MECH366 : Modeling of Mechatronic Systems

L6 : Linear graph

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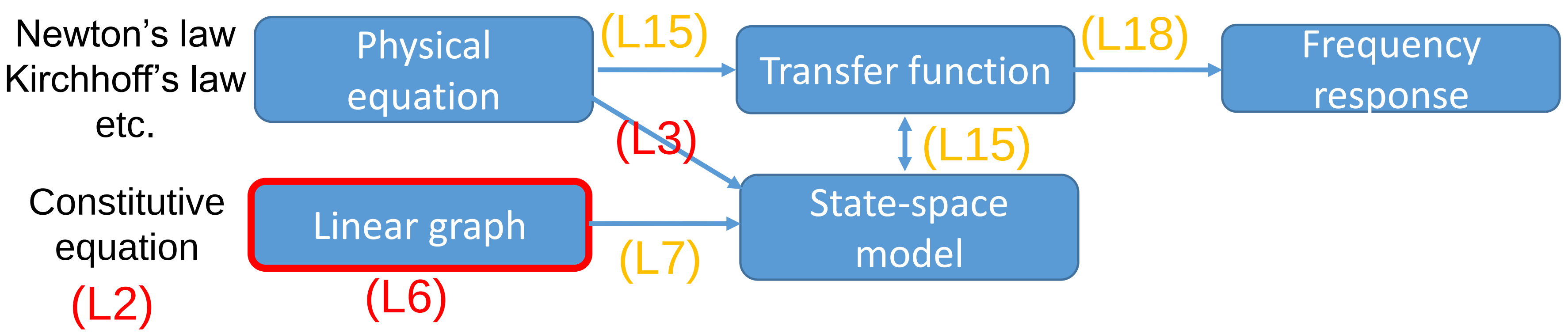
Review and today's topic

- Up to now, we have studied
  - Constitutive relations of basic elements (See next slide)
  - State-space model, linearization
- Today, we will study the linear graph
- Various models and their relations (Lecture number)



___ : State variable

	Constitutive relation for		
	Energy storage element		Energy dissipating element
System type	A-Type	T-Type	D-Type
Mechanical	Mass	Spring	Viscous Damper
(translational)			
v: velocity across var.	$m\underline{\dot{v}} = f$	$\underline{\dot{f}} = kv$	$f = bv$
f: force through var.	m: mass	k: stiffness	b: damping const.
Electrical	Capacitor	Inductor	Resistor
v: voltage	$C\dot{v} = i$	$L\dot{i} = v$	$v = Ri$
i: current	C: capacitance	L: inductance	R: resistance
Thermal	Thermal capacitor	None	Thermal resistor
T: temperature	$C_t\dot{T} = Q$		$T = R_t Q$
Q: heat transfer rate	C: thermal capacitance		R_t: thermal resistance
Fluid	Fluid capacitor	Fluid inertor	Fluid resistor
P: pressure difference	$C_f\dot{P} = Q$	$I_f\dot{Q} = P$	$P = R_f Q$
Q: volume flow rate	C_f: fluid capacitance	I_f: fluid inertance	R_f: fluid resistance

power

$$\mathcal{P} = fv$$

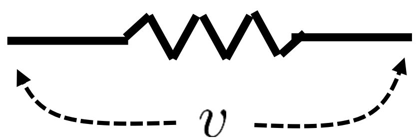
Across and through variables

- Across variable is measured “across” the element.

Mechanical spring

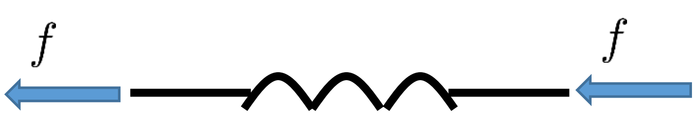
$$v := v_1 - v_2$$

Electrical resistor



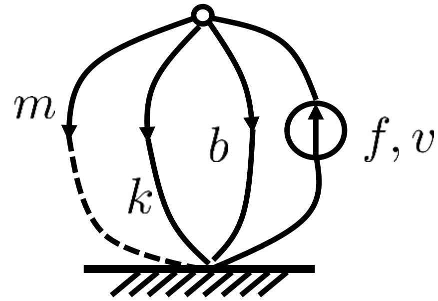
- Through variable passes “through” the element.

