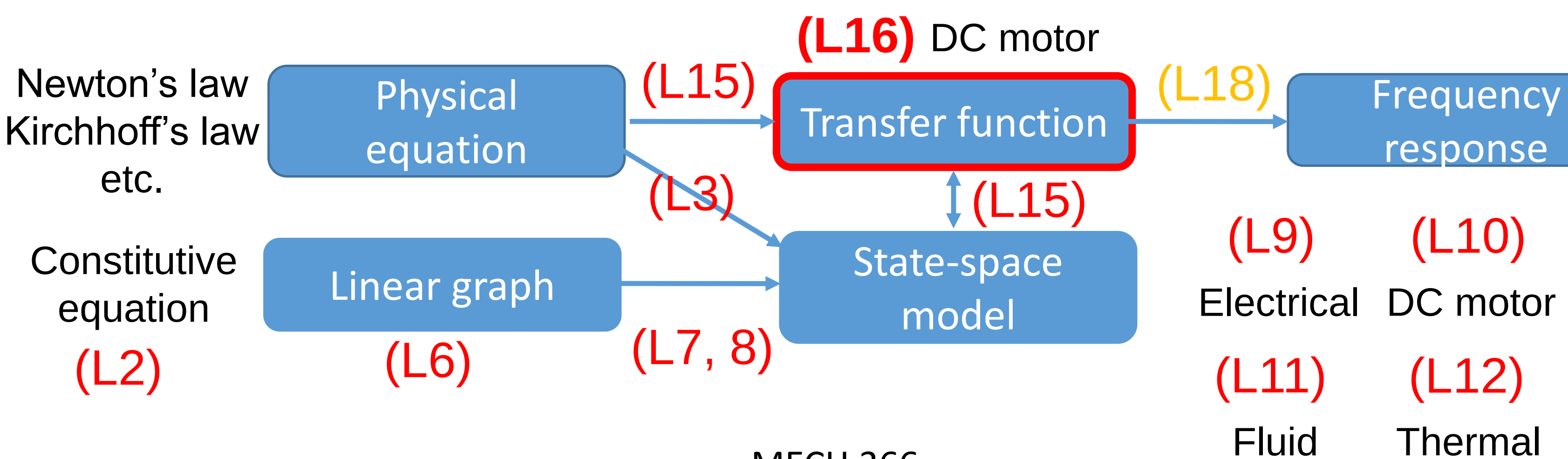
MECH366 : Modeling of Mechatronic Systems

L16 : Transfer function of DC motors

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Review and today's topic

- Up to now, we have studied state-space modeling based on linear graphs, and transfer function.
- Today, we will learn transfer function modeling of DC motors.
- Various models and their relations



Transfer function (review)

- A transfer function is defined by

$$G(s) := \frac{Y(s)}{R(s)}$$

Laplace transform of system output

Laplace transform of system input

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow \qquad Y(s) = G(s)R(s)$$

input *output*

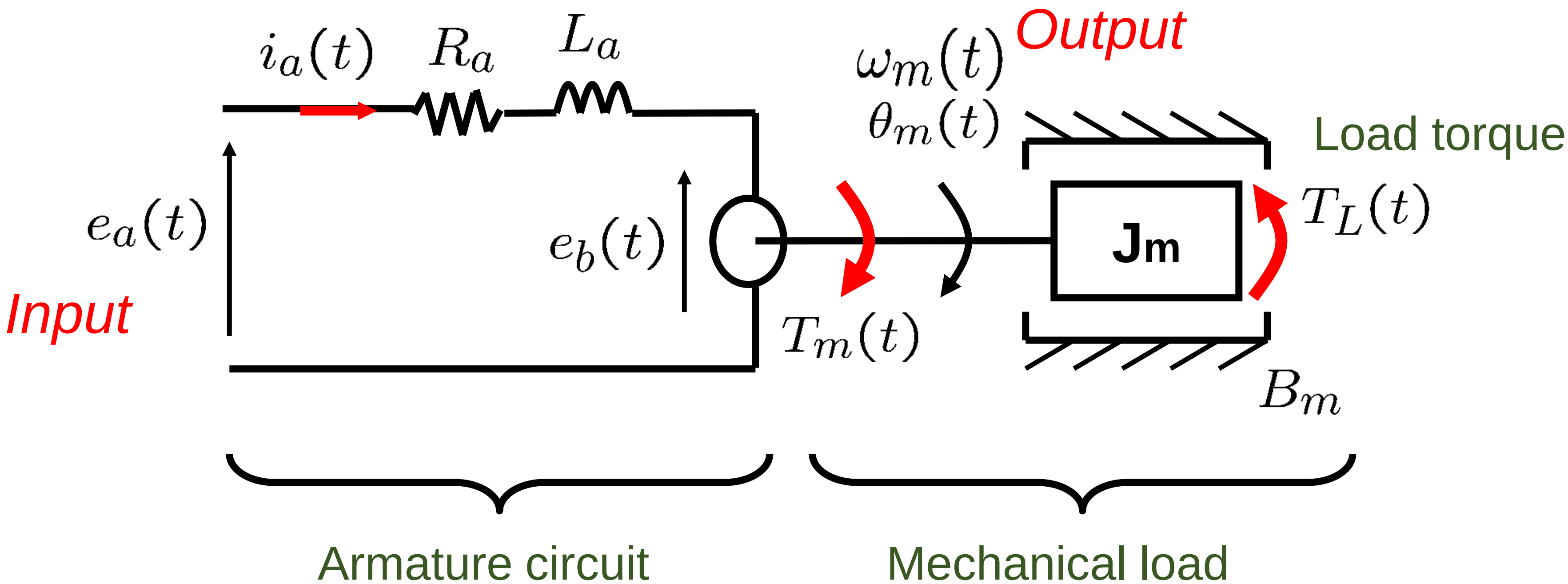
- A system is assumed to be at rest. (zero initial condition)
- Transfer function is a generalization of “gain” concept.

