1. Answer the following questions **concisely (if possible, by a few sentences or even by one-word or two-words if appropriate)**.

   (a) Give **only one** reason why a linear model is preferred to a nonlinear model? (1pt)

   Write your answer here.

   Linear models are easier to $\begin{cases} \text{analyze} \\ \text{design a controller} \\ \text{deal with theoretically} \end{cases}$ than nonlinear models.

   (b) What is the difference between **the static system** and **the dynamic system**? (1pt)

   Write your answer here.

   Static system : Output at time t depends on only input at time t.

   Dynamic system : Output at time t depends on past inputs.

   (c) Is a system represented by $y = 2u + 1$ linear? Here, u is the input and y is the output of the system. **Motivate your answer properly**, rather than just answering 'Yes' or 'No'. (1pt)

   Write your answer here.

   'No' because $u = 1 \Rightarrow y = 3$

   twice

   $u = 2 \Rightarrow y = 2 \cdot 2 + 1 = 5$

   NOT twice!

(d) In **electrical** systems, write the constitutive equation for the **T-type** element. (1pt)

Write your answer here.

$$L \frac{di}{dt} = V$$

(e) **Using the relation between the energy and the power**, derive the **energy formula** for the mechanical **mass** element. (1pt)

Write your answer here.

$$E = \int P\, dt = \int fv\, dt = \int m \frac{dv}{dt} v\, dt$$

$$= \int m v\, dv$$

$$= \frac{1}{2} m v^2$$

2. Consider the following normalized equation of motion for a pendulum system:

$$\ddot{\theta} = \tau - b(\dot{\theta}) - \sin\theta.$$

Here, the input is the torque τ and the outputs are the angle θ and the angular acceleration $\ddot{\theta}$, and the term $b(\dot{\theta})$ is a differentiable nonlinear function and $b(0) \neq 0$.

(a) Obtain the nonlinear state-space model. (1pt)

(b) Linearize the nonlinear state-space model (i.e., both state equation and output equation) around an equilibrium **point** (θ_0, τ_0). (2pt)

(c) For the equilibrium **point** $\theta_0 = \pi/6$, obtain the corresponding torque input τ_0. (Note: $\sin(\pi/6) = 1/2$.) (1pt)

(d) For the equilibrium **trajectory** $\theta_0(t) = \sin(2t)$, obtain the corresponding torque input trajectory $\tau_0(t)$. (1pt)

Write your answer here.

(a) Define $x_1 := \theta$

$x_2 := \dot{\theta}$

Then, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \tau - b(x_2) - \sin x_1 \end{bmatrix} \quad (=: f(x,\tau))$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \tau - b(x_2) - \sin x_1 \end{bmatrix} \quad (=: h(x,\tau))$

(b) For deviation variables

$\delta x := x - x_0 \qquad x_0 = \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix}$

$\delta \tau := \tau - \tau_0$

$\delta y := y - y_0 \qquad y_0 = \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix}$

$$\begin{cases} \delta \dot{x} = A\,\delta x + B\,\delta\tau \\ \delta y = C\,\delta x + D\,\delta\tau \end{cases}$$

where

$A = \left.\dfrac{\partial f}{\partial x}\right|_{(x_0,\tau_0)} = \begin{bmatrix} 0 & 1 \\ -\cos x_0 & -\underbrace{b'(x_{20})}_{=0} \end{bmatrix}$

$B = \left.\dfrac{\partial f}{\partial \tau}\right|_{(x_0,\tau_0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = \left.\dfrac{\partial h}{\partial x}\right|_{(x_0,\tau_0)} = \begin{bmatrix} 1 & 0 \\ -\cos x_0 & -b'(x_{20}) \end{bmatrix}$

$D = \left.\dfrac{\partial h}{\partial \tau}\right|_{(x_0,\tau_0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Write your answer here.

(c) $\tau_0 = \ddot{\theta}_0 + b(\dot{\theta}) + \sin\theta_0$

$= 0 + b(0) + \frac{1}{2} = b(0) + \frac{1}{2}$

(d) $\tau_0(t) = \ddot{\theta}_0(t) + b(\dot{\theta}_0(t)) + \sin\theta_0(t)$

$= -4\sin(2t) + b\left(2\cos(2t)\right) + \sin\left(\sin(2t)\right)$

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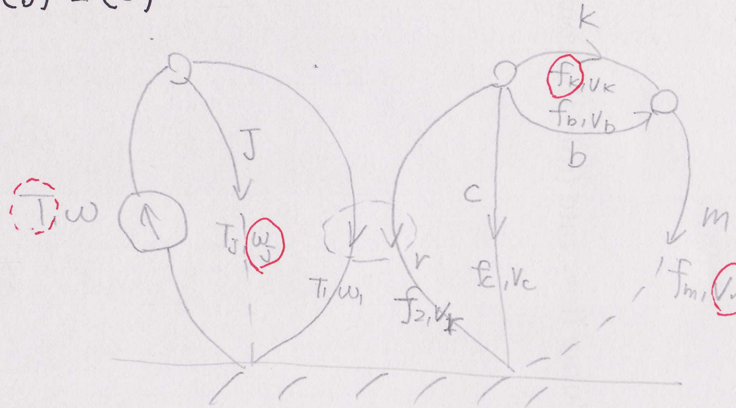
(a) $V_r = \frac{d}{2}\omega \Rightarrow r = \frac{d}{2}$

(b) & (c)



$$X = \begin{bmatrix} \omega_J \\ f_k \\ V_m \end{bmatrix}$$

(d) $J\dot{\omega}_J = T_J$

$f_c = cV_c \quad \dot{f}_k = kV_k \quad m\dot{V}_m = f_m$

$\begin{cases} V_r = r\omega_1 \\ f_2 = -\frac{1}{r}T_1 \end{cases} \quad f_b = bV_b$

(e) Loop $\omega = \omega_J = \omega_1 \quad V_r = V_c = V_b + V_m$

$V_k = V_b$

Node $T = T_J + T_1 \quad f_2 + f_c + f_k + f_b = 0$

$f_k + f_b = f_m$

(f)

$$\dot{\omega}_J = \frac{1}{J}T_J = \frac{1}{J}(T - T_1) = \frac{1}{J}(T + rf_2)$$

$$= \frac{1}{J}(T + r(-f_c - f_k - f_b))$$

$$= \frac{1}{J}(T - r(c\underbrace{V_c}_{V_r} + f_k + bV_b))$$

$$= \frac{1}{J}(T - r\{cr\omega_J + f_k + b(\underbrace{V_c}_{r\omega_J} - V_m)\})$$

$$\dot{f}_k = kV_k = k(\underbrace{V_c}_{r\omega_J} - V_m)$$

$$\dot{V}_m = \frac{1}{m}f_m = \frac{1}{m}(f_k + f_b) = \frac{1}{m}(f_k + bV_b)$$

$$= \frac{1}{m}(f_k + b(\underbrace{V_c}_{r\omega_J} - V_m))$$

$$\begin{bmatrix} \dot{\omega}_J \\ \dot{f}_k \\ \dot{V}_m \end{bmatrix} = \begin{bmatrix} \frac{-r^2(b+c)}{J} & \frac{-r}{J} & \frac{rb}{J} \\ kr & 0 & -k \\ \frac{br}{m} & \frac{1}{m} & \frac{-b}{m} \end{bmatrix} \begin{bmatrix} \omega_J \\ f_k \\ V_m \end{bmatrix} + \begin{bmatrix} \frac{1}{J} \\ 0 \\ 0 \end{bmatrix} T$$