MECH366 : Modeling of Mechatronic Systems

L11 : Modeling of fluid systems

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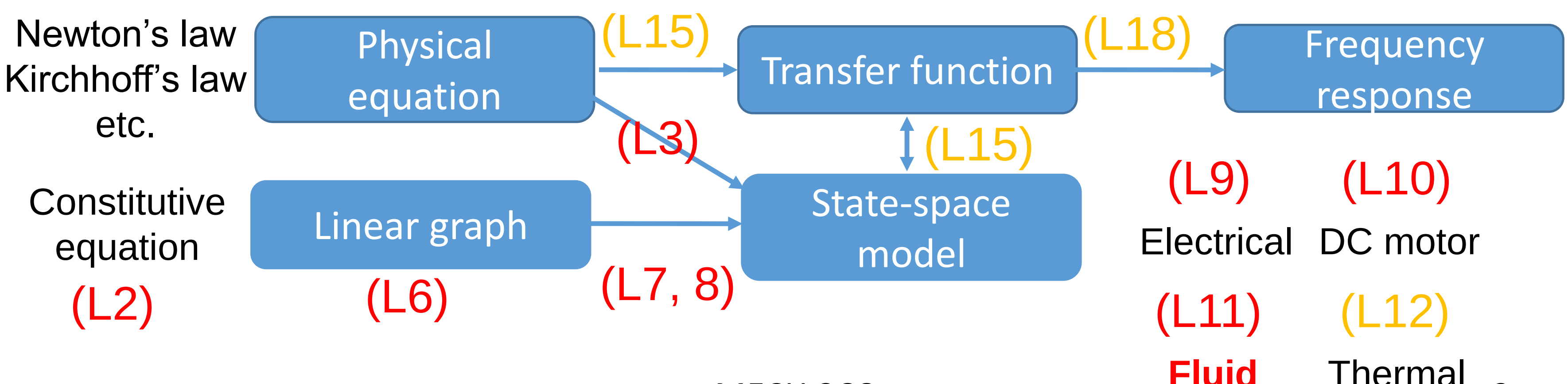
Review and today's topic

- Up to now, we have studied for mechanical and electrical systems:
  - How to draw linear graphs & derive state-space models
- Today, we will study modeling of fluid systems.
- Various models and their relations



$\underline{\quad}$: State variable

	Constitutive relation for			
	Energy storage element		Energy dissipating element	
System type	A-Type	T-Type	D-Type	
Mechanical	Mass	Spring	Viscous Damper	
(translational)				power
v: velocity across var.	$m\underline{\dot{v}} = f$	$\underline{\dot{f}} = kv$	$f = bv$	$\mathcal{P} = fv$
f: force through var.	m: mass	k: stiffness	b: damping const.	
Electrical	Capacitor	Inductor	Resistor	
v: voltage across var.	$C\underline{\dot{v}} = i$	$L\underline{\dot{i}} = v$	$v = Ri$	$\mathcal{P} = iv$
i: current through	C: capacitance	L: inductance	R: resistance	
Thermal	Thermal capacitor	None	Thermal resistor	
T: temperature	$C_t\dot{T} = Q$		$T = R_tQ$	
Q: heat transfer rate	C_t: thermal capacitance		R_t: thermal resistance	
Fluid across var.	Fluid capacitor	Fluid inertor	Fluid resistor	
P: pressure difference $[N/m^2]$	$C_f\underline{\dot{P}} = Q$	$I_f\underline{\dot{Q}} = P$	$P = R_fQ$	$\mathcal{P} = QP$
Q: volume flow rate $[m^3/s]$ through var.	C_f: fluid capacitance	I_f: fluid inertance	R_f: fluid resistance	

Energy expressions based on across and through variables

	A-type element	T-type element
Mechanical v : Across variable f : Through variable	Kinetic energy $\frac{1}{2}mv^2$	Potential energy $\left(\frac{1}{2}kx^2 =\right)\frac{1}{2}\frac{f^2}{k}$
Electrical v : Across variable i : Through variable	Electrostatic energy $\frac{1}{2}Cv^2$	Electromagnetic energy $\frac{1}{2}Li^2$
Thermal T : Across variable Q : Through variable	Thermal energy $\int Q = C_t T$	N/A
Fluid P : Across variable Q : Through variable	Potential energy $\frac{1}{2}C_f P^2$	Kinetic energy $\frac{1}{2}I_f Q^2$

