a place of mind

MECH366 : Modeling of Mechatronic Systems

L5 : Illustrative examples for linearization

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Review and today's topic

- Last lecture was about linearization of nonlinear systems.

$$\left\{\begin{array}{rcl} \dot{x} & = & f(x,u) \\ y & = & h(x,u) \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \delta\dot{x} & = & \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)}}_{A}\delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)}}_{B}\delta u \\ \delta y & = & \underbrace{\left.\frac{\partial h}{\partial x}\right|_{(x_0,u_0)}}_{C}\delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{(x_0,u_0)}}_{D}\delta u \end{array}\right.$$

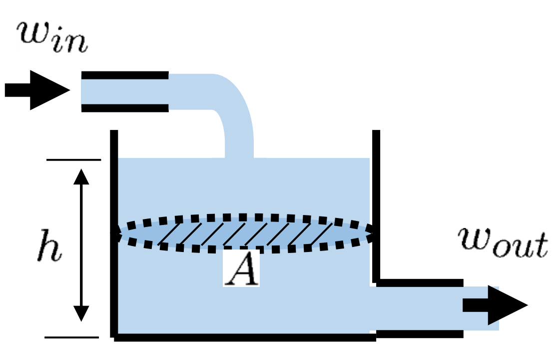
  - A pendulum
  - Water level in a tank
- Today, we will give other illustrative examples of linearization. But before that, some remarks are presented.

Remark 1: Deviation variables (perturbation)

- Linearized models are linear w.r.t. deviation variables

$$\begin{aligned} \delta x &:= x - x_0 \\ \delta u &:= u - u_0 \\ \delta y &:= y - y_0 \end{aligned} \quad \left(\Leftrightarrow \quad \begin{aligned} x &= x_0 + \delta x \\ u &= u_0 + \delta u \\ y &= y_0 + \delta y \end{aligned} \right)$$

