a place of mind

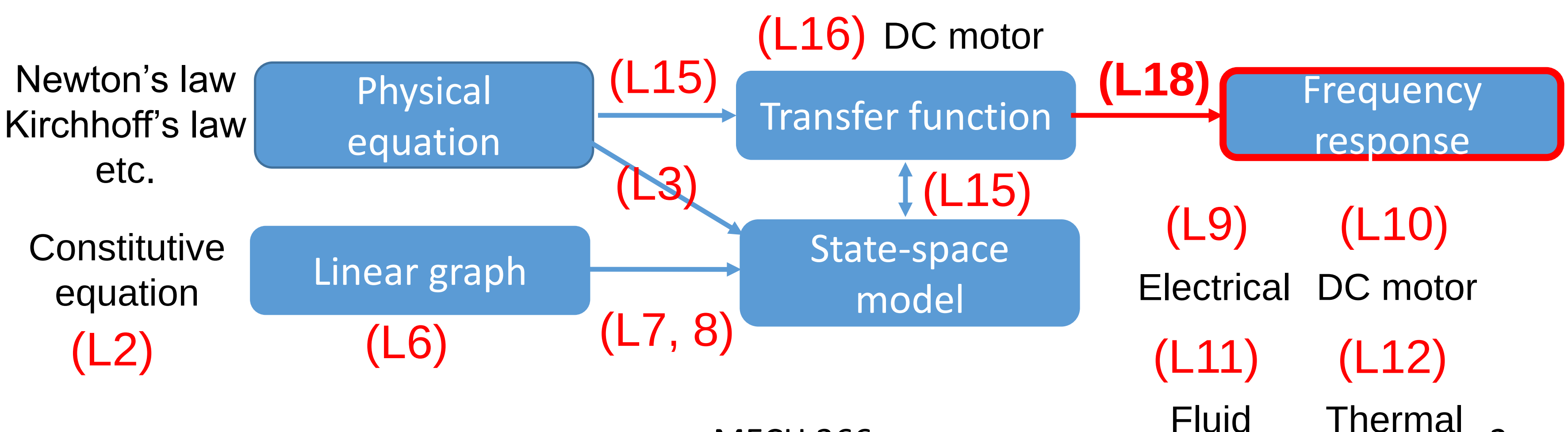
MECH366 : Modeling of Mechatronic Systems

L18 : Frequency response

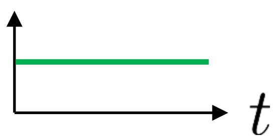
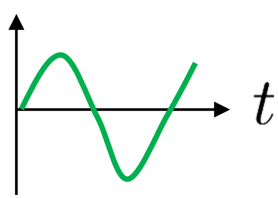
Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Today's topic & class schedule

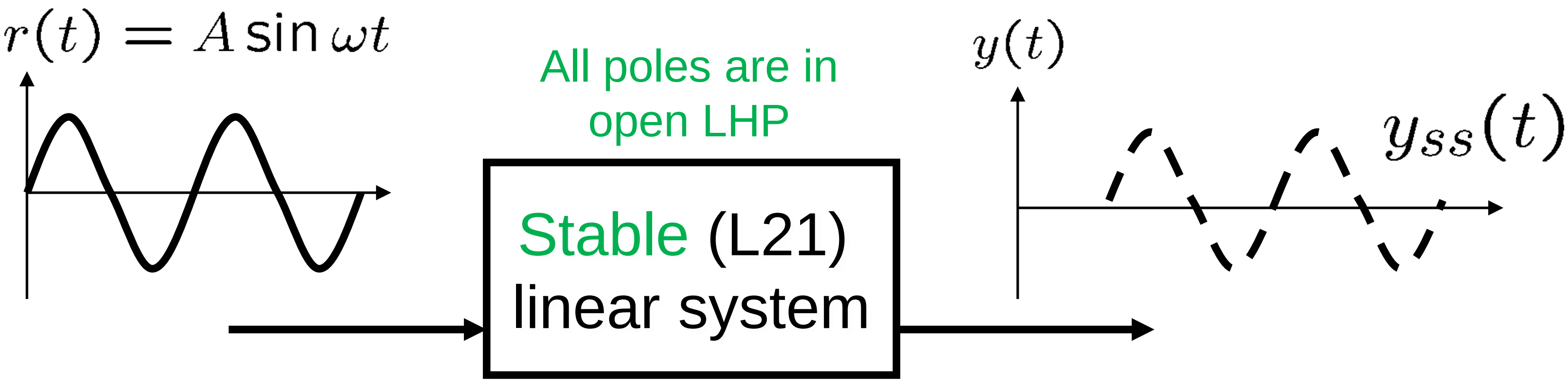
- L18: Nov 15 (Fri): Frequency response
- L19: Nov 18 (Mon): Bode diagram (Lab 4 report content, report due Nov 25, 6pm)
- L20: Nov 22 (Fri): Simulink, overdamped system
- L21: Nov 25 (Mon): Stability, course summary