Linear graph representation

- Single-port elements
  - Energy storage elements
    - Fluid capacitor (3 examples)
    - Fluid inerter (1 example)
  - Energy dissipation elements
  - Energy sources
- Two-port elements (Energy transfer elements)
  - Transformer

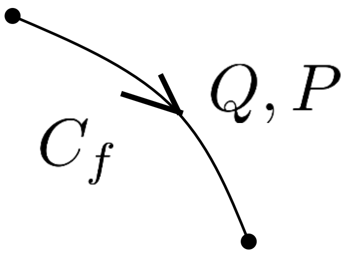
Energy storage element
Fluid capacitor

- Constitutive equation

$$C_f \frac{dP}{dt} = Q$$

$C_f \ [m^5/N]$: fluid capacitance

Linear graph



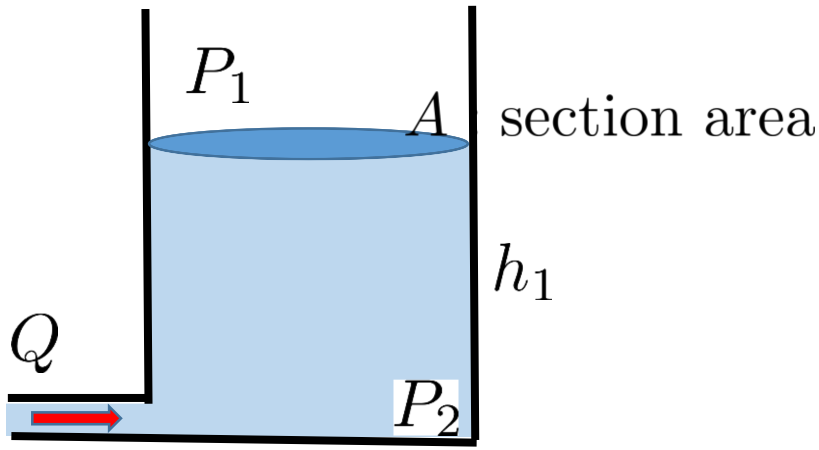
- Potential energy stored

$$\mathcal{E} = \frac{1}{2} C_f P^2$$

- Behave like a “fluid spring”.

Energy storage element
Fluid capacitor: Example

- Incompressible fluid in an open tank



$$\boxed{C_f = \frac{A}{\rho g}}$$

- Derivation of C_f
  - Pressure at tank bottom

$$P = P_2 - P_1 = \rho g h$$

$$\Rightarrow \quad h = \frac{P}{\rho g}$$

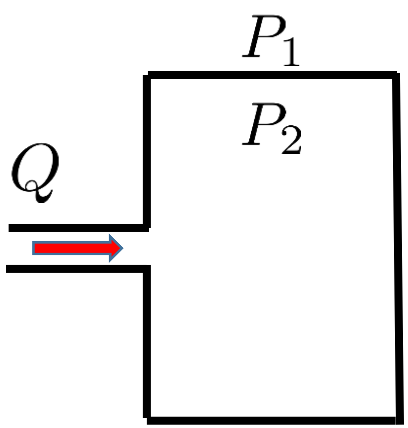
  - Flow rate

$$Q = \frac{d(Ah)}{dt} = A\frac{dh}{dt}$$

$$\Rightarrow \quad Q = \frac{A}{\rho g}\frac{dP}{dt}$$

Energy storage element
Fluid capacitor: Example

- Compressible fluid in a rigid container



$$P = P_2 - P_1$$

$$\boxed{C_f = \frac{V}{\beta}}$$

β : fluid bulk modulus

"degree of compressibility"
How much pressure is necessary to change the density?

