling, q''_x(L,t) = h[T(L,t) - T_∞]; at t = 0, q''_x(L,0) = -1875 W/m^2. For steady-state conditions, the heat flux q''_x(x,∞) is everywhere constant at q''_o.

Continued...

PROBLEM 5.100 (Cont.)



Comments: The IHT workspace using the *Finite-Difference Equations Tool* to determine the temperature distributions and heat fluxes is shown below. Some lines of code were omitted to save space on the page.

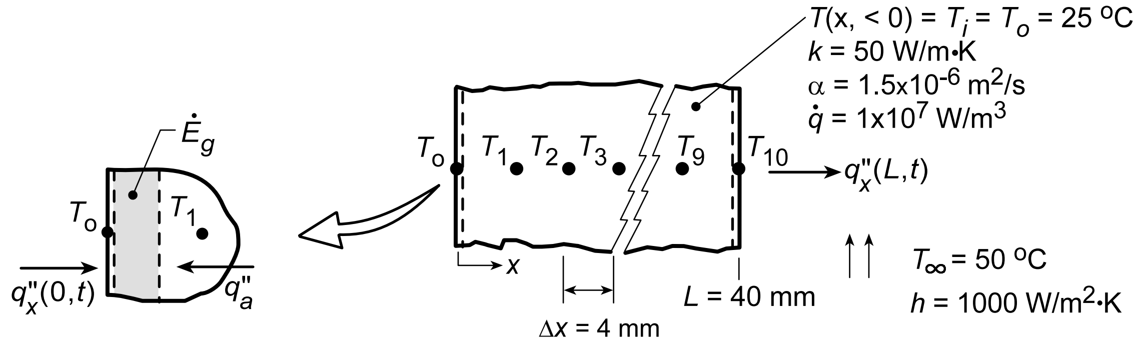
```
// Finite-Difference Equations, One-Dimensional, Transient Tool:
// Node 0 - Applied heater flux
/* Node 0: surface node (w-orientation); transient conditions; e labeled 1. */
rho * cp * der(T0,t) = fd_1d_sur_w(T0,T1,k,qdot,deltax,Tinf0,h0,q''a0)
q''a0 = 2000        // Applied heat flux, W/m^2;
Tinf0 = 25          // Fluid temperature, C; arbitrary value since h0 is zero; no convection process
h0 = 1e-20          // Convection coefficient, W/m^2.K; made zero since no convection process

// Interior Nodes 1 - 9:
/* Node 1: interior node; e and w labeled 2 and 0. */
rho*cp*der(T1,t) = fd_1d_int(T1,T2,T0,k,qdot,deltax)
/* Node 2: interior node; e and w labeled 3 and 1. */
rho*cp*der(T2,t) = fd_1d_int(T2,T3,T1,k,qdot,deltax)
......
......
/* Node 9: interior node; e and w labeled 10 and 8. */
rho*cp*der(T9,t) = fd_1d_int(T9,T10,T8,k,qdot,deltax)

// Node 10 - Convection process:
/* Node 10: surface node (e-orientation); transient conditions; w labeled 9. */
rho * cp * der(T10,t) = fd_1d_sur_e(T10,T9,k,qdot,deltax,Tinf,h,q''a)
q''a = 0        // Applied heat flux, W/m^2; zero flux shown

// Heat Flux Distribution at Interior Nodes, q''m:
q''1 = k / deltax * (T0 - T2) / 2
q''2 = k / deltax * (T1 - T3) / 2
......
......
q''9 = k / deltax * (T8 - T10) / 2

// Heat flux at boundary x= L, q''10
q''xL = h * (T10 - Tinf)

// Assigned Variables:
deltax = 0.005              // Spatial increment, m
k = 1.5                     // thermal conductivity, W/m.K
alpha = 7.5e-6              // Thermal diffusivity, m^2/s
cp = 1000                   // Specific heat, J/kg.K; arbitrary value
alpha = k / (rho * cp)      // Defintion from which rho is calculated
qdot = 0                    // Volumetric heat generation rate, W/m^3
Ti = 25                     // Initial temperature, C; used also for plotting initial distribution
Tinf = 50                   // Fluid temperature, K
h = 75                      // Convection coefficient, W/m^2.K

// Solver Conditions: integrated t from 0 to 1200 with 1 s step, log every 2nd value
```

PROBLEM 5.101

KNOWN: Plane wall, initially at a uniform temperature $T_o = 25°C$, has one surface (x = L) suddenly exposed to a convection process with $T_\infty = 50°C$ and $h = 1000\ W/m^2\cdot K$, while the other surface (x = 0) is maintained at T_o. Also, the wall suddenly experiences uniform volumetric heating with $\dot{q} = 1 \times 10^7\ W/m^3$. See also Problem 2.44.

FIND: (a) Using spatial and time increments of $\Delta x = 4$ mm and $\Delta t = 1s$, compute and plot the temperature distributions in the wall for the initial condition, the steady-state condition, and two intermediate times, and (b) On q''_x-t coordinates, plot the heat flux at x = 0 and x = L. At what elapsed time is there zero heat flux at x = L?

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction and (2) Constant properties.

ANALYSIS: (a) Using the *IHT Finite-Difference Equations, One-Dimensional, Transient Tool*, the temperature distributions were obtained and plotted below.

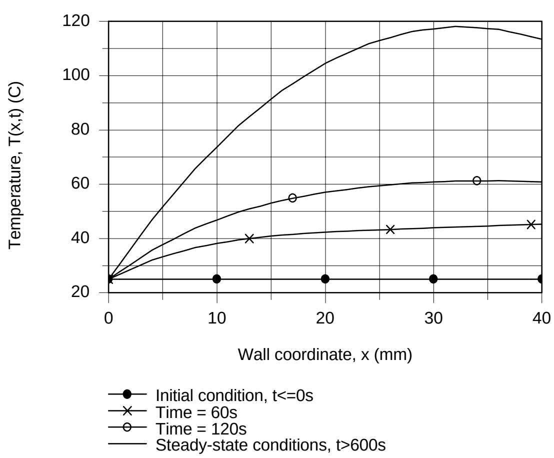
(b) The heat flux, $q''_x(L,t)$, can be expressed in terms of Newton's law of cooling,

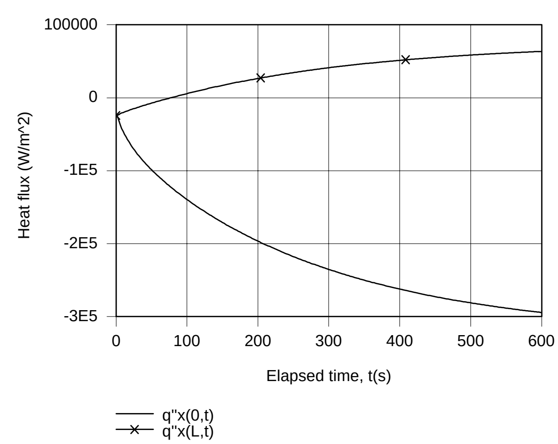
$$q''_x(L,t) = h\left(T_{10}^p - T_\infty\right).$$

From the energy balance on the control volume about node 0 shown above,

$$q''_x(0,t) + \dot{E}_g + q''_a = 0 \qquad\qquad q''_x(0,t) = -\dot{q}(\Delta x/2) - k\left(T_1^p - T_o\right)\Big/\Delta x$$

From knowledge of the temperature distribution, the heat fluxes are computed and plotted.





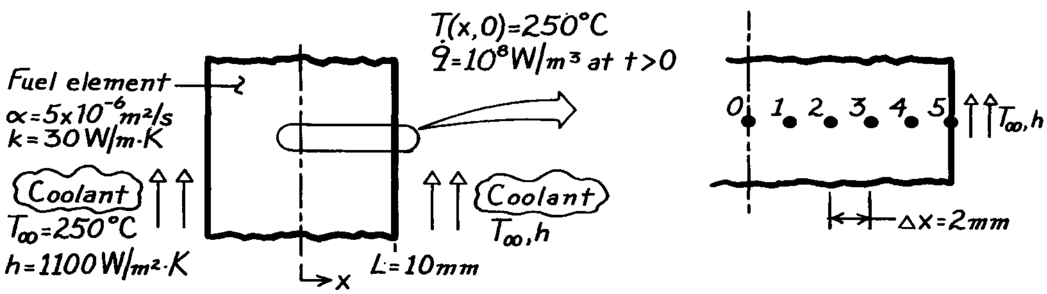
COMMENTS: The steady-state analytical solution has the form of Eq. 3.40 where $C_1 = 6500$ m-1/°C and $C_2 = 25°C$. Find $q''_x(0,\infty) = -3.25\times10^5\ W/m^2$ and $q''_x(L) = +7.5\times10^4\ W/m^2$. Comparing with the graphical results above, we conclude that steady-state conditions are not reached in 600 x.

PROBLEM 5.102

KNOWN: Fuel element of Example 5.8 is initially at a uniform temperature of 250°C with no internal generation; suddenly a uniform generation, $\dot{q} = 10^8 W/m^3$, occurs when the element is inserted into the core while the surfaces experience convection (T_∞,h).

FIND: Temperature distribution 1.5s after element is inserted into the core.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Constant properties, (3) $\dot{q} = 0$, initially; at t > 0, $\dot{q}$ is uniform.

ANALYSIS: As suggested, the explicit method with a space increment of 2mm will be used. Using the nodal network of Example 5.8, the same finite-difference equations may be used.

Interior nodes, m = 1, 2, 3, 4

$$T_m^{p+1} = Fo\left[T_{m-1}^p + T_{m+1}^p + \frac{\dot{q}(\Delta x)^2}{2}\right] + (1 - 2\ Fo)T_m^p. \tag{1}$$

Midplane node, m = 0

Same as Eq. (1), but with $T_{m-1}^p = T_{m+1}^p$.

Surface node, m = 5

$$T_5^{p+1} = 2\ Fo\left[T_4^p + Bi \cdot T_\infty + \frac{\dot{q}(\Delta x)^2}{2k}\right] + (1 - 2Fo - 2Bi \cdot Fo)T_5^p. \tag{2}$$

The most restrictive stability criterion is associated with Eq. (2), Fo(1+Bi) ≤ 1/2. Consider the following parameters:

$$Bi = \frac{h\Delta x}{k} = \frac{1100 W/m^2 \cdot K \times (0.002m)}{30 W/m \cdot K} = 0.0733$$

$$Fo \le \frac{1/2}{(1 + Bi)} = 0.466$$

$$\Delta t \le \frac{Fo(\Delta x)^2}{\alpha} = 0.466\frac{(0.002m)^2}{5 \times 10^{-6} m^2/s} = 0.373s.$$

Continued …..

To be well within the stability limit, select $\Delta t = 0.3s$, which corresponds to

$$Fo = \frac{\alpha \Delta t}{\Delta x^2} = \frac{5\times 10^{-6} m^2/s \times 0.3s}{(0.002m)^2} = 0.375$$
$$t = p\Delta t = 0.3p(s).$$

Substituting numerical values with $\dot{q} = 10^8 W/m^3$, the nodal equations become

$$T_0^{p+1} = 0.375\left[2T_1^p + 10^8 W/m^3 (0.002m)^2 / 30 W/m\cdot K\right] + (1-2\times 0.375)T_0^p$$

$$T_0^{p+1} = 0.375\left[2T_1^p + 13.33\right] + 0.25\ T_0^p \qquad (3)$$

$$T_1^{p+1} = 0.375\left[T_0^p + T_2^p + 13.33\right] + 0.25\ T_1^p \qquad (4)$$

$$T_2^{p+1} = 0.375\left[T_1^p + T_3^p + 13.33\right] + 0.25\ T_2^p \qquad (5)$$

$$T_3^{p+1} = 0.375\left[T_2^p + T_4^p + 13.33\right] + 0.25\ T_3^p \qquad (6)$$

$$T_4^{p+1} = 0.375\left[T_3^p + T_5^p + 13.33\right] + 0.25\ T_4^p \qquad (7)$$

$$T_5^{p+1} = 2\times 0.375\left[T_4^p + 0.0733\times 250 + \frac{13.33}{2}\right] + (1-2\times 0.375 - 2\times 0.0733\times 0.375)T_5^p$$

$$T_5^{p+1} = 0.750\ \left[T_4^p + 24.99\right] + 0.195\ T_5^p. \qquad (8)$$

The initial temperature distribution is $T_i = 250°C$ at all nodes. The marching solution, following the procedure of Example 5.8, is represented in the table below.

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5(°C)
0	0	250	250	250	250	250	250
1	0.3	255.00	255.00	255.00	255.00	255.00	254.99
2	0.6	260.00	260.00	260.00	260.00	260.00	259.72
3	0.9	265.00	265.00	265.00	265.00	264.89	264.39
4	1.2	270.00	270.00	270.00	269.96	269.74	268.97
5	1.5	275.00	275.00	274.98	274.89	274.53	273.50 <

The desired temperature distribution T(x, 1.5s), corresponds to p = 5.

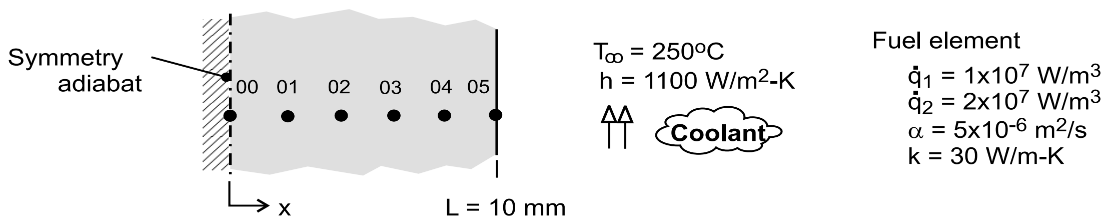
COMMENTS: Note that the nodes near the midplane (0,1) do not feel any effect of the coolant during the first 1.5s time period.

PROBLEM 5.103

KNOWN: Conditions associated with heat generation in a rectangular fuel element with surface cooling. See Example 5.8.

FIND: (a) The temperature distribution 1.5 s after the change in operating power; compare your results with those tabulated in the example, (b) Calculate and plot temperature histories at the mid-plane (00) and surface (05) nodes for $0 \le t \le 400$ s; determine the new steady-state temperatures, and approximately how long it will take to reach the new steady-state condition after the step change in operating power. Use the IHT *Tools | Finite-Difference Equations | One-Dimensional | Transient* conduction model builder as your solution tool.

SCHEMATIC:



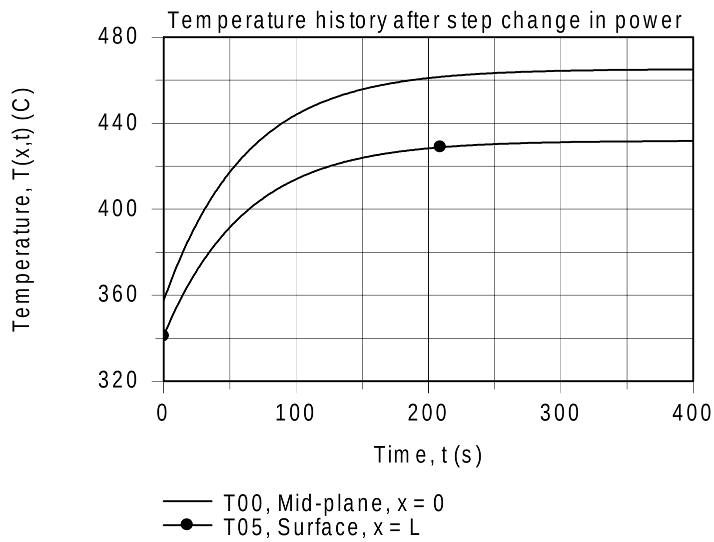
ASSUMPTIONS: (1) One dimensional conduction in the x-direction, (2) Uniform generation, and (3) Constant properties.

ANALYIS: The IHT model builder provides the transient finite-difference equations for the implicit method of solution. Selected portions of the IHT code used to obtain the results tabulated below are shown in the Comments.

(a) Using the IHT code, the temperature distribution (°C) as a function of time (s) up to 1.5 s after the step power change is obtained from the summarized results copied into the workspace

	t	T00	T01	T02	T03	T04	T05
1	0	357.6	356.9	354.9	351.6	346.9	340.9
2	0.3	358.1	357.4	355.4	352.1	347.4	341.4
3	0.6	358.6	357.9	355.9	352.6	347.9	341.9
4	0.9	359.1	358.4	356.4	353.1	348.4	342.3
5	1.2	359.6	358.9	356.9	353.6	348.9	342.8
6	1.5	360.1	359.4	357.4	354.1	349.3	343.2

(b) Using the code, the mid-plane (00) and surface (05) node temperatures are plotted as a function of time.



Continued …..

PROBLEM 5.103 (Cont.)

Note that at t ≈ 240 s, the wall has nearly reached the new steady-state condition for which the nodal temperatures (°C) were found as:

T00	T01	T02	T03	T04	T05
465	463.7	459.7	453	443.7	431.7

COMMENTS: (1) Can you validate the new steady-state nodal temperatures from part (b) by comparison against an analytical solution?

(2) Will using a smaller time increment improve the accuracy of the results? Use your code with Δt = 0.15 s to justify your explanation.

(3) Selected portions of the IHT code to obtain the nodal temperature distribution using spatial and time increments of Δx = 2 mm and Δt = 0.3 s, respectively, are shown below. For the solve-integration step, the initial condition for each of the nodes corresponds to the steady-state temperature distribution with $\dot{q}_1$.

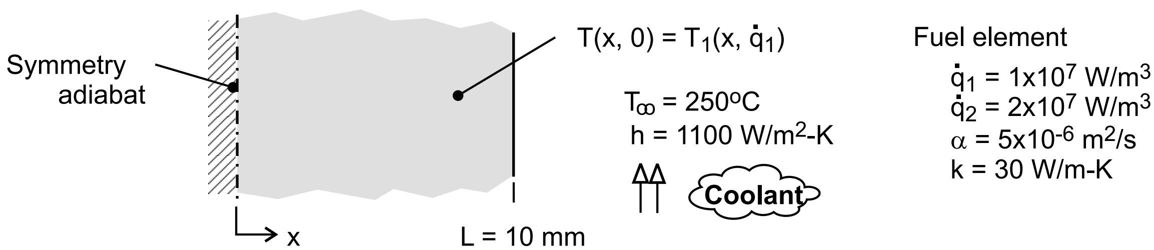
```
// Tools | Finite-Difference Equations | One-Dimensional | Transient
/* Node 00: surface node (w-orientation); transient conditions; e labeled 01. */
rho * cp * der(T00,t) = fd_1d_sur_w(T00,T01,k,qdot,deltax,Tinf01,h01,q''a00)
q''a00 = 0          // Applied heat flux, W/m^2; zero flux shown
Tinf01 = 20          // Arbitrary value
h01 = 1e-8        // Causes boundary to behave as adiabatic
/* Node 01: interior node; e and w labeled 02 and 00. */
rho*cp*der(T01,t) = fd_1d_int(T01,T02,T00,k,qdot,deltax)
/* Node 02: interior node; e and w labeled 03 and 01. */
rho*cp*der(T02,t) = fd_1d_int(T02,T03,T01,k,qdot,deltax)
/* Node 03: interior node; e and w labeled 04 and 02. */
rho*cp*der(T03,t) = fd_1d_int(T03,T04,T02,k,qdot,deltax)
/* Node 04: interior node; e and w labeled 05 and 03. */
rho*cp*der(T04,t) = fd_1d_int(T04,T05,T03,k,qdot,deltax)
/* Node 05: surface node (e-orientation); transient conditions; w labeled 04. */
rho * cp * der(T05,t) = fd_1d_sur_e(T05,T04,k,qdot,deltax,Tinf05,h05,q''a05)
q''a05 = 0            // Applied heat flux, W/m^2; zero flux shown
Tinf05 = 250        // Coolant temperature, C
h05 = 1100          // Convection coefficient, W/m^2.K

// Input parameters
qdot = 2e7           // Volumetric rate, W/m^3, step change
deltax = 0.002      // Space increment
k = 30                // Thermophysical properties
alpha = 5e-6
rho = 1000
alpha = k / (rho * cp)

/* Steady-state conditions,  with qdot1 = 1e7 W/m^3; initial conditions for step change
T_x = 16.67 * (1 - x^2/L^2) + 340.91              // See text
Seek T_x for x = 0, 2, 4, 6, 8, 10 mm; results used for Ti are
Node    T_x
00       357.6
01       356.9
02       354.9
03       351.6
04       346.9
05       340.9         */
```

PROBLEM 5.104

KNOWN: Conditions associated with heat generation in a rectangular fuel element with surface cooling. See Example 5.8.

FIND: (a) The temperature distribution 1.5 s after the change in the operating power; compare results with those tabulated in the Example, and (b) Plot the temperature histories at the midplane, x = 0, and the surface, x = L, for $0 \le t \le 400$ s; determine the new steady-state temperatures, and approximately how long it takes to reach this condition. Use the finite-element software *FEHT* as your solution tool.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the x-direction, (2) Uniform generation, (3) Constant properties.

ANALYSIS: Using *FEHT*, an outline of the fuel element is drawn of thickness 10 mm in the x-direction and arbitrary length in the y-direction. The boundary conditions are specified as follows: on the y-planes and the x = 0 plane, treat as adiabatic; on the x = 10 mm plane, specify the convection option. Specify the material properties and the internal generation with $\dot{q}_1$. In the *Setup* menu, click on *Steady-state*, and then *Run* to obtain the temperature distribution corresponding to the initial temperature distribution, $T_i(x,0) = T(x,\dot{q}_1)$, before the change in operating power to $\dot{q}_2$.

Next, in the *Setup* menu, click on *Transient*; in the *Specify | Internal Generation* box, change the value to $\dot{q}_2$; and in the *Run* command, click on *Continue* (not *Calculate*).

(a) The temperature distribution 1.5 s after the change in operating power from the FEHT analysis and from the FDE analysis in the Example are tabulated below.

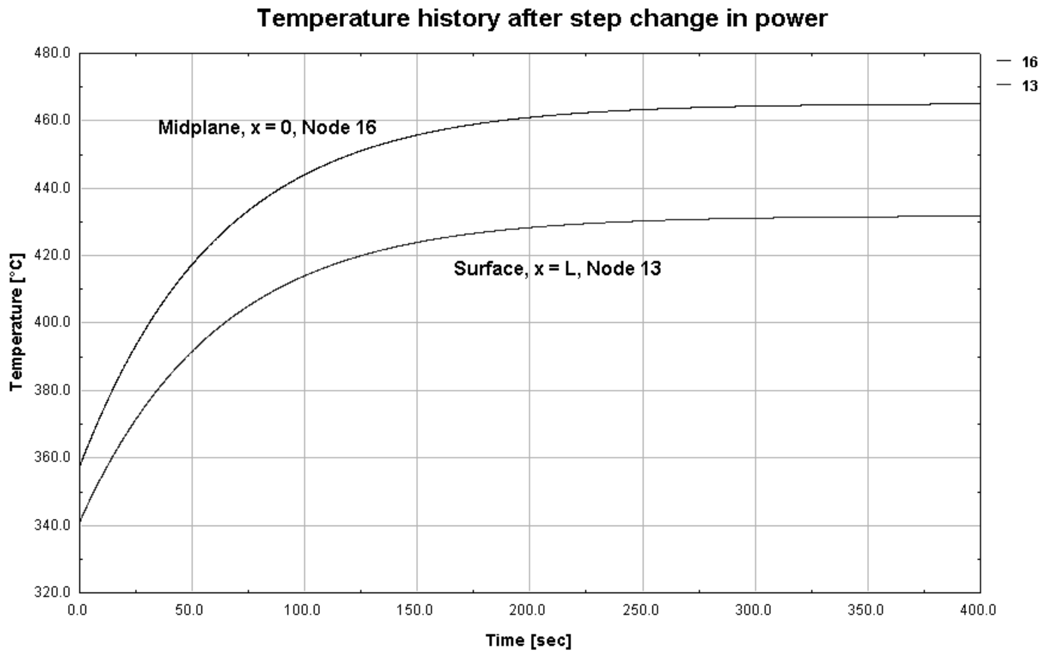
x/L	0	0.2	0.4	0.6	0.8	1.0
T(x/L, 1.5 s)						
FEHT (°C)	360.1	359.4	357.4	354.1	349.3	343.2
FDE (°C)	360.08	359.41	357.41	354.07	349.37	343.27

The mesh spacing for the FEHT analysis was 0.5 mm and the time increment was 0.005 s. For the FDE analyses, the spatial and time increments were 2 mm and 0.3 s. The agreement between the results from the two numerical methods is within 0.1°C.

(b) Using the FEHT code, the temperature histories at the mid-plane (x = 0) and the surface (x = L) are plotted as a function of time.

Continued …..

PROBLEM 5.104 (Cont.)



From the distribution, the steady-state condition (based upon 98% change) is approached in 215 s. The steady-state temperature distributions after the step change in power from the FEHT and FDE analysis in the Example are tabulated below. The agreement between the results from the two numerical methods is within 0.1°C

x/L	0	0.2	0.4	0.6	0.8	1.0
$T(x/L, \infty)$						
FEHT (°C)	465.0	463.7	459.6	453.0	443.6	431.7
FDE (°C)	465.15	463.82	459.82	453.15	443.82	431.82

COMMENTS: (1) For background information on the *Continue* option, see the *Run* menu in the *FEHT Help* section. Using the *Run/Calculate* command, the steady-state temperature distribution was determined for the $\dot{q}_1$ operating power. Using the *Run*|*Continue* command (after re-setting the generation to $\dot{q}_2$ and clicking on *Setup* | *Transient*), this steady-state distribution automatically becomes the initial temperature distribution for the $\dot{q}_2$ operating power. This feature allows for conveniently prescribing a non-uniform initial temperature distribution for a transient analysis (rather than specifying values on a node-by-node basis).

(2) Use the *View* | *Tabular Output* command to obtain nodal temperatures to the maximum number of significant figures resulting from the analysis.

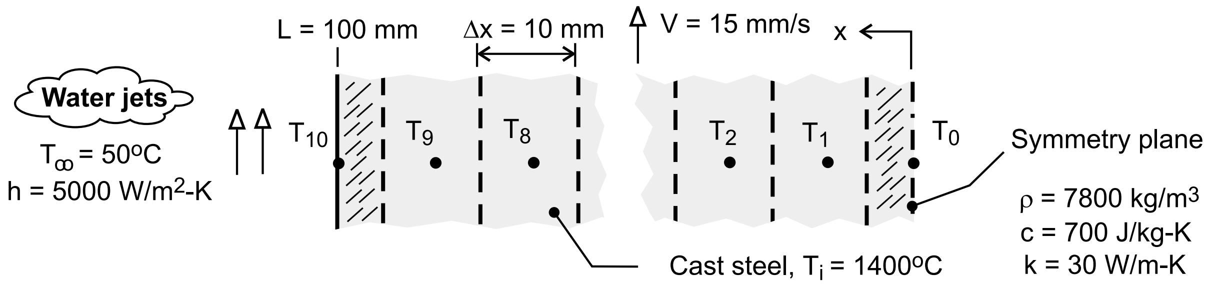
(3) Can you validate the new steady-state nodal temperatures from part (b) (with $\dot{q}_2,\ t \to \infty$) by comparison against an analytical solution?

PROBLEM 5.105

KNOWN: Thickness, initial temperature, speed and thermophysical properties of steel in a thin-slab continuous casting process. Surface convection conditions.

FIND: Time required to cool the outer surface to a prescribed temperature. Corresponding value of the midplane temperature and length of cooling section.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible radiation at quenched surfaces, (3) Symmetry about the midplane, (4) Constant properties.

ANALYSIS: Adopting the implicit scheme, the finite-difference equaiton for the cooled surface node is given by Eq. (5.88), from which it follows that

$$(1+2\,\mathrm{Fo}+2\,\mathrm{FoBi})T_{10}^{p+1}-2\,\mathrm{Fo}\,T_9^{p+1}=2\,\mathrm{FoBi}\,T_\infty+T_{10}^{p}$$

The general form of the finite-difference equation for any interior node (1 to 9) is given by Eq. (5.89),

$$(1+2\,\mathrm{Fo})T_m^{p+1}-\mathrm{Fo}\left(T_{m-1}^{p+1}+T_{m+1}^{p+1}\right)=T_m^p$$

The finite-difference equation for the midplane node may be obtained by applying the symmetry requirement to Eq. (5.89); that is, $T_{m+1}^p=T_{m-1}^p$. Hence,

$$(1+2\,\mathrm{Fo})T_0^{p+1}-2\,\mathrm{Fo}\,T_1^{p+1}=T_0^p$$

For the prescribed conditions, Bi = $h\Delta x/k$ = 5000 W/m^2·K (0.010m)/30 W/m·K = 1.67. If the explicit method were used, the stability requirement would be given by Eq. (5.79). Hence, for Fo(1 + Bi) ≤ 0.5, Fo ≤ 0.187. With Fo = $\alpha\Delta t/\Delta x^2$ and $\alpha = k/\rho c = 5.49\times10^{-6}$ m^2/s, the corresponding restriction on the time increment would be $\Delta t \le 3.40$s. Although no such restriction applies for the implicit method, a value of Δt = 1s is chosen, and the set of 11 finite-difference equations is solved using the *Tools* option designated as *Finite-Difference Equations*, *One-Dimensional* and *Transient* from the IHT Toolpad. For T_{10} (t) = 300°C, the solution yields

$$t = 161\text{s}$$ <

Continued …..

