University of British Columbia
Department of Mechanical Engineering

MECH 305/306. Test 2, March 13, 2018

Suggested Time: 60 min **Allowed Time:** 70 min

Materials admitted: Pen, pencil, eraser, straightedge, Mech 2 style calculator, one letter-size sheet of paper (both sides) for hand-written notes.

There are 4 questions on this test. You are asked to answer all the questions on this test paper.

The purpose of this test is to evaluate your knowledge of the course material. Orderly presentation demonstrates your knowledge most clearly, while disorganized and unprofessional work creates serious doubt. Marks are assigned accordingly. A bonus of up to 5 marks will be given for exemplary presentation.

If you use any of the attached graphs, be sure to include them in the materials handed in.

Complete the section below **during** the examination time **only.**

NAME:______________________________ Section (305/306) ________

SIGNATURE:____

STUDENT NUMBER:__________________________________

	Mark Received	**Maximum Mark**
1		15
2		15
3		12
4		18
Presentation		5 bonus
Total		60

1. [*12 marks*] **Short answer (**2-3 sentences for each).

a. What does the Central Limit Theorem state and why is this useful?

A **linear combination of random variables [2]**, say z, is itself a random variable, and if there are **sufficient terms in the linear combination, then z is distributed normally [2] even if the component variables are not normal**. This is useful because **many real situations are affected by factors that are themselves not normal, but the aggregate result is often normal** – which means that **we can use the machinery of the well-known normal distribution** (such as z and t tests, chi-squared, least squares optimality) [2].

b. Suppose $w=F(x,y)=x^2+y^2$, where x and y are random variables with mean μ_x and μ_y. Is the mean value of w (that is, μ_w) equal to, larger than or smaller than $F(\mu_x,\mu_y)= \mu_x^2 +\mu_y^2$? Explain briefly.

The mean of w is larger than $F(\mu_x,\mu_y)$ [3]. Linearity of expectation says that $\mu_w = F(\mu_x,\mu_y)$, but that is only for linear functions F. Why larger? [3] The high extremes will add more than the low extremes will take away. For example, suppose $w=x^2$, and we find a sample of x=[1,2,3,4,5]. Equating sample means to population means just to illustrate the point, $\mu_x=3$ and $\mu_x^2=9$. However, the corresponding sample of w values would be w=[1,4, 9, 16, 25] with mean of (1+4+9+16+25)/5=55/5=11.

Suggested marks given here are approximate, and we may assign extra weight to parts of the question that are answered particularly well. Some answers were vaguely related to the actual question, but not wrong, and might get 1 or 2 points.

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2. [*18 marks*] In a wind tunnel experiment, the velocity V at some location is determined from the pressure difference ΔP measured with a pitot probe using the relation,

$$V[\frac{m}{s}] = \sqrt{\frac{2\Delta P}{\rho}} \qquad (1)$$

The density ρ is not known exactly, but determined with the ideal gas law

$$\rho\,(\frac{kg}{m^3}) = \frac{P_a}{287\,T} \qquad (2)$$

For a particular experiment, the direct measurements and their associated uncertainties (estimated from long experience with the experiment) are:

ΔP=50 ±3 Pa
P_a= 101,000 ±1000 Pa
T=299 ±0.8 K

Calculate the relative uncertainty in velocity σ_V/V.

Combine (1) & (2) $\Rightarrow$ $V = \sqrt{\frac{2(287)\Delta P T}{P_a}} = \sqrt{2(287)}\left(\frac{T\Delta P}{P_a}\right)^{1/2} = A\,T^{1/2}\Delta P^{1/2}P_a^{-1/2}$

$\frac{\partial V}{\partial T} = \frac{1}{2}AT^{1/2-1}\Delta P^{1/2}P_a^{-1/2} = \frac{1}{2}\frac{V}{T}$ and similarly $\frac{\partial V}{\partial \Delta P} = \frac{1}{2}\frac{V}{\Delta P}$ & $\frac{\partial V}{\partial P_a} = \frac{-1}{2}\frac{V}{P_a}$ } 6

Relative errors are $\frac{\sigma_V}{V}$, σ_T/T, $\sigma_{\Delta P}/\Delta P$, $\frac{\sigma_{P_a}}{P_a}$

$\sigma_V^2 = \left(\frac{1}{2}\frac{V}{T}\sigma_T\right)^2 + \left(\frac{1}{2}\frac{V}{\Delta P}\sigma_{\Delta P}\right)^2 + \left(-\frac{1}{2}\frac{V}{P_a}\sigma_{P_a}\right)^2$] 4

