1. Select **only one** correct statement, by **circling one of the numbers i, ii, iii, or iv**, for the following sentences. **No need to motivate your answers.**
(1pt each)

(a) A mathematical model is useful for:

i. analyzing system properties.
ii. designing feedback controllers.
iii. predicting system responses for excitation signals.
iv. All of i, ii, iii.

(b) Which of the following can be a through variable?

i. pressure
ii. voltage
iii. temperature
iv. None of i, ii, iii.

(c) For thermal systems, which of the following defines power?

i. heat transfer rate
ii. temperature
iii. heat transfer rate times temperature
iv. None of i, ii, iii.

(d) For fluid systems, which of the following defines power?

i. mass flow rate
ii. pressure
~~iii.~~ mass flow rate times pressure
iv. None of i, ii, iii. (volume flow rate) x (pressure)

(e) For fluid systems, which of the following elements stores kinetic energy?

i. fluid capacitance
ii. fluid inerter
iii. fluid resistance
iv. None of i, ii, iii.

(f) A loop equation in a linear graph is:

i. a balance of across variables.
ii. a balance of through variables.
iii. a constitutive equation.
iv. None of i, ii, iii.

(g) Which of the following transfer functions has the largest steady-state value for a unit step input?

i. $G_1(s) = \dfrac{1}{s^2+s+1}$

ii. $G_2(s) = \dfrac{1}{s^2+s+10}$

iii. $G_3(s) = \dfrac{1}{s^2+s+100}$

iv. $G_1(s)$, $G_2(s)$ and $G_3(s)$ has the same steady-state value.

(h) Which of the following transfer functions has the shortest peak time for a unit step input?

i. $G_1(s) = \dfrac{1}{s^2+s+1}$

ii. $G_2(s) = \dfrac{1}{s^2+s+10}$

iii. $G_3(s) = \dfrac{1}{s^2+s+100}$

iv. $G_1(s)$, $G_2(s)$ and $G_3(s)$ has the same peak time.

(i) Which of the following transfer functions has the smallest percent overshoot for a unit step input?

i. $G_1(s) = \dfrac{1}{s^2+s+1}$

ii. $G_2(s) = \dfrac{1}{s^2+s+10}$

iii. $G_3(s) = \dfrac{1}{s^2+s+100}$

iv. $G_1(s)$, $G_2(s)$ and $G_3(s)$ has the same percent overshoot.

(j) Which of the following transfer functions has the shortest settling time for a unit step input?

