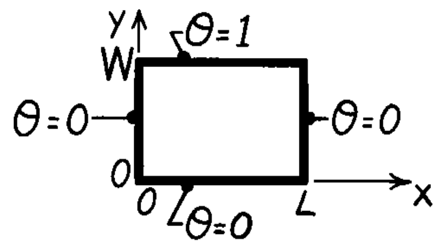
PROBLEM 4.1

KNOWN: Method of separation of variables (Section 4.2) for two-dimensional, steady-state conduction.

FIND: Show that negative or zero values of λ^2, the separation constant, result in solutions which cannot satisfy the boundary conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, identification of the separation constant λ^2 leads to the two ordinary differential equations, 4.6 and 4.7, having the forms

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad (1,2)$$

and the temperature distribution is

$$\theta(x,y) = X(x) \cdot Y(y). \qquad (3)$$

Consider now the situation when $\lambda^2 = 0$. From Eqs. (1), (2), and (3), find that

$$X = C_1 + C_2 x, \quad Y = C_3 + C_4 y \quad \text{and} \quad \theta(x,y) = (C_1 + C_2 x)\ (C_3 + C_4 y). \qquad (4)$$

Evaluate the constants - C_1, C_2, C_3 and C_4 - by substitution of the boundary conditions:

$$\begin{array}{lll}
x = 0: & \theta(0,y) = (C_1 + C_2 \cdot 0)(C_3 + C_4 y) = 0 & C_1 = 0 \\
y = 0: & \theta(x,0) = (0 + C_2 X)(C_3 + C_4 \cdot 0) = 0 & C_3 = 0 \\
x = L: & \theta(L,0) = (0 + C_2 L)(0 + C_4 y) = 0 & C_2 = 0 \\
y = W: & \theta(x,W) = (0 + 0 \cdot x)(0 + C_4 W) = 1 & 0 \neq 1
\end{array}$$

The last boundary condition leads to an impossibility ($0 \neq 1$). We therefore conclude that a λ^2 value of zero will not result in a form of the temperature distribution which will satisfy the boundary conditions. Consider now the situation when $\lambda^2 < 0$. The solutions to Eqs. (1) and (2) will be

$$X = C_5 e^{-\lambda x} + C_6 e^{+\lambda x}, \qquad Y = C_7 \cos \lambda y + C_8 \sin \lambda y \qquad (5,6)$$

and

$$\theta(x,y) = \left[C_5 e^{-\lambda x} + C_6 e^{+\lambda x}\right]\ \left[C_7 \cos \lambda y + C_8 \sin \lambda y\right]. \qquad (7)$$

Evaluate the constants for the boundary conditions.

$$\begin{array}{lll}
y = 0: & \theta(x,0) = \left[C_5 e^{-\lambda x} + C_6 e^{-\lambda x}\right]\ \left[C_7 \cos 0 + C_8 \sin 0\right] = 0 & C_7 = 0 \\
x = 0: & \theta(0,y) = \left[C_5 e^{0} + C_6 e^{0}\right]\ \left[0 + C_8 \sin \lambda y\right] = 0 & C_8 = 0
\end{array}$$

If $C_8 = 0$, a trivial solution results or $C_5 = -C_6$.

$$x = L: \quad \theta(L,y) = C_5\left[e^{-xL} - e^{+xL}\right] C_8 \sin \lambda y = 0.$$

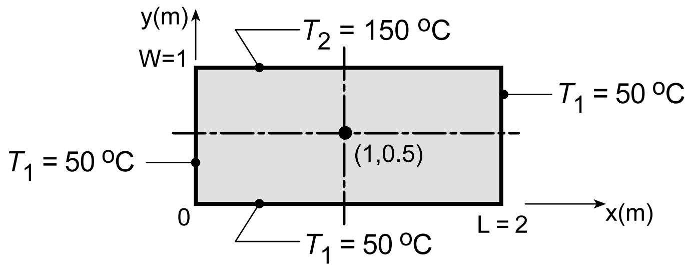
From the last boundary condition, we require C_5 or C_8 is zero; either case leads to a trivial solution with either no x or y dependence.

PROBLEM 4.2

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperature at the mid-point using the exact solution considering the first five non-zero terms; assess error resulting from using only first three terms. Plot the temperature distributions T(x,0.5) and T(1,y).

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(x,y) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\cdot\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (1,4.19)$$

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi}{2}\right)\cdot\frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}.$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{2\sin\left(\frac{\pi}{2}\right)\frac{\sinh(\pi/4)}{\sinh(\pi/2)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh(3\pi/4)}{\sinh(3\pi/2)} + \right.$$

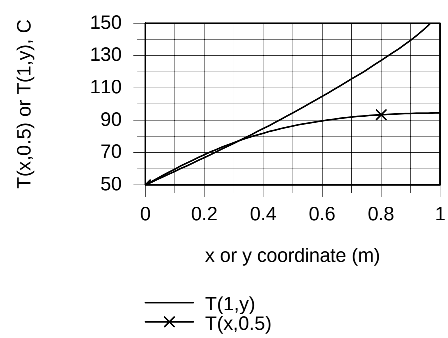
$$\left.\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh(5\pi/4)}{\sinh(5\pi/2)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh(7\pi/4)}{\sinh(7\pi/2)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh(9\pi/4)}{\sinh(9\pi/2)}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}[0.755 - 0.063 + 0.008 - 0.001 + 0.000] = 0.445 \qquad (2)$$

$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150-50) + 50 = 94.5^\circ C.$$ <

If only the first three terms of the series, Eq. (2), are considered, the result will be θ(1,0.5) = 0.46; that is, there is less than a 0.2% effect.

Using Eq. (1), and writing out the first five terms of the series, expressions for θ(x,0.5) or T(x,0.5) and θ(1,y) or T(1,y) were keyboarded into the IHT workspace and evaluated for sweeps over the x or y variable. Note that for T(1,y), that as $y \rightarrow 1$, the upper boundary, T(1,1) is greater than 150°C. Upon examination of the magnitudes of terms, it becomes evident that more than 5 terms are required to provide an accurate solution.

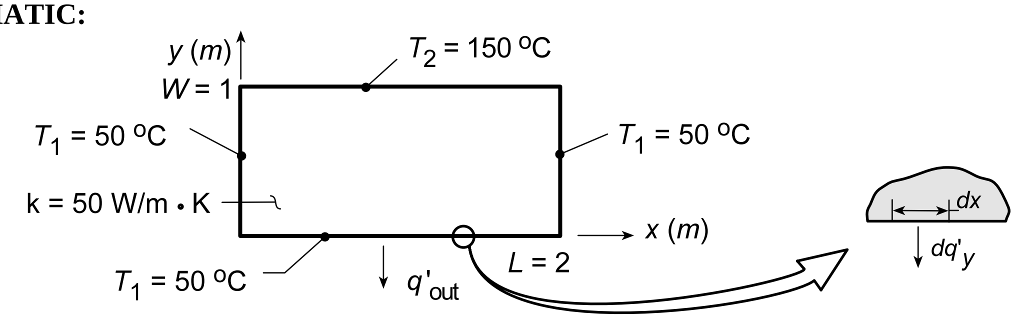


PROBLEM 4.3

KNOWN: Temperature distribution in the two-dimensional rectangular plate of Problem 4.2.

FIND: Expression for the heat rate per unit thickness from the lower surface ($0 \le x \le 2, 0$) and result based on first five non-zero terms of the infinite series.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: The heat rate per unit thickness *from the plate* along the lower surface is

$$q'_{out} = -\int_{x=0}^{x=2} dq'_y(x,0) = -\int_{x=0}^{x=2} -k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx = k(T_2 - T_1)\int_{x=0}^{x=2}\left.\frac{\partial \theta}{\partial y}\right|_{y=0} dx \qquad (1)$$

where from the solution to Problem 4.2,

$$\theta \equiv \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (2)$$

Evaluate the gradient of θ from Eq. (2) and substitute into Eq. (1) to obtain

$$q'_{out} = k(T_2 - T_1)\int_{x=0}^{x=2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\left.\frac{(n\pi/L)\cosh(n\pi y/L)}{\sinh(n\pi W/L)}\right|_{y=0} dx$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi W/L)}\left[-\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{2}\right]$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi/L)}\left[1-\cos(n\pi)\right] \qquad <$$

To evaluate the first five, non-zero terms, recognize that since $\cos(n\pi) = 1$ for $n = 2, 4, 6$ ..., only the n-odd terms will be non-zero. Hence,

Continued …..

$$q'_{out} = 50\,\mathrm{W/m\cdot K}(150-50)^\circ\mathrm{C}\,\frac{2}{\pi}\left\{\frac{(-1)^2+1}{1}\cdot\frac{1}{\sinh(\pi/2)}(2)+\frac{(-1)^4+1}{3}\cdot\frac{1}{\sinh(3\pi/2)}\cdot(2)\right.$$

$$\left.+\frac{(-1)^6+1}{5}\cdot\frac{1}{\sinh(5\pi/2)}(2)+\frac{(-1)^8+1}{7}\cdot\frac{1}{\sinh(7\pi/2)}(2)+\frac{(-1)^{10}+1}{9}\cdot\frac{1}{\sinh(9\pi/2)}(2)\right\}$$

$$q'_{out} = 3.183\,\mathrm{kW/m}\left[1.738+0.024+0.00062+(...)\right] = 5.611\,\mathrm{kW/m} \qquad <$$

COMMENTS: If the foregoing procedure were used to evaluate the heat rate into the upper surface, $q'_{in} = -\int_{x=0}^{x=2} dq'_y(x,W)$, it would follow that

$$q'_{in} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\coth(n\pi/2)\left[1-\cos(n\pi)\right]$$

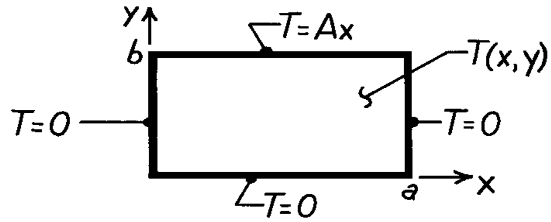
However, with coth(nπ/2) ≥ 1, irrespective of the value of n, and with $\sum_{n=1}^{\infty}\left[(-1)^{n+1}+1\right]/n$ being a divergent series, the complete series does not converge and $q'_{in} \to \infty$. This physically untenable condition results from the temperature discontinuities imposed at the upper left and right corners.

PROBLEM 4.4

KNOWN: Rectangular plate subjected to prescribed boundary conditions.

FIND: Steady-state temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: The solution follows the method of Section 4.2. The product solution is

$$T(x,y) = X(x)\cdot Y(y) = (C_1 \cos l\, x + C_2 \sin l\, x)\left(C_3 e^{-l\, y} + C_4 e^{+l\, y}\right)$$

and the boundary conditions are: T(0,y) = 0, T(a,y) = 0, T(x,0) = 0, T(x.b) = Ax. Applying BC#1, T(0,y) = 0, find $C_1 = 0$. Applying BC#2, T(a,y) = 0, find that $\lambda = n\pi/a$ with n = 1,2,…. Applying BC#3, T(x,0) = 0, find that $C_3 = -C_4$. Hence, the product solution is

$$T(x,y) = X(x)\cdot Y(y) = C_2 C_4 \sin\left[\frac{np}{a}x\right]\left(e^{+l\, y} - e^{-l\, y}\right).$$

Combining constants and using superposition, find

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{np\, x}{a}\right] \sinh\left[\frac{np\, y}{a}\right].$$

To evaluate C_n, use orthogonal functions with Eq. 4.16 to find

$$C_n = \int_0^a Ax\cdot \sin\left[\frac{np\, x}{a}\right]\cdot dx / \sinh\left[\frac{np\, b}{a}\right] \int_0^a \sin^2\left[\frac{np\, x}{a}\right] dx,$$

noting that y = b. The numerator, denominator and C_n, respectively, are:

$$A\int_0^a x\cdot \sin\frac{np\, x}{a}\cdot dx = A\left[\left[\frac{a}{np}\right]^2 \sin\left[\frac{np\, x}{a}\right] - \frac{ax}{np}\cos\left[\frac{np\, x}{a}\right]\right]_0^a = \frac{Aa^2}{np}\left[-\cos(np)\right] = \frac{Aa^2}{np}(-1)^{n+1},$$

$$\sinh\left[\frac{np\, b}{a}\right]\int_0^a \sin^2\frac{np\, x}{a}\cdot dx = \sinh\left[\frac{np\, b}{a}\right]\left[\frac{1}{2}x - \frac{1}{4np}\sin\left[\frac{2np\, x}{a}\right]\right]_0^a = \frac{a}{2}\cdot \sinh\left[\frac{np\, b}{a}\right],$$

$$C_n = \frac{Aa^2}{np}(-1)^{n+1} / \frac{a}{2}\sinh\left[\frac{np\, b}{a}\right] = 2Aa\,(-1)^{n+1}/np\ \sinh\left[\frac{np\, b}{a}\right].$$

Hence, the temperature distribution is

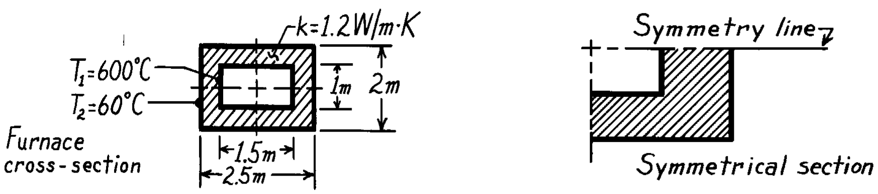
$$T(x,y) = \frac{2\,Aa}{p}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot \sin\left[\frac{np\, x}{a}\right]\frac{\sinh\left[\frac{np\, y}{a}\right]}{\sinh\left[\frac{np\, b}{a}\right]}.$$ <

PROBLEM 4.5

KNOWN: Long furnace of refractory brick with prescribed surface temperatures and material thermal conductivity.

FIND: Shape factor and heat transfer rate per unit length using the flux plot method

SCHEMATIC:

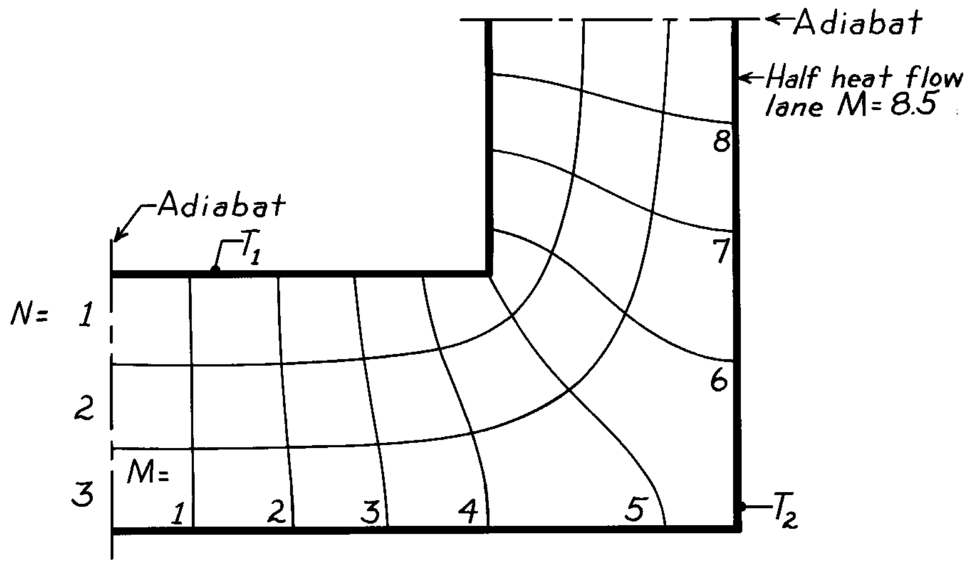


ASSUMPTIONS: (1) Furnace length normal to page, ℓ, >> cross-sectional dimensions, (2) Two-dimensional, steady-state conduction, (3) Constant properties.

ANALYSIS: Considering the cross-section, the cross-hatched area represents a symmetrical element. Hence, the heat rte for the entire furnace per unit length is

$$q' = \frac{q}{\ell} = 4\frac{S}{\ell}k\left(T_1 - T_2\right) \tag{1}$$

where S is the shape factor for the symmetrical section. Selecting three temperature increments (N = 3), construct the flux plot shown below.



From Eq. 4.26, $S = \frac{M\ell}{N}$ or $\frac{S}{\ell} = \frac{M}{N} = \frac{8.5}{3} = 2.83$ <

and from Eq. (1), $q' = 4 \times 2.83 \times 1.2 \frac{W}{m \cdot K}\left(600 - 60\right)^\circ C = 7.34$ kW/m. <

COMMENTS: The shape factor can also be estimated from the relations of Table 4.1. The symmetrical section consists of two plane walls (horizontal and vertical) with an adjoining edge. Using the appropriate relations, the numerical values are, in the same order,

$$S = \frac{0.75m}{0.5m}\ell + 0.54\ell + \frac{0.5m}{0.5m}\ell = 3.04\ell$$

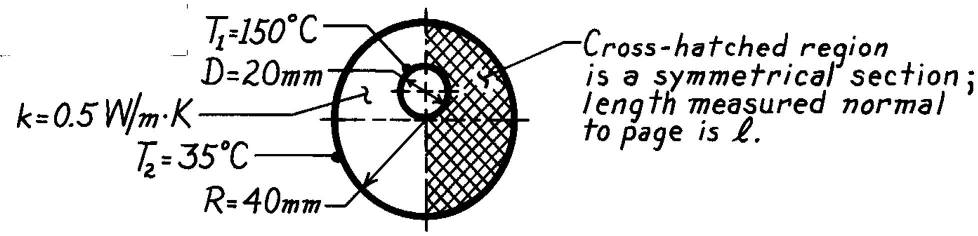
Note that this result compares favorably with the flux plot result of 2.83ℓ.

PROBLEM 4.6

KNOWN: Hot pipe embedded eccentrically in a circular system having a prescribed thermal conductivity.

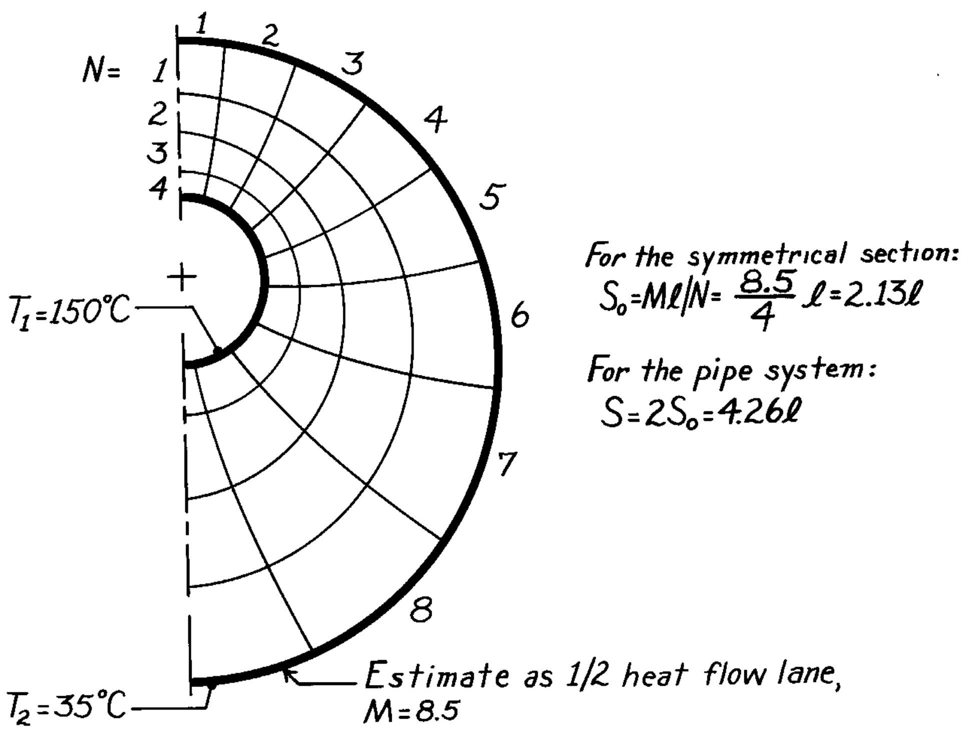
FIND: The shape factor and heat transfer per unit length for the prescribed surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Length ℓ >> diametrical dimensions.

ANALYSIS: Considering the cross-sectional view of the pipe system, the symmetrical section shown above is readily identified. Selecting four temperature increments (N = 4), construct the flux plot shown below.



For the pipe system, the heat rate per unit length is

$$q' = \frac{q}{\ell} = kS(T_1 - T_2) = 0.5\frac{W}{m \cdot K} \times 4.26(150-35)^\circ C = 245\ W/m. \qquad <$$

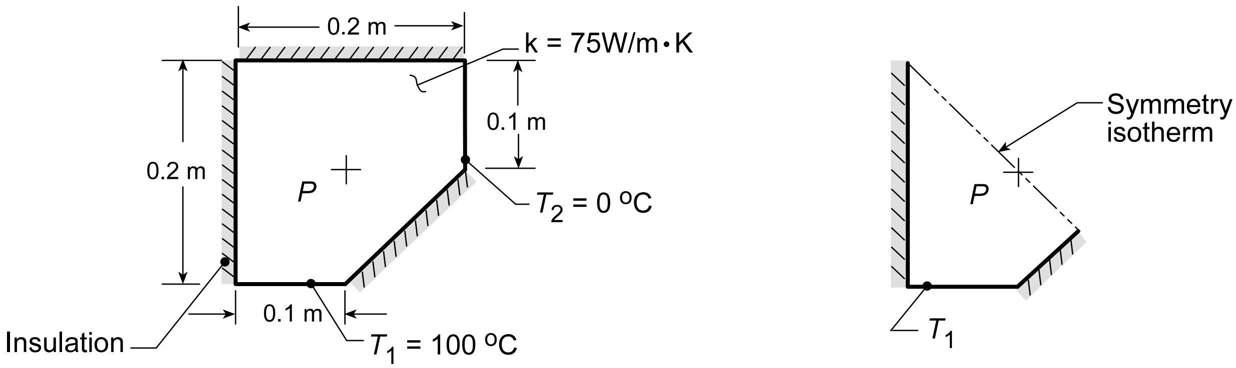
COMMENTS: Note that in the lower, right-hand quadrant of the flux plot, the curvilinear squares are irregular. Further work is required to obtain an improved plot and, hence, obtain a more accurate estimate of the shape factor.

PROBLEM 4.7

KNOWN: Structural member with known thermal conductivity subjected to a temperature difference.

FIND: (a) Temperature at a prescribed point P, (b) Heat transfer per unit length of the strut, (c) Sketch the 25, 50 and 75°C isotherms, and (d) Same analysis on the shape but with adiabatic-isothermal boundary conditions reversed.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties.

ANALYSIS: (a) Using the methodology of Section 4.3.1, construct a flux plot. Note the line of symmetry which passes through the point P is an isotherm as shown above. It follows that

$$T(P) = (T_1 + T_2)/2 = (100 + 0)^\circ C/2 = 50^\circ C. \qquad <$$

(b) The flux plot on the symmetrical section is now constructed to obtain the shape factor from which the heat rate is determined. That is, from Eq. 4.25 and 4.26,

$$q = kS(T_1 - T_2) \quad \text{and} \quad S = M\ell/N. \qquad (1,2)$$

From the plot of the symmetrical section,

$$S_o = 4.2\ell/4 = 1.05\ell.$$

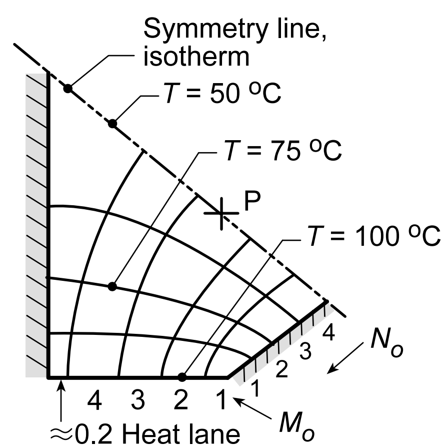
For the full section of the strut,

$$M = M_o = 4.2$$

but N = $2N_o$ = 8. Hence,

$$S = S_o/2 = 0.53\ell$$

and with $q' = q/\ell$, giving



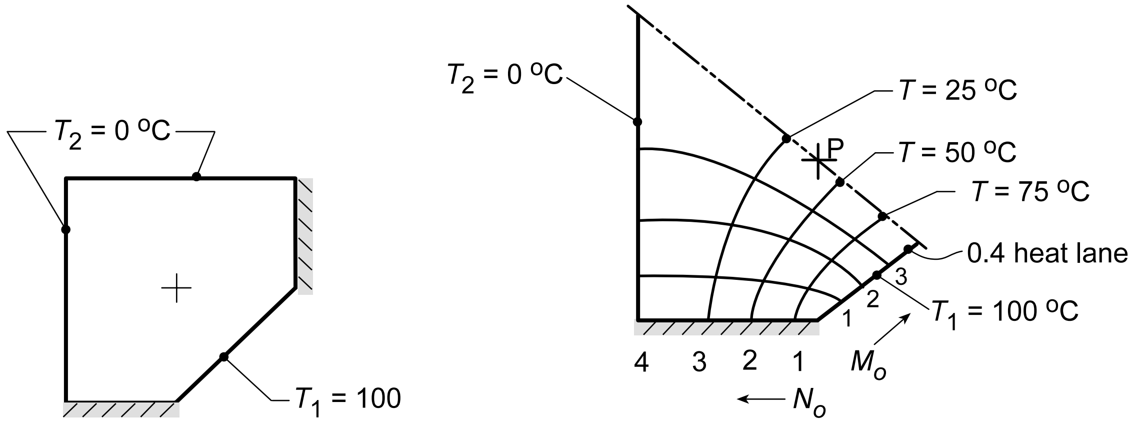
$$q'/\ell = 75\,W/m\cdot K \times 0.53(100-0)^\circ C = 3975\,W/m. \qquad <$$

(c) The isotherms for T = 50, 75 and 100°C are shown on the flux plot. The T = 25°C isotherm is symmetric with the T = 75°C isotherm.

(d) By reversing the adiabatic and isothermal boundary conditions, the two-dimensional shape appears as shown in the sketch below. The symmetrical element to be flux plotted is the same as for the strut, except the symmetry line is now an adiabat.

Continued...

PROBLEM 4.7 (Cont.)



From the flux plot, find $M_o = 3.4$ and $N_o = 4$, and from Eq. (2)

$$S_o = M_o \ell / N_o = 3.4\ell/4 = 0.85\ell \qquad S = 2S_o = 1.70\ell$$

and the heat rate per unit length from Eq. (1) is

$$q' = 75\,\text{W/m}\cdot\text{K} \times 1.70(100-0)^\circ\,\text{C} = 12{,}750\,\text{W/m}$$ <

From the flux plot, estimate that

$$T(P) \approx 40^\circ\text{C}.$$ <

COMMENTS: (1) By inspection of the shapes for parts (a) and (b), it is obvious that the heat rate for the latter will be greater. The calculations show the heat rate is greater by more than a factor of three.

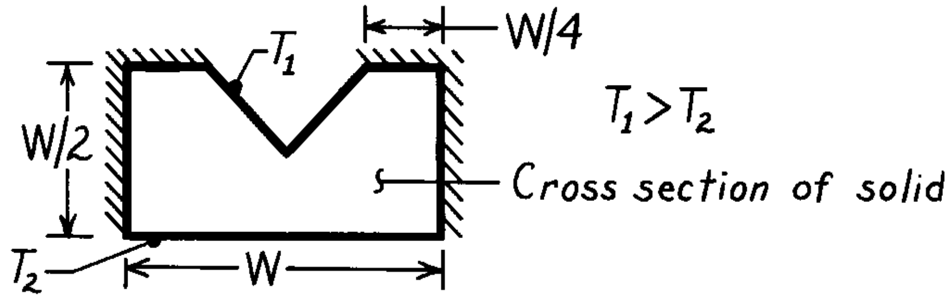
(2) By comparing the flux plots for the two configurations, and corresponding roles of the adiabats and isotherms, would you expect the shape factor for parts (a) to be the reciprocal of part (b)?

PROBLEM 4.8

KNOWN: Relative dimensions and surface thermal conditions of a V-grooved channel.

FIND: Flux plot and shape factor.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties.

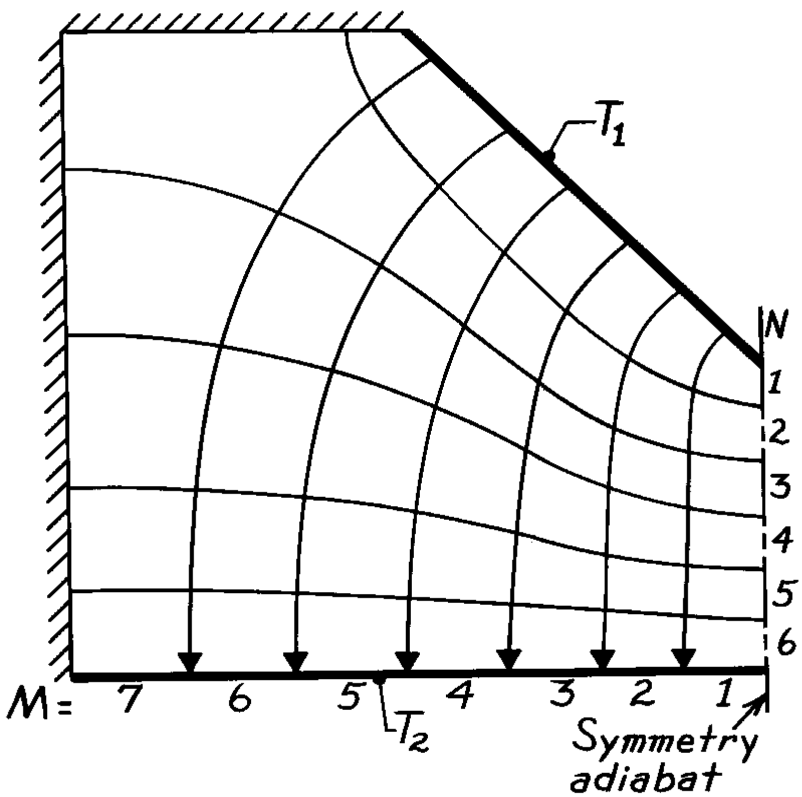
ANALYSIS: With symmetry about the midplane, only one-half of the object need be considered as shown below.