PROBLEM 5.105 (Cont.)

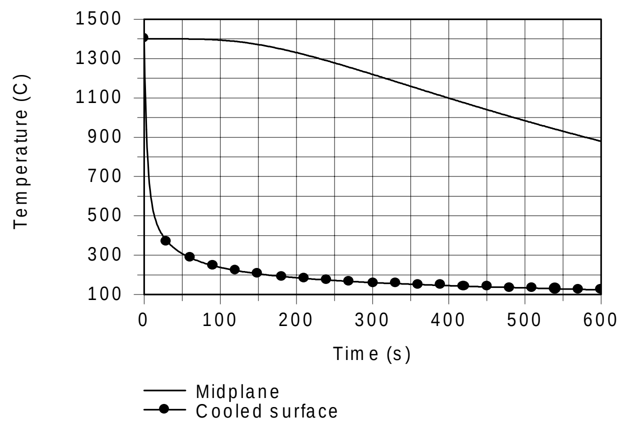
$$T_0(t) = 1364°C \qquad <$$

With a casting speed of V = 15 mm/s, the length of the cooling section is

$$L_{cs} = Vt = 0.015\,m/s(161s) = 2.42m \qquad <$$

COMMENTS: (1) With Fo = $\alpha t/L^2$ = 0.088 < 0.2, the one-term approximation to the exact solution for one-dimensional conduction in a plane wall cannot be used to confirm the foregoing results. However, using the exact solution from the *Models, Transient Conduction*, *Plane Wall* Option of IHT, values of T_0 = 1366°C and T_s = 200.7°C are obtained and are in good agreement with the finite-difference predictions. The accuracy of these predictions could still be improved by reducing the value of Δx.

(2) Temperature histories for the surface and midplane nodes are plotted for 0 < t < 600s.



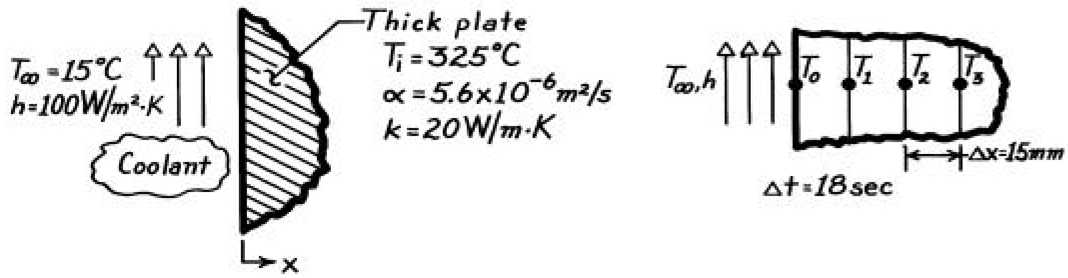
While T_{10} (600s) = 124°C, T_o (600s) has only dropped to 879°C. The much slower thermal response at the midplane is attributable to the small value of α and the large value of Bi = 16.67.

PROBLEM 5.106

KNOWN: Very thick plate, initially at a uniform temperature, T_i, is suddenly exposed to a convection cooling process (T_∞,h).

FIND: Temperatures at the surface and a 45mm depth after 3 minutes using finite-difference method with space and time increments of 15mm and 18s.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Plate approximates semi-infinite medium, (3) Constant properties.

ANALYSIS: The grid network representing the plate is shown above. The finite-difference equation for node 0 is given by Eq. 5.82 for one-dimensional conditions or Eq. 5.77,

$$T_0^{p+1} = 2\ \mathrm{Fo}\left(T_1^p + \mathrm{Bi} \cdot T_\infty\right) + \left(1 - 2\ \mathrm{Fo} - 2\ \mathrm{Bi} \cdot \mathrm{Fo}\right) T_0^p. \tag{1}$$

The numerical values of Fo and Bi are

$$\mathrm{Fo} = \frac{a\Delta t}{\Delta x^2} = \frac{5.6\times10^{-6}\,\mathrm{m}^2/\mathrm{s}\times 18\mathrm{s}}{(0.015\mathrm{m})^2} = 0.448$$

$$\mathrm{Bi} = \frac{h\Delta x}{k} = \frac{100\ \mathrm{W/m^2 \cdot K} \times \left(15\times10^{-3}\mathrm{m}\right)}{20\ \mathrm{W/m \cdot K}} = 0.075.$$

Recognizing that T_∞ = 15°C, Eq. (1) has the form

$$T_0^{p+1} = 0.0359\ T_0^p + 0.897\ T_1^p + 1.01. \tag{2}$$

It is important to satisfy the stability criterion, Fo (1+Bi) ≤ 1/2. Substituting values, 0.448 (1+0.075) = 0.482 ≤ 1/2, and the criterion is satisfied.

The finite-difference equation for the interior nodes, m = 1, 2…, follows from Eq. 5.73,

$$T_m^{p+1} = \mathrm{Fo}\left(T_{m+1}^p + T_{m-1}^p\right) + (1 - 2\mathrm{Fo})\ T_m^p. \tag{3}$$

Recognizing that the stability criterion, Fo ≤ 1/2, is satisfied with Fo = 0.448,

$$T_m^{p+1} = 0.448\left(T_{m+1}^p + T_{m-1}^p\right) + 0.104 T_m^p. \tag{4}$$

Continued …..

PROBLEM 5.106 (Cont.)

The time scale is related to p, the number of steps in the calculation procedure, and Δt, the time increment,

$$t = p\Delta t. \tag{5}$$

The finite-difference calculations can now be performed using Eqs. (2) and (4). The results are tabulated below.

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7(K)
0	0	325	325	325	325	325	325	325	325
1	18	304.2	324.7	325	325	325	325	325	325
2	36	303.2	315.3	324.5	325	325	325	325	325
3	54	294.7	313.7	320.3	324.5	325	325	325	325
4	72	293.0	307.8	318.9	322.5	324.5	325	325	325
5	90	287.6	305.8	315.2	321.5	323.5	324.5	325	325
6	108	285.6	301.6	313.5	319.3	322.7	324.0	324.5	325
7	126	281.8	299.5	310.5	317.9	321.4	323.3	324.2	
8	144	279.8	296.2	308.6	315.8	320.4	322.5		
9	162	276.7	294.1	306.0	314.3	319.0			
10	180	274.8	291.3	304.1	312.4				

Hence, find

$$T(0,\ 180\text{s}) = T_0^{10} = 275^\circ\text{C} \qquad T(45\text{mm},\ 180\text{s}) = T_3^{10} = 312^\circ\text{C}.$$ <

COMMENTS: (1) The above results can be readily checked against the analytical solution represented in Fig. 5.8 (see also Eq. 5.60). For x = 0 and t = 180s, find

$$\frac{x}{2(a\ t)^{1/2}} = 0$$

$$\frac{h(a\ t)^{1/2}}{k} = \frac{100\ \text{W/m}^2\cdot\text{K}\left(5.60\times10^{-6}\text{m}^2/\text{s}\times180\text{s}\right)^{1/2}}{20\ \text{W/m}\cdot\text{K}} = 0.16$$

for which the figure gives

$$\frac{T - T_i}{T_\infty - T_i} = 0.15$$

so that,

$$T(0,\ 180\text{s}) = 0.15(T_\infty - T_i) + T_i = 0.15(15 - 325)^\circ\text{C} + 325^\circ\text{C}$$

$$T(0,\ 180\text{s}) = 278^\circ\text{C}.$$

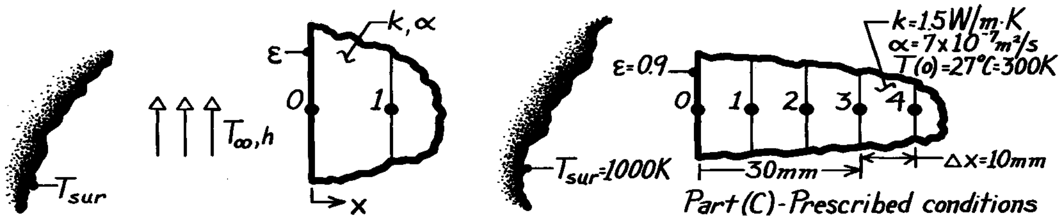
For x = 45mm, the procedure yields T(45mm, 180s) = 316°C. The agreement with the numerical solution is nearly within 1%.

PROBLEM 5.107

KNOWN: Sudden exposure of the surface of a thick slab, initially at a uniform temperature, to convection and to surroundings at a high temperature.

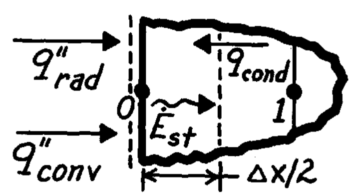
FIND: (a) Explicit, finite-difference equation for the surface node in terms of Fo, Bi, Bi_r, (b) Stability criterion; whether it is more restrictive than that for an interior node and does it change with time, and (c) Temperature at the surface and at 30mm depth for prescribed conditions after 1 minute exposure.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction, (2) Thick slab may be approximated as semi-infinite medium, (3) Constant properties, (4) Radiation exchange is between small surface and large surroundings.

ANALYSIS: (a) The explicit form of the FDE for the surface node may be obtained by applying an energy balance to a control volume about the node.



$$\dot{E}''_{in} - \dot{E}''_{out} = q''_{conv} + q''_{rad} + q''_{cond} = \dot{E}''_{st}$$

$$h\left(T_\infty - T_o^p\right) + h_r\left(T_{sur} - T_o^p\right) + k \cdot 1 \cdot \frac{T_1^p - T_o^p}{\Delta x}$$

$$= \rho\, c\left[\frac{\Delta x}{2} \cdot 1\right]\frac{T_o^{p+1} - T_o^p}{\Delta t} \qquad (1)$$

where the radiation process has been linearized, Eq. 1.8. (See also Comment 4, Example 5.9),

$$h_r = h_r^p\left(T_o^p,\ T_{sur}\right) = \varepsilon\sigma\left(T_o^p + T_{sur}\right)\left(\left[T_0^p\right]^2 + T_{sur}^2\right). \qquad (2)$$

Divide Eq. (1) by $\rho c\Delta x/2\Delta t$ and regroup using these definitions to obtain the FDE:

$$Fo \equiv (k/\rho\ c)\Delta t/\Delta x^2 \qquad Bi \equiv h\Delta x/k \qquad Bi_r \equiv h_r\Delta x/k \qquad (3,4,5)$$

$$T_o^{p+1} = 2Fo\left(Bi \cdot T_\infty + Bi_r \cdot T_{sur} + T_1^p\right) + \left(1 - 2\ Bi \cdot Fo - 2Bi_r \cdot Fo - 2Fo\right)T_o^p. \qquad (6) <$$

(b) The stability criterion for Eq. (6) requires that the coefficient of T_o^p be positive.

$$1 - 2Fo\left(Bi + Bi_r + 1\right) \geq 0 \qquad \text{or} \qquad Fo \leq 1/2\left(Bi + Bi_r + 1\right). \qquad (7) <$$

The stability criterion for an interior node, Eq. 5.74, is Fo ≤ 1/2. Since Bi + Bi_r > 0, the stability criterion of the surface node is more restrictive. Note that Bi_r is not constant but depends upon h_r which increases with increasing T_o^p (time). Hence, the restriction on Fo increases with increasing T_o^p (time).

Continued …..

PROBLEM 5.107 (Cont.)

(c) Consider the prescribed conditions with negligible convection (Bi = 0). The FDEs for the thick slab are:

Surface (0) $$T_o^{p+1} = 2Fo\left(Bi \cdot Fo + Bi_r \cdot T_{sur} + T_1^p\right) + \left(1 - 2Bi \cdot Fo - 2Bi_r \cdot Fo - 2Fo\right)T_o^p \quad (8)$$

Interior (m≥1) $$T_m^{p+1} = Fo\left(T_{m+1}^p + T_{m-1}^p\right) + \left(1 - 2Fo\right)T_m^p \quad (9,5,7,3)$$

The stability criterion from Eq. (7) with Bi = 0 is,

$$Fo \le 1/2\left(1 + Bi_r\right) \quad (10)$$

To proceed with the explicit, marching solution, we need to select a value of Δt (Fo) that will satisfy the stability criterion. A few trial calculations are helpful. A value of Δt = 15s provides Fo = 0.105, and using Eqs. (2) and (5), h_r(300K, 1000K) = 72.3 W/m^2·K and Bi_r = 0.482. From the stability criterion, Eq. (10), find Fo ≤ 0.337. With increasing T_o^p, h_r and Bi_r increase: h_r(800K, 1000K) = 150.6 W/m^2·K, Bi_r = 1.004 and Fo ≤ 0.249. Hence, if $T_o^p < 800K, \Delta t = 15s \text{ or } Fo = 0.105$ satisfies the stability criterion.

Using Δt = 15s or Fo = 0.105 with the FDEs, Eqs. (8) and (9), the results of the solution are tabulated below. Note how h_r^p and Bi_r^p are evaluated at each time increment. Note that t = p·Δt, where Δt = 15s.

p	t(s)	T_o / h_r^p / Bi_r	T1(K)	T_2	T_3	T_4	….
0	0	300 72.3 0.482	300	300	300	300	
1	15	370.867 79.577 0.5305	300	300	300	300	
2	30	426.079 85.984 0.5733	307.441	300	300	300	
3	45	470.256 91.619 0.6108	319.117	300.781	300	300	
4	60	502.289	333.061	302.624	300.082	300	

After 60s(p = 4), T_o(0, 1 min) = 502.3K and T_3(30mm, 1 min) = 300.1K. <

COMMENTS: (1) The form of the FDE representing the surface node agrees with Eq. 5.82 if this equation is reduced to one-dimension.

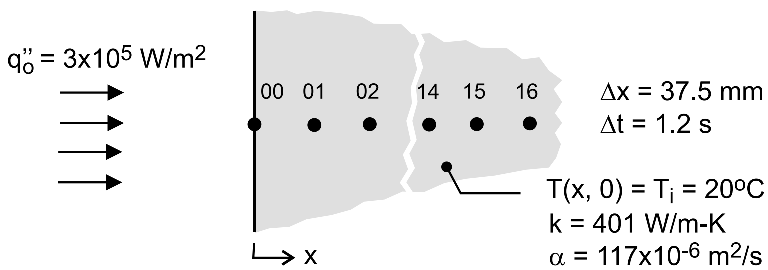
(2) We should recognize that the Δt = 15s time increment represents a coarse step. To improve the accuracy of the solution, a smaller Δt should be chosen.

PROBLEM 5.108

KNOWN: Thick slab of copper, initially at a uniform temperature, is suddenly exposed to a constant net radiant flux at one surface. See Example 5.9.

FIND: (a) The nodal temperatures at nodes 00 and 04 at t = 120 s; that is, T00(0, 120 s) and T04(0.15 m, 120 s); compare results with those given by the exact solution in Comment 1; will a time increment of 0.12 s provide more accurate results?; and, (b) Plot the temperature histories for x = 0, 150 and 600 mm, and explain key features of your results. Use the *IHT Tools | Finite-Difference Equations | One-Dimensional | Transient* conduction model builder to obtain the implicit form of the FDEs for the interior nodes. Use space and time increments of 37.5 mm and 1.2 s, respectively, for a 17-node network. For the surface node 00, use the FDE derived in Section 2 of the Example.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the x-direction, (2) Slab of thickness 600 mm approximates a semi-infinite medium, and (3) Constant properties.

ANALYSIS: The IHT model builder provides the implicit-method FDEs for the interior nodes, 01 – 15. The +x boundary condition for the node-16 control volume is assumed adiabatic. The FDE for the surface node 00 exposed to the net radiant flux was derived in the Example analysis. Selected portions of the IHT code used to obtain the following results are shown in the Comments.

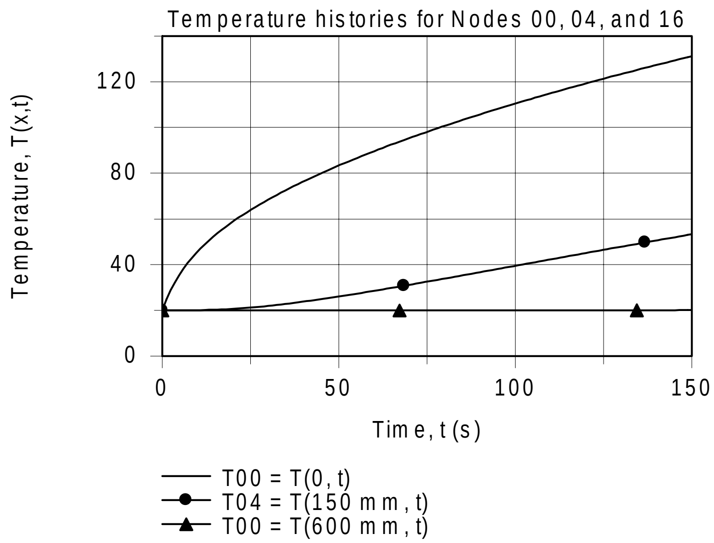
(a) The 00 and 04 nodal temperatures for t = 120 s are tabulated below using a time increment of Δt = 1.2 s and 0.12 s, and compared with the results given from the exact analytical solution, Eq. 5.59.

Node	FDE results (°C)		Analytical result (°C)
	Δt = 1.2 s	Δt = 0.12 s	Eq. 5.59
00	119.3	119.4	120.0
04	45.09	45.10	45.4

The numerical FDE-based results with the different time increments agree quite closely with one another. At the surface, the numerical results are nearly 1 °C less than the result from the exact analytical solution. This difference represents an error of -1% (-1 °C / (120 – 20) °C x 100). At the x = 150 mm location, the difference is about -0.4 °C, representing an error of –1.5%. For this situation, the smaller time increment (0.12 s) did not provide improved accuracy. To improve the accuracy of the numerical model, it would be necessary to reduce the space increment, in addition to using the smaller time increment.

(b) The temperature histories for x = 0, 150 and 600 mm (nodes 00, 04, and 16) for the range 0 ≤ t ≤ 150 s are as follows.

Continued …..



As expected, the surface temperature, T00 = T(0,t), increases markedly at early times. As thermal penetration increases with increasing time, the temperature at the location x = 150 mm, T04 = T(150 mm, t), begins to increase after about 20 s. Note, however, the temperature at the location x = 600 mm, T16 = T(600 mm, t), does not change significantly within the 150 s duration of the applied surface heat flux. Our assumption of treating the +x boundary of the node 16 control volume as adiabatic is justified. A copper plate of 600-mm thickness is a good approximation to a semi-infinite medium at times less than 150 s.

COMMENTS: Selected portions of the *IHT* code with the nodal equations to obtain the temperature distribution are shown below. Note how the FDE for node 00 is written in terms of an energy balance using the *der (T,t)* function. The FDE for node 16 assumes that the "east" boundary is adiabatic.

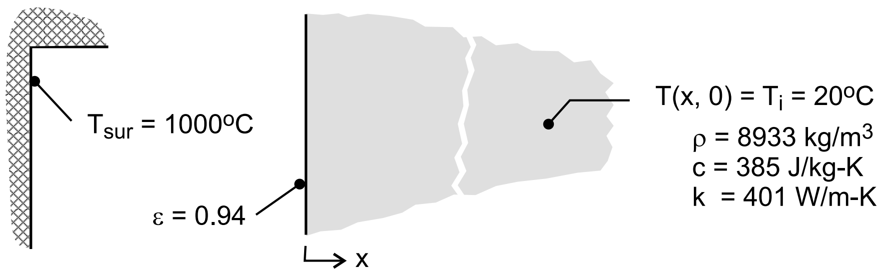
```
// Finite-difference equation, node 00; from Examples solution derivation; implicit method
q''o + k * (T01 - T00) / deltax = rho * (deltax / 2) *cp * der (T00,t)

// Finite-difference equations, interior nodes 01-15; from Tools
/* Node 01: interior node; e and w labeled 02 and 00. */
rho*cp*der(T01,t) = fd_1d_int(T01,T02,T00,k,qdot,deltax)
rho*cp*der(T02,t) = fd_1d_int(T02,T03,T01,k,qdot,deltax)
…..
…..
rho*cp*der(T14,t) = fd_1d_int(T14,T15,T13,k,qdot,deltax)
rho*cp*der(T15,t) = fd_1d_int(T15,T16,T14,k,qdot,deltax)

// Finite-difference equation node 16; from Tools, adiabatic surface
 /* Node 16: surface node (e-orientation); transient conditions; w labeled 15. */
rho * cp * der(T16,t) = fd_1d_sur_e(T16,T15,k,qdot,deltax,Tinf16,h16,q''a16)
q''a16 = 0          // Applied heat flux, W/m^2; zero flux shown
Tinf16 = 20         // Arbitrary value
h16 = 1e-8          // Causes boundary to behave as adiabatic
```

PROBLEM 5.109

KNOWN: Thick slab of copper as treated in Example 5.9, initially at a uniform temperature, is suddenly exposed to large surroundings at 1000°C (instead of a net radiant flux).

FIND: (a) The temperatures T(0, 120 s) and T(0.15 m, 120s) using the finite-element software *FEHT* for a surface emissivity of 0.94 and (b) Plot the temperature histories for x = 0, 150 and 600 mm, and explain key features of your results.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the x-direction, (2) Slab of thickness 600 mm approximates a semi-infinite medium, (3) Slab is small object in large, isothermal surroundings.

ANALYSIS: (a) Using FEHT, an outline of the slab is drawn of thickness 600 mm in the x-direction and arbitrary length in the y-direction. Click on *Setup* | *Temperatures in K*, to enter all temperatures in kelvins. The boundary conditions are specified as follows: on the y-planes and the x = 600 mm plane, treat as adiabatic; on the surface (0,y), select the convection coefficient option, enter the linearized radiation coefficient after Eq. 1.9 written as

0.94 * 5.67e-8 * (T + 1273) * (T^2 + 1273^2)

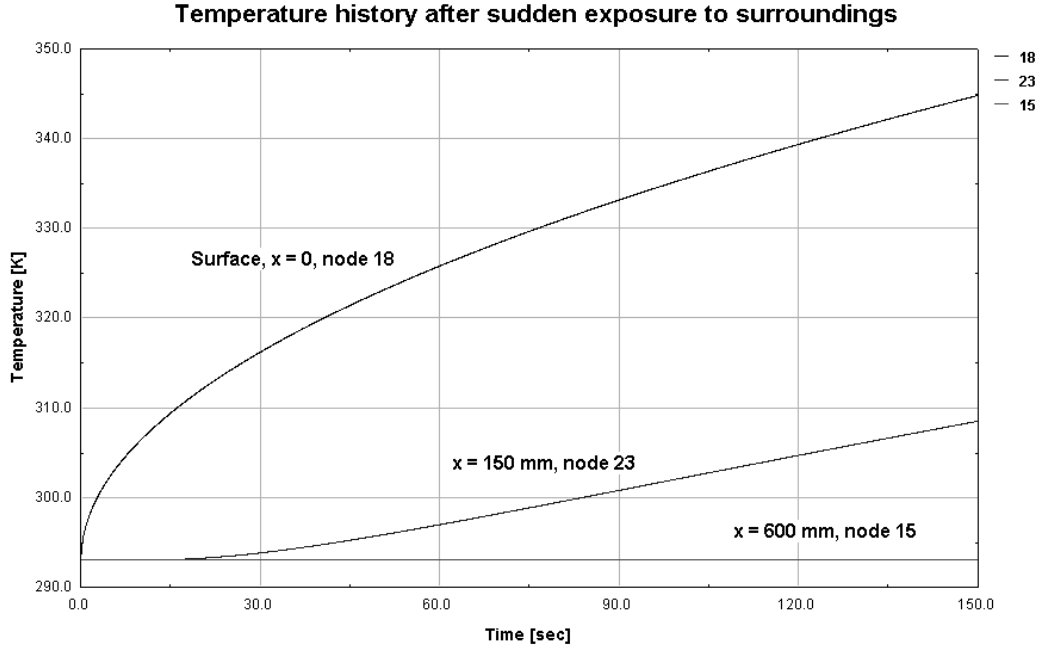
and enter the surroundings temperature, 1273 K, in the fluid temperature box. See the Comments for a view of the input screen. From *View*|*Temperatures*, find the results:

T(0, 120 s) = 339 K = 66°C   T(150 mm, 120 s) = 305K = 32°C   <

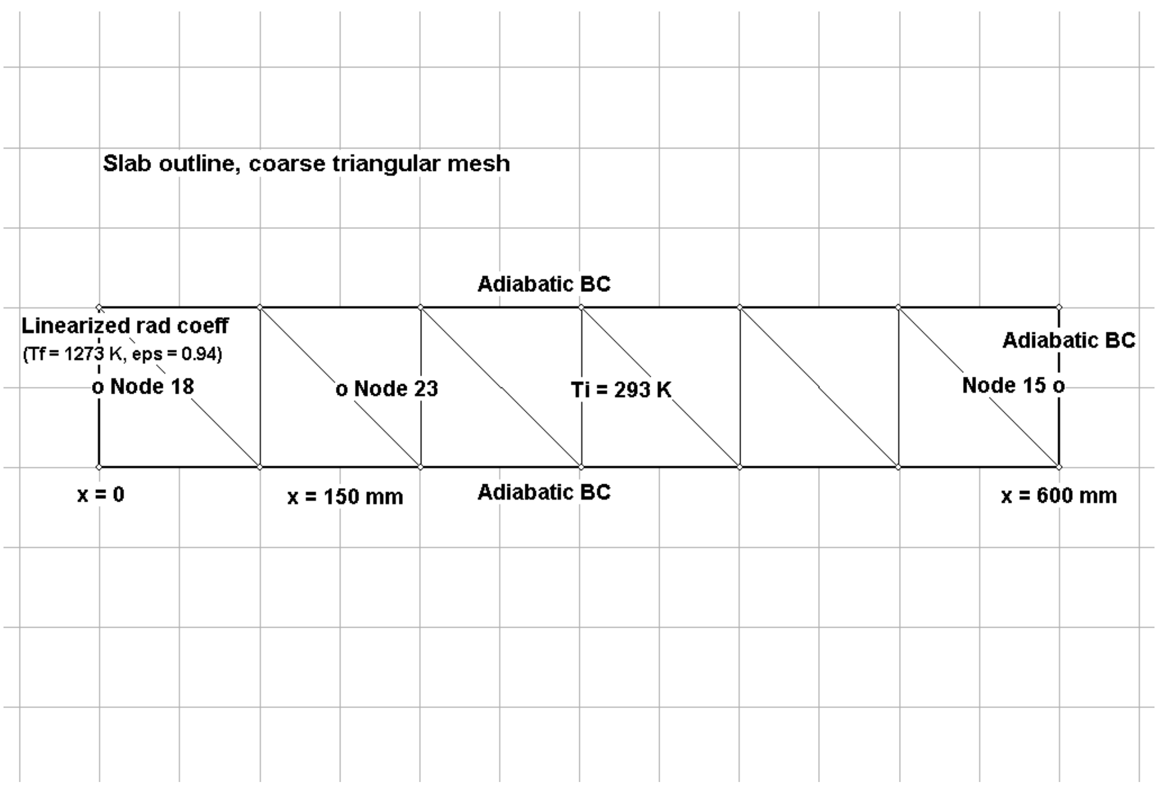
(b) Using the *View* | *Temperatures* command, the temperature histories for x = 0, 150 and 600 mm (10 mm mesh, Nodes 18, 23 and 15, respectively) are plotted. As expected, the surface temperature increases markedly at early times. As thermal penetration increases with increasing time, the temperature at the location x = 150 mm begins to increase after about 30 s. Note, however, that the temperature at the location x = 600 mm does not change significantly within the 150 s exposure to the hot surroundings. Our assumption of treating the boundary at the x = 600 mm plane as adiabatic is justified. A copper plate of 600 mm is a good approximation to a semi-infinite medium at times less than 150 s.

Continued …..

PROBLEM 5.109 (Cont.)



COMMENTS: The annotated *Input* screen shows the outline of the slab, the boundary conditions, and the triangular mesh before using the *Reduce-mesh* option.

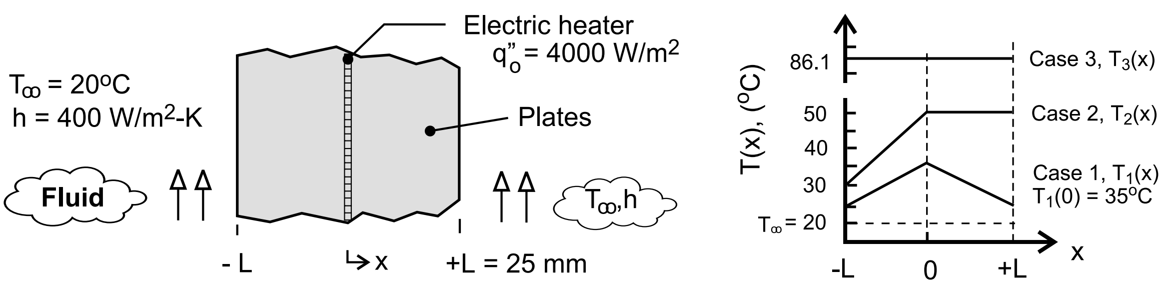


PPROBLEM 5.110

KNOWN: Electric heater sandwiched between two thick plates whose surfaces experience convection. Case 2 corresponds to steady-state operation with a loss of coolant on the x = -L surface. Suddenly, a second loss of coolant condition occurs on the x = +L surface, but the heater remains energized for the next 15 minutes. Case 3 corresponds to the eventual steady-state condition following the second loss of coolant event. See Problem 2.53.