- Example

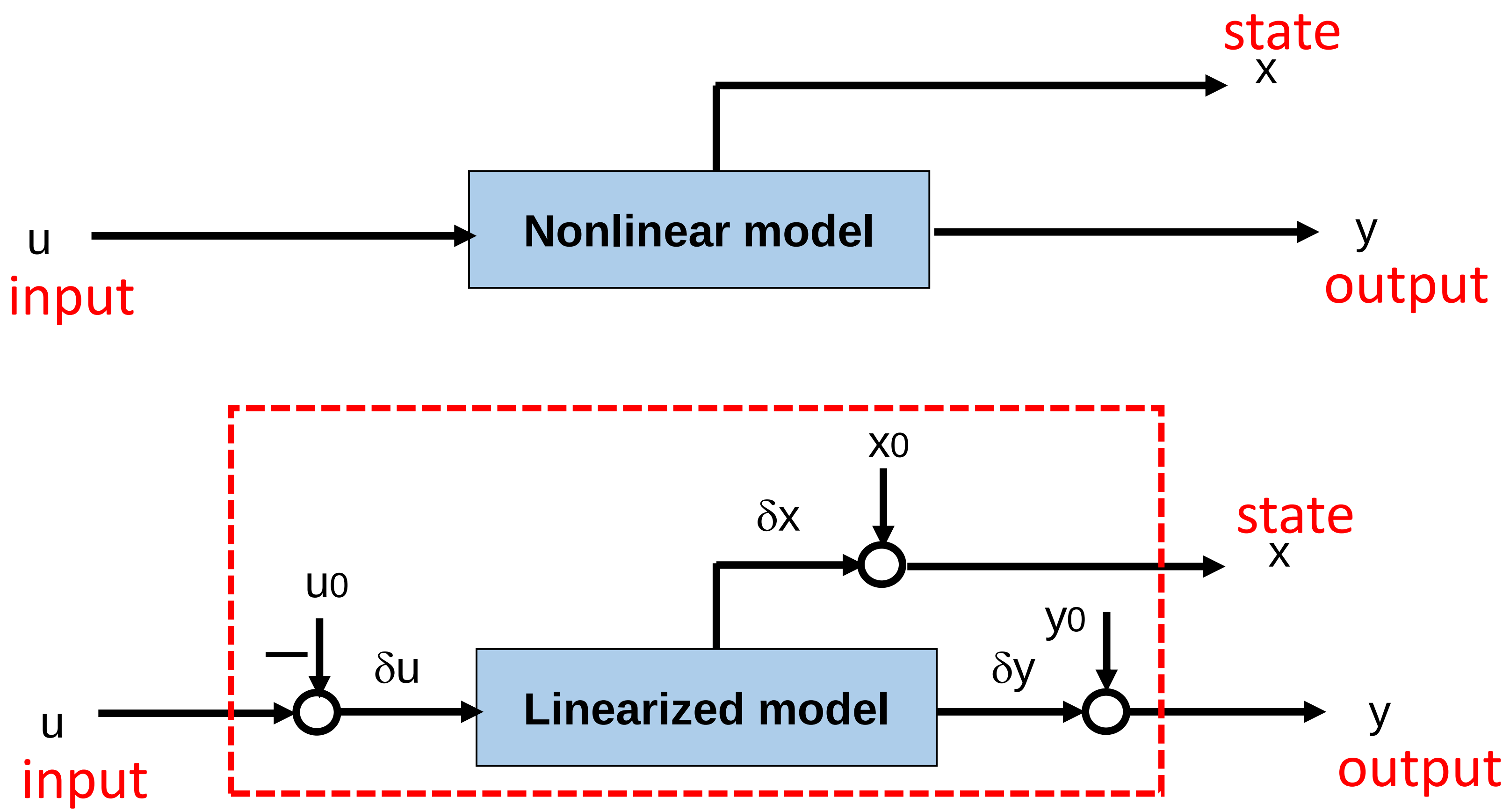


$$h_0 = 10 \ [\mathrm{m}]$$

$$\delta h = 1 \ [\mathrm{m}] \text{ means } h = 11 \ [\mathrm{m}]$$

$$\delta h = -1 \ [\mathrm{m}] \text{ means } h = 9 \ [\mathrm{m}]$$

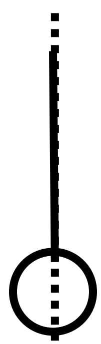
Remark 2: Comparison between nonlinear and its linearized models



Remark 3: Equilibrium point selection

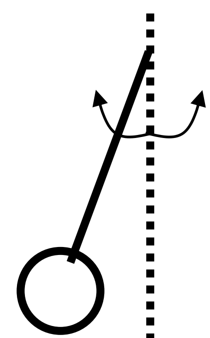
- Select an equilibrium point (or trajectory) around which:
  - you want to analyze the system, and
  - you want to design a feedback controller.
- To consider a regulation problem at:

Ex. $\theta = 0$



$\theta_0 = 0$

Ex. $\theta(t) = \sin \omega t$



$\theta_0(t) = \sin \omega t$

Remark 4: Validity of linearized models

- A linearized model is considered to be valid around the operating (linearization) point.
- How far from the operating point can the deviation variables deviate?
  - It depends on required accuracy, nonlinearity.
- What if the operating condition changes a lot during the operation?
  - Gain scheduling control: Design local controllers and switch/interpolate them based on operating points.

Two examples

Taken from de Silva's book (optional book in this course)

