MECH366 : Modeling of Mechatronic Systems

L7 : Derivation of state-space models from linear graphs

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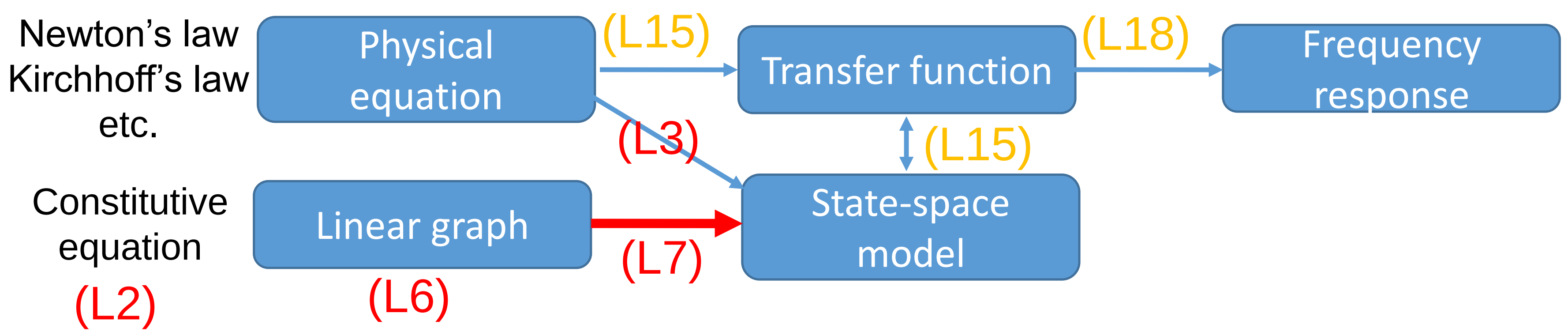
Review and today's topic

- In the last class, we studied how to represent mechanical elements in linear graphs:
  - Single-port elements, including energy source elements
  - Two-port elements (transformer & gyrator)
- Today, we will study how to derive state-space models from linear graphs.



$\underline{\quad}$: State variable

	Constitutive relation for		
	Energy storage element		Energy dissipating element
System type	A-Type	T-Type	D-Type
Mechanical	Mass	Spring	Viscous Damper
(translational)			
v: velocity across var.	$m\underline{\dot{v}} = f$	$\underline{\dot{f}} = kv$	$f = bv$
f: force through var.	m: mass	k: stiffness	b: damping const.
Electrical	Capacitor	Inductor	Resistor
v: voltage	$C\dot{v} = i$	$L\dot{i} = v$	$v = Ri$
i: current	C: capacitance	L: inductance	R: resistance
Thermal	Thermal capacitor	None	Thermal resistor
T: temperature	$C_t\dot{T} = Q$		$T = R_tQ$
Q: heat transfer rate	C: thermal capacitance		R_t: thermal resistance
Fluid	Fluid capacitor	Fluid inertor	Fluid resistor
P: pressure difference	$C_f\dot{P} = Q$	$I_f\dot{Q} = P$	$P = R_fQ$
Q: volume flow rate	C_f: fluid capacitance	I_f: fluid inertance	R_f: fluid resistance

power

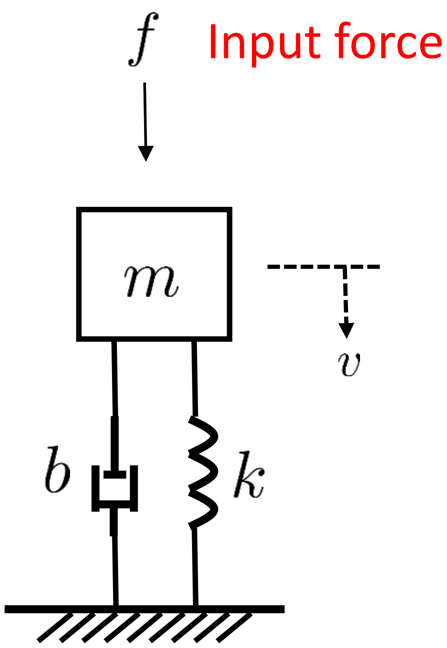
$$\mathcal{P} = fv$$

How to derive state-space models from linear graphs

- Key steps
  1. Draw a linear graph.
  2. Define state variables.
  3. Write a constitutive equation for each element.
  4. Write loop equations and node equations.
     - Loop equations are similar to Kirchhoff voltage law.
     - Node equations are similar to Kirchhoff current law.
  5. Eliminate redundant variables.