Choosing 6 temperature increments (N = 6), it follows from the plot that M ≈ 7. Hence from Eq. 4.26, the shape factor for the half section is

$$S = \frac{M}{N}\ell = \frac{7}{6}\ell = 1.17\ell.$$

For the complete system, the shape factor is then

$$S = 2.34\ell.$$ <

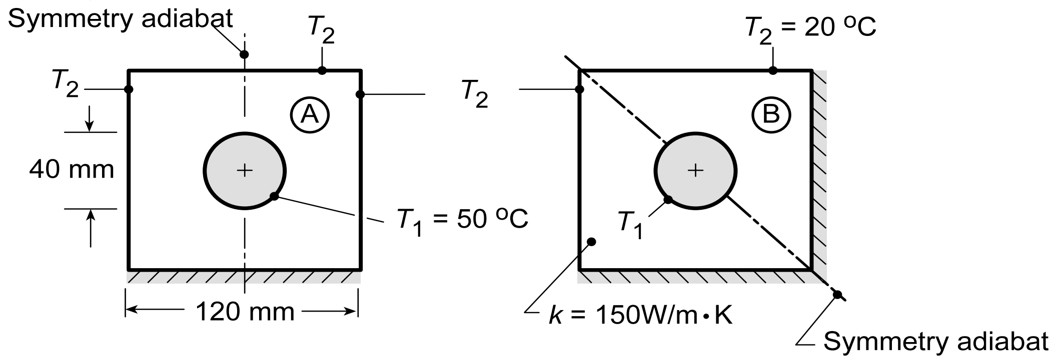


PROBLEM 4.9

KNOWN: Long conduit of inner circular cross section and outer surfaces of square cross section.

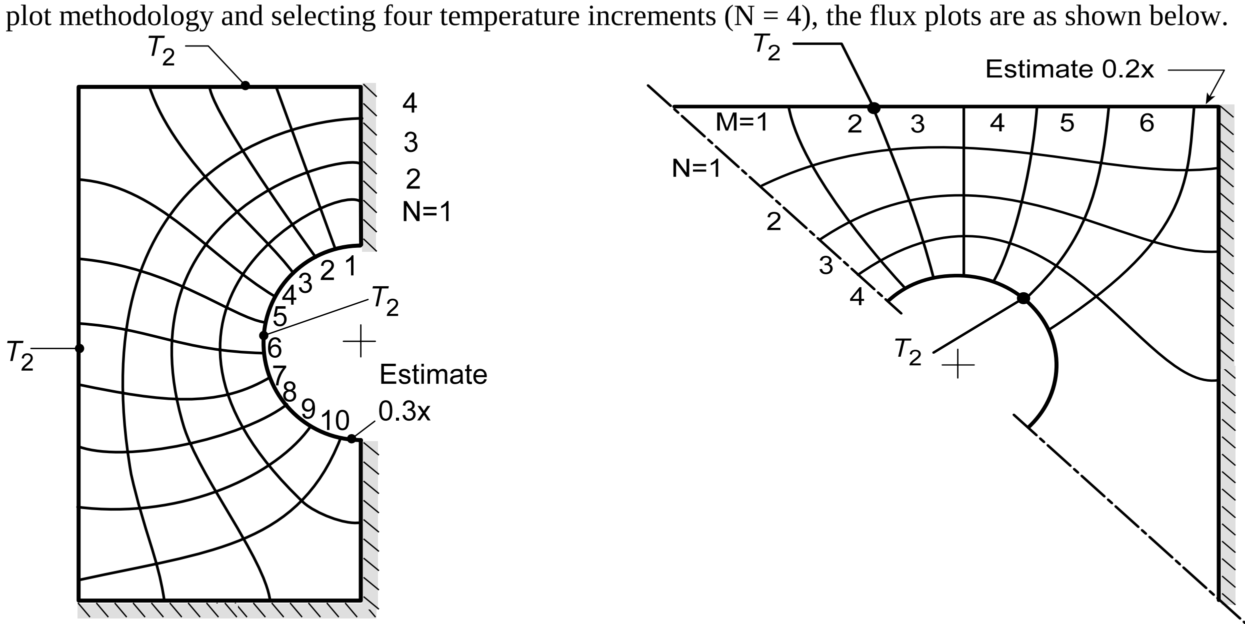
FIND: Shape factor and heat rate for the two applications when outer surfaces are insulated or maintained at a uniform temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties and (3) Conduit is very long.

ANALYSIS: The adiabatic symmetry lines for each of the applications is shown above. Using the flux plot methodology and selecting four temperature increments (N = 4), the flux plots are as shown below.



For the symmetrical sections, $S = 2S_o$, where $S_o = M\ell/N$ and the heat rate for each application is $q = 2(S_o/\ell)k(T_1 - T_2)$.

Application	M	N	S_o/ℓ	q' (W/m)	
A	10.3	4	2.58	11,588	<
B	6.2	4	1.55	6,975	<

COMMENTS: (1) For application A, most of the heat lanes leave the inner surface (T_1) on the upper portion.

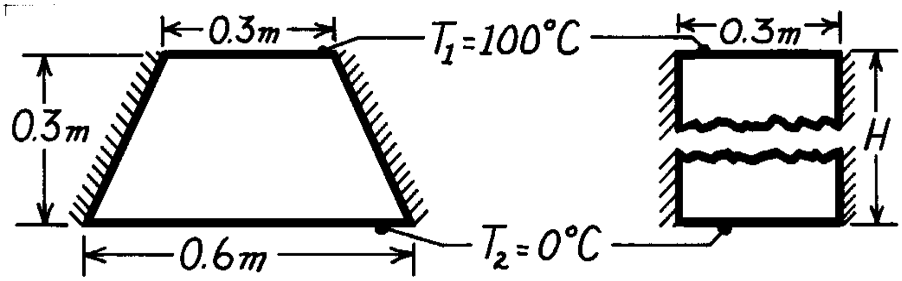
(2) For application B, most of the heat flow lanes leave the inner surface on the upper portion (that is, lanes 1-4). Because the lower, right-hand corner is insulated, the entire section experiences small heat flows (lane 6 + 0.2). Note the shapes of the isotherms near the right-hand, insulated boundary and that they intersect the boundary normally.

PROBLEM 4.10

KNOWN: Shape and surface conditions of a support column.

FIND: (a) Heat transfer rate per unit length. (b) Height of a rectangular bar of equivalent thermal resistance.

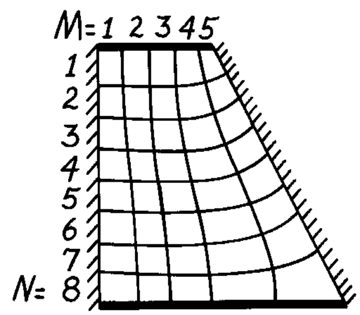
SCHEMATIC:



ASSUMPTIONS: (1)Steady-state conditions, (2) Negligible three-dimensional conduction effects, (3) Constant properties, (4) Adiabatic sides.

PROPERTIES: *Table A-1*, Steel, AISI 1010 (323K): k = 62.7 W/m·K.

ANALYSIS: (a) From the flux plot for the half section, M ≈ 5 and N ≈ 8. Hence for the full section



$$S = 2\frac{M\ell}{N} \approx 1.25\ell$$

$$q = Sk\left(T_1 - T_2\right)$$

$$q' \approx 1.25 \times 62.7 \frac{W}{m \cdot K}(100 - 0)^\circ C$$

$$q' \approx 7.8 \text{ kW/m.}$$ <

(b) The rectangular bar provides for one-dimensional heat transfer. Hence,

$$q = k\,A\frac{\left(T_1 - T_2\right)}{H} = k\left(0.3\ell\right)\frac{\left(T_1 - T_2\right)}{H}$$

Hence, $$H = \frac{0.3k\left(T_1 - T_2\right)}{q'} = \frac{0.3m\left(62.7 \text{ W/m}\cdot\text{K}\right)\left(100^\circ C\right)}{7800 \text{ W/m}} = 0.24m.$$ <

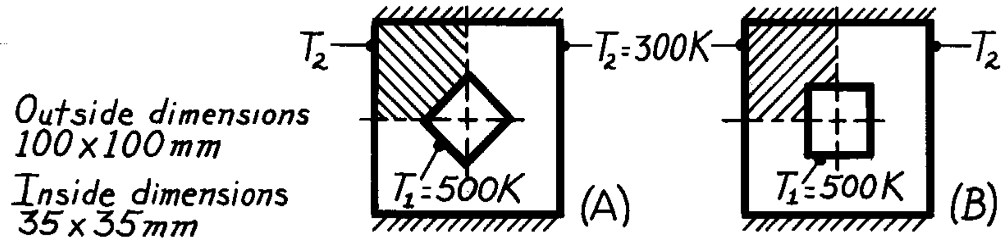
COMMENTS: The fact that H < 0.3m is consistent with the requirement that the thermal resistance of the trapezoidal column must be less than that of a rectangular bar of the same height and top width (because the width of the trapezoidal column increases with increasing distance, x, from the top). Hence, if the rectangular bar is to be of equivalent resistance, it must be of smaller height.

PROBLEM 4.11

KNOWN: Hollow prismatic bars fabricated from plain carbon steel, 1m in length with prescribed temperature difference.

FIND: Shape factors and heat rate per unit length.

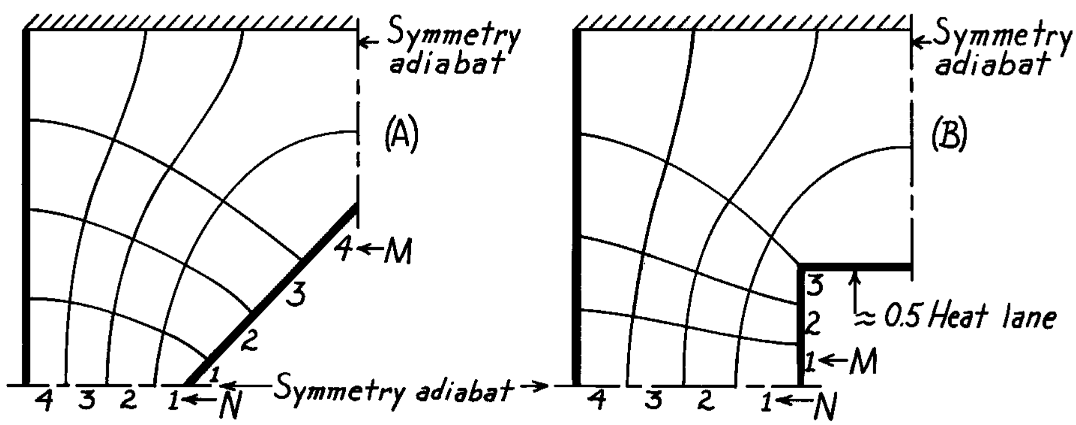
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

PROPERTIES: *Table A-1*, Steel, Plain Carbon (400K), k = 57 W/m·K.

ANALYSIS: Construct a flux plot on the symmetrical sections (shaded-regions) of each of the bars.



The shape factors for the symmetrical sections are,

$$S_{o,A} = \frac{M\ell}{N} = \frac{4}{4}\ell = 1\ell \qquad S_{o,B} = \frac{M\ell}{N} = \frac{3.5}{4}\ell = 0.88\ell.$$

Since each of these sections is ¼ of the bar cross-section, it follows that

$$S_A = 4\times 1\ell = 4\ell \qquad S_B = 4\times 0.88\ell = 3.5\ell. \qquad <$$

The heat rate per unit length is $q' = q/\ell = k(S/\ell)(T_1 - T_2)$,

$$q'_A = 57\frac{W}{m\cdot K}\times 4(500-300)K = 45.6\text{ kW/m} \qquad <$$

$$q'_B = 57\frac{W}{m\cdot K}\times 3.5(500-300)K = 39.9\text{ kW/m}. \qquad <$$

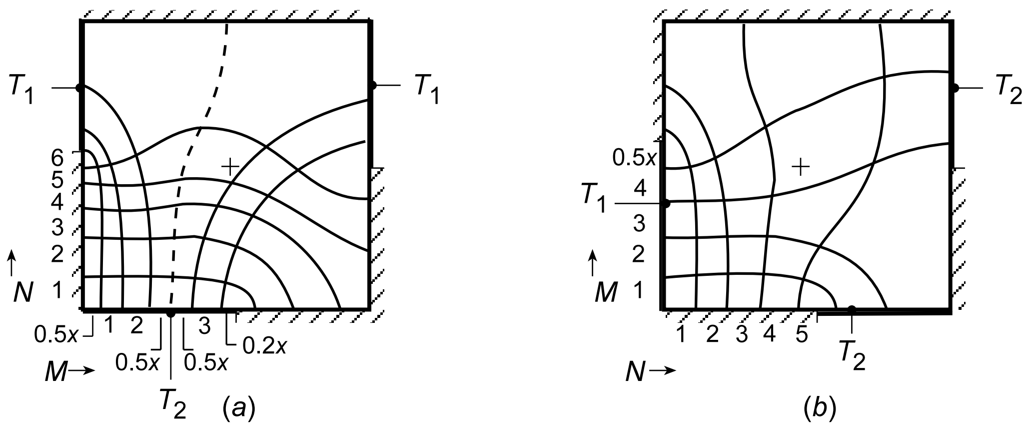
PROBLEM 4.12

KNOWN: Two-dimensional, square shapes, 1 m to a side, maintained at uniform temperatures as prescribed, perfectly insulated elsewhere.

FIND: Using the flux plot method, estimate the heat rate per unit length normal to the page if the thermal conductivity is 50 W/m·K

ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: Use the methodology of Section 4.3.1 to construct the flux plots to obtain the shape factors from which the heat rates can be calculated. With Figure (a), begin at the lower-left side making the isotherms almost equally spaced, since the heat flow will only slightly spread toward the right. Start sketching the adiabats in the vicinity of the T_2 surface. The dashed line represents the adiabat which separates the shape into two segments. Having recognized this feature, it was convenient to identify partial heat lanes. Figure (b) is less difficult to analyze since the isotherm intervals are nearly regular in the lower left-hand corner.



The shape factors are calculated from Eq. 4.26 and the heat rate from Eq. 4.25.

$$S' = \frac{M}{N} = \frac{0.5 + 3 + 0.5 + 0.5 + 0.2}{6} \qquad S' = \frac{M}{N} = \frac{4.5}{5} = 0.90$$

$$S' = 0.70$$

$$q' = kS'(T_1 - T_2) \qquad q' = kS'(T_1 - T_2)$$

$$q' = 10\,\mathrm{W/m\cdot K} \times 0.70(100 - 0)\,\mathrm{K} = 700\,\mathrm{W/m} \quad q' = 10\,\mathrm{W/m\cdot K} \times 0.90(100 - 0)\,\mathrm{K} = 900\,\mathrm{W/m} \quad <$$

COMMENTS: Using a finite-element package with a fine mesh, we determined heat rates of 956 and 915 W/m, respectively, for Figures (a) and (b). The estimate for the less difficult Figure (b) is within 2% of the numerical method result. For Figure (a), our flux plot result was 27% low.

PROBLEM 4.13

KNOWN: Uniform media of prescribed geometry.

FIND: (a) Shape factor expressions from thermal resistance relations for the plane wall, cylindrical shell and spherical shell, (b) Shape factor expression for the isothermal sphere of diameter D buried in an infinite medium.

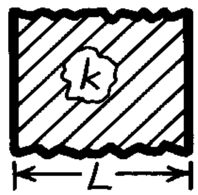
ASSUMPTIONS: (1) Steady-state conditions, (2) Uniform properties.

ANALYSIS: (a) The relationship between the shape factor and thermal resistance of a shape follows from their definitions in terms of heat rates and overall temperature differences.

$$q = kS\Delta T \quad (4.25), \qquad q = \frac{\Delta T}{R_t} \quad (3.19), \qquad S = 1/kR_t \qquad (4.27)$$

Using the thermal resistance relations developed in Chapter 3, their corresponding shape factors are:

Plane wall:



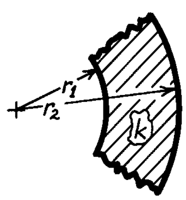
$$R_t = \frac{L}{kA} \qquad S = \frac{A}{L}. \qquad <$$

Cylindrical shell:

$$R_t = \frac{\ln(r_2/r_1)}{2\pi Lk} \qquad S = \frac{2\pi L}{\ln r_2/r_1}. \qquad <$$

(*L into the page*)

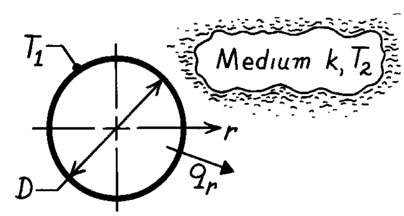
Spherical shell:



$$R_t = \frac{1}{4\pi k}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \qquad S = \frac{4\pi}{1/r_1 - 1/r_2}.$$

<

(b) The shape factor for the sphere of diameter D in an infinite medium can be derived using the alternative conduction analysis of Section 3.1. For this situation, q_r is a constant and Fourier's law has the form



$$q_r = -k\left(4\pi r^2\right)\frac{dT}{dr}.$$

Separate variables, identify limits and integrate.

$$-\frac{q_r}{4\pi k}\int_{D/2}^{\infty}\frac{dr}{r^2} = \int_{T_1}^{T_2} dT \qquad -\frac{q_r}{4\pi k}\left[-\frac{1}{r}\right]_{D/2}^{\infty} = -\frac{q_r}{4\pi k}\left[0 - \frac{2}{D}\right] = \left(T_2 - T_1\right)$$

$$q_r = 4\pi k\left[\frac{D}{2}\right]\left(T_1 - T_2\right) \qquad \text{or} \qquad S = 2\pi D. \qquad <$$

COMMENTS: Note that the result for the buried sphere, $S = 2\pi D$, can be obtained from the expression for the spherical shell with $r_2 = \infty$. Also, the shape factor expression for the "isothermal sphere buried in a semi-infinite medium" presented in Table 4.1 provides the same result with $z \to \infty$.

PROBLEM 4.14

KNOWN: Heat generation in a buried spherical container.

FIND: (a) Outer surface temperature of the container, (b) Representative isotherms and heat flow lines.

SCHEMATIC:

ASSUMPTIONS: (1) Steady-state conditions, (2) Soil is a homogeneous medium with constant properties.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K.

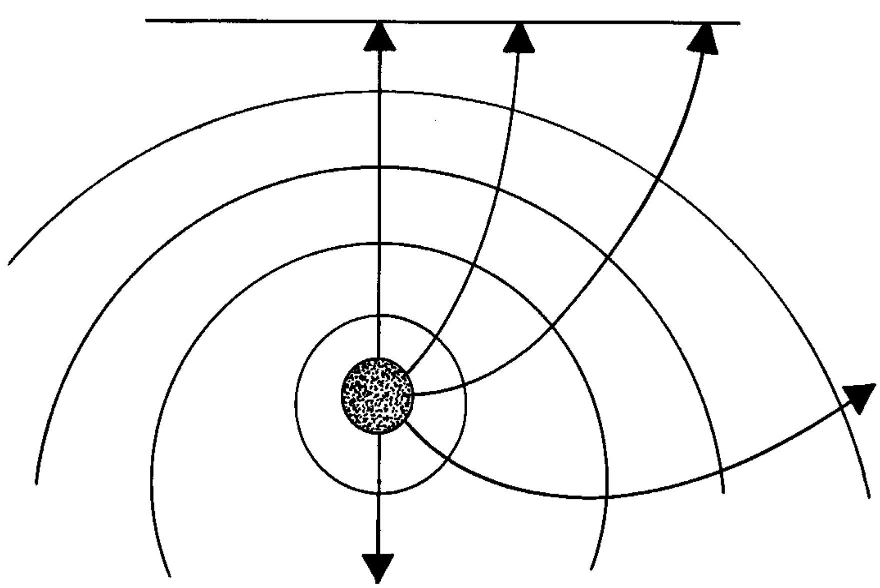
ANALYSIS: (a) From an energy balance on the container, $q = \dot{E}_g$ and from the first entry in Table 4.1,

$$q = \frac{2\pi D}{1 - D/4z} k\left(T_1 - T_2\right).$$

Hence,

$$T_1 = T_2 + \frac{q}{k}\frac{1 - D/4z}{2\pi D} = 20^\circ\text{C} + \frac{500\text{W}}{0.52\frac{\text{W}}{\text{m}\cdot\text{K}}}\frac{1 - 2\text{m}/40\text{m}}{2\pi\,(2\text{m})} = 92.7^\circ\text{C} \qquad <$$

(b) The isotherms may be viewed as spherical surfaces whose center moves downward with increasing radius. The surface of the soil is an isotherm for which the center is at z = ∞.

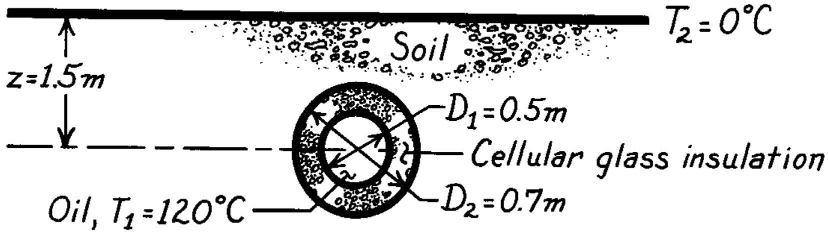


PROBLEM 4.15

KNOWN: Temperature, diameter and burial depth of an insulated pipe.

FIND: Heat loss per unit length of pipe.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction through insulation, two-dimensional through soil, (3) Constant properties, (4) Negligible oil convection and pipe wall conduction resistances.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K; *Table A-3,* Cellular glass (365K): k = 0.069 W/m·K.

ANALYSIS: The heat rate can be expressed as

$$q = \frac{T_1 - T_2}{R_{tot}}$$

where the thermal resistance is $R_{tot} = R_{ins} + R_{soil}$. From Eq. 3.28,

$$R_{ins} = \frac{\ell n\left(D_2 / D_1\right)}{2p \;\; Lk_{ins}} = \frac{\ell n\left(0.7\text{m}/0.5\text{m}\right)}{2p \;\; L \times 0.069 \text{ W/m} \cdot \text{K}} = \frac{0.776\text{m} \cdot \text{K/W}}{L}.$$

From Eq. 4.27 and Table 4.1,

$$R_{soil} = \frac{1}{SK_{soil}} = \frac{\cosh^{-1}\left(2z/D_2\right)}{2p \;\; Lk_{soil}} = \frac{\cosh^{-1}\left(3/0.7\right)}{2p \times \left(0.52 \text{ W/m} \cdot \text{K}\right) L} = \frac{0.653}{L} \text{m} \cdot \text{K/W}.$$

Hence,

$$q = \frac{(120 - 0)^{\circ} \text{ C}}{\frac{1}{L}\left(0.776 + 0.653\right)\frac{\text{m} \cdot \text{K}}{\text{W}}} = 84 \frac{\text{W}}{\text{m}} \times L$$

$$q' = q/L = 84 \text{ W/m}.$$ <

COMMENTS: (1) Contributions of the soil and insulation to the total resistance are approximately the same. The heat loss may be reduced by burying the pipe deeper or adding more insulation.

(2) The convection resistance associated with the oil flow through the pipe may be significant, in which case the foregoing result would overestimate the heat loss. A calculation of this resistance may be based on results presented in Chapter 8.

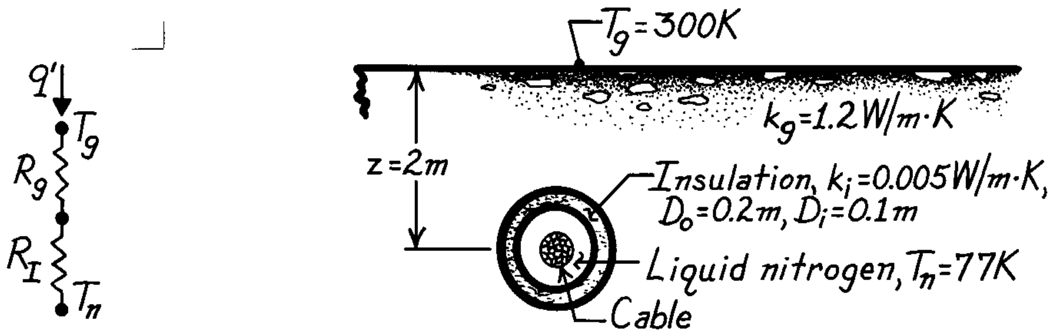
(3) Since z > 3D/2, the shape factor for the soil can also be evaluated from $S = 2\pi L/\ell n\,(4z/D)$ of Table 4.1, and an equivalent result is obtained.

PROBLEM 4.16

KNOWN: Operating conditions of a buried superconducting cable.

FIND: Required cooling load.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Two-dimensional conduction in soil, (4) One-dimensional conduction in insulation.

ANALYSIS: The heat rate per unit length is

$$q' = \frac{T_g - T_n}{R'_g + R'_I}$$

$$q' = \frac{T_g - T_n}{\left[k_g\left(2\pi/\ln\left(4z/D_o\right)\right)\right]^{-1} + \ln\left(D_o/D_i\right)/2\pi\; k_i}$$

where Tables 3.3 and 4.1 have been used to evaluate the insulation and ground resistances, respectively. Hence,

$$q' = \frac{\left(300-77\right)\text{K}}{\left[\left(1.2\ \text{W/m}\cdot\text{K}\right)\left(2\pi\ /\ln\left(8/0.2\right)\right)\right]^{-1} + \ln\left(2\right)/\ 2\pi\times 0.005\ \text{W/m}\cdot\text{K}}$$

$$q' = \frac{223\ \text{K}}{\left(0.489+22.064\right)\text{m}\cdot\text{K/W}}$$

$$q' = 9.9\ \text{W/m}. \qquad <$$

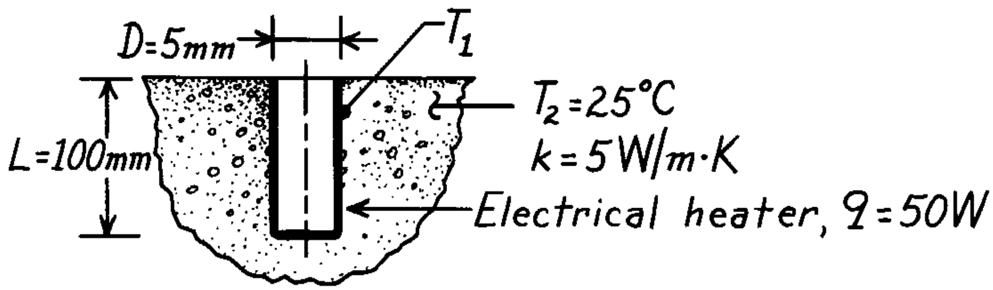
COMMENTS: The heat gain is small and the dominant contribution to the thermal resistance is made by the insulation.

PROBLEM 4.17

KNOWN: Electrical heater of cylindrical shape inserted into a hole drilled normal to the surface of a large block of material with prescribed thermal conductivity.

FIND: Temperature reached when heater dissipates 50 W with the block at 25°C.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Block approximates semi-infinite medium with constant properties, (3) Negligible heat loss to surroundings above block surface, (4) Heater can be approximated as isothermal at T_1.

ANALYSIS: The temperature of the heater surface follows from the rate equation written as

$$T_1 = T_2 + q/kS$$

where S can be estimated from the conduction shape factor given in Table 4.1 for a "vertical cylinder in a semi-infinite medium,"

$$S = 2\pi L/\ell n(4L/D).$$

Substituting numerical values, find

$$S = 2\pi \times 0.1\text{m}/\ell n\left[\frac{4\times 0.1\text{m}}{0.005\text{m}}\right] = 0.143\text{m}.$$

The temperature of the heater is then

$$T_1 = 25°\text{C} + 50\ \text{W}/(5\ \text{W/m}\cdot\text{K} \times 0.143\text{m}) = 94.9°\text{C}.$$ <

COMMENTS: (1) Note that the heater has L >> D, which is a requirement of the shape factor expression.

(2) Our calculation presumes there is negligible thermal contact resistance between the heater and the medium. In practice, this would not be the case unless a conducting paste were used.

(3) Since L >> D, assumption (3) is reasonable.

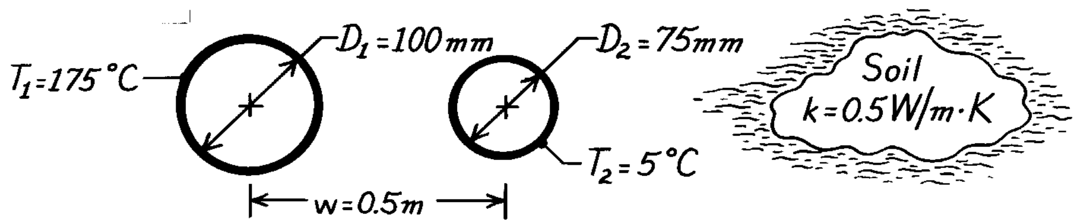
(4) This configuration has been used to determine the thermal conductivity of materials from measurement of q and T_1.

PROBLEM 4.18

KNOWN: Surface temperatures of two parallel pipe lines buried in soil.

FIND: Heat transfer per unit length between the pipe lines.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) Pipe lines are buried very deeply, approximating burial in an infinite medium, (5) Pipe length >> D_1 or D_2 and w > D_1 or D_2.