i. $G_1(s) = \dfrac{1}{s^2+s+1}$

ii. $G_2(s) = \dfrac{1}{s^2+s+10}$

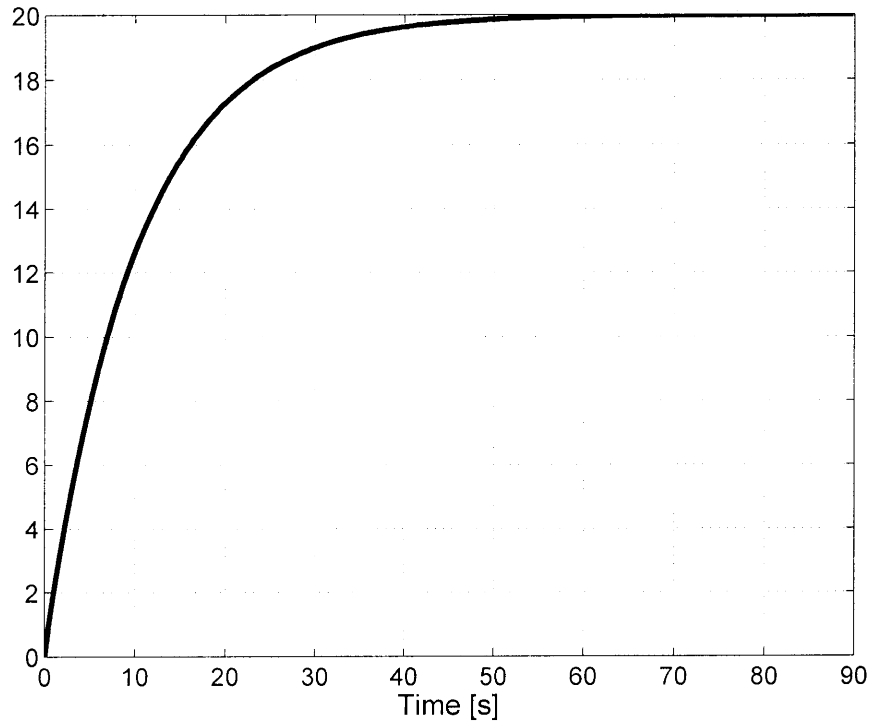
iii. $G_3(s) = \dfrac{1}{s^2+s+100}$

iv. $G_1(s)$, $G_2(s)$ and $G_3(s)$ has the same settling time.

2. Answer the following questions on system identification and system analysis.

   (a) For the following response to a step input with amplitude 2, estimate the corresponding first-order transfer function. (2pt)

   (b) For the transfer function estimated in (a), sketch the Bode plot using the straight-line approximation. (3pt)



Write your answer here.

(a) DC gain $= \frac{20}{2} = 10$

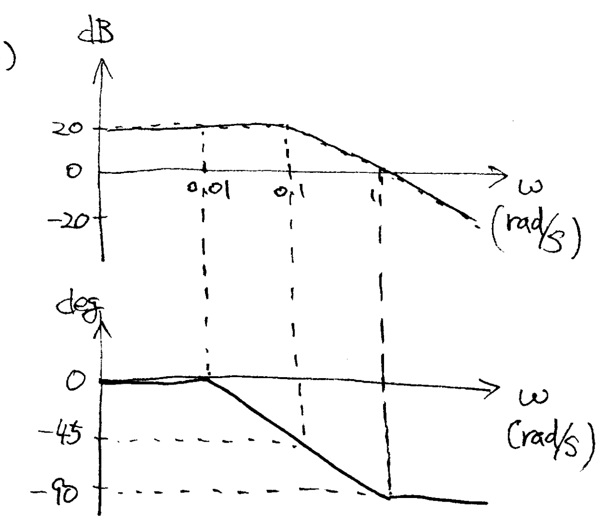
Time constant $=$ time for 20×0.63

≈ 12.6

$= 10$

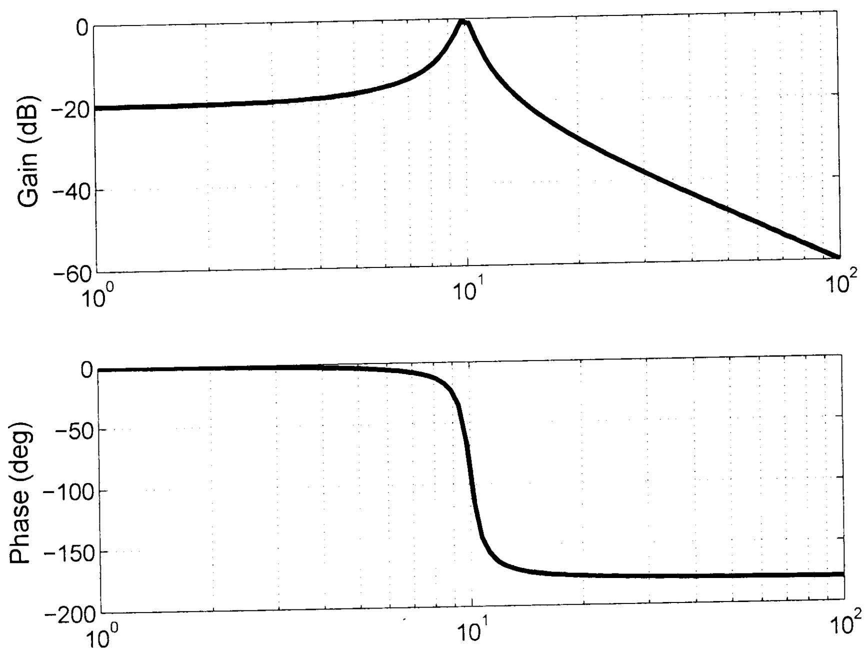
$\Rightarrow G(s) = \frac{10}{10s+1}$

(b)