$\frac{\sigma_V}{V} = \sqrt{\left(\frac{\sigma_T}{2T}\right)^2 + \left(\frac{\sigma_{\Delta P}}{2\Delta P}\right)^2 + \left(\frac{\sigma_{P_a}}{2P_a}\right)^2} = \frac{1}{2}\sqrt{\left(\frac{0.8}{299}\right)^2 + \left(\frac{3}{50}\right)^2 + \left(\frac{1000}{101,000}\right)^2}$ } 4

$= \frac{1}{2}\sqrt{(.00276)^2 + (0.06)^2 + (0.0099)^2} = \frac{1}{2}\sqrt{7.2\times10^{-6} + 0.0036 + 9.7\times10^{-5}}$ } 4

$\frac{\sigma_V}{V} = \frac{1}{2}(0.0609) = 0.0304$ or 3.0%

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3. [*12 marks*] A large dairy packages milk into cartons with a nominal size of 1000 ml. To check that the average package volume is correctly supplied, a sample of 25 cartons was taken and the volume of each carton was carefully measured. The average volume was found to be 995ml with a standard deviation of 10ml.

   (a) For a critical region 996 to 1004ml what is the probability α of a false negative?

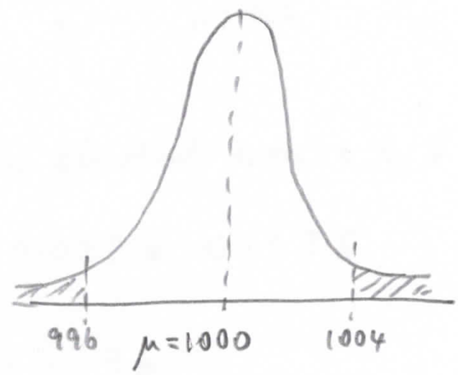
   (b) What is the p-value for the sample?

   (c) What critical region would you establish to test whether the carton volume is actually 1000 ml with a confidence of 5%. Draw a diagram to explain the concept of your procedure. Does your sample pass or fail this test?

(a) The sample standard deviation

$\sigma = 10$ ml.

The standard deviation of the average $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$ ml.



A false negative could occur if the mean actually is 1000mL but the average occurs in the shaded areas

$H_0: \mu = 1000$

$H_1: \mu \neq 1000$

Assume normal distribution.

$$z_1 = \frac{996 - 1000}{2} = -2.0 \qquad z_2 = \frac{1004 - 1000}{2} = 2.0$$

From graph, $P(2.0) = 0.977 \rightarrow P(-2.0) = 1 - P(2.0) = 0.02$

$$\alpha = 1 - P(z_1) + P(z_2) = 0.046 = \underline{4.6\%}$$

(b) p-value = risk of rejecting Ho when it is true.

$$= 2 \times P\left(\frac{995 - 1000}{2}\right) = 2 \times P(-2.5) = 2 \times (1 - P(2.5))$$

two tails

$$= 2 \times (1 - 0.994) = 0.016 = \underline{1.6\%}$$

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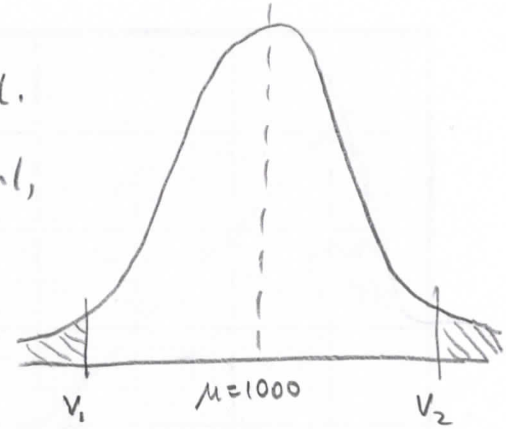
(c) If the measurements are normally distributed with a mean $\mu = 1000$ ml. and a standard deviation $\sigma_{\bar{x}} = 2$ ml, then there would be a 5% chance of a false negative if the measurement falls in one of the two shaded regions. Since the distribution is symmetrical, each shaded area = 2.5%



$\rightarrow$ $P(V_1) = 0.025$ $\quad P(V_2) = 1 - 0.025 = 0.975$

From graph, $z_2 = 1.96$ $\rightarrow$ $z_1 = -1.96$

Critical region $\quad V_1 = \mu + z_1 \sigma_{\bar{x}} = 1000 - 1.96 \times 2 = \underline{996.1\text{ ml.}}$

$V_2 = \mu + z_2 \sigma_{\bar{x}} = 1000 + 1.96 \times 2 = \underline{1003.9\text{ ml}}$

The measured value 995 ml. lies outside this range, so the sample results indicate that the average milk volume is not 1000 ml with a probability of false result less than 5%. The hypothesis $H_0: \mu = 1000$ ml is rejected.