Mechanical spring



Electrical resistor

Linear graph

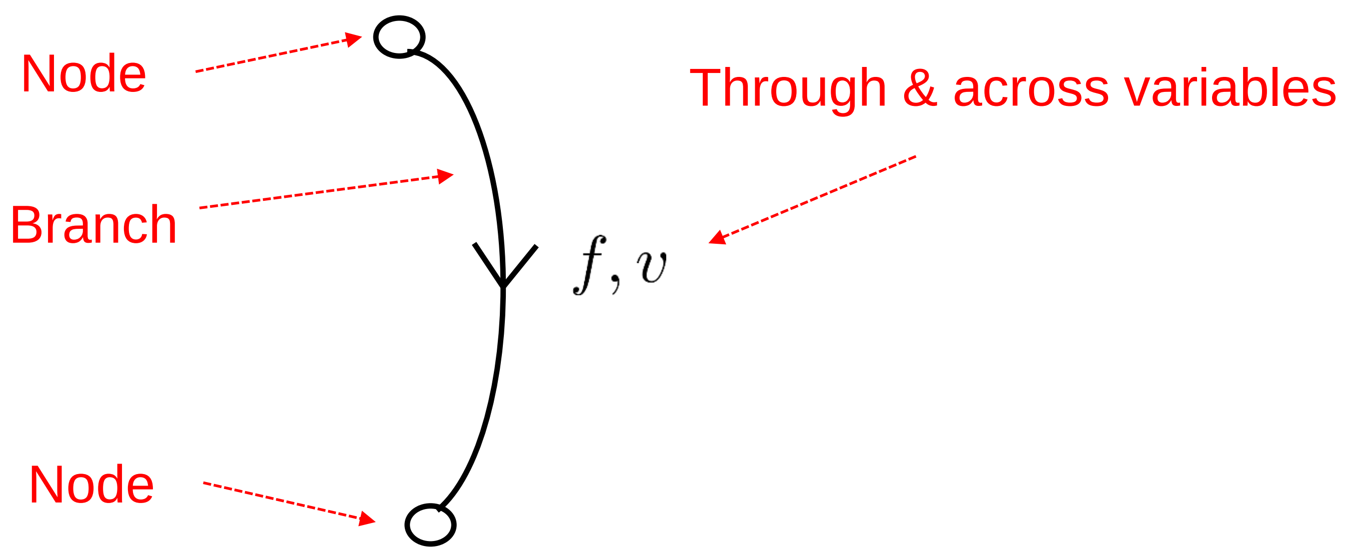


- Topological relations of lumped element interconnections within a system
- The term *linear* denotes a graphical line segment, and is not related to the mathematical linearity.
- Linear graphs are a unified method of representing systems that involve more than one energy medium.
- They can be used to derive state-space models systematically and in a computer-automated way.
- They are similar in form to electrical circuit diagrams.

Let's first focus on linear graphs for mechanical systems.

Linear graph representation of each passive element

- A branch with an arrow (an oriented line segment) and two nodes
- One through variable and one across variable

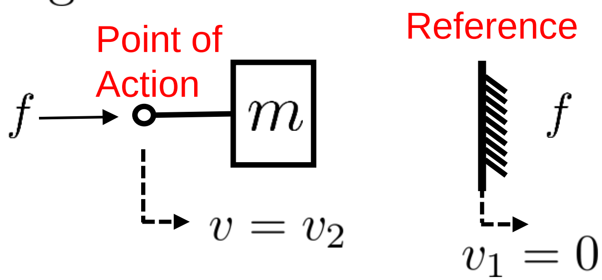


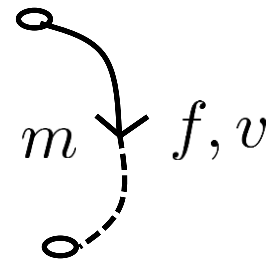
Linear graph representation

- Single-port elements
  - Energy storage elements
  - Energy dissipation elements
  - Energy source
- Two-port elements (Energy transfer elements)
  - Transformer
  - Gyrator

