MECH366 Modeling of Mechatronic Systems
Exercise solutions for ODE solutions

1. Inverse Laplace transform: $f(t) = \mathcal{L}^{-1}\{F(s)\}$

   (a) $F(s) = \dfrac{5}{s(s+1)(s+2)} = \dfrac{5/2}{s} - \dfrac{5}{s+1} + \dfrac{5/2}{s+2}$

   $$\Rightarrow f(t) = \frac{5}{2} - 5e^{-t} + \frac{5}{2}e^{-2t}$$

   Final value theorem:

   $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \frac{5}{2}$$

   (b) $F(s) = \dfrac{1}{s^2(s+1)} = -\dfrac{1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}$

   $$\Rightarrow f(t) = -1 + t + e^{-t}$$

   Final value theorem does not apply in this case, since $sF(s)$ has a pole (i.e., denominator root) at the origin. In fact, $f(t)$ goes to infinity as t does.

   (c) $F(s) = \dfrac{2s+1}{s^2+2s+10} = \dfrac{2(s+1)}{(s+1)^2+3^2} - \dfrac{1}{3}\cdot\dfrac{3}{(s+1)^2+3^2}$

   $$\Rightarrow f(t) = 2e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t$$

   Final value theorem:

   $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 0$$

   (d) $F(s) = \dfrac{s-30}{s(s^2+4s+29)} = -\dfrac{30}{29}\cdot\dfrac{1}{s} + \dfrac{1}{29}\cdot\dfrac{30s+149}{(s+2)^2+5^2} = -\dfrac{30}{29}\cdot\dfrac{1}{s} + \dfrac{1}{29}\cdot \dfrac{30(s+2)+89}{(s+2)^2+5^2}$

   $$\Rightarrow f(t) = \frac{30}{29}\left(-1 + e^{-2t}\cos 5t\right) + \frac{89}{29\cdot 5}e^{-2t}\sin 5t$$

   Final value theorem:

   $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = -\frac{30}{29}$$

2. $F(s) = \dfrac{s+5}{s(s^2+4s+13)} = \dfrac{5}{13}\cdot\dfrac{1}{s} - \dfrac{1}{13}\cdot\dfrac{5s+7}{(s+2)^2+3^2}.$

   $$\Rightarrow f(t) = \frac{5}{13}\left(1 - e^{-2t}\cos 3t\right) + \frac{1}{13}e^{-2t}\sin 3t$$

3. Laplace transform

$$\begin{aligned}
&\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 4x(t) = 10u(t) \\
&\Rightarrow (s^2X(s) - sx(0) - x'(0)) + 5(sX(s) - x(0)) + 4X(s) = \frac{10}{s} \\
&\Rightarrow X(s) = \frac{1}{s^2+5s+4}\left(\frac{10}{s} + (s+5)x(0) + x'(0)\right)
\end{aligned}$$

(a) $x(0) = x'(0) = 0$.

$$\begin{aligned}
&X(s) = \frac{1}{s^2+5s+4}\cdot\frac{10}{s} = \frac{5}{2}\cdot\frac{1}{s} - \frac{10}{3}\cdot\frac{1}{s+1} + \frac{5}{6}\cdot\frac{1}{s+4} \\
&\Rightarrow x(t) = \frac{5}{2} - \frac{10}{3}e^{-t} + \frac{5}{6}e^{-4t}
\end{aligned}$$

(b) $x(0) = x'(0) = 1$.

$$\begin{aligned}
&X(s) = \frac{1}{s^2+5s+4}\cdot\frac{s^2+6s+10}{s} = \frac{5}{2}\cdot\frac{1}{s} - \frac{5}{3}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+4} \\
&\Rightarrow x(t) = \frac{5}{2} - \frac{5}{3}e^{-t} + \frac{1}{6}e^{-4t}
\end{aligned}$$

4. Laplace transform

$$\begin{aligned}
&\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} + x(t) = 5u(t) \\
&\Rightarrow (s^2X(s) - sx(0) - x'(0)) + 2(sX(s) - x(0)) + X(s) = \frac{5}{s} \\
&\Rightarrow X(s) = \frac{1}{(s+1)^2}\left(\frac{5}{s} + (s+2)x(0) + x'(0)\right)
\end{aligned}$$

(a) $x(0) = x'(0) = 0$.

$$\begin{aligned}
&X(s) = \frac{1}{(s+1)^2}\cdot\frac{5}{s} = 5\cdot\frac{1}{s} - 5\cdot\frac{1}{s+1} - 5\cdot\frac{1}{(s+1)^2} \\
&\Rightarrow x(t) = 5(1 - e^{-t} - te^{-t})
\end{aligned}$$

(b) $x(0) = 0$, $x'(0) = 2$.

$$\begin{aligned}
&X(s) = \frac{1}{(s+1)^2}\cdot\frac{2s+5}{s} = 5\cdot\frac{1}{s} - 5\cdot\frac{1}{s+1} - 3\cdot\frac{1}{(s+1)^2} \\
&\Rightarrow x(t) = 5(1 - e^{-t}) - 3te^{-t}
\end{aligned}$$

5. (a) $c''(t) + 10c'(t) + 60c(t) = 60r(t)$, $c(0) = c'(0) = 0$

   (b) $c'''(t) + 4c''(t) + 8c'(t) + 20c(t) = 3r'(t) + 20r(t)$, $c(0) = c'(0) = c''(0) = r(0) = 0$

   (c) $c''(t) = r'(t) + r(t)$, $c(0) = c'(0) = r(0) = 0$

   (d) $c''(t) + 5c'(t) + 32c(t) = 7r(t - 0.2)$, $c(0) = c'(0) = 0$