Today’s topics

- Modeling of DC motor
- Step response of the first-order system

These topics are relevant to Lab 4-A.

Model of DC motor



$''a''$: armature

e_a : applied voltage

i_a : armature current

$''b''$: back EMF

$''m''$: mechanical

θ_m : angular position

ω_m : angular velocity

J_m : total inertia

B_m : viscous friction

Modeling of DC motor: *t*-domain

- Armature circuit $e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_b(t)$

Load torque

- Mechanical load $J_m \dot{\omega}_m(t) = T_m(t) - B_m \omega_m(t) - T_L(t)$

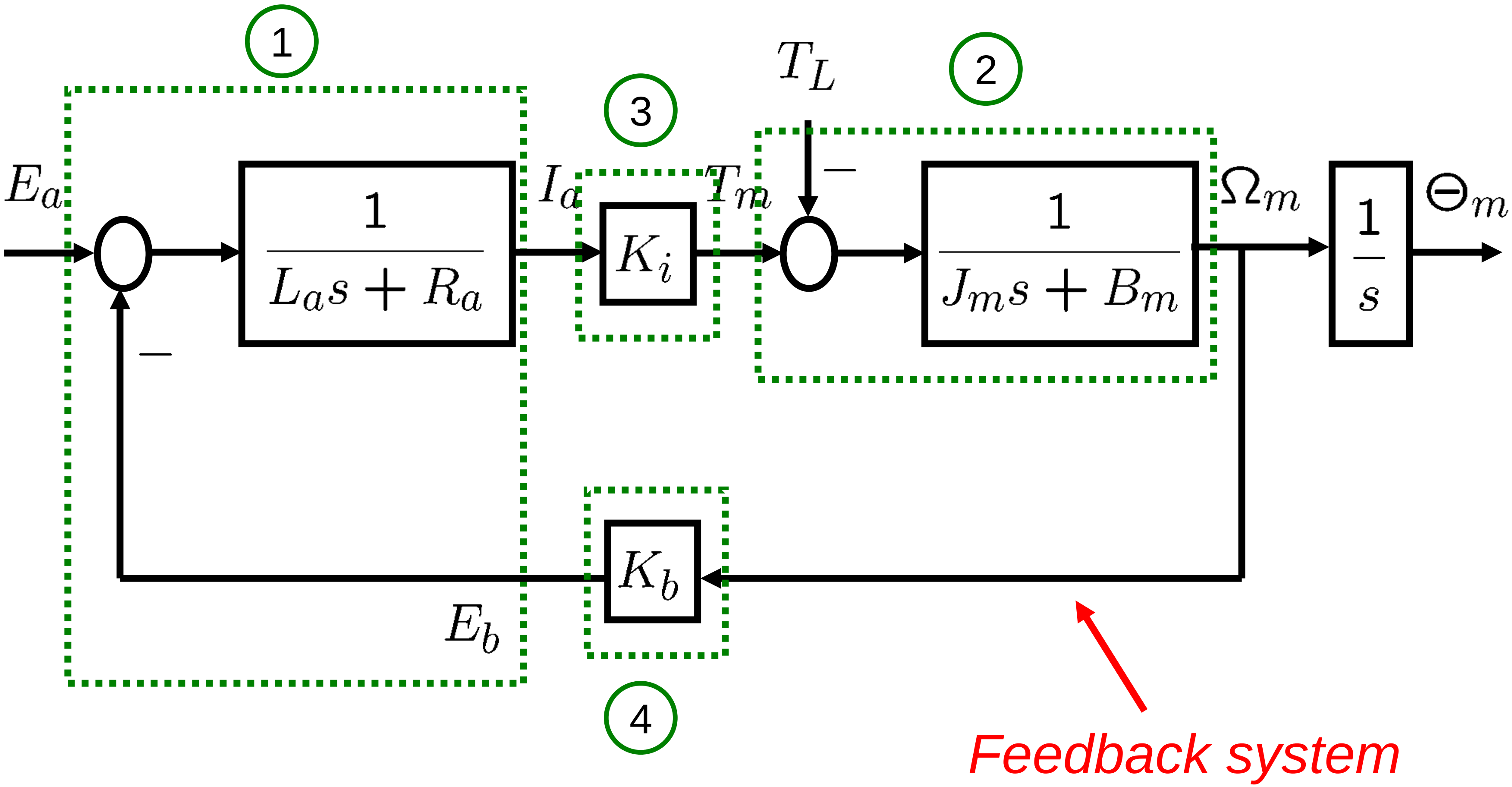
- Connection between mechanical/electrical parts
  - Motor torque $T_m(t) = K_i i_a(t)$
  - Back EMF $e_b(t) = K_b \omega_m(t)$