(c) For the following Bode plot, estimate the corresponding <u>second-order</u> transfer function, with damping ratio $\zeta = 0.05$. (2pt)

(d) For the transfer function estimated in (c), plot roughly the response to a unit step input. In the plot, indicate the **steady-state value**, **2% settling time**, and **peak time**. There is **no need** to obtain the **percent overshoot**. (3pt)
<u>Hint</u>: Complicated calculations (such as partial fraction expansion or inverse Laplace transform) are NOT necessary for this plotting.



<u>Write your answer here.</u>

(c) $\omega_n \simeq 10$ (resonant frequency), DC gain $= -20$ dB $= 0.1$

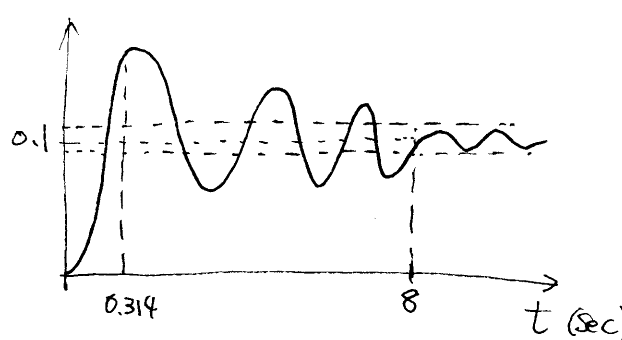
$$\Rightarrow G(s) = \frac{0.1 \cdot 10^2}{s^2 + 2 \cdot 0.05 \cdot 10 s + 10^2} = \frac{10}{s^2 + s + 100}$$

(d) poles $= \dfrac{-1 \pm \sqrt{399}\, j}{2} \approx \dfrac{-1 \pm 20 j}{2}$

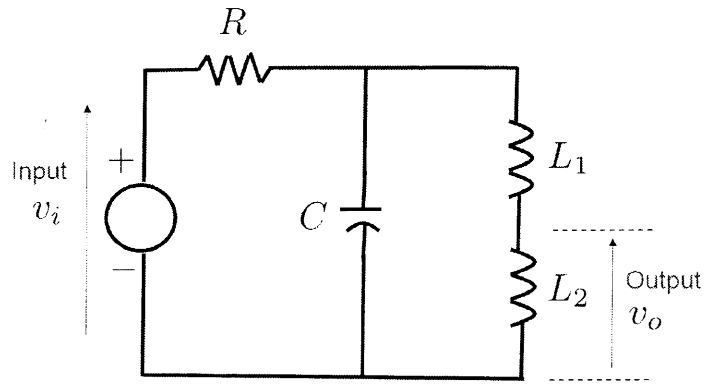
$$T_s \approx \frac{4}{|Re|} = 8$$

$$T_p \approx \frac{\pi}{Im} = \frac{\pi}{10} \approx 0.314$$

$$G(0) = \frac{10}{100} = 0.1$$



3. Consider the following electric circuit. Here, R is the resistance, C is the capacitance, and L_1 and L_2 are the inductances. The input is the voltage v_i and the output is the voltage v_o (i.e., voltage across L_2).

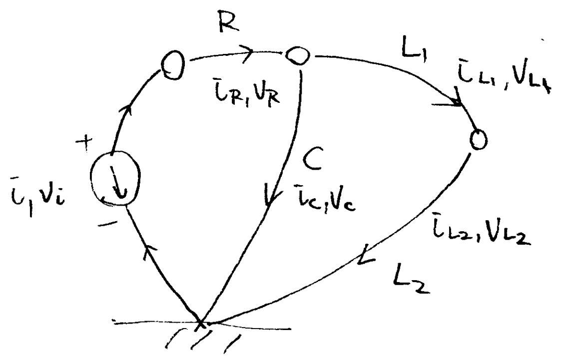


(a) Draw the linear graph. (2pt)

(b) Using the linear graph, obtain a state-space model <u>with two states</u>. (3pt)

Write your answer here.