Linear graph representation Mechanical single-port elements

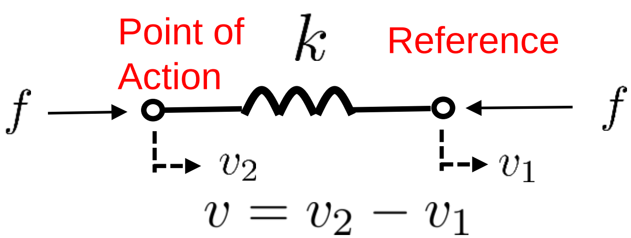
Energy storage element

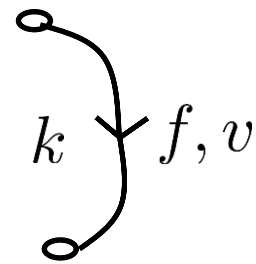




$$\dot{v} = \frac{1}{m} f$$

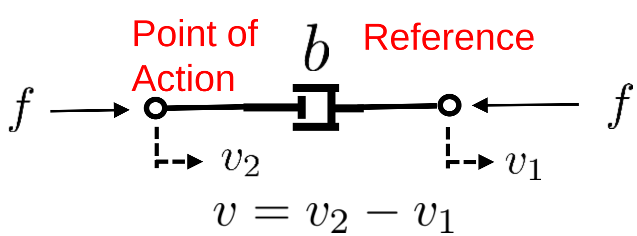
Energy storage element

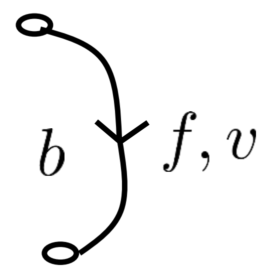




$$\dot{f} = kv$$

Energy dissipation element





$$f = bv$$

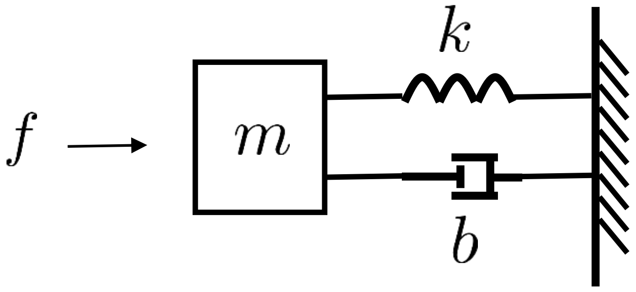
Linear graph: Remarks

- Note the dashed line for mass. It means
  - the force does not physically travel from one end to the other end.
  - the velocity is measured with respect to reference 0.
- Arrow in the linear graph denotes the direction of positive power flow
  - Start of arrow: Point of action (Power into element).
  - End of arrow: Point of reference (Power out of element)
- Rotational mechanical elements are represented analogously. (*f*: torque, *v*: angular velocity)

Mass-spring-damper example

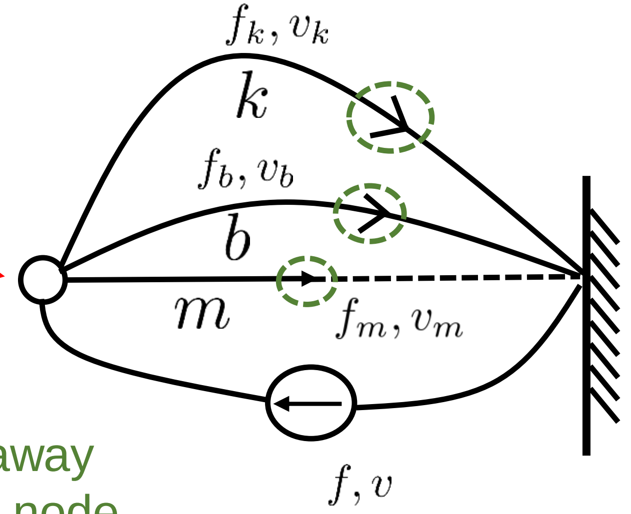
- Lumped model

External
force source



- Linear graph

Take a node for each
different velocity point.