Response analyses (useful for modeling and controller design)

$G(s)$	$\dfrac{K}{Ts+1}$	$\dfrac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
Step response 	(L16) 	(L17) underdamped  (L20) overdamped 
Frequency response (L18) 	(L19)	(L19)

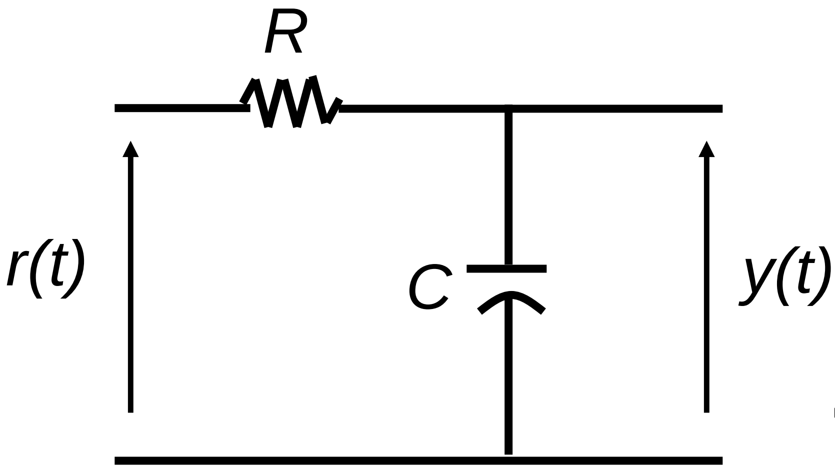
What is frequency response?



- We would like to analyze a system property by applying a *sinusoidal input* *r(t)* and observing a response *y(t)*.
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