FIND: Calculate and plot the temperature time histories at the plate locations x = 0, ±L during the transient period between steady-state distributions for Case 2 and Case 3 using the finite-element approach with *FEHT* and the finite-difference method of solution with *IHT* (Δx = 5 mm and Δt = 1 s).

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Heater has negligible thickness, and (4) Negligible thermal resistance between the heater surfaces and the plates.

PROPERTIES: Plate material (*given*); ρ = 2500 kg/m^3, c = 700 J/kg·K, k = 5 W/m·K.

ANALYSIS: The temperature distribution for Case 2 shown in the above graph represents the initial condition for the period of time following the second loss of coolant event. The boundary conditions at x = ±L are adiabatic, and the heater flux is maintained at q''_o = 4000 W/m^2 for $0 \le t \le 15$ min.

Using *FEHT*, the heater is represented as a plate of thickness L_h = 0.5 mm with very low thermal capacitance (ρ = 1 kg/m and c = 1 J/kg·K), very high thermal conductivity (k= 10,000 W/m·K), and a uniform volumetric generation rate of $\dot{q} = q''_o / L_h = 4000\ \text{W/m}^2 / 0.0005\ \text{m} = 8.0 \times 10^6\ \text{W/m}^3$ for $0 \le t \le$ 900 s. In the *Specify* | *Generation* box, the generation was prescribed by the *lookup file* (see *FEHT* Help): 'hfvst',1,2,Time. This *Notepad* file is comprised of four lines, with the values on each line separated by a single tab space:

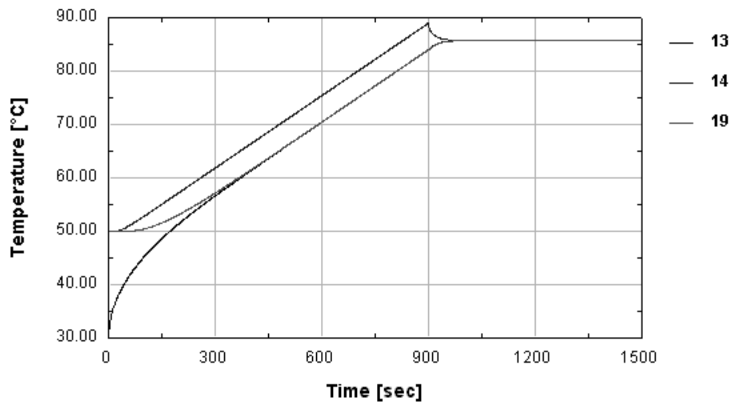
0	8e6
900	8e6
901	0
5000	0

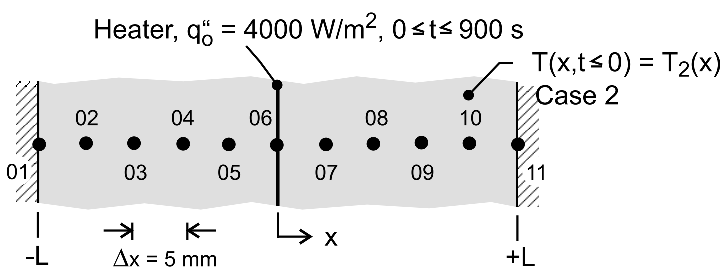
The temperature-time histories are shown in the graph below for the surfaces x = - L (lowest curve, 13) and x = +L (19) and the center point x = 0 (highest curve, 14). The center point experiences the maximum temperature of 89°C at the time the heater is deactivated, t = 900 s.

Continued …..

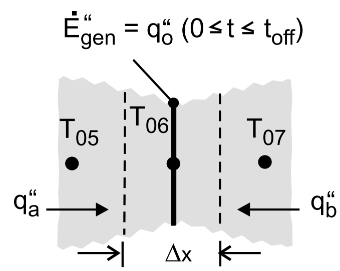
PROBLEM 5.110

For the finite-difference method of solution, the nodal arrangement for the system is shown below. The *IHT* model builder *Tools* | *Finite-Difference Equations* | *One Dimensional* can be used to obtain the FDEs for the internal nodes (02-04, 07-10) and the adiabatic boundary nodes (01, 11).





Nodal arrangement, FDE solution



Control volume, node 06

For the heater-plate interface node 06, the FDE for the implicit method is derived from an energy balance on the control volume shown in the schematic above.

$$\dot{E}''_{in} - \dot{E}''_{out} + \dot{E}''_{gen} = \dot{E}''_{st}$$

$$q''_a + q''_b + q''_o = \dot{E}''_{st}$$

$$k\frac{T_{05}^{p+1} - T_{06}^{p+1}}{\Delta x} + k\frac{T_{07}^{p+1} - T_{06}^{p+1}}{\Delta x} + q''_o = \rho c \Delta x \frac{T_{06}^{p+1} - T_{06}^{p}}{\Delta t}$$

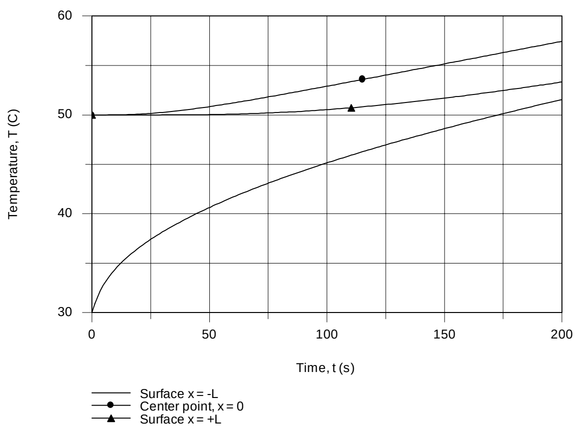
The *IHT* code representing selected nodes is shown below for the adiabatic boundary node 01, interior node 02, and the heater-plates interface node 06. Note how the foregoing derived finite-difference equation in implicit form is written in the *IHT Workspace*. Note also the use of a *Lookup Table* for representing the heater flux *vs*. time.

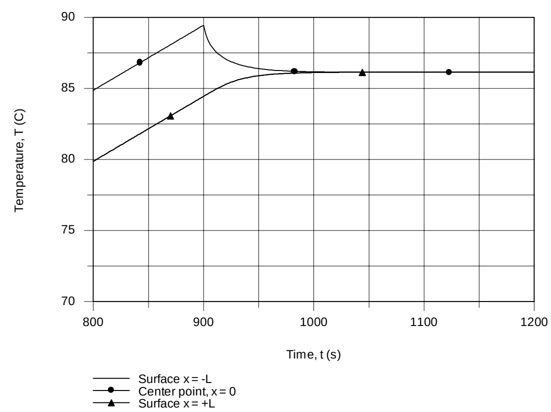
Continued …..

PROBLEM 5.110 (Cont.)

```
//  Finite-difference equations from Tools, Nodes 01, 02
/* Node 01: surface node (w-orientation); transient conditions; e labeled 02. */
rho * cp * der(T01,t) = fd_1d_sur_w(T01,T02,k,qdot,deltax,Tinf01,h01,q''a01)
q''a01 = 0              // Applied heat flux, W/m^2; zero flux shown
qdot = 0                // No internal generation
Tinf01 = 20             // Arbitrary value
h01 = 1e-6                // Causes boundary to behave as adiabatic

/* Node 02: interior node; e and w labeled 03 and 01. */
rho*cp*der(T02,t) = fd_1d_int(T02,T03,T01,k,qdot,deltax)

//  Finite-difference equation from energy balance on CV, Node 06
k * (T05 - T06) / deltax + k * (T07 - T06)/ deltax + q''h = rho * cp * deltax * der(T06,t)
q''h = LOOKUPVAL(qhvst,1,t,2)           // Heater flux, W/m^2; specified by Lookup Table

/* See HELP (Solver, Lookup Tables).  The Look-up table file name "qhvst" contains
        0       4000
        900     4000
        900.5   0
        5000    0               */
```

The temperature-time histories using the *IHT* code for the plate locations x = 0, ±L are shown in the graphs below. We chose to show expanded presentations of the histories at early times, just after the second loss of coolant event, t = 0, and around the time the heater is deactivated, t = 900 s.





COMMENTS: (1) The maximum temperature during the transient period is at the center point and occurs at the instant the heater is deactivated, T(0, 900s) = 89°C. After 300 s, note that the two surface temperatures are nearly the same, and never rise above the final steady-state temperature.

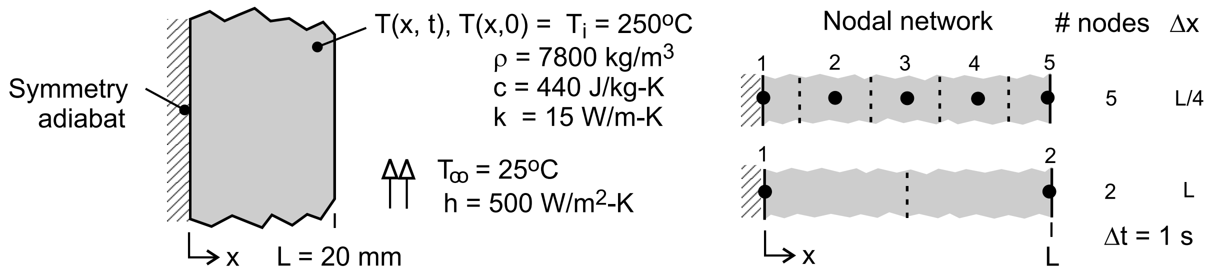
(2) Both the FEHT and IHT methods of solution give identical results. Their steady-state solutions agree with the result of an energy balance on a time interval basis yielding T_{ss} = 86.1°C.

PROBLEM 5.111

KNOWN: Plane wall of thickness 2L, initially at a uniform temperature, is suddenly subjected to convection heat transfer.

FIND: The mid-plane, T(0,t), and surface, T(L,t), temperatures at t = 50, 100, 200 and 500 s, using the following methods: (a) the one-term series solution; determine also the Biot number; (b) the lumped capacitance solution; and (c) the two- and 5-node finite-difference numerical solutions. Prepare a table summarizing the results and comment on the relative differences of the predicted temperatures.

SCHEMATIC:



(a) Plane wall, thickness 2L **(b) Nodal networks**

ASSUMPTIONS: (1) One-dimensional conduction in the x-direction, and (2) Constant properties.

ANALYSIS: (a) The results are tabulated below for the mid-plane and surface temperatures using the one-term approximation to the series solution, Eq. 5.40 and 5.41. The Biot number for the heat transfer process is

$$\mathrm{Bi} = h\ L/k = 500\ \mathrm{W/m^2 \cdot K} \times 0.020\ \mathrm{m} / 15\ \mathrm{W/m \cdot K} = 0.67$$

Since Bi >> 0.1, we expect an appreciable temperature difference between the mid-plane and surface as the tabulated results indicate (Eq. 5.10).

(b) The results are tabulated below for the wall temperatures using the lumped capacitance method (LCM) of solution, Eq. 5.6. The LCM neglects the internal conduction resistance and since Bi = 0.67 >> 0.1, we expect this method to predict systematically lower temperatures (faster cooling) at the midplane compared to the one-term approximation.

Solution method/Time(s)	50	100	200	500
Mid-plane, T(0,t) (°C)				
One-term, Eqs. 5.40, 5.41	207.1	160.5	99.97	37.70
Lumped capacitance	181.7	133.9	77.69	30.97
2-node FDE	210.6	163.5	100.5	37.17
5-node FDE	207.5	160.9	100.2	37.77
Surface, T(L,t) (°C)				
One-term, Eqs. 5.40, 5.41	160.1	125.4	80.56	34.41
Lumped capacitance	181.7	133.9	77.69	30.97
2-node FDE	163.7	125.2	79.40	33.77
5-node FDE	160.2	125.6	80.67	34.45

(c) The 2- and 5-node nodal networks representing the wall are shown in the schematic above. The implicit form of the finite-difference equations for the mid-plane, interior (if present) and surface nodes can be derived from energy balances on the nodal control volumes. The time-rate of change of the temperature is expressed in terms of the *IHT* integral intrinsic function, *der*(*T*,*t*).

Continued …..

PROBLEM 5.111 (Cont.)

Mid-plane node

$$k\left(T2-T1\right)/\Delta x=\rho c\left(\Delta x/2\right)\cdot der\left(T1,t\right)$$

Interior node (5-node network)

$$k\left(T1-T2\right)/\Delta x+k\left(T3-T2\right)/\Delta x=\rho c\,\Delta x\cdot der\left(T2,t\right)$$

Surface node (shown for 5-node network)

$$k\left(T4-T5\right)/\Delta x+h\left(T\inf-T5\right)=\rho c\left(\Delta x/2\right)\cdot der\left(T5,t\right)$$

With appropriate values for Δx, the foregoing FDEs were entered into the *IHT* workspace and solved for the temperature distributions as a function of time over the range $0 \le t \le 500$ s using an integration time step of 1 s. Selected portions of the *IHT* codes for each of the models are shown in the Comments. The results of the analysis are summarized in the foregoing table.

COMMENTS: (1) Referring to the table above, we can make the following observations about the relative differences and similarities of the estimated temperatures: (a) The one-term series model estimates are the most reliable, and can serve as the benchmark for the other model results; (b) The LCM model over estimates the rate of cooling, and poorly predicts temperatures since the model neglects the effect of internal resistance and Bi = 0.67 >> 0.1; (c) The 5-node model results are in excellent agreement with those from the one-term series solution; we can infer that the chosen space and time increments are sufficiently small to provide accurate results; and (d) The 2-node model under estimates the rate of cooling for early times when the time-rate of change is high; but for late times, the agreement is improved.

(2) See the *Solver | Intrinsic Functions* section of *IHT|Help* or the *IHT Examples* menu (Example 5.3) for guidance on using the *der(T,t)* function.

(3) Selected portions of the *IHT* code for the 2-node network model are shown below.

```
// Writing the finite-difference equations – 2-node model
//  Node 1
k * (T2 - T1)/ deltax  = rho * cp * (deltax / 2) * der(T1,t)
// Node 2
k * (T1 - T2)/ deltax  + h * (Tinf - T2) = rho * cp * (deltax / 2) * der(T2,t)

// Input parameters
L = 0.020
deltax = L
rho = 7800          // density, kg/m^3
cp =   440          // specific heat, J/kg·K
k = 15              // thermal conductivity, W/m.K
h =   500           // convection coefficient, W/m^2·K
Tinf = 25           // fluid temperature, K
```

(4) Selected portions of the IHT code for the 5-node network model are shown below.

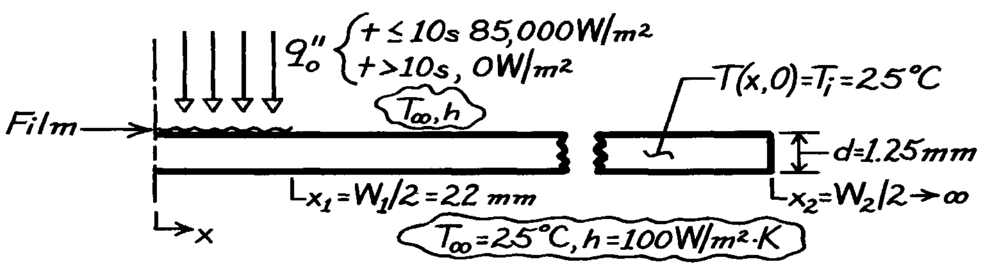
```
// Writing the finite-difference equations – 5-node model
//  Node 1 - midplane
k * (T2 - T1)/ deltax  = rho * cp * (deltax / 2) * der(T1,t)
// Interior nodes
k * (T1 - T2)/ deltax +  k * (T3 - T2 )/ deltax = rho * cp * deltax  * der(T2,t)
k * (T2 - T3)/ deltax +  k * (T4 - T3 )/ deltax = rho * cp * deltax  * der(T3,t)
k * (T3 - T4)/ deltax +  k * (T5 - T4 )/ deltax = rho * cp * deltax  * der(T4,t)
// Node5 - surface
k * (T4 - T5)/ deltax + h * (Tinf - T5) = rho * cp * (deltax / 2) * der(T5,t)

// Input parameters
L = 0.020
deltax = L / 4
……..
……..
```

PROBLEM 5.112

KNOWN: Plastic film on metal strip initially at 25°C is heated by a laser (85,000 W/m^2 for Δt_{on} = 10 s), to cure adhesive; convection conditions for ambient air at 25°C with coefficient of 100 W/m^2·K.

FIND: Temperature histories at center and film edge, T(0,t) and T(x_1,t), for $0 \le t \le 30$ s, using an implicit, finite-difference method with Δx = 4mm and Δt = 1 s; determine whether adhesive is cured ($T_c \ge 90$°C for Δt_c = 10s) and whether the degradation temperature of 200°C is exceeded.

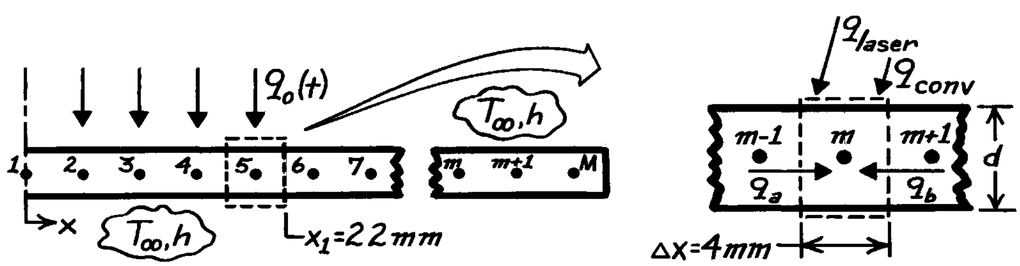
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Uniform convection coefficient on upper and lower surfaces, (4) Thermal resistance and mass of plastic film are negligible, (5) All incident laser flux is absorbed.

PROPERTIES: Metal strip (given): ρ = 7850 kg/m^3, c_p = 435 J/kg·K, k = 60 W/m·K, $\alpha = k/\rho c_p = 1.757 \times 10^{-5}$ m^2/s.

ANALYSIS: (a) Using a space increment of Δx = 4mm, set up the nodal network shown below. Note that the film half-length is 22mm (rather than 20mm as in Problem 3.97) to simplify the finite-difference equation derivation.



Consider the general control volume and use the conservation of energy requirement to obtain the finite-difference equation.

$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st}$$

$$q_a + q_b + q_{laser} + q_{conv} = Mc_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

Continued …..

PROBLEM 5.112 (Cont.)

$$k(d \cdot 1)\frac{T_{m-1}^{p+1} - T_m^{p+1}}{\Delta x} + k(d \cdot 1)\frac{T_{m+1}^{p+1} - T_m^{p+1}}{\Delta x}$$

$$+ q_o''(\Delta x \cdot 1) + 2h(\Delta x \cdot 1)\left(T_\infty - T_m^{p+1}\right) = \rho(\Delta x \cdot d \cdot 1)c_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$T_m^p = (1 + 2Fo + 2Fo \cdot Bi)T_m^{p+1} \tag{1}$$

$$-Fo\left(T_{m+1}^{p+1} + T_{m+1}^{p+1}\right) - 2Fo \cdot Bi \cdot T_\infty - Fo \cdot Q$$

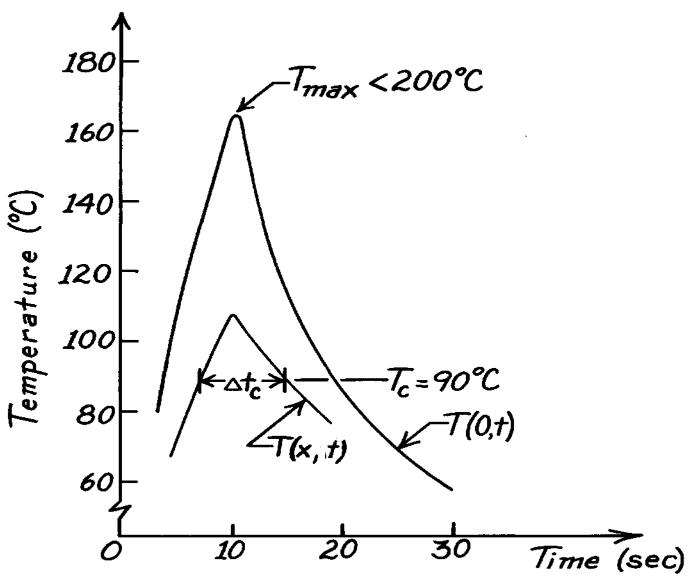
where

$$Fo = \frac{\alpha \Delta t}{\Delta x^2} = \frac{1.757 \times 10^{-5} m^2/s \times 1s}{(0.004\ m)^2} = 1.098 \tag{2}$$

$$Bi = \frac{h\left(\Delta x^2/d\right)}{k} = \frac{100\ W/m^2 \cdot K\left(0.004^2/0.00125\right)m}{60\ W/m \cdot K} = 0.0213 \tag{3}$$

$$Q = \frac{q_o''\left(\Delta x^2/d\right)}{k} = \frac{85,000\ W/m^2\left(0.004^2/0.0015\right)m}{60\ W/m \cdot K} = 18.133. \tag{4}$$

The results of the matrix inversion numerical method of solution (Δx = 4mm, Δt = 1s) are shown below. The temperature histories for the center (m = 1) and film edge (m = 5) nodes, T(0,t) and $T(x_1,t)$, respectively, permit determining whether the adhesive has cured (T ≥ 90°C for 10 s).



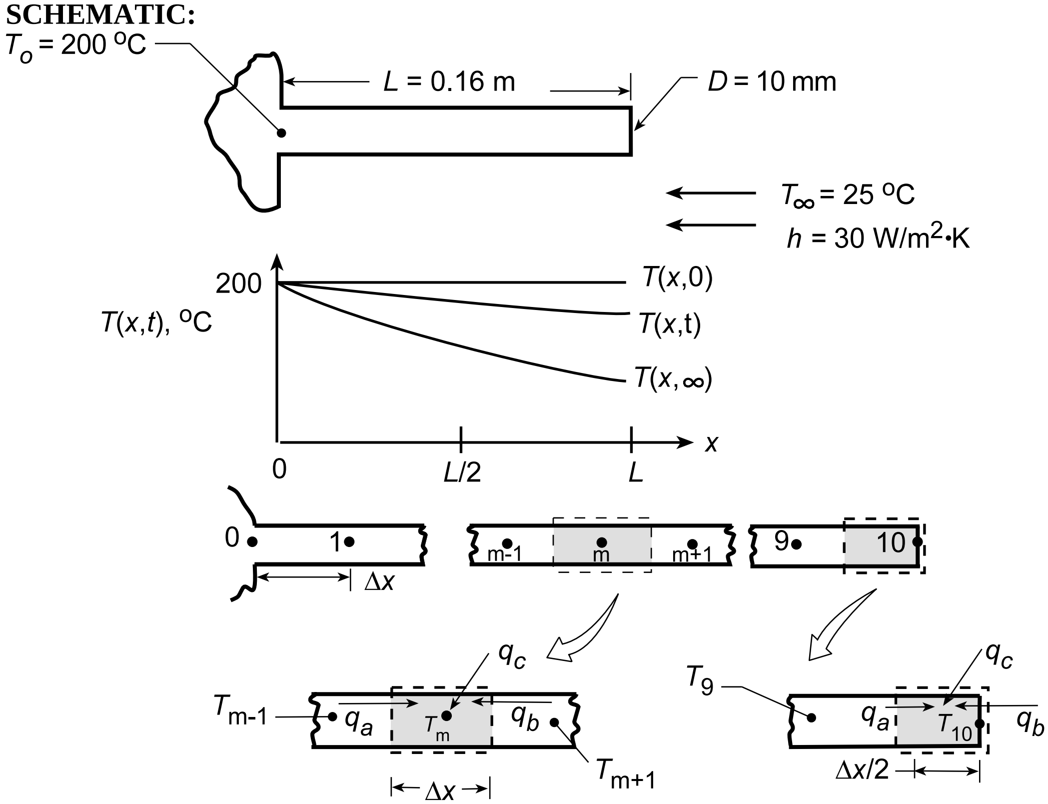
Certainly the center region, T(0,t), is fully cured and furthermore, the degradation temperature (200°C) has not been exceeded. From the $T(x_1,t)$ distribution, note that $\Delta t_c \approx 8$ sec, which is 20% less than the 10 s interval sought. Hence, the laser exposure (now 10 s) should be slightly increased and quite likely, the maximum temperature will not exceed 200°C.

PROBLEM 5.113

KNOWN: Insulated rod of prescribed length and diameter, with one end in a fixture at 200°C, reaches a uniform temperature. Suddenly the insulating sleeve is removed and the rod is subjected to a convection process.

FIND: (a) Time required for the mid-length of the rod to reach 100°C, (b) Temperature history $T(x,t \leq t_1)$, where t_1 is time at which the midlength reaches 50°C. Temperature distribution at 0, 200s, 400s and t_1.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional transient conduction in rod, (2) Uniform h along rod and at end, (3) Negligible radiation exchange between rod and surroundings, (4) Constant properties.

ANALYSIS: (a) Choosing $\Delta x = 0.016$ m, the finite-difference equations for the interior and end nodes are obtained.

Interior Point, m: $$q_a + q_b + q_c = \rho \cdot A_c \Delta x \cdot c_p \cdot \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$k \cdot A_c \frac{T_{m-1}^p - T_m^p}{\Delta x} + kA_c \frac{T_{m+1}^p - T_m^p}{\Delta x} + hP\Delta x \left(T_\infty - T_m^p\right) = \rho A_c \Delta x c_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

Regrouping,

$$T_m^{p+1} = T_m^p \left(1 - 2Fo - Bi \cdot Fo\right) + Fo\left(T_{m-1}^p + T_{m+1}^p\right) + Bi \cdot Fo T_\infty \qquad (1)$$

where

$$Fo = \frac{\alpha \Delta t}{\Delta x^2} \qquad (2) \qquad\qquad Bi = h\left[\Delta x^2 / \left(A_c/P\right)\right] / k . \qquad (3)$$

From Eq. (1), recognize that the stability of the numerical solution will be assured when the first term on the RHS is positive; that is

Continued...

$$\left(1-2\mathrm{Fo}-\mathrm{Bi}\cdot\mathrm{Fo}\right)\geq 0 \qquad \text{or} \qquad \mathrm{Fo}\leq 1/\left(2+\mathrm{Bi}\right). \tag{4}$$

Nodal Point 1: Consider Eq. (1) for the special case that $T_{m-1}^{p} = T_o$, which is independent of time. Hence,

$$T_1^{p+1} = T_1^{p}\left(1-2\mathrm{Fo}-\mathrm{Bi}\cdot\mathrm{Fo}\right)+\mathrm{Fo}\left(T_o+T_2^{p}\right)+\mathrm{Bi}\cdot\mathrm{Fo}T_\infty . \tag{5}$$

End Nodal Point 10: $\quad q_a+q_b+q_c=\rho\cdot A_c\dfrac{\Delta x}{2}\cdot c_p\dfrac{T_{10}^{p+1}-T_{10}^{p}}{\Delta t}$

$$k\cdot A_c\frac{T_9^{p}-T_{10}^{p}}{\Delta x}+hA_c\left(T_\infty-T_{10}^{p}\right)+hP\frac{\Delta x}{2}\left(T_\infty-T_{10}^{p}\right)=\rho A_c\frac{\Delta x}{2}c_p\frac{T_{10}^{p+1}-T_{10}^{p}}{\Delta t}$$

Regrouping, $T_{10}^{p+1}=T_{10}^{p}\left(1-2\mathrm{Fo}-2N\cdot\mathrm{Fo}-\mathrm{Bi}\cdot\mathrm{Fo}\right)+2\mathrm{Fo}T_9^{p}+T_\infty\left(2N\cdot\mathrm{Fo}+\mathrm{Bi}\cdot\mathrm{Fo}\right)$ (6)

where $N = h\Delta x/k$. (7)

The stability criterion is $\mathrm{Fo} \leq 1/2(1 + N + \mathrm{Bi}/2)$. (8)

With the finite-difference equations established, we can now proceed with the numerical solution. Having already specified $\Delta x = 0.016$ m, Bi can now be evaluated. Noting that $A_c = \pi D^2/4$ and $P = \pi D$, giving $A_c/P = D/4$, Eq. (3) yields

$$\mathrm{Bi}=30\,\mathrm{W/m^2\cdot K}\left[\left(0.016\,\mathrm{m}\right)^2\Big/\frac{0.010\,\mathrm{m}}{4}\right]\Big/14.8\,\mathrm{W/m\cdot K}=0.208 \tag{9}$$

From the stability criteria, Eqs. (4) and (8), for the finite-difference equations, it is recognized that Eq. (8) requires the greater value of Fo. Hence

$$\mathrm{Fo}=\frac{1}{2}\left(1+0.0324+\frac{0.208}{2}\right)=0.440 \tag{10}$$

where from Eq. (7), $N=\dfrac{30\,\mathrm{W/m^2\cdot K}\times 0.016\,\mathrm{m}}{14.8\,\mathrm{W/m\cdot K}}=0.0324$. (11)

From the definition of Fo, Eq. (2), we obtain the time increment

$$\Delta t=\frac{\mathrm{Fo}\left(\Delta x\right)^2}{\alpha}=0.440\left(0.016\,\mathrm{m}\right)^2\Big/3.63\times 10^{-6}\,\mathrm{m^2/s}=31.1\mathrm{s} \tag{12}$$

and the time relation is $t = p\Delta t = 31.1t$. (13)

Using the numerical values for Fo, Bi and N, the finite-difference equations can now be written (°C).