- Derivation of C_f
  - Conservation of mass

$$\rho Q = \frac{d}{dt}(\rho V) = \rho \underbrace{\frac{dV}{dt}}_{=0} + V\frac{d\rho}{dt}$$

  - Fluid bulk modulus

$$\frac{d\rho}{\rho} = \frac{dP}{\beta}$$

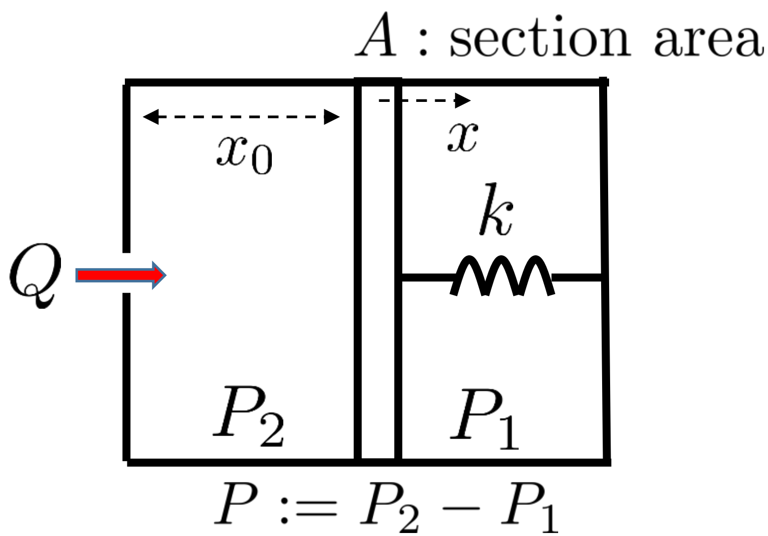
$$\Rightarrow \quad Q = \frac{V}{\rho}\frac{d\rho}{dt} = \frac{V}{\beta}\frac{dP}{dt}$$

$$\beta \approx 2.1 \times 10^9 \ [Pa]$$

for oil and water

Energy storage element
Fluid capacitor: Example

- Incompressible fluid in a flexible container



$$\boxed{C_f = \frac{A^2}{k}}$$

- Derivation of C_f
  - Conservation of flow

$$Q = \frac{d(A(x_0 + x))}{dt} = A\frac{dx}{dt}$$

  - Spring equilibrium

$$AP = kx$$

$$\Rightarrow \quad Q = A\frac{dx}{dt} = \frac{A^2}{k}\frac{dP}{dt}$$

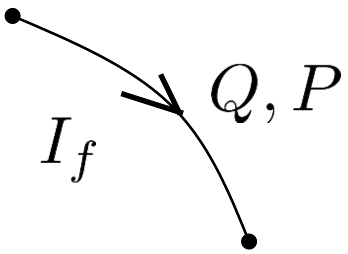
Energy storage element
Fluid inerter

- Constitutive equation

$$I_f \frac{dQ}{dt} = P$$

$$I_f \; [Ns^2/m^5] : \text{fluid inertance}$$

Linear graph



- Kinetic energy stored

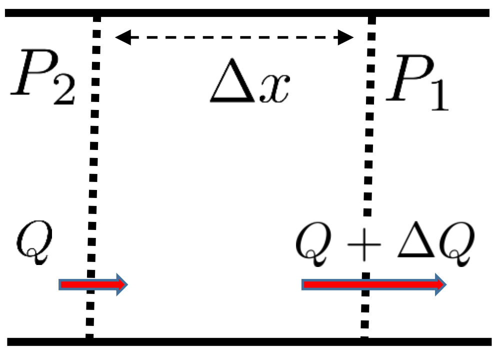
$$\mathcal{E} = \frac{1}{2} I_f Q^2$$

- Behave like a "fluid inertia".