A simple example

- RC circuit



$$G(s) = \frac{1}{RCs + 1}$$

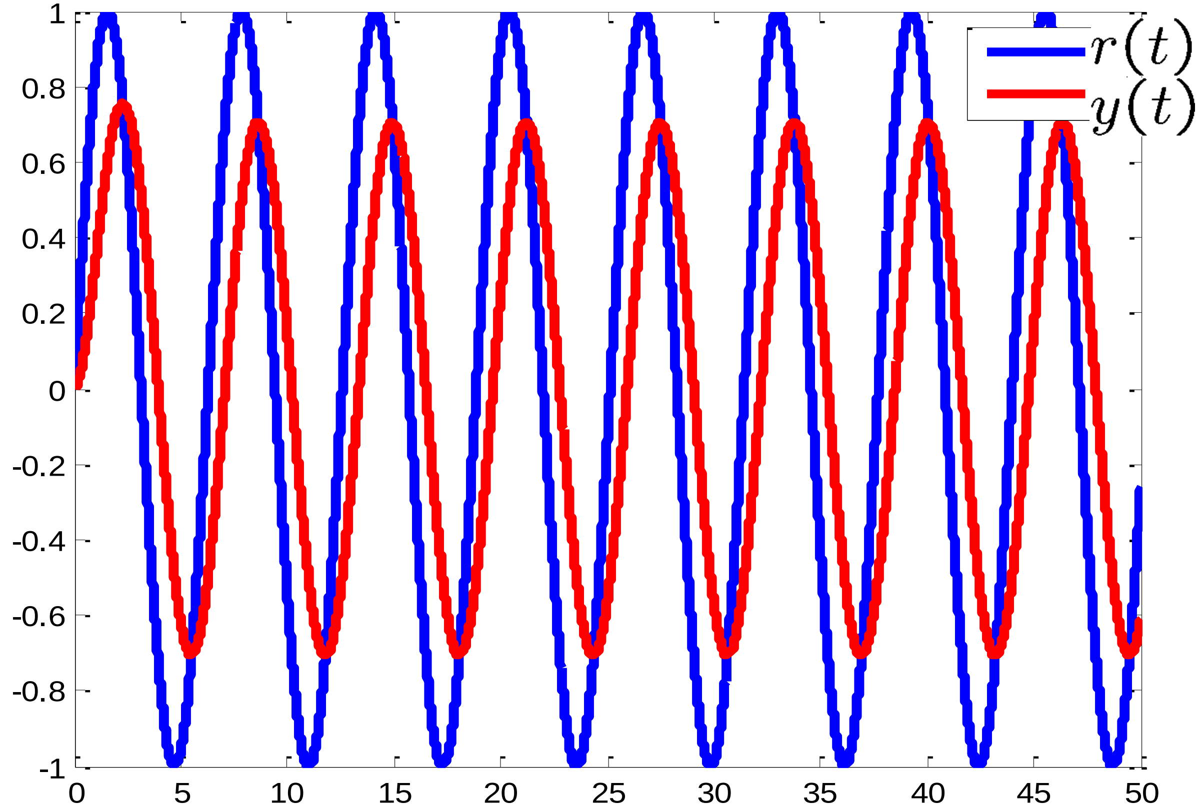
- Input a sinusoidal voltage *r(t)*
- What is the output voltage *y(t)*?

An example (cont'd)

- TF (*R=C=1*)

$$G(s) = \frac{1}{s+1}$$

- *r(t)=sin(t)*



At steady-state, *r(t)* and *y(t)* has same frequency, but different amplitude and phase!

An example (cont'd)

- Derivation of *y(t)*

$$Y(s) = G(s)R(s) = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

Partial fraction expansion

- Inverse Laplace

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

0 as *t* goes to infinity.

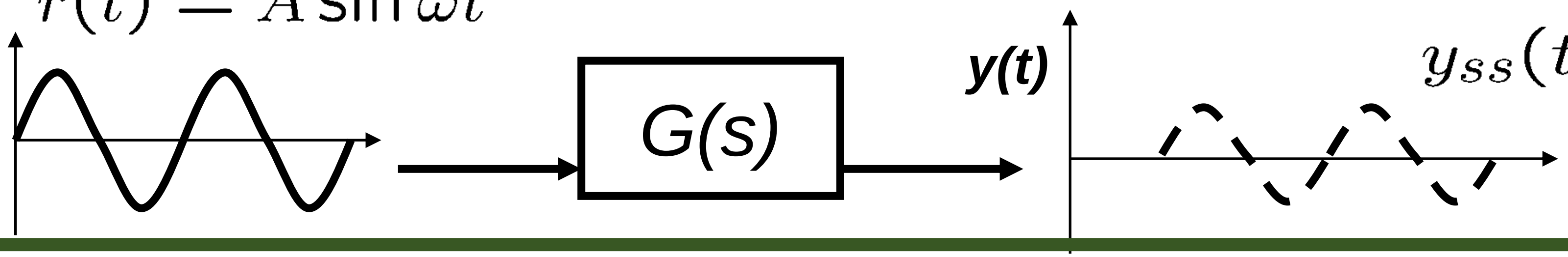
$$y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

(Derivation for general *G(s)* is given at the end of lecture slide.)

Response to sinusoidal input

- What is the steady state output of a stable linear system when the input is sinusoidal?