ANALYSIS: The heat transfer rate per unit length from the hot pipe to the cool pipe is

$$q' = \frac{q}{L} = \frac{S}{L} k \left(T_1 - T_2\right).$$

The shape factor S for this configuration is given in Table 4.1 as

$$S = \frac{2p L}{\cosh^{-1}\left[\frac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right]}.$$

Substituting numerical values,

$$\frac{S}{L} = 2p \;/\cosh^{-1}\left[\frac{4\times(0.5\text{m})^2 - (0.1\text{m})^2 - (0.075\text{m})^2}{2\times 0.1\text{m}\times 0.075\text{m}}\right] = 2p \;/\cosh^{-1}(65.63)$$

$$\frac{S}{L} = 2p \;/4.88 = 1.29.$$

Hence, the heat rate per unit length is

$$q' = 1.29\times 0.5\text{W/m}\cdot\text{K}\left(175 - 5\right)^{\circ}\text{C} = 110\ \text{W/m}.$$ <

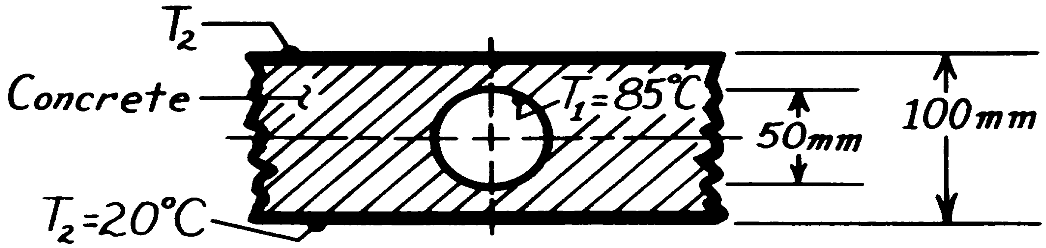
COMMENTS: The heat gain to the cooler pipe line will be larger than 110 W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

PROBLEM 4.19

KNOWN: Tube embedded in the center plane of a concrete slab.

FIND: (a) The shape factor and heat transfer rate per unit length using the appropriate tabulated relation, (b) Shape factor using flux plot method.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties, (4) Concrete slab infinitely long in horizontal plane, L >> z.

PROPERTIES: *Table A-3*, Concrete, stone mix (300K): k = 1.4 W/m·K.

ANALYSIS: (a) If we relax the restriction that z >> D/2, the embedded tube-slab system corresponds to the fifth case of Table 4.1. Hence,

$$S = \frac{2\pi L}{\ell n\left(8z/\pi D\right)}$$

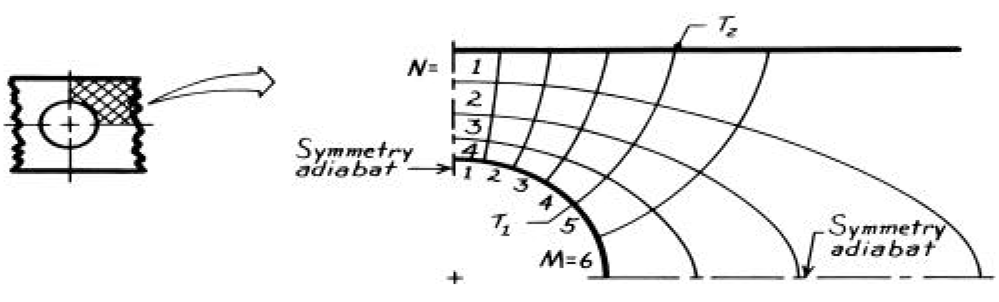
where L is the length of the system normal to the page, z is the half-thickness of the slab and D is the diameter of the tube. Substituting numerical values, find

$$S = 2\pi L/\ell n\left(8\times 50\text{mm}/\pi 50\text{mm}\right) = 6.72\text{L}.$$

Hence, the heat rate per unit length is

$$q' = \frac{q}{L} = \frac{S}{L}k\left(T_1 - T_2\right) = 6.72\times 1.4\frac{W}{m\cdot K}\left(85-20\right)^\circ C = 612\ W.$$

(b) To find the shape factor using the flux plot method, first identify the symmetrical section bounded by the symmetry adiabats formed by the horizontal and vertical center lines. Selecting four temperature increments (N = 4), the flux plot can then be constructed.



From Eq. 4.26, the shape factor of the symmetrical section is

$$S_o = ML/N = 6L/4 = 1.5L.$$

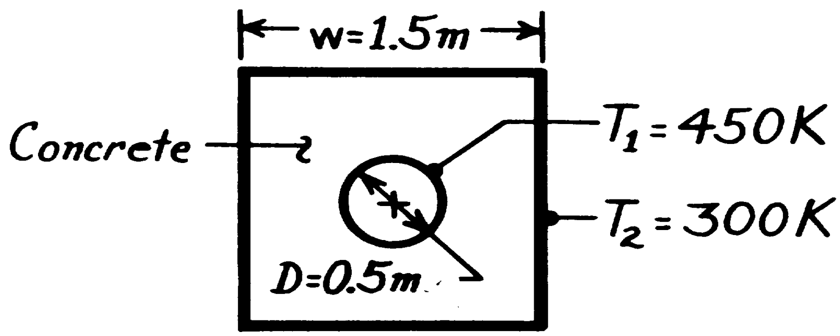
For the tube-slab system, it follows that $S = 4S_o = 6.0L$, which compares favorably with the result obtained from the shape factor relation.

PROBLEM 4.20

KNOWN: Dimensions and boundary temperatures of a steam pipe embedded in a concrete casing.

FIND: Heat loss per unit length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible steam side convection resistance, pipe wall resistance and contact resistance (T_1 = 450K), (3) Constant properties.

PROPERTIES: *Table A-3*, Concrete (300K): k = 1.4 W/m·K.

ANALYSIS: The heat rate can be expressed as

$$q = Sk\Delta T_{1\text{-}2} = Sk\left(T_1 - T_2\right)$$

From Table 4.1, the shape factor is

$$S = \frac{2\pi L}{\ell n\left[\frac{1.08\ w}{D}\right]}.$$

Hence,

$$q' = \left[\frac{q}{L}\right] = \frac{2\pi k\left(T_1 - T_2\right)}{\ell n\left[\frac{1.08\ w}{D}\right]}$$

$$q' = \frac{2\pi \times 1.4\text{W/m}\cdot\text{K} \times \left(450 - 300\right)\text{K}}{\ell n\left[\frac{1.08 \times 1.5\text{m}}{0.5\text{m}}\right]} = 1122\ \text{W/m}.$$ <

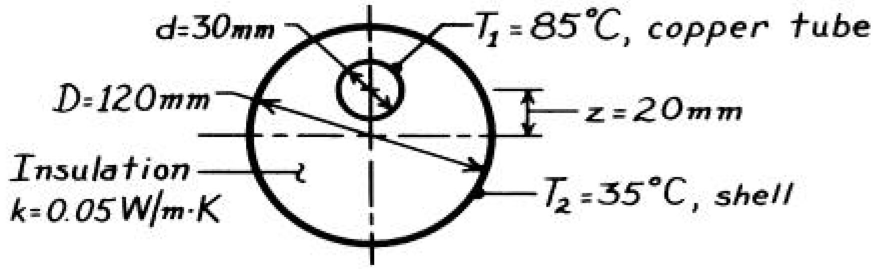
COMMENTS: Having neglected the steam side convection resistance, the pipe wall resistance, and the contact resistance, the foregoing result overestimates the actual heat loss.

PROBLEM 4.21

KNOWN: Thin-walled copper tube enclosed by an eccentric cylindrical shell; intervening space filled with insulation.

FIND: Heat loss per unit length of tube; compare result with that of a concentric tube-shell arrangement.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Thermal resistances of copper tube wall and outer shell wall are negligible, (4) Two-dimensional conduction in insulation.

ANALYSIS: The heat loss per unit length written in terms of the shape factor S is $q' = k(S/\ell)(T_1 - T_2)$ and from Table 4.1 for this geometry,

$$\frac{S}{\ell} = 2p\ /\cosh^{-1}\left[\frac{D^2 + d^2 - 4z^2}{2Dd}\right].$$

Substituting numerical values, all dimensions in mm,

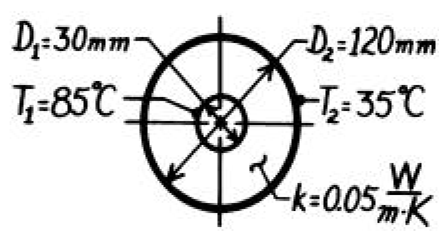
$$\frac{S}{\ell} = 2p\ /\cosh^{-1}\left[\frac{120^2 + 30^2 - 4(20)^2}{2\times 120\times 30}\right] = 2p\ /\cosh^{-1}(1.903) = 4.991.$$

Hence, the heat loss is

$$q' = 0.05\text{W/m}\cdot\text{K}\times 4.991(85-35)^{\circ}\text{C} = 12.5\ \text{W/m}.$$ <

If the copper tube were concentric with the shell, but all other conditions were the same, the heat loss would be

$$q'_c = \frac{2p\ k(T_1 - T_2)}{\ell n(D_2/D_1)}$$



using Eq. 3.27. Substituting numerical values,

$$q'_c = 2p\times 0.05\frac{\text{W}}{\text{m}\cdot\text{K}}(85-35)^{\circ}\text{C}/\ell n\left({}^{120}/_{30}\right)$$

$$q'_c = 11.3\ \text{W/m}.$$

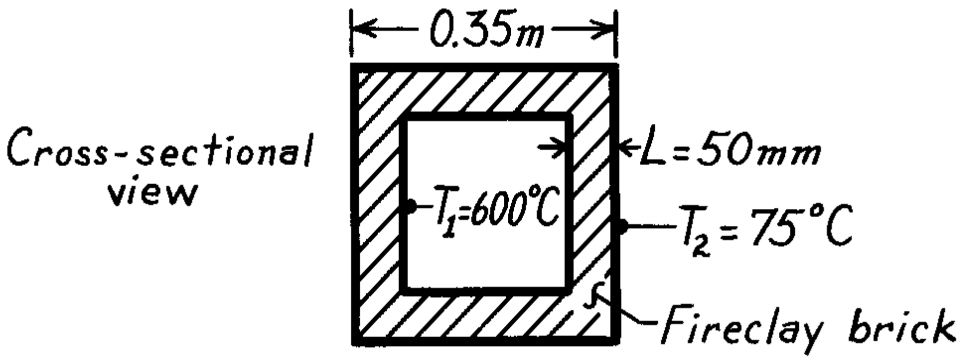
COMMENTS: As expected, the heat loss with the eccentric arrangement is larger than that for the concentric arrangement. The effect of the eccentricity is to increase the heat loss by (12.5 - 11.3)/11.3 ≈ 11%.

PROBLEM 4.22

KNOWN: Cubical furnace, 350 mm external dimensions, with 50 mm thick walls.

FIND: The heat loss, q(W).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

PROPERTIES: *Table A-3*, Fireclay brick $\left(\overline{T} = (T_1 + T_2)/2 = 610\text{K}\right)$: $k \approx 1.1$ W/m·K.

ANALYSIS: Using relations for the shape factor from Table 4.1,

Plane Walls (6) $\quad S_W = \dfrac{A}{L} = \dfrac{0.25 \times 0.25\text{m}^2}{0.05\text{m}} = 1.25\text{m}$

Edges (12) $\quad S_E = 0.54D = 0.54 \times 0.25\text{m} = 0.14\text{m}$

Corners (8) $\quad S_C = 0.15L = 0.15 \times 0.05\text{m} = 0.008\text{m}.$

The heat rate in terms of the shape factors is

$$q = kS(T_1 - T_2) = k(6S_W + 12S_E + 8S_C)\ (T_1 - T_2)$$

$$q = 1.1\frac{\text{W}}{\text{m}\cdot\text{K}}(6\times1.25\text{m}+12\times0.14\text{m}+0.15\times0.008\text{m})\ (600-75)^{\circ}\text{C}$$

$$q = 5.30\ \text{kW}.$$ <

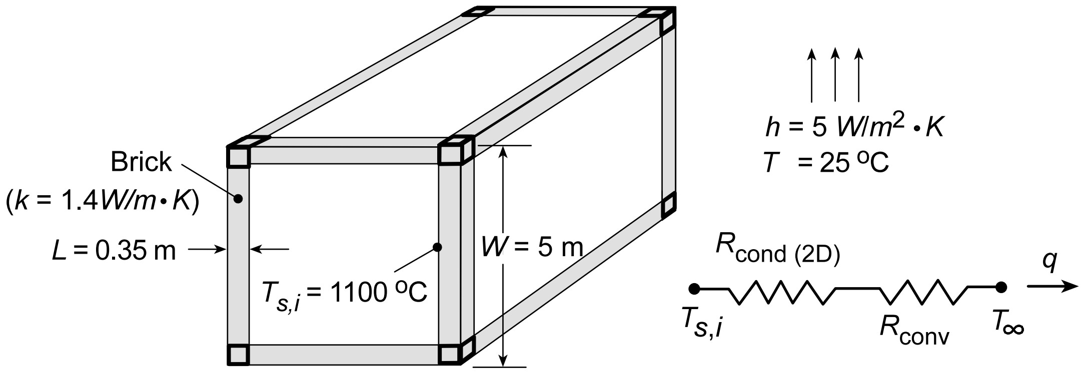
COMMENTS: Note that the restrictions for S_E and S_C have been met.

PROBLEM 4.23

KNOWN: Dimensions, thermal conductivity and inner surface temperature of furnace wall. Ambient conditions.

FIND: Heat loss.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Uniform convection coefficient over entire outer surface of container.

ANALYSIS: From the thermal circuit, the heat loss is

$$q = \frac{T_{s,i} - T_\infty}{R_{cond(2D)} + R_{conv}}$$

where $R_{conv} = 1/hA_{s,o} = 1/6(hW^2) = 1/6[5\ W/m^2\cdot K(5\ m)^2] = 0.00133\ K/W$. From Eq. (4.27), the two-dimensional conduction resistance is

$$R_{cond(2D)} = \frac{1}{Sk}$$

where the shape factor S must include the effects of conduction through the 8 corners, 12 edges and 6 plane walls. Hence, using the relations for Cases 8 and 9 of Table 4.1,

$$S = 8(0.15L) + 12\times 0.54(W - 2L) + 6A_{s,i}/L$$

where $A_{s,i} = (W - 2L)^2$. Hence,

$$S = [8(0.15\times 0.35) + 12\times 0.54(4.30) + 6(52.83)]m$$

$$S = (0.42 + 27.86 + 316.98)m = 345.26m$$

and $R_{cond(2D)} = 1/(345.26\ m \times 1.4\ W/m\cdot K) = 0.00207\ K/W$. Hence

$$q = \frac{(1100 - 25)^\circ C}{(0.00207 + 0.00133)K/W} = 316\,kW$$ <

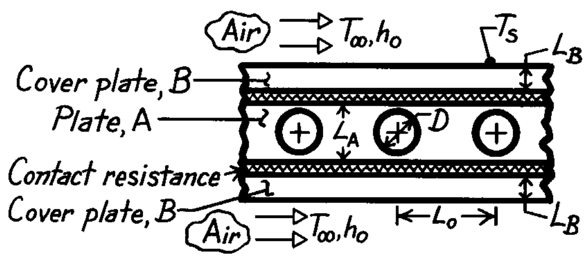
COMMENTS: The heat loss is extremely large and measures should be taken to insulate the furnace.

PROBLEM 4.24

KNOWN: Platen heated by passage of hot fluid in poor thermal contact with cover plates exposed to cooler ambient air.

FIND: (a) Heat rate per unit thickness from each channel, q_i', (b) Surface temperature of cover plate, T_s, (c) q_i' and T_s if lower surface is perfectly insulated, (d) Effect of changing centerline spacing on q_i' and T_s

SCHEMATIC:



$D=15$ mm $\quad L_o=60$ mm

$L_A=30$ mm $\quad L_B=7.5$ mm

$T_i=150°C \quad h_i=1000\ W/m^2{\cdot}K$

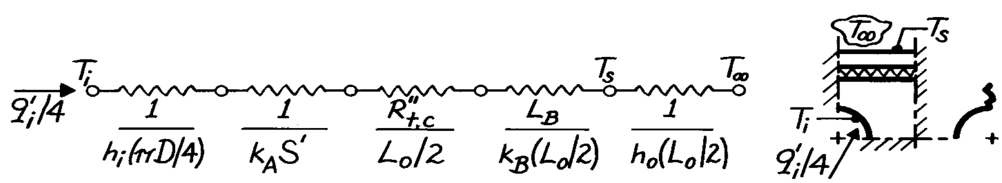
$T_\infty=25°C \quad h_o=200\ W/m^2{\cdot}K$

$k_A=20$ W/m·K $\quad k_B=75$ W/m·K

$R''_{t,c} = 2.0\times10^{-4}\ m^2\cdot K/W$

ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction in platen, but one-dimensional in coverplate, (3) Temperature of interfaces between A and B is uniform, (4) Constant properties.

ANALYSIS: (a) The heat rate per unit thickness from each channel can be determined from the following thermal circuit representing the quarter section shown.



The value for the shape factor is $S'=1.06$ as determined from the flux plot shown on the next page. Hence, the heat rate is

$$q_i' = 4\left(T_i - T_\infty\right)/R'_{tot} \tag{1}$$

$$R'_{tot} = [1/1000\ W/m^2\cdot K\left(p0.015m/4\right) + 1/20\ W/m\cdot K\times1.06$$
$$+2.0\times10^{-4}m^2\cdot K/W\left(0.060m/2\right) + 0.0075m/75\ W/m\cdot K\left(0.060m/2\right)$$
$$+1/200\ W/m^2\cdot K\left(0.060m/2\right)]$$

$$R'_{tot} = \left[0.085 + 0.047 + 0.0067 + 0.0033 + 0.1667\right]m\cdot K/W$$
$$R'_{tot} = 0.309\ m\cdot K/W$$

$$q_i' = 4\left(150-25\right)K/0.309\ m\cdot K/W = 1.62\ kW/m. \quad <$$

(b) The surface temperature of the cover plate also follows from the thermal circuit as

$$q_i'/4 = \frac{T_s - T_\infty}{1/h_o\left(L_o/2\right)} \tag{2}$$

Continued …..

PROBLEM 4.24 (Cont.)

$$T_s = T_\infty + \frac{q'_i}{4}\frac{1}{h_o(L_o/2)} = 25^\circ C + \frac{1.62\ kW}{4} \times 0.167\ m \cdot K/W$$

$$T_s = 25^\circ C + 67.6^\circ C \approx 93^\circ C.$$ <

(c,d) The effect of the centerline spacing on q'_i and T_s can be understood by examining the relative magnitudes of the thermal resistances. The dominant resistance is that due to the ambient air convection process which is inversely related to the spacing L_o. Hence, from Eq. (1), the heat rate will increase nearly linearly with an increase in L_o,

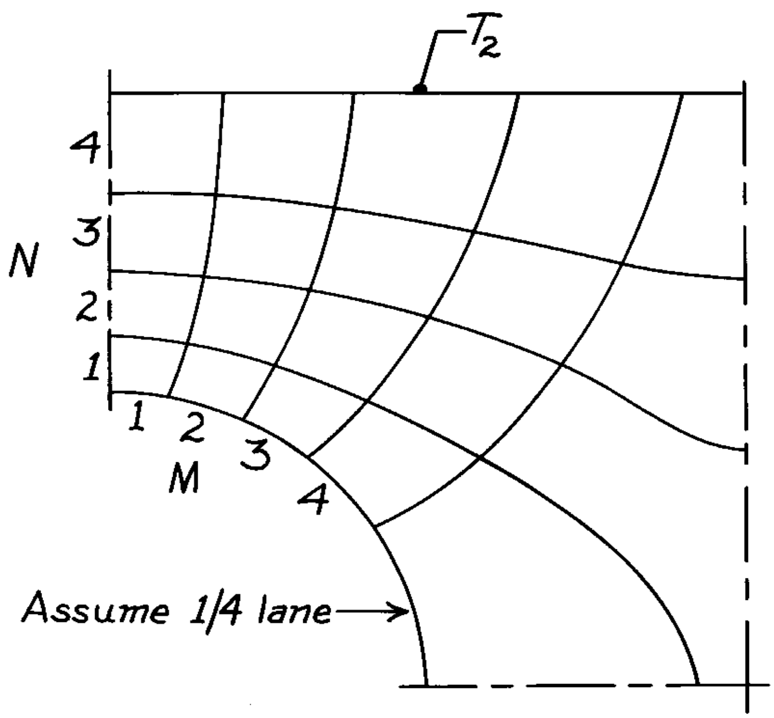
$$q'_i \sim \frac{1}{R'_{tot}} \approx \frac{1}{1/h_o(L_o/2)} \sim L_o.$$

From Eq. (2), find

$$\Delta T = T_s - T_\infty = \frac{q'_i}{4}\frac{1}{h_o(L_o/2)} \sim q'_i \cdot L_o^{-1} \sim L_o \cdot L_o^{-1} \approx 1.$$

Hence we conclude that ΔT will not increase with a change in L_o. Does this seem reasonable? What effect does L_o have on Assumptions (2) and (3)?

If the lower surface were insulated, the heat rate would be decreased nearly by half. This follows again from the fact that the overall resistance is dominated by the surface convection process. The temperature difference, T_s - T_∞, would only increase slightly.



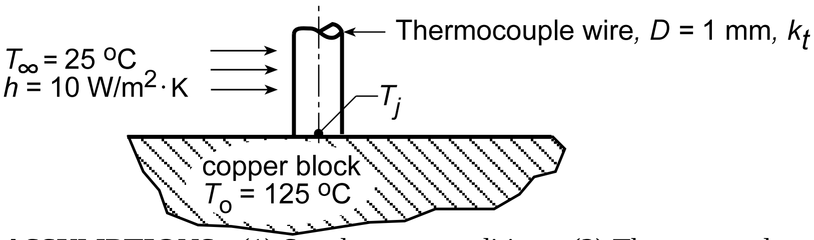
$S' = S/l = M/N = 4.25/4 = 1.06$

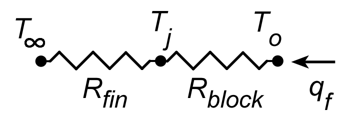
PROBLEM 4.25

KNOWN: Long constantan wire butt-welded to a large copper block forming a thermocouple junction on the surface of the block.

FIND: (a) The measurement error (T_j - T_o) for the thermocouple for prescribed conditions, and (b) Compute and plot (T_j - T_o) for h = 5, 10 and 25 W/m^2·K for block thermal conductivity $15 \le k \le 400$ W/m·K. When is it advantageous to use smaller diameter wire?

SCHEMATIC:





ASSUMPTIONS: (1) Steady-state conditions, (2) Thermocouple wire behaves as a fin with constant heat transfer coefficient, (3) Copper block has uniform temperature, except in the vicinity of the junction.

PROPERTIES: *Table A-1*, Copper (pure, 400 K), k_b = 393 W/m·K; Constantan (350 K), $k_t \approx 25$ W/m·K.

ANALYSIS: The thermocouple wire behaves as a long fin permitting heat to flow from the surface thereby depressing the sensing junction temperature below that of the block T_o. In the block, heat flows into the circular region of the wire-block interface; the thermal resistance to heat flow within the block is approximated as a disk of diameter D on a semi-infinite medium (k_b, T_o). The thermocouple-block combination can be represented by a thermal circuit as shown above. The thermal resistance of the fin follows from the heat rate expression for an infinite fin, $R_{fin} = (hPk_tA_c)^{-1/2}$.

From Table 4.1, the shape factor for the disk-on-a-semi-infinite medium is given as S = 2D and hence $R_{block} = 1/k_bS = 1/2k_bD$. From the thermal circuit,

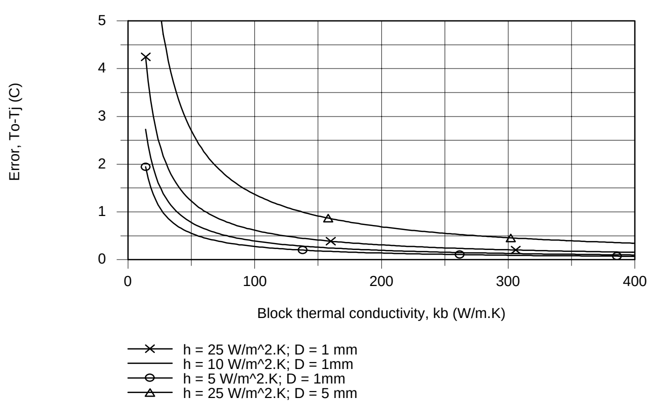
$$T_o - T_j = \frac{R_{block}}{R_{fin} + R_{block}}\left(T_o - T_\infty\right) = \frac{1.27}{1273 + 1.27}(125-25)^\circ\text{C} \approx 0.001(125-25)^\circ\text{C} = 0.1^\circ\text{C}. <$$

with $P = \pi D$ and $A_c = \pi D^2/4$ and the thermal resistances as

$$R_{fin} = \left(10\,\text{W/m}^2\cdot\text{K}\,(\pi/4)\,25\,\text{W/m}\cdot\text{K} \times \left(1\times10^{-3}\,\text{m}\right)^3\right)^{-1/2} = 1273\,\text{K/W}$$

$$R_{block} = (1/2) \times 393\,\text{W/m}\cdot\text{K} \times 10^{-3}\,\text{m} = 1.27\,\text{K/W}.$$

(b) We keyed the above equations into the IHT workspace, performed a sweep on k_b for selected values of h and created the plot shown. When the block thermal conductivity is low, the error (T_o - T_j) is larger, increasing with increasing convection coefficient. A smaller diameter wire will be advantageous for low values of k_b and higher values of h.

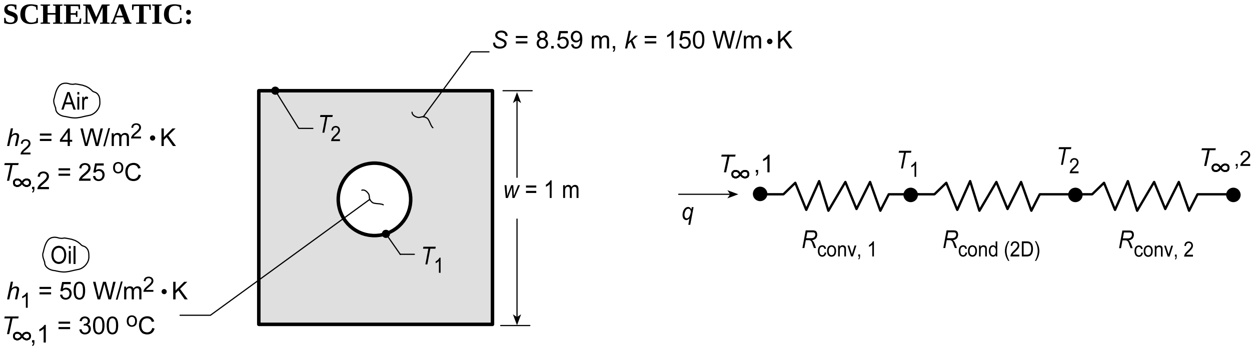


PROBLEM 4.26

KNOWN: Dimensions, shape factor, and thermal conductivity of square rod with drilled interior hole. Interior and exterior convection conditions.

FIND: Heat rate and surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) Uniform convection coefficients at inner and outer surfaces.

ANALYSIS: The heat loss can be expressed as

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{R_{conv,1} + R_{cond(2D)} + R_{conv,2}}$$

where

$$R_{conv,1} = (h_1 \pi D_1 L)^{-1} = \left(50\,\mathrm{W/m^2 \cdot K} \times \pi \times 0.25\,\mathrm{m} \times 2\,\mathrm{m}\right)^{-1} = 0.01273\,\mathrm{K/W}$$

$$R_{cond(2D)} = (Sk)^{-1} = (8.59\,\mathrm{m} \times 150\,\mathrm{W/m \cdot K})^{-1} = 0.00078\,\mathrm{K/W}$$

$$R_{conv,2} = (h_2 \times 4wL)^{-1} = \left(4\,\mathrm{W/m^2 \cdot K} \times 4\,\mathrm{m} \times 1\mathrm{m}\right)^{-1} = 0.0625\,\mathrm{K/W}$$

Hence,

$$q = \frac{(300 - 25)^\circ\mathrm{C}}{0.076\,\mathrm{K/W}} = 3.62\,\mathrm{kW} \qquad <$$

$$T_1 = T_{\infty,1} - qR_{conv,1} = 300^\circ\mathrm{C} - 46^\circ\mathrm{C} = 254^\circ\mathrm{C} \qquad <$$

$$T_2 = T_{\infty,2} + qR_{conv,2} = 25^\circ\mathrm{C} + 226^\circ\mathrm{C} = 251^\circ\mathrm{C} \qquad <$$

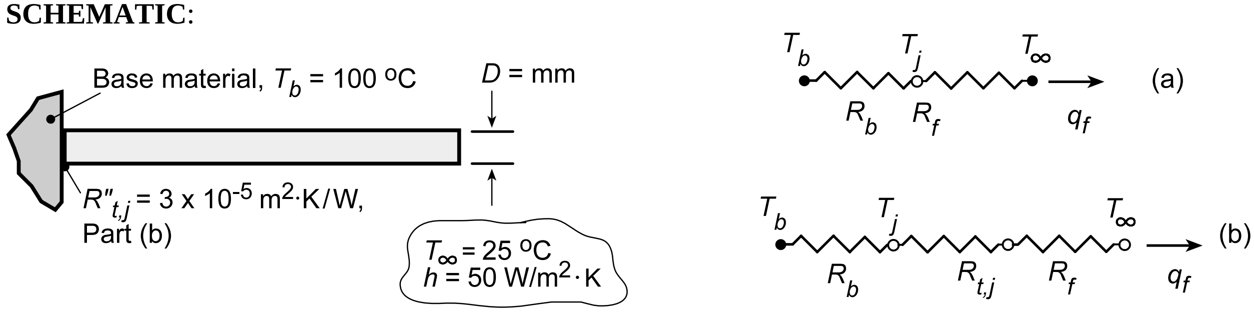
COMMENTS: The largest resistance is associated with convection at the outer surface, and the conduction resistance is much smaller than both convection resistances. Hence, $(T_2 - T_{\infty,2}) > (T_{\infty,1} - T_1) >> (T_1 - T_2)$.

PROBLEM 4.27

KNOWN: Long fin of aluminum alloy with prescribed convection coefficient attached to different base materials (aluminum alloy or stainless steel) with and without thermal contact resistance $R''_{t,j}$ at the junction.

FIND: (a) Heat rate q_f and junction temperature T_j for base materials of aluminum and stainless steel, (b) Repeat calculations considering thermal contact resistance, $R''_{t,j}$, and (c) Plot as a function of h for the range $10 \le h \le 1000\ W/m^2 \cdot K$ for each base material.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Infinite fin.

PROPERTIES: (Given) Aluminum alloy, k = 240 W/m·K, Stainless steel, k = 15 W/m·K.