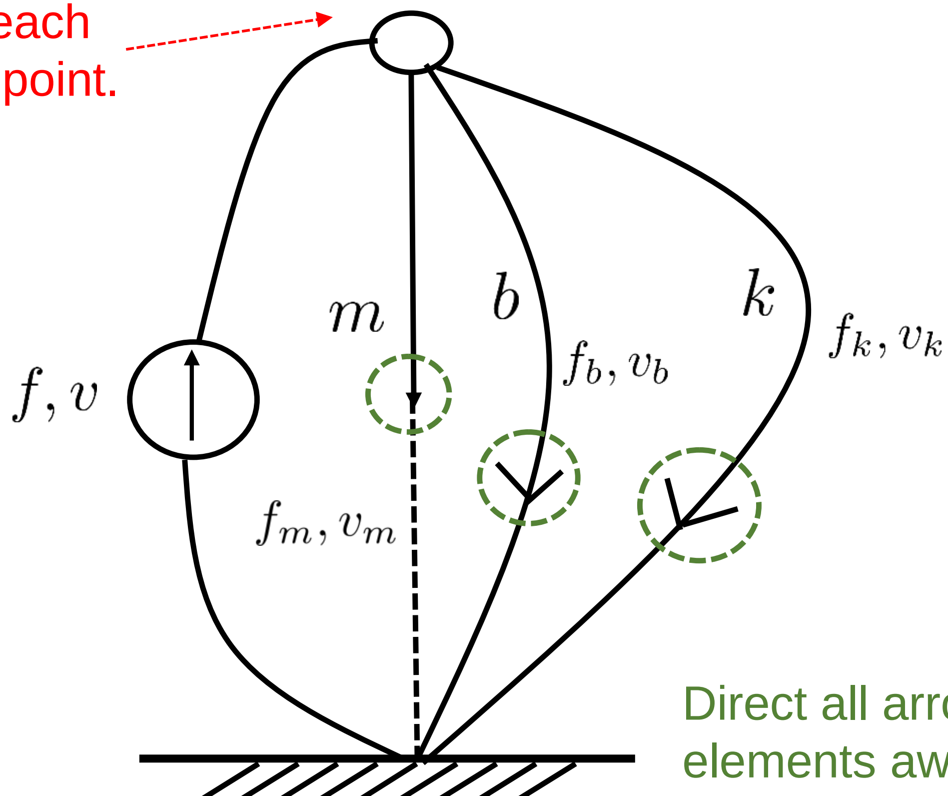
Example 1
System description

- 1-DOF mass-spring-damper



Example 1
Linear graph drawing

Take a node for each different velocity point.



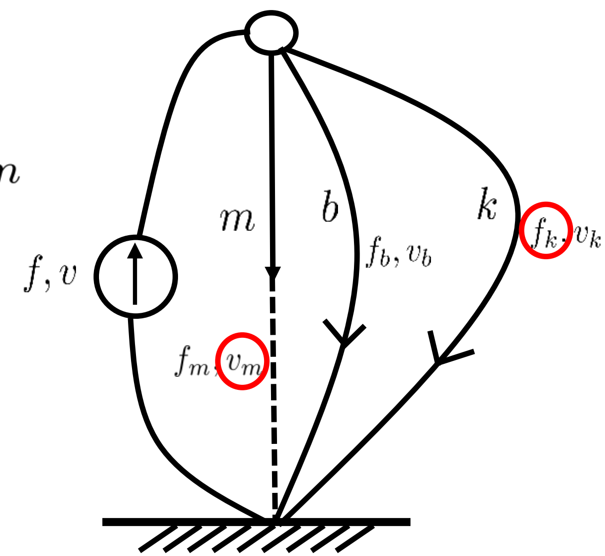
Direct all arrows on passive elements away from sources and toward the reference node.