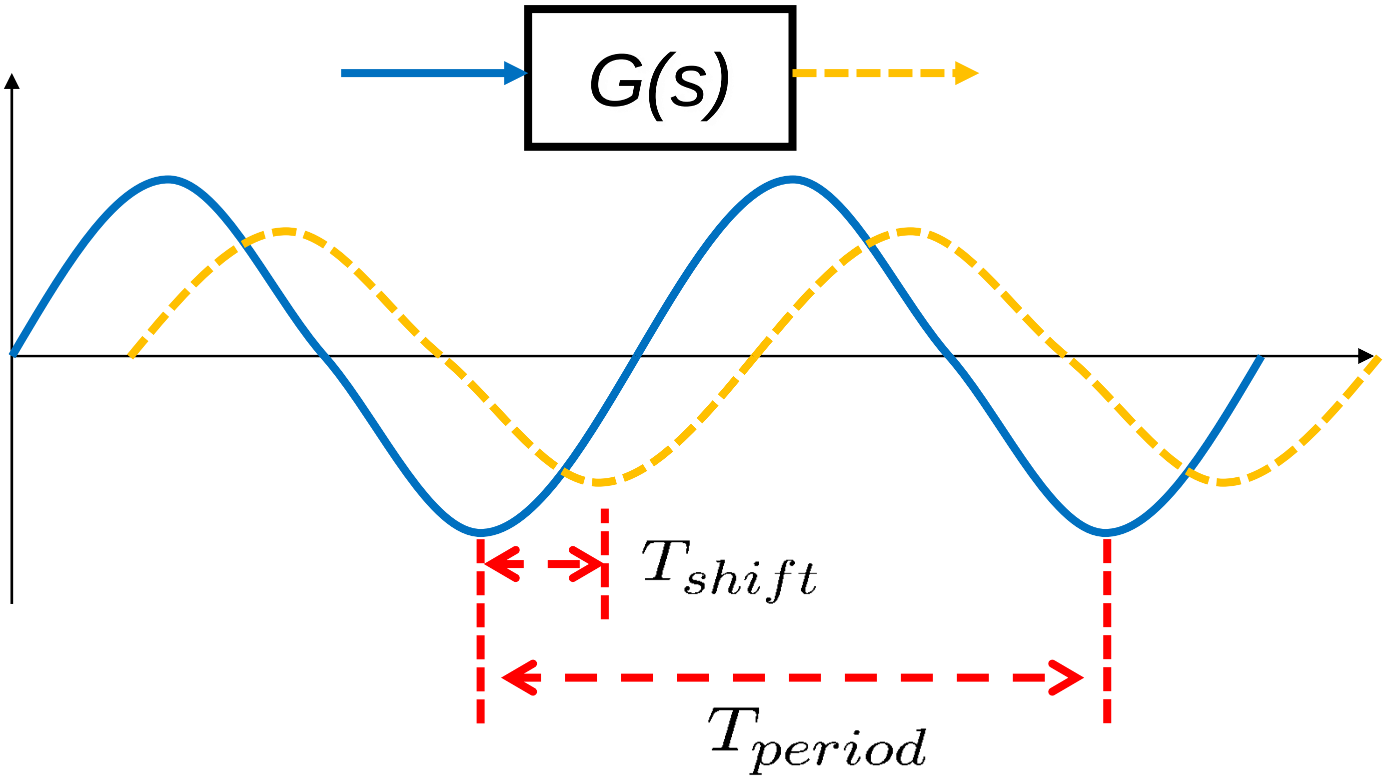
$r(t) = A \sin \omega t$



- Steady state output $y_{ss}(t) = A\left|G(j\omega)\right| \sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $\left|G(j\omega)\right|$
  - Phase shifts $\angle G(j\omega)$