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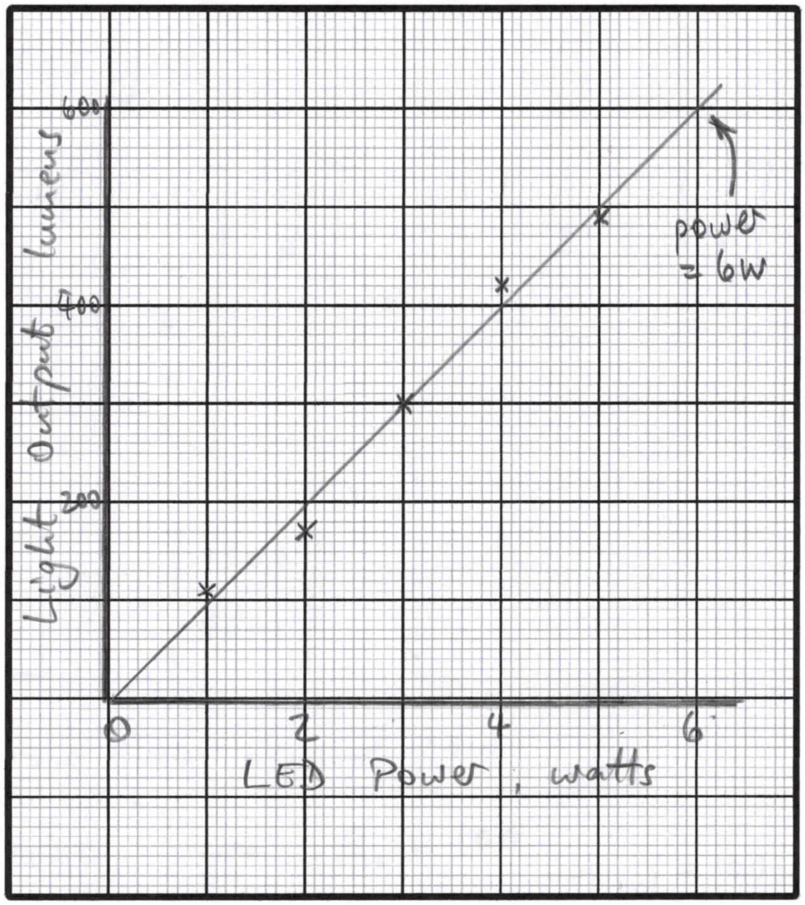
4. [*18 marks*] The table shows the measured light outputs of various sizes of LED bulbs.

Power, watts	Light output, lumens
1	110
2	180
3	300
4	420
5	490



(a) Draw a scatter plot of the data in the box provided

(b) Use the Normal Equations to determine the best-fit straight line.

(c) What would be the expected light output of a 6 watt bulb?

(d) Draw your best-fit line on your scatter plot. Is the result reasonable?

(e) Based on your graph, *estimate* the standard deviation of the residual.

(b) Fit data to $y = \beta_0 + \beta_1 x$ where x = LED power, watts; y = light output, lumens

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 \\
y_2 &= \beta_0 + \beta_1 x_2 \\
y_3 &= \beta_0 + \beta_1 x_3 \\
y_4 &= \beta_0 + \beta_1 x_4 \\
y_5 &= \beta_0 + \beta_1 x_5
\end{aligned}
\rightarrow
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \end{bmatrix}
\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}
$$

$$\rightarrow \underset{\sim}{G}\, \underline{\beta} = \underline{y}$$

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Least-squares solution is $\underset{\sim}{G}^T \underset{\sim}{G} \underline{\beta} = \underset{\sim}{G}^T \underline{y}$

Substituting the numbers gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 110 \\ 180 \\ 300 \\ 420 \\ 490 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1500 \\ 5500 \end{bmatrix}$$

Subtract $3 \times$ first row from second row

$\rightarrow 10\beta_1 = 1000 \rightarrow \beta_1 = 100 \rightarrow y = 100x$

Sub. β_1 into first row $\rightarrow \beta_0 = 0$

$\rightarrow$ Best-fit line is lumens = 100 × watts (no intercept)

(c) Expected output of 6 watt bulb = 100 × watts = 600 w

(d) Computed line passes among the points nicely.

(e) The residuals are +10, −20, 0, +20, −10

$\rightarrow$ standard deviation $= \sqrt{\dfrac{\Sigma (y_i - \hat{y}_i)^2}{n-2}}$ $\quad n = 5$

$= \sqrt{\dfrac{1000}{3}} = 6.1$ lumens