ANALYSIS: (a,b) From the thermal circuits, the heat rate and junction temperature are

$$q_f = \frac{T_b - T_\infty}{R_{tot}} = \frac{T_b - T_\infty}{R_b + R_{t,j} + R_f} \qquad (1)$$

$$T_j = T_\infty + q_f R_f \qquad (2)$$

and, with $P = \pi D$ and $A_c = \pi D^2/4$, from Tables 4.1 and 3.4 find

$$R_b = 1/Sk_b = 1/(2Dk_b) = (2 \times 0.005\,m \times k_b)^{-1}$$

$$R_{t,j} = R''_{t,j}/A_c = 3\times10^{-5} m^2 \cdot K/W \Big/ \pi(0.005\,m)^2/4 = 1.528\,K/W$$

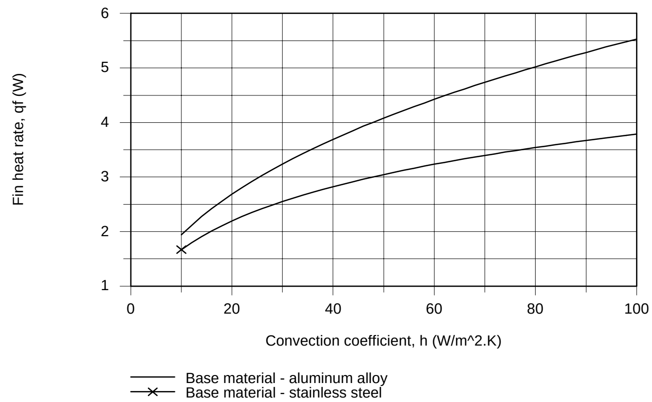
$$R_f = (hPkA_c)^{-1/2} = \left[50\,W/m^2 \cdot K\,\pi^2 (0.005\,m)^2\, 240\,W/m \cdot K\Big/4\right]^{-1/2} = 16.4\,K/W$$

		Without $R''_{t,j}$		With $R''_{t,j}$	
Base	R_b (K/W)	q_f (W)	T_j (°C)	q_f (W)	T_j (°C)
Al alloy	0.417	4.46	98.2	4.09	92.1
St. steel	6.667	3.26	78.4	3.05	75.1

(c) We used the *IHT Model for Extended Surfaces, Performance Calculations, Rectangular Pin Fin* to calculate q_f for $10 \le h \le 100\ W/m^2 \cdot K$ by replacing R''_{tc} (thermal resistance at fin base) by the sum of the contact and spreading resistances, $R''_{t,j} + R''_b$.

Continued...

PROBLEM 4.27 (Cont.)



COMMENTS: (1) From part (a), the aluminum alloy base material has negligible effect on the fin heat rate and depresses the base temperature by only 2°C. The effect of the stainless steel base material is substantial, reducing the heat rate by 27% and depressing the junction temperature by 25°C.

(2) The contact resistance reduces the heat rate and increases the temperature depression relatively more with the aluminum alloy base.

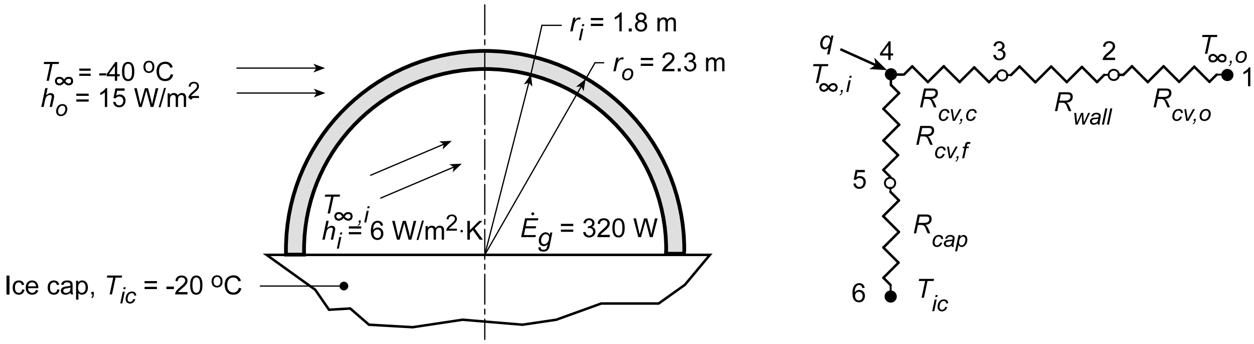
(3) From the plot of q_f vs. h, note that at low values of h, the heat rates are nearly the same for both materials since the fin is the dominant resistance. As h increases, the effect of R''_b becomes more important.

PROBLEM 4.28

KNOWN: Igloo constructed in hemispheric shape sits on ice cap; igloo wall thickness and inside/outside convection coefficients (h_i, h_o) are prescribed.

FIND: (a) Inside air temperature $T_{\infty,i}$ when outside air temperature is $T_{\infty,o}$ = -40°C assuming occupants provide 320 W within igloo, (b) Perform parameter sensitivity analysis to determine which variables have significant effect on T_i.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Convection coefficient is the same on floor and ceiling of igloo, (3) Floor and ceiling are at uniform temperature, (4) Floor-ice cap resembles disk on semi-infinite medium, (5) One-dimensional conduction through igloo walls.

PROPERTIES: Ice and compacted snow (given): k = 0.15 W/m·K.

ANALYSIS: (a) The thermal circuit representing the heat loss from the igloo to the outside air and through the floor to the ice cap is shown above. The heat loss is

$$q = \frac{T_{\infty,i} - T_{\infty,o}}{R_{cv,c} + R_{wall} + R_{cv,o}} + \frac{T_{\infty,i} - T_{ic}}{R_{cv,f} + R_{cap}}.$$

Convection, ceiling:
$$R_{cv,c} = \frac{2}{h_i\left(4\pi r_i^2\right)} = \frac{2}{6\,\mathrm{W/m^2\cdot K} \times 4\pi\,(1.8\,\mathrm{m})^2} = 0.00819\,\mathrm{K/W}$$

Convection, outside:
$$R_{cv,o} = \frac{2}{h_o\left(4\pi r_o^2\right)} = \frac{2}{15\,\mathrm{W/m^2\cdot K} \times 4\pi\,(2.3\,\mathrm{m})^2} = 0.00201\,\mathrm{K/W}$$

Convection, floor:
$$R_{cv,f} = \frac{1}{h_i\left(\pi r_i^2\right)} = \frac{1}{6\,\mathrm{W/m^2\cdot K} \times \pi\,(1.8\,\mathrm{m})^2} = 0.01637\,\mathrm{K/W}$$

Conduction, wall:
$$R_{wall} = 2\left[\frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)\right] = 2\left[\frac{1}{4\pi \times 0.15\,\mathrm{W/m\cdot K}}\left(\frac{1}{1.8} - \frac{1}{2.3}\right)\mathrm{m}\right] = 0.1281\,\mathrm{K/W}$$

Conduction, ice cap:
$$R_{cap} = \frac{1}{kS} = \frac{1}{4kr_i} = \frac{1}{4 \times 0.15\,\mathrm{W/m\cdot K} \times 1.8\,\mathrm{m}} = 0.9259\,\mathrm{K/W}$$

where S was determined from the shape factor of Table 4.1. Hence,

$$q = 320\,\mathrm{W} = \frac{T_{\infty,i} - (-40)^\circ\mathrm{C}}{(0.00818 + 0.1281 + 0.0020)\,\mathrm{K/W}} + \frac{T_{\infty,i} - (-20)^\circ\mathrm{C}}{(0.01637 + 0.9259)\,\mathrm{K/W}}$$

$$320\ \mathrm{W} = 7.232(T_{\infty,i} + 40) + 1.06(T_{\infty,i} + 20) \qquad T_{\infty,i} = 1.1^\circ\mathrm{C}. \qquad <$$

Continued...

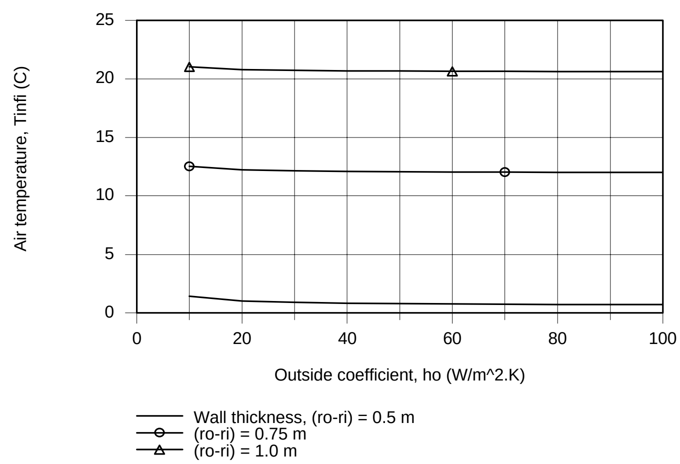
PROBLEM 4.28 (Cont.)

(b) Begin the parameter sensitivity analysis to determine important variables which have a significant influence on the inside air temperature by examining the thermal resistances associated with the processes present in the system and represented by the network.

Process	Symbols		Value (K/W)
Convection, outside	$R_{cv,o}$	R21	0.0020
Conduction, wall	R_{wall}	R32	0.1281
Convection, ceiling	$R_{cv,c}$	R43	0.0082
Convection, floor	$R_{cv,f}$	R54	0.0164
Conduction, ice cap	R_{cap}	R65	0.9259

It follows that the convection resistances are negligible relative to the conduction resistance across the igloo wall. As such, only changes to the wall thickness will have an appreciable effect on the inside air temperature relative to the outside ambient air conditions. We don't want to make the igloo walls thinner and thereby allow the air temperature to dip below freezing for the prescribed environmental conditions.

Using the *IHT Thermal Resistance Network Model*, we used the circuit builder to construct the network and perform the energy balances to obtain the inside air temperature as a function of the outside convection coefficient for selected increased thicknesses of the wall.



COMMENTS: (1) From the plot, we can see that the influence of the outside air velocity which controls the outside convection coefficient h_o is negligible.

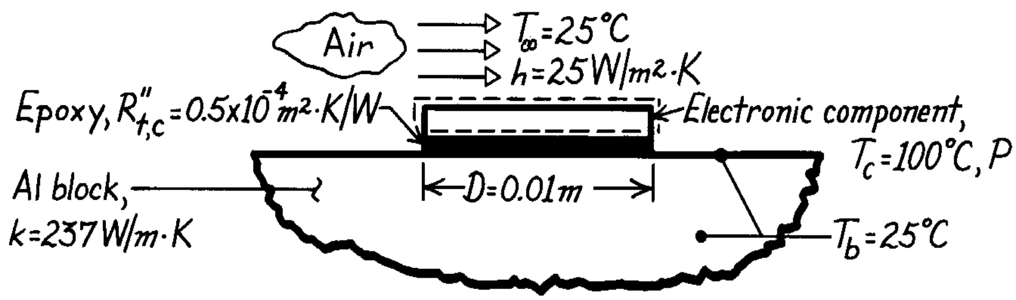
(2) The thickness of the igloo wall is the dominant thermal resistance controlling the inside air temperature.

PROBLEM 4.29

KNOWN: Diameter and maximum allowable temperature of an electronic component. Contact resistance between component and large aluminum heat sink. Temperature of heat sink and convection conditions at exposed component surface.

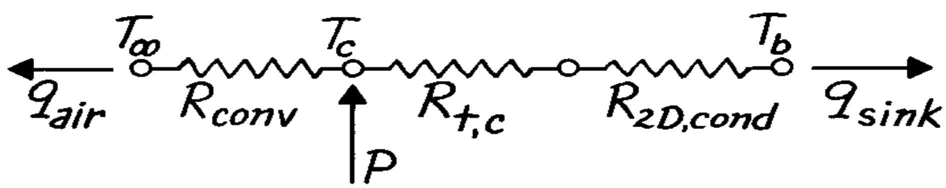
FIND: (a) Thermal circuit, (b) Maximum operating power of component.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Negligible heat loss from sides of chip.

ANALYSIS: (a) The thermal circuit is:



where $R_{2D,cond}$ is evaluated from the shape factor S = 2D of Table 4.1.

(b) Performing an energy balance for a control surface about the component,

$$P = q_{air} + q_{sink} = h\left(\pi\ D^2/4\right)\left(T_c - T_\infty\right) + \frac{T_c - T_b}{R''_{t,c}/\left(\pi\ D^2/4\right) + 1/2Dk}$$

$$P = 25\ \text{W/m}^2\cdot\text{K}\left(\pi/4\right)\left(0.01\ \text{m}\right)^2 75^\circ\text{C} + \frac{75^\circ\text{C}}{\left\{\left[0.5\times10^{-4}/\left(\pi/4\right)\left(0.01\right)^2\right] + \left(0.02\times237\right)^{-1}\right\}\text{K/W}}$$

$$P = 0.15\ \text{W} + \frac{75^\circ\text{C}}{\left(0.64+0.21\right)\text{K/W}} = 0.15\ \text{W} + 88.49\ \text{W} = 88.6\ \text{W}. \qquad <$$

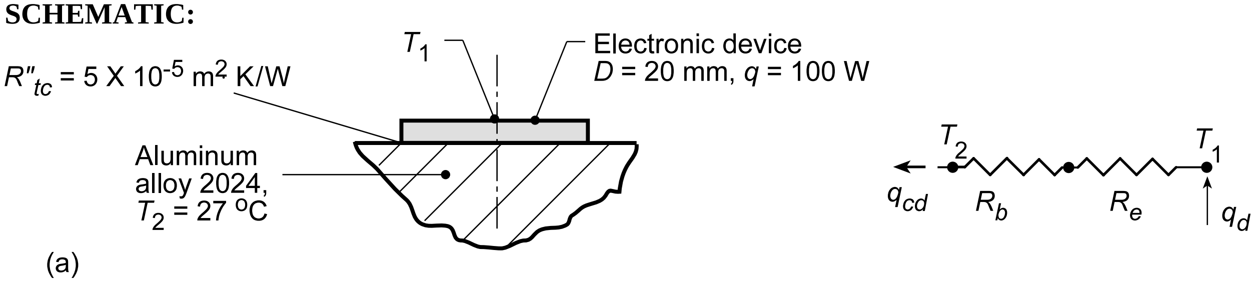
COMMENTS: The convection resistance is much larger than the cumulative contact and conduction resistance. Hence, virtually all of the heat dissipated in the component is transferred through the block. The two-dimensional conduction resistance is significantly underestimated by use of the shape factor S = 2D. Hence, the maximum allowable power is less than 88.6 W.

PROBLEM 4.30

KNOWN: Disc-shaped electronic devices dissipating 100 W mounted to aluminum alloy block with prescribed contact resistance.

FIND: (a) Temperature device will reach when block is at 27°C assuming all the power generated by the device is transferred by conduction to the block and (b) For the operating temperature found in part (a), the permissible operating power with a 30-pin fin heat sink.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Device is at uniform temperature, T_1, (3) Block behaves as semi-infinite medium.

PROPERTIES: *Table A.1*, Aluminum alloy 2024 (300 K): k = 177 W/m·K.

ANALYSIS: (a) The thermal circuit for the conduction heat flow between the device and the block shown in the above Schematic where R_e is the thermal contact resistance due to the epoxy-filled interface,

$$R_e = R''_{t,c}/A_c = R''_{t,c}\Big/\left(\pi D^2/4\right)$$

$$R_e = 5\times10^{-5}\ \mathrm{K\cdot m^2/W}\Big/\left(\pi\left(0.020\,\mathrm{m}\right)^2\right)\Big/4 = 0.159\,\mathrm{K/W}$$

The thermal resistance between the device and the block is given in terms of the conduction shape factor, Table 4.1, as

$$R_b = 1/Sk = 1/\left(2Dk\right)$$

$$R_b = 1/\left(2\times0.020\,\mathrm{m}\times177\,\mathrm{W/m\cdot K}\right) = 0.141\,\mathrm{K/W}$$

From the thermal circuit,

$$T_1 = T_2 + q_d\left(R_b + R_e\right)$$

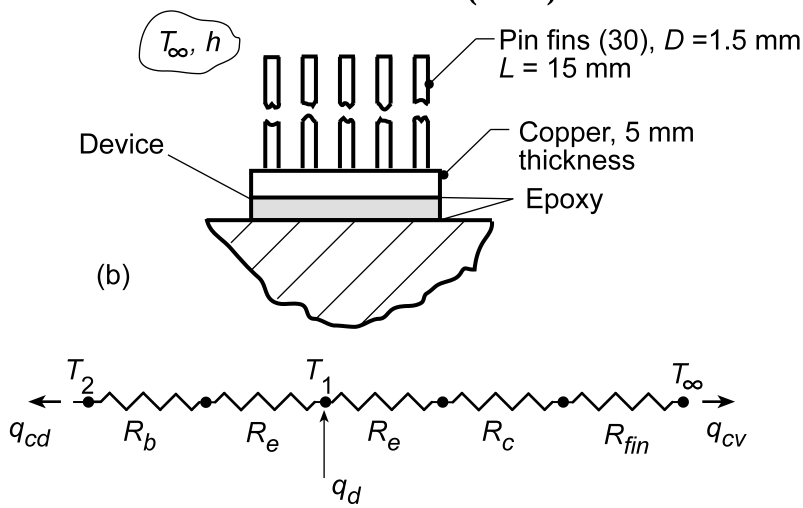
$$T_1 = 27^\circ\mathrm{C} + 100\,\mathrm{W}\left(0.141+0.159\right)\mathrm{K/W}$$

$$T_1 = 27^\circ\mathrm{C} + 30^\circ\mathrm{C} = 57^\circ\mathrm{C} \qquad <$$

(b) The schematic below shows the device with the 30-pin fin heat sink with fins and base material of copper (k = 400 W/m·K). The airstream temperature is 27°C and the convection coefficient is 1000 $W/m^2 \cdot K$.

Continued...

PROBLEM 4.30 (Cont.)



The thermal circuit for this system has two paths for the device power: to the block by conduction, q_{cd}, and to the ambient air by conduction to the fin array, q_{cv},

$$q_d = \frac{T_1 - T_2}{R_b + R_e} + \frac{T_1 - T_\infty}{R_e + R_c + R_{fin}} \tag{3}$$

where the thermal resistance of the fin base material is

$$R_c = \frac{L_c}{k_c A_c} = \frac{0.005\,\mathrm{m}}{400\,\mathrm{W/m\cdot K}\left(\pi 0.02^2/4\right)\mathrm{m}^2} = 0.03979\,\mathrm{K/W} \tag{4}$$

and R_{fin} represents the thermal resistance of the fin array (see Section 3.6.5),

$$R_{fin} = R_{t,o} = \frac{1}{\eta_o h A_t} \tag{5, 3.103}$$

$$\eta_o = 1 - \frac{NA_f}{A_t}\left(1 - \eta_f\right) \tag{6, 3.102}$$

where the fin and prime surface area is

$$A_t = NA_f + A_b \tag{3.99}$$

$$A_t = N\left(\pi D_f L\right) + \left[\pi D_d^2/4 - N\left(\pi D_f^2/4\right)\right]$$

where A_f is the fin surface area, D_d is the device diameter and D_f is the fin diameter.

$$A_t = 30\left(\pi \times 0.0015\,\mathrm{m} \times 0.015\,\mathrm{m}\right) + \left[\pi\left(0.020\,\mathrm{m}\right)^2/4 - 30\left(\pi\left(0.0015\,\mathrm{m}\right)^2/4\right)\right]$$

A_t = 0.06362 m^2 + 0.0002611 m^2 = 0.06388 m^2

Using the *IHT Model*, *Extended Surfaces*, *Performance Calculations*, *Rectangular Pin Fin*, find the fin efficiency as

$$\eta_f = 0.8546 \tag{7}$$

Continued...

Substituting numerical values into Eq. (6), find

$$\eta_o = 1 - \frac{30 \times \pi \times 0.0015\,\text{m} \times 0.015\,\text{m}}{0.06388\,\text{m}^2}(1 - 0.8546)$$

$$\eta_o = 0.8552$$

and the fin array thermal resistance is

$$R_{fin} = \frac{1}{0.8552 \times 1000\,\text{W/m}^2 \cdot \text{K} \times 0.06388\,\text{m}^2} = 0.01831\,\text{K/W}$$

Returning to Eq. (3), with T_1 = 57°C from part (a), the permissible heat rate is

$$q_d = \frac{(57-27)^\circ\text{C}}{(0.141 + 0.159)\,\text{K/W}} + \frac{(57-27)^\circ\text{C}}{(0.159 + 0.03979 + 0.01831)\,\text{K/W}}$$

$$q_d = 100\,\text{W} + 138.2\,\text{W} = 238\,\text{W}$$ <

COMMENTS: (1) Recognize in the part (b) analysis, that thermal resistances of the fin base and array are much smaller than the resistance due to the epoxy contact interfaces.

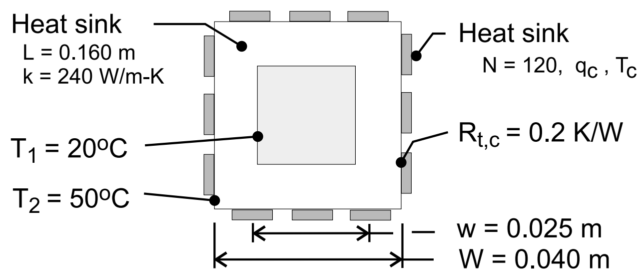
(2) In calculating the fin efficiency, η_f, using the IHT Model it is not necessary to know the base temperature as η_f depends only upon geometric parameters, thermal conductivity and the convection coefficient.

PROBLEM 4.31

KNOWN: Dimensions and surface temperatures of a square channel. Number of chips mounted on outer surface and chip thermal contact resistance.

FIND: Heat dissipation per chip and chip temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Steady state, (2) Approximately uniform channel inner and outer surface temperatures, (3) Two-dimensional conduction through channel wall (negligible end-wall effects), (4) Constant thermal conductivity.

ANALYSIS: The total heat rate is determined by the two-dimensional conduction resistance of the channel wall, $q = (T_2 - T_1)/R_{t,cond(2D)}$, with the resistance determined by using Eq. 4.27 with Case 11 of Table 4.1. For W/w = 1.6 > 1.4

$$R_{t,cond(2D)} = \frac{0.930 \ln(W/w) - 0.050}{2\pi \ L \ k} = \frac{0.387}{2\pi(0.160\text{m})240 \ \text{W/m}\cdot\text{K}} = 0.00160 \ \text{K/W}$$

The heat rate per chip is then

$$q_c = \frac{T_2 - T_1}{N \ R_{t,cond(2D)}} = \frac{(50-20)^\circ\text{C}}{120(0.0016 \ \text{K/W})} = 156.3 \ \text{W} \qquad <$$

and, with $q_c = (T_c - T_2)/R_{t,c}$, the chip temperature is

$$T_c = T_2 + R_{t,c} \ q_c = 50^\circ\text{C} + (0.2 \ \text{K/W})156.3 \ \text{W} = 81.3^\circ\text{C} \qquad <$$

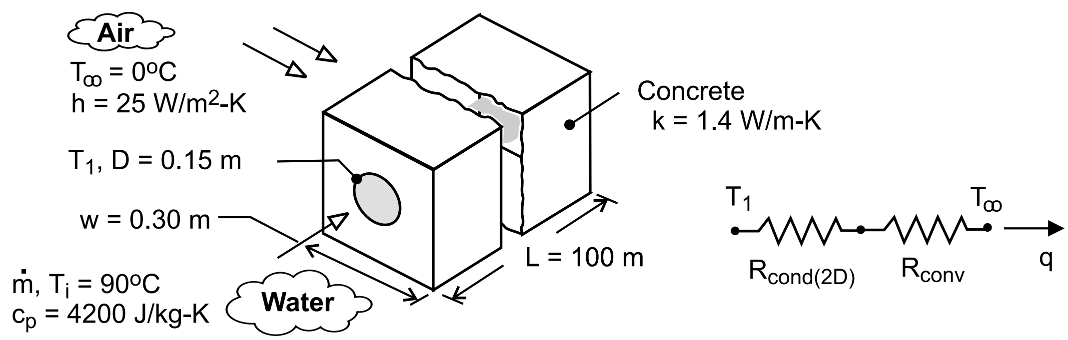
COMMENTS: (1) By acting to *spread* heat flow lines away from a chip, the channel wall provides an excellent *heat sink* for dissipating heat generated by the chip. However, recognize that, in practice, there will be temperature variations on the inner and outer surfaces of the channel, and if the prescribed values of T_1 and T_2 represent minimum and maximum inner and outer surface temperatures, respectively, the rate is overestimated by the foregoing analysis. (2) The shape factor may also be determined by combining the expression for a plane wall with the result of Case 8 (Table 4.1). With S = [4(wL)/(W-w)/2] + 4(0.54 L) = 2.479 m, $R_{t,cond(2D)}$ = 1/(Sk) = 0.00168 K/W.

PROBLEM 4.32

KNOWN: Dimensions and thermal conductivity of concrete duct. Convection conditions of ambient air. Inlet temperature of water flow through the duct.

FIND: (a) Heat loss per duct length near inlet, (b) Minimum allowable flow rate corresponding to maximum allowable temperature rise of water.

SCHEMATIC:



ASSUMPTIONS: (1) Steady state, (2) Negligible water-side convection resistance, pipe wall conduction resistance, and pipe/concrete contact resistance (temperature at inner surface of concrete corresponds to that of water), (3) Constant properties, (4) Negligible flow work and kinetic and potential energy changes.

ANALYSIS: (a) From the thermal circuit, the heat loss per unit length near the entrance is

$$q' = \frac{T_i - T_\infty}{R'_{cond(2D)} + R'_{conv}} = \frac{T_i - T_\infty}{\dfrac{\ln(1.08\,w/D)}{2\pi k} + \dfrac{1}{h(4w)}}$$

where $R'_{cond(2D)}$ is obtained by using the shape factor of Case 6 from Table 4.1 with Eq. (4.27).

Hence,

$$q' = \frac{(90-0)°C}{\dfrac{\ln(1.08\times 0.3m/0.15m)}{2\pi(1.4\,W/m\cdot K)} + \dfrac{1}{25\,W/m^2\cdot K(1.2m)}} = \frac{90°C}{(0.0876+0.0333)K\cdot m/W} = 745\ W/m$$ <

(b) From Eq. (1.11e), with $q = q'L$ and $(T_i - T_o) = 5°C$,

$$\dot{m} = \frac{q'L}{u_i - u_o} = \frac{q'L}{c(T_i - T_o)} = \frac{745\,W/m(100m)}{4207\,J/kg\cdot K(5°C)} = 3.54\ kg/s$$ <

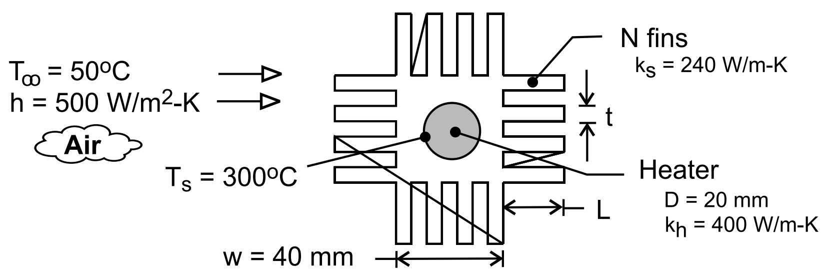
COMMENTS: The small reduction in the temperature of the water as it flows from inlet to outlet induces a slight departure from two-dimensional conditions and a small reduction in the heat rate per unit length. A slightly conservative value (upper estimate) of $\dot{m}$ is therefore obtained in part (b).

PROBLEM 4.33

KNOWN: Dimensions and thermal conductivities of a heater and a finned sleeve. Convection conditions on the sleeve surface.

FIND: (a) Heat rate per unit length, (b) Generation rate and centerline temperature of heater, (c) Effect of fin parameters on heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady state, (2) Constant properties, (3) Negligible contact resistance between heater and sleeve, (4) Uniform convection coefficient at outer surfaces of sleeve, (5) Uniform heat generation, (6) Negligible radiation.

ANALYSIS: (a) From the thermal circuit, the desired heat rate is

$$q' = \frac{T_s - T_\infty}{R'_{cond(2D)} + R'_{t,o}} = \frac{T_s - T_\infty}{R'_{tot}}$$

The two-dimensional conduction resistance, may be estimated from Eq. (4.27) and Case 6 of Table 4.2

$$R'_{cond(2D)} = \frac{1}{S'k_s} = \frac{\ln(1.08w/D)}{2\pi\, k_s} = \frac{\ln(2.16)}{2\pi(240\,W/m\cdot K)} = 5.11\times10^{-4}\ m\cdot K/W$$

The thermal resistance of the fin array is given by Eq. (3.103), where η_o and A_t are given by Eqs. (3.102) and (3.99) and η_f is given by Eq. (3.89). With $L_c = L + t/2 = 0.022$ m, $m = (2h/k_s t)^{1/2} = 32.3$ m^{-1} and $mL_c = 0.710$,

$$\eta_f = \frac{\tanh\, mL_c}{mL_c} = \frac{0.61}{0.71} = 0.86$$

$$A'_t = NA'_f + A'_b = N(2L+t) + (4w - Nt) = 0.704m + 0.096m = 0.800m$$

$$\eta_o = 1 - \frac{NA'_f}{A'_t}(1-\eta_f) = 1 - \frac{0.704m}{0.800m}(0.14) = 0.88$$

$$R'_{t,o} = (\eta_o h\, A'_t)^{-1} = \left(0.88\times500\,W/m^2\cdot K\times0.80m\right)^{-1} = 2.84\times10^{-3} m\cdot K/W$$

$$q' = \frac{(300-50)°C}{\left(5.11\times10^{-4} + 2.84\times10^{-3}\right)m\cdot K/W} = 74{,}600\,W/m$$ <

(b) Eq. (3.55) may be used to determine $\dot{q}$, if h is replaced by an overall coefficient based on the surface area of the heater. From Eq. (3.32),

$$U_s A_s' = U_s \pi D = \left(R'_{tot}\right)^{-1} = (3.35\,\mathrm{m \cdot K/W})^{-1} = 298\,\mathrm{W/m \cdot K}$$

$$U_s = 298\,\mathrm{W/m \cdot K}/(\pi \times 0.02\mathrm{m}) = 4750\,\mathrm{W/m^2 \cdot K}$$

$$\dot{q} = 4 U_s (T_s - T_\infty)/D = 4\left(4750\,\mathrm{W/m^2 \cdot K}\right)(250°\mathrm{C})/0.02\mathrm{m} = 2.38 \times 10^8\,\mathrm{W/m^3} \quad <$$

From Eq. (3.53) the centerline temperature is

$$T(0) = \frac{\dot{q}(D/2)^2}{4k_h} + T_s = \frac{2.38 \times 10^8\,\mathrm{W/m^3}\,(0.01\mathrm{m})^2}{4(400\,\mathrm{W/m \cdot K})} + 300°\mathrm{C} = 315°\mathrm{C} \quad <$$

(c) Subject to the prescribed constraints, the following results have been obtained for parameter variations corresponding to 16 ≤ N ≤ 40, 2 ≤ t ≤ 8 mm and 20 ≤ L ≤ 40 mm.