Gain

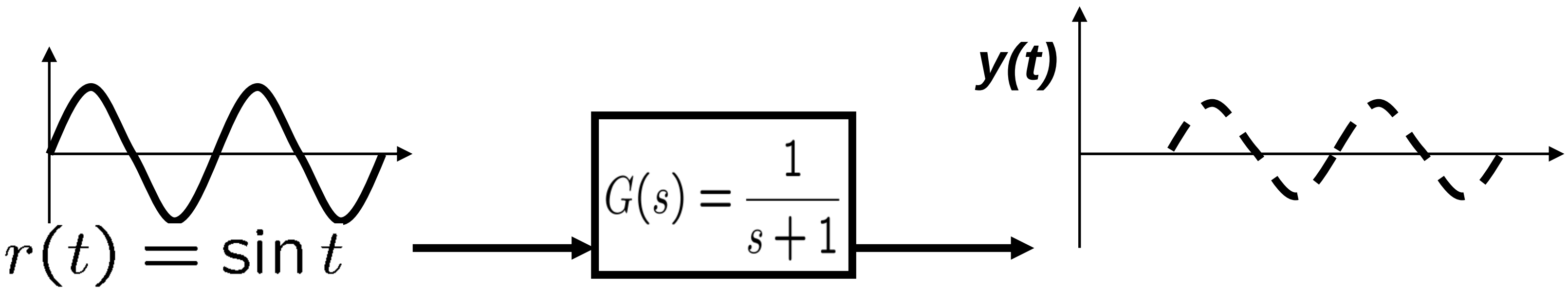
Phase shift



$$\frac{T_{shift}}{T_{period}} = \frac{-\angle G(j\omega)}{360^\circ} \Rightarrow \angle G(j\omega) = -\frac{T_{shift}}{T_{period}} \times 360^\circ$$

Revisit to the example

- What is the steady state output?



$$r(t) = \sin t \qquad G(s) = \frac{1}{s+1}$$

$$\omega = 1$$

$$y_{ss}(t) = \underbrace{\frac{1}{\sqrt{2}}}_{|G(j\cdot 1)|} \sin(t \underbrace{-45^\circ}_{\angle G(j\cdot 1)})$$

Gain

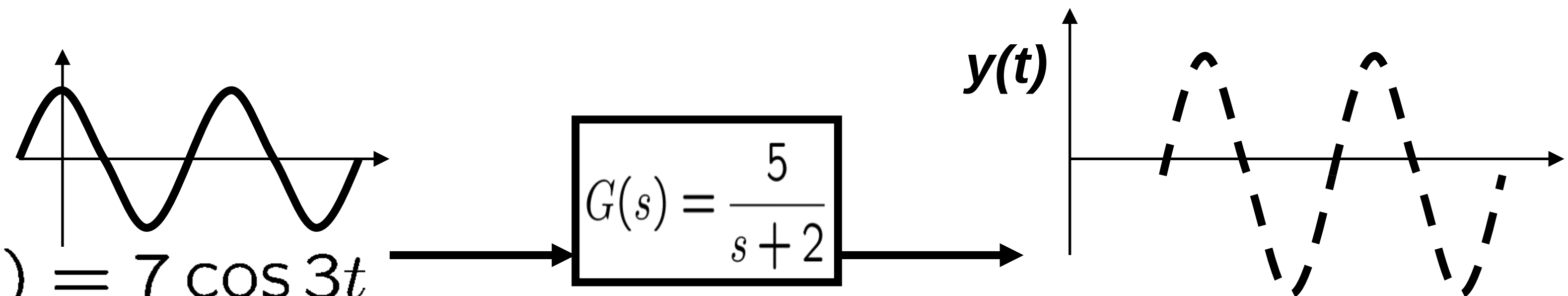
Phase

$$\boxed{\begin{aligned} |G(j\cdot 1)| &= \left|\frac{1}{j+1}\right| \\ &= \frac{1}{|j+1|} = \frac{1}{\sqrt{2}} \end{aligned}}$$

$$\boxed{\begin{aligned} \angle G(j\cdot 1) &= \angle \frac{1}{j+1} \\ &= \underbrace{\angle 1}_{=0} - \underbrace{\angle (j+1)}_{=45^\circ} \end{aligned}}$$

Another example

- What is the steady state output?