- Angular position $\omega_m(t) = \dot{\theta}_m(t)$

Modeling of DC motor: *s*-domain

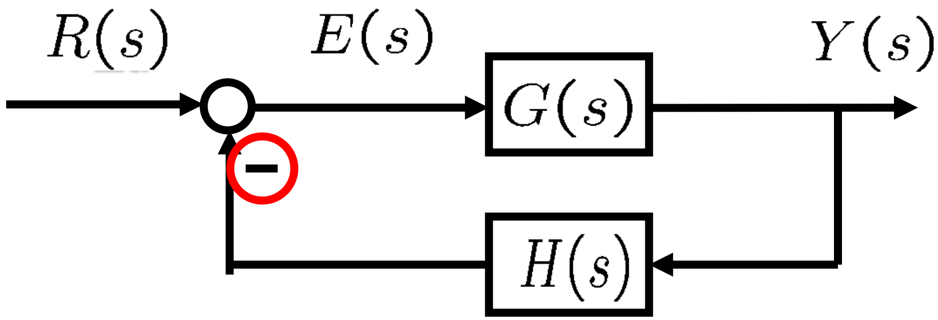
- Armature circuit $$I_a(s) = \frac{1}{L_a s + R_a}(E_a(s) - E_b(s))$$ ①
- Mechanical load $$\Omega_m(s) = \frac{1}{J_m s + B_m}(T_m(s) - T_L(s))$$ ②
- Connection between mechanical/electrical parts
  - Motor torque $$T_m(s) = K_i I_a(s)$$ ③
  - Back EMF $$E_b(s) = K_b \Omega_m(s)$$ ④
- Angular position $$\Theta_m(s) = \frac{1}{s}\Omega_m(s)$$ ⑤

DC motor: Block diagram



Transfer function (TF) with feedback Black's formula

- Negative feedback system



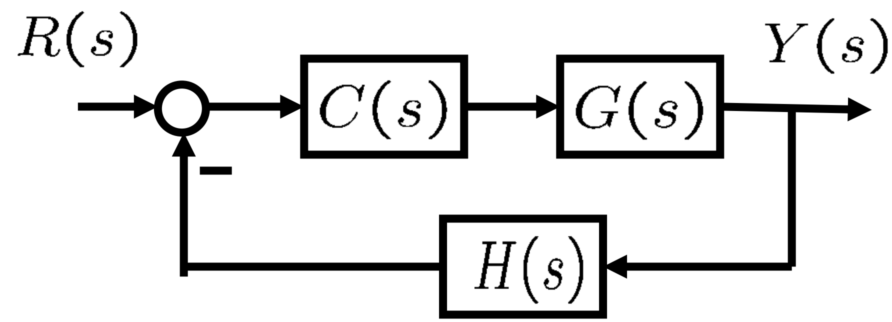
$$E(s) = R(s) - H(s)G(s)E(s) \quad \Rightarrow \quad E(s) = \frac{1}{1+G(s)H(s)}R(s)$$

