University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Midterm exam

Examiner: Dr. Ryozo Nagamune
October 11 (Friday), 2019, 3pm-3:50pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: One-page letter-size hand-written cheat-sheet (both sides).
- Not-allowed: PC, calculators.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 20 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are <u>NOT</u> allowed.

If you finish early ...

- Please stay at your seat until the end of exam, i.e., 3:50pm. (You are not allowed to leave the room before the end of exam, except going to washroom.)

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		5
2		5
3		10
Total		20

1. Answer the following questions **concisely, by a few sentences and/or equations, or even by one-word or two-words if appropriate**.

   (a) For what purposes can a mathematical model of a physical system be used? Give **exactly two** such purposes. (If you write more than two purposes, you will lose some mark.) (1pt)

   Write your answer here.

   - Prediction
   - Analysis
   - Simulation
   - Controller design etc.

   (b) Explain why 'voltage' in electrical systems is called 'across variable'. (1pt)

   Write your answer here.

   Because the voltage is measured across electrical elements.

   (c) Explain the 'model validation' step in the modeling procedure. (1pt)

   Write your answer here.

   After getting a model, we apply the same inputs to both the real physical system and the model, and check if the outputs are similar. If they are similar, we can increase our confidence about the model validity.

(d) Give the **definition** of a linear system, by using the notations u, x and y as the input, the state and the output of the system, respectively. (**Hint:** 'State-space model' representation is NOT the definition of a linear system.)

Write your answer here.

If a system has the following superposition property, then it is a linear system.

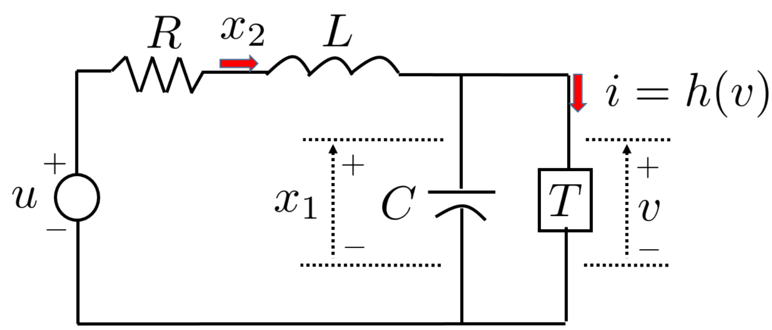
Suppose that, for zero initial condition $x(0) = 0$, the inputs $u_1(t)$ and $u_2(t)$ generate the outputs $y_1(t)$ and $y_2(t)$, respectively. Then, for any scalars α_1 and α_2, if we input $\alpha_1 u_1(t) + \alpha_2 u_2(t)$, then we can get the output $\alpha_1 y_1(t) + \alpha_2 y_2(t)$.

(e) **Using the relation between the energy and the power**, derive the **energy formula** for the electrical **inductor** element. (1pt)

Write your answer here.

$$E = \int i \cdot v dt = \int i \cdot L \frac{di}{dt} dt = L \int i di = \frac{1}{2} L i^2$$

2. Consider the electric circuit depicted below. Here, the notations R, L and C respectively denote the resistance, inductance and capacitance, and u is the voltage source. An electrical element T has the characteristic $i = h(v)$, where i is the current through T and v is the voltage across T, and h is a nonlinear function which is differentiable with respect to v (i.e., $h'(v)$ exists).



(a) Let x_1 be the voltage across the capacitance, and x_2 be the current through the inductance. Prove that the state equation for this system is described as follows. (You don't need to use the linear graph.) (2pt)

$$
\begin{aligned}
\dot{x}_1(t) &= -\frac{1}{C}h(x_1(t)) + \frac{1}{C}x_2(t) \\
\dot{x}_2(t) &= -\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}u(t)
\end{aligned}
$$

(b) Linearize the state equation above around the operating point $(x_1, x_2, u) = (x_{10}, x_{20}, u_0)$. (2pt)

(c) Express x_{20} and u_0 as functions of x_{10}. (1pt)

Write your answer here.

Write your answer here.

(a) By Kirchhoff's current and voltage laws, we have

$$\begin{aligned} x_2 &= C\dot{x}_1 + h(x_1) \\ u &= Rx_2 + L\dot{x}_2 + x_1 \end{aligned}$$

By manipulating these equations, we can reach the state equation.