Example 1
State-variable selection

- Select the following as state variables:
  - Across variable (*v*) for A-type element (*m*)
  - Through variable (*f*) for T-type element (*k*)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_m \\ f_k \end{bmatrix}$$

$$f_m = m\dot{v}_m$$
$$\dot{f}_k = kv_k$$
$$f_b = bv_b$$



- By constitutive equations

$$\begin{cases} \dot{x}_1 = \dot{v}_m = \frac{1}{m} f_m \\ \dot{x}_2 = \dot{f}_k = kv_k \end{cases}$$

Example 1
Loop and node equations

States

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_m \\ f_k \end{bmatrix}$$

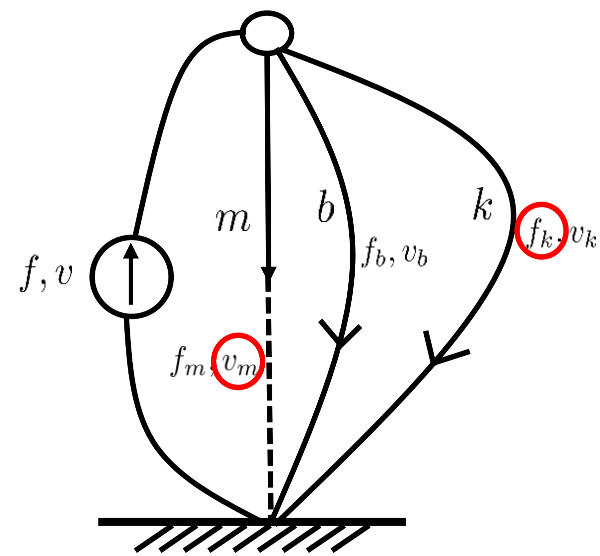
- Loop equation: In each loop,

$$\sum (\text{across variable}) = 0$$

(Similar to Kirchhoff voltage law)

$$\Rightarrow \begin{cases} v - v_m = 0 \\ v_m - v_b = 0 \\ v_b - v_k = 0 \end{cases}$$

$$\Rightarrow v_m = v_b = v_k = v$$



- Node equation: At each node,

$$\sum (\text{through variable}) = 0$$

(Similar to Kirchhoff current law)

Write all terms by state variables

$$\Rightarrow f - f_m - f_b - f_k = 0 \Rightarrow f_m = f - f_k - \underbrace{bv_b}_{f_b}$$

Example 1
State-space model

States

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_m \\ f_k \end{bmatrix}$$

- State equation

$$\begin{cases} \dot{x}_1 = \frac{1}{m} f_m = \frac{1}{m}(f - f_k - b v_b) = \frac{1}{m}(u - x_2 - b x_1) \\ \dot{x}_2 = k v_k = k v_m = k x_1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -b/m & -1/m \\ k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} u$$

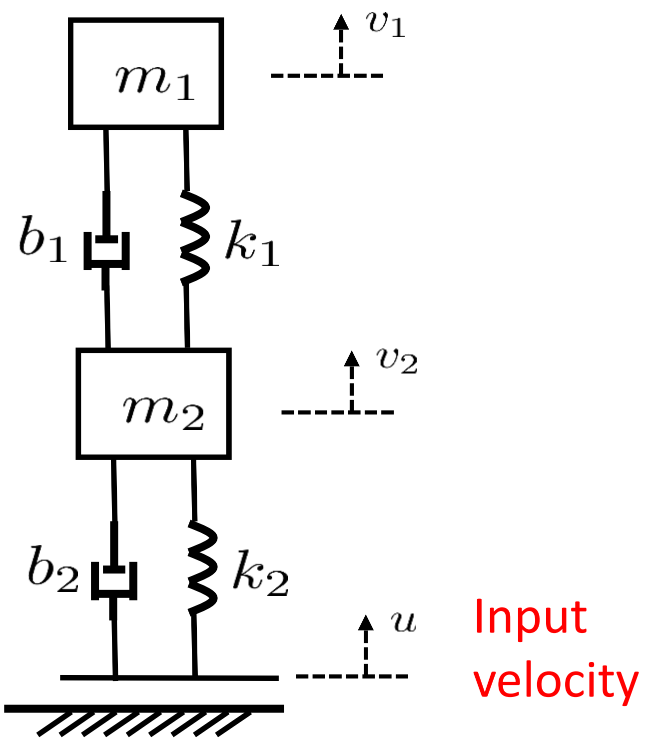
- Output equation
  - If $y = v_m$ then $y = [1 \ 0]\, x$
  - If $y = z_m$ then $y = [0 \ 1/k]\, x \quad (f_k = k z_k = k z_m)$

    (z_m: displacement)

(See Lecture 3, Slide 10.)