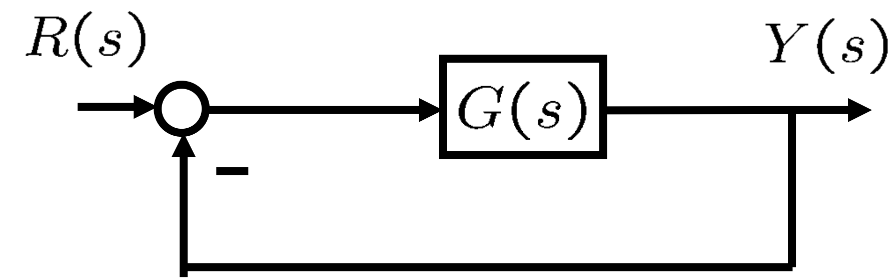
$$Y(s) = G(s)E(s) \quad \Rightarrow \quad Y(s) = \boxed{\frac{G(s)}{1+G(s)H(s)}}R(s)$$

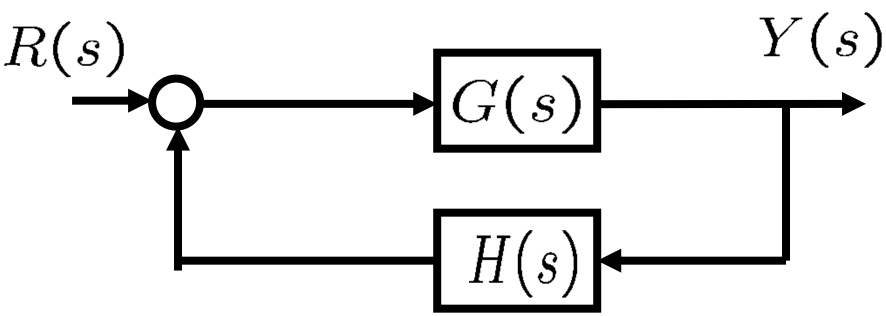
$$\left(\begin{array}{ll} G(s) & \text{: forward path TF} \\ G(s)H(s) & \text{: open-loop TF} \end{array} \right)$$

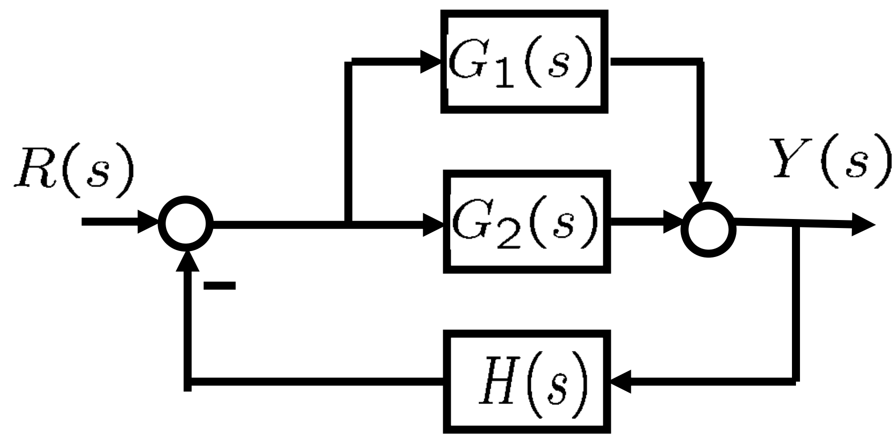
Ex: TF of feedback systems

- Compute transfer functions from *R(s)* to *Y(s)*.









DC motor: Transfer functions

If $T_L = 0$, then $$\frac{\Omega_m(s)}{E_a(s)} = \frac{\dfrac{K_i}{(L_a s + R_a)(J_m s + B_m)}}{1 + \dfrac{K_b K_i}{(L_a s + R_a)(J_m s + B_m)}} = \underbrace{\frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i}}_{G_1(s)}$$