Example 3.3: Elevator

Example 3.4: Rocket-propelled spacecraft

- Setting
- Modeling
- Operating point/trajectory
- Linearization

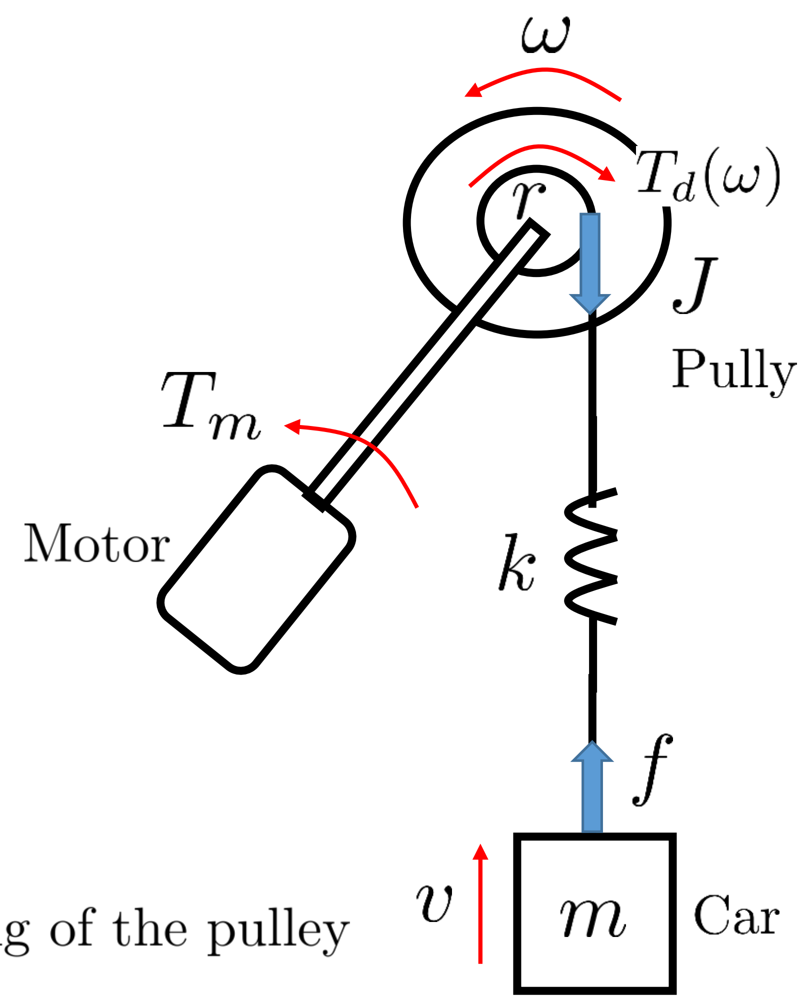
Simplified model of an elevator Setting

- Parameters

J	moment of inertia of the pulley
r	radius of the pulley
k	stiffness of the cable
m	mass of the car and occupants

- Signals

T_m	torque (input)
v	velocity of the car (output)
ω	angular velocity of the pulley
f	tension force in the cable
$T_d(\omega)$	damping torque at the bearing of the pulley



Simplified model of an elevator Modeling

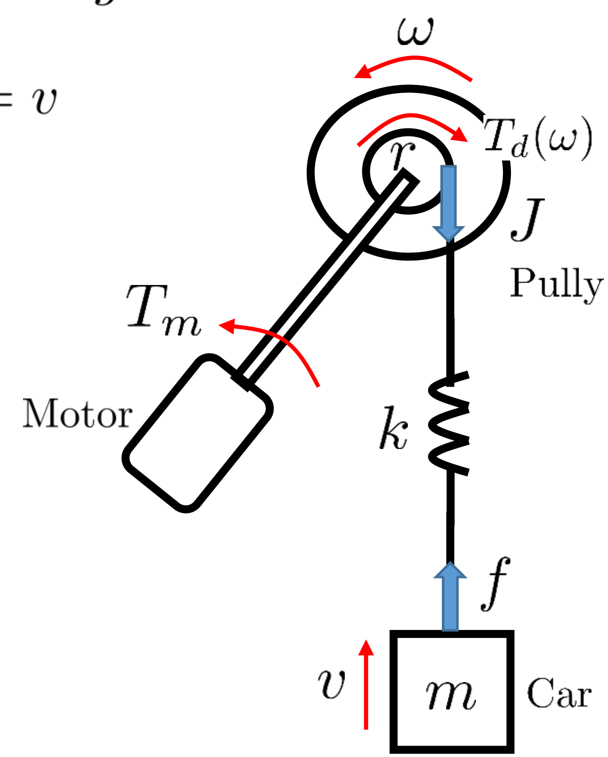
- Equations of motion
$$\begin{cases} J\dot{\omega} &= T_m - rf - T_d(\omega) \\ \dot{f} &= k(r\omega - v) \\ m\dot{v} &= f - mg \end{cases}$$