N	t(mm)	L(mm)	η_f	$q'(\mathrm{W/m})$
16	4	20	0.86	74,400
16	8	20	0.91	77,000
28	4	20	0.86	107,900
32	3	20	0.83	115,200
40	2	20	0.78	127,800
40	2	40	0.51	151,300

Clearly there is little benefit to simply increasing t, since there is no change in A_t' and only a marginal increase in η_f. However, due to an attendant increase in A_t', there is significant benefit to increasing N for fixed t (no change in η_f) and additional benefit in concurrently increasing N while decreasing t. In this case the effect of increasing A_t' exceeds that of decreasing η_f. The same is true for increasing L, although there is an upper limit at which diminishing returns would be reached. The upper limit to L could also be influenced by manufacturing constraints.

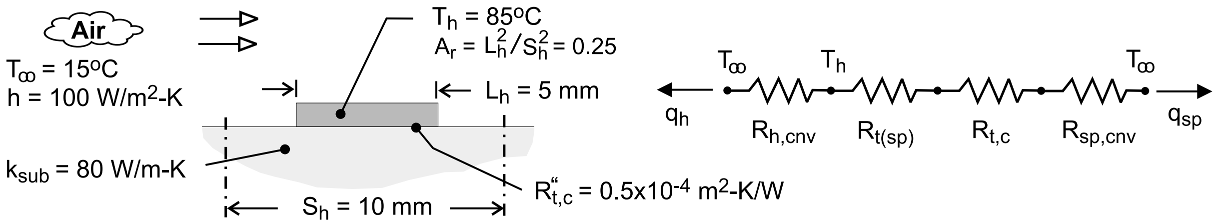
COMMENTS: Without the sleeve, the heat rate would be $q' = \pi D h (T_s - T_\infty) = 7850\,\mathrm{W/m}$, which is well below that achieved by using the increased surface area afforded by the sleeve.

PROBLEM 4.34

KNOWN: Dimensions of chip array. Conductivity of substrate. Convection conditions. Contact resistance. Expression for resistance of spreader plate. Maximum chip temperature.

FIND: Maximum chip heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Constant thermal conductivity, (3) Negligible radiation, (4) All heat transfer is by convection from the chip and the substrate surface (negligible heat transfer from bottom or sides of substrate).

ANALYSIS: From the thermal circuit,

$$q = q_h + q_{sp} = \frac{T_h - T_\infty}{R_{h,cnv}} + \frac{T_h - T_\infty}{R_{t(sp)} + R_{t,c} + R_{sp,cnv}}$$

$$R_{h,cnv} = \left(h\,A_{s,n}\right)^{-1} = \left(hL_h^2\right)^{-1} = \left[100\,W/m^2\cdot K\left(0.005m\right)^2\right]^{-1} = 400\,K/W$$

$$R_{t(sp)} = \frac{1-1.410\,A_r + 0.344\,A_r^3 + 0.043\,A_r^5 + 0.034\,A_r^7}{4\,k_{sub}\,L_h} = \frac{1-0.353+0.005+0+0}{4\left(80\,W/m\cdot K\right)\left(0.005m\right)} = 0.408\,K/W$$

$$R_{t,c} = \frac{R''_{t,c}}{L_h^2} = \frac{0.5\times10^{-4}m^2\cdot K/W}{\left(0.005m\right)^2} = 2.000\,K/W$$

$$R_{sp,cnv} = \left[h\left(A_{sub} - A_{s,h}\right)\right]^{-1} = \left[100\,W/m^2\cdot K\left(0.010m^2 - 0.005m^2\right)\right]^{-1} = 133.3\,K/W$$

$$q = \frac{70°C}{400\,K/W} + \frac{70°C}{\left(0.408+2+133.3\right)K/W} = 0.18\,W + 0.52\,W = 0.70\,W \qquad <$$

COMMENTS: (1) The thermal resistances of the substrate and the chip/substrate interface are much less than the substrate convection resistance. Hence, the heat rate is increased almost in proportion to the additional surface area afforded by the substrate. An increase in the spacing between chips (S_h) would increase q correspondingly.

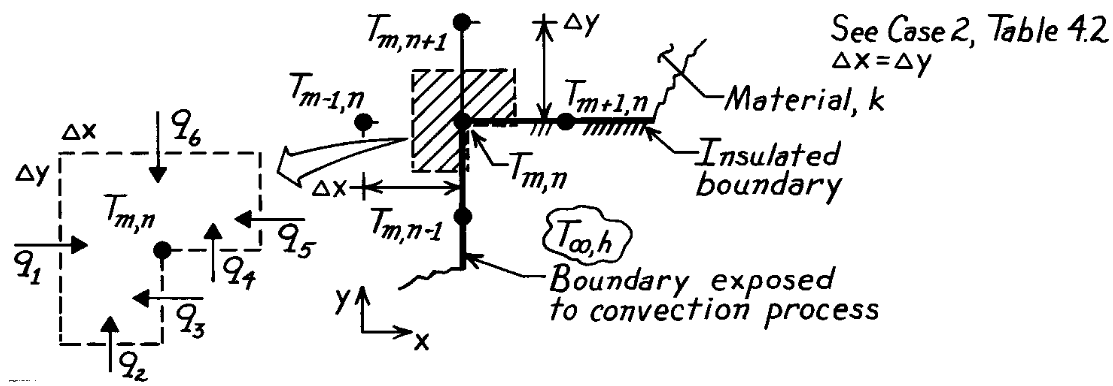
(2) In the limit $A_r \to 0$, $R_{t(sp)}$ reduces to $2\pi^{1/2}k_{sub}D_h$ for a circular heat source and $4k_{sub}L_h$ for a square source.

PROBLEM 4.35

KNOWN: Internal corner of a two-dimensional system with prescribed convection boundary conditions.

FIND: Finite-difference equations for these situations: (a) Horizontal boundary is perfectly insulated and vertical boundary is subjected to a convection process (T_∞,h), (b) Both boundaries are perfectly insulated; compare result with Eq. 4.45.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) No internal generation.

ANALYSIS: Consider the nodal network shown above and also as Case 2, Table 4.2. Having defined the control volume – the shaded area of unit thickness normal to the page – next identify the heat transfer processes. Finally, perform an energy balance wherein the processes are expressed using appropriate rate equations.

(a) With the horizontal boundary insulated and the vertical boundary subjected to a convection process, the energy balance results in the following finite-difference equation:

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_3 + q_4 + q_5 + q_6 = 0$$

$$k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left[\frac{\Delta x}{2}\cdot 1\right]\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left[\frac{\Delta y}{2}\cdot 1\right]\left(T_\infty - T_{m,n}\right)$$

$$+ 0 + k\left[\frac{\Delta y}{2}\cdot 1\right]\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} = 0.$$

Letting $\Delta x = \Delta y$, and regrouping, find

$$2\left(T_{m-1,n} + T_{m,n+1}\right) + \left(T_{m+1,n} + T_{m,n-1}\right) + \frac{h\Delta x}{k}T_\infty - \left[6 + \frac{h\Delta x}{k}\right]T_{m,n} = 0. \qquad <$$

(b) With both boundaries insulated, the energy balance would have $q_3 = q_4 = 0$. The same result would be obtained by letting h = 0 in the previous result. Hence,

$$2\left(T_{m-1,n} + T_{m,n+1}\right) + \left(T_{m+1,n} + T_{m,n-1}\right) - 6\,T_{m,n} = 0. \qquad <$$

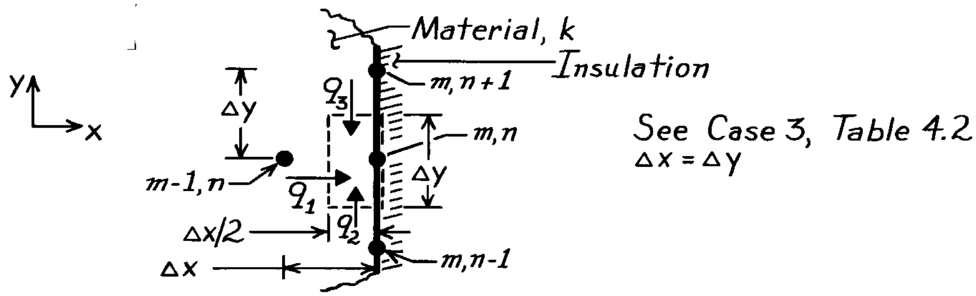
Note that this expression compares exactly with Eq. 4.45 when h = 0, which corresponds to insulated boundaries.

PROBLEM 4.36

KNOWN: Plane surface of two-dimensional system.

FIND: The finite-difference equation for nodal point on this boundary when (a) insulated; compare result with Eq. 4.46, and when (b) subjected to a constant heat flux.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction with no generation, (2) Constant properties, (3) Boundary is adiabatic.

ANALYSIS: (a) Performing an energy balance on the control volume, $(\Delta x/2)\cdot\Delta y$, and using the conduction rate equation, it follows that

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_3 = 0 \tag{1,2}$$

$$k\left(\Delta y \cdot 1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left[\frac{\Delta x}{2}\cdot 1\right]\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k\left[\frac{\Delta x}{2}\cdot 1\right]\frac{T_{m,n+1} - T_{m,n}}{\Delta y} = 0. \tag{3}$$

Note that there is no heat rate across the control volume surface at the insulated boundary. Recognizing that $\Delta x = \Delta y$, the above expression reduces to the form

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} = 0. \tag{4}$$ <

The Eq. 4.46 of Table 4.3 considers the same configuration but with the boundary subjected to a convection process. That is,

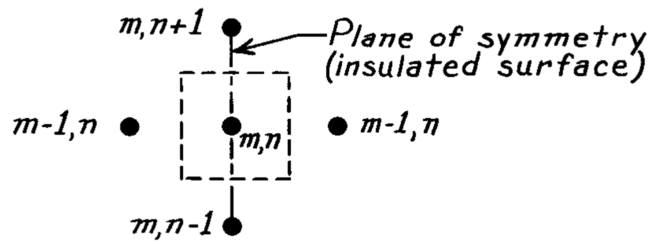
$$\left(2T_{m-1,n} + T_{m,n-1} + T_{m,n+1}\right) + \frac{2h\Delta x}{k}T_\infty - 2\left[\frac{h\Delta x}{k} + 2\right]T_{m,n} = 0. \tag{5}$$

Note that, if the boundary is insulated, h = 0 and Eq. 4.46 reduces to Eq. (4).

(b) If the surface is exposed to a constant heat flux, q_o'', the energy balance has the form $q_1 + q_2 + q_3 + q_o'' \cdot \Delta y = 0$ and the finite difference equation becomes

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} = -\frac{q_o''\Delta x}{k}.$$ <

COMMENTS: Equation (4) can be obtained by using the "interior node" finite-difference equation, Eq. 4.33, where the insulated boundary is treated as a symmetry plane as shown below.

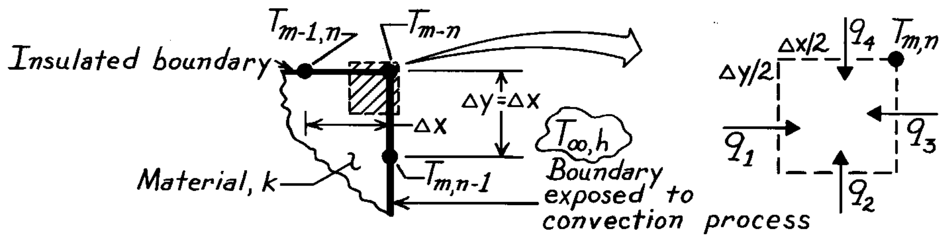


PROBLEM 4.37

KNOWN: External corner of a two-dimensional system whose boundaries are subjected to prescribed conditions.

FIND: Finite-difference equations for these situations: (a) Upper boundary is perfectly insulated and side boundary is subjected to a convection process, (b) Both boundaries are perfectly insulated; compare result with Eq. 4.47.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) No internal generation.

ANALYSIS: Consider the nodal point configuration shown in the schematic and also as Case 4, Table 4.2. The control volume about the node – shaded area above of unit thickness normal to the page – has dimensions, $(\Delta x/2)(\Delta y/2)\cdot 1$. The heat transfer processes at the surface of the CV are identified as q_1, q_2 ···. Perform an energy balance wherein the processes are expressed using the appropriate rate equations.

(a) With the upper boundary insulated and the side boundary subjected to a convection process, the energy balance has the form

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_3 + q_4 = 0 \tag{1,2}$$

$$k\left[\frac{\Delta y}{2}\cdot 1\right]\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left[\frac{\Delta x}{2}\cdot 1\right]\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left[\frac{\Delta y}{2}\cdot 1\right]\left(T_{\infty} - T_{m,n}\right) + 0 = 0.$$

Letting $\Delta x = \Delta y$, and regrouping, find

$$T_{m,n-1} + T_{m-1,n} + \frac{h\Delta x}{k}T_{\infty} - 2\left[\frac{1}{2}\frac{h\Delta x}{k} + 1\right]T_{m,n} = 0. \tag{3}$$ <

(b) With both boundaries insulated, the energy balance of Eq. (2) would have $q_3 = q_4 = 0$. The same result would be obtained by letting $h = 0$ in the finite-difference equation, Eq. (3). The result is

$$T_{m,n-1} + T_{m-1,n} - 2T_{m,n} = 0.$$ <

Note that this expression is identical to Eq. 4.47 when $h = 0$, in which case both boundaries are insulated.

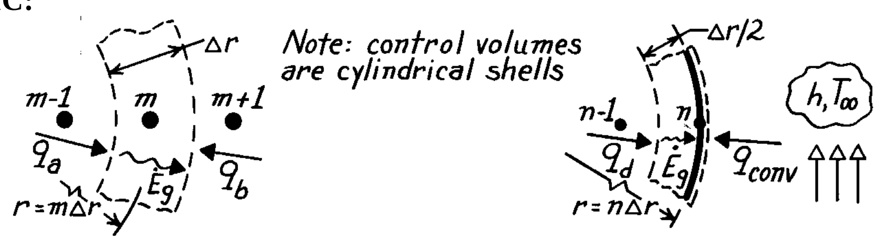
COMMENTS: Note the convenience resulting from formulating the energy balance by *assuming* that all the heat flow is *into the node*.

PROBLEM 4.38

KNOWN: Conduction in a one-dimensional (radial) *cylindrical* coordinate system with volumetric generation.

FIND: Finite-difference equation for (a) Interior node, m, and (b) Surface node, n, with convection.

SCHEMATIC:



(a) Interior node, m (b) Surface node with convection, n

ASSUMPTIONS: (1) Steady-state, one-dimensional (radial) conduction in *cylindrical* coordinates, (2) Constant properties.

ANALYSIS: (a) The network has nodes spaced at equal Δr increments with m = 0 at the center; hence, r = mΔr (or nΔr). The control volume is $V = 2p\ r \cdot \Delta r \cdot \ell = 2p\ (m\Delta r)\,\Delta r \cdot \ell$. The energy balance is $\dot{E}_{in} + \dot{E}_g = q_a + q_b + \dot{q}V = 0$

$$k\left[2p\left[r - \frac{\Delta r}{2}\right]\ell\right]\frac{T_{m-1} - T_m}{\Delta r} + k\left[2p\left[r + \frac{\Delta r}{2}\right]\ell\right]\frac{T_{m+1} - T_m}{\Delta r} + \dot{q}\left[2p\ (m\Delta r)\,\Delta r\ell\right] = 0.$$

Recognizing that r = mΔr, canceling like terms, and regrouping find

$$\left[m - \frac{1}{2}\right]T_{m-1} + \left[m + \frac{1}{2}\right]T_{m+1} - 2mT_m + \frac{\dot{q}m\Delta r^2}{k} = 0.$$ <

(b) The control volume for the surface node is $V = 2p\ r \cdot (\Delta r/2) \cdot \ell$. The energy balance is

$\dot{E}_{in} + \dot{E}_g = q_d + q_{conv} + \dot{q}V = 0$. Use Fourier's law to express q_d and Newton's law of cooling for q_{conv} to obtain

$$k\left[2p\left[r - \frac{\Delta r}{2}\right]\ell\right]\frac{T_{n-1} - T_n}{\Delta r} + h\left[2p\ r\ell\right](T_\infty - T_n) + \dot{q}\left[2p\ (n\Delta r)\frac{\Delta r}{2}\ell\right] = 0.$$

Let r = nΔr, cancel like terms and regroup to find

$$\left[n - \frac{1}{2}\right]T_{n-1} - \left[\left[n - \frac{1}{2}\right] + \frac{hn\Delta r}{k}\right]T_n + \frac{\dot{q}n\Delta r^2}{2k} + \frac{hn\Delta r}{k}T_\infty = 0.$$

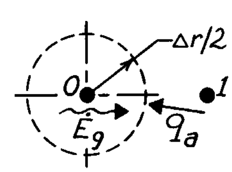
<

COMMENTS: (1) Note that when m or n becomes very large compared to ½, the finite-difference equation becomes independent of m or n. Then the cylindrical system approximates a rectangular one.

(2) The finite-difference equation for the center node (m = 0) needs to be treated as a special case. The control volume is

$V = p\ (\Delta r/2)^2\ \ell$ and the energy balance is

$$\dot{E}_{in} + \dot{E}_g = q_a + \dot{q}V = k\left[2p\left[\frac{\Delta r}{2}\right]\ell\right]\frac{T_1 - T_0}{\Delta r} + \dot{q}\left[p\left[\frac{\Delta r}{2}\right]^2\ell\right] = 0.$$



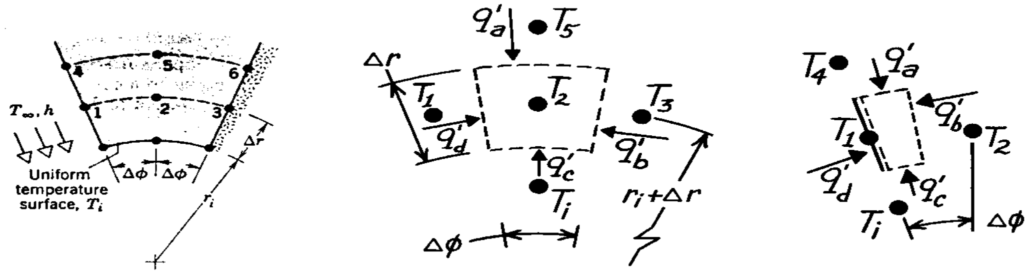
Regrouping, the finite-difference equation is $-T_o + T_1 + \frac{\dot{q}\Delta r^2}{4k} = 0.$

PROBLEM 4.39

KNOWN: Two-dimensional cylindrical configuration with prescribed radial (Δr) and angular ($\Delta\phi$) spacings of nodes.

FIND: Finite-difference equations for nodes 2, 3 and 1.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction in cylindrical coordinates (r,ϕ), (3) Constant properties.

ANALYSIS: The method of solution is to define the appropriate control volume for each node, to identify relevant processes and then to perform an energy balance.

(a) Node 2. This is an *interior* node with control volume as shown above. The energy balance is $\dot{E}_{in} = q'_a + q'_b + q'_c + q'_d = 0.$ Using Fourier's law for each process, find

$$k\left[\left[r_i + \frac{3}{2}\Delta r\right]\Delta f\right]\frac{(T_5 - T_2)}{\Delta r} + k(\Delta r)\frac{(T_3 - T_2)}{(r_i + \Delta r)\Delta f} +$$

$$+k\left[\left[r_i + \frac{1}{2}\Delta r\right]\Delta f\right]\frac{(T_i - T_2)}{\Delta r} + k(\Delta r)\frac{(T_1 - T_2)}{(r_i + \Delta r)\Delta f} = 0.$$

Canceling terms and regrouping yields,

$$-2\left[(r_i + \Delta r) + \frac{(\Delta r)^2}{(\Delta f)^2}\frac{1}{(r_i + \Delta r)}\right]T_2 + \left[r_i + \frac{3}{2}\Delta r\right]T_5 + \frac{(\Delta r)^2}{(r_i + \Delta r)(\Delta f)^2}(T_3 + T_1) + \left[r_i + \frac{1}{2}\Delta r\right]T_i = 0.$$

(b) Node 3. The adiabatic surface behaves as a symmetry surface. We can utilize the result of Part (a) to write the finite-difference equation by inspection as

$$-2\left[(r_i + \Delta r) + \frac{(\Delta r)^2}{(\Delta f)^2}\frac{1}{(r_i + \Delta r)}\right]T_3 + \left[r_i + \frac{3}{2}\Delta r\right]T_6 + \frac{2(\Delta r)^2}{(r_i + \Delta r)(\Delta f)^2}\cdot T_2 + \left[r_i + \frac{1}{2}\Delta r\right]T_i = 0.$$

(c) Node 1. The energy balance is $q'_a + q'_b + q'_c + q'_d = 0.$ Substituting,

$$k\left[\left[r_i + \frac{3}{2}\Delta r\right]\frac{\Delta f}{2}\right]\frac{(T_4 - T_1)}{\Delta r} + k(\Delta r)\frac{(T_2 - T_1)}{(r_i + \Delta r)\Delta f} +$$

$$+k\left[\left[r_i + \frac{1}{2}\Delta r\right]\frac{\Delta f}{2}\right]\frac{(T_i - T_1)}{\Delta r} + h(\Delta r)(T_\infty - T_1) = 0$$ <

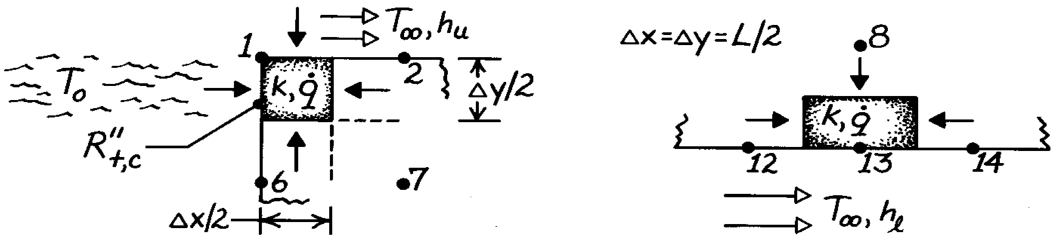
This expression could now be rearranged.

PROBLEM 4.40

KNOWN: Heat generation and thermal boundary conditions of bus bar. Finite-difference grid.

FIND: Finite-difference equations for selected nodes.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: (a) Performing an energy balance on the control volume, $(\Delta x/2)(\Delta y/2)\cdot 1$, find the FDE for node 1,

$$\frac{T_o - T_1}{R''_{t,c}/(\Delta y/2)1} + h_u\left(\frac{\Delta x}{2}\cdot 1\right)(T_\infty - T_1) + \frac{k(\Delta y/2\cdot 1)}{\Delta x}(T_2 - T_1)$$
$$+\frac{k(\Delta x/2\cdot 1)}{\Delta y}(T_6 - T_1) + \dot{q}\left[(\Delta x/2)(\Delta y/2)1\right] = 0$$
$$\left(\Delta x/kR''_{t,c}\right)T_o + \left(h_u\Delta x/k\right)T_\infty + T_2 + T_6$$
$$+\dot{q}(\Delta x)^2/2k - \left[\left(\Delta x/kR''_{t,c}\right) + \left(h_u\Delta x/k\right) + 2\right]T_1 = 0. \qquad <$$

(b) Performing an energy balance on the control volume, $(\Delta x)(\Delta y/2)\cdot 1$, find the FDE for node 13,

$$h_l(\Delta x\cdot 1)(T_\infty - T_{13}) + (k/\Delta x)(\Delta y/2\cdot 1)(T_{12} - T_{13})$$
$$+(k/\Delta y)(\Delta x\cdot 1)(T_8 - T_{13}) + (k/\Delta x)(\Delta y/2\cdot 1)(T_{14} - T_{13}) + \dot{q}(\Delta x\cdot\Delta y/2\cdot 1) = 0$$

$$\left(h_l\Delta x/k\right)T_\infty + 1/2\left(T_{12} + 2T_8 + T_{14}\right) + \dot{q}(\Delta x)^2/2k - \left(h_l\Delta x/k + 2\right)T_{13} = 0. \qquad <$$

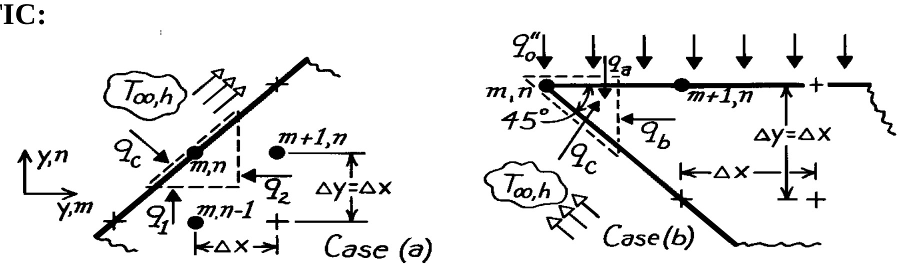
COMMENTS: For fixed T_o and T_∞, the relative amounts of heat transfer to the air and heat sink are determined by the values of h and $R''_{t,c}$.

PROBLEM 4.41

KNOWN: Nodal point configurations corresponding to a diagonal surface boundary subjected to a convection process and to the tip of a machine tool subjected to constant heat flux and convection cooling.

FIND: Finite-difference equations for the node m,n in the two situations shown.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: (a) The control volume about node m,n has triangular shape with sides Δx and Δy while the diagonal (surface) length is $\sqrt{2}\ \Delta x$. The heat rates associated with the control volume are due to conduction, q_1 and q_2, and to convection, q_c. Performing an energy balance, find

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_c = 0$$

$$k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h\left(\sqrt{2}\ \Delta x \cdot 1\right)\left(T_\infty - T_{m,n}\right) = 0.$$

Note that we have considered the tool to have unit depth normal to the page. Recognizing that Δx = Δy, dividing each term by k and regrouping, find

$$T_{m,n-1} + T_{m+1,n} + \sqrt{2}\ \cdot \frac{h\Delta x}{k} T_\infty - \left[2 + \sqrt{2}\ \cdot \frac{h\Delta x}{k}\right] T_{m,n} = 0. \qquad <$$

(b) The control volume about node m,n has triangular shape with sides Δx/2 and Δy/2 while the lower diagonal surface length is $\sqrt{2}\,(\Delta x/2)$. The heat rates associated with the control volume are due to the constant heat flux, q_a, to conduction, q_b, and to the convection process, q_c. Perform an energy balance,

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_a + q_b + q_c = 0$$

$$q_o'' \cdot \left[\frac{\Delta x}{2} \cdot 1\right] + k \cdot \left[\frac{\Delta y}{2} \cdot 1\right] \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h \cdot \left[\sqrt{2} \cdot \frac{\Delta x}{2}\right]\left(T_\infty - T_{m,n}\right) = 0.$$

Recognizing that Δx = Δy, dividing each term by k/2 and regrouping, find

$$T_{m+1,n} + \sqrt{2} \cdot \frac{h\Delta x}{k} \cdot T_\infty + q_o'' \cdot \frac{\Delta x}{k} - \left(1 + \sqrt{2} \cdot \frac{h\Delta x}{k}\right) T_{m,n} = 0. \qquad <$$

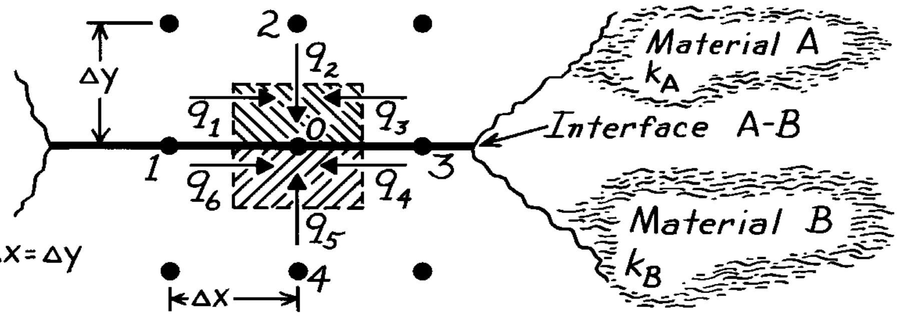
COMMENTS: Note the appearance of the term hΔx/k in both results, which is a dimensionless parameter (the *Biot number*) characterizing the relative effects of convection and conduction.

PROBLEM 4.42

KNOWN: Nodal point on boundary between two materials.

FIND: Finite-difference equation for steady-state conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) No internal heat generation, (5) Negligible thermal contact resistance at interface.