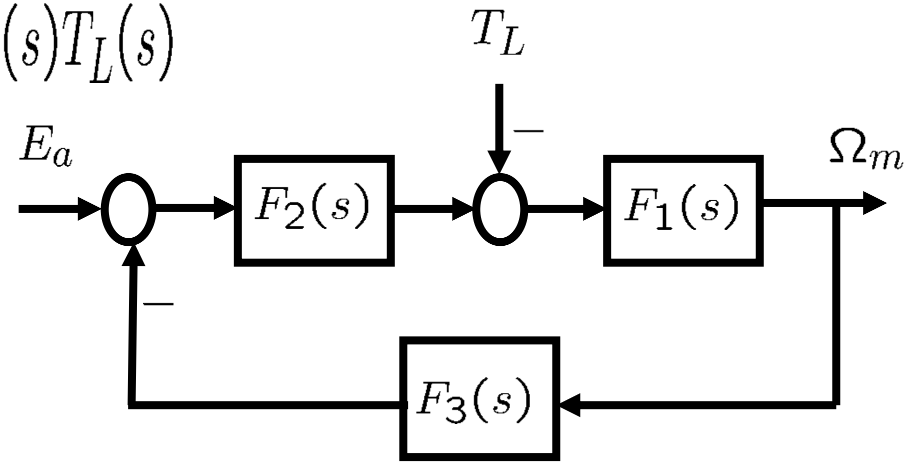
If $E_a = 0$, then $$\frac{\Omega_m(s)}{T_L(s)} = -\frac{\dfrac{1}{J_m s + B_m}}{1 + \dfrac{K_b K_i}{(L_a s + R_a)(J_m s + B_m)}} = -\underbrace{\frac{L_a s + R_a}{(L_a s + R_a)(J_m s + B_m) + K_b K_i}}_{G_2(s)}$$

➡ $$\Omega_m(s) = G_1(s)E_a(s) + G_2(s)T_L(s)$$

➡ $$\Theta_m(s) = \frac{1}{s}\Omega_m(s) = \frac{1}{s}\left(G_1(s)E_a(s) + G_2(s)T_L(s)\right)$$

DC motor: Derivation of TFs

- Why $\Omega_m(s) = G_1(s)E_a(s) + G_2(s)T_L(s)$



$$\Omega_m(s) = F_1(s)\left[-T_L(s) + F_2(s)\left\{E_a(s) - F_3(s)\Omega_m(s)\right\}\right]$$

$$\Longrightarrow \quad \left\{1 + F_1(s)F_2(s)F_3(s)\right\}\Omega_m(s) = F_1(s)\left\{-T_L(s) + F_2(s)E_a(s)\right\}$$

$$\Longrightarrow \quad \Omega_m(s) = \frac{F_1(s)F_2(s)}{1 + F_1(s)F_2(s)F_3(s)}E_a(s) - \frac{F_1(s)}{1 + F_1(s)F_2(s)F_3(s)}T_L(s)$$

DC motor: TFs (cont'd)

- Note: For DC motors, *La<<Ra*. Then, an approximated TF is obtained by setting *La=0*.

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i} \approx \frac{K_i}{R_a(J_m s + B_m) + K_b K_i}$$

$$=: \frac{K}{Ts+1} \qquad \left(K := \frac{K_i}{R_a B_m + K_b K_i},\ T = \frac{R_a J_m}{R_a B_m + K_b K_i}\right)$$

2nd order system → *1st order system*

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K}{s(Ts+1)}$$

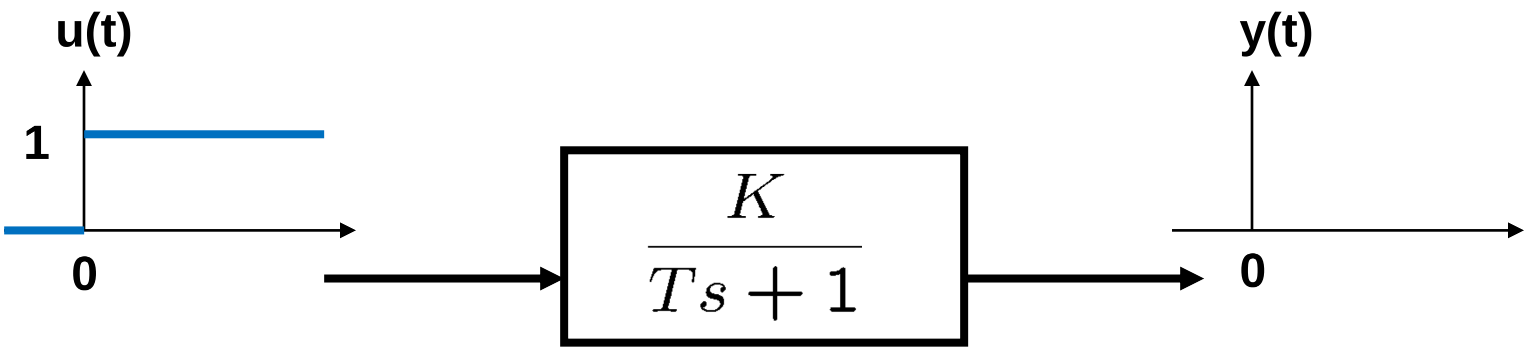
Today’s topics

- Modeling of DC motor
- Step response of the first-order system

These topics are relevant to Lab 4-A.