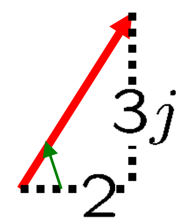
$$r(t) = 7\cos 3t$$

$$\omega = 3$$

$$y_{ss}(t) = 7 \cdot \underbrace{\frac{5}{\sqrt{13}}}_{|G(j\cdot 3)|} \cos(3t + \underbrace{\theta}_{\angle G(j\cdot 3)})$$

Gain

Phase



$$\angle G(j\cdot 3) = \angle \frac{5}{3j+2}$$

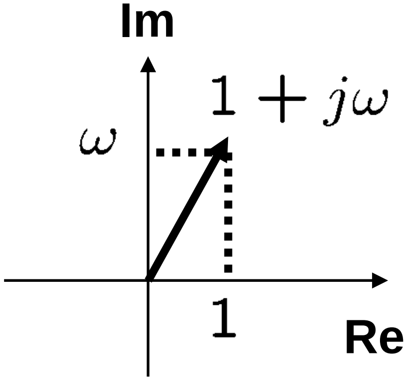
$$= \underbrace{\angle 5}_{=0} - \underbrace{\angle(3j+2)}_{=\tan^{-1}\frac{3}{2}}$$

Frequency response function

- For a stable system *G(s)*, *G(jω)* (*ω* is positive) is called *frequency response function (FRF)*.
- For each *ω*, FRF takes a complex number *G(jω)*, which has a gain and a phase.
- First order example

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$



First order example (cont'd)

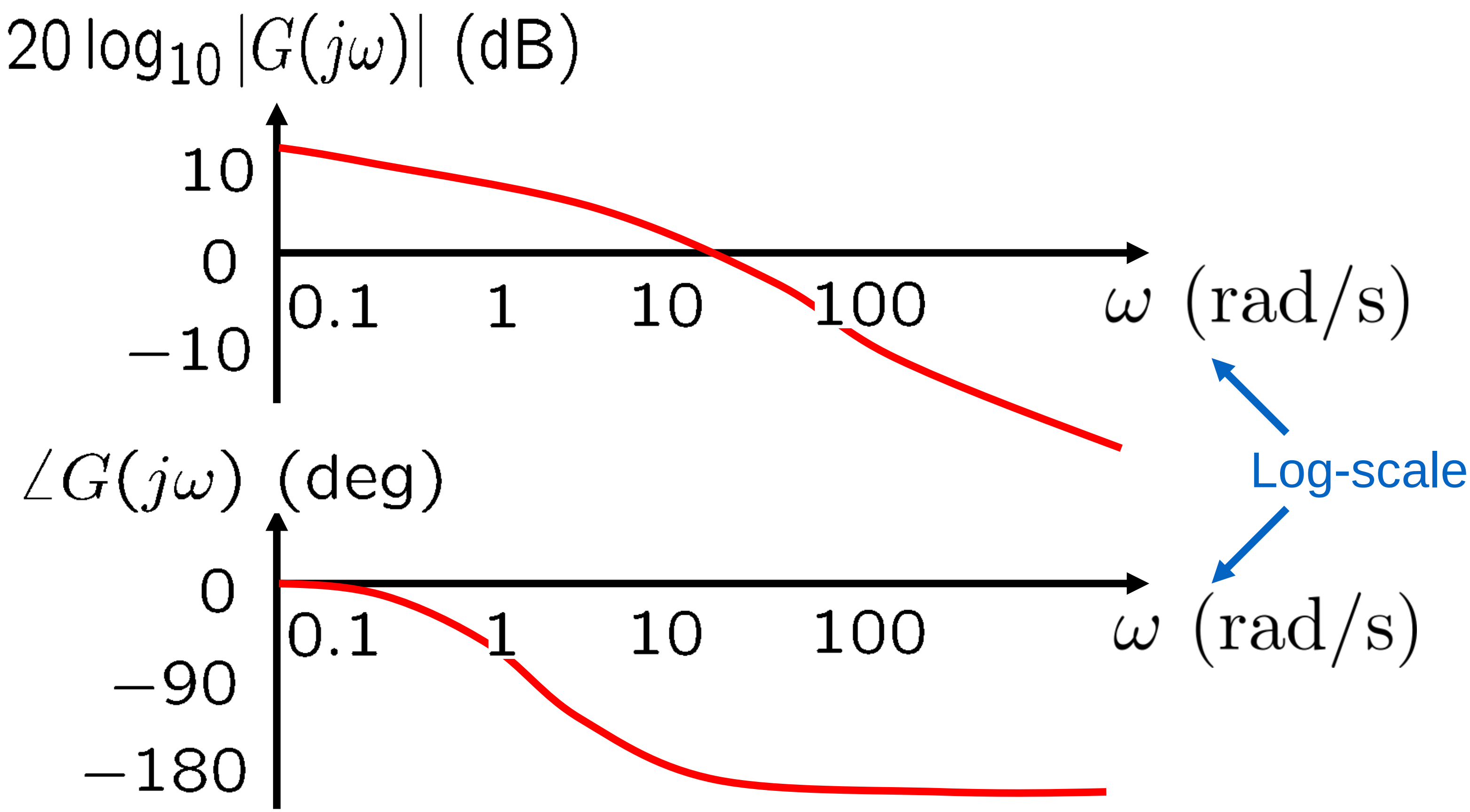
- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