Energy storage element
Fluid inerter: Example

- Incompressible fluid in a pipe with uniform velocity distribution

$$P := P_2 - P_1$$



$$\boxed{I_f = \frac{\rho \Delta x}{A}}$$

ρ : density
Δx : pipe length
A : uniform section area

- Derivation of I_f
  - Mass $\rho A \Delta x$
  - Net force AP
  - Flow velocity Q/A
  - Newton's 2nd law

$$AP = (\rho A \Delta x)(\dot{Q}/A)$$

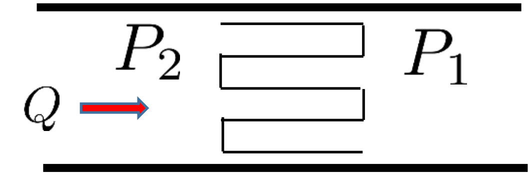
$$\Rightarrow \quad P = \frac{\rho \Delta x}{A}\frac{dQ}{dt}$$

Linear graph representation

- Single-port elements
  - Energy storage elements
  - Energy dissipation elements
    - Fluid resistor (1 example)
  - Energy sources
- Two-port elements (Energy transfer elements)
  - Transformer

Energy dissipation element
Fluid resistor

$$P := P_2 - P_1$$

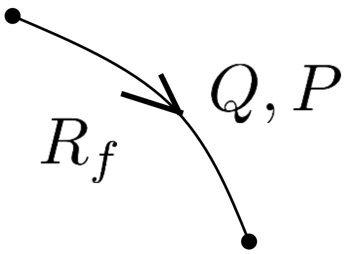


- Constitutive equation

$$R_f Q = P$$

$$R_f \; [Ns/m^5] : \text{fluid resistance}$$

Linear graph



- Power dissipated

$$\mathcal{P} = QP = R_f Q^2 = \frac{1}{R_f} P^2$$

- Behave like a “fluid friction”.

Energy dissipation element
Fluid resistor: Example

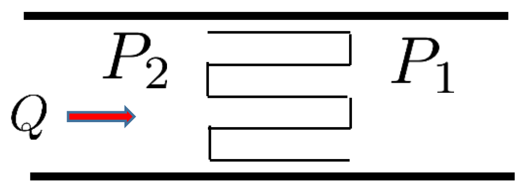
- Fluid resistance is associated with flow through pipes, orifices and valve openings.
- For a long, uniform area circular pipe with laminar flow, (due to Hagen-Poiseuille flow law)

$$\boxed{R_f = 128\frac{\mu\ell}{\pi d^4}}$$

ℓ : pipe length
μ : fluid viscocity
d : pipe diameter

$$R_f Q = P$$

$$P := P_2 - P_1$$



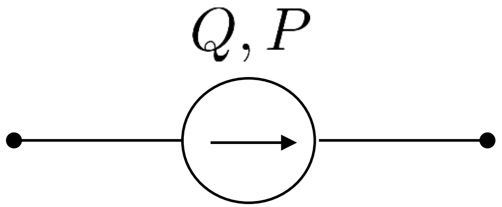
Linear graph representation

- Single-port elements
  - Energy storage elements
  - Energy dissipation elements
  - Energy sources
- Two-port elements (Energy transfer elements)
  - Transformer

Linear graph representation
Fluid energy sources

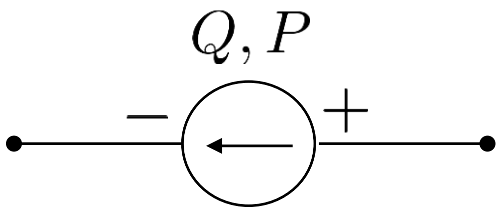
- Flow source
  - Regulated pump

Linear graph



- Pressure source
  - Regulated pump
  - Large reservoir
  - Accumulator

Linear graph



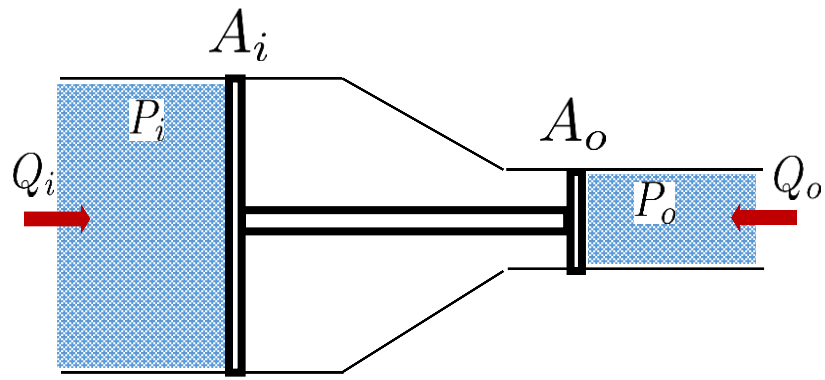
Linear graph representation

- Single-port elements
    - Energy storage elements
    - Energy dissipation elements
    - Energy sources
- Two-port elements (Energy transfer elements)
    - Transformer