Direct all arrows on passive elements away
from sources and toward the reference node.

Linear graph representation

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  - Energy dissipation elements
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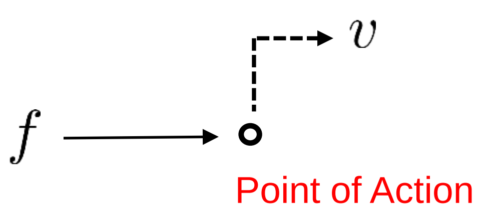
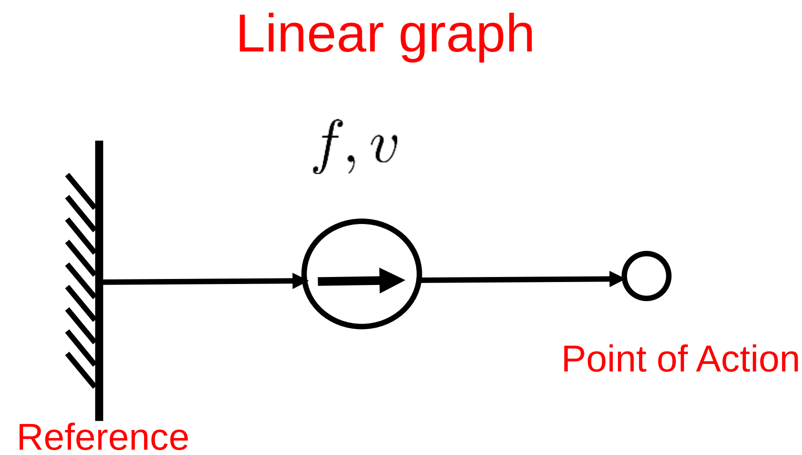
- Two-port elements (Energy transfer elements)
  - Transformer
  - Gyrator

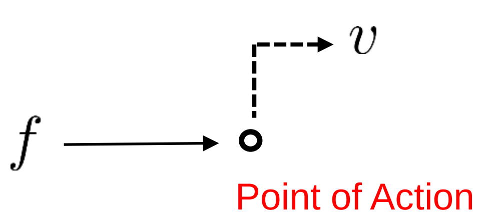
Linear graph representation Mechanical energy sources

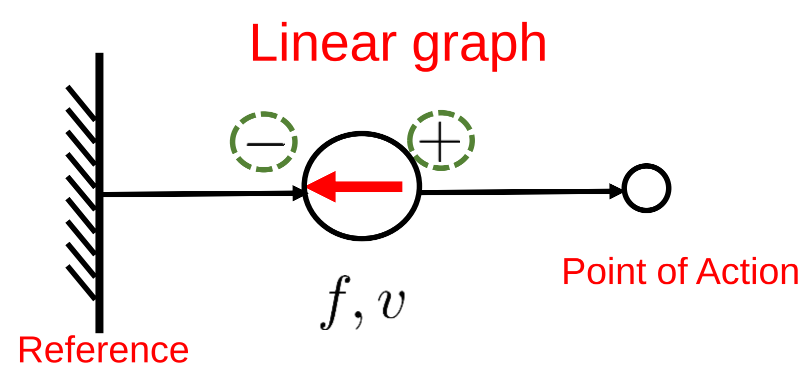
- Force source





- Velocity source ("+": point of action, "-": reference)





Linear graph representation

- Single-port elements
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- Two-port elements (Energy transfer elements)
  - Transformer