(a)



(b) From a node equation, $i_{L_1} = i_{L_2} (=: i_L)$

<u>States</u>

$$X := \begin{bmatrix} V_c \\ i_L \end{bmatrix}$$

<u>Constitutive equations</u>

$$R i_R = V_R$$
$$C \dot V_c = i_c$$
$$L_1 \dot i_L = V_{L_1}$$
$$L_2 \dot i_L = V_{L_2} (= V_o)$$

<u>Loop eq.</u>

$$V_i = V_R + V_c$$
$$V_c = V_{L_1} + V_{L_2}$$

<u>Node eq.</u>

$$i = i_R = i_c + i_L$$

$$\dot V_c = \frac{1}{C} i_c = \frac{1}{C}(i - i_L) = \frac{1}{C}\left(\frac{1}{R}V_R - i_L\right)$$
$$= \frac{1}{C}\left(\frac{1}{R}(V_i - V_c) - i_L\right)$$

$$\dot i_L = \frac{1}{L_1+L_2}(V_{L_1} + V_{L_2}) = \frac{1}{L_1+L_2} V_c$$

$$V_o = V_{L_2} = L_2 \dot i_L = \frac{L_2}{L_1+L_2} V_c$$

$$\Rightarrow \left\{ \begin{aligned} \begin{bmatrix} \dot V_c \\ \dot i_L \end{bmatrix} &= \begin{bmatrix} -\frac{1}{CR} & -\frac{1}{C} \\ \frac{1}{L_1+L_2} & 0 \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{CR} \\ 0 \end{bmatrix} V_i \\ V_o &= \begin{bmatrix} \frac{L_2}{L_1+L_2} & 0 \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix} \end{aligned} \right.$$

(c) Using any method, obtain a state-space model with two states, which is different from the model obtained in (b). (2pt)

(d) For the state-space model obtained in (b), derive the corresponding transfer function. (3pt)

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Write your answer here.

(c) By exchanging the states,

$$\begin{cases} \begin{bmatrix} \dot{i}_L \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L_1+L_2} \\ -\frac{1}{C} & -\frac{1}{CR} \end{bmatrix}\begin{bmatrix} i_L \\ V_c \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{CR} \end{bmatrix} V_i \\ V_o = \begin{bmatrix} 0 & \frac{L_2}{L_1+L_2} \end{bmatrix}\begin{bmatrix} i_L \\ V_c \end{bmatrix} \end{cases}$$

(d) For the state-space model in (b),

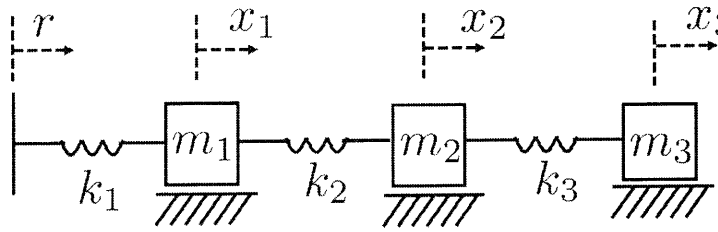
$$TF = C(SI-A)^{-1}B$$

$$= \begin{bmatrix} \frac{L_2}{L_1+L_2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} S+\frac{1}{CR} & \frac{1}{C} \\ -\frac{1}{L_1+L_2} & S \end{bmatrix}^{-1}} \begin{bmatrix} \frac{1}{CR} \\ 0 \end{bmatrix}$$