- Notation $x := [\omega, f, v]^T, \; u := T_m, \; y := v$

- Nonlinear state-space model

$$\dot{x} = \begin{bmatrix} \frac{1}{J}(u - rx_2 - T_d(x_1)) \\ k(rx_1 - x_3) \\ \frac{1}{m}x_2 - g \end{bmatrix}$$

$$y = x_3$$

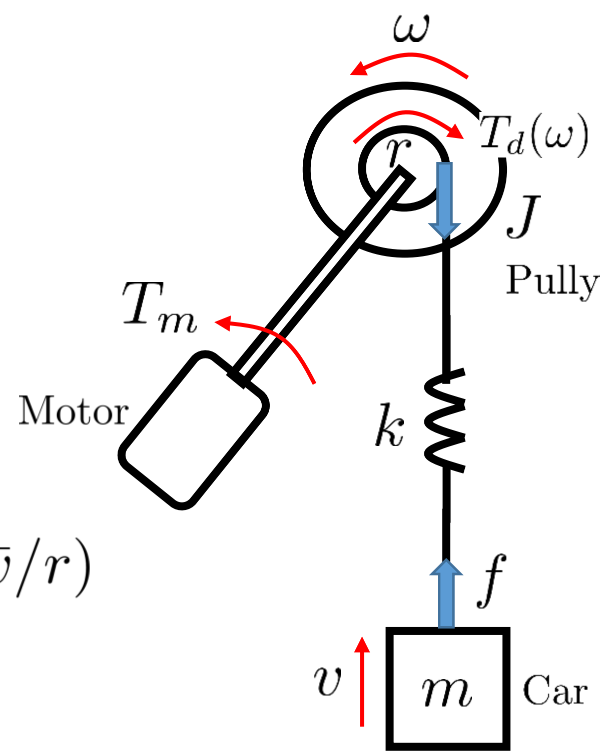


Simplified model of an elevator
Operating point

- Operating point for constant car velocity $x_3 = \bar{v}$

$$\dot{x} = \begin{bmatrix} \frac{1}{J}(u - rx_2 - T_d(x_1)) \\ k(rx_1 - x_3) \\ \frac{1}{m}x_2 - g \end{bmatrix} = 0$$

$$\Rightarrow \quad x_0 = \begin{bmatrix} \bar{v}/r \\ mg \\ \bar{v} \end{bmatrix}, \; u_0 = rmg + T_d(\bar{v}/r)$$



Simplified model of an elevator Linearization

- Linearization around operating point (x_0, u_0)

$$\left\{ \begin{array}{lcl} \dot{\delta x} & = & A\delta x + B\delta u \\ \delta y & = & C\delta x \end{array} \right.$$

where

$$A := \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} = \begin{bmatrix} -\frac{1}{J}\frac{\partial T_d}{\partial x_1}(\bar{v}/r) & -\frac{r}{J} & 0 \\ kr & 0 & -k \\ 0 & \frac{1}{m} & 0 \end{bmatrix}$$

$$B := \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} = \begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix}$$

$$C := [0 \ 0 \ 1]$$

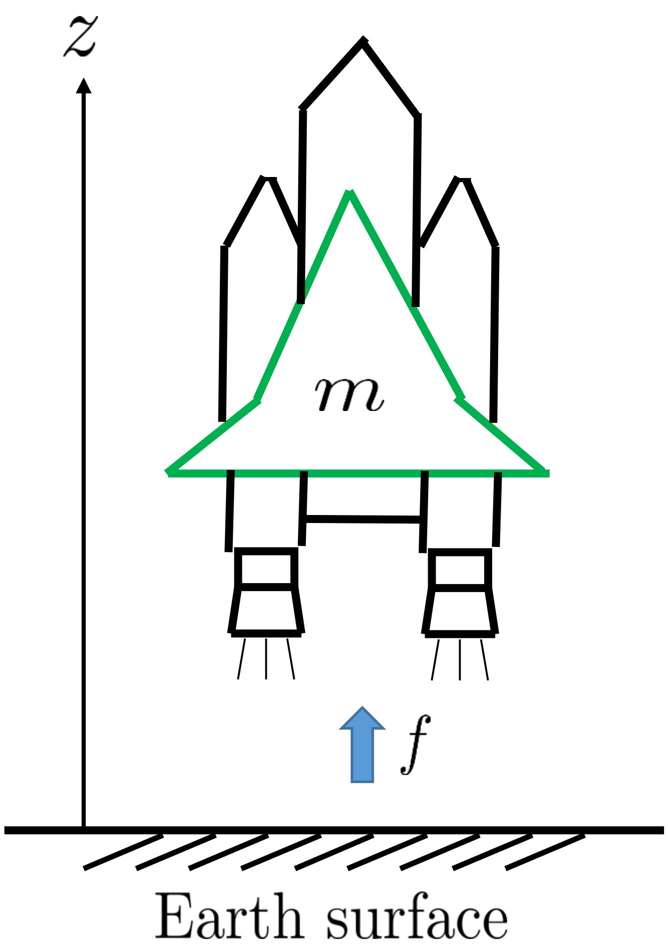
Rocket-propelled spacecraft Setting

- Parameters

m mass of the spacecraft
R average radius of the earth ($\approx 6370km$)