Step response of first-order system

- Input a unit step function to a first-order system. Then, what is the output?



$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
&= \frac{K/T}{s+1/T} \cdot \frac{1}{s} \\
&= \frac{K}{s} + \frac{-K}{s+1/T}
\end{aligned}
$$

(Partial fraction expansion)

$\mathcal{L}^{-1}$

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= K(1 - e^{-t/T}) \\
& (t > 0)
\end{aligned}
$$

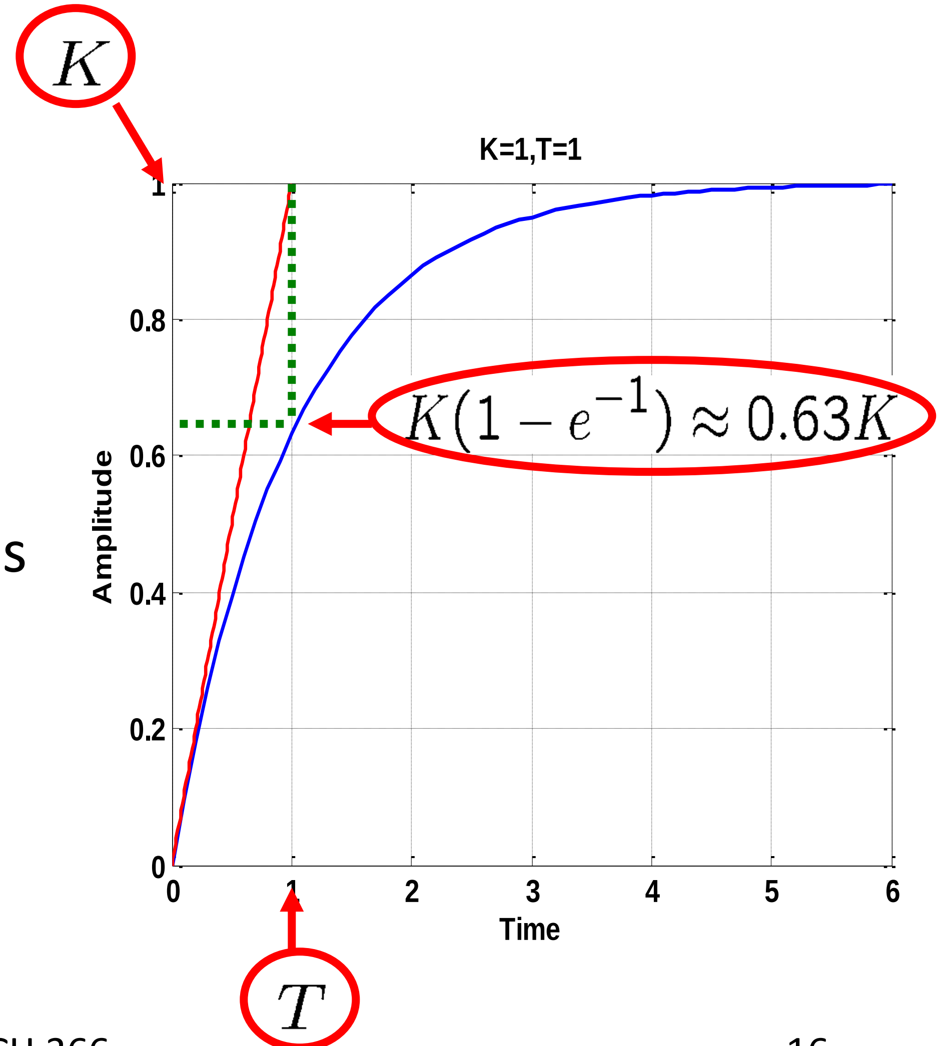
Meaning of *K* and *T*