Example 2
System description

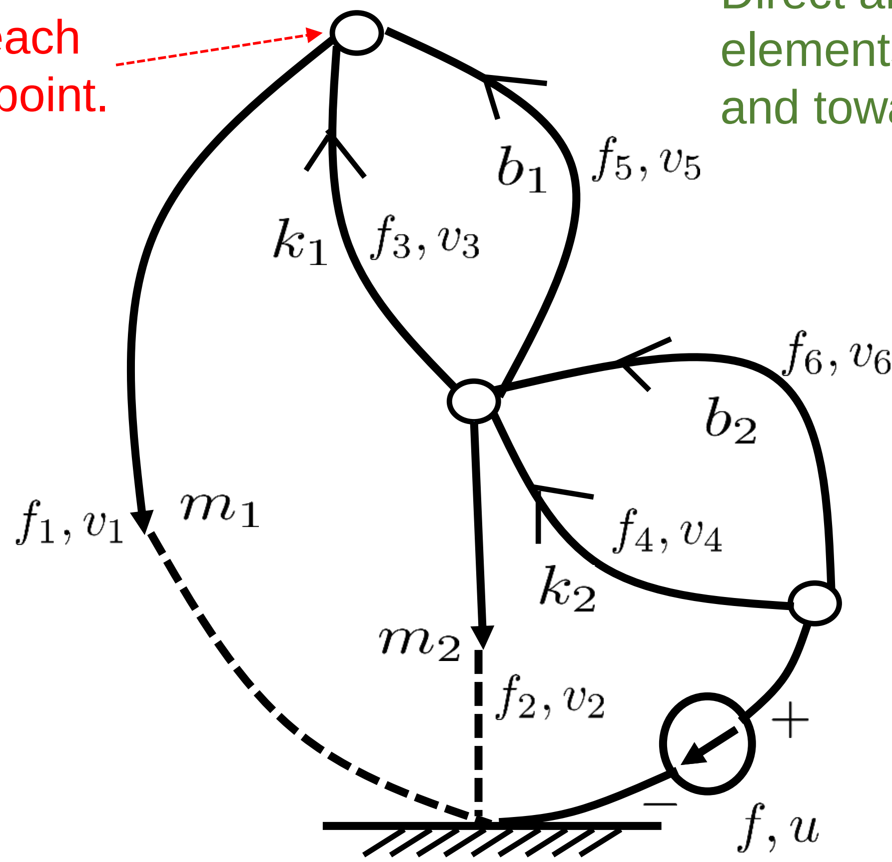
- 2-DOF mass-spring-damper



Example 2
Linear graph drawing

Take a node for each different velocity point.

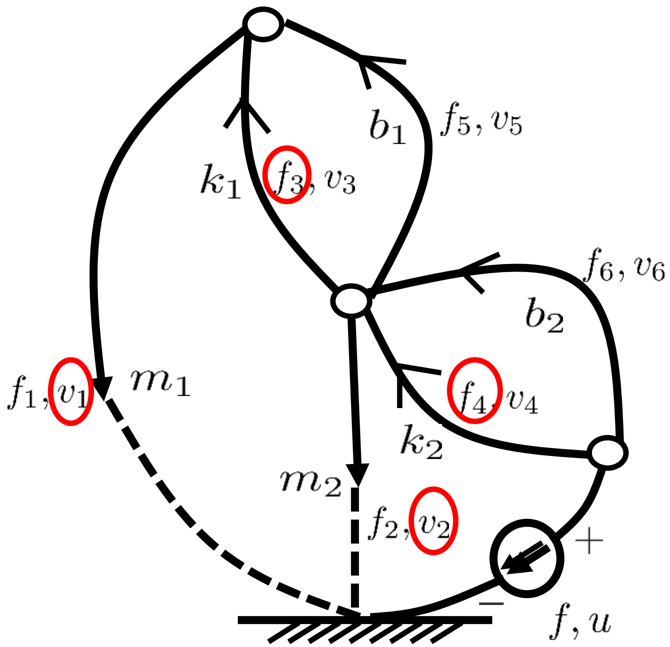
Direct all arrows on passive elements away from sources and toward the reference node.