(b) By introducing the deviation variables

$$\delta x_1 := x_1 - x_{10},\ \delta x_2 := x_2 - x_{20},\ u := u - u_0,$$

the linearized state equation can be written as

$$\begin{aligned} \dot{\delta x}_1(t) &= -\frac{1}{C}h'(x_{10})\delta x_1(t) + \frac{1}{C}\delta x_2(t) \\ \dot{\delta x}_2(t) &= -\frac{1}{L}\delta x_1(t) - \frac{R}{L}\delta x_2(t) + \frac{1}{L}\delta u(t). \end{aligned}$$

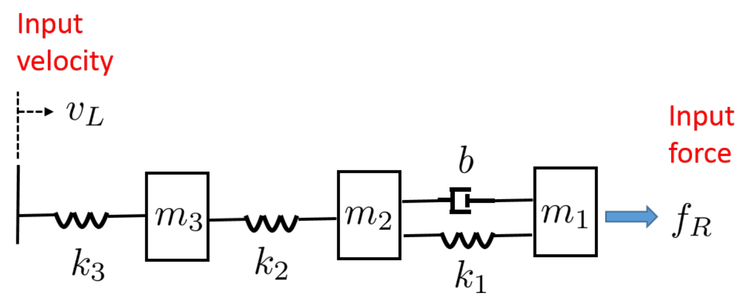
It can also be written in a matrix form as

$$\frac{d}{dt}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} -\frac{h'(x_{10})}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

(c) In the nonlinear state equation, we set the derivative terms to be zero. Then, we have

$$x_{20} = h(x_{10}), \quad u_0 = x_{10} + Rx_{20} = x_{10} + Rh(x_{10}).$$

3. Consider a 3-DOF mass-spring-damper system in the figure below. Here, m, b, and k (with subscripts) are respectively mass, viscous damping constant, and spring constant. Two inputs are the velocity v_L and the force f_R as indicated in the figure, and the outputs are the **displacement and acceleration** of m_1 (i.e., right-most mass in the figure).



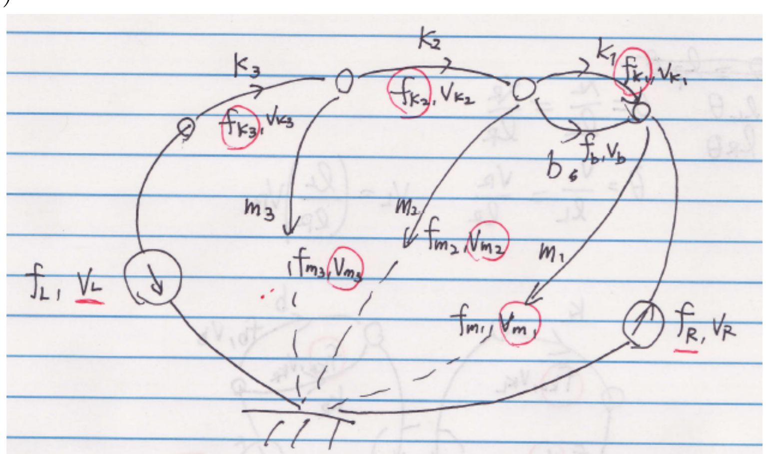
(a) Draw a linear graph, by introducing notations appropriately. (2pt)

(b) Select the state variables. (1pt)

(c) Write the constitutive equations for the passive elements in the linear graph. (1pt)

Write your answer here.

(a) (Add '+' and '−' signs above and below the velocity source, respectively.)



(b)

$$x := \begin{bmatrix} f_{k_1} & f_{k_2} & f_{k_3} & v_{m_1} & v_{m_2} & v_{m_3} & z_{m_1} \end{bmatrix}^T$$

where z_{m_1} is the displacement of mass m_1. (This last state is necessary to have an displacement output for m_1.)

(c)

$$\dot{f}_{k_i} = k_i v_{k_i} \qquad i = 1, 2, 3$$

$$f_{m_i} = m_i \dot{v}_{m_i} \quad i = 1, 2, 3$$

$$f_b = b v_b$$

(d) By using the linear graph, derive a state-space model in a matrix-vector form, i.e., in the form of '$\dot{x} = Ax + Bu$' and '$y = Cx + Du$'. (There is no need to write down the loop and node equations.) (3pt)

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Write your answer here.