ANALYSIS: The control volume is defined about nodal point 0 as shown above. The conservation of energy requirement has the form

$$\sum_{i=1}^{6} q_i = q_1 + q_2 + q_3 + q_4 + q_5 + q_6 = 0$$

since all heat rates are shown as *into* the CV. Each heat rate can be written using Fourier's law,

$$k_A \cdot \frac{\Delta y}{2} \cdot \frac{T_1 - T_0}{\Delta x} + k_A \cdot \Delta x \cdot \frac{T_2 - T_0}{\Delta y} + k_A \cdot \frac{\Delta y}{2} \cdot \frac{T_3 - T_0}{\Delta x}$$
$$+ k_B \cdot \frac{\Delta y}{2} \cdot \frac{T_3 - T_0}{\Delta x} + k_B \cdot \Delta x \cdot \frac{T_4 - T_0}{\Delta y} + k_B \cdot \frac{\Delta y}{2} \cdot \frac{T_1 - T_0}{\Delta x} = 0.$$

Recognizing that $\Delta x = \Delta y$ and regrouping gives the relation,

$$-T_0 + \frac{1}{4} T_1 + \frac{k_A}{2(k_A + k_B)} T_2 + \frac{1}{4} T_3 + \frac{k_B}{2(k_A + k_B)} T_4 = 0. \qquad <$$

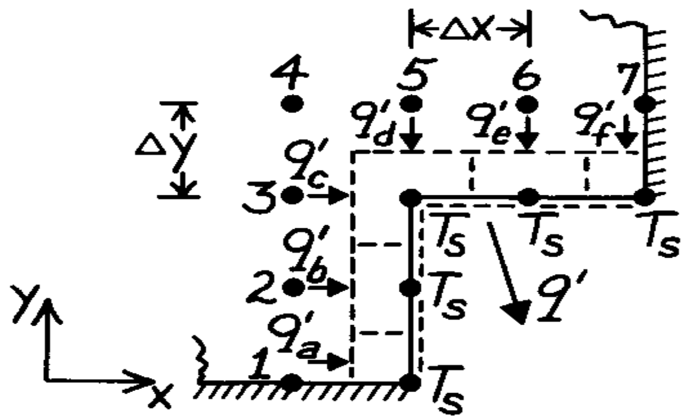
COMMENTS: Note that when $k_A = k_B$, the result agrees with Eq. 4.33 which is appropriate for an interior node in a medium of fixed thermal conductivity.

PROBLEM 4.43

KNOWN: Two-dimensional grid for a system with no internal volumetric generation.

FIND: Expression for heat rate per unit length normal to page crossing the isothermal boundary.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional heat transfer, (3) Constant properties.

ANALYSIS: Identify the surface nodes (T_s) and draw control volumes about these nodes. Since there is no heat transfer in the direction parallel to the isothermal surfaces, the heat rate out of the constant temperature surface boundary is

$$q' = q'_a + q'_b + q'_c + q'_d + q'_e + q'_f$$

For each q'_i, use Fourier's law and pay particular attention to the manner in which the cross-sectional area and gradients are specified.

$$q' = k(\Delta y/2)\frac{T_1 - T_s}{\Delta x} + k(\Delta y)\frac{T_2 - T_s}{\Delta x} + k(\Delta y)\frac{T_3 - T_s}{\Delta x}$$
$$+ k(\Delta x)\frac{T_5 - T_s}{\Delta y} + k(\Delta x)\frac{T_6 - T_s}{\Delta y} + k(\Delta x/2)\frac{T_7 - T_s}{\Delta y}$$

Regrouping with $\Delta x = \Delta y$, find

$$q' = k\left[0.5T_1 + T_2 + T_3 + T_5 + T_6 + 0.5T_7 - 5T_s\right].$$ <

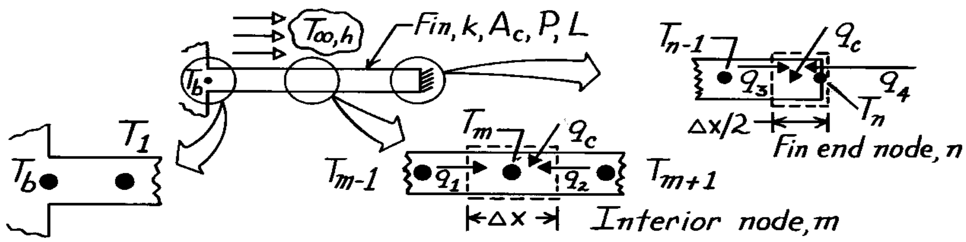
COMMENTS: Looking at the corner node, it is important to recognize the areas associated with q'_c and q'_d (Δy and Δx, respectively).

PROBLEM 4.44

KNOWN: One-dimensional fin of uniform cross section insulated at one end with prescribed base temperature, convection process on surface, and thermal conductivity.

FIND: Finite-difference equation for these nodes: (a) Interior node, m and (b) Node at end of fin, n, where x = L.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction.

ANALYSIS: (a) The control volume about node m is shown in the schematic; the node spacing and control volume length in the x direction are both Δx. The uniform cross-sectional area and fin perimeter are A_c and P, respectively. The heat transfer process on the control surfaces, q_1 and q_2, represent conduction while q_c is the convection heat transfer rate between the fin and ambient fluid. Performing an energy balance, find

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_c = 0$$
$$kA_c \frac{T_{m-1} - T_m}{\Delta x} + kA_c \frac{T_{m+1} - T_m}{\Delta x} + hP\Delta x \left(T_{\infty} - T_m\right) = 0.$$

Multiply the expression by $\Delta x/kA_c$ and regroup to obtain

$$T_{m-1} + T_{m+1} + \frac{hP}{kA_c} \cdot \Delta x^2 T_{\infty} - \left[2 + \frac{hP}{kA_c}\Delta x^2\right] T_m = 0 \qquad 1<m<n \qquad <$$

Considering now the special node m = 1, then the m-1 node is T_b, the base temperature. The finite-difference equation would be

$$T_b + T_2 + \frac{hP}{kA_c}\Delta x^2 T_{\infty} - \left[2 + \frac{hP}{kA_c}\Delta x^2\right] T_1 = 0 \qquad m=1 \qquad <$$

(b) The control volume of length $\Delta x/2$ about node n is shown in the schematic. Performing an energy balance,

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_3 + q_4 + q_c = 0$$
$$kA_c \frac{T_{n-1} - T_n}{\Delta x} + 0 + hP\frac{\Delta x}{2}\left(T_{\infty} - T_n\right) = 0.$$

Note that $q_4 = 0$ since the end (x = L) is insulated. Multiplying by $\Delta x/kA_c$ and regrouping,

$$T_{n-1} + \frac{hP}{kA_c} \cdot \frac{\Delta x^2}{2} T_{\infty} - \left[\frac{hP}{kA_c} \cdot \frac{\Delta x^2}{2} + 1\right] T_n = 0. \qquad <$$

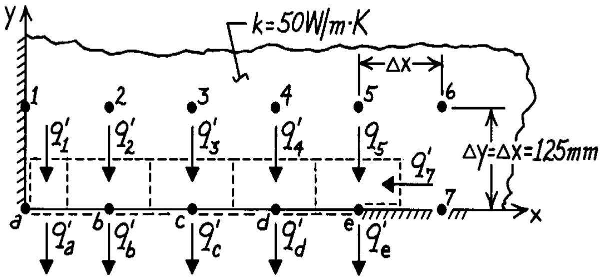
COMMENTS: The value of Δx will be determined by the selection of n; that is, $\Delta x = L/n$. Note that the grouping, hP/kA_c, appears in the finite-difference and differential forms of the energy balance.

PROBLEM 4.45

KNOWN: Two-dimensional network with prescribed nodal temperatures and thermal conductivity of the material.

FIND: Heat rate per unit length normal to page, q'.

SCHEMATIC:



Node	T_i(°C)
1	120.55
2	120.64
3	121.29
4	123.89
5	134.57
6	150.49
7	147.14

ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional heat transfer, (3) No internal volumetric generation, (4) Constant properties.

ANALYSIS: Construct control volumes around the nodes on the surface maintained at the uniform temperature T_s and indicate the heat rates. The heat rate per unit length is $q' = q'_a + q'_b + q'_c + q'_d + q'_e$ or in terms of conduction terms between nodes,

$$q' = q'_1 + q'_2 + q'_3 + q'_4 + q'_5 + q'_7.$$

Each of these rates can be written in terms of nodal temperatures and control volume dimensions using Fourier's law,

$$q' = k \cdot \frac{\Delta x}{2} \cdot \frac{T_1 - T_s}{\Delta y} + k \cdot \Delta x \cdot \frac{T_2 - T_s}{\Delta y} + k \cdot \Delta x \frac{T_3 - T_s}{\Delta y} + k \cdot \Delta x \frac{T_4 - T_s}{\Delta y}$$
$$+ k \cdot \Delta x \frac{T_5 - T_s}{\Delta y} + k \cdot \frac{\Delta y}{2} \cdot \frac{T_7 - T_s}{\Delta x}.$$

and since $\Delta x = \Delta y$,

$$q' = k[(1/2)(T_1 - T_s) + (T_2 - T_s) + (T_3 - T_s)$$
$$+ (T_4 - T_s) + (T_5 - T_s) + (1/2)(T_7 - T_s)].$$

Substituting numerical values, find

$$q' = 50\ \text{W/m} \cdot \text{K}[(1/2)(120.55 - 100) + (120.64 - 100) + (121.29 - 100)$$
$$+ (123.89 - 100) + (134.57 - 100) + (1/2)(147.14 - 100)]$$

$$q' = 6711\ \text{W/m}.$$ <

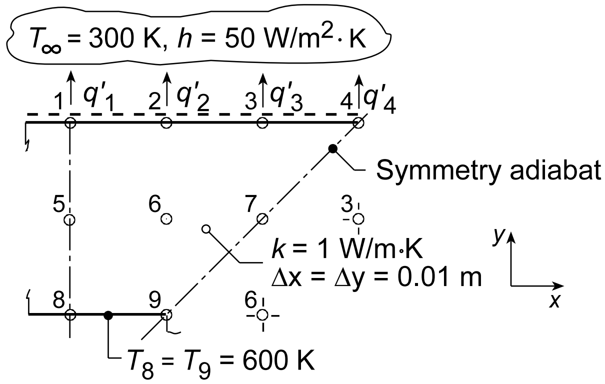
COMMENTS: For nodes a through d, there is no heat transfer into the control volumes in the x-direction. Look carefully at the energy balance for node e, $q'_e = q'_5 + q'_7$, and how q'_5 and q'_7 are evaluated.

PROBLEM 4.46

KNOWN: Nodal temperatures from a steady-state, finite-difference analysis for a one-eighth symmetrical section of a square channel.

FIND: (a) Beginning with properly defined control volumes, derive the finite-difference equations for nodes 2, 4 and 7, and determine T_2, T_4 and T_7, and (b) Heat transfer loss per unit length from the channel, q'.

SCHEMATIC:



Node	T(°C)
1	430
3	394
6	492
8,9	600

ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) No internal volumetric generation, (4) Constant properties.

ANALYSIS: (a) Define control volumes about the nodes 2, 4, and 7, taking advantage of symmetry where appropriate and performing energy balances, $\dot{E}_{in} - \dot{E}_{out} = 0$, with Δx = Δy,

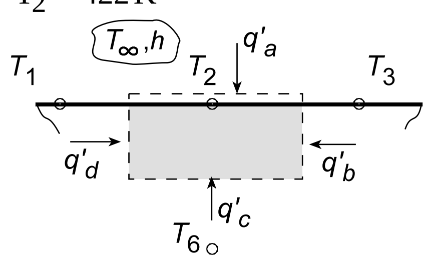
Node 2: $q'_a + q'_b + q'_c + q'_d = 0$

$$h\Delta x\left(T_\infty - T_2\right) + k\left(\Delta y/2\right)\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_6 - T_2}{\Delta y} + k\left(\Delta y/2\right)\frac{T_1 - T_2}{\Delta x} = 0$$

$$T_2 = \left[0.5T_1 + 0.5T_3 + T_6 + \left(h\Delta x/k\right)T_\infty\right] / \left[2 + \left(h\Delta x/k\right)\right]$$

$$T_2 = \left[0.5\times 430 + 0.5\times 394 + 492 + \left(50\,\mathrm{W/m^2\cdot K}\times 0.01\,\mathrm{m}/1\,\mathrm{W/m\cdot K}\right)300\right]\mathrm{K} \Big/ \left[2 + 0.50\right]$$

$$T_2 = 422\,\mathrm{K}$$ <



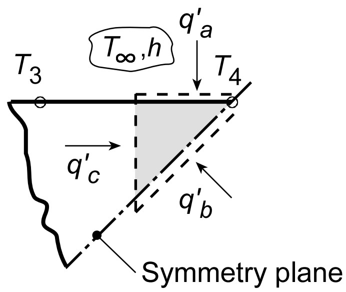
Node 4: $q'_a + q'_b + q'_c = 0$

$$h\left(\Delta x/2\right)\left(T_\infty - T_4\right) + 0 + k\left(\Delta y/2\right)\frac{T_3 - T_4}{\Delta x} = 0$$

$$T_4 = \left[T_3 + \left(h\Delta x/k\right)T_\infty\right] / \left[1 + \left(h\Delta x/k\right)\right]$$

$$T_4 = \left[394 + 0.5\times 300\right]\mathrm{K} / \left[1 + 0.5\right] = 363\,\mathrm{K}$$ <

Continued...



Node 7: From the first schematic, recognizing that the diagonal is a symmetry adiabat, we can treat node 7 as an interior node, hence

$$T_7 = 0.25(T_3 + T_3 + T_6 + T_6) = 0.25(394 + 394 + 492 + 492)\,\text{K} = 443\,\text{K} \qquad <$$

(b) The heat transfer loss from the upper surface can be expressed as the sum of the convection rates from each node as illustrated in the first schematic,

$$q'_{cv} = q'_1 + q'_2 + q'_3 + q'_4$$

$$q'_{cv} = h(\Delta x/2)(T_1 - T_\infty) + h\Delta x(T_2 - T_\infty) + h\Delta x(T_3 - T_\infty) + h(\Delta x/2)(T_4 - T_\infty)$$

$$q'_{cv} = 50\,\text{W/m}^2\cdot\text{K} \times 0.1\,\text{m}\left[(430 - 300)/2 + (422 - 300) + (394 - 300) + (363 - 300)/2\right]\text{K}$$

$$q'_{cv} = 156\,\text{W/m} \qquad <$$

COMMENTS: (1) Always look for symmetry conditions which can greatly simplify the writing of the nodal equation as was the case for Node 7.

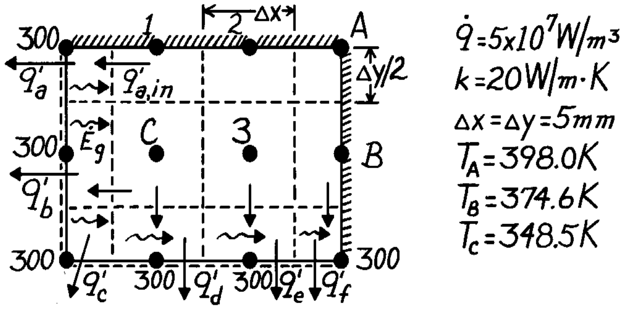
(2) Consider using the *IHT Tool*, *Finite-Difference Equations*, for *Steady-State*, *Two-Dimensional* heat transfer to determine the nodal temperatures T_1 - T_7 when only the boundary conditions T_8, T_9 and (T_∞,h) are specified.

PROBLEM 4.47

KNOWN: Steady-state temperatures (K) at three nodes of a long rectangular bar.

FIND: (a) Temperatures at remaining nodes and (b) heat transfer per unit length from the bar using nodal temperatures; compare with result calculated using knowledge of $\dot{q}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: (a) The finite-difference equations for the nodes (1,2,3,A,B,C) can be written by inspection using Eq. 4.39 and recognizing that the adiabatic boundary can be represented by a symmetry plane.

$$\sum T_{neighbors} - 4T_i + \dot{q}\Delta x^2/k = 0 \quad \text{and} \quad \frac{\dot{q}\Delta x^2}{k} = \frac{5\times10^7\ \text{W/m}^3(0.005\text{m})^2}{20\ \text{W/m}\cdot\text{K}} = 62.5\text{K}.$$

Node A (to find T_2*):*

$$2T_2 + 2T_B - 4T_A + \dot{q}\Delta x^2/k = 0$$

$$T_2 = \frac{1}{2}(-2\times374.6 + 4\times398.0 - 62.5)\,\text{K} = 390.2\text{K} \qquad <$$

Node 3 (to find T_3*):*

$$T_c + T_2 + T_B + 300\text{K} - 4T_3 + \dot{q}\Delta x^2/k = 0$$

$$T_3 = \frac{1}{4}(348.5 + 390.2 + 374.6 + 300 + 62.5)\,\text{K} = 369.0\text{K} \qquad <$$

Node 1 (to find T_1*):*

$$300 + 2T_C + T_2 - 4T_1 + \dot{q}\Delta x^2/k = 0$$

$$T_1 = \frac{1}{4}(300 + 2\times348.5 + 390.2 + 62.5) = 362.4\text{K} \qquad <$$

(b) The heat rate out of the bar is determined by calculating the heat rate out of each control volume around the 300K nodes. Consider the node in the upper left-hand corner; from an energy balance

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad \text{or} \qquad q'_a = q'_{a,in} + \dot{E}_g \qquad \text{where} \qquad \dot{E}_g = \dot{q}V.$$

Hence, for the entire bar $\quad q'_{bar} = q'_a + q'_b + q'_c + q'_d + q'_e + q'_f$, or

$$q'_{bar} = \left[k\frac{\Delta y}{2}\frac{T_1 - 300}{\Delta x} + \dot{q}\left[\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}\right]\right]_a + \left[k\Delta y\frac{T_C - 300}{\Delta x} + \dot{q}\left[\frac{\Delta x}{2}\cdot\Delta y\right]\right]_b + \left[\dot{q}\left[\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}\right]\right]_c +$$

$$\left[k\Delta x\frac{T_C - 300}{\Delta y} + \dot{q}\left[\Delta x\cdot\frac{\Delta y}{2}\right]\right]_d + \left[k\Delta x\frac{T_3 - 300}{\Delta y} + \dot{q}\left[\Delta x\cdot\frac{\Delta y}{2}\right]\right]_e + \left[k\frac{\Delta x}{2}\frac{T_B - 300}{\Delta y} + \dot{q}\left[\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}\right]\right]_f.$$

Substituting numerical values, find $q'_{bar} = 7{,}502.5$ W/m. From an overall energy balance on the bar,

$$q'_{bar} = \dot{E}'_g = \dot{q}V/\ell = \dot{q}(3\Delta x\cdot 2\Delta y) = 5\times10^7\ \text{W/m}^3\times6(0.005\text{m})^2 = 7{,}500\ \text{W/m}. \qquad <$$

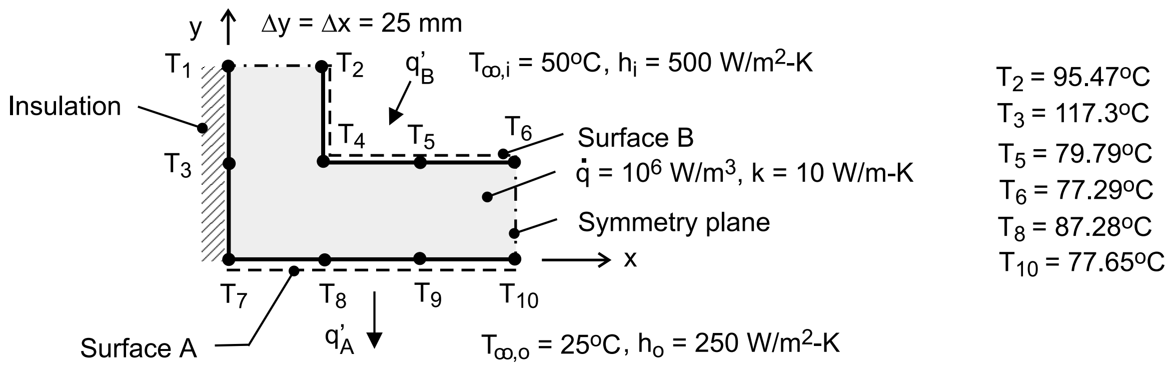
As expected, the results of the two methods agree. Why must that be?

PROBLEM 4.48

KNOWN: Steady-state temperatures at selected nodal points of the symmetrical section of a flow channel with uniform internal volumetric generation of heat. Inner and outer surfaces of channel experience convection.

FIND: (a) Temperatures at nodes 1, 4, 7, and 9, (b) Heat rate per unit length (W/m) from the outer surface A to the adjacent fluid, (c) Heat rate per unit length (W/m) from the inner fluid to surface B, and (d) Verify that results are consistent with an overall energy balance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) The nodal finite-difference equations are obtained from energy balances on control volumes about the nodes shown in the schematics below.

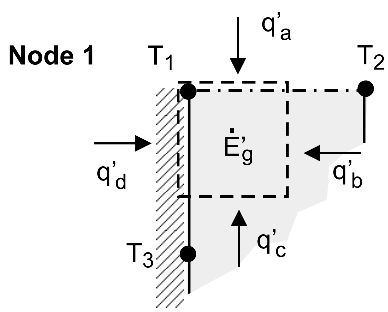
Node 1

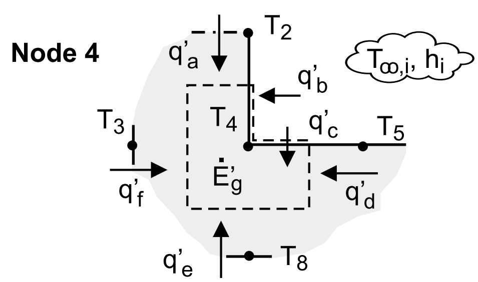
$$q'_a + q'_b + q'_c + q'_d + \dot{E}'_g = 0$$

$$0 + k(\Delta y/2)\frac{T_2 - T_1}{\Delta x} + k(\Delta x/2)\frac{T_3 - T_1}{\Delta y} + 0 + \dot{q}(\Delta x \cdot \Delta y/4) = 0$$

$$T_1 = (T_2 + T_3)/2 + \dot{q}\Delta x^2/2k$$

$$T_1 = (95.47 + 117.3)°\text{C}/2 + 10^6\ \text{W/m}^3\,(25\times25)\times10^{-6}\ \text{m}^2/(2\times10\ \text{W/m}\cdot\text{K}) = 122.0°\text{C}$$





Node 4

$$q'_a + q'_b + q'_c + q'_d + q'_e + q'_f + \dot{E}'_g = 0$$

$$k(\Delta x/2)\frac{T_2 - T_4}{\Delta y} + h_i(\Delta y/2)(T_{\infty,i} - T_4) + h_i(\Delta x/2)(T_\infty - T_4) +$$

Continued …..

$$k(\Delta y/2)\frac{T_5 - T_4}{\Delta x} + k(\Delta x)\frac{T_8 - T_4}{\Delta y} + k(\Delta y)\frac{T_3 - T_4}{\Delta x} + \dot{q}(3\Delta x \cdot \Delta y/4) = 0$$

$$T_4 = \left[T_2 + 2T_3 + T_5 + 2T_8 + 2(h_i \Delta x/k)T_{\infty,i} + \left(3\dot{q}\Delta x^2/2k\right)\right] \Big/ \left[6 + 2(h_i \Delta x/k)\right]$$

$$T_4 = 94.50°C \qquad <$$

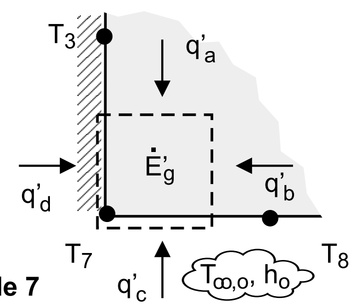
Node 7

$$q'_a + q'_b + q'_c + q'_d + \dot{E}'_g = 0$$

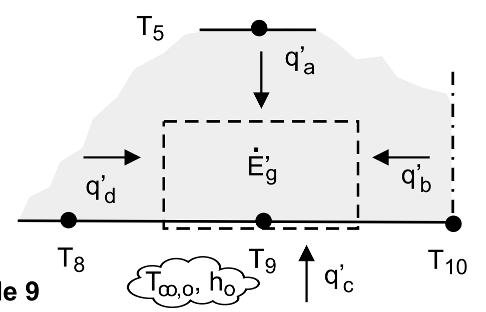
$$k(\Delta x/2)\frac{T_3 - T_7}{\Delta y} + k(\Delta y/2)\frac{T_8 - T_7}{\Delta x} + h_o(\Delta x/2)\left(T_{\infty,o} - T_7\right) + 0 + \dot{q}(\Delta x \cdot \Delta y/4) = 0$$

$$T_7 = \left[T_3 + T_8 + (h_o \Delta x/k)T_{\infty,o} + \dot{q}\Delta x^2/2k\right] \Big/ (2 + h_o \Delta x/k)$$

$$T_7 = 95.80°C \qquad <$$



Node 7



Node 9

Node 9

$$q'_a + q'_b + q'_c + q'_d + \dot{E}'_g = 0$$

$$k(\Delta x)\frac{T_5 - T_9}{\Delta y} + k(\Delta y/2)\frac{T_{10} - T_9}{\Delta y} + h_o(\Delta x)\left(T_{\infty,o} - T_9\right)$$

$$+k(\Delta y/2)\frac{T_8 - T_9}{\Delta x} + \dot{q}(\Delta x \cdot \Delta y/2) = 0$$

$$T_9 = \left[T_5 + 0.5T_8 + 0.5T_{10} + (h_o \Delta x/k)T_{\infty,o} + \dot{q}\Delta x^2/2k\right] / (2 + h_o \Delta x/k)$$

$$T_9 = 79.67°C \qquad <$$

(b) The heat rate per unit length from the outer surface A to the adjacent fluid, q'_A, is the sum of the convection heat rates from the outer surfaces of nodes 7, 8, 9 and 10.

$$q'_A = h_o\left[(\Delta x/2)\left(T_7 - T_{\infty,o}\right) + \Delta x\left(T_8 - T_{\infty,o}\right) + \Delta x\left(T_9 - T_{\infty,o}\right) + (\Delta x/2)\left(T_{10} - T_{\infty,o}\right)\right]$$

$$q'_A = 250\ \text{W/m}^2 \cdot \text{K}\left[(25/2)(95.80 - 25) + 25(87.28 - 25)\right.$$

$$\left. +25(79.67 - 25) + (25/2)(77.65 - 25)\right] \times 10^{-3}\text{m} \cdot \text{K}$$

Continued …..

$$q'_A = 1117\ \text{W/m} \qquad <$$

(c) The heat rate per unit length from the inner fluid to the surface B, q'_B, is the sum of the convection heat rates from the inner surfaces of nodes 2, 4, 5 and 6.

$$q'_B = h_i\left[(\Delta y/2)\left(T_{\infty,i} - T_2\right) + (\Delta y/2 + \Delta x/2)\left(T_{\infty,i} - T_4\right) + \Delta x\left(T_{\infty,i} - T_5\right) + (\Delta x/2)\left(T_{\infty,i} - T_6\right)\right]$$

$$q'_B = 500\ \text{W/m}^2\cdot\text{K}\left[(25/2)(50-95.47) + (25/2+25/2)(50-94.50)\right.$$

$$\left. +25(50-79.79) + (25/2)(50-77.29)\right]\times 10^{-3}\ \text{m}\cdot\text{K}$$

$$q'_B = -1383\ \text{W/m} \qquad <$$

(d) From an overall energy balance on the section, we see that our results are consistent since the conservation of energy requirement is satisfied.

$$\dot{E}'_{in} - \dot{E}'_{out} + \dot{E}'_{gen} = -q'_A + q'_B + \dot{E}'_{gen} = (-1117 - 1383 + 2500)\text{W/m} = 0$$

where $\dot{E}'_{gen} = \dot{q}\forall' = 10^6\ \text{W/m}^3\,[25\times 50 + 25\times 50]\times 10^{-6}\ \text{m}^2 = 2500\ \text{W/m}$

COMMENTS: The nodal finite-difference equations for the four nodes can be obtained by using IHT Tool *Finite-Difference Equations* | *Two-Dimensional* | *Steady-state*. Options are provided to build the FDEs for interior, corner and surface nodal arrangements including convection and internal generation. The IHT code lines for the FDEs are shown below.

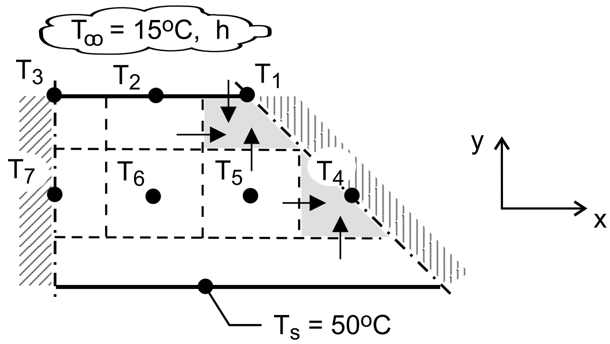
```
/* Node 1: interior node; e, w, n, s labeled 2, 2, 3, 3. */
0.0   = fd_2d_int(T1,T2,T2,T3,T3,k,qdot,deltax,deltay)

/* Node 4: internal corner node, e-n orientation; e, w, n, s labeled 5, 3, 2, 8. */
0.0   = fd_2d_ic_en(T4,T5,T3,T2,T8,k,qdot,deltax,deltay,Tinfi,hi,q•  a4
q•  a4 = 0                  // Applied heat flux, W/m^2; zero flux shown

/* Node 7: plane surface node, s-orientation; e, w, n labeled 8, 8, 3. */
0.0   = fd_2d_psur_s(T7,T8,T8,T3,k,qdot,deltax,deltay,Tinfo,ho,q•  a7
q•  a7=0                    // Applied heat flux, W/m^2; zero flux shown

/* Node 9: plane surface node, s-orientation; e, w, n labeled 10, 8, 5. */
0.0 = fd_2d_psur_s(T9, T10, T8, T5,k,qdot,deltax,deltay,Tinfo,ho,q•  a9
q•  a9 = 0                  // Applied heat flux, W/m^2; zero flux shown
```

PROBLEM 4.49

KNOWN: Outer surface temperature, inner convection conditions, dimensions and thermal conductivity of a heat sink.

FIND: Nodal temperatures and heat rate per unit length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Two-dimensional conduction, (3) Uniform outer surface temperature, (4) Constant thermal conductivity.