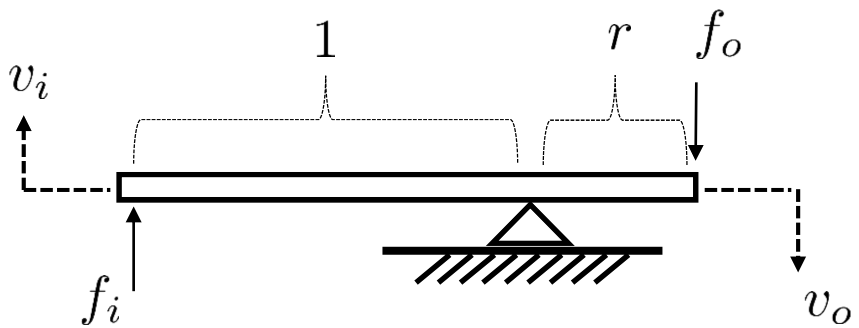
$$\begin{cases} v_o = r v_i \\ f_o = -\dfrac{1}{r} f_i \end{cases} \qquad \begin{bmatrix} v_o \\ f_o \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & -1/r \end{bmatrix} \begin{bmatrix} v_i \\ f_i \end{bmatrix}$$

  - Gyrator

$$\begin{cases} v_o = M f_i \\ f_o = -\dfrac{1}{M} v_i \end{cases} \qquad \begin{bmatrix} v_o \\ f_o \end{bmatrix} = \begin{bmatrix} 0 & M \\ -1/M & 0 \end{bmatrix} \begin{bmatrix} v_i \\ f_i \end{bmatrix}$$