frequency ω	gain $\lvert G(j\omega) \rvert$	phase $\angle G(j\omega)$
0	1	0°
0.5	0.894	-26.6°
1.0	0.707	-45°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°

- Two graphs representing FRF
  - Bode diagram/plot
  - Nyquist diagram/plot (to be covered in MECH467)

Bode plot (Bode diagram) of *G(jω)*

- Bode diagram consists of gain plot & phase plot



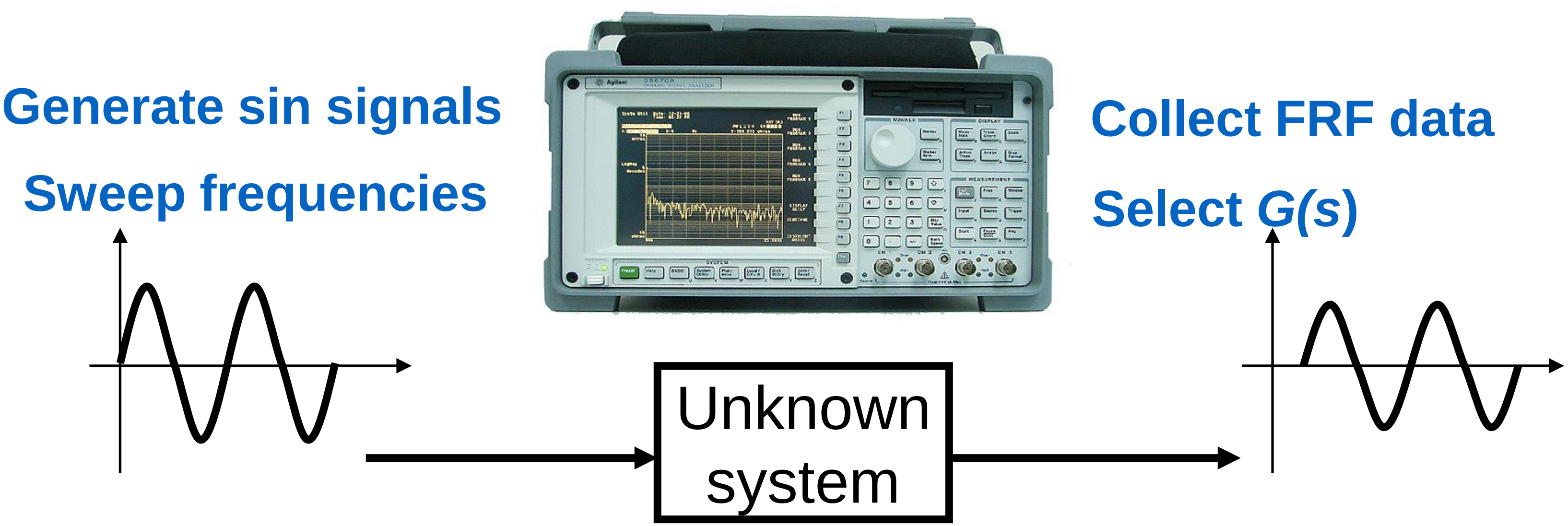
Remarks on Bode diagram

- Bode diagram shows gain and phase shift of a system output for sinusoidal inputs with various frequencies.
- Bode diagram is very useful and important in analysis and design of mechatronics systems.
- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)

System identification

- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select *G(s)* so that $G(j\omega)$ fits the FRF data.