$$G(s) = \frac{K}{Ts+1}$$

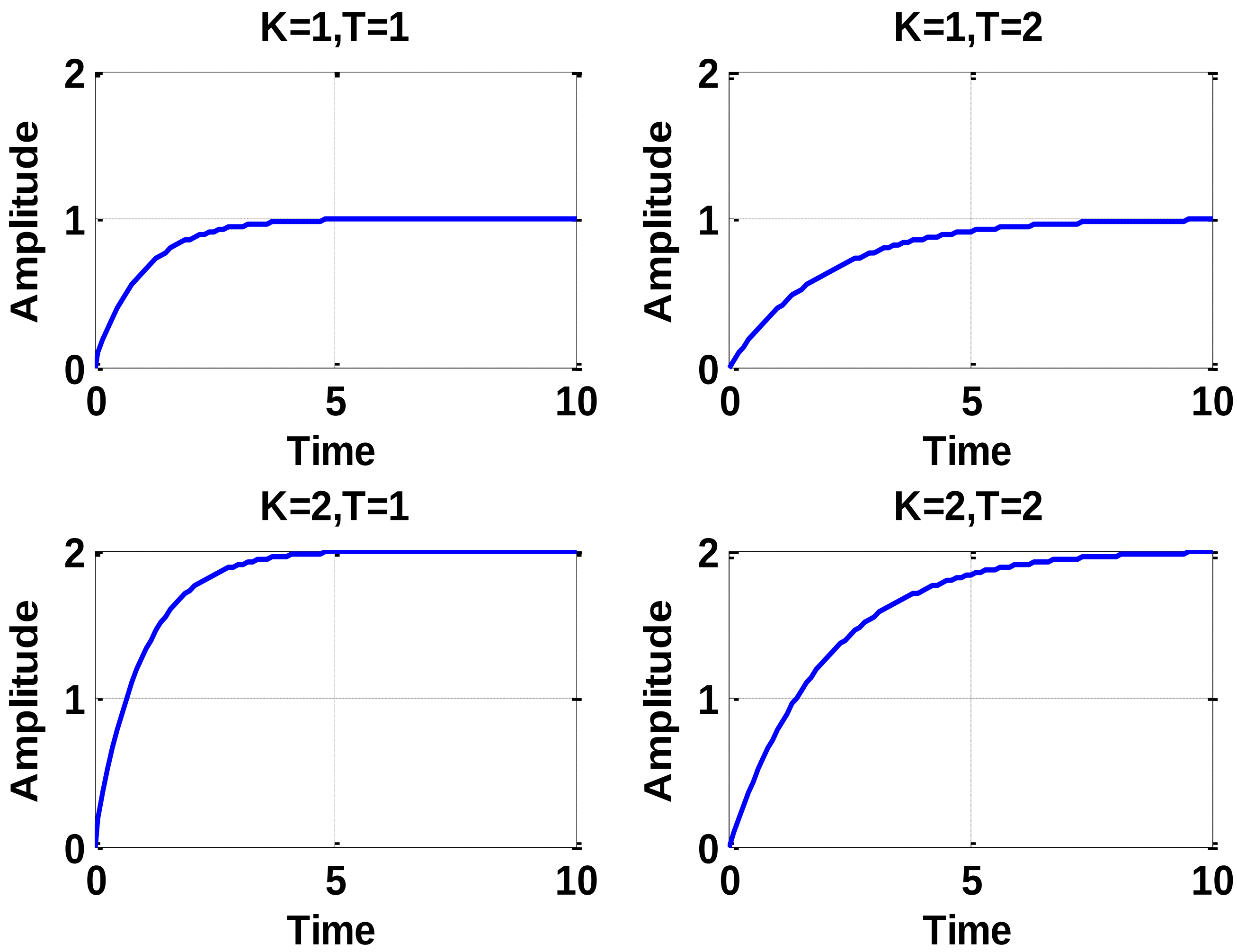
- *K* : DC gain
  - Final (steady-state) value

$$\lim_{t\to\infty} y(t) = K$$

- *T* : Time constant
  - Time when response rises 63% of final value
  - Indication of speed of response (convergence)
  - Response is faster as *T* becomes smaller.



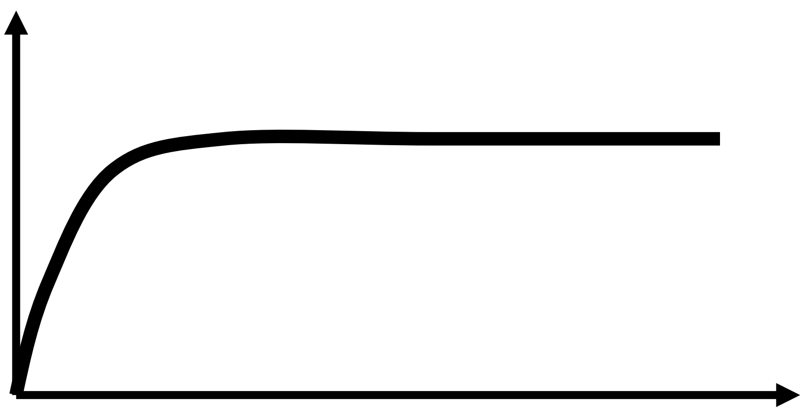
Step response for some *K* & *T*



System identification (Empirical modeling technique)

- Suppose that we have a “black-box” system.