Linear graph representation
Two-port elements: Transformer

- Example: Lever with length ratio *r*



  - *Velocity ratio*

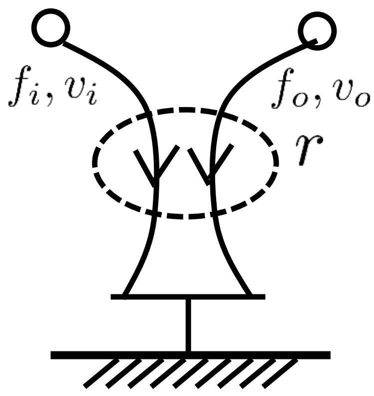
$$v_o = r v_i$$

  - *Conservation of power*

$$f_i v_i + f_o v_o = 0$$

$$\Rightarrow \quad f_0 = -\frac{v_i}{v_o} f_i = -\frac{1}{r} f_i$$

Linear graph

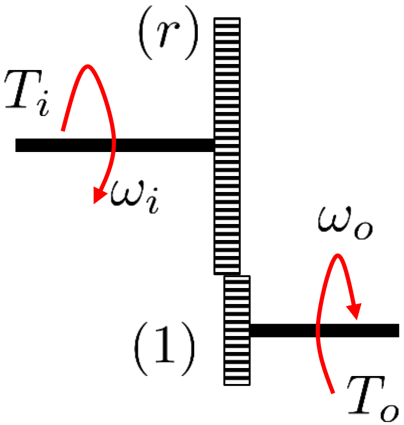


$$v_o = r v_i$$

$$f_o = -\frac{1}{r} f_i$$

Linear graph representation
Two-port elements: Transformer

- Example: Gear with the number of teeth ratio 1/*r*



  - *Angular velocity ratio*

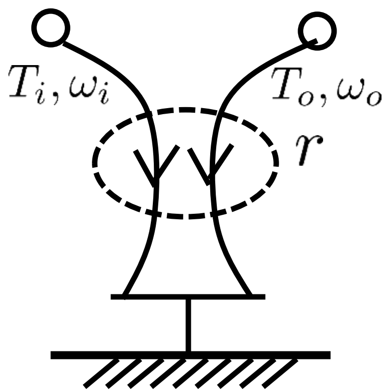
$$\omega_o = r\omega_i$$

  - *Conservation of power*

$$T_i\omega_i + T_o\omega_o = 0$$

$$\Rightarrow T_0 = -\frac{\omega_i}{\omega_o}T_i = -\frac{1}{r}T_i$$

Linear graph



$$\omega_o = r\omega_i$$

$$T_o = -\frac{1}{r}T_i$$

Linear graph representation

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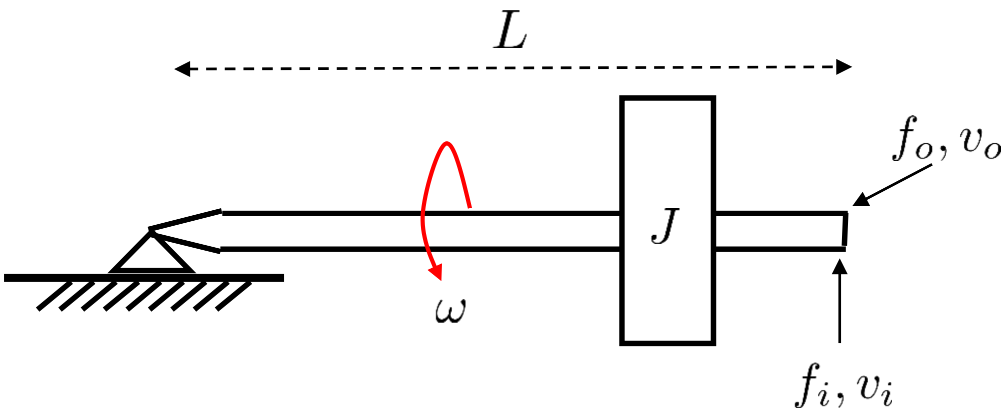
  - Gyrator

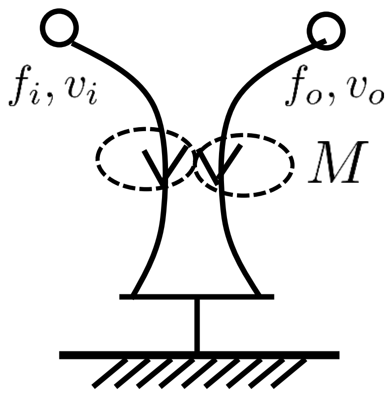
$$\begin{cases} v_o = M f_i \\ f_o = -\dfrac{1}{M} v_i \end{cases} \qquad \begin{bmatrix} v_o \\ f_o \end{bmatrix} = \begin{bmatrix} 0 & M \\ -1/M & 0 \end{bmatrix} \begin{bmatrix} v_i \\ f_i \end{bmatrix}$$

Linear graph representation Two-port elements: Gyrator

- Example: Gyroscope (spinning top)



Linear graph



$$v_0 = M f_i$$

$$f_0 = -\frac{1}{M} v_i$$

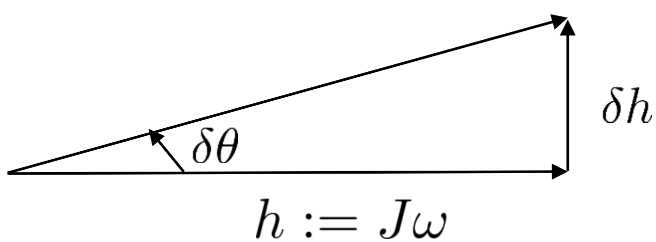
$$M := \frac{L^2}{J\omega}$$

Derivations for gyrator equations (Not covered in the course)

1. Angular momentum about positive *x*-axis is

$$h := J\omega$$

2. Angular momentum vector is given rotation about positive *z*-axis



$$\delta h := h \tan \delta\theta \approx J\omega\delta\theta$$

3. Rate of change of angular momentum

$$\frac{\delta h}{\delta t} = J\omega \frac{\delta \theta}{\delta t} \Rightarrow \frac{dh}{dt} = J\omega \frac{d\theta}{dt} = J\omega \frac{v_i}{L}$$

4. Newton's 2nd law

$$-f_0 L = J\omega \frac{v_i}{L} \Rightarrow f_0 = -\frac{J\omega}{L^2} v_i = -\frac{1}{M} v_i$$

5. Conservation of power

$$f_i v_i + f_o v_o = 0 \Rightarrow v_0 = M f_i$$

Summary

- Linear graph
  - Single-port elements
    - Energy storage elements
    - Energy dissipation elements
    - Energy source
  - Two-port elements (Energy transfer elements)
    - Transformer
    - Gyrator
- Next, the derivation of state-spaces model from linear graphs