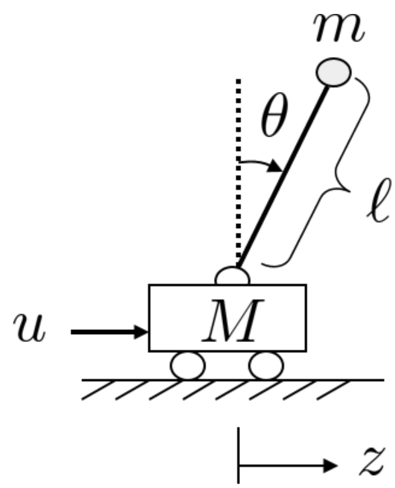
University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Homework 2

Due: September 30 (Monday), 2019, 3pm

Consider the inverted pendulum system below. Here, the input is the force u [N] and the two outputs are the position of the cart z [m] and the pendulum angular position θ [rad]. Other parameters are shown in the figure and below.



ℓ [m]	: length of the pendulum
m [kg]	: mass lumped at the top of the pendulum
M [kg]	: mass of the cart

The equations of motion for this system can be derived as follows:

$$\begin{cases} (M+m)\ddot{z} + (m\ell\cos\theta)\ddot{\theta} = u + m\ell\left(\dot{\theta}\right)^2 \sin\theta \\ (\cos\theta)\ddot{z} + (\ell)\ddot{\theta} = g\sin\theta \end{cases}$$

To answer the following questions, use the equations of motion above. (There is no need to re-derive them. The derivation is given in Appendix.)

1. By defining the states as

$$x_1 := z,\ x_2 = \dot{z},\ x_3 := \theta,\ x_4 := \dot{\theta},$$

obtain the nonlinear state-space model.

Solution: By solving the equations of motion with respect to $\ddot{z}$ and $\ddot{\theta}$, we have

$$\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} \ell & -m\ell\cos\theta \\ -\cos\theta & M+m \end{bmatrix}\begin{bmatrix} u + m\ell\left(\dot{\theta}\right)^2\sin\theta \\ g\sin\theta \end{bmatrix},$$

where

$$d := (M+m)\ell - m\ell\cos^2\theta = \ell(M + m\sin^2\theta).$$

Thus, the nonlinear state-space model is written by

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} x_2 \\ \dfrac{\ell}{d(x_3)}\left\{u + m(\ell x_4^2 - g\cos x_3)\sin x_3\right\} \\ x_4 \\ \dfrac{1}{d(x_3)}\left\{-u\cos x_3 + \big((M+m)g - m\ell x_4^2\cos x_3\big)\sin x_3\right\} \end{bmatrix}, \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x,
\end{aligned}$$

where $d(x_3) := \ell(M + m\sin^2 x_3)$.

2. For an operating point

$$x_0 := \begin{bmatrix} z_0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where z_0 is a constant displacement, derive a linearized state-space model.

Solution: Jacobian computations, with the substitution of the operating point x_0 above and $u_0 = 0$, are as follows.

$$\left.\frac{\partial f}{\partial x_1}\right|_{(x_0,u_0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \left.\frac{\partial f}{\partial x_2}\right|_{(x_0,u_0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} = \begin{bmatrix} 0 \\ \frac{\ell}{d(x_{30})} \\ 0 \\ -\frac{\cos x_{30}}{d(x_3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x_3}\right|_{(x_0,u_0)} = \begin{bmatrix} 0 \\ (a) \\ 0 \\ (b) \end{bmatrix}, \quad \left.\frac{\partial f}{\partial x_4}\right|_{(x_0,u_0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
(a) &= \frac{\ell}{d(x_{30})^2}\left\{(-mg)d(x_{30}) - (u_0 + m(\cdots)\sin x_{30})d'(x_{30})\right\} \\
&= -\frac{mg\ell}{M\ell} = -\frac{mg}{M} \\
(b) &= \frac{1}{d(x_{30})^2}\left\{(M+m)g\cos x_{30} d(x_{30}) - (0)d'(x_{30})\right\} \\
&= \frac{(M+m)g}{M\ell}
\end{aligned}$$

In summary, the linearized state-space model around the operating point x_0 becomes

$$\begin{aligned}
\delta\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix} \delta u, \\
\delta y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \delta x,
\end{aligned}$$

where the deviation variables are $\delta x := x - x_0$, $\delta u := u - u_0 = u$, $\delta y = y - Cx_0$.