$$
\begin{aligned}
\dot{f}_{k_1} &= k_1 v_{k_1} = k_1(v_{m_2} - v_{m_1}) \\
\dot{f}_{k_2} &= k_2 v_{k_2} = k_2(v_{m_3} - v_{m_2}) \\
\dot{f}_{k_3} &= k_3 v_{k_3} = k_3(v_L - v_{m_3}) \\
\dot{v}_{m_1} &= \frac{1}{m_1} f_{m_1} = \frac{1}{m_1}(f_R + f_{k_1} + f_b) = \frac{1}{m_1}(f_R + f_{k_1} + b v_b) \\
&= \frac{1}{m_1}(f_R + f_{k_1} + b(v_{m_2} - v_{m_1})) \\
\dot{v}_{m_2} &= \frac{1}{m_2} f_{m_2} = \frac{1}{m_2}(f_{k_2} - f_{k_1} - f_b) = \frac{1}{m_2}(f_{k_2} - f_{k_1} - b v_b) \\
&= \frac{1}{m_2}(f_{k_2} - f_{k_1} - b(v_{m_2} - v_{m_1})) \\
\dot{v}_{m_3} &= \frac{1}{m_3} f_{m_3} = \frac{1}{m_3}(f_{k_3} - f_{k_2}) \\
\dot{z}_{m_1} &= v_{m_1}
\end{aligned}
$$

The outputs are z_1 and

$$
\ddot{z}_{m_1} = \dot{v}_{m_1} = \frac{1}{m_1}(f_R + f_{k_1} + b(v_{m_2} - v_{m_1}))
$$

In a matrix form,

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix}
0 & 0 & 0 & -k_1 & k_1 & 0 & 0 \\
0 & 0 & 0 & 0 & -k_2 & k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -k_3 & 0 \\
1/m_1 & 0 & 0 & -b/m_1 & b/m_1 & 0 & 0 \\
-1/m_2 & 1/m_2 & 0 & b/m_2 & -b/m_2 & 0 & 0 \\
0 & -1/m_3 & 1/m_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
k_3 & 0 \\
0 & 1/m_1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix} v_L \\ f_R \end{bmatrix} \\
y &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1/m_1 & 0 & 0 & -b/m_1 & b/m_1 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 1/m_1
\end{bmatrix} \begin{bmatrix} v_L \\ f_R \end{bmatrix}
\end{aligned}
$$

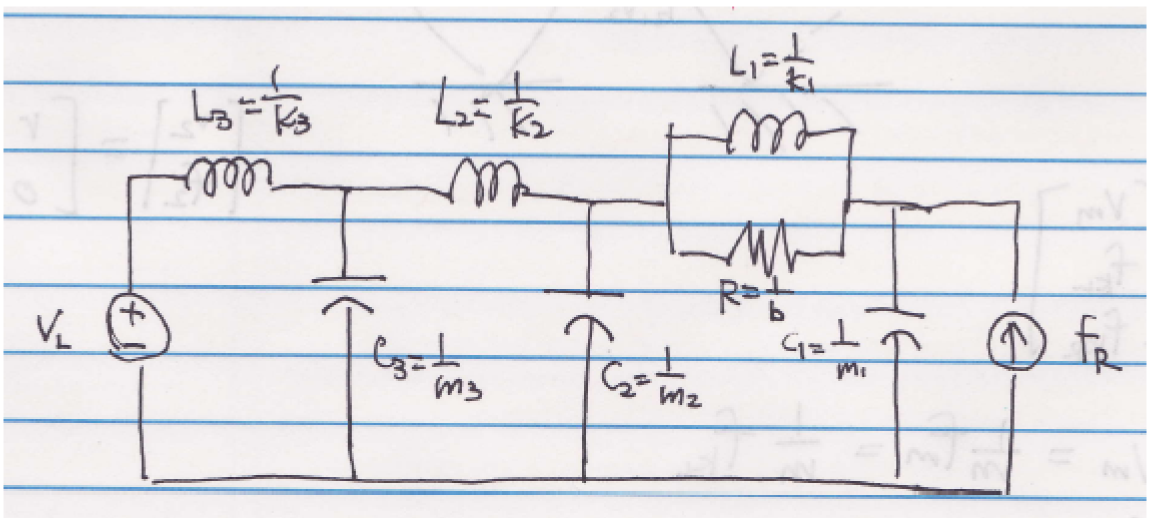
(e) Make an analogous electrical circuit for this mechanical system. (2pt)

(f) Instead of the input velocity v_L, when the input force f_L is applied at the same location as v_L, explain the reason why k_3 is not necessary in the state-space model. (1pt)

——— (End of Midterm Exam) ———

Write your answer here.

(e)



(f) The force f_L goes through the k_3 element, and it is applied to m_3. Therefore, we can consider that the force input is applied directly to m_3, and thus k_3 is not necessary for the state-space modeling.

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