ANALYSIS: (a) To determine the heat rate, the nodal temperatures must first be computed from the corresponding finite-difference equations. From an energy balance for node 1,

$$h(\Delta x/2\cdot 1)(T_\infty - T_1) + k(\Delta y/2\cdot 1)\frac{T_2 - T_1}{\Delta x} + k(\Delta x\cdot 1)\frac{T_5 - T_1}{\Delta y} = 0$$

$$-\left(3 + \frac{h\Delta x}{k}\right)T_1 + T_2 + 2T_5 + \frac{h\Delta x}{k}T_\infty = 0 \tag{1}$$

With nodes 2 and 3 corresponding to Case 3 of Table 4.2,

$$T_1 - 2\left(\frac{h\Delta x}{k} + 2\right)T_2 + T_3 + 2T_6 + \frac{2h\Delta x}{k}T_\infty = 0 \tag{2}$$

$$T_2 - \left(\frac{h\Delta x}{k} + 2\right)T_3 + T_7 + \frac{h\Delta x}{k}T_\infty = 0 \tag{3}$$

where the symmetry condition is invoked for node 3. Applying an energy balance to node 4, we obtain

$$-2T_4 + T_5 + T_s = 0 \tag{4}$$

The interior nodes 5, 6 and 7 correspond to Case 1 of Table 4.2. Hence,

$$T_1 + T_4 - 4T_5 + T_6 + T_s = 0 \tag{5}$$

$$T_2 + T_5 - 4T_6 + T_7 + T_s = 0 \tag{6}$$

$$T_3 + 2T_6 - 4T_7 + T_s = 0 \tag{7}$$

where the symmetry condition is invoked for node 7. With $T_s = 50°C$, $T_\infty = 20°C$, and $h\Delta x/k = 5000\,W/m^2\cdot K(0.005m)/240\,W/m\cdot K = 0.1042$, the solution to Eqs. (1) – (7) yields

$$T_1 = 46.61°C,\ T_2 = 45.67°C,\ T_3 = 45.44°C,\ T_4 = 49.23°C$$

$$T_5 = 48.46°C,\ T_6 = 48.00°C,\ T_7 = 47.86°C$$ <

Continued …..

PROBLEM 4.49 (Cont.)

The heat rate per unit length of channel may be evaluated by computing convection heat transfer from the inner surface. That is,

$$q' = 8h\left[\Delta x/2\left(T_1 - T_\infty\right) + \Delta x\left(T_2 - T_\infty\right) + \Delta x/2\left(T_3 - T_\infty\right)\right]$$

$$q' = 8\times 5000\,W/m^2\cdot K\left[0.0025m\left(46.61-20\right)°C + 0.005m\left(45.67-20\right)°C\right.$$

$$\left. +0.0025m\left(45.44-20\right)°C\right] = 10,340\ W/m \qquad <$$

(b) Since $h = 5000\ W/m^2\cdot K$ is at the high end of what can be achieved through forced convection, we consider the effect of reducing h. Representative results are as follows

$h\left(W/m^2\cdot K\right)$	$T_1\,(°C)$	$T_2\,(°C)$	$T_3\,(°C)$	$T_4\,(°C)$	$T_5\,(°C)$	$T_6\,(°C)$	$T_7\,(°C)$	$q'\,(W/m)$
200	49.84	49.80	49.79	49.96	49.93	49.91	49.90	477
1000	49.24	49.02	48.97	49.83	49.65	49.55	49.52	2325
2000	48.53	48.11	48.00	49.66	49.33	49.13	49.06	4510
5000	46.61	45.67	45.44	49.23	48.46	48.00	47.86	10,340

There are two resistances to heat transfer between the outer surface of the heat sink and the fluid, that due to conduction in the heat sink, $R_{cond(2D)}$, and that due to convection from its inner surface to the fluid, R_{conv}. With decreasing h, the corresponding increase in R_{conv} reduces heat flow and increases the uniformity of the temperature field in the heat sink. The nearly 5-fold reduction in q' corresponding to the 5-fold reduction in h from 1000 to 200 $W/m^2\cdot K$ indicates that the convection resistance is dominant $\left(R_{conv} >> R_{cond(2D)}\right)$.

COMMENTS: To check our finite-difference solution, we could assess its consistency with conservation of energy requirements. For example, an energy balance performed at the inner surface requires a balance between convection from the surface and conduction to the surface, which may be expressed as

$$q' = k\left(\Delta x\cdot 1\right)\frac{\left(T_5 - T_1\right)}{\Delta y} + k\left(\Delta x\cdot 1\right)\frac{T_6 - T_2}{\Delta y} + k\left(\Delta x/2\cdot 1\right)\frac{T_7 - T_3}{\Delta y}$$

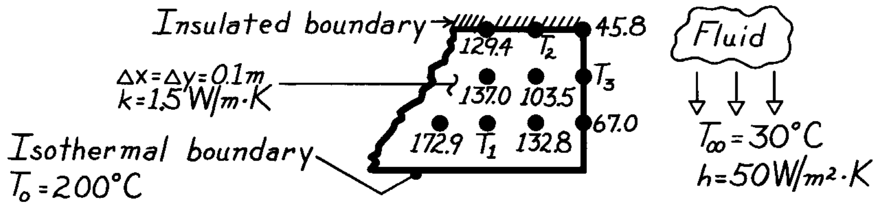
Substituting the temperatures corresponding to $h = 5000\ W/m^2\cdot K$, the expression yields $q' = 10,340\,W/m$, and, as it must be, conservation of energy is precisely satisfied. Results of the analysis may also be checked by using the expression $q' = \left(T_s - T_\infty\right)/\left(R'_{cond(2D)} + R'_{conv}\right)$, where, for $h = 5000\ W/m^2\cdot K$, $R'_{conv} = \left(1/4hw\right) = 2.5\times 10^{-3}\,m\cdot K/W$, and from Eq. (4.27) and Case 11 of Table 4.1, $R'_{cond} = \left[0.930\ \ln\left(W/w\right) - 0.05\right]/2\pi k = 3.94\times 10^{-4}\,m\cdot K/W$. Hence, $q' = \left(50-20\right)°C/\left(2.5\times 10^{-3} + 3.94\times 10^{-4}\right)m\cdot K/W = 10,370\ W/m$, and the agreement with the finite-difference solution is excellent. Note that, even for $h = 5000\ W/m^2\cdot K$, $R'_{conv} >> R'_{cond(2D)}$.

PROBLEM 4.50

KNOWN: Steady-state temperatures (°C) associated with selected nodal points in a two-dimensional system.

FIND: (a) Temperatures at nodes 1, 2 and 3, (b) Heat transfer rate per unit thickness from the system surface to the fluid.

SCHEMATIC:



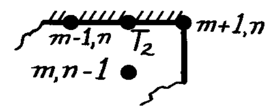
ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) Using the finite-difference equations for Nodes 1, 2 and 3:

Node 1, Interior node, Eq. 4.33: $T_1 = \frac{1}{4} \cdot \sum T_{neighbors}$

$$T_1 = \frac{1}{4}(172.9 + 137.0 + 132.8 + 200.0)^\circ C = 160.7^\circ C \qquad <$$

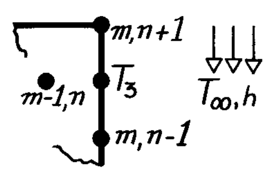
Node 2, Insulated boundary, Eq. 4.46 with h = 0, $T_{m,n} = T_2$



$$T_2 = \frac{1}{4}\left(T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1}\right)$$

$$T_2 = \frac{1}{4}(129.4 + 45.8 + 2 \times 103.5)^\circ C = 95.6^\circ C \qquad <$$

Node 3, Plane surface with convection, Eq. 4.46, $T_{m,n} = T_3$



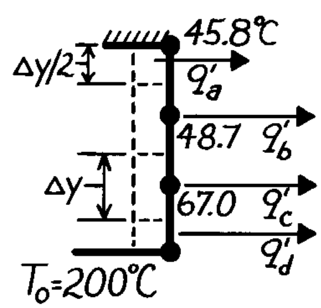
$$2\left[\frac{h\Delta x}{k} + 2\right]T_3 = \left(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) + \frac{2h\Delta x}{k}T_\infty$$

$$h\Delta x/k = 50 W/m^2 \cdot K \times 0.1 m/1.5 W/m \cdot K = 3.33$$

$$2(3.33 + 2)T_3 = (2 \times 103.5 + 45.8 + 67.0)^\circ C + 2 \times 3.33 \times 30^\circ C$$

$$T_3 = \frac{1}{10.66}(319.80 + 199.80)^\circ C = 48.7^\circ C \qquad <$$

(b) The heat rate per unit thickness from the surface to the fluid is determined from the sum of the convection rates from each control volume surface.



$$q'_{conv} = q'_a + q'_b + q'_c + q'_d$$

$$q_i = h\Delta y_i\,(T_i - T_\infty)$$

$$q'_{conv} = 50\frac{W}{m^2 \cdot K}\left[\frac{0.1}{2}m\,(45.8 - 30.0)^\circ C + 0.1m\,(48.7 - 30.0)^\circ C + 0.1m\,(67.0 - 30.0)^\circ C + \frac{0.1m}{2}(200.0 - 30.0)^\circ C\right]$$

$$q'_{conv} = (39.5 + 93.5 + 185.0 + 425)\ W/m = 743\ W/m. \qquad <$$

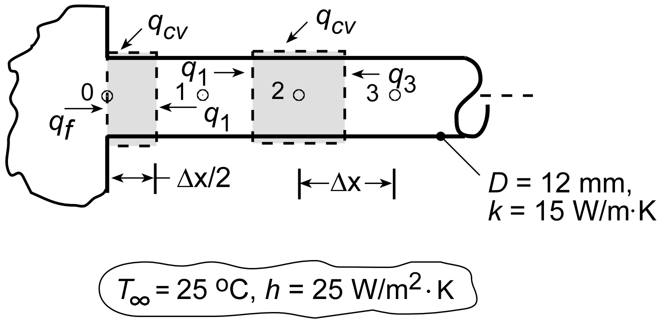
PROBLEM 4.51

KNOWN: Nodal temperatures from a steady-state finite-difference analysis for a cylindrical fin of prescribed diameter, thermal conductivity and convection conditions (T_∞, h).

FIND: (a) The fin heat rate, q_f, and (b) Temperature at node 3, T_3.

SCHEMATIC:

$T_0 = 100.0°C$
$T_1 = 93.4°C$
$T_2 = 89.5°C$



ASSUMPTIONS: (a) The fin heat rate, q_f, is that of conduction at the base plane, x = 0, and can be found from an energy balance on the control volume about node 0, $\dot{E}_{in} - \dot{E}_{out} = 0$,

$$q_f + q_1 + q_{conv} = 0 \qquad \text{or} \qquad q_f = -q_1 - q_{conv}.$$

Writing the appropriate rate equation for q_1 and q_{conv}, with $A_c = \pi D^2/4$ and $P = \pi D$,

$$q_f = -kA_c \frac{T_1 - T_0}{\Delta x} - hP(\Delta x/2)(T_\infty - T_0) = -\frac{\pi k D^2}{4\Delta x}(T_1 - T_0) - (\pi/2)Dh\Delta x(T_\infty - T_0)$$

Substituting numerical values, with Δx = 0.010 m, find

$$q_f = -\frac{\pi \times 15\,\text{W/m}\cdot\text{K}(0.012\,\text{m})^2}{4 \times 0.010\,\text{m}}(93.4 - 100)^\circ\text{C}$$
$$-\frac{\pi}{2} \times 0.012\,\text{m} \times 25\,\text{W/m}^2\cdot\text{K} \times 0.010\,\text{m}(25 - 100)^\circ\text{C}$$

$$q_f = (1.120 + 0.353)\,\text{W} = 1.473\,\text{W}.$$ <

(b) To determine T_3, derive the finite-difference equation for node 2, perform an energy balance on the control volume shown above, $\dot{E}_{in} - \dot{E}_{out} = 0$,

$$q_{cv} + q_3 + q_1 = 0$$

$$hP\Delta x(T_\infty - T_2) + kA_c \frac{T_3 - T_2}{\Delta x} + kA_c \frac{T_1 - T_2}{\Delta x} = 0$$

$$T_3 = -T_1 + 2T_2 - \frac{hP\Delta x^2}{kA_c}\Delta x^2 [T_\infty - T_2]$$

Substituting numerical values, find

$$T_2 = 89.2^\circ\text{C}$$ <

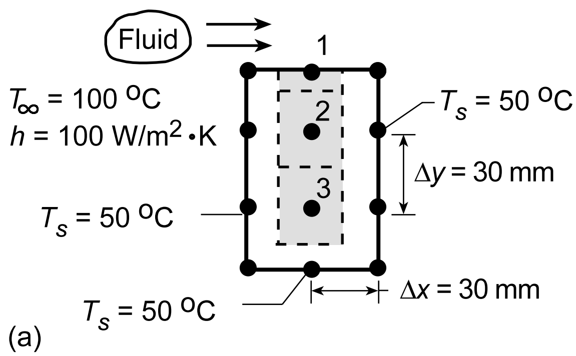
COMMENTS: Note that in part (a), the convection heat rate from the outer surface of the control volume is significant (25%). It would have been poor approximation to ignore this term.

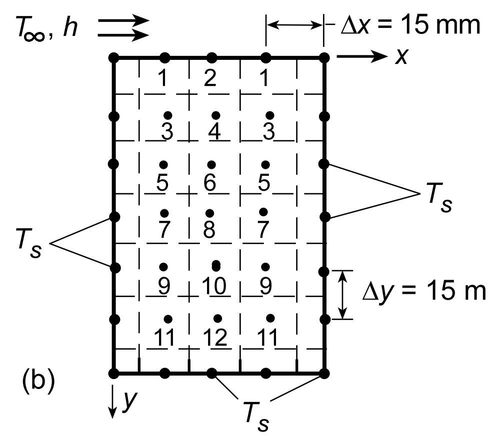
PROBLEM 4.52

KNOWN: Long rectangular bar having one boundary exposed to a convection process (T_∞, h) while the other boundaries are maintained at a constant temperature (T_s).

FIND: (a) Using a grid spacing of 30 mm and the Gauss-Seidel method, determine the nodal temperatures and the heat rate per unit length into the bar from the fluid, (b) Effect of grid spacing and convection coefficient on the temperature field.

SCHEMATIC:





ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) With the grid spacing $\Delta x = \Delta y = 30$ mm, three nodes are created. Using the finite-difference equations as shown in Table 4.2, but written in the form required of the Gauss-Seidel method (see Section 4.5.2), and with $Bi = h\Delta x/k = 100\ W/m^2\cdot K \times 0.030\ m/1\ W/m\cdot K = 3$, we obtain:

Node 1: $$T_1 = \frac{1}{(Bi+2)}\left(T_2 + T_s + BiT_\infty\right) = \frac{1}{5}\left(T_2 + 50 + 3\times 100\right) = \frac{1}{5}\left(T_2 + 350\right) \quad (1)$$

Node 2: $$T_2 = \frac{1}{4}\left(T_1 + 2T_s + T_3\right) = \frac{1}{4}\left(T_1 + T_3 + 2\times 50\right) = \frac{1}{4}\left(T_1 + T_3 + 100\right) \quad (2)$$

Node 3: $$T_3 = \frac{1}{4}\left(T_2 + 3T_s\right) = \frac{1}{4}\left(T_2 + 3\times 50\right) = \frac{1}{4}\left(T_2 + 150\right) \quad (3)$$

Denoting each nodal temperature with a superscript to indicate iteration step, e.g. T_1^k, calculate values as shown below.

k	T_1	T_2	T_3 (°C)	
0	85	60	55	← initial guess
1	82.00	59.25	52.31	
2	81.85	58.54	52.14	
3	81.71	58.46	52.12	
4	81.69	58.45	52.11	

By the 4th iteration, changes are of order 0.02°C, suggesting that further calculations may not be necessary.

Continued...

PROBLEM 4.52 (Cont.)

In finite-difference form, the heat rate from the fluid to the bar is

$$q'_{conv} = h\left(\Delta x/2\right)\left(T_\infty - T_s\right) + h\Delta x\left(T_\infty - T_1\right) + h\left(\Delta x/2\right)\left(T_\infty - T_s\right)$$

$$q'_{conv} = h\Delta x\left(T_\infty - T_s\right) + h\Delta x\left(T_\infty - T_1\right) = h\Delta x\left[\left(T_\infty - T_s\right) + \left(T_\infty - T_1\right)\right]$$

$$q'_{conv} = 100\,W/m^2\cdot K \times 0.030\,m\left[\left(100-50\right)+\left(100-81.7\right)\right]^\circ C = 205\,W/m\,. \qquad <$$

(b) Using the *Finite-Difference Equations* option from the *Tools* portion of the IHT menu, the following two-dimensional temperature field was computed for the grid shown in schematic (b), where x and y are in mm and the temperatures are in °C.

y\x	0	15	30	45	60
0	50	80.33	85.16	80.33	50
15	50	63.58	67.73	63.58	50
30	50	56.27	58.58	56.27	50
45	50	52.91	54.07	52.91	50
60	50	51.32	51.86	51.32	50
75	50	50.51	50.72	50.51	50
90	50	50	50	50	50

The improved prediction of the temperature field has a significant influence on the heat rate, where, accounting for the symmetrical conditions,

$$q' = 2h\left(\Delta x/2\right)\left(T_\infty - T_s\right) + 2h\left(\Delta x\right)\left(T_\infty - T_1\right) + h\left(\Delta x\right)\left(T_\infty - T_2\right)$$

$$q' = h\left(\Delta x\right)\left[\left(T_\infty - T_s\right) + 2\left(T_\infty - T_1\right) + \left(T_\infty - T_2\right)\right]$$

$$q' = 100\,W/m^2\cdot K\left(0.015\,m\right)\left[50 + 2\left(19.67\right) + 14.84\right]^\circ C = 156.3\,W/m \qquad <$$

Additional improvements in accuracy could be obtained by reducing the grid spacing to 5 mm, although the requisite number of finite-difference equations would increase from 12 to 108, significantly increasing problem *set-up* time.

An increase in h would increase temperatures everywhere within the bar, particularly at the heated surface, as well as the rate of heat transfer by convection to the surface.

COMMENTS: (1) Using the matrix-inversion method, the exact solution to the system of equations (1, 2, 3) of part (a) is T_1 = 81.70°C, T_2 = 58.44°C, and T_3 = 52.12°C. The fact that only 4 iterations were required to obtain agreement within 0.01°C is due to the close initial guesses.

(2) Note that the rate of heat transfer by convection to the top surface of the rod must balance the rate of heat transfer by conduction to the sides and bottom of the rod.

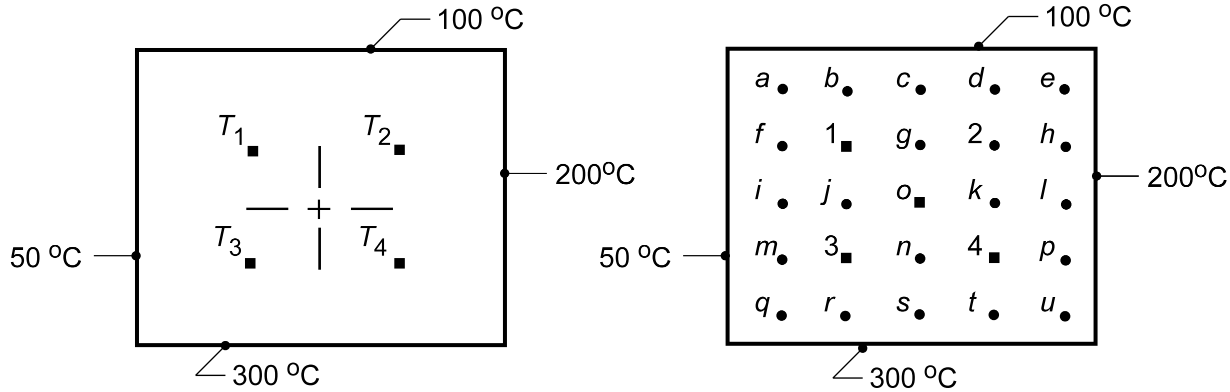
NOTE TO INSTRUCTOR: Although the problem statement calls for calculations with $\Delta x = \Delta y$ = 5 mm and for plotting associated isotherms, the instructional value and benefit-to-effort ratio are small. Hence, it is recommended that this portion of the problem not be assigned.

PROBLEM 4.53

KNOWN: Square shape subjected to uniform surface temperature conditions.

FIND: (a) Temperature at the four specified nodes; estimate the midpoint temperature T_o, (b) Reducing the mesh size by a factor of 2, determine the corresponding nodal temperatures and compare results, and (c) For the finer grid, plot the 75, 150, and 250°C isotherms.

SCHEMATIC:



(a) Coarse grid (b) Finer grid

ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) The finite-difference equation for each node follows from Eq. 4.33 for an interior point written in the form, $T_i = 1/4\Sigma T_{neighbors}$. Using the Gauss-Seidel iteration method, Section 4.5.2, the finite-difference equations for the four nodes are:

$$T_1^k = 0.25\left(100 + T_2^{k-1} + T_3^{k-1} + 50\right) = 0.25T_2^{k-1} + 0.25T_3^{k-1} + 37.5$$

$$T_2^k = 0.25\left(100 + 200 + T_4^{k-1} + T_1^{k-1}\right) = 0.25T_1^{k-1} + 0.25T_4^{k-1} + 75.0$$

$$T_3^k = 0.25\left(T_1^{k-1} + T_4^{k-1} + 300 + 50\right) = 0.25T_1^{k-1} + 0.25T_4^{k-1} + 87.5$$

$$T_4^k = 0.25\left(T_2^{k-1} + 200 + 300 + T_3^{k-1}\right) = 0.25T_2^{k-1} + 0.25T_3^{k-1} + 125.0$$

The iteration procedure using a hand calculator is implemented in the table below. Initial estimates are entered on the k = 0 row.

k	T_1 (°C)	T_2 (°C)	T_3 (°C)	T_4 (°C)
0	100	150	150	250
1	112.50	165.63	178.13	210.94
2	123.44	158.60	171.10	207.43
3	119.93	156.40	169.34	206.55
4	119.05	156.40	168.90	206.33
5	118.83	156.29	168.79	206.27
6	118.77	156.26	168.76	206.26
7	118.76	156.25	168.76	206.25

<

Continued...

PROBLEM 4.53 (Cont.)

By the seventh iteration, the convergence is approximately 0.01°C. The midpoint temperature can be estimated as

$$T_o = (T_1 + T_2 + T_3 + T_4)/2 = (118.76 + 156.25 + 168.76 + 206.25)^\circ C/4 = 162.5^\circ C$$

(b) Because all the nodes are interior ones, the nodal equations can be written by inspection directly into the IHT workspace and the set of equations solved for the nodal temperatures (°C).

Mesh	T_o	T_1	T_2	T_3	T_4
Coarse	162.5	118.76	156.25	168.76	206.25
Fine	162.5	117.4	156.1	168.9	207.6

The maximum difference for the interior points is 1.5°C (node 4), but the estimate at the center, T_o, is the same, independently of the mesh size. In terms of the boundary surface temperatures,

$$T_o = (50 + 100 + 200 + 300)^\circ C/4 = 162.5^\circ C$$

Why must this be so?

(c) To generate the isotherms, it would be necessary to employ a contour-drawing routine using the tabulated temperature distribution (°C) obtained from the finite-difference solution. Using these values as a guide, try sketching a few isotherms.

-	100	100	100	100	100	-
50	86.0	105.6	119	131.7	151.6	200
50	88.2	117.4	138.7	156.1	174.6	200
50	99.6	137.1	162.5	179.2	190.8	200
50	123.0	168.9	194.9	207.6	209.4	200
50	173.4	220.7	240.6	246.8	239.0	200
-	300	300	300	300	300	-

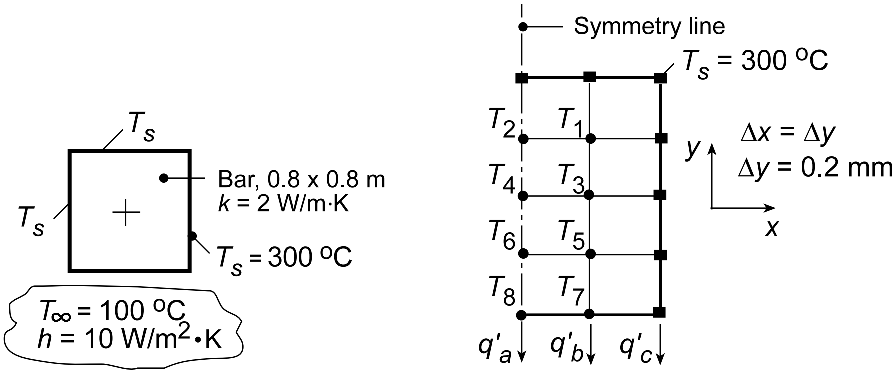
COMMENTS: Recognize that this finite-difference solution is only an approximation to the temperature distribution, since the heat conduction equation has been solved for only four (or 25) discrete points rather than for all points if an analytical solution had been obtained.

PROBLEM 4.54

KNOWN: Long bar of square cross section, three sides of which are maintained at a constant temperature while the fourth side is subjected to a convection process.

FIND: (a) The mid-point temperature and heat transfer rate between the bar and fluid; a numerical technique with grid spacing of 0.2 m is suggested, and (b) Reducing the grid spacing by a factor of 2, find the midpoint temperature and the heat transfer rate. Also, plot temperature distribution across the surface exposed to the fluid.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) Considering symmetry, the nodal network is shown above. The matrix inversion method of solution will be employed. The finite-difference equations are:

Nodes 1, 3, 5 -	Interior nodes, Eq. 4.33; written by inspection.
Nodes 2, 4, 6 -	Also can be treated as interior points, considering symmetry.
Nodes 7, 8 -	On a plane with convection, Eq. 4.46; noting that $h\Delta x/k = 10\ W/m^2\cdot K \times 0.2\ m/2W/m\cdot K = 1$, find
	Node 7: $(2T_5 + 300 + T_8) + 2\times1\cdot100 - 2(1+2)T_7 = 0$
	Node 8: $(2T_6 + T_7 + T_7) + 2\times1\cdot100 - 2(1+2)T_8 = 0$

The solution matrix [T] can be found using a stock matrix program using the [A] and [C] matrices shown below to obtain the solution matrix [T] (Eq. 4.52). Alternatively, the set of equations could be entered into the IHT workspace and solved for the nodal temperatures.

$$A = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} -600 \\ -300 \\ -300 \\ 0 \\ -300 \\ 0 \\ -500 \\ -200 \end{bmatrix} \qquad T = \begin{bmatrix} 292.2 \\ 289.2 \\ 279.7 \\ 272.2 \\ 254.5 \\ 240.1 \\ 198.1 \\ 179.4 \end{bmatrix}$$

From the solution matrix, [T], find the mid-point temperature as

$T_4 = 272.2°C$ <

Continued...

The heat rate by convection between the bar and fluid is given as,

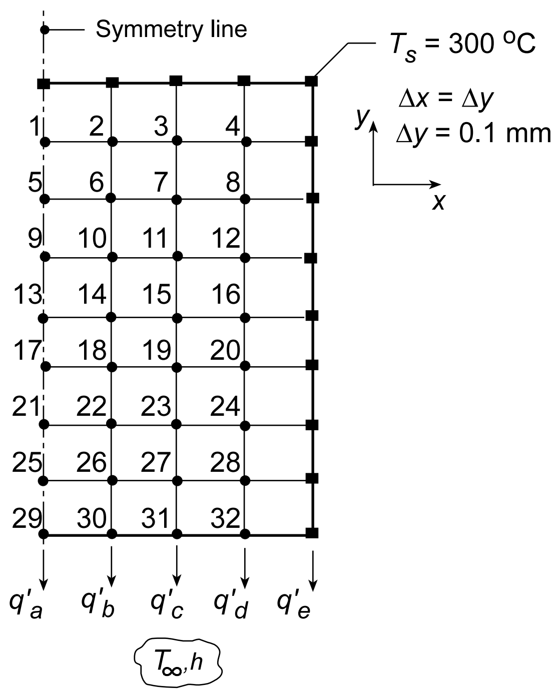
$$q'_{conv} = 2\left(q'_a + q'_b + q'_c\right)$$

$$q'_{conv} = 2x\left[h\left(\Delta x/2\right)\left(T_8 - T_\infty\right) + h\left(\Delta x\right)\left(T_7 - T_\infty\right) + h\left(\Delta x/2\right)\left(300 - T_\infty\right)\right]$$

$$q'_{conv} = 2x\left[10\,W/m^2 \cdot K \times \left(0.2\,m/2\right)\left[\left(179.4 - 100\right) + 2\left(198.1 - 100\right) + \left(300 - 100\right)\right]K\right]$$

$$q'_{conv} = 952\,W/m\,.$$ <

(b) Reducing the grid spacing by a factor of 2, the nodal arrangement will appear as shown. The finite-difference equation for the interior and centerline nodes were written by inspection and entered into the IHT workspace. The *IHT Finite-Difference Equations Tool* for 2-D, SS conditions, was used to obtain the FDE for the nodes on the exposed surface.



The midpoint temperature T_{13} and heat rate for the finer mesh are

T_{13} = 271.0°C $\qquad$ q' = 834 W/m <

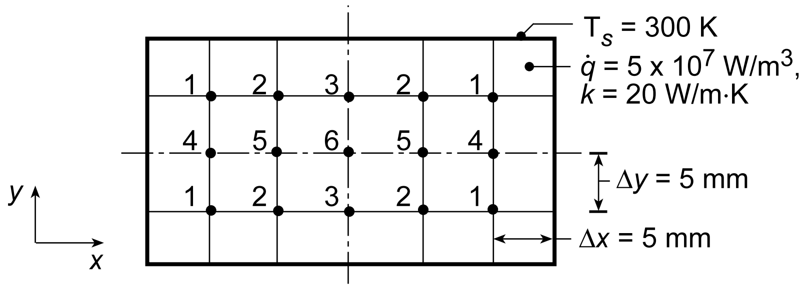
COMMENTS: The midpoint temperatures for the coarse and finer meshes agree closely, T_4 = 272°C vs. T_{13} = 271.0°C, respectively. However, the estimate for the heat rate is substantially influenced by the mesh size; q' = 952 vs. 834 W/m for the coarse and finer meshes, respectively.

PROBLEM 4.55

KNOWN: Volumetric heat generation in a rectangular rod of uniform surface temperature.

FIND: (a) Temperature distribution in the rod, and (b) With boundary conditions unchanged, heat generation rate causing the midpoint temperature to reach 600 K.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) Uniform volumetric heat generation.