Example 2
State-variable selection

- Select the following as state variables:
  - Across variable (*v*) for A-type element (*m*)
  - Through variable (*f*) for T-type element (*k*)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} v_1 \\ v_2 \\ f_3 \\ f_4 \end{bmatrix}$$



- Constitutive equations

$$\dot{v}_1 = \frac{1}{m_1} f_1, \quad \dot{v}_2 = \frac{1}{m_2} f_2$$

$$\dot{f}_3 = k_1 v_3, \quad \dot{f}_4 = k_2 v_4$$

$$f_5 = b_1 v_5, \quad f_6 = b_2 v_6$$

Example 2
Loop and node equations

- Loop equations

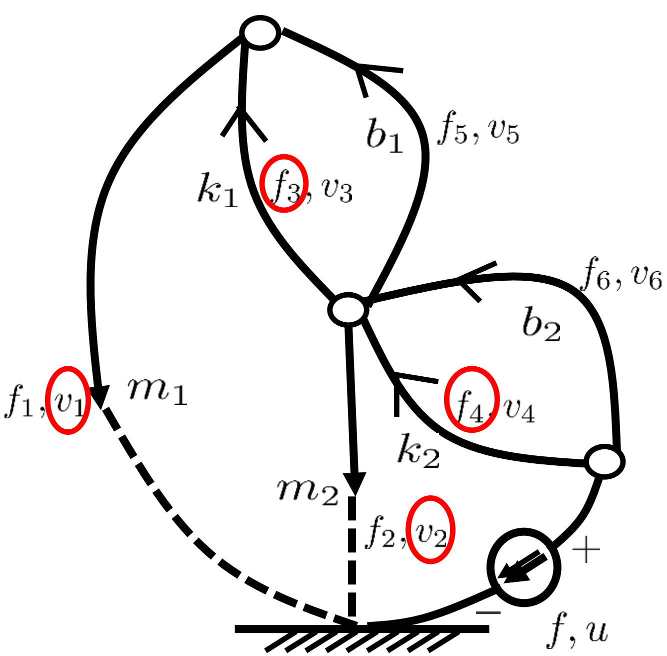
$$\begin{cases} v_1 - v_2 + v_3 = 0 \\ v_2 - u + v_4 = 0 \\ -v_4 + v_6 = 0 \\ -v_3 + v_5 = 0 \end{cases}$$

- Node equations

$$\begin{cases} -f_1 + f_3 + f_5 = 0 \\ -f_3 - f_5 - f_2 + f_4 + f_6 = 0 \\ -f_4 - f_6 + f = 0 \end{cases}$$

States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} v_1 \\ v_2 \\ f_3 \\ f_4 \end{bmatrix}$$



Example 2
State equation

States

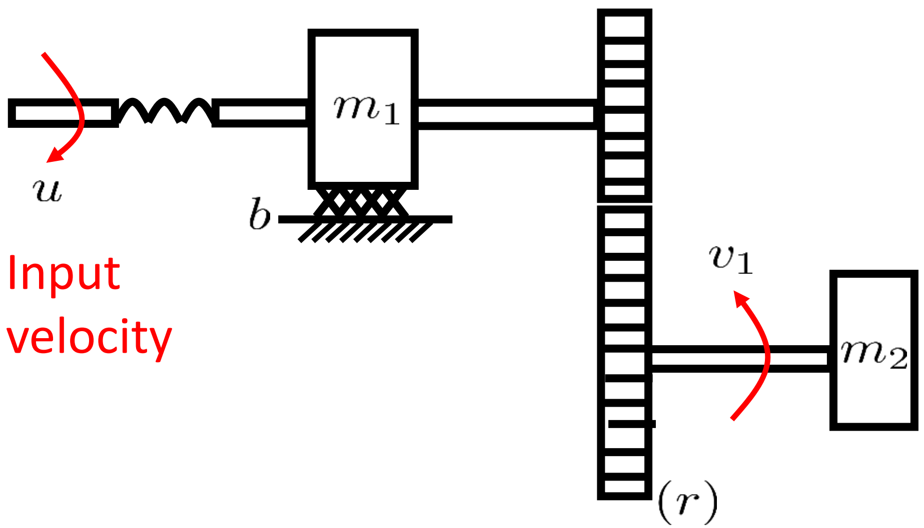
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} v_1 \\ v_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -b_1/m_1 & b_1/m_1 & 1/m_1 & 0 \\ b_1/m_2 & -(b_1+b_2)/m_2 & -1/m_2 & 1/m_2 \\ -k_1 & k_1 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2/m_2 \\ 0 \\ k_2 \end{bmatrix} u$$