Nodal Point m ($2 \leq m \leq 9$):

$$T_m^{p+1}=T_m^{p}\left(1-2\times 0.440-0.208\times 0.440\right)+0.440\left(T_{m-1}^{p}+T_{m+1}^{p}\right)+0.208\times 0.440\times 25$$

$$T_m^{p+1}=0.029T_m^{p}+0.440\left(T_{m-1}^{p}+T_{m+1}^{p}\right)+2.3 \tag{14}$$

Nodal Point 1:

$$T_1^{p+1}=0.029T_1^{p}+0.440\left(200+T_2^{p}\right)+2.3=0.029T_1^{p}+0.440T_2^{p}+90.3 \tag{15}$$

Nodal Point 10:

$$T_{10}^{p+1}=0\times T_{10}^{p}+2\times 0.440T_9^{p}+25\left(2\times 0.0324\times 0.440+0.208\times 0.440\right)=0.880T_9^{p}+3.0 \tag{16}$$

Continued...

PROBLEM 5.113 (Cont.)

Using finite-difference equations (14-16) with Eq. (13), the calculations may be performed to obtain

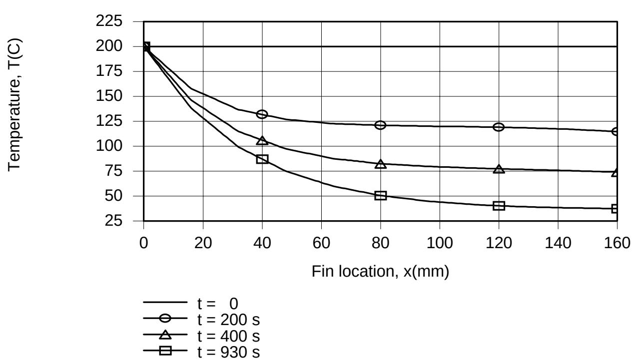
p	t(s)	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}(°C)
0	0	200	200	200	200	200	200	200	200	200	200
1	31.1	184.1	181.8	181.8	181.8	181.8	181.8	181.8	181.8	181.8	179.0
2	62.2	175.6	166.3	165.3	165.3	165.3	165.3	165.3	165.3	164.0	163.0
3	93.3	168.6	154.8	150.7	150.7	150.7	150.7	150.7	149.7	149.2	147.3
4	124.4	163.3	145.0	138.8	137.0	137.0	137.0	136.5	136.3	135.0	134.3
5	155.5	158.8	137.1	128.1	125.3	124.5	124.3	124.2	123.4	123.0	121.8
6	186.6	155.2	130.2	119.2	114.8	113.4	113.0	112.6	112.3	111.5	111.2
7	217.7	152.1	124.5	111.3	105.7	103.5	102.9	102.4			
8	248.8	145.1	119.5	104.5	97.6	94.8					

Using linear interpolation between rows 7 and 8, we obtain $T(L/2, 230s) = T_5 \approx 100°C$. <

(b) Using the option concerning *Finite-Difference Equations* for *One-Dimensional Transient Conduction* in *Extended Surfaces* from the IHT Toolpad, the desired temperature histories were computed for $0 \le t \le t_1 = 930s$. A *Lookup Table* involving data for T(x) at t = 0, 200, 400 and 930s was created.

t(s)/x(mm)	0	16	32	48	64	80	96	112	128	144	160
0	200	200	200	200	200	200	200	200	200	200	200
200	200	157.8	136.7	127.0	122.7	121.0	120.2	119.6	118.6	117.1	114.7
400	200	146.2	114.9	97.32	87.7	82.57	79.8	78.14	76.87	75.6	74.13
930	200	138.1	99.23	74.98	59.94	50.67	44.99	41.53	39.44	38.2	37.55

and the *LOOKUPVAL2* interpolating function was used with the *Explore* and *Graph* feature of IHT to create the desired plot.



Temperatures decrease with increasing x and t, and except for early times (t < 200s) and locations in proximity to the fin tip, the magnitude of the temperature gradient, |dT/dx|, decreases with increasing x. The slight increase in |dT/dx| observed for t = 200s and x → 160 mm is attributable to significant heat loss from the fin tip.

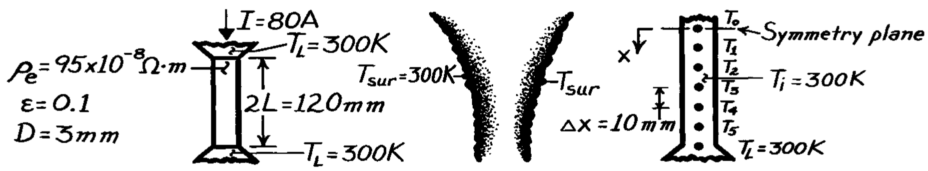
COMMENTS: The steady-state condition may be obtained by extending the finite-difference calculations in time to t ≈ 2650s or from Eq. 3.70.

PROBLEM 5.114

KNOWN: Tantalum rod initially at a uniform temperature, 300K, is suddenly subjected to a current flow of 80A; surroundings (vacuum enclosure) and electrodes maintained at 300K.

FIND: (a) Estimate time required for mid-length to reach 1000K, (b) Determine the steady-state temperature distribution and estimate how long it will take to reach steady-state. Use a finite-difference method with a space increment of 10mm.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction in rod, (2) Surroundings are much larger than rod, (3) Properties are constant and evaluated at an average temperature.

PROPERTIES: *Table A-1*, Tantalum $\left(\bar{T} = (300+1000)\text{K}/2 = 650\text{K}\right)$: ρ = 16,600 kg/m^3, c = 147 J/kg·K, k = 58.8 W/m·K, and α = k/ρc = 58.8 W/m·K/16,600 kg/m^3 × 147 J/kg·K = 2.410 × 10^{-5}m^2/s.

ANALYSIS: From the derivation of the previous problem, the finite-difference equation was found to be

$$T_m^{p+1} = \text{Fo}\left(T_{m-1}^p + T_{m+1}^p\right) + (1-2\text{Fo})T_m^p - \frac{\varepsilon P\sigma\Delta x^2}{kA_c}\text{Fo}\left(T_m^{4,p} - T_{sur}^4\right) + \frac{I^2\rho_e\Delta x^2}{kA_c^2}\cdot\text{Fo} \qquad (1)$$

where $\quad \text{Fo} = \alpha\Delta t/\Delta x^2 \qquad A_c = \pi\, D^2/4 \qquad P = \pi\, D. \qquad (2,3,4)$

From the stability criterion, let Fo = 1/2 and numerically evaluate terms of Eq. (1).

$$T_m^{p+1} = \frac{1}{2}\left(T_{m-1}^p + T_{m+1}^p\right) - \frac{0.1\times 5.67\times 10^{-8}\,\text{W/m}^2\cdot\text{K}^4\times(0.01\text{m})^2\,4}{58.8\ \text{W/m}\cdot\text{K}\times(0.003\text{m})}\cdot\frac{1}{2}\left(T_m^{4,p} - [300\text{K}]^4\right) +$$

$$+\frac{(80\text{A})^2\times 95\times 10^{-8}\,\Omega\cdot\text{m}\,(0.01\text{m})^2}{58.8\ \text{W/m}\cdot\text{K}\left(\pi\,[0.003\text{m}]^2/4\right)^2}\cdot\frac{1}{2}$$

$$T_m^{p+1} = \frac{1}{2}\left(T_{m-1}^p + T_{m+1}^p\right) - 6.4285\times 10^{-12}\,T_m^{4,p} + 103.53. \qquad (5)$$

Note that this form applies to nodes 0 through 5. For node 0, $T_{m-1} = T_{m+1} = T_1$. Since Fo = 1/2, using Eq. (2), find that

$$\Delta t = \Delta x^2\text{Fo}/\alpha = (0.01\text{m})^2\times 1/2/2.410\times 10^{-5}\,\text{m}^2/\text{s} = 2.07\text{s}. \qquad (6)$$

Hence, t = pΔt = 2.07p. (7)

Continued …..

PROBLEM 5.114 (Cont.)

(a) To estimate the time required for the mid-length to reach 1000K, that is T_o = 1000K, perform the forward-marching solution beginning with T_i = 300K at p = 0. The solution, as tabulated below, utilizes Eq. (5) for successive values of p. Elapsed time is determined by Eq. (7).

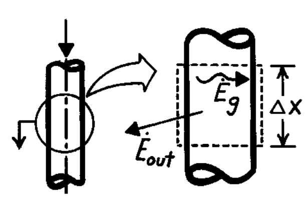
P	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6(°C)
0	0	300	300	300	300	300	300	300
1		403.5	403.5	403.5	403.5	403.5	403.5	300
2		506.9	506.9	506.9	506.9	506.9	455.1	300
3		610.0	610.0	610.0	610.0	584.1	506.7	300
4		712.6	712.6	712.6	699.7	661.1	545.2	300
5	10.4	814.5	814.5	808.0	788.8	724.7	583.5	300
6		915.2	911.9	902.4	867.4	787.9	615.1	300
7		1010.9	1007.9	988.9	945.0	842.3	646.6	300
8		1104.7	1096.8	1073.8	1014.0	896.1	673.6	300
9		1190.9	1183.5	1150.4	1081.7	943.2	700.3	300
10	20.7	1274.1	1261.6	1224.9	1141.5	989.4	723.6	300
11		1348.2	1336.7	1290.6	1199.8	1029.9	746.5	300
12		1419.7	1402.4	1353.9	1250.5	1069.4	766.5	300
13		1479.8	1465.5	1408.4	1299.8	1103.6	786.0	300
14		1542.6	1538.2	1460.9	1341.2	1136.9	802.9	300
15	31.1	1605.3	1569.3	1514.0	1381.6	1164.8	819.3	300

Note that, at p ≈ 6.9 or t = 6.9 × 2.07 = 14.3s, the mid-point temperature is $T_o \approx$ 1000K. <

(b) The steady-state temperature distribution can be obtained by continuing the marching solution until only small changes in T_m are noted. From the table above, note that at p = 15 or t = 31s, the temperature distribution is still changing with time. It is likely that at least 15 more calculation sets are required to see whether steady-state is being approached.

COMMENTS: (1) This problem should be solved with a computer rather than a hand-calculator. For such a situation, it would be appropriate to decrease the spatial increment in order to obtain better estimates of the temperature distribution.

(2) If the rod were very long, the steady-state temperature distribution would be very flat at the mid-length x = 0. Performing an energy balance on the small control volume shown to the right, find



$$\dot{E}_g - \dot{E}_{out} = 0$$
$$I^2 \frac{\rho_e \Delta x}{A_c} - \varepsilon\sigma P \Delta x \left(T_o^4 - T_{sur}^4\right) = 0.$$

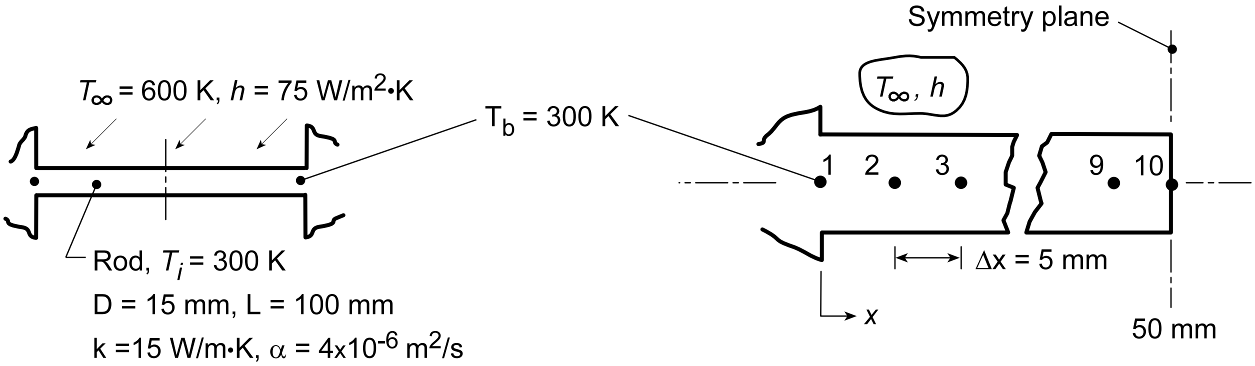
Substituting numerical values, find T_o = 2003K. It is unlikely that the present rod would ever reach this steady-state, maximum temperature. That is, the effect of conduction along the rod will cause the center temperature to be less than this value.

PROBLEM 5.115

KNOWN: Support rod spanning a channel whose walls are maintained at $T_b = 300$ K. Suddenly the rod is exposed to cross flow of hot gases with $T_\infty = 600$ K and $h = 75$ W/m^2·K. After the rod reaches steady-state conditions, the hot gas flow is terminated and the rod cools by free convection and radiation exchange with surroundings.

FIND: (a) Compute and plot the midspan temperature as a function of elapsed heating time; compare the steady-state temperature distribution with results from an analytical model of the rod and (b) Compute the midspan temperature as a function of elapsed cooling time and determine the time required for the rod to reach the safe-to-touch temperature of 315 K.

SCHEMATIC:

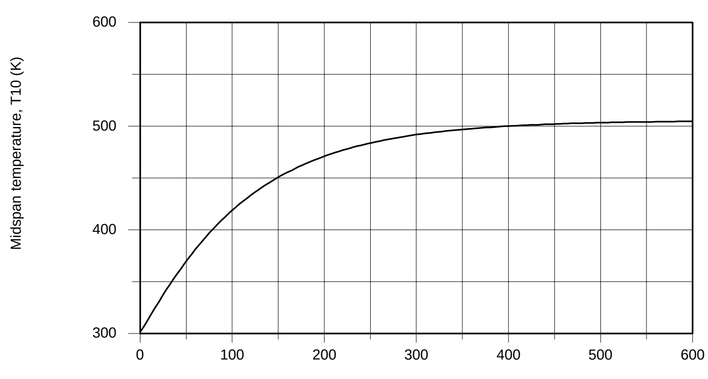


ASSUMPTIONS: (1) One-dimensional, transient conduction in rod, (2) Constant properties, (3) During heating process, uniform convection coefficient over rod, (4) During cooling process, free convection coefficient is of the form $h = C\Delta T^n$ where $C = 4.4$ W/m^2·K$^{1.188}$ and n = 0.188, and (5) During cooling process, surroundings are large with respect to the rod.

ANALYSIS: (a) The finite-difference equations for the 10-node mesh shown above can be obtained using the *IHT Finite-Difference Equation, One-Dimensional, Transient Extended Surfaces Tool*. The temperature-time history for the midspan position T_{10} is shown in the plot below. The steady-state temperature distribution for the rod can be determined from Eq. 3.75, Case B, Table 3.4. This case is treated in the *IHT Extended Surfaces Model, Temperature Distribution and Heat Rate, Rectangular Pin Fin*, for the adiabatic tip condition. The following table compares the steady-state temperature distributions for the numerical and analytical methods.

Method	Temperatures (K) vs. Position x (mm)					
	0	10	20	30	40	50
Analytical	300	386.1	443.4	479.5	499.4	505.8
Numerical	300	386.0	443.2	479.3	499.2	505.6

The comparison is excellent indicating that the nodal mesh is sufficiently fine to obtain precise results.



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PROBLEM 5.115 (Cont.)

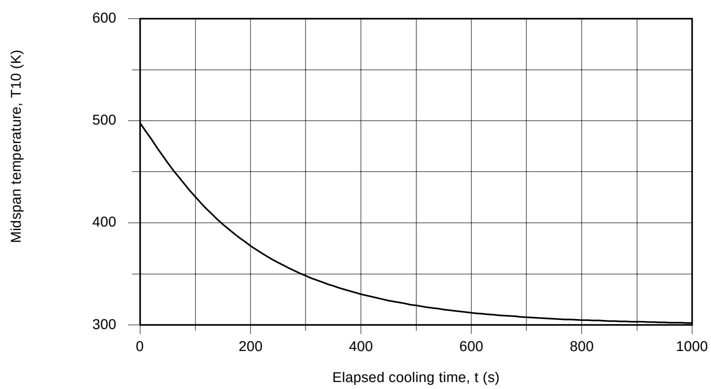
(b) The same finite-difference approach can be used to model the cooling process. In using the IHT tool, the following procedure was used: (1) Set up the FDEs with the convection coefficient expressed as $h_m = h_{fc,m} + h_{r,m}$, the sum of the free convection and linearized radiation coefficients based upon nodal temperature T_m.

$$h_{fc,m} = C\left(T_m^p - T_\infty\right)$$

$$h_{r,m} = \varepsilon\sigma\left(T_m^p + T_{sur}\right)\left(\left(T_m^p\right)^2 + T_{sur}^2\right)$$

(2) For the initial solve, set $h_{fc,m} = h_{r,m} = 5\ W/m^2\cdot K$ and solve, (3) Using the solved results as the Initial Guesses for the next solve, allow $h_{fc,m}$ and $h_{r,m}$ to be unknowns. The temperature-time history for the midspan during the cooling process is shown in the plot below. The time to reach the safe-to-touch temperature, $T_{10}^p = 315\,K$, is

$t = 550\ s$ <

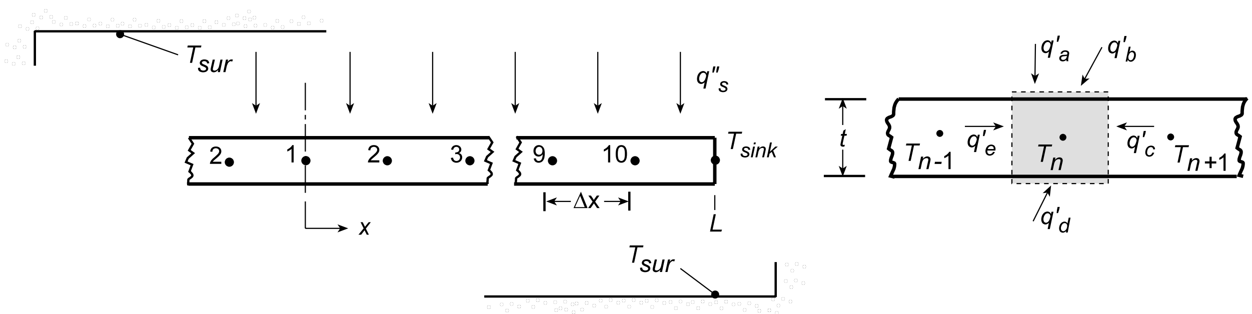


PROBLEM 5.116

KNOWN: Thin metallic foil of thickness, w, whose edges are thermally coupled to a sink at temperature, T_{sink}, initially at a uniform temperature $T_i = T_{sink}$, is suddenly exposed on the top surface to an ion beam heat flux, q_s'', and experiences radiation exchange with the vacuum enclosure walls at T_{sur}. Consider also the situation when the foil is operating under steady-state conditions when suddenly the ion beam is deactivated.

FIND: (a) Compute and plot the midspan temperature-time history during the *heating* process; determine the elapsed time that this point on the foil reaches a temperature within 1 K of the steady-state value, and (b) Compute and plot the midspan temperature-time history during the *cooling* process from steady-state operation; determine the elapsed time that this point on the foil reaches the *safe-to-touch* temperature of 315 K.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, transient conduction in the foil, (2) Constant properties, (3) Upper and lower surfaces of foil experience radiation exchange with the large surroundings, (4) Ion beam incident on upper surface only, (4) Foil is of unit width normal to the page.

ANALYSIS: (a) The finite-difference equations for the 10-node mesh shown above can be obtained using the *IHT Finite-Difference Equation, One-Dimensional, Transient, Extended Surfaces Tool*. In formulating the energy-balance functions, the following steps were taken: (1) the FDE function coefficient h must be identified for each node, e.g., h_1 and (2) coefficient can be represented by the linearized radiation coefficient, e.g., $h_1 = \varepsilon\sigma\left(T_1 + T_{sur}\right)\left(T_1^2 + T_{sur}^2\right)$, (3) set $q_a'' = q_o''/2$ since the ion beam is incident on only the top surface of the foil, and (4) when solving, the initial condition corresponds to T_i = 300 K for each node. The temperature-time history of the midspan position is shown below. The time to reach within 1 K of the steady-state temperature (374.1 K) is

$$T_{10}\left(t_h\right) = 373\,\text{K} \qquad t_h = 136\,\text{s}$$ <

(b) The same IHT workspace may be used to obtain the temperature-time history for the cooling process by taking these steps: (1) set q_s'' = 0, (2) specify the initial conditions as the steady-state temperature (K) distribution tabulated below,

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
374.1	374.0	373.5	372.5	370.9	368.2	363.7	356.6	345.3	327.4

(3) when performing the integration of the independent time variable, set the start value as 200 s and (4) save the results for the heating process in Data Set A. The temperature-time history for the heating and cooling processes can be made using Data Browser results from the Working and A Data Sets. The time required for the midspan to reach the *safe-to-touch* temperature is

$$T_{10}\left(t_c\right) = 315\,\text{K} \qquad t_c = 73\,\text{s}$$ <

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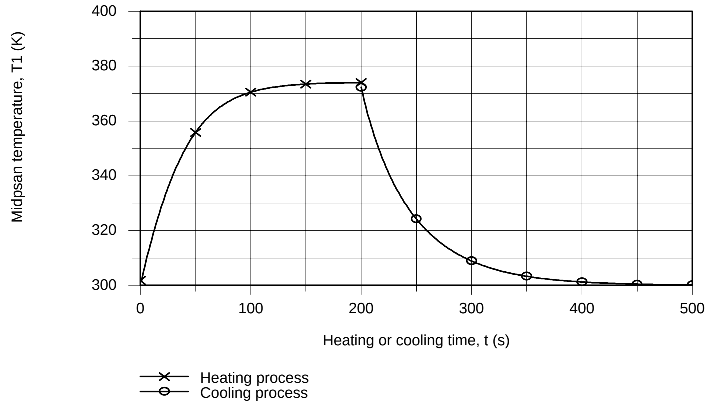


COMMENTS: The IHT workspace using the Finite-Difference Equations Tool to determine the temperature-time distributions is shown below. Some of the lines of code were omitted to save space on the page.

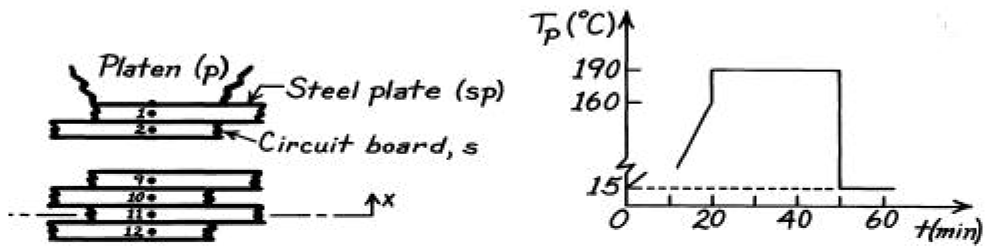
```
// Finite Difference Equations Tool: One-Dimensional, Transient, Extended Surface
/* Node 1: extended surface interior node;  transient conditions; e and w labeled 2 and 2. */
rho * cp * der(T1,t) = fd_1d_xsur_i(T1,T2,T2,k,qdot,Ac,P,deltax,Tinf, h1,q''a)
q''a1 = q''s / 2                // Applied heat flux, W/m^2;  on the upper surface only
h1 = eps * sigma * (T1 + Tsur) * (T1^2 + Tsur^2)
sigma = 5.67e-8                 // Boltzmann constant, W/m^2.K^4
/* Node 2: extended surface interior node;  transient conditions; e and w labeled 3 and 1. */
rho * cp * der(T2,t) = fd_1d_xsur_i(T2,T3,T1,k,qdot,Ac,P,deltax,Tinf, h2,q''a2)
q''a2 = 0                       // Applied heat flux, W/m^2; zero flux shown
h2 = eps * sigma * (T2+ Tsur) * (T2^2 + Tsur^2)
.......
.......
/* Node 10: extended surface interior node;  transient conditions; e and w labeled sk and 9. */
rho * cp * der(T10,t) = fd_1d_xsur_i(T10,Tsk,T9,k,qdot,Ac,P,deltax,Tinf, h10,q''a)
q''a10 = 0                      // Applied heat flux, W/m^2; zero flux shown
h10 = eps * sigma * (T10 + Tsur) * (T10^2 + Tsur^2)

// Assigned variables
deltax = L / 10                 // Spatial increment, m
Ac = w * 1                      // Cross-sectional area, m^2
P = 2 * 1                       // Perimeter, m
L = 0.150                       // Overall length, m
w = 0.00025                     // Foil thickness, m
eps = 0.45                      // Foil emissivity
Tinf = Tsur                     // Fluid temperature, K
Tsur = 300                      // Surroundings temperature, K
k = 40                          // Foil thermal conductivity
Tsk = 300                       // Sink temperature, K
q''s = 600                      // Ion beam heat flux, W/m^2; for heating process
q''s = 0                        // Ion beam heat flux, W/m^2; for cooling process
qdot = 0                        // Foil volumetric generation rate, W/m^3
alpha = 3e-5                    // Thermal diffusivity, m^2/s
rho = 1000                      // Density, kg.m^3; arbitrary value
alpha = k / (rho * cp)          // Definition
```

PROBLEM 5.117

KNOWN: Stack or book of steel plates (sp) and circuit boards (b) subjected to a prescribed platen heating schedule $T_p(t)$. See Problem 5.42 for other details of the book.

FIND: (a) Using the implicit numerical method with Δx = 2.36mm and Δt = 60s, find the mid-plane temperature T(0,t) of the book and determine whether curing will occur (> 170°C for 5 minutes), (b) Determine how long it will take T(0,t) to reach 37°C following reduction of the platen temperature to 15°C (at t = 50 minutes), (c) Validate code by using a sudden change of platen temperature from 15 to 190°C and compare with the solution of Problem 5.38.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible contact resistance between plates, boards and platens.

PROPERTIES: Steel plates (sp, given): ρ_{sp} = 8000 kg/m^3, $c_{p,sp}$ = 480 J/kg·K, k_{sp} = 12 W/m·K; Circuit boards (b, given): ρ_b = 1000 kg/m^3, $c_{p,b}$ = 1500 J/kg·K, k_b = 0.30 W/m·K.

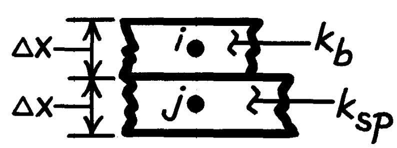
ANALYSIS: (a) Using the suggested space increment Δx = 2.36mm, the model grid spacing treating the steel plates (sp) and circuit boards (b) as discrete elements, we need to derive the nodal equations for the interior nodes (2-11) and the node next to the platen (1). Begin by defining appropriate control volumes and apply the conservation of energy requirement.

Effective thermal conductivity, k_e: Consider an adjacent steel plate-board arrangement. The thermal resistance between the nodes i and j is

$$R''_{ij} = \frac{\Delta x}{k_e} = \frac{\Delta x/2}{k_b} + \frac{\Delta x/2}{k_{sp}}$$

$$k_e = \frac{2}{1/k_{b+} + 1/k_{sp}} = \frac{2}{1/0.3 + 1/12} \text{W/m}\cdot\text{K}$$

$$k_e = 0.585 \text{ W/m}\cdot\text{K}.$$



Odd-numbered nodes, 3 £ m £ 11 - steel plates (sp): Treat as interior nodes using Eq. 5.89 with

$$a_{sp} \equiv \frac{k_e}{r_{sp}c_{sp}} = \frac{0.585 \text{ W/m}\cdot\text{K}}{8000 \text{ kg/m}^3 \times 480 \text{ J/kg}\cdot\text{K}} = 1.523\times10^{-7} \text{ m}^2/\text{s}$$

$$Fo_m = \frac{a_{sp}\Delta t}{\Delta x^2} = \frac{1.523\times10^{-7} \text{ m}^2/\text{s}\times 60\text{s}}{(0.00236 \text{ m})^2} = 1.641$$

Continued …..

to obtain, with m as odd-numbered,

$$\left(1+2\mathrm{Fo}_m\right)T_m^{p+1}-\mathrm{Fo}_m\left(T_{m-1}^{p+1}+T_{m+1}^{p+1}\right)=T_m^p \qquad (1)$$

Even-numbered nodes, 2 £ n £ 10 - circuit boards (b): Using Eq. 5.89 and evaluating α_b and Fo_n

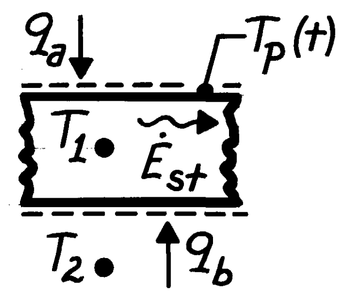
$$a_b=\frac{k_e}{r_b c_b}=3.900\times10^{-7}\ \mathrm{m}^2/\mathrm{s} \qquad \mathrm{Fo}_n=4.201$$

$$\left(1+2\mathrm{Fo}_n\right)T_n^{p+1}-\mathrm{Fo}_n\left(T_{n-1}^{p+1}+T_{n+1}^{p+1}\right)=T_n^p \qquad (2)$$

Plate next to platen, n = 1 - steel plate (sp): The finite-difference equation for the plate node (n = 1) next to the platen follows from a control volume analysis.

$$\dot{E}_{in}-\dot{E}_{out}=\dot{E}_{st}$$

$$q_a''+q_b''=r_{sp}\Delta x c_{sp}\frac{T_1^{p+1}-T_1^p}{\Delta t}$$



where

$$q_a''=k_{sp}\frac{T_p(t)-T_1^{p+1}}{\Delta x/2} \qquad q_b''=k_e\frac{T_2^{p+1}-T_1^{p+1}}{\Delta x}$$

and $T_p(t) = T_p(p)$ is the platen temperature which is changed with time according to the heating schedule. Regrouping find,

$$\left(1+\mathrm{Fo}_m\left(1+\frac{2k_{sp}}{k_e}\right)\right)T_1^{p+1}-\mathrm{Fo}_m T_2^{p+1}-\frac{2k_{sp}}{k_e}\mathrm{Fo}_m T_p(p)=T_1^p \qquad (3)$$

where $2k_{sp}/k_e$ = 2 × 12 W/m·K/0.585 W/m·K = 41.03.