Linear graph representation
Two-port element: Fluid transformer

- Section area ratio

$$r := \frac{A_i}{A_o}$$

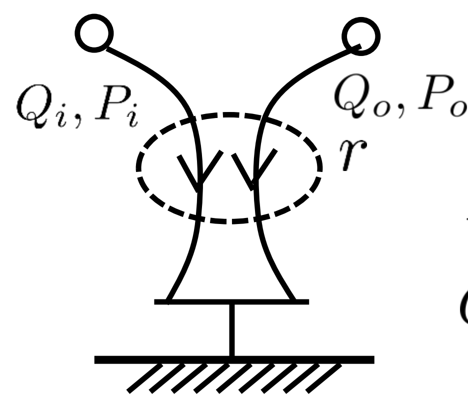


- Pressure ratio $P_o = rP_i$
- Conservation of power

$$Q_iP_i + Q_oP_o = 0$$

$$\Rightarrow \quad Q_o = -\frac{P_i}{P_o}Q_i = -\frac{1}{r}Q_i$$

Linear graph

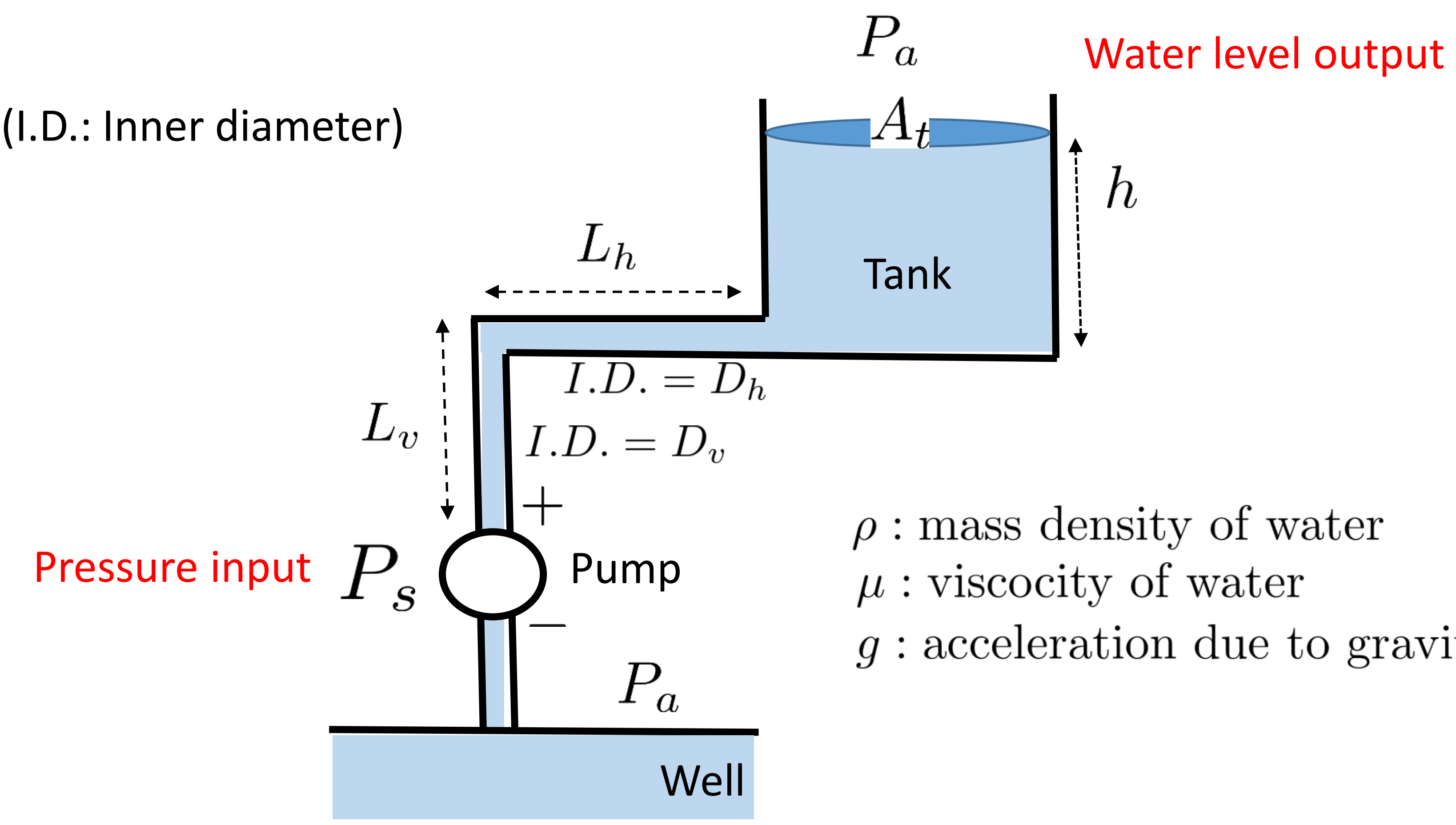


$$P_o = rP_i$$

$$Q_o = -\frac{1}{r}Q_i$$

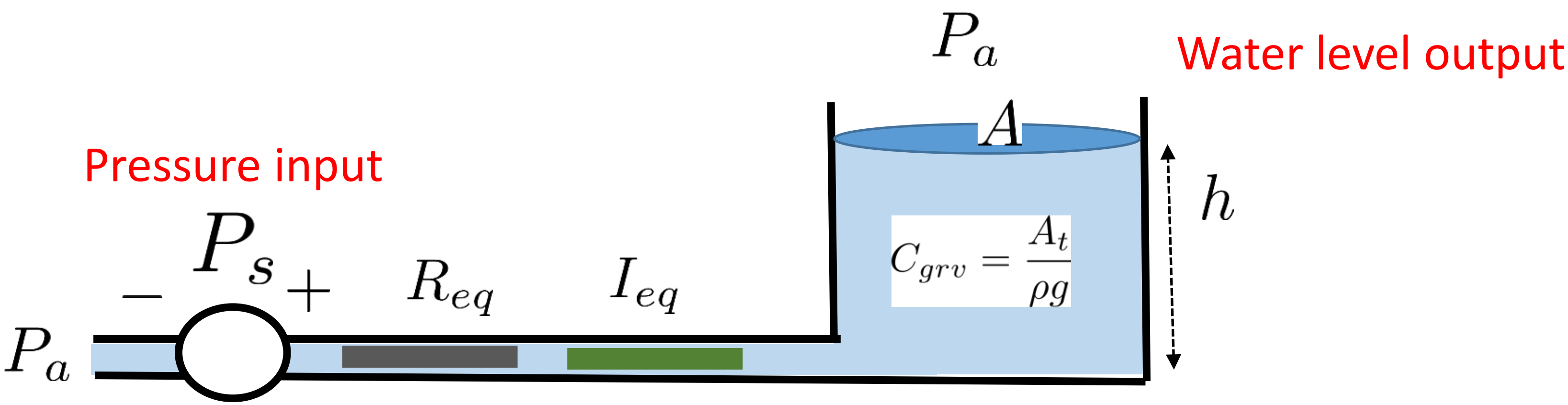
Example (taken from de Silva's book) Pumping water from well to tank

(I.D.: Inner diameter)