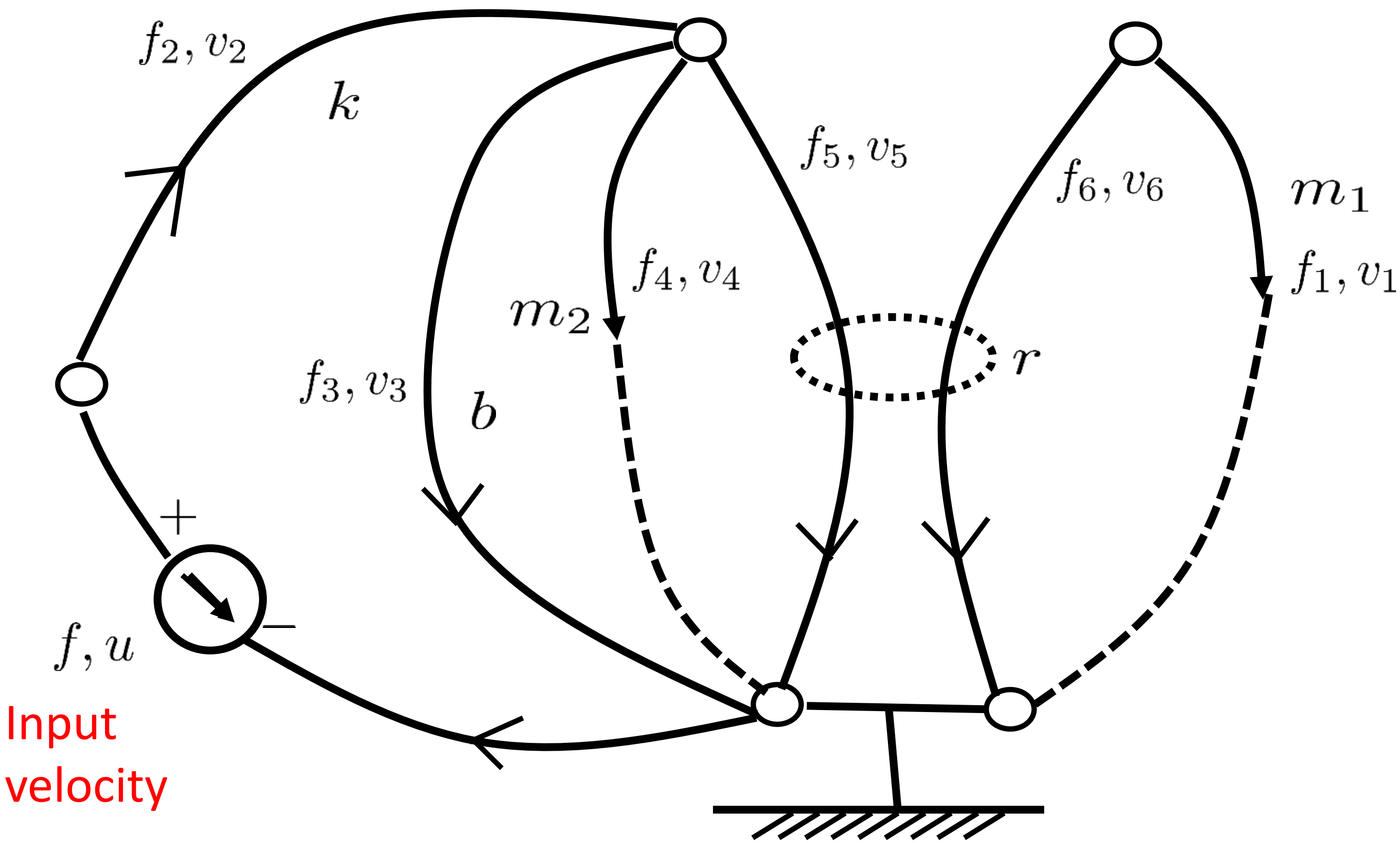
$$\left(\begin{aligned} &\dot{v}_1 = \frac{1}{m_1} f_1 = \frac{1}{m_1}(f_3+f_5) = \frac{1}{m_1}(f_3+b_1 v_5) = \frac{1}{m_1}(f_3+b_1 v_3) = \frac{1}{m_1}(f_3+b_1(-v_1+v_2)) \\ &\dot{v}_2 = \frac{1}{m_2} f_2 = \frac{1}{m_2}(f_4+f_6-f_3-f_5) = \frac{1}{m_2}(f_4+b_2 v_6 - f_3 - b_1 v_5) \\ &= \frac{1}{m_2}(f_4 + b_2 v_4 - f_3 - b_1 v_3) = \frac{1}{m_2}(f_4 + b_2(u - v_2) - f_3 - b_1(v_2 - v_1)) \\ &\dot{f}_3 = k_1 v_3 = k_1(v_2 - v_1) \\ &\dot{f}_4 = k_2 v_4 = k_2(u - v_2) \end{aligned} \right)$$