Using the nodal Eqs. (1) -(3), an inversion method of solution was effected and the temperature distributions are shown on the following page.

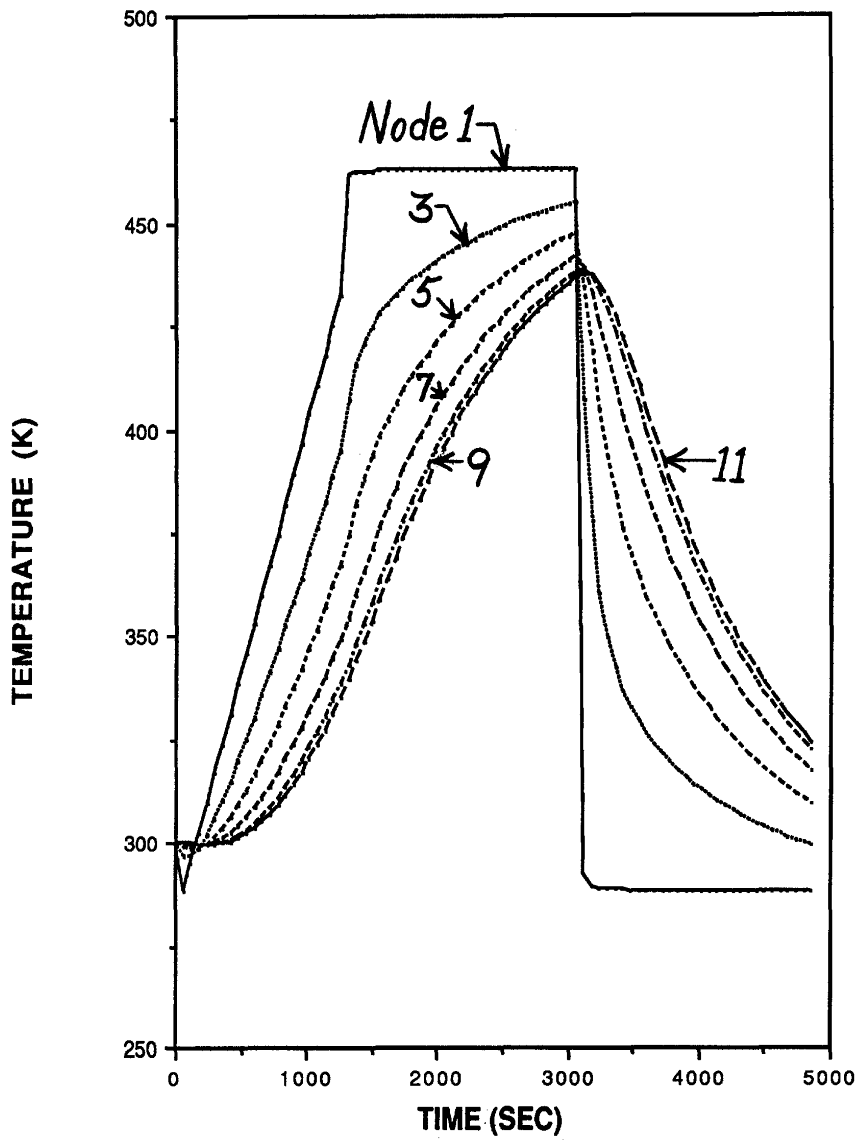
Temperature distributions - discussion: As expected, the temperatures of the nodes near the center of the book considerably lag those nearer the platen. The criterion for cure is T ≥ 170°C = 443 K for Δt_c = 5 min = 300 sec. From the temperature distributions, note that node 10 just reaches 443 K after 50 minutes and will not be cured. It appears that the region about node 5 will be cured.

(b) The time required for the book to reach 37°C = 310 K can likewise be seen from the temperature distribution results. The plates/boards nearest the platen will cool to the safe handling temperature with 1000 s = 16 min, but those near the center of the stack will require in excess of 2000 s = 32 min.

Continued …..

PROBLEM 5.117 (Cont.)

(c) It is important when validating computer codes to have the program work a "problem" which has an exact analytical solution. You should select the problem such that all features of the code are tested.

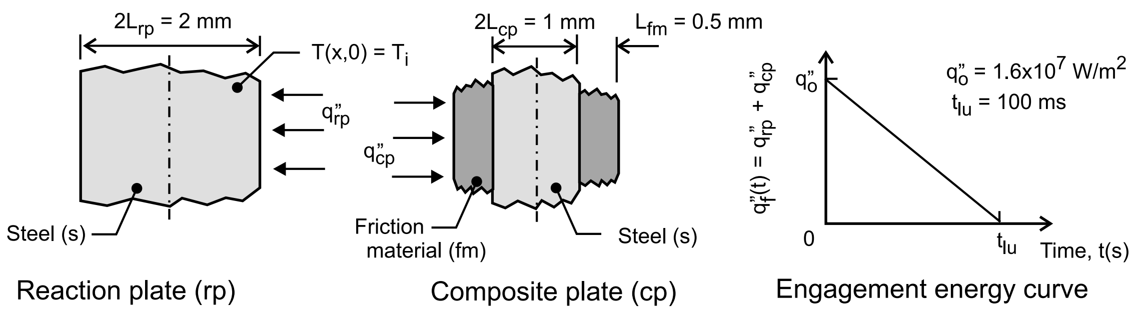


PROBLEM 5.118

KNOWN: Reaction and composite clutch plates, initially at a uniform temperature, $T_i = 40°C$, are subjected to the frictional-heat flux shown in the engagement energy curve, q''_f vs. t .

FIND: (a) On T-t coordinates, sketch the temperature histories at the mid-plane of the reaction plate, at the interface between the clutch pair, and at the mid-plane of the composite plate; identify key features; (b) Perform an energy balance on the clutch pair over a time interval basis and calculate the steady-state temperature resulting from a clutch engagement; (c) Obtain the temperature histories using the finite-element approach with *FEHT* and the finite-difference method of solution with *IHT* ($\Delta x = 0.1$ mm and $\Delta t = 1$ ms). Calculate and plot the frictional heat fluxes to the reaction and composite plates, q''_{rp} and q''_{cp}, respectively, as a function of time. Comment on the features of the temperature and frictional-heat flux histories.

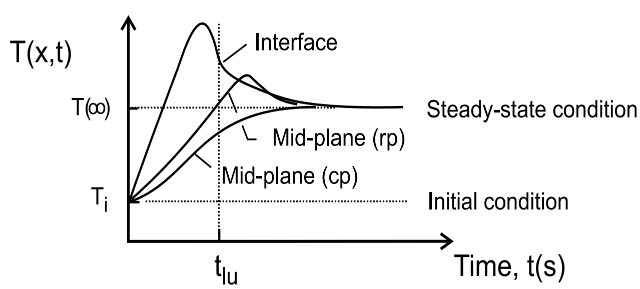
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible heat transfer to the surroundings.

PROPERTIES: Steel, $\rho_s = 7800$ kg/m^3, $c_s = 500$ J/kg·K, $k_s = 40$ W/m·K; Friction material, $\rho_{fm} = 1150$ kg/m^3, $c_{fm} = 1650$ J/kg·K, and $k_{fm} = 4$ W/m·K.

ANALYSIS: (a) The temperature histories for specified locations in the system are sketched on T-t coordinates below.

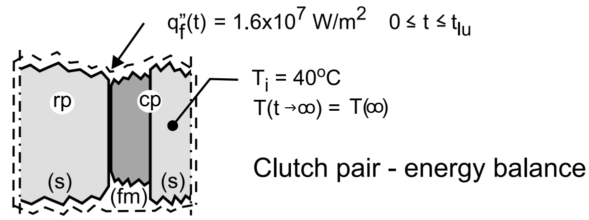


Initially, the temperature at all locations is uniform at T_i. Since there is negligible heat transfer to the surroundings, eventually the system will reach a uniform, steady-state temperature $T(\infty)$. During the engagement period, the interface temperature increases much more rapidly than at the mid-planes of the reaction (rp) and composite (cp) plates. The interface temperature should be the maximum within the system and could occur before lock-up, $t = t_{lu}$.

Continued …..

PROBLEM 5.118 (Cont.)

(b) To determine the steady-state temperature following the engagement period, apply the conservation of energy requirement on the clutch pair on a time-interval basis, Eq. 1.11b.



The final and initial states correspond to uniform temperatures of $T(\infty)$ and T_i, respectively. The energy input is determined from the engagement energy curve, q''_f vs. t .

$$E''_{n} - E''_{out} + E''_{gen} = \Delta E''_{st} \qquad E''_{in} = E''_{out} = 0$$

$$\int_0^{t_{lu}} q''_f(t)dt = E''_f - E''_i = \left[\rho_s c_s \left(L_{rp}/2 + L_{cp}/2\right) + \rho_{fm} c_{fm} L_{fm}\right]\left(T_f - T_i\right)$$

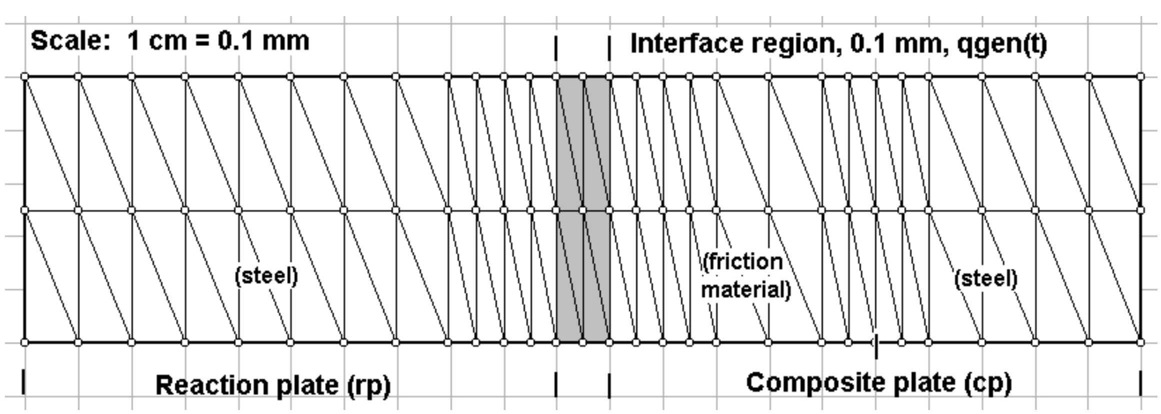
Substituting numerical values, with T_i = 40°C and $T_f = T(\infty)$.

$$0.5\, q''_o\, t_{lu} = \left[\rho_s c_s \left(L_{rp}/2 + L_{cp}/2\right) + \rho_{fm} c_{fm} L_{fm}\right]\left(T(\infty) - T_i\right)$$

$$0.5 \times 1.6 \times 10^7 \text{ W/m}^2 \times 0.100 \text{ s} = \Big[7800 \text{ kg/m}^3 \times 500 \text{ J/kg}\cdot\text{K}\left(0.001 + 0.0005\right)\text{m}$$
$$+1150 \text{ kg/m}^3 \times 1650 \text{ J/kg}\cdot\text{K} \times 0.0005 \text{ m}\Big]\left(T(\infty) - 40\right)°\text{C}$$

$$T(\infty) = 158°\text{C}$$ $<$

(c) *Finite-element method of solution, FEHT*. The clutch pair is comprised of the reaction plate (1 mm), an interface region (0.1 mm), and the composite plate (cp) as shown below.



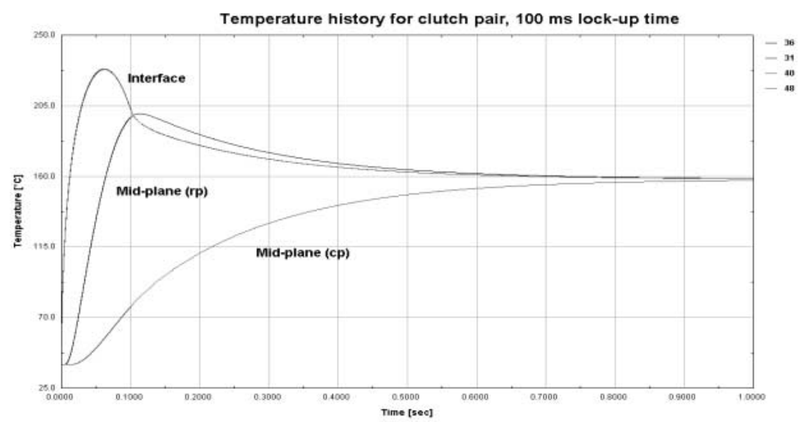
Continued (2)...

PROBLEM 5.118 (Cont.)

The external boundaries of the system are made adiabatic. The interface region provides the means to represent the frictional heat flux, specified with negligible thermal resistance and capacitance. The generation rate is prescribed as

$$\dot{q} = 1.6\times10^{11}\left(1-\text{Time}/0.1\right)\text{W}/\text{m}^3 \qquad\qquad 0 \le \text{Time} \le t_{lu}$$

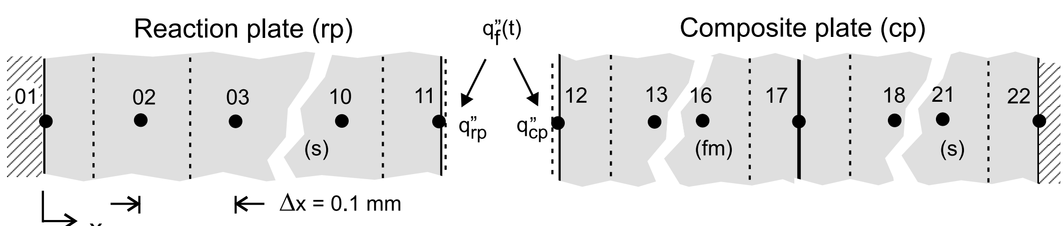
where the first coefficient is evaluated as $q_o'' \;/\, 0.1\times10^{-3}\text{m}$ and the 0.1 mm parameter is the thickness of the region. Using the *Run* command, the integration is performed from 0 to 0.1 s with a time step of 1×10^{-6} s. Then, using the *Specify|Generation* command, the generation rate is set to zero and the *Run|Continue* command is executed. The temperature history is shown below.



(c) *Finite-difference method of solution*, IHT. The nodal arrangement for the clutch pair is shown below with Δx = 0.1 mm and Δt = 1 ms. Nodes 02-10, 13-16 and 18-21 are interior nodes, and their finite-difference equations (FDE) can be called into the *Workspace* using *Tools|Finite Difference Equations|One-Dimenisonal|Transient*. Nodes 01 and 22 represent the mid-planes for the reaction and composite plates, respectively, with adiabatic boundaries. The FDE for node 17 is derived from an energy balance on its control volume (CV) considering different properties in each half of the CV. The FDE for node 11 and 12 are likewise derived using energy balances on their CVs. At the interface, the following conditions must be satisfied

$$T_{11} = T_{12} \qquad\qquad\qquad q_f'' = q_{rp}'' + q_{cp}''$$

The frictional heat flux is represented by a *Lookup Table,* which along with the FDEs, are shown in the *IHT* code listed in Comment 2.

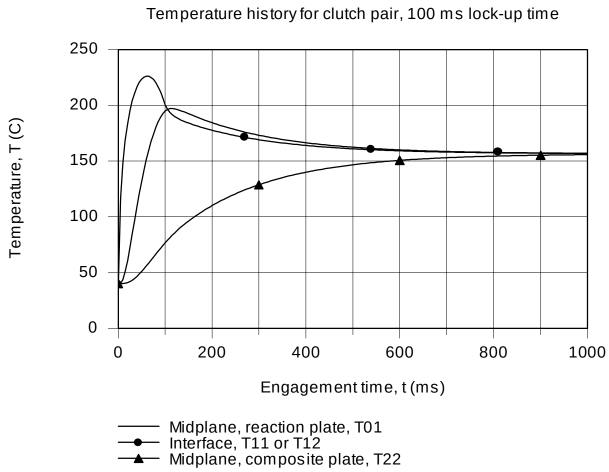


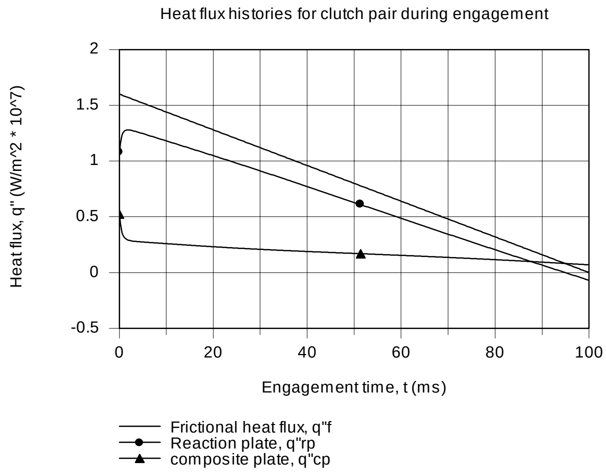
Nodal arrangement, FDE solution

Continued (3)...

PROBLEM 5.118 (Cont.)

The temperature and heat flux histories are plotted below. The steady-state temperature was found as 156.5° C, which is in reasonable agreement with the energy balance result from part (a).





COMMENTS: (1) The temperature histories resulting from the FEHT and IHT based solutions are in agreement. The interface temperature peaks near 225°C after 75 ms, and begins dropping toward the steady-state condition. The mid-plane of the reaction plate peaks around 100 ms, nearly reaching 200°C. The temperature of the mid-plane of the composite plate increases slowly toward the steady-state condition.

(2) The calculated temperature-time histories for the clutch pair display similar features as expected from our initial sketches on T vs. t coordinates, part a. The maximum temperature for the composite is very high, subjecting the bonded frictional material to high thermal stresses as well as accelerating deterioration. For the reaction steel plate, the temperatures are moderate, but there is a significant gradient that could give rise to thermal stresses and hence, warping. Note that for the composite plate, the steel section is nearly isothermal and is less likely to experience warping.

(2) The *IHT* code representing the FDE for the 22 nodes and the frictional heat flux relation is shown below. Note use of the *Lookup Table* for representing the frictional heat flux *vs*. time boundary condition for nodes 11 and 12.

```
// Nodal equations, reaction plate (steel)
/* Node 01: surface node (w-orientation); transient conditions; e labeled 02. */
rhos * cps * der(T01,t) = fd_1d_sur_w(T01,T02,ks,qdot,deltax,Tinf01,h01,q''a01)
q''a01 = 0            // Applied heat flux, W/m^2; zero flux shown
Tinf01 = 40           // Arbitrary value
h01 = 1e-5            // Causes boundary to behave as adiabatic
qdot = 0
/* Node 02: interior node; e and w labeled 03 and 01. */
rhos*cps*der(T02,t) = fd_1d_int(T02,T03,T01,ks,qdot,deltax)
………………………………
/* Node 10: interior node; e and w labeled 11 and 09. */
rhos*cps*der(T10,t) = fd_1d_int(T10,T11,T09,ks,qdot,deltax)
/* Node 11: From an energy on the CV about node 11 */
ks * (T10 - T11) / deltax + q''rp =  rhos * cps * deltax / 2 * der(T11,t)
```

Continued (4)...

PROBLEM 5.118 (Cont.)

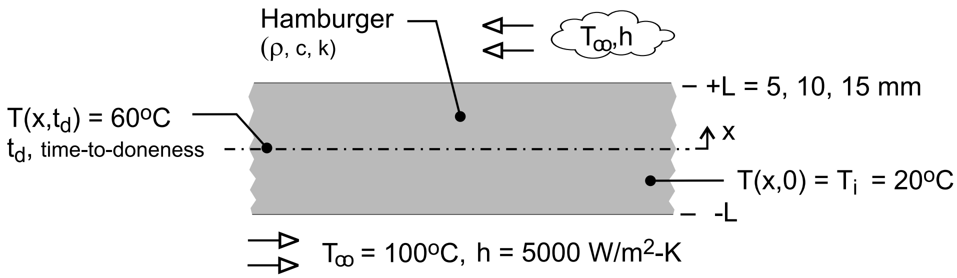
```
// Friction-surface interface conditions
T11 = T12
q''f = LOOKUPVAL(HFVST16,1,t,2)      // Applied heat flux, W/m^2; specified by Lookup Table
/* See HELP (Solver, Lookup Tables).  The look-up table, file name "HFVST16'  contains
            0       16e6
            0.1     0
            100     0           */
q''rp + q''cp = q''f            // Frictional heat flux

// Nodal equations - composite plate
// Frictional material, nodes 12-16
/* Node 12: From an energy on the CV about node 12 */
 kfm * (T13 - T12) / deltax + q''cp =  rhofm * cpfm * deltax / 2 * der(T12,t)
/* Node 13: interior node; e and w labeled 08 and 06. */
rhofm*cpfm*der(T13,t) = fd_1d_int(T13,T14,T12,kfm,qdot,deltax)
……………………………….
/* Node 16: interior node; e and w labeled 11 and 09. */
rhofm*cpfm*der(T16,t) = fd_1d_int(T16,T17,T15,kfm,qdot,deltax)
// Interface between friction material and steel, node 17
/* Node 17: From an energy on the CV about node 17 */
kfm * (T16 - T17) / deltax + ks * (T18 - T17) / deltax = RHS
RHS = ( (rhofm * cpfm * deltax /2) + (rhos * cps * deltax /2) ) * der(T17,t)
// Steel, nodes 18-22
/* Node 18: interior node; e and w labeled 03 and 01. */
rhos*cps*der(T18,t) = fd_1d_int(T18,T19,T17,ks,qdot,deltax)
……………………………….
/* Node 22: interior node; e and w labeled 21 and 21.  Symmetry condition. */
rhos*cps*der(T22,t) = fd_1d_int(T22,T21,T21,ks,qdot,deltax)
// qdot = 0

// Input variables
// Ti = 40                  // Initial temperature; entered during Solve
deltax = 0.0001
rhos = 7800                 // Steel properties
cps = 500
ks = 40
rhofm = 1150                //Friction material properties
cpfm = 1650
kfm = 4

// Conversions, to facilitate graphing
t_ms = t * 1000
qf_7 = q''f / 1e7
qrp_7 = q''rp / 1e7
qcp_7 = q''cp / 1e7
```

PROBLEM 5.119

KNOWN: Hamburger patties of thickness 2L = 10, 20 and 30 mm, initially at a uniform temperature T_i = 20°C, are grilled on both sides by a convection process characterized by T_∞ = 100°C and h = 5000 W/m^2·K.

FIND: (a) Determine the relationship between *time-to-doneness*, t_d, and patty thickness. Doneness criteria is 60°C at the center. Use *FEHT* and the *IHT Models|Transient Conduction|Plane Wall.* (b) Using the results from part (a), estimate the *time-to-doneness* if the initial temperature is 5 °C rather than 20°C. Calculate values using the *IHT* model, and determine the relationship between time-to-doneness and initial temperature.

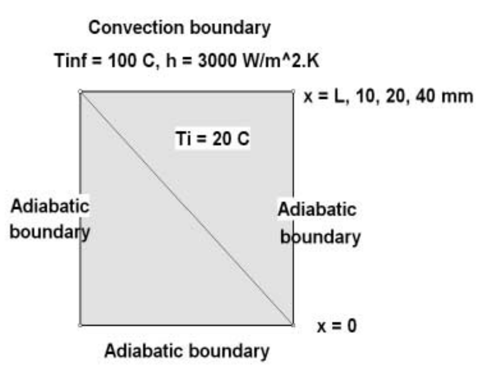
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, and (2) Constant properties are approximated as those of water at 300 K.

PROPERTIES: *Table A-6*, Water (300K), ρ = 1000 kg/m^3, c = 4179 J/kg·K, k = 0.613 W/m·K.

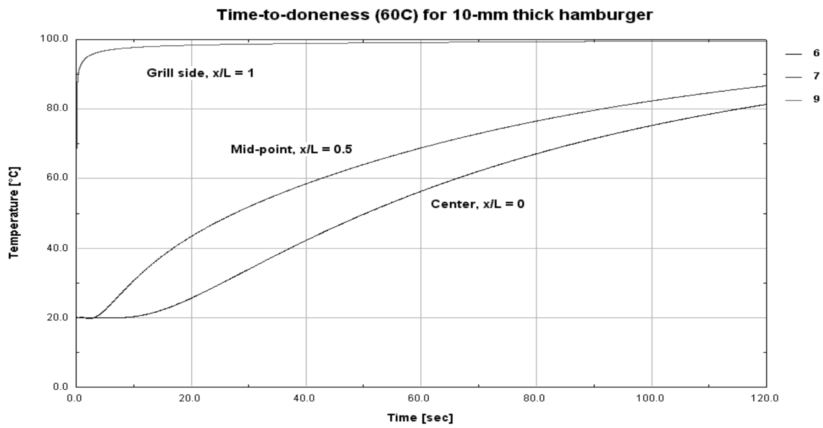
ANALYSIS: (a) To determine T(0, t_d), the center point temperature at the *time-to-doneness* time, t_d, a one-dimensional shape as shown in the *FEHT* screen below is drawn, and the material properties, boundary conditions, and initial temperature are specified. With the *Run|Calculate* command, the early integration steps are made very fine to accommodate the large temperature-time changes occurring near x = L. Use the *Run|Continue* command (see *FEHT HELP*) for the second and subsequent steps of the integration. This sequence of *Start-(Step)-Stop* values was used: 0 (0.001) 0.1 (0.01) 1 (0.1) 120 (1.0) 840 s.



Continued …..

PROBLEM 5.119 (Cont.)

Using the *View|Temperature vs. Time* command, the temperature-time histories for the x/L = 0 (center), 0.5, and 1.0 (grill side) are plotted and shown below for the 2L = 10 mm thick patty.



Using the *View|Temperatures* command, the time slider can be adjusted to read t_d, when the center point, x = 0, reaches 60°C. See the summary table below.

The *IHT ready-to-solve* model in *Models|Transient Conduction|Plane Wall* is based upon Eq. 5.40 and permits direct calculation of t_d when $T(0,t_d) = 60°C$ for patty thickness 2L = 10, 20 and 30 mm and initial temperatures of 20 and 5°C. The *IHT* code is provided in Comment 3, and the results are tabulated below.

	Solution method	Time-to-doneness, t (s)			T_i (C)
		Patty thickness, 2L (mm)			
		10	20	30	
	FEHT	66.2	264.5	591	20
	IHT	67.7 80.2	264.5 312.2	590.4 699.1	20 5
	Eq. 5.40 (see Comment 4)	x 	x x	 x	5 20

Considering the *IHT* results for $T_i = 20°C$, note that when the thickness is doubled from 10 to 20 mm, t_d is (264.5/67.7=) 3.9 times larger. When the thickness is trebled, from 10 to 30 mm, t_d is (590.4/67.7=) 8.7 times larger. We conclude that, t_d is nearly proportional to L^2, rather than linearly proportional to thickness.

Continued …..

(b) The temperature span for the cooking process ranges from T_∞ = 100 to T_i = 20 or 5°C. The differences are (100-20 =) 80 or (100-5 =) 95°C. If t_d is proportional to the overall temperature span, then we expect t_d for the cases with T_i = 5°C to be a factor of (95/80 =) 1.19 higher (approximately 20%) than with T_i = 20°C. From the tabulated results above, for the thickness 2L = 10, 20 and 30 mm, the t_d with T_i = 5°C are (80.2/67.7 =) 1.18, (312 / 264.5 =) 1.18, and (699.1/590.4 =) 1.18, respectively, higher than with T_i = 20°C. We conclude that t_d is nearly proportional to the temperature span (T_∞ - T_i).

COMMENTS: (1) The results from the *FEHT* and *IHT* calculations are in excellent agreement. For this analysis, the *FEHT* model is more convenient to use as it provides direct calculations of the time-to-doneness. The *FEHT* tool allows the user to watch the cooking process. Use the *View*|*Temperature Contours* command, click on the *from start-to-stop* button, and observe how color band changes represent the temperature distribution as a function of time.

(2) It is good practice to check software tool analyses against hand calculations. Besides providing experience with the basic equations, you can check whether the tool was used or functioned properly. Using the one-term series solution, Eq. 5.40:

$$\theta_o^* = \frac{T(0,t_d) - T_\infty}{T_i - T_\infty} = C_1 \exp\left(-\zeta^2 Fo\right)$$

$$Fo = \alpha t_d / L^2 \qquad\qquad C_1, \zeta = (Bi), \text{ Table 5.1}$$

T_i (°C)	2L (mm)	θ_o^*	Bi	C_1	ζ_1	Fo	t_d (s)
20	10	0.5000	24.47	1.2707	1.5068	0.4108	70.0
5	30	0.4211	73.41	1.2729	1.5471	0.4622	709

The results are slightly higher than those from the *IHT* model, which is based upon a multiple- rather than single-term series solution.