Agilent Technologies: FFT Dynamic Signal Analyzer



Summary

- Frequency response
  - For a linear stable system, a sinusoidal input generates a sinusoidal output with same frequency but different amplitude and phase.
- Bode plot is a graphical representation of frequency response function. (MATLAB command “bode.m”)
- Next, we learn how to sketch Bode plots.
- Project: Fridays Nov 22, 29 (presentation)
- Homework 7: Due Nov 18 (Monday), 3pm
- Lab 4 report: Due Nov 25 (Monday), 6pm

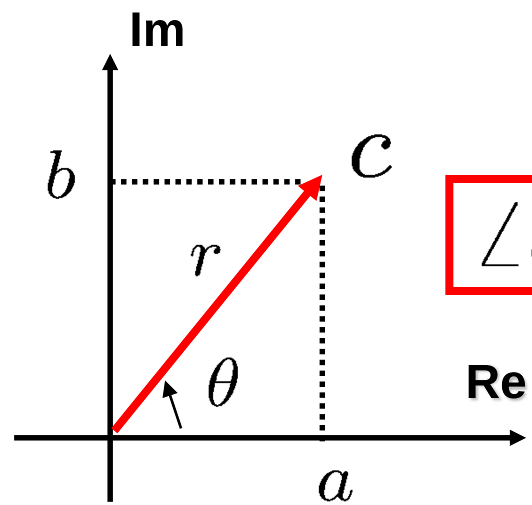
Appendix
Complex numbers (review)

- Representation
  - Cartesian form

$$c = a + bj$$

  - Polar form

$$c = re^{j\theta}$$



$$\boxed{\angle c = \theta}$$

- Multiplication & division in the polar form

$$\left.\begin{aligned} c_1 &= r_1 e^{j\theta_1} \\ c_2 &= r_2 e^{j\theta_2} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 c_2 &= r_1 r_2 e^{j(\theta_1+\theta_2)} \\ \frac{c_1}{c_2} &= \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \end{aligned}$$

Appendix
Derivation of frequency response

$$Y(s) = G(s)R(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{k_1}{s+j\omega} + \frac{k_2}{s-j\omega} + C_g(s)$$

Term having denominator of stable *G(s)*

$$\begin{cases} k_1 = \lim_{s\to -j\omega}(s+j\omega)G(s)\dfrac{A\omega}{s^2+\omega^2} = G(-j\omega)\dfrac{A\omega}{-2j\omega} = -\dfrac{AG(-j\omega)}{2j} \\ k_2 = \lim_{s\to j\omega}(s-j\omega)G(s)\dfrac{A\omega}{s^2+\omega^2} = G(j\omega)\dfrac{A\omega}{2j\omega} = \dfrac{AG(j\omega)}{2j} \end{cases}$$

0 as *t* goes to infinity.

$$\Rightarrow \quad y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} + \cancel{\mathcal{L}^{-1}\{C_g(s)\}}$$

$$\Rightarrow \quad y_{ss}(t) = A|G(j\omega)|\underbrace{\frac{e^{j(\omega t+\angle G(j\omega))} - e^{-j(\omega t+\angle G(j\omega))}}{2j}}_{\sin(\omega t+\angle G(j\omega))}$$

Appendix
Why deg(den)>=deg(num)?

- All the transfer functions we encountered so far have the property $\deg(\mathrm{den}) \geq \deg(\mathrm{num})$

  Ex: $\dfrac{1}{Ms^2 + Bs + K}$ $\quad \dfrac{K}{Ts+1}$

- What if deg(num) is larger than deg(den)?
  - Then, $|G(j\omega)| \to \infty \text{ as } \omega \to \infty$
  - However, there is no such system in reality that has increasing gain as input frequency increases to infinity.
- That is why all the transfer function needs to meet

$$\deg(\mathrm{den}) \geq \deg(\mathrm{num})$$