Example 3
System description

- Rotary motion control with a gearbox



Example 3
Linear graph drawing



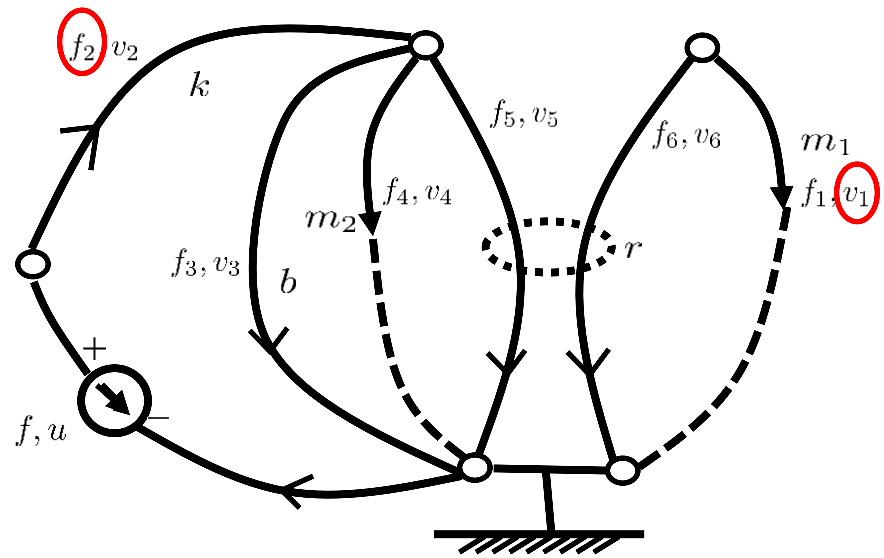
Example 3
State-variable selection

- Select the following as state variables:
  - Across variable (*v*) for A-type element (Inertia *m*)
  - Through variable (*f*) for T-type element (Rot. Spring *k*)
- Two masses are not independent.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_1 \\ f_2 \end{bmatrix}$$