(3) The *IHT* code used to obtain the tabulated results is shown below. Note that T_xt_trans is an intrinsic heat transfer function dropped into the *Workspace* from the *Models* window (see *IHT Help|Solver|Intrinsic Functions|Heat Transfer Functions*).

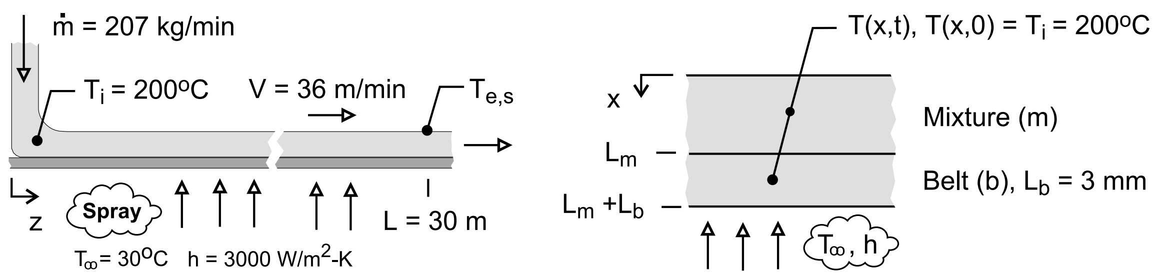
```
// Models | Transient Conduction | Plane Wall
/* Model: Plane wall of thickness 2L, initially with a uniform temperature T(x,0) = Ti, suddenly subjected
to convection conditions (Tinf,h).  */
// The temperature distribution is
T_xt = T_xt_trans("Plane Wall",xstar,Fo,Bi,Ti,Tinf)            // Eq 5.39
// The dimensionless parameters are
xstar = x / L
Bi = h * L / k                    // Eq 5.9
Fo= alpha * t / L^2               // Eq 5.33
alpha = k/ (rho * cp)

// Input parameters
x = 0                // Center point of meat
L = 0.005            // Meat half-thickness, m
//L = 0.010
//L = 0.015
T_xt = 60             // Doneness temperature requirement at center, x = 0; C
Ti = 20               // Initial uniform temperature
//Ti = 5
rho = 1000           // Water properties at 300 K
cp = 4179
k = 0.613
h = 5000              // Convection boundary conditions
Tinf = 100
```

PROBLEM 5.120

KNOWN: A process mixture at 200°C flows at a rate of 207 kg/min onto a 1-m wide conveyor belt traveling with a velocity of 36 m/min. The underside of the belt is cooled by a water spray.

FIND: The surface temperature of the mixture at the end of the conveyor belt, $T_{e,s,}$ using (a) *IHT* for writing and solving the FDEs, and (b) *FEHT*. Validate your numerical codes against an appropriate analytical method of solution.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the x-direction at any z-location, (2) Negligible heat transfer from mixture upper surface to ambient air, and (3) Constant properties.

PROPERTIES: Process mixture (m), $\rho_m = 960$ kg/m^3, c_m = 1700 J/kg·K, and k_m = 1.5 W/m·K; Conveyor belt (b), $\rho_b = 8000$ kg/ m^3, c_b = 460 J/kg·K, and k_b = 15 W/m·K.

ANALYSIS: From the conservation of mass requirement, the thickness of the mixture on the conveyor belt can be determined.

$$\dot{m} = \rho_m A_c V \qquad \text{where} \qquad A_c = W\,L_m$$

$$207\ \text{kg}/\text{min} \times 1\ \text{min}/60\,\text{s} = 960\ \text{kg}/\text{m}^3 \times 1\,\text{m} \times L_m \times 36\,\text{m}/\text{min} \times 1\,\text{min}/60\,\text{s}$$

$$L_m = 0.0060\,\text{m} = 6\ \text{mm}$$

The time that the mixture is in contact with the steel conveyor belt, referred to as the residence time, is

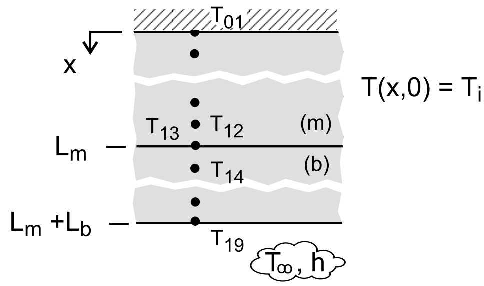
$$t_{res} = L_c / V = 30\ \text{m} / \left(36\ \text{m}/\text{min} \times 1\ \text{min}/60\ \text{s}\right) = 50\ \text{s}$$

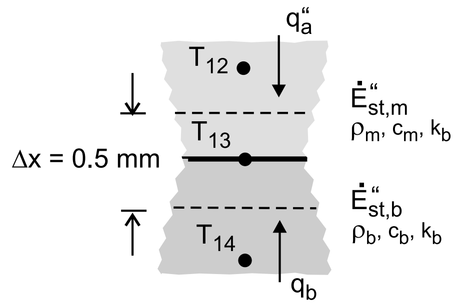
The composite system comprised of the belt, L_b = 3 mm, and mixture, L_m = 6 mm, as represented in the schematic above, is initially at a uniform temperature $T(x,0) = T_i = 200°C$ while at location z = 0, and suddenly is exposed to convection cooling (T_∞, h). We will calculate the mixture upper surface temperature after 50 s, $T(0, t_{res}) = T_{e,s}$.

(a) The nodal arrangement for the composite system is shown in the schematic below. The *IHT* model builder *Tools|Finite-Difference Equations|Transient* can be used to obtain the FDEs for nodes 01-12 and 14-19.

Continued …..

PROBLEM 5.120 (Cont.)





For the mixture-belt interface node 13, the FDE for the implicit method is derived from an energy balance on the control volume about the node as shown above.

$$\dot{E}''_{in} - \dot{E}''_{out} = \dot{E}''_{st}$$

$$q''_a + q''_b = \dot{E}''_{st,m} + \dot{E}''_{st,b}$$

$$k_m \frac{T_{12}^{p+1} - T_{13}^{p+1}}{\Delta x} + k_b \frac{T_{14}^{p+1} - T_{13}^{p+1}}{\Delta x} = \left(\rho_m c_m + \rho_b c_b\right)\left(\Delta x / 2\right)\frac{T_{13}^{p+1} - T_{13}^{p}}{\Delta t}$$

IHT code representing selected FDEs, nodes 01, 02, 13 and 19, is shown in Comment 4 below (Δx = 0.5 mm, Δt = 0.1 s). Note how the FDE for node 13 derived above is written in the *Workspace*. From the analysis, find

$T_{e,s} = T(0, 50s) = 54.8°C$ <

(b) Using *FEHT*, the composite system is drawn and the material properties, boundary conditions, and initial temperature are specified. The screen representing the system is shown below in Comment 5 with annotations on key features. From the analysis, find

$T_{e,s} = T(0, 50s) = 54.7°C$ <

COMMENTS: (1) Both numerical methods, *IHT* and *FEHT*, yielded the same result, 55°C. For the safety of plant personnel working in the area of the conveyor exit, the mixture exit temperature should be lower, like 43°C.

(2) By giving both regions of the composite the same properties, the analytical solution for the plane wall with convection, Section 5.5, Eq. 5.40, can be used to validate the *IHT* and *FEHT* codes. Using the *IHT Models|Transient Conduction|Plane Wall* for a 9-mm thickness wall with mixture thermophysical properties, we calculated the temperatures after 50 s for three locations: T(0, 50s) = 91.4°C; T(6 mm, 50s) = 63.6°C; and T(3 mm, 50s) = 91.4°C. The results from the *IHT* and *FEHT* codes agreed exactly.

(3) In view of the high heat removal rate on the belt lower surface, it is reasonable to assume that negligible heat loss is occurring by convection on the top surface of the mixture.

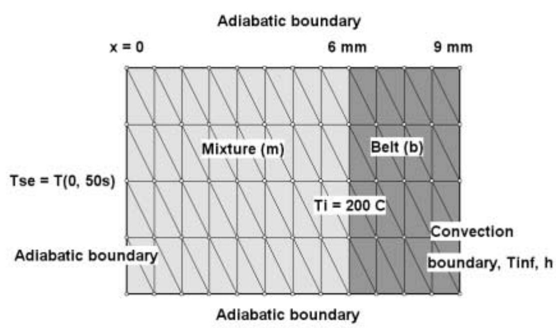
Continued …..

PROBLEM 5.120 (Cont.)

(4) The *IHT* code representing selected FDEs, nodes 01, 02, 13 and 19, is shown below. The FDE for node 13 was derived from an energy balance, while the others are written from the *Tools* pad.

```
// Finite difference equations from Tools, Nodes 01 -12 (mixture) and 14-19 (belt)
/* Node 01: surface node (w-orientation); transient conditions; e labeled 02. */
rhom * cm * der(T01,t) = fd_1d_sur_w(T01,T02,km,qdot,deltax,Tinf01,h01,q''a01)
q''a01 = 0        // Applied heat flux, W/m^2; zero flux shown
qdot = 0
Tinf01 = 20      // Arbitrary value
h01 = 1e-6       // Causes boundary to behave as adiabatic

/* Node 02: interior node; e and w labeled 03 and 01. */
rhom*cm*der(T02,t) = fd_1d_int(T02,T03,T01,km,qdot,deltax)

/* Node 19: surface node (e-orientation); transient conditions; w labeled 18. */
rhob * cb * der(T19,t) = fd_1d_sur_e(T19,T18,kb,qdot,deltax,Tinf19,h19,q''a19)
q''a19 = 0        // Applied heat flux, W/m^2; zero flux shown
Tinf19 = 30
h19 = 3000

// Finite-difference equation from energy balance on CV, Node 13
km*(T12 - T13)/deltax + kb*(T14 - T13)/deltax = (rhom*cm + rhob*cb) *(deltax/2)*der(T13,t)
```

(5) The screen from the *FEHT* analysis is shown below. It is important to use small time steps in the integration at early times. Use the *View|Temperatures* command to find the temperature of the mixture surface at t_{res} = 50 s.

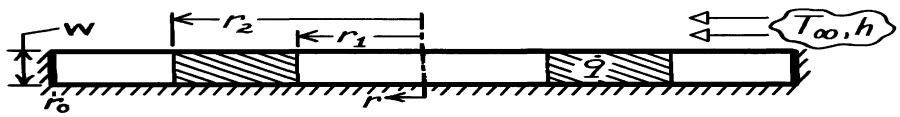


PROBLEM 5.121

KNOWN: Thin, circular-disc subjected to induction heating causing a uniform heat generation in a prescribed region; upper surface exposed to convection process.

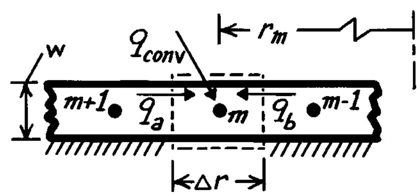
FIND: (a) Transient finite-difference equation for a node in the region subjected to induction heating, (b) Sketch the steady-state temperature distribution on T-r coordinates; identify important features.

SCHEMATIC:



ASSUMPTIONS: (1) Thickness w << r_o, such that conduction is one-dimensional in r-direction, (2) In prescribed region, $\dot{q}$ is uniform, (3) Bottom surface of disc is insulated, (4) Constant properties.

ANALYSIS: (a) Consider the nodal point arrangement for the region subjected to induction heating. The size of the control volume is $V = 2\pi\ r_m \cdot \Delta r \cdot w$. The energy conservation requirement for the node m has the form



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

with $\quad q_a + q_b + q_{conv} + \dot{q}V = \dot{E}_{st}.$

Recognizing that q_a and q_b are conduction terms and q_{conv} is the convection process,

$$k\left[2\pi\left[r_m - \frac{\Delta r}{2}\right]w\right]\frac{T_{m-1}^p - T_m^p}{\Delta r} + k\left[2\pi\left[r_m + \frac{\Delta r}{2}\right]w\right]\frac{T_{m+1}^p - T_m^p}{\Delta r}$$

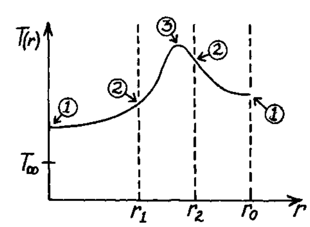
$$+h\left[2\pi\ r_m \cdot \Delta r\right]\left(T_\infty - T_m^p\right) + \dot{q}\left[2\pi\ r_m \cdot \Delta r \cdot w\right] = \rho\ c_p\left[2\pi\ r_m \cdot \Delta r \cdot w\right]\frac{T_m^{p+1} - T_m^p}{\Delta t}.$$

Upon regrouping, the finite-difference equation has the form,

$$T_m^{p+1} = Fo\left[\left[1 - \frac{\Delta r}{2r_m}\right]T_{m-1}^p + \left[1 + \frac{\Delta r}{2r_m}\right]T_{m+1}^p + Bi\left[\frac{\Delta r}{w}\right]T_\infty + \frac{\dot{q}\Delta r^2}{k}\right] + \left[1 - 2Fo - Bi \cdot Fo\left[\frac{\Delta r}{w}\right]\right]T_m^p \qquad <$$

where $\qquad Fo = \alpha\Delta t/\Delta r^2 \qquad Bi = h\Delta r/k.$

(b) The steady-state temperature distribution has these features:



1. Zero gradient at r = 0, r_0
2. No discontinuity at r_1, r_2
3. T_{max} occurs in region $r_1 < r < r_2$

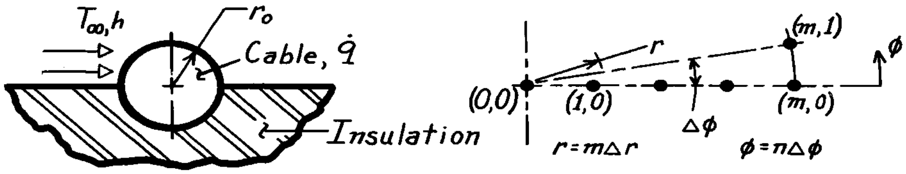
Note also, distribution will not be linear anywhere; distribution is <u>not</u> parabolic in $r_1 < r < r_2$ region.

PROBLEM 5.122

KNOWN: An electrical cable experiencing uniform volumetric generation; the lower half is well insulated while the upper half experiences convection.

FIND: (a) Explicit, finite-difference equations for an interior node (m,n), the center node (0,0), and an outer surface node (M,n) for the convective and insulated boundaries, and (b) Stability criterion for each FDE; identify the most restrictive criterion.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional (r,ϕ), transient conduction, (2) Constant properties, (3) Uniform $\dot{q}$.

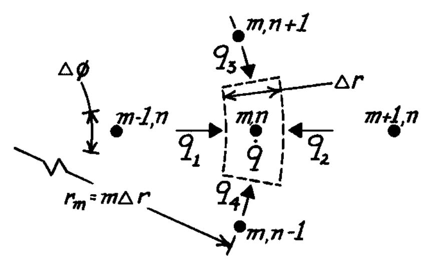
ANALYSIS: The explicit, finite-difference equations may be obtained by applying energy balances to appropriate control volumes about the node of interest. Note the coordinate system defined above where $(r,\phi) \rightarrow (m\Delta r, n\Delta\phi)$. The stability criterion is determined from the coefficient associated with the node of interest.

Interior Node (m,n). The control volume for an interior node is

$$V = r_m \Delta\phi \cdot \Delta r \cdot \ell$$

(with $r_m = m\Delta r$, $\ell = 1$) where ℓ is the length normal to the page. The conservation of energy requirement is

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$



$$\left(q_1 + q_2\right)_r + \left(q_3 + q_4\right)_\theta + \dot{q}V = \rho\, cV \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}$$

$$k \cdot \left[m - \frac{1}{2}\right]\Delta r \cdot \Delta\phi \cdot \frac{T_{m-1,n}^{p} - T_{m,n}^{p}}{\Delta r} + k \cdot \left[m + \frac{1}{2}\right]\Delta r \cdot \Delta\phi \cdot \frac{T_{m+1,n}^{p} - T_{m,n}^{p}}{\Delta r} + k \cdot \Delta r \cdot \frac{T_{m,n+1}^{p} - T_{m,n}^{p}}{\left(m\Delta r\right)\Delta\phi}$$

$$+ k \cdot \Delta r \cdot \frac{T_{m,n-1}^{p} - T_{m,n}^{p}}{\left(m\Delta r\right)\Delta\phi} + \dot{q}\left(m\Delta r \cdot \Delta\phi\right) \cdot \Delta r = \rho\, c\left(m\Delta r \cdot \Delta\phi\right) \cdot \Delta r \cdot \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} \qquad (1)$$

Define the Fourier number as

$$Fo = \frac{k}{\rho\, c} \cdot \frac{\Delta t}{\Delta r^2} = \frac{\alpha \Delta t}{\Delta r^2} \qquad (2)$$

and then regroup the terms of Eq. (1) to obtain the FDE,

$$T_{m,n}^{p+1} = Fo\left\{\frac{m - 1/2}{m} T_{m-1,n}^{p} + \frac{m + 1/2}{m} T_{m+1,n}^{p} + \frac{1}{\left(m\Delta\phi\right)^2}\left(T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + \frac{\dot{q}}{k}\Delta r^2\right\}$$

$$+ \left\{-Fo\left[2 + \frac{2}{\left(m\Delta\phi\right)^2}\right] + 1\right\} T_{m,n}^{p}. \qquad (3) <$$

Continued …..

PROBLEM 5.122 (Cont.)

The stability criterion requires that the last term on the right-hand side in braces be positive. That is, the coefficient of $T^p_{m,n}$ must be positive and the stability criterion is

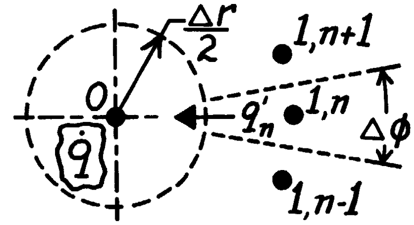
$$\text{Fo} \le 1/2\left[1 + 1/(m\Delta\phi)^2\right] \qquad (4)$$

Note that, for m >> 1/2 and $(m\Delta\phi)^2$ >>1, the FDE takes the form of a 1-D cartesian system.

Center Node (0,0). For the control volume,

$V = \pi(\Delta r/2)^2 \cdot 1$. The energy balance is

$\dot{E}'_{in} - \dot{E}'_{out} + \dot{E}'_g = \dot{E}'_{st}$ where $\dot{E}'_{in} = \Sigma q'_n$.



$$\sum_{n=0}^{N} k \cdot \left[\frac{\Delta r}{2}\Delta\phi\right] \cdot \frac{T^p_{1,n} - T^p_o}{\Delta r} + \dot{q}\pi\left[\frac{\Delta r}{2}\right]^2$$

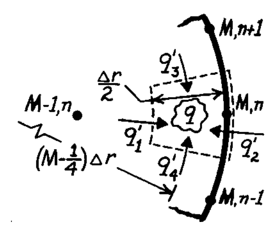
$$= \rho\, c \cdot \pi\left[\frac{\Delta r}{2}\right]^2 \frac{T^{p+1}_o - T^p_o}{\Delta t} \qquad (5)$$

where N = $(2\pi/\Delta\phi) - 1$, the total number of q_n. Using the definition of Fo, find

$$T^{p+1}_o = 4\text{Fo}\left\{\frac{1}{N+1}\sum_{n=0}^{N} T^p_{1,n} + \frac{\dot{q}}{4k}\Delta r^2\right\} + (1 - 4\text{Fo})T^p_o. \quad <$$

By inspection, the stability criterion is Fo ≤ 1/4. (7)

Surface Nodes (M,n). The control volume for the surface node is V = (M - ¼)ΔrΔϕ·Δr/2.1.



From the energy balance,

$$\dot{E}'_{in} - \dot{E}'_{out} + \dot{E}'_g = (q'_1 + q'_2)_r + (q'_3 + q'_4)_\phi + \dot{q}V = \dot{E}'_{st}$$

$$k \cdot (M - 1/2)\Delta r \cdot \Delta\phi \frac{T^p_{M-1,n} - T^p_{M,n}}{\Delta r} + h(M\Delta r \cdot \Delta\phi)\left(T_\infty - T^p_{M,n}\right) + k \cdot \frac{\Delta r}{2} \cdot \frac{T^p_{M,n+1} - T^p_{M,n}}{(M\Delta r)\Delta\phi}$$

$$+k \cdot \frac{\Delta r}{2} \cdot \frac{T^p_{M,n-1} - T^p_{M,n}}{(M\Delta r)\Delta\phi} + \dot{q}\left[(M - 1/4)\Delta r \cdot \Delta\phi \cdot \frac{\Delta r}{2}\right] = \rho\, c\left[(M - 1/4)\Delta r \cdot \Delta\phi \cdot \frac{\Delta r}{2}\right]\frac{T^{p+1}_{M,n} - T^p_{M,n}}{\Delta t}.$$

Regrouping and using the definitions for Fo = $\alpha\Delta t/\Delta r^2$ and Bi = hΔr/k,

$$T^{p+1}_{m,n} = \text{Fo}\left\{2\frac{M - 1/2}{M - 1/4}T^p_{M-1,n} + \frac{1}{(M-1/4)M(\Delta\phi)^2}\left(T^p_{M,n+1} - T^p_{M,n-1}\right) + 2\text{Bi} \cdot T_\infty + \frac{\dot{q}}{k}\Delta r^2\right\}$$

$$+\left\{1 - 2\text{Fo}\left[\frac{M-1/2}{M-1/4} + \text{Bi} \cdot \frac{M}{M - 1/4} + \frac{1}{(M - 1/4)M(\Delta\phi)^2}\right]\right\}T^p_{M,n}. \qquad (8) \quad <$$

The stability criterion is
$$\text{Fo} \le \frac{1}{2}\left[\frac{M - 1/2}{M - 1/4} + \text{Bi}\frac{M}{M - 1/4} + \frac{1}{(M - 1/4)M(\Delta\phi)^2}\right]. \qquad (9)$$

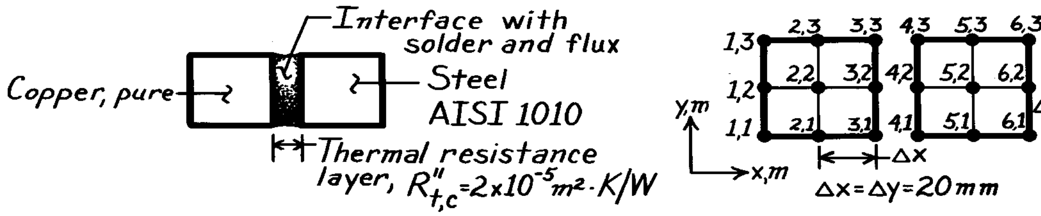
To determine which stability criterion is most restrictive, compare Eqs. (4), (7) and (9). The most restrictive (lowest Fo) has the largest denominator. For small values of m, it is not evident whether Eq. (7) is more restrictive than Eq. (4); Eq. (4) depends upon magnitude of Δϕ. Likewise, it is not clear whether Eq. (9) will be more or less restrictive than Eq. (7). Numerical values must be substituted.

PROBLEM 5.123

KNOWN: Initial temperature distribution in two bars that are to be soldered together; interface contact resistance.

FIND: (a) Explicit FDE for $T_{4,2}$ in terms of Fo and $Bi = \Delta x/k\,R''_{t,c}$; stability criterion, (b) $T_{4,2}$ one time step after contact is made if Fo = 0.01 and value of Δt; whether the stability criterion is satisfied.

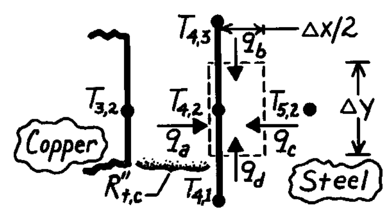
SCHEMATIC:



PROPERTIES: *Table A-1*, Steel, AISI 1010 (1000K): k = 31.3 W/m·K, c = 1168 J/kg·K, ρ = 7832 kh/m^3.

ASSUMPTIONS: (1) Two-dimensional transient conduction, (2) Constant properties, (3) Interfacial solder layer has negligible thickness.

ANALYSIS: (a) From an energy balance on the control volume V = (Δx/2)·Δy·1.



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$q_a + q_b + q_c + q_d = \rho\, cV \frac{T_{4,2}^{p+1} - T_{4,2}^{p}}{\Delta t}.$$

Note that $q_a = \left(\Delta T/R''_{t,c}\right)A_c$ while the remaining q_i are conduction terms,

$$\frac{1}{R''_{t,c}}\left(T_{3,2}^p - T_{4,2}^p\right)\Delta y + k\left(\Delta x/2\right)\frac{\left(T_{4,3}^p - T_{4,2}^p\right)}{\Delta y} + k\left(\Delta y\right)\frac{\left(T_{5,2}^p - T_{4,2}^p\right)}{\Delta x}$$

$$+ k\left(\Delta x/2\right)\frac{\left(T_{4,1}^p - T_{4,2}^p\right)}{\Delta y} = \rho\, c\left[\left(\Delta x/2\right)\cdot\Delta y\right]\frac{T_{4,2}^{p+1} - T_{4,2}^p}{\Delta t}.$$

Defining $Fo \equiv \left(k/\rho\, c\right)\Delta t/\Delta x^2$ and $Bi_c \equiv \Delta y/R''_{t,c}k$, regroup to obtain

$$T_{4,2}^{p+1} = Fo\left(T_{4,3}^p + 2T_{5,2}^p + T_{4,1}^p + 2Bi\,T_{3,2}^p\right) + \left(1 - 4Fo - 2FoBi\right)T_{4,2}^p. \qquad \text{<}$$

The stability criterion requires the coefficient of the $T_{4,2}^p$ term be zero or positive,

$$\left(1 - 4Fo - 2FoBi\right) \ge 0 \qquad \text{or} \qquad Fo \le 1/\left(4 + 2Bi\right) \qquad \text{<}$$

(b) For $Fo = 0.01$ and $Bi = 0.020\text{m}/\left(2\times10^{-5}\text{m}^2\cdot\text{K/W}\times31.3\text{W/m}\cdot\text{K}\right) = 31.95$,

$$T_{4,2}^{p+1} = 0.01\left(1000 + 2\times900 + 1000 + 2\times31.95\times700\right)\text{K} + \left(1 - 4\times0.01 - 2\times0.01\times31.95\right)1000\text{K}$$

$$T_{4,2}^{p+1} = 485.30\text{K} + 321.00\text{K} = 806.3\text{K}. \qquad \text{<}$$

With Fo = 0.01, the time step is

$$\Delta t = Fo\,\Delta x^2\left(\rho\, c/k\right) = 0.01\left(0.020\text{m}\right)^2\left(7832\text{kg/m}^3\times1168\text{J/kg}\cdot\text{K}/31.3\text{W/m}\cdot\text{K}\right) = 1.17\text{s}. \qquad \text{<}$$

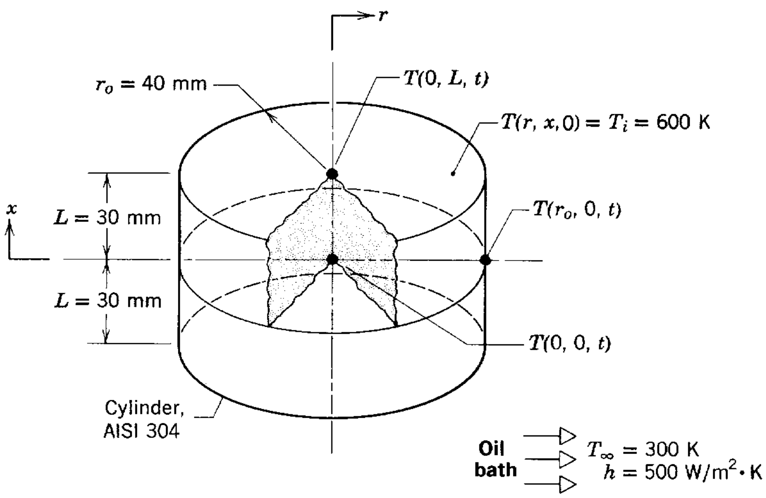
With Bi = 31.95 and Fo = 0.01, the stability criterion, Fo ≤ 0.015, is satisfied. <

PROBLEM 5.124

KNOWN: Stainless steel cylinder of Ex. 5.7, 80-mm diameter by 60-mm length, initially at 600 K, suddenly quenched in an oil bath at 300 K with h = 500 $W/m^2 \cdot K$. Use the ready-to-solve model in the *Examples* menu of *FEHT* to obtain the following solutions.

FIND: (a) Calculate the temperatures T(r, x ,t) after 3 min: at the cylinder center, T(0, 0, 3 mm), at the center of a circular face, T(0, L, 3 min), and at the midheight of the side, $T(r_o, 0, 3\text{ min})$; compare your results with those in the example; (b) Calculate and plot temperature histories at the cylinder center, T(0, 0, t), the mid-height of the side, $T(r_o, 0, t)$, for $0 \le t \le 10$ min; use the *View/Temperature vs. Time* command; comment on the gradients and what effect they might have on phase transformations and thermal stresses; (c) Using the results for the total integration time of 10 min, use the *View/Temperature Contours* command; describe the major features of the cooling process shown in this display; create and display a 10-isotherm temperature distribution for t = 3 min; and (d) For the locations of part (a), calculate the temperatures after 3 min if the convection coefficient is doubled (h = 1000 $W/m^2 \cdot K$); for these two conditions, determine how long the cylinder needs to remain in the oil bath to achieve a safe-to touch surface temperature of 316 K. Tabulate and comment on the results of your analysis.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction in r- and x-coordinates, (2) Constant properties.