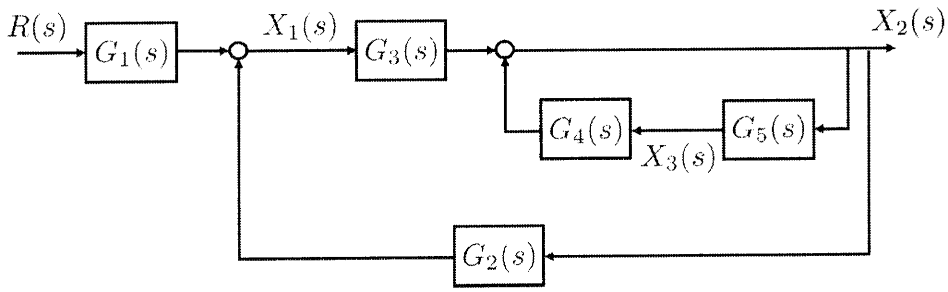
$$\frac{1}{S^2+\frac{1}{CR}S+\frac{1}{C(L_1+L_2)}}\begin{bmatrix} S & -\frac{1}{C} \\ \frac{1}{L_1+L_2} & S+\frac{1}{CR} \end{bmatrix}$$

$$= \frac{\frac{L_2}{CR(L_1+L_2)}S}{S^2+\frac{1}{CR}S+\frac{1}{C(L_1+L_2)}}$$

4. Consider the three degrees-of-freedom mass-spring system below. Here, m_1, m_2 and m_3 [kg] are masses, and k_1, k_2 and k_3 [N/m] are spring constants. The signals x_1, x_2 and x_3 [m] are displacements. The signal r [m] is the displacement input. Capital letters (for example, $R(s)$) denote the Laplace transform of signals (for example, $r(t)$). Ignore the damping and friction.



(a) The block diagram for this system can be depicted as below. Obtain the transfer functions $G_i(s)$, $i = 1, 2, 3, 4, 5$. (5pt)



Write your answer here.

(a)

$$\begin{cases} m_1\ddot{x}_1 = -k_1(x_1 - r) - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3(x_2 - x_3) \\ m_3\ddot{x}_3 = -k_3(x_3 - x_2) \end{cases}$$

$$\overset{\mathcal{L}}{\Longrightarrow} \begin{cases} (m_1 s^2 + k_1 + k_2)\, X_1(s) = k_2 X_2(s) + k_1 R(s) \\ (m_2 s^2 + k_2 + k_3) X_2(s) = k_2 X_1(s) + k_3 X_3(s) \\ (m_3 s^2 + k_3)\, X_3(s) = k_3 X_2(s) \end{cases} \Rightarrow \begin{cases} X_1(s) = \overbrace{\dfrac{k_1}{m_1 s^2 + k_1 + k_2}}^{G_1} R(s) + \overbrace{\dfrac{k_2}{m_1 s^2 + k_1 + k_2}}^{G_2} X_2(s) \\ X_2(s) = \overbrace{\dfrac{k_2}{m_2 s^2 + k_2 + k_3}}^{G_3} X_1(s) + \overbrace{\dfrac{k_3}{m_2 s^2 + k_2 + k_3}}^{G_4} X_3(s) \\ X_3(s) = \underbrace{\dfrac{k_3}{m_3 s^2 + k_3}}_{G_5} X_2(s) \end{cases}$$

(b) Using the notations $G_i(s)$, $i = 1, 2, 3, 4, 5$, obtain:

i. the transfer function from $X_1(s)$ to $X_2(s)$. (You can **name this transfer function as $G_6(s)$, and use in questions below.**) (2pt)

ii. the transfer function from $R(s)$ to $X_2(s)$. (2pt)

iii. the transfer function from $R(s)$ to $X_3(s)$. (1pt)

Write your answer here.

(b)