- Obtain step response with amplitude A.



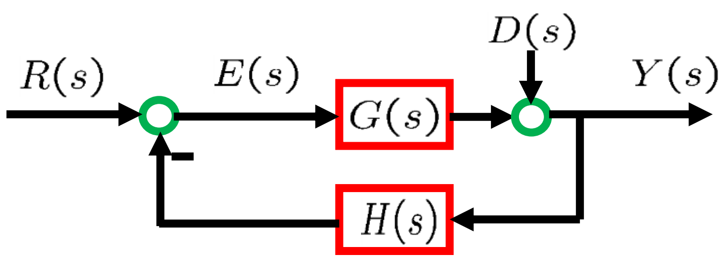
- Can you obtain a transfer function? How?

Summary

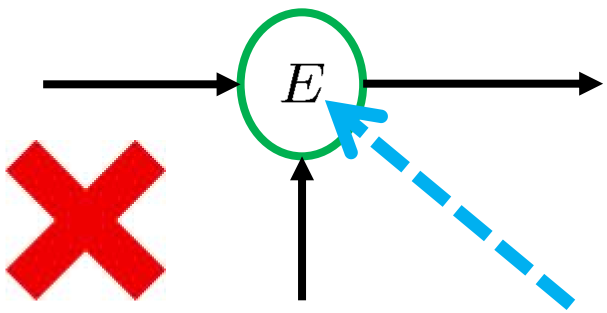
- Transfer function modeling of DC motors
  - Block diagram
  - Black's formula
  - Step response
- Next,
  - Performance specification
  - Step response of second order systems
- Lab 4: Nov 8 (report due Nov 25 (Monday), 6pm)
- Project: Fridays Nov 15, 22, 29 (presentation)
- Homework 6: Due Nov 12 (Tuesday), 6pm

Block diagram

- Represents relations among signals and systems
- Very useful in representing control systems
- Also useful in computer simulations (Simulink)
- Elements
  - Block: transfer function ("gain" block)
  - Arrow: signal
  - Node: summation (or subtraction) of signals

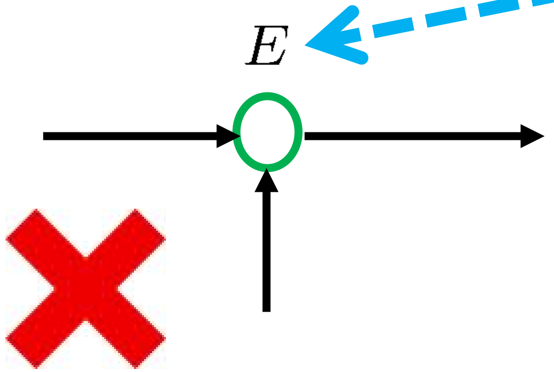


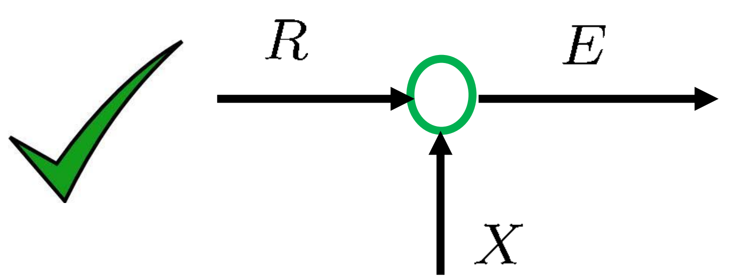
Typical mistakes



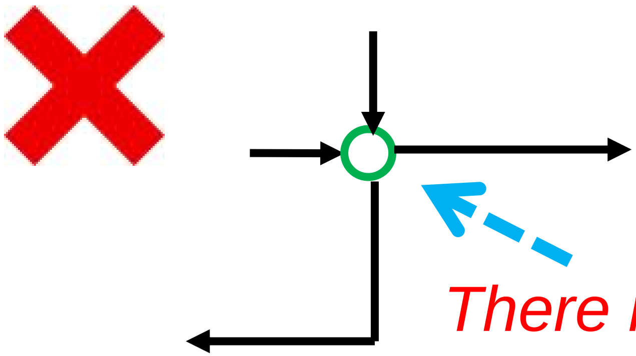
Unclear which signal is "E"

Signal must be indicated on an arrow.





Typical mistakes (cont'd)



There must be only one output from a node.

Both are fine, but they have different meanings!