PROPERTIES: Stainless steel (*Example 5.7*): ρ = 7900 kg/m^3, c = 256 J/kg·K, k = 17.4 W/m·K.

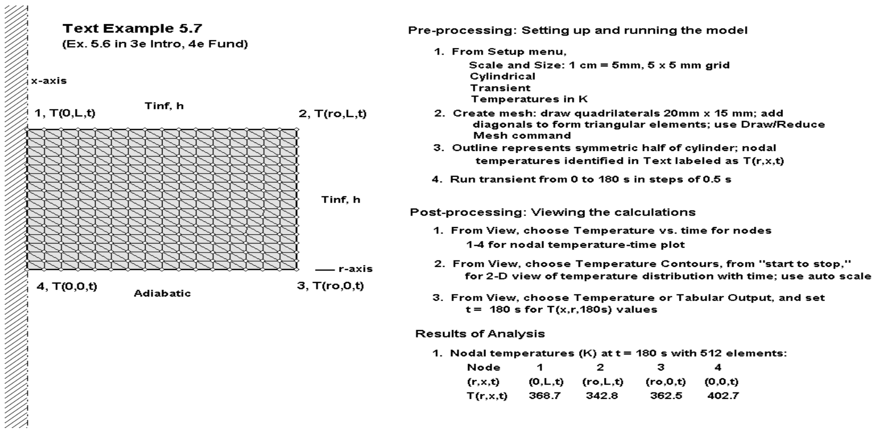
ANALYSIS: (a) The *FEHT ready-to-solve* model for Example 5.7 is accessed through the *Examples* menu and the annotated *Input* page is shown below. The following steps were used to obtain the solution: (1) Use the *Draw* |*Reduce Mesh* command three times to create the 512-element mesh; (2) In *Run*, click on *Check*, (3) In *Run*, press *Calculate* and hit *OK* to initiate the solver; and (4) Go to the *View* menu, select *Tabular Output* and read the nodal temperatures 4, 1, and 3 at $t = t_o$ = 180 s. The tabulated results below include those from the n-term series solution used in the *IHT* software.

Continued …..

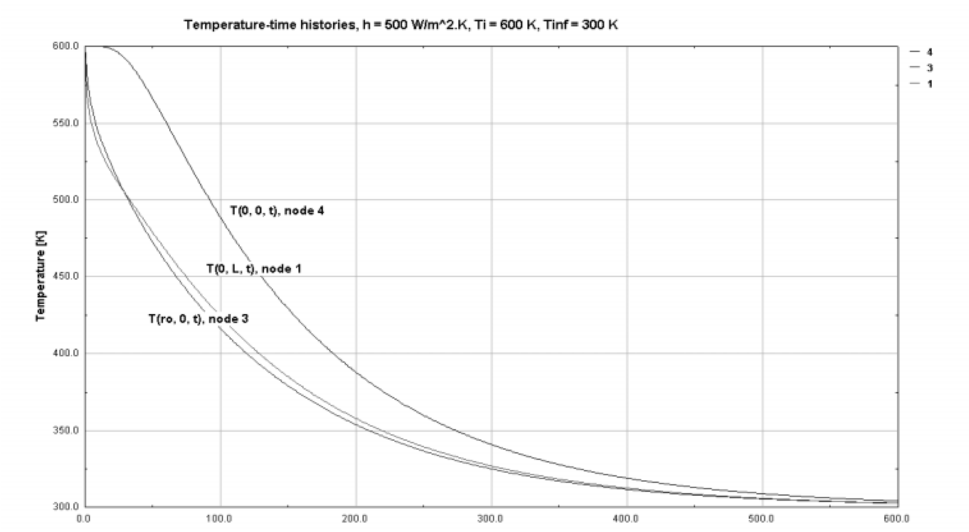
PROBLEM 5.124 (Cont.)

(r, x, t_o)	FEHT node	$T(r, x, t_o)$ (K) *FEHT*	$T(r, x, t_o)$ (K) 1-term series	$T(r, x, t_o)$ (K) n-term series
$0, 0, t_o$	4	402.7	405	402.7
$0, L, t_o$	1	368.7	372	370.5
$ro, 0, t_o$	3	362.5	365	362.4

Note that the one-term series solution results of Example 5.7 are systematically lower than those from the 512-element, finite-difference *FEHT* analyses. The *FEHT* results are in excellent agreement with the *IHT* n-term series solutions for the x = 0 plane nodes (4,3), except for the x = L plane node (1).



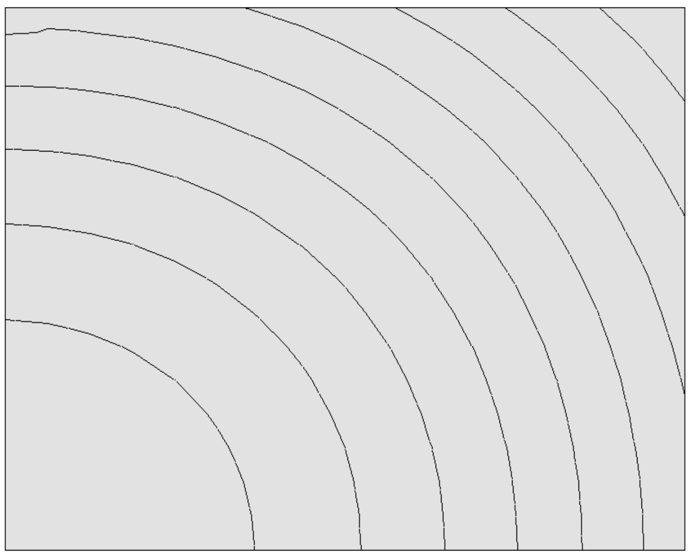
(b) Using the *View Temperature vs. Time* command, the temperature histories for nodes 4, 1, and 3 are plotted in the graph shown below. There is very small temperature difference between the locations on the surface, (node 1; 0, L) and (node 3; r_o, 0). But, the temperature difference between these surface locations and the cylinder center (node 4; 0, 0) is large at early times. Such differences wherein locations cool at considerably different rates could cause variations in microstructure and hence, mechanical properties, as well as induce thermal stresses.



Continued …..

PROBLEM 5.124 (Cont.)

(c) Use the *View|Temperature Contours* command with the shaded band option for the isotherm contours. Selecting the *From Start to Stop* time option, see the display of the contours as the cylinder cools during the quench process. The "movie" shows that cooling initiates at the corner (r_o,L,t) and the isotherms quickly become circular and travel toward the center (0,0,t). The 10-isotherm distribution for t = 3 min is shown below.



(d) Using the *FEHT* model with convection coefficients of 500 and 1000 $W/m^2 \cdot K$, the temperatures at $t = t_o = 180$ s for the three locations of part (a) are tabulated below.

	$h = 500\ W/m^2 \cdot K$	$h = 1000\ W/m^2 \cdot K$
$T(0, 0, t_o)$, K	402.7	352.8
$T(0, L, t_o)$, K	368.7	325.8
$T(r_o, 0, t_o)$, K	362.5	322.1

Note that the effect of doubling the convection coefficient is to reduce the temperature at these locations by about 40°C. The time the cylinder needs to remain in the oil bath to achieve the *safe-to-touch* surface temperature of 316 K can be determined by examining the temperature history of the location (node1; 0, L). For the two convection conditions, the results are tabulated below. Doubling the coefficient reduces the cooling process time by 40 %.

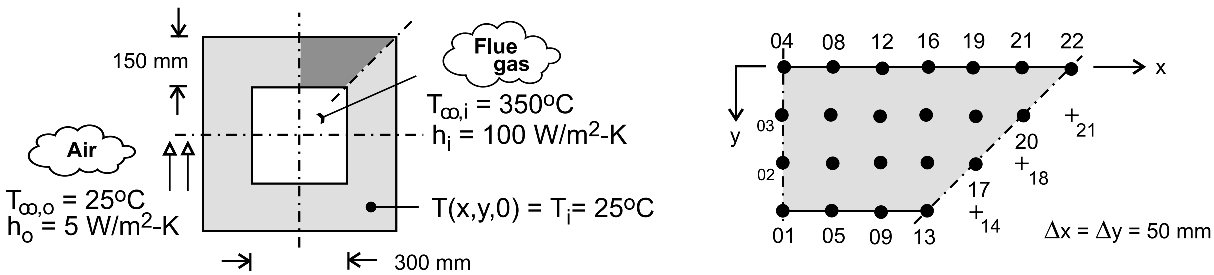
$T(0, L, t_o)$	$h\ (W/m^2 \cdot K)$	t_o (s)
316	500	370
316	1000	219

PROBLEM 5.125

KNOWN: Flue of square cross-section, initially at a uniform temperature is suddenly exposed to hot flue gases. See Problem 4.57.

FIND: Temperature distribution in the wall 5, 10, 50 and 100 hours after introduction of gases using the *implicit* finite-difference method.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional transient conduction, (2) Constant properties.

PROPERTIES: Flue (given): k = 0.85 W/m·K, $\alpha = 5.5 \times 10^{-7}$ m²/s.

ANALYSIS: The network representing the flue cross-sectional area is shown with Δx = Δy = 50mm. Initially all nodes are at T_i = 25°C when suddenly the interior and exterior surfaces are exposed to convection processes, ($T_{\infty,i}$, h_i) and ($T_{\infty,o}$, h_o), respectively Referring to the network above, note that there are four types of nodes: interior (02, 03, 06, 07, 10, 11, 14, 15, 17, 18, 20); plane surfaces with convection (interior – 01, 05, 09); interior corner with convection (13), plane surfaces with convection (exterior – 04, 08, 12, 16, 19, 21); and, exterior corner with convection. The system of finite-difference equations representing the network is obtained using *IHT|Tools|Finite-difference equations|Two-dimensional|Transient*. The *IHT* code is shown in Comment 2 and the results for t = 5, 10, 50 and 100 hour are tabulated below.

Node 17 $$(1+4\text{Fo})T_{17}^{p+1} - \text{Fo}\left(T_{18}^{p+1} + T_{14}^{p+1} + T_{18}^{p+1} + T_{14}^{p+1}\right) = T_{17}^{p}$$

Node 13 $$\left[1+4\text{Fo}\left[1+\frac{1}{3}\text{Bi}_i\right]\right]T_{13}^{p+1} - \frac{2}{3}\text{Fo}\left(2T_{14}^{p+1} + T_9 + 2T_{14}^{p+1} + T_9^{p+1}\right) = T_{13}^{p} + \frac{4}{3}\text{Bi}_i \cdot \text{Fo} \cdot T_{\infty,i}$$

Node 12 $$\left(1+2\text{Fo}\left(2+\text{Bi}_o\right)\right)T_{12}^{p+1} - \text{Fo}\left(2T_{11}^{p+1} + T_{16}^{p+1} + T_8^{p+1}\right) = T_{12}^{p} + 2\text{Bi}_o \cdot \text{Fo} \cdot T_{\infty,o}$$

Node 22 $$\left(1+4\text{Fo}\left(1+\text{Bi}_o\right)\right)T_{22}^{p+1} - 2\text{Fo}\left(T_{21}^{p+1} + T_{21}^{p+1}\right) = T_{22}^{p} + 4\text{Bi}_o \cdot \text{Fo} \cdot T_{\infty,o}$$

Numerical values for the relevant parameters are:

$$\text{Fo} = \frac{\alpha\Delta t}{\Delta x^2} = \frac{5.5\times10^{-6}\text{m}^2/\text{s}\times3600\text{s}}{(0.050\text{m})^2} = 7.92000$$

$$\text{Bi}_o = \frac{h_o\Delta x}{k} = \frac{5\ \text{W/m}^2\cdot\text{K}\times0.050\text{m}}{0.85\ \text{W/m}\cdot\text{K}} = 0.29412$$

$$\text{Bi}_i = \frac{h_i\Delta x}{k} = \frac{100\ \text{W/m}^2\cdot\text{K}\times0.050\text{m}}{0.85\ \text{W/m}\cdot\text{K}} = 5.88235$$

The system of FDEs can be represented in matrix notation, [A][T] = [C]. The coefficient matrix [A] and terms for the right-hand side matrix [C] are given on the following page.

Continued …..

PROBLEM 5.125 (Cont.)

	The coefficient matrix [A]																						*RHS matrix [C]*
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
1	E	2	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_1^p - 7331.1765$
2	1	F	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_2^p$
3	0	1	F	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_3^p$
4	0	0	2	G	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_4^p - 175.38235$
5	1	0	0	0	E	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_5^p - 7331.1765$
6	0	1	0	0	1	F	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_6^p$
7	0	0	1	0	0	1	F	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	$-0.12626T_7^p$
8	0	0	0	1	0	0	2	G	0	0	0	1	0	0	0	0	0	0	0	0	0	0	$-0.12626T_8^p - 175.37235$
9	0	0	0	0	1	0	0	0	E	2	0	0	1	0	0	0	0	0	0	0	0	0	$-0.12626T_9^p - 7331.1765$
10	0	0	0	0	0	1	0	0	1	F	1	0	0	1	0	0	0	0	0	0	0	0	$-0.12626T_{10}^p$
11	0	0	0	0	0	0	1	0	0	1	F	1	0	0	1	0	0	0	0	0	0	0	$-0.12626T_{11}^p$
12	0	0	0	0	0	0	0	1	0	0	2	G	0	0	0	1	0	0	0	0	0	0	$-0.12626T_{12}^p - 175.38235$
13	0	0	0	0	0	0	0	0	4	0	0	0	H	8	0	0	0	0	0	0	0	0	$-0.37879T_{13}^p - 14,658.824$
14	0	0	0	0	0	0	0	0	0	1	0	0	1	F	1	0	1	0	0	0	0	0	$-0.12626T_{14}^p$
15	0	0	0	0	0	0	0	0	0	0	1	0	0	1	F	1	0	1	0	0	0	0	$-0.12626T_{15}^p$
16	0	0	0	0	0	0	0	0	0	0	0	1	0	0	2	G	0	0	1	0	0	0	$-0.12626T_{16}^p - 175.38235$
17	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	F	2	0	0	0	0	$-0.12626T_{17}^p$
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	F	1	1	0	0	$-0.12626T_{18}^p$
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	G	0	1	0	$-0.12626T_{19}^p - 175.38235$
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	F	2	0	$-0.12626T_{20}^p$
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	G	1	$-0.12626T_{21}^p - 175.38235$
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	K	$-0.12626T_{22}^p - 350.76471$

E = −15.89096 F = −4.12626 G = −4.71450 H = −35.90819 K = −5.30274

For this problem a stock computer program was used to obtain the solution matrix [T]. The initial temperature distribution was $T_m^0 = 298\text{K}$. The results are tabulated below.

Node/time (h)	T(m,n) (C) 0	5	10	50	100
T01	25	335.00	338.90	340.20	340.20
T02	25	248.00	274.30	282.90	282.90
T03	25	179.50	217.40	229.80	229.80
T04	25	135.80	170.30	181.60	181.60
T05	25	334.50	338.50	339.90	339.90
T06	25	245.30	271.90	280.80	280.80
T07	25	176.50	214.60	227.30	227.30
T08	25	133.40	168.00	179.50	179.50
T09	25	332.20	336.60	338.20	338.20
T10	25	235.40	263.40	273.20	273.20
T11	25	166.40	205.40	219.00	219.00
T12	25	125.40	160.40	172.70	172.70
T13	25	316.40	324.30	327.30	327.30
T14	25	211.00	243.00	254.90	254.90
T15	25	146.90	187.60	202.90	202.90
T16	25	110.90	146.70	160.20	160.20
T17	25	159.80	200.50	216.20	216.20
T18	25	117.40	160.50	177.50	177.50
T19	25	90.97	127.40	141.80	141.80
T20	25	90.62	132.20	149.00	149.00
T21	25	72.43	106.70	120.60	120.60
T22	25	59.47	87.37	98.89	98.89

COMMENTS: (1) Note that the steady-state condition is reached by t = 5 hours; this can be seen by comparing the distributions for t = 50 and 100 hours. Within 10 hours, the flue is within a few degrees of the steady-state condition.

Continued …..

PROBLEM 5.125 (Cont.)

(2) The *IHT* code for performing the numerical solution is shown in its entirety below. Use has been made of symmetry in writing the FDEs. The tabulated results above were obtained by copying from the *IHT Browser* and pasting the desired columns into EXCEL.

```
// From Tools|Finite-difference equations|Two-dimensional|Transient
// Interior surface nodes, 01, 05, 09, 13
/* Node 01: plane surface node, s-orientation; e, w, n labeled 05, 05, 02 . */
rho * cp * der(T01,t) = fd_2d_psur_s(T01,T05,T05,T02,k,qdot,deltax,deltay,Tinfi,hi,q''a)
q''a = 0          // Applied heat flux, W/m^2; zero flux shown
qdot = 0
rho * cp * der(T05,t) = fd_2d_psur_s(T05,T09,T01,T06,k,qdot,deltax,deltay,Tinfi,hi,q''a)
rho * cp * der(T09,t) = fd_2d_psur_s(T09,T13,T05,T10,k,qdot,deltax,deltay,Tinfi,hi,q''a)
/* Node 13: internal corner node, w-s orientation; e, w, n, s labeled 14, 09, 14, 09. */
rho * cp * der(T13,t) = fd_2d_ic_ws(T13,T14,T09,T14,T09,k,qdot,deltax,deltay,Tinfi,hi,q''a)

// Interior nodes, 02, 03, 06, 07, 10, 11, 14, 15, 18, 20
/* Node 02: interior node; e, w, n, s labeled 06, 06, 03, 01. */
rho * cp * der(T02,t) = fd_2d_int(T02,T06,T06,T03,T01,k,qdot,deltax,deltay)
rho * cp * der(T03,t) = fd_2d_int(T03,T07,T07,T04,T02,k,qdot,deltax,deltay)
rho * cp * der(T06,t) = fd_2d_int(T06,T10,T02,T07,T05,k,qdot,deltax,deltay)
rho * cp * der(T07,t) = fd_2d_int(T07,T11,T03,T08,T06,k,qdot,deltax,deltay)
rho * cp * der(T10,t) = fd_2d_int(T10,T14,T06,T11,T09,k,qdot,deltax,deltay)
rho * cp * der(T11,t) = fd_2d_int(T11,T15,T07,T12,T10,k,qdot,deltax,deltay)
rho * cp * der(T14,t) = fd_2d_int(T14,T17,T10,T15,T13,k,qdot,deltax,deltay)
rho * cp * der(T15,t) = fd_2d_int(T15,T18,T11,T16,T14,k,qdot,deltax,deltay)
rho * cp * der(T17,t) = fd_2d_int(T17,T18,T14,T18,T14,k,qdot,deltax,deltay)
rho * cp * der(T18,t) = fd_2d_int(T18,T20,T15,T19,T17,k,qdot,deltax,deltay)
rho * cp * der(T20,t) = fd_2d_int(T20,T21,T18,T21,T18,k,qdot,deltax,deltay)

// Exterior surface nodes, 04, 08, 12, 16, 19, 21, 22
/* Node 04: plane surface node, n-orientation; e, w, s labeled 08, 08, 03. */
rho * cp * der(T04,t) = fd_2d_psur_n(T04,T08,T08,T03,k,qdot,deltax,deltay,Tinfo,ho,q''a)
rho * cp * der(T08,t) = fd_2d_psur_n(T08,T12,T04,T07,k,qdot,deltax,deltay,Tinfo,ho,q''a)
rho * cp * der(T12,t) = fd_2d_psur_n(T12,T16,T08,T11,k,qdot,deltax,deltay,Tinfo,ho,q''a)
rho * cp * der(T16,t) = fd_2d_psur_n(T16,T19,T12,T15,k,qdot,deltax,deltay,Tinfo,ho,q''a)
rho * cp * der(T19,t) = fd_2d_psur_n(T19,T21,T16,T18,k,qdot,deltax,deltay,Tinfo,ho,q''a)
rho * cp * der(T21,t) = fd_2d_psur_n(T21,T22,T19,T20,k,qdot,deltax,deltay,Tinfo,ho,q''a)
/* Node 22: external corner node, e-n orientation; w, s labeled 21, 21. */
rho * cp * der(T22,t) = fd_2d_ec_en(T22,T21,T21,k,qdot,deltax,deltay,Tinfo,ho,q''a)

// Input variables
deltax = 0.050
deltay = 0.050
Tinfi = 350
hi = 100
Tinfo = 25
ho = 5
k = 0.85
alpha = 5.55e-7
alpha = k / (rho * cp)
rho = 1000                    // arbitrary value
```

(3) The results for t = 50 hour, representing the steady-state condition, are shown below, arranged according to the coordinate system.

	Tmn (C)						
x/y (mm)	0	50	100	150	200	250	300
0	181.60	179.50	172.70	160.20	141.80	120.60	98.89
50	229.80	227.30	219.00	202.90	177.50	149.00	
100	282.90	280.80	273.20	172.70	216.20		
150	340.20	339.90	338.20	327.30			

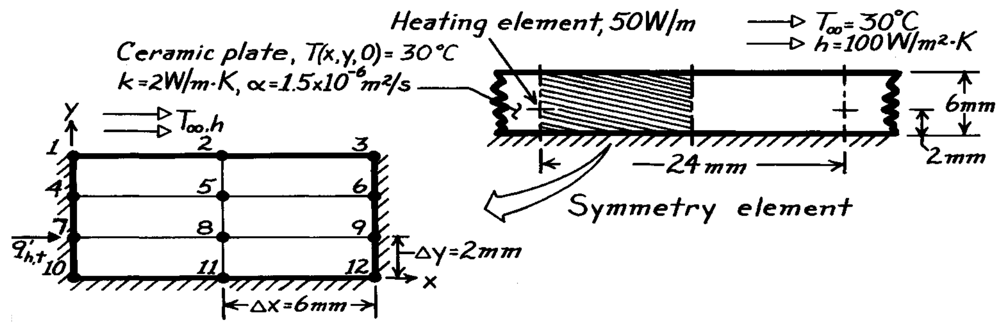
In Problem 4.57, the temperature distribution was determined using the FDEs written for steady-state conditions, but with a finer network, $\Delta x = \Delta y = 25$ mm. By comparison, the results for the coarser network are slightly higher, within a fraction of 1°C, along the mid-section of the flue, but notably higher in the vicinity of inner corner. (For example, node 13 is 2.6°C higher with the coarser mesh.)

PROBLEM 5.126

KNOWN: Electrical heating elements embedded in a ceramic plate as described in Problem 4.75; initially plate is at a uniform temperature and suddenly heaters are energized.

FIND: Time required for the difference between the surface and initial temperatures to reach 95% of the difference for steady-state conditions using the implicit, finite-difference method.

SCHEMATIC:

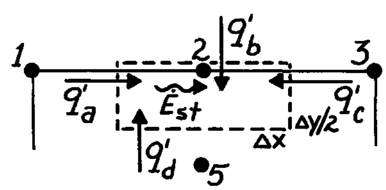


ASSUMPTIONS: (1) Two-dimensional conduction, (2) Constant properties, (3) No internal generation except for Node 7, (4) Heating element approximates a line source; wire diameter is negligible.

ANALYSIS: The grid for the symmetry element above consists of 12 nodes. Nodes 1-3 are points on a surface experiencing convection; nodes 4-12 are interior nodes; node 7 is a special case with internal generation and because of symmetry, $q'_{ht} = 25$ W/m. Their finite-difference equations are derived as follows

Surface Node 2. From an energy balance on the prescribed control volume with $\Delta x/\Delta y = 3$,

$$\dot{E}_{in} = \dot{E}_{st} = q'_a + q'_b + q'_c + q'_d = \rho\, cV \frac{T_2^{p+1} - T_2^p}{\Delta t}$$



$$k\frac{\Delta y}{2}\frac{T_1^{p+1} - T_2^{p+1}}{\Delta x} + h\Delta x\left(T_\infty - T_2^{p+1}\right)$$

$$+ k\frac{\Delta y}{2}\frac{T_3^{p+1} - T_2^{p+1}}{\Delta x} + k\Delta x\frac{T_5^{p+1} - T_2^{p+1}}{\Delta y} = \rho\, c\left[\Delta x\frac{\Delta y}{2}\right]\frac{T_2^{p+1} - T_2^p}{\Delta t}.$$

Continued …..

PROBLEM 5.126 (Cont.)

Divide by k, use the following definitions, and regroup to obtain the finite-difference equations.

$$N \equiv h\Delta x/k = 100\ \text{W/m}^2\cdot\text{K} \times 0.006\text{m}/2\ \text{W/m}\cdot\text{K} = 0.3000 \qquad (1)$$

$$Fo \equiv (k/\rho\, c)\Delta t/\Delta x\cdot\Delta y = \alpha\Delta t/\Delta x\cdot\Delta y =$$

$$1.5\times10^{-6}\text{m}^2/\text{s}\times1\text{s}/(0.006\times0.002)\text{m}^2 = 0.1250 \qquad (2)$$

$$\frac{1}{2}\left[\frac{\Delta y}{\Delta x}\right]\left(T_1^{p+1}-T_2^{p+1}\right)+N\left(T_\infty-T_2^{p+1}\right)+\frac{1}{2}\left[\frac{\Delta y}{\Delta x}\right]\left(T_3^{p+1}-T_2^{p+1}\right)$$

$$+\left[\frac{\Delta x}{\Delta y}\right]\left(T_5^{p+1}-T_2^{p+1}\right)=\frac{1}{2Fo}\left(T_2^{p+1}-T_2^{p}\right)$$

$$\frac{1}{2}\left[\frac{\Delta y}{\Delta x}\right]T_1^{p+1}-\left[\left[\frac{\Delta x}{\Delta y}\right]+N+\left[\frac{\Delta y}{\Delta x}\right]+\frac{1}{2Fo}\right]T_2^{p+1}+\frac{1}{2}\left[\frac{\Delta x}{\Delta y}\right]T_3^{p+1}$$

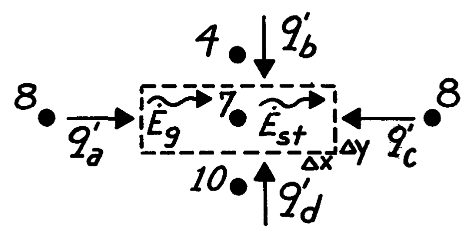
$$+\left[\frac{\Delta x}{\Delta y}\right]T_5^{p+1}=-NT_\infty-\frac{1}{2Fo}T_2^{p}. \qquad (3)$$

Substituting numerical values for Fo and N, and using T_∞ = 30°C and $\Delta x/\Delta y$ = 3, find

$$0.16667T_1^{p+1}-7.63333T_2^{p+1}+0.16667T_3^{p+1}+3.00000T_5^{p+1}=9.0000-4.0000T_2^{p}. \qquad (4)$$

By inspection and use of Eq. (3), the FDEs for Nodes 1 and 3 can be inferred.

Interior Node 7. From an energy balance on the prescribed control volume with $\Delta x/\Delta y$ = 3,



$$\dot{E}'_{in}+\dot{E}'_{g}=\dot{E}'_{st}$$

where $\dot{E}'_g = 2q'_{ht}$ and $\dot{E}'_{in}$ represents the conduction terms $-q'_a+q'_b+q'_c+q'_d$,

$$k\Delta y\frac{T_8^{p+1}-T_7^{p+1}}{\Delta x}+k\Delta x\frac{T_4^{p+1}-T_7^{p+1}}{\Delta y}+k\Delta y\frac{T_8^{p+1}-T_7^{p+1}}{\Delta x}$$

$$+k\Delta x\frac{T_{10}^{p+1}-T_7^{p+1}}{\Delta y}+2q'_{ht}=\rho\, c(\Delta x\cdot\Delta y)\frac{T_7^{p+1}-T_7^{p}}{\Delta t}.$$

Using the definition of Fo, Eq. (2), and regrouping, find

$$\frac{1}{2}\left[\frac{\Delta x}{\Delta y}\right]T_4^{p+1}-\left[\left[\frac{\Delta x}{\Delta y}\right]+\left[\frac{\Delta y}{\Delta x}\right]+\frac{1}{2Fo}\right]T_7^{p+1}$$

$$+\left[\frac{\Delta y}{\Delta x}\right]T_8^{p+1}+\frac{1}{2}\left[\frac{\Delta x}{\Delta y}\right]T_{10}^{p+1}=-\frac{q'_{ht}}{k}-\frac{1}{2Fo}T_7^{p} \qquad (5)$$

$$1.50000T_4^{p+1}-7.33333T_7^{p+1}+0.33333T_8^{p+1}+1.50000T_{10}^{p+1}=-12.5000-4.0000T_7^{p}. \qquad (6)$$

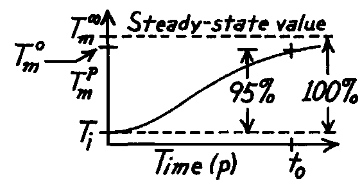
Continued …..

PROBLEM 5.126 (Cont.)