$$\dot{x}_1 = \dot{v}_1 = \frac{1}{m_1} f_1$$

$$\dot{x}_2 = \dot{f}_2 = k v_2$$

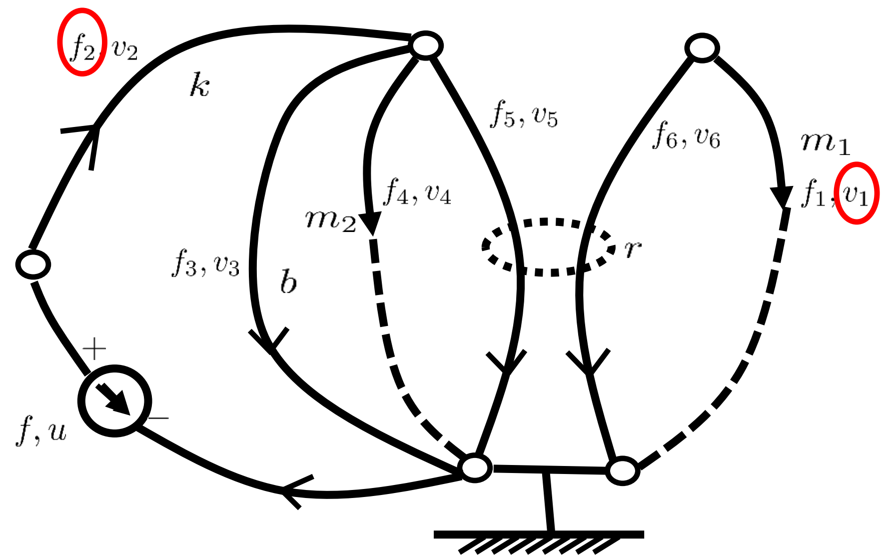


Example 3
Constitutive equations

- Remaining constitutive equations
  - Damper: $f_3 = bv_3$
  - Mass 2: $\dot{v}_4 = \frac{1}{m_2} f_4$
- Transformer

$$v_6 = rv_5$$

$$f_6 = -\frac{1}{r} f_5$$



Example 3
Loop and node equations

States

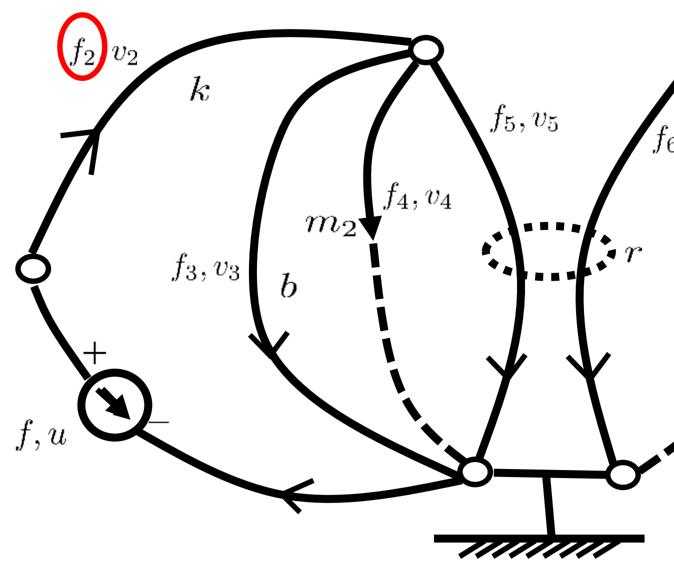
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_1 \\ f_2 \end{bmatrix}$$

- Loop equations

$$\begin{cases} v_6 - v_1 = 0 \\ v_4 - v_5 = 0 \\ v_3 - v_4 = 0 \\ -v_2 + u - v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = v_6 \\ v_3 = v_4 = v_5 \\ v_2 + v_3 = u \end{cases}$$

- Node equations

$$\begin{cases} -f_1 - f_6 = 0 \\ f_2 - f_3 - f_4 - f_5 = 0 \\ f - f_2 = 0 \end{cases}$$



Example 3
State equation

States

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_1 \\ f_2 \end{bmatrix}$$

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{m_1} f_1 = \frac{1}{m_1}(-f_6) = \frac{1}{m_1}\left(\frac{1}{r} f_5\right) \\
&= \frac{1}{m_1 r}(f_2 - f_3 - f_4) = \frac{1}{m_1 r}(f_2 - b v_3 - m_2 \dot{v}_4) \\
&= \frac{1}{m_1 r}\left(f_2 - b\frac{1}{r} v_6 - m_2 \frac{1}{r}\dot{v}_1\right) = \frac{1}{m_1 r}\left(f_2 - b\frac{1}{r} v_1 - m_2 \frac{1}{r}\dot{v}_1\right)
\end{aligned}$$

$$\Rightarrow \left(1 + \frac{m_2}{m_1 r^2}\right)\dot{v}_1 = \frac{1}{m_1 r}\left(f_2 - \frac{b}{r} v_1\right) \Rightarrow \dot{v}_1 = \frac{1}{m}(r f_2 - b v_1)$$

$$(m := m_1 r^2 + m_2)$$

$$\dot{f}_2 = k v_2 = k(u - v_3) = k\left(u - \frac{1}{r} v_1\right)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -b/m & r/m \\ -k/r & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} u$$

Summary

- Derivation of state-space models from linear graphs
- Three examples
  - 1-DOF mass spring damper
  - 2-DOF mass spring damper
  - Rotary motion control with a gearbox
- Next, two more examples
- Homework 2: Due Sep 30 (Monday), 3pm