i) $\dfrac{G_3}{1 - G_4 G_5} \quad (=: G_6)$

ii) $\dfrac{G_1 G_6}{1 - G_2 G_6}$

iii) $\dfrac{G_1 G_6}{1 - G_2 G_6} \times G_5$

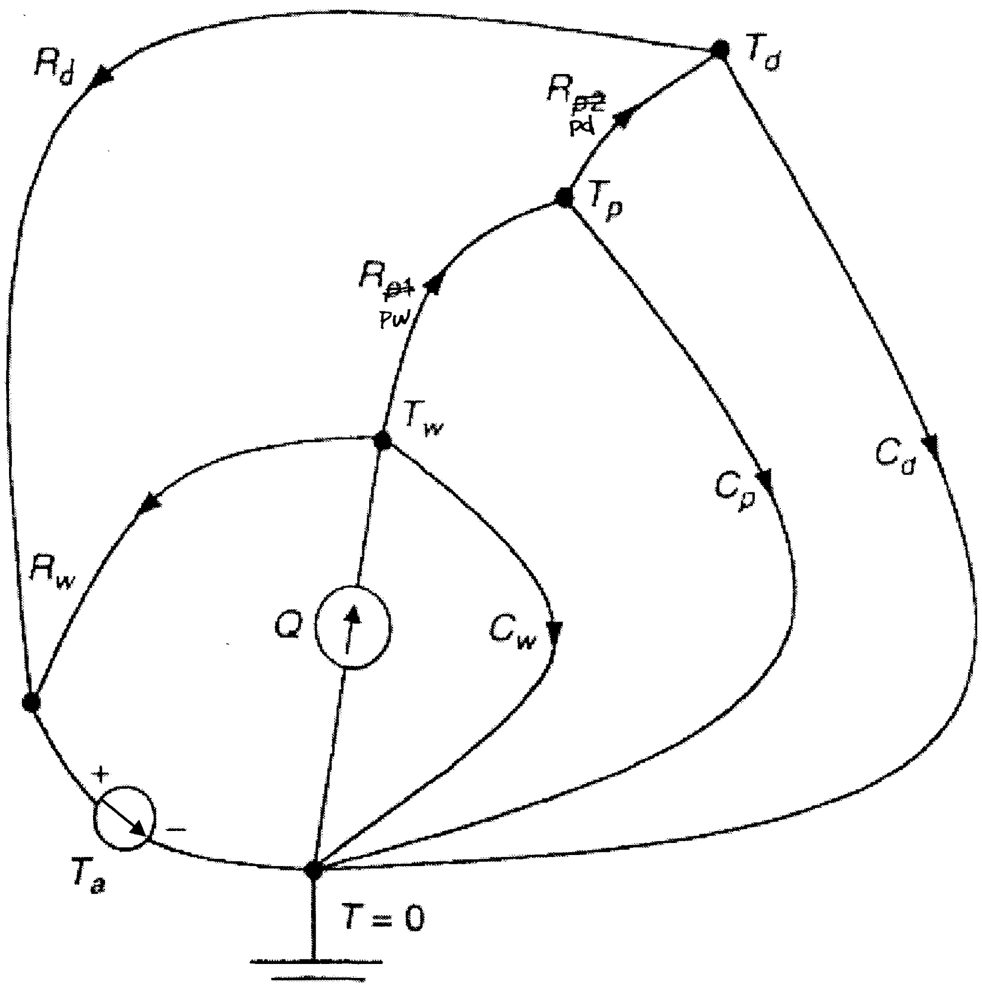
Q5. Solution

a place of mind

UBC

Example 2.15 Linear graph drawing

- States T_w, T_p, T_d



- Constitutive eq.

$$R_{\cancel{p1}\ pw} Q_{R_{\cancel{p1}\ pw}} = T_w - T_p$$
$$R_{\cancel{p2}\ pd} Q_{R_{\cancel{p2}\ pw}} = T_p - T_d$$
$$R_d Q_{R_d} = T_d - T_a$$
$$R_w Q_{R_w} = T_w - T_a$$
$$C_w \dot{T}_w = Q_{C_w} \quad C_d \dot{T}_d = Q_{C_d}$$
$$C_p \dot{T}_p = Q_{C_p}$$

- Node eq.

$$Q_{R_{p2}} = Q_{R_d} + Q_{C_d}$$
$$Q_{R_{\cancel{p1}\ pw}} = Q_{R_{\cancel{p2}\ pd}} + Q_{C_p}$$
$$Q_{R_{\cancel{p1}\ pw}} + Q_{R_w} + Q_{C_w} = Q$$
$$Q_{R_d} + Q_{R_w} + Q_a = 0$$