Recognizing the form of Eq. (5), it is a simple matter to infer the FDE for the remaining interior points for which $\dot{q}_{ht} = 0$. In matrix notation [A][T] = [C], the coefficient matrix [A] and RHS matrix [C] are:

THE COEFFICIENT MATRIX, [A] **[C]**

-7.633330	0.333330	0	3.000000	0	0	0	0	0	0	0	0	$-4.0T_1^p - 9.0$
0.166670	-7.633330	0.166670	0	3.000000	0	0	0	0	0	0	0	$-4.0T_2^p - 9.0$
0	0.333330	-7.633330	0	0	3.000000	0	0	0	0	0	0	$-4.0T_3^p - 9.0$
1.500000	0	0	-7.333330	0.333330	0	1.500000	0	0	0	0	0	$-4.0T_4^p$
0	3.000000	0	0.333330	-14.666670	0.333330	0	3.000000	0	0	0	0	$-8.0T_5^p$
0	0	1.500000	0	0.333330	-7.333330	0	0	1.500000	0	0	0	$-4.0T_6^p$
0	0	0	1.500000	0	0	-7.333330	0.333330	0	1.500000	0	0	$-4.0T_7^p - 12.5$
0	0	0	0	3.000000	0	0.333330	-14.666670	0.333330	0	3.000000	0	$-8.0T_8^p$
0	0	0	0	0	1.500000	0	0.333330	-7.333330	0	0	1.500000	$-4.0T_9^p$
0	0	0	0	0	0	3.000000	0	0	-7.333330	0.333330	0	$-4.0T_{10}^p$
0	0	0	0	0	0	0	3.000000	0	0.166670	-7.333330	0.166670	$-4.0T_{11}^p$
0	0	0	0	0	0	0	0	3.000000	0	0.333330	-7.333330	$-4.0T_{12}^p$



Recall that the problem asks for the time required to reach 95% of the difference for steady-state conditions. This provides information on approximately how long it takes for the plate to come to a steady operating condition. If you worked Problem 4.71, you know the steady-state temperature distribution. Then you can proceed to find the T_m^p values with increasing time until the *first* node reaches the required limit. We should not expect the nodes to reach their limit at the same time.

Not knowing the steady-state temperature distribution, use the implicit FDE in matrix form above to step through time $\rightarrow \infty$ to the steady-state solution; that is, proceed to p $\rightarrow$ 10,20…100 until the solution matrix [T] does not change. The results of the analysis are tabulated below. Column 1 labeled $T_m(\infty)$ is the steady-state distribution. Column 2, $T_m(95\%)$, is the 95% limit being sought as per the graph directly above. The third column is the temperature distribution at t = to = 248s, $T_m(248s)$; at this elapsed time, Node 1 has reached its limit. Can you explain why this node was the first to reach this limit? Which nodes will be the last to reach their limits?

$T_m(\infty)$	$T_m(95\%)$	$T_m(248s)$
55.80	54.51	54.51
49.93	48.93	48.64
47.67	46.78	46.38
59.03	57.58	57.64
51.72	50.63	50.32
49.19	48.23	47.79
63.89	62.20	62.42
52.98	51.83	51.52
50.14	49.13	48.68
62.84	61.20	61.35
53.35	52.18	51.86
50.46	49.43	48.98

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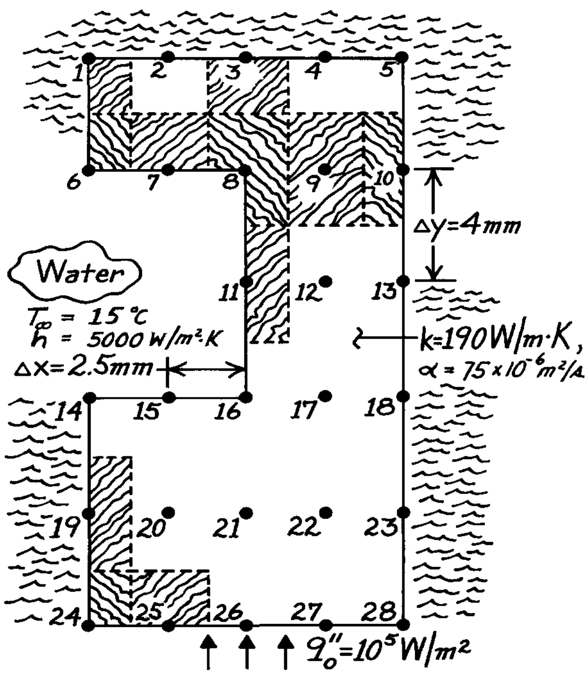
PROBLEM 5.127

KNOWN: Nodal network and operating conditions for a water-cooled plate.

FIND: Transient temperature response.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-sate conditions, (2) Two-dimensional conduction.

ANALYSIS: The energy balance method must be applied to each nodal region. Grouping similar regions, the following results are obtained.

Nodes 1 and 5:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_1^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_2^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_6^{p+1}=T_1^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_5^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_4^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{10}^{p+1}=T_5^p$$

Nodes 2, 3, 4:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{m-1,n}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{m+1,n}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{m,n-1}^{p+1}=T_{m,n}^p$$

Nodes 6 and 14:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_6^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_1^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_7^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_6^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_{14}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{15}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{19}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_{14}^p$$

Continued …..

Nodes 7 and 15:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_7^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_2^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_6^{p+1}-\frac{\alpha\Delta t}{k\Delta x^2}T_8^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_7^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_{15}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{14}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{16}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{20}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_{15}^p$$

Nodes 8 and 16:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2}{3}\;\frac{h\alpha\Delta t}{k\Delta x}+\frac{2}{3}\frac{h\alpha\Delta t}{k\Delta y}\right)T_8^{p+1}-\frac{4}{3}\frac{\alpha\Delta t}{\Delta y^2}T_3^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta x^2}T_7^{p+1}$$
$$-\frac{4}{3}\frac{\alpha\Delta t}{\Delta x^2}T_9^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{11}^{p+1}=\frac{2}{3}\frac{h\alpha\Delta t}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty+T_8^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2}{3}+\frac{h\alpha\Delta t}{k\Delta x}+\frac{2}{3}\frac{h\alpha\Delta t}{k\Delta y}\right)T_{16}^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{11}^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta x^2}T_{15}^{p+1}$$
$$-\frac{4}{3}\frac{\alpha\Delta t}{\Delta x^2}T_{17}^{p+1}-\frac{4}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{21}^{p+1}=\frac{2}{3}\frac{h\alpha\Delta t}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty+T_{16}^p$$

Node 11:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta x}\right)T_{11}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}T_8^{p+1}-2\alpha\frac{\Delta t}{\Delta x^2}T_{12}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}T_{16}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta x}T_\infty+T_{11}^p$$

Nodes 9, 12, 17, 20, 21, 22:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)-\frac{\alpha\Delta t}{\Delta x^2}\left(T_{m-1,n}^{p+1}+T_{m+1,n}^{p+1}\right)=T_{m,n}^p$$

Nodes 10, 13, 18, 23:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)-\frac{2\alpha\Delta t}{\Delta x^2}T_{m-1,n}^{p+1}=T_{m,n}^p$$

Node 19:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{19}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{14}^{p+1}+T_{24}^{p+1}\right)-\frac{2\alpha\Delta t}{\Delta x^2}T_{20}^{p+1}=T_{19}^p$$

Nodes 24, 28:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{24}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{19}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{25}^{p+1}=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{24}^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{28}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{23}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{27}^{p+1}=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{28}^p$$

Continued …..

PROBLEM 5.127 (Cont.)

Nodes 25, 26, 27:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{m,n+1}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}\left(T_{m-1,n}^{p+1}+T_{m+1,n}^{p+1}\right)=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{m,n}^{p+1}$$

The convection heat rate is

$$q'_{conv}=h\left[(\Delta x/2)(T_6-T_\infty)+\Delta x(T_7-T_\infty)+(\Delta x+\Delta y)(T_8-T_\infty)/2+\Delta y(T_{11}-T_\infty)+(\Delta x+\Delta y)(T_{16}-T_\infty)/2+\Delta x(T_{15}-T_\infty)+(\Delta x/2)(T_{14}-T_\infty)\right]=q_{out}.$$

The heat input is

$$q'_{in}=q''_o(4\Delta x)$$

and, on a percentage basis, the ratio is

$$n\equiv\left(q'_{conv}/q'_{in}\right)\times 100.$$

Results of the calculations (in °C) are as follows:

Time: 5.00 sec; n = 60.57%

19.612	19.712	19.974	20.206	20.292
19.446	19.597	20.105	20.490	20.609
		21.370	21.647	21.730
24.217	24.074	23.558	23.494	23.483
25.658	25.608	25.485	25.417	25.396
27.581	27.554	27.493	27.446	27.429

Time: 10.00 sec; n = 85.80%

22.269	22.394	22.723	23.025	23.137
21.981	22.167	22.791	23.302	23.461
		24.143	24.548	24.673
27.216	27.075	26.569	26.583	26.598
28.898	28.851	28.738	28.690	28.677
30.901	30.877	30.823	30.786	30.773

Time: 15.0 sec; n = 94.89%

23.228	23.363	23.716	24.042	24.165
22.896	23.096	23.761	24.317	24.491
		25.142	25.594	25.733
28.294	28.155	27.652	27.694	27.719
30.063	30.018	29.908	29.867	29.857
32.095	32.072	32.021	31.987	31.976

Time: 20.00 sec; n = 98.16%

23.574	23.712	24.073	24.409	24.535
23.226	23.430	24.110	24.682	24.861
		25.502	25.970	26.115
28.682	28.543	28.042	28.094	28.122
30.483	30.438	30.330	30.291	30.282
32.525	32.502	32.452	32.419	32.409

Time: 23.00 sec; n = 99.00%

23.663	23.802	24.165	24.503	24.630
23.311	23.516	24.200	24.776	24.957
		25.595	26.067	26.214
28.782	28.644	28.143	28.198	28.226
30.591	30.546	30.438	30.400	30.392
32.636	32.613	32.563	32.531	32.520

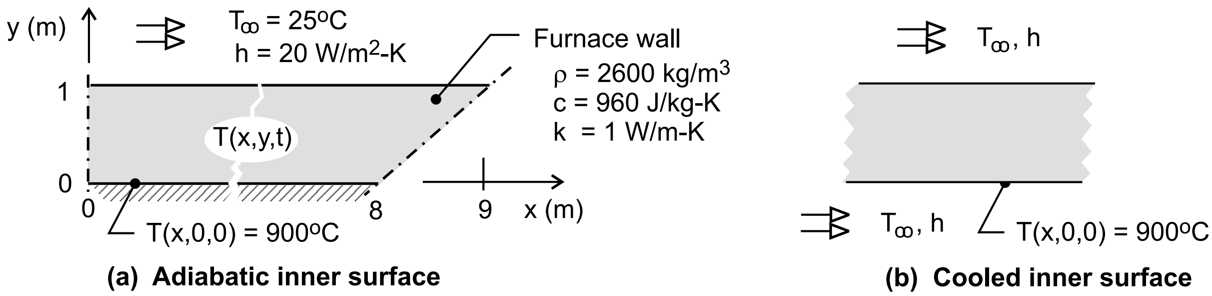
COMMENTS: Temperatures at t = 23 s are everywhere within 0.13°C of the final steady-state values.

PROBLEM 5.128

KNOWN: Cubic-shaped furnace, with prescribed operating temperature and convection heat transfer on the exterior surfaces.

FIND: Time required for the furnace to cool to a safe working temperature corresponding to an inner wall temperature of 35°C considering convection cooling on (a) the exterior surfaces and (b) on both the exterior and interior surfaces.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction through the furnace walls and (2) Constant properties.

ANALYSIS: Assuming two-dimensional conduction through the walls and taking advantage of symmetry for the cubical shape, the analysis considers the quarter section shown in the schematic above. For part (a), with no cooling on the interior during the cool-down process, the inner surface boundary condition is adiabatic. For part (b), with cooling on both the exterior and interior, the boundary conditions are prescribed by the convection process. The boundaries through the centerline of the wall and the diagonal through the corner are symmetry planes and considered as adiabatic. We have chosen to use the finite-element software *FEHT* as the solution tool.

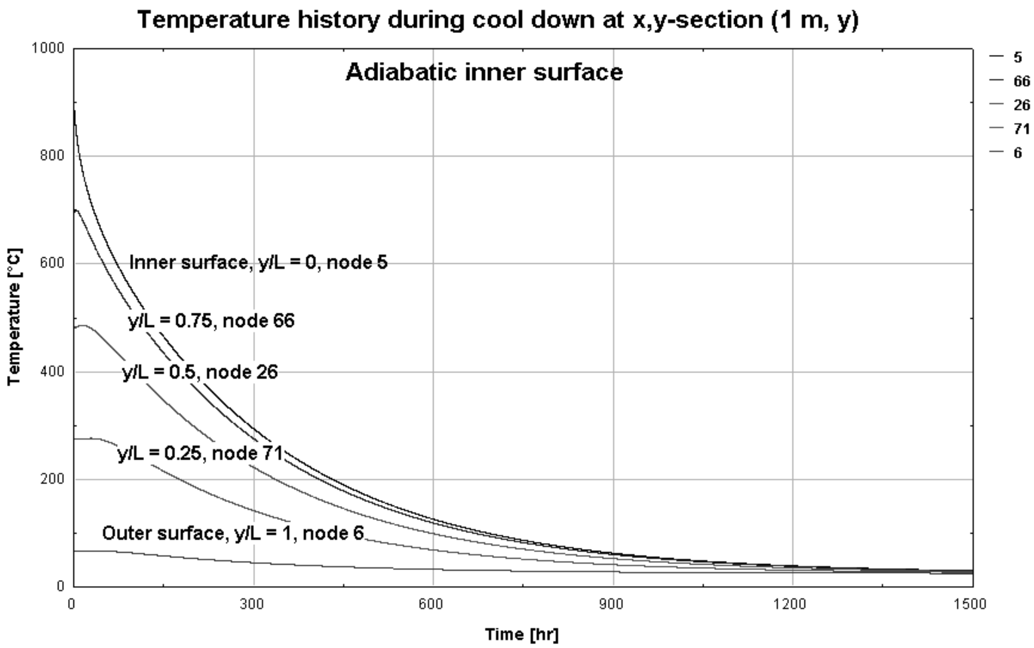
Using *FEHT*, an outline of the symmetrical wall section is drawn, and the material properties are specified. To determine the initial conditions for the cool-down process, we will first find the temperature distribution for steady-state operation. As such, specify the boundary condition for the inner surface as a constant temperature of 900°C; the other boundaries are as earlier described. In the *Setup* menu, click *on Steady-State*, and then *Run* to obtain the steady-state temperature distribution. This distribution represents the initial temperature distribution, T_i (x, y, 0), for the wall at the onset of the cool-down process.

Next, in the *Setup* menu, click on *Transient*; for the nodes on the inner surface, in the *Specify | Boundary Conditions* menu, deselect the *Temperature* box (900°C) and set the *Flux* box to zero for the adiabatic condition (part (a)); and, in the *Run* command, click on *Continue* (not *Calculate*). Be sure to change the integration time scale from *seconds* to *hours*.

Because of the high ratio of wall section width (nearly 8.5 m) to the thickness (1 m), the conduction heat transfer through the section is nearly one-dimensional. We chose the x,y-section 1 m to the right of the centerline (1 m, y) as the location for examining the temperature-time history, and determining the cool-down time for the inner surface to reach the safe working temperature of 35°C.

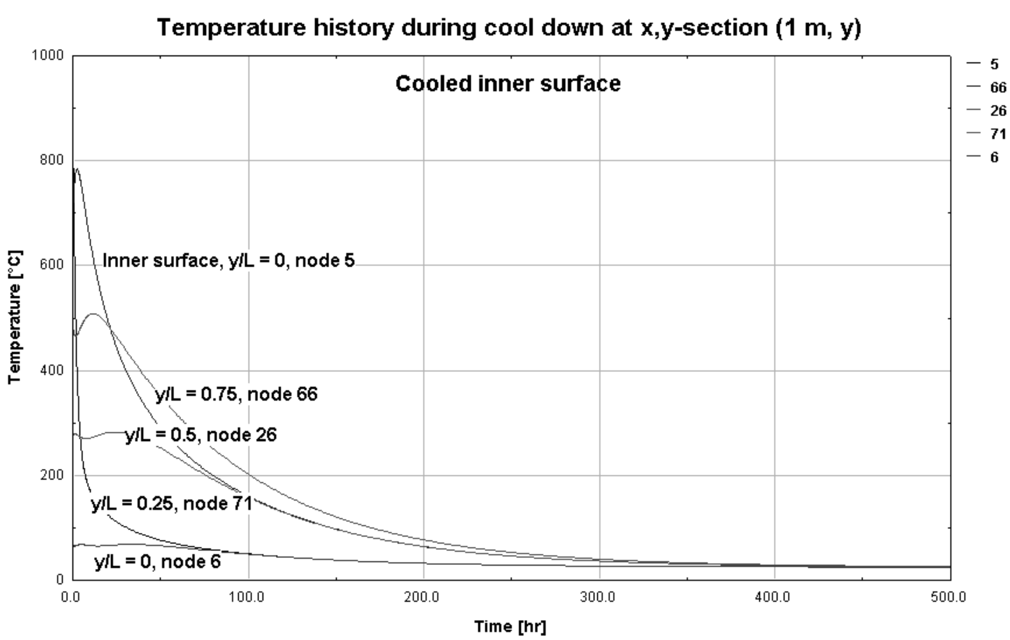
Continued …..

PROBLEM 5.128 (Cont.)



Time-to-cool, Part (a), Adiabatic inner surface. From the above temperature history, the cool-down time, t_a, corresponds to the condition when T_a (1 m, 0, t_a) = 35°C. As seen from the history, this location is the last to cool. From the *View | Tabular Output*, find that

$$t_a = 1306 \text{ h} = 54 \text{ days}$$ <



Continued …..

Time-to-cool, Part (b), Cooled inner surface. From the above temperature history, note that the center portion of the wall, and not the inner surface, is the last to cool. The inner surface cools to 35°C in approximately 175 h or 7 days. However, if the cooling process on the inner surface were discontinued, its temperature would increase and eventually exceed the desired safe working temperature. To assure the safe condition will be met, estimate the cool down time as, t_b, corresponding to the condition when T_b (1 m, 0.75 m, t_b) = 35°C. From the *View* | *Tabular Output*, find that

$$t_b = 311 \text{ h} = 13 \text{ days} \qquad <$$

COMMENTS: (1) Assuming the furnace can be approximated by a two-dimensional symmetrical section greatly simplifies our analysis by not having to deal with three-dimensional corner effects. We justify this assumption on the basis that the corners represent a much shorter heat path than the straight wall section. Considering corner effects would reduce the cool-down time estimates; hence, our analysis provides a conservative estimate.

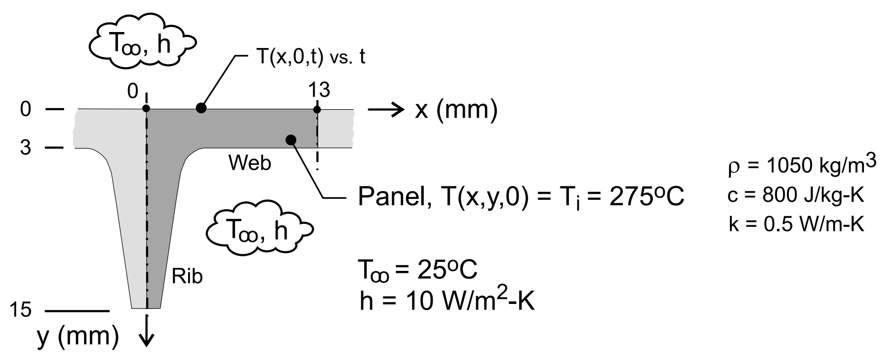
(2) For background information on the *Continue* option, see the *Run* menu in the *FEHT Help* section. Using the *Run* | *Calculate* command, the steady-state temperature distribution was determined for the normal operating condition of the furnace. Using the *Run* | *Continue* command (after clicking on *Setup* | *Transient*), this steady-state distribution automatically becomes the initial temperature distribution for the cool-down transient process. This feature allows for conveniently prescribing a non-uniform initial temperature distribution for a transient analysis (rather than specifying values on a node-by-node basis.

PROBLEM 5.129

KNOWN: Door panel with ribbed cross-section, initially at a uniform temperature of 275°C, is ejected from the hot extrusion press and experiences convection cooling with ambient air at 25°C and a convection coefficient of 10 W/m^2·K.

FIND: (a) Using the *FEHT View|Temperature vs. Time* command, create a graph with temperature-time histories of selected locations on the panel surface, T(x,0,t). Comment on whether you see noticeable differential cooling in the region above the rib that might explain the appearance defect; and Using the *View|Temperature Contours* command with the shaded-band option for the isotherm contours, select the *From start to stop* time option, and view the temperature contours as the panel cools. Describe the major features of the cooling process you have seen. Use other options of this command to create a 10-isotherm temperature distribution at some time that illustrates important features. How would you re-design the ribbed panel in order to reduce this thermally induced paint defect situation, yet retain the stiffening function required of the ribs?

SCHEMATIC:



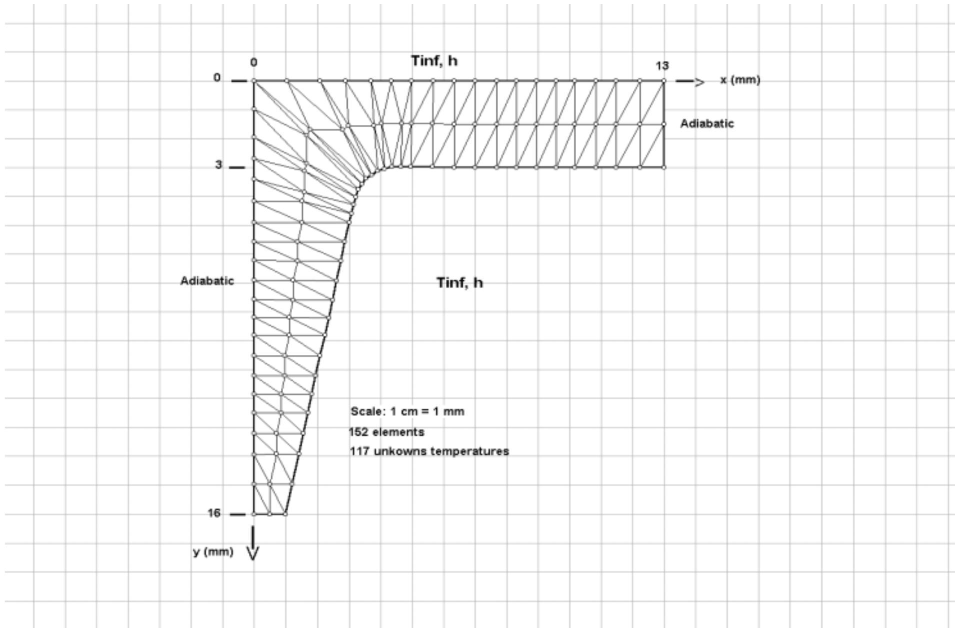
ASSUMPTIONS: (1) Two-dimensional conduction in the panel, (2) Uniform convection coefficient over the upper and lower surfaces of the panel, (3) Constant properties.

PROPERTIES: Door panel material (*given*): ρ = 1050 kg/m^3, c = 800 J/kg·K, k = 0.5 W/m·K.

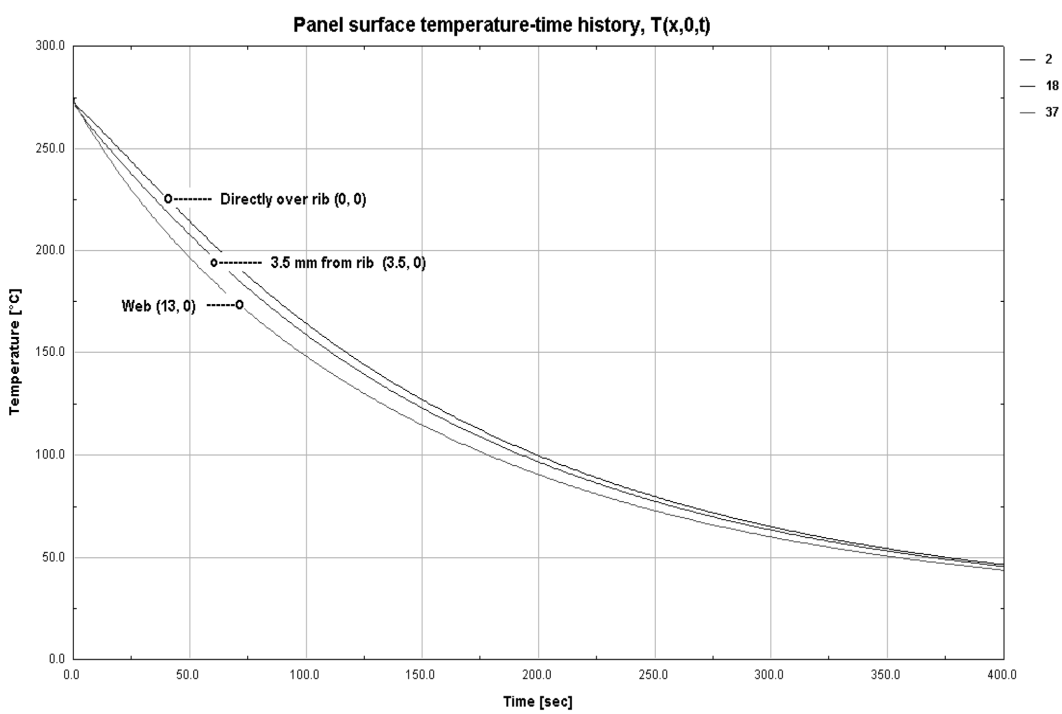
ANALYSIS:

(a) Using the *Draw* command, the shape of the symmetrical element of the panel (darkened region in schematic) was generated and elements formed as shown below. The symmetry lines represent adiabatic surfaces, while the boundary conditions for the exposed web and rib surfaces are characterized by (T_∞, h).

Continued …..



After running the calculation for the time period 0 to 400 s with a 1-second time step, the temperature-time histories for three locations were obtained and the graph is shown below.

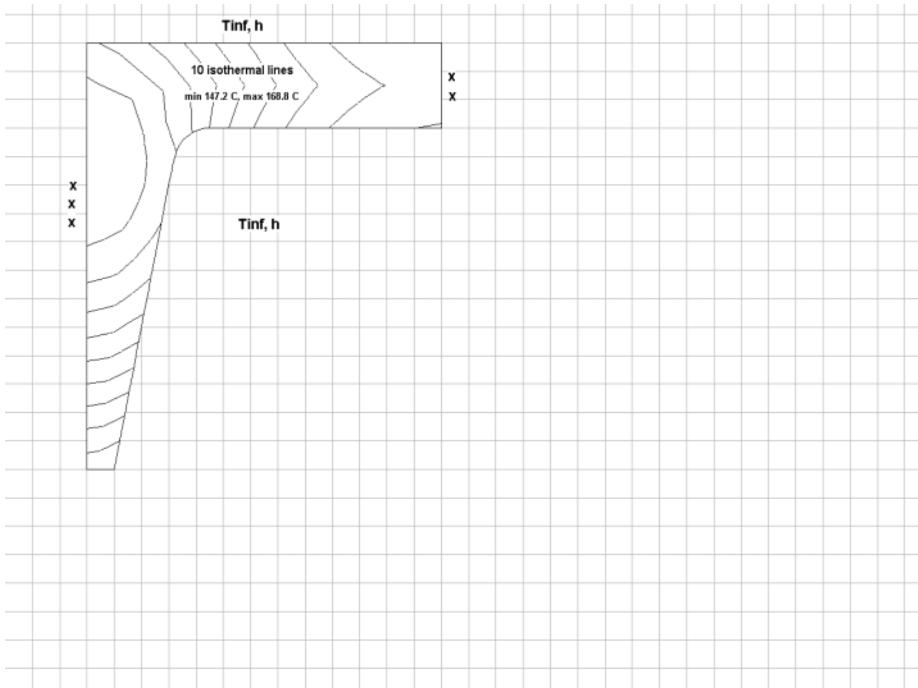


As expected, the region directly over the rib (0,0) cooled the slowest, while the extreme portion of the web (0, 13 mm) cooled the fastest. The largest temperature differences between these two locations occur during the time period 50 to 150 s. The maximum difference does not exceed 25°C.

Continued …..

PROBLEM 5.129 (Cont.)

(b) It is possible that the temperature gradients within the web-rib regions – rather than just the upper surface temperature differentials – might be important for understanding the panel's response to cooling. Using the *Temperature Contours* command (with the *From start to stop* option), we saw that the center portion of the web and the end of the rib cooled quickly, but that the region on the rib centerline (0, 3-5 mm), was the hottest region. The isotherms corresponding to t = 100 s are shown below. For this condition, the temperature differential is about 21°C.



From our analyses, we have identified two possibilities to consider. First, there is a significant surface temperature distribution across the panel during the cooling process. Second, the web and the extended portion of the rib cool at about the same rate, and with only a modest normal temperature gradient. The last region to cool is at the location where the rib is thickest (0, 3-5 mm). The large temperature gradient along the centerline toward the surface may be the cause of microstructure variations, which could influence the adherence of paint. An obvious re-design consideration is to reduce the thickness of the rib at the web joint, thereby reducing the temperature gradients in that region. This fix comes at the expense of decreasing the spacing between the ribs.