- Signals

f upward thrust force (input)
z vertical distance of the centroid of the spacecraft from the earth's surface (output)



Rocket-propelled spacecraft Modeling

- Equations of motion $m\ddot{z} = f - mg\dfrac{R^2}{(R+z)^2} - k\dot{z}^2 e^{-z/r}$

Aerodynamic drag

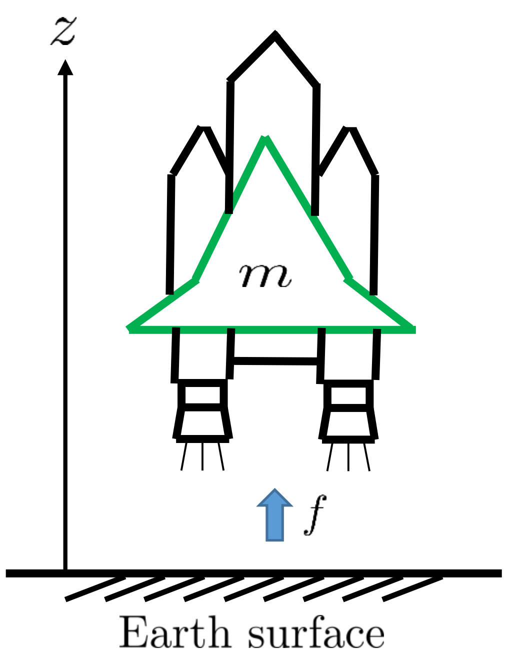
Gravitational force

$k > 0,\ r > 0$

- Notation $x := [z, \dot{z}]^T,\ u := f,\ y := z$
- Nonlinear state-space model

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{1}{m}\left(u - mg\frac{R^2}{(R+x_1)^2} - kx_2^2 e^{-x_1/r}\right) \end{bmatrix}$$

$$y = x_1$$



Rocket-propelled spacecraft Operating trajectory

- Operating trajectory for constant spacecraft velocity v_0 starting at *t=0* and height z_0

$$x_0(t) = \begin{bmatrix} v_0 t + z_0 \\ v_0 \end{bmatrix}$$

- By plugging this into nonlinear state equation:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{1}{m}\left(u - mg\frac{R^2}{(R+x_1)^2} - kx_2^2 e^{-x_1/r}\right) \end{bmatrix}$$

we can obtain

$$u_0(t) = \frac{mgR^2}{(R + v_0 t + z_0)^2} + kv_0^2 e^{-(v_0 t + r_0)/r}$$

$$(u_0(t) \to 0 \text{ as } t \to \infty)$$

Rocket-propelled spacecraft Linearization

- Linearization around operating trajectory $(x_0(t), u_0(t))$

$$\begin{cases} \dot{\delta x} &= A(t)\delta x + B\delta u \\ \delta y &= C\delta x \end{cases}$$

$$\begin{cases} \delta x(t) &:= x(t) - x_0(t) \\ \delta u(t) &:= u(t) - u_0(t) \\ \delta y(t) &:= y(t) - y_0(t) \end{cases}$$

where $A(t) := \left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t))} = \begin{bmatrix} 0 & 1 \\ (*) & -\frac{2k}{m} v_0 e^{-(v_0 t + z_0)/r} \end{bmatrix}$

$$(*) = 2g\frac{R^2}{(R + v_0 t + z_0)^3} + \frac{k}{mr} v_0^2 e^{-(v_0 t + z_0)/r}$$

$$B := \left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t))} = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} \qquad C := [1 \;\; 0]$$

Summary

- Remarks on linearization
- Illustrative examples for linearization
  - Elevator
  - Rocket-propelled spacecraft
- Next, linear graph
- Homework 1: Due Sep 23 (Monday), 3pm