ρ : mass density of water
μ : viscocity of water
g : acceleration due to gravity

Example (taken from de Silva's book) Pumping water from well to tank



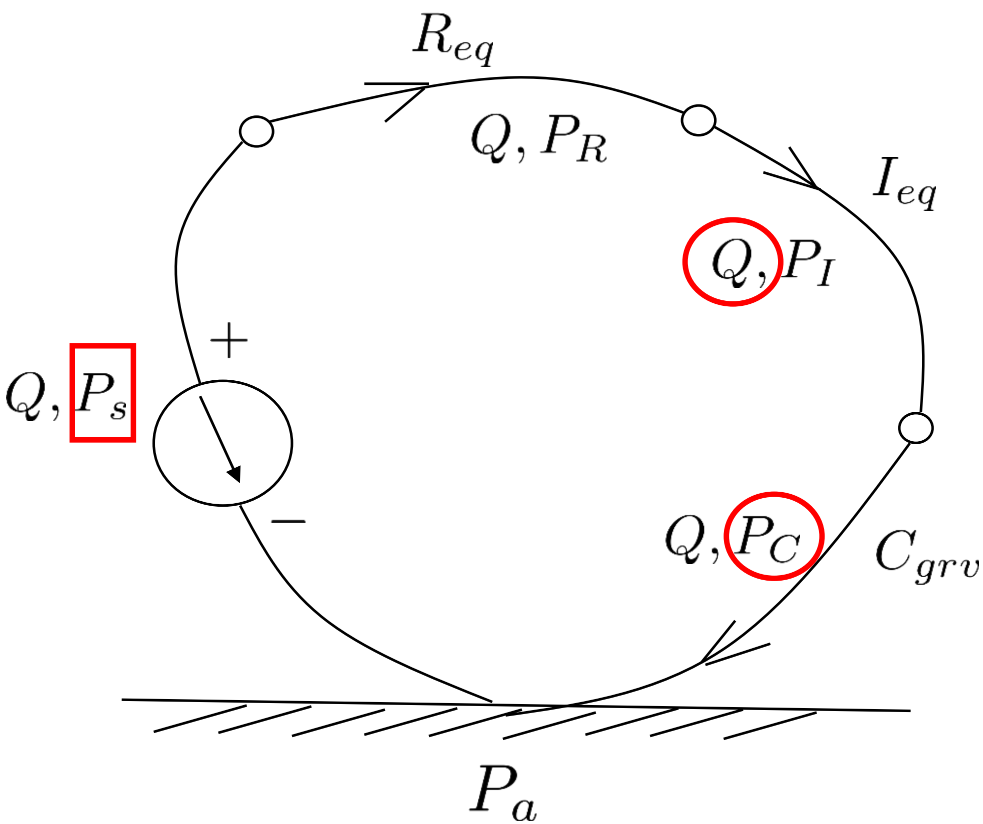
- Equivalent fluid resistance of the overall pipe

$$R_{eq} = 128\frac{\mu}{\pi}\left(\frac{L_v}{d_v^4} + \frac{L_h}{d_h^4}\right)$$

- Equivalent fluid inertance within the overall pipe (assuming uniform velocity)

$$I_{eq} = \frac{\rho L_v}{\pi d_v^2/4} + \frac{\rho L_h}{\pi d_h^2/4}$$

Example
Linear graph drawing



- State variables

$$x := \begin{bmatrix} P_C \\ Q \end{bmatrix}$$

- Constitutive eq.

$$P_R = R_{eq} Q$$

$$I_{eq} \dot{Q} = P_I$$

$$C_{grv} \dot{P}_C = Q$$

- Loop equation

$$P_s = P_R + P_I + P_C$$

Example
State-space model derivation

$$\dot{P}_C = \frac{1}{C_{grv}} Q$$

$$\dot{Q} = \frac{1}{I_{eq}} P_I = \frac{1}{I_{eq}}(P_s - P_R - P_C) = \frac{1}{I_{eq}}(P_s - R_{eq}Q - P_C)$$

$$P_C = \rho g h$$

$$\Rightarrow \left\{ \begin{array}{lcl} \begin{bmatrix} \dot{P}_C \\ \dot{Q} \end{bmatrix} & = & \begin{bmatrix} 0 & 1/C_{grv} \\ -1/I_{eq} & -R_{eq}/I_{eq} \end{bmatrix} \begin{bmatrix} P_C \\ Q \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I_{eq} \end{bmatrix} P_s \\ h & = & \begin{bmatrix} \frac{1}{\rho g} & 0 \end{bmatrix} \begin{bmatrix} P_C \\ Q \end{bmatrix} \end{array} \right.$$

Summary

- Linear graph for fluid systems
  - Single-port elements
    - Energy storage elements
    - Energy dissipation elements
    - Energy sources
  - Two-port elements (Energy transfer elements)
    - Transformer
- Next, thermal systems