ANALYSIS: (a) From symmetry it follows that six unknown temperatures must be determined. Since all nodes are interior ones, the finite-difference equations may be obtained from Eq. 4.39 written in the form

$$T_i = 1/2 \sum T_{neighbors} + 1/4\left(\dot{q}\left(\Delta x \Delta y 1\right)/k\right).$$

With $\dot{q}\left(\Delta x \Delta y\right)/4k$ = 62.5 K, the system of finite-difference equations is

$$T_1 = 0.25\left(T_s + T_2 + T_4 + T_s\right) + 15.625 \quad (1)$$

$$T_2 = 0.25\left(T_s + T_3 + T_5 + T_1\right) + 15.625 \quad (2)$$

$$T_3 = 0.25\left(T_s + T_2 + T_6 + T_2\right) + 15.625 \quad (3)$$

$$T_4 = 0.25\left(T_1 + T_5 + T_1 + T_s\right) + 15.625 \quad (4)$$

$$T_5 = 0.25\left(T_2 + T_6 + T_2 + T_4\right) + 15.625 \quad (5)$$

$$T_6 = 0.25\left(T_3 + T_5 + T_3 + T_5\right) + 15.625 \quad (6)$$

With T_s = 300 K, the set of equations was written directly into the IHT workspace and solved for the nodal temperatures,

T_1	T_2	T_3	T_4	T_5	T_6 (K)
348.6	368.9	374.6	362.4	390.2	398.0

<

(b) With the boundary conditions unchanged, the $\dot{q}$ required for T_6 = 600 K can be found using the same set of equations in the IHT workspace, but with these changes: (1) replace the last term on the RHS (15.625) of Eqs. (1-6) by $\dot{q}$ $(\Delta x\Delta y)/4k = (0.005\ m)^2\ \dot{q}/4\times20$ W/m·K = $3.125 \times 10^{-7}\ \dot{q}$ and (2) set T_6 = 600 K. The set of equations has 6 unknown, five nodal temperatures plus $\dot{q}$. Solving find

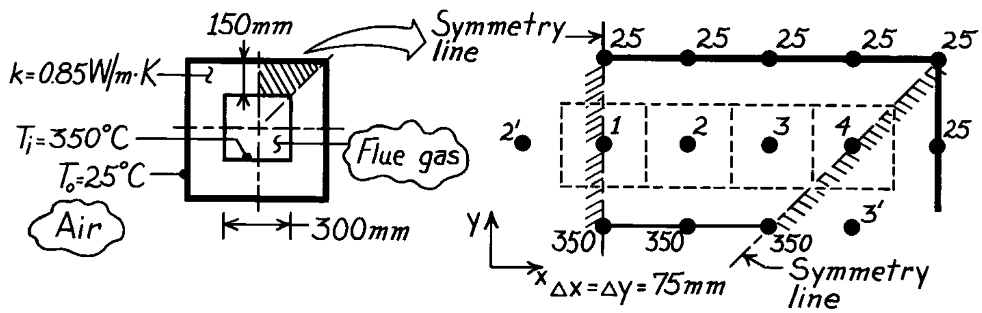
$$\dot{q} = 1.53\times10^8\ \mathrm{W/m^3}$$ <

PROBLEM 4.56

KNOWN: Flue of square cross section with prescribed geometry, thermal conductivity and inner and outer surface temperatures.

FIND: Heat loss per unit length from the flue, q'.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) No internal generation.

ANALYSIS: Taking advantage of symmetry, the nodal network using the suggested 75mm grid spacing is shown above. To obtain the heat rate, we first need to determine the unknown temperatures T_1, T_2, T_3 and T_4. Recognizing that these nodes may be treated as interior nodes, the nodal equations from Eq. 4.33 are

$$(T_2 + 25 + T_2 + 350) - 4T_1 = 0$$
$$(T_1 + 25 + T_3 + 350) - 4T_2 = 0$$
$$(T_2 + 25 + T_4 + 350) - 4T_3 = 0$$
$$(T_3 + 25 + 25 + T_3) - 4T_4 = 0.$$

The Gauss-Seidel iteration method is convenient for this system of equations and following the procedures of Section 4.5.2, they are rewritten as,

$$T_1^k = 0.50\, T_2^{k-1} + 93.75$$
$$T_2^k = 0.25\, T_1^k + 0.25\, T_3^{k-1} + 93.75$$
$$T_3^k = 0.25\, T_2^k + 0.25\, T_4^{k-1} + 93.75$$
$$T_4^k = 0.50\, T_3^k + 12.5.$$

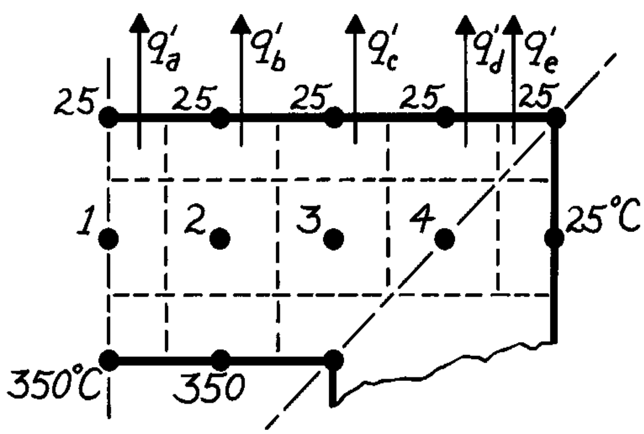
The iteration procedure is implemented in the table on the following page, one row for each iteration k. The initial estimates, for k = 0, are all chosen as (350 + 25)/2 ≈ 185°C. Iteration is continued until the maximum temperature difference is less than 0.2°C, i.e., $\varepsilon < 0.2°C$.

Note that if the system of equations were organized in matrix form, Eq. 4.52, diagonal dominance would exist. Hence there is no need to reorder the equations since the magnitude of the diagonal element is greater than that of other elements in the same row.

Continued …..

k	T_1(°C)	T_2(°C)	T_3(°C)	T_4(°C)	
0	185	185	185	185	← initial estimate
1	186.3	186.6	186.6	105.8	
2	187.1	187.2	167.0	96.0	
3	187.4	182.3	163.3	94.2	
4	184.9	180.8	162.5	93.8	
5	184.2	180.4	162.3	93.7	
6	184.0	180.3	162.3	93.6	
7	183.9	180.3	162.2	93.6	← ε <0.2°C

From knowledge of the temperature distribution, the heat rate may be obtained by summing the heat rates across the nodal control volume surfaces, as shown in the sketch.



The heat rate leaving the outer surface of this flue section is,

$$q' = q'_a + q'_b + q'_c + q'_d + q'_e$$

$$q' = k\frac{\Delta x}{\Delta y}\left[\frac{1}{2}(T_1 - 25) + (T_2 - 25) + (T_3 - 25) + (T_4 - 25) + 0\right]$$

$$q' = 0.85\frac{W}{m\cdot K}\left[\frac{1}{2}(183.9 - 25) + (180.3 - 25) + (162.2 - 26) + (93.6 - 25)\right]$$

$$q' = 374.5 \text{ W/m.}$$

Since this flue section is 1/8 the total cross section, the total heat loss from the flue is

$$q' = 8 \times 374.5 \text{ W/m} = 3.00 \text{ kW/m.}$$ <

COMMENTS: The heat rate could have been calculated at the inner surface, and from the above sketch has the form

$$q' = k\frac{\Delta x}{\Delta y}\left[\frac{1}{2}(350 - T_1) + (350 - T_2) + (350 - T_3)\right] = 374.5 \text{ W/m.}$$

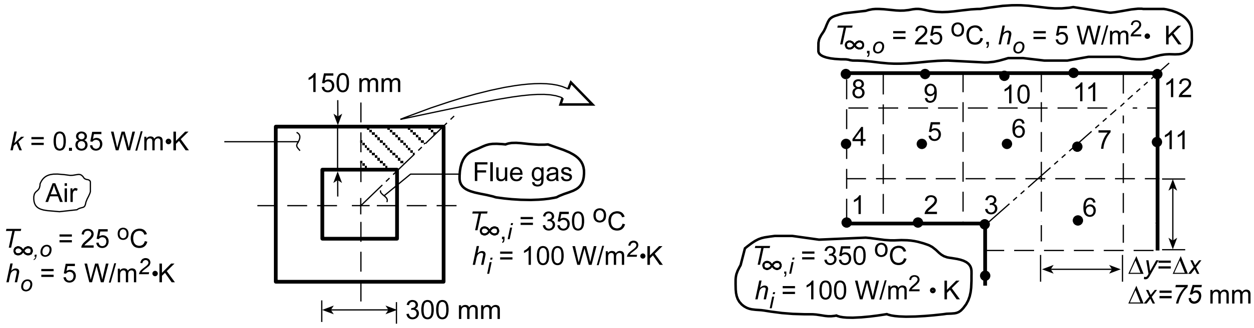
This result should compare very closely with that found for the outer surface since the conservation of energy requirement must be satisfied in obtaining the nodal temperatures.

PROBLEM 4.57

KNOWN: Flue of square cross section with prescribed geometry, thermal conductivity and inner and outer surface convective conditions.

FIND: (a) Heat loss per unit length, q', by convection to the air, (b) Effect of grid spacing and convection coefficients on temperature field; show isotherms.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) Taking advantage of symmetry, the nodal network for a 75 mm grid spacing is shown in schematic (a). To obtain the heat rate, we need first to determine the temperatures T_i. Recognize that there are four types of nodes: interior (4-7), plane surface with convection (1, 2, 8-11), internal corner with convection (3), and external corner with convection (12). Using the appropriate relations from Table 4.2, the finite-difference equations are

Node		Equation
1	$(2T_4 + T_2 + T_2) + \dfrac{2h_i\Delta x}{k}T_{\infty,i} - \left(\dfrac{h_i\Delta x}{k} + 2\right)T_1 = 0$	4.46
2	$(2T_5 + T_3 + T_1) + \dfrac{2h_i\Delta x}{k}T_{\infty,i} - 2\left(\dfrac{h_i\Delta x}{k} + 2\right)T_2 = 0$	4.46
3	$2(T_6 + T_6) + (T_2 + T_2) + \dfrac{2h_i\Delta x}{k}T_{\infty,i} - 2\left(3 + \dfrac{h_i\Delta x}{k}\right)T_3 = 0$	4.45
4	$(T_8 + T_5 + T_1 + T_5) - 4T_4 = 0$	4.33
5	$(T_9 + T_6 + T_2 + T_4) - 4T_5 = 0$	4.33
6	$(T_{10} + T_7 + T_3 + T_5) - 4T_6 = 0$	4.33
7	$(T_{11} + T_{11} + T_6 + T_6) - 4T_7 = 0$	4.33
8	$(2T_4 + T_9 + T_9) + \dfrac{2h_o\Delta x}{k}T_{\infty,o} - 2\left(\dfrac{h_o\Delta x}{k} + 2\right)T_8 = 0$	4.46
9	$(2T_5 + T_{10} + T_8) + \dfrac{2h_o\Delta x}{k}T_{\infty,o} - 2\left(\dfrac{h_o\Delta x}{k} + 2\right)T_9 = 0$	4.46
10	$(2T_6 + T_{11} + T_9) + \dfrac{2h_o\Delta x}{k}T_{\infty,o} - 2\left(\dfrac{h_o\Delta x}{k} + 2\right)T_{10} = 0$	4.46
11	$(2T_7 + T_{12} + T_{10}) + \dfrac{2h_o\Delta x}{k}T_{\infty,o} - 2\left(\dfrac{h_o\Delta x}{k} + 2\right)T_{11} = 0$	4.46
12	$(T_{11} + T_{11}) + \dfrac{2h_o\Delta x}{k}T_{\infty,o} - 2\left(\dfrac{h_o\Delta x}{k} + 1\right)T_{12} = 0$	4.47

Continued...

PROBLEM 4.57 (Cont.)

The Gauss-Seidel iteration is convenient for this system of equations. Following procedures of Section 4.5.2, the system of equations is rewritten in the proper form. Note that diagonal dominance is present; hence, no re-ordering is necessary.

$$T_1^k = 0.09239T_2^{k-1} + 0.09239T_4^{k-1} + 285.3$$

$$T_2^k = 0.04620T_1^k + 0.04620T_3^{k-1} + 0.09239T_5^{k-1} + 285.3$$

$$T_3^k = 0.08457T_2^k + 0.1692T_6^{k-1} + 261.2$$

$$T_4^k = 0.25T_1^k + 0.50T_5^{k-1} + 0.25T_8^{k-1}$$

$$T_5^k = 0.25T_2^k + 0.25T_4^k + 0.25T_6^{k-1} + 0.25T_9^{k-1}$$

$$T_6^k = 0.25T_3^k + 0.25T_5^k + 0.25T_7^{k-1} + 0.25T_9^{k-1}$$

$$T_7^k = 0.50T_6^k + 0.50T_{11}^{k-1}$$

$$T_8^k = 0.4096T_4^k + 0.4096T_9^{k-1} + 4.52$$

$$T_9^k = 0.4096T_5^k + 0.2048T_8^k + 0.2048T_{10}^{k-1} + 4.52$$

$$T_{10}^k = 0.4096T_6^k + 0.2048T_9^k + 0.2048T_{11}^{k-1} + 4.52$$

$$T_{11}^k = 0.4096T_7^k + 0.2048T_{10}^k + 0.2048T_{12}^{k-1} + 4.52$$

$$T_{12}^k = 0.6939T_{11}^k + 7.65$$

The initial estimates (k = 0) are carefully chosen to minimize calculation labor; let $\varepsilon < 1.0$.

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
0	340	330	315	250	225	205	195	160	150	140	125	110
1	338.9	336.3	324.3	237.2	232.1	225.4	175.2	163.1	161.7	155.6	130.7	98.3
2	338.3	337.4	328.0	241.4	241.5	226.6	178.6	169.6	170.0	158.9	130.4	98.1
3	338.8	338.4	328.2	247.7	245.7	230.6	180.5	175.6	173.7	161.2	131.6	98.9
4	339.4	338.8	328.9	251.6	248.7	232.9	182.3	178.7	176.0	162.9	132.8	99.8
5	339.8	339.2	329.3	254.0	250.5	234.5	183.7	180.6	177.5	164.1	133.8	100.5
6	340.1	339.4	329.7	255.4	251.7	235.7	184.7	181.8	178.5	164.7	134.5	101.0
7	340.3	339.5	329.9	256.4	252.5	236.4	185.5	182.7	179.1	165.6	135.1	101.4

The heat loss to the outside air for the upper surface (Nodes 8 through 12) is of the form

$$q' = h_o \Delta x \left[\frac{1}{2}\left(T_8 - T_{\infty,o}\right) + \left(T_9 - T_{\infty,o}\right) + \left(T_{10} - T_{\infty,o}\right) + \left(T_{11} - T_{\infty,o}\right) + \frac{1}{2}\left(T_{12} - T_{\infty,o}\right) \right]$$

$$q' = 5\,\mathrm{W/m^2 \cdot K} \times 0.075\,\mathrm{m} \left[\frac{1}{2}(182.7 - 25) + (179.1 - 25) + (165.6 - 25) + (135.1 - 25) + \frac{1}{2}(101.4 - 25) \right] {}^\circ\mathrm{C} = 195\,\mathrm{W/m}$$

Hence, for the entire flue cross-section, considering symmetry,

$$q'_{tot} = 8 \times q' = 8 \times 195\,\mathrm{W/m} = 1.57\,\mathrm{kW/m}$$ <

The convection heat rate at the inner surface is

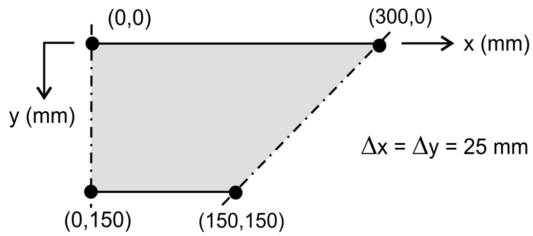
$$q'_{tot} = 8 \times h_i \Delta x \left[\frac{1}{2}\left(T_{\infty,i} - T_1\right) + \left(T_{\infty,i} - T_2\right) + \frac{1}{2}\left(T_{\infty,i} - T_3\right) \right] = 8 \times 190.5\,\mathrm{W/m} = 1.52\,\mathrm{kW/m}$$

which is within 2.5% of the foregoing result. The calculation would be identical if $\varepsilon = 0$.

Continued...

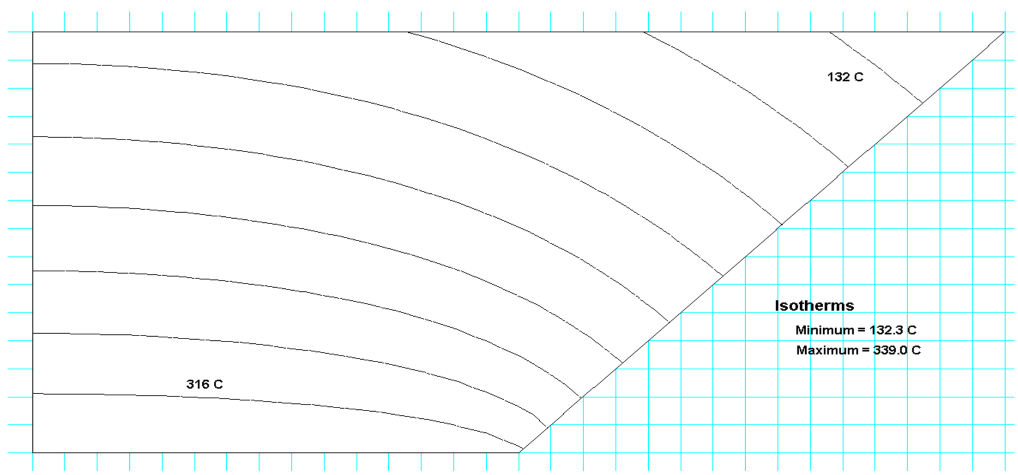
PROBLEM 4.57 (Cont.)

(b) Using the *Finite-Difference Equations* option from the *Tools* portion of the IHT menu, the following two-dimensional temperature field was computed for the grid shown in the schematic below, where x and y are in mm and the temperatures are in °C.



y\x	0	25	50	75	100	125	150	175	200	225	250	275	300
0	180.7	180.2	178.4	175.4	171.1	165.3	158.1	149.6	140.1	129.9	119.4	108.7	98.0
25	204.2	203.6	201.6	198.2	193.3	186.7	178.3	168.4	157.4	145.6	133.4	121.0	
50	228.9	228.3	226.2	222.6	217.2	209.7	200.1	188.4	175.4	161.6	147.5		
75	255.0	254.4	252.4	248.7	243.1	235.0	223.9	209.8	194.1	177.8			
100	282.4	281.8	280.1	276.9	271.6	263.3	250.5	232.8	213.5				
125	310.9	310.5	309.3	307.1	303.2	296.0	282.2	257.5					
150	340.0	340.0	339.6	339.1	337.9	335.3	324.7						

Agreement between the temperature fields for the (a) and (b) grids is good, with the largest differences occurring at the interior and exterior corners. Ten isotherms generated using *FEHT* are shown on the symmetric section below. Note how the heat flow is nearly normal to the flue wall around the mid-section. In the corner regions, the isotherms are curved and we'd expect that grid size might influence the accuracy of the results. Convection heat transfer to the inner surface is



$$q' = 8h_i\Delta x\Big[\left(T_{\infty,i} - T_1\right)/2 + \left(T_{\infty,i} - T_2\right) + \left(T_{\infty,i} - T_3\right) + \left(T_{\infty,i} - T_4\right)$$
$$+ \left(T_{\infty,i} - T_5\right) + \left(T_{\infty,i} - T_6\right) + \left(T_{\infty,i} - T_7\right)/2\Big] = 1.52\,\text{kW/m}$$

and the agreement with results of the coarse grid is excellent.

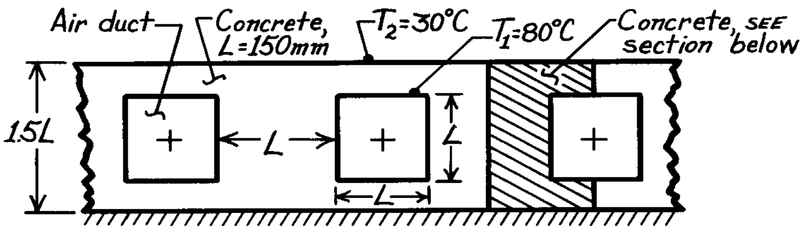
The heat rate increases with increasing h_i and h_o, while temperatures in the wall increase and decrease, respectively, with increasing h_i and h_o.

PROBLEM 4.58

KNOWN: Rectangular air ducts having surfaces at 80°C in a concrete slab with an insulated bottom and upper surface maintained at 30°C.

FIND: Heat rate from each duct per unit length of duct, q'.

SCHEMATIC:



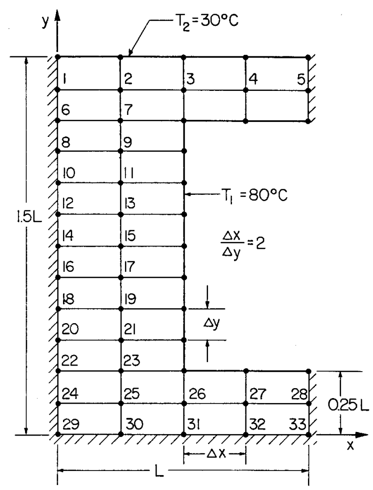
ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) No internal volumetric generation, (4) Constant properties.

PROPERTIES: Concrete (given): k = 1.4 W/m·K.

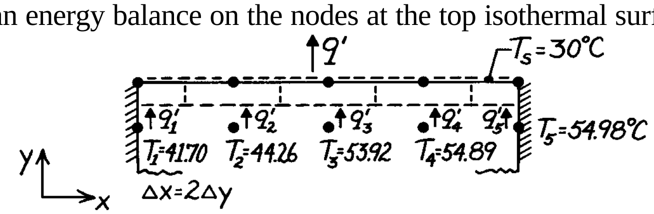
ANALYSIS: Taking advantage of symmetry, the nodal network, using the suggested grid spacing

$$\Delta x = 2\Delta y = 37.50 \text{ mm}$$
$$\Delta y = 0.125L = 18.75 \text{ mm}$$

where L = 150 mm, is shown in the sketch. To evaluate the heat rate, we need the temperatures T_1, T_2, T_3, T_4, and T_5. All the nodes may be treated as interior nodes (considering symmetry for those nodes on insulated boundaries), Eq. 4.33. Use matrix notation, Eq. 4.52, [A][T] = [C], and perform the inversion.



The heat rate per unit length from the prescribed section of the duct follows from an energy balance on the nodes at the top isothermal surface.



$$q' = q_1' + q_2' + q_3' + q_4' + q_5'$$
$$q' = k(\Delta x/2)\frac{T_1 - T_s}{\Delta y} + k\cdot\Delta x\frac{T_2 - T_s}{\Delta y} + k\cdot\Delta x\frac{T_3 - T_s}{\Delta y} + k\cdot\Delta x\frac{T_4 - T_s}{\Delta y} + k(\Delta x/2)\frac{T_5 - T_s}{\Delta y}$$
$$q' = k\left[(T_1 - T_s) + 2(T_2 - T_s) + 2(T_3 - T_s) + 2(T_4 - T_s) + (T_5 - T_s)\right]$$
$$q' = 1.4 \text{ W/m}\cdot\text{K}\left[(41.70-30) + 2(44.26-30) + 2(53.92-30) + 2(54.89-30) + (54.98-30)\right]$$
$$q' = 228 \text{ W/m.}$$

Since the section analyzed represents one-half of the region about an air duct, the heat loss per unit length for each duct is,

$$q'_{duct} = 2xq' = 456 \text{ W/m.}$$ <

Continued …..

PROBLEM 4.58 (Cont.)

Coefficient matrix [A]

-1.0	.2	0	0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.1	-1.0	.1	0	0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	.1	-1.0	.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	.1	-1.0	.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	.2	-1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.4	0	0	0	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	.4	0	0	0	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	.4	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	0	.4	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	0	-1.0	.2	0	0	0	.4	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.4	.1	-1.0	.1	0	0	0	.4	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.1	-1.0	.1	0	0	0	.4	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.1	-1.0	.1	0	0	0	.4	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.2	-1.0	0	0	0	0	.4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.8	0	0	0	0	-1.0	.2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.8	0	0	0	.1	-1.0	.1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.8	0	0	0	.1	-1.0	.1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.8	0	0	0	.1	-1.0	.1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.8	0	0	0	.2	-1.0

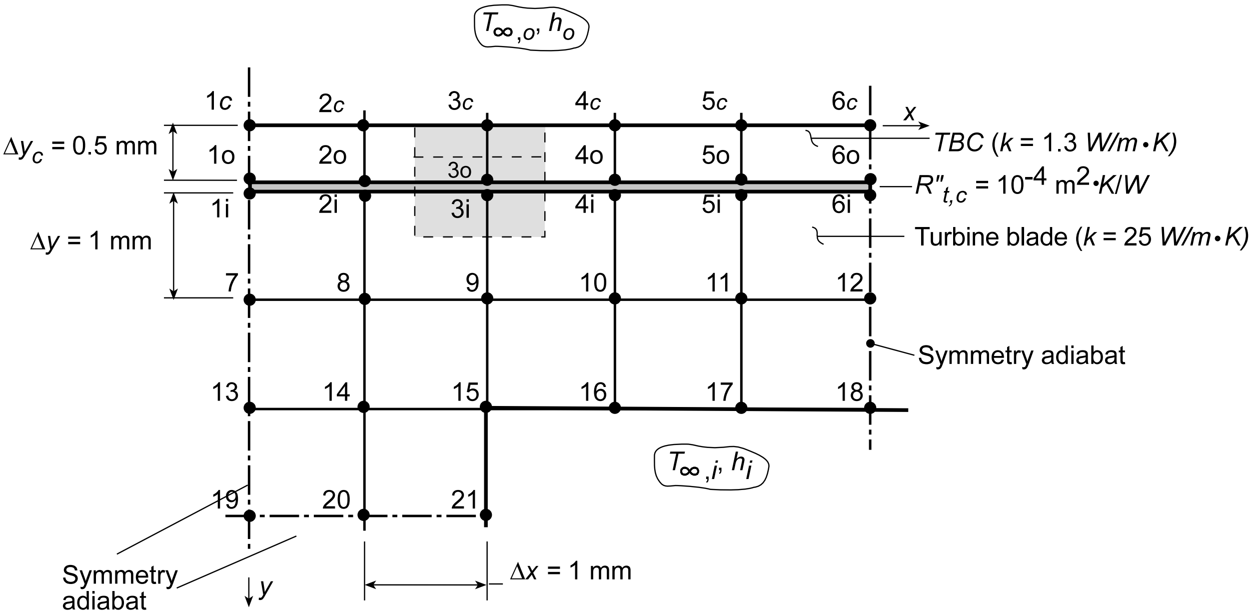
RHS Vector	*Solution Vector*
-12.0	41.70
-12	44.26
-44.0	53.92
-44.0	54.89
44	54.98
0	52.13
-80.0	56.75
0	60.24
-80.0	64.58
0	66.19
-80.0	69.64
0	70.41
-80.0	72.98
0	73.35
-80.0	75.20
0	75.37
-80.0	76.68
0	76.73
-80.0	77.66
0	77.62
-80.0	78.30
0	78.16
-80.0	78.68
0	78.45
0	78.85
-32.0	79.75
-32.0	79.94
-32.0	79.97
0	78.54
0	78.91
0	79.68
0	79.92
0	79.96

PROBLEM 4.59

KNOWN: Dimensions and operating conditions for a gas turbine blade with embedded channels.

FIND: Effect of applying a zirconia, thermal barrier coating.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) Negligible radiation.

ANALYSIS: Preserving the nodal network of Example 4.4 and adding surface nodes for the TBC, finite-difference equations previously developed for nodes 7 through 21 are still appropriate, while new equations must be developed for nodes 1c-6c, 1o-6o, and 1i-6i. Considering node 3c as an example, an energy balance yields

$$h_o \Delta x \left(T_{\infty,o} - T_{3c}\right) + \frac{k_c\left(\Delta y_c/2\right)}{\Delta x}\left(T_{2c} - T_{3c}\right) + \frac{k_c\left(\Delta y_c/2\right)}{\Delta x}\left(T_{4c} - T_{3c}\right) + \frac{k_c \Delta x}{\Delta y_c}\left(T_{3o} - T_{3c}\right) = 0$$

or, with Δx = 1 mm and Δy_c = 0.5 mm,

$$0.25\left(T_{2c} + T_{4c}\right) + 2T_{3o} - \left(2.5 + \frac{h_o \Delta x}{k_c}\right)T_{3c} = -\frac{h_o \Delta x}{k_c} T_{\infty,o}$$

Similar expressions may be obtained for the other 5 nodal points on the outer surface of the TBC.

Applying an energy balance to node 3o at the inner surface of the TBC, we obtain

$$\frac{k_c \Delta x}{\Delta y_c}\left(T_{3c} - T_{3o}\right) + \frac{k_c\left(\Delta y_c/2\right)}{\Delta x}\left(T_{2o} - T_{3o}\right) + \frac{k_c\left(\Delta y_c/2\right)}{\Delta x}\left(T_{4o} - T_{3o}\right) + \frac{\Delta x}{R''_{t,c}}\left(T_{3i} - T_{3o}\right) = 0$$

or,

$$2T_{3c} + 0.25\left(T_{2o} + T_{4o}\right) + \frac{\Delta x}{k_c R''_{t,c}} T_{3i} - \left(2.5 + \frac{\Delta x}{k_c R''_{t,c}}\right)T_{3o} = 0$$

Similar expressions may be obtained for the remaining nodal points on the inner surface of the TBC (outer region of the contact resistance).

Continued...

Applying an energy balance to node 3i at the outer surface of the turbine blade, we obtain

$$\frac{\Delta x}{R''_{t,c}}\left(T_{3o}-T_{3i}\right)+\frac{k\left(\Delta y/2\right)}{\Delta x}\left(T_{2i}-T_{3i}\right)+\frac{k\left(\Delta y/2\right)}{\Delta x}\left(T_{4i}-T_{3i}\right)+\frac{k\Delta x}{\Delta y}\left(T_9-T_{3i}\right)=0$$

or,

$$\frac{\Delta x}{kR''_{t,c}}T_{3o}+0.5\left(T_{2,i}+T_{4,i}\right)+T_9-\left(2+\frac{\Delta x}{kR''_{t,c}}\right)T_{3i}=0$$

Similar expressions may be obtained for the remaining nodal points on the inner region of the contact resistance.

The 33 finite-difference equations were entered into the workspace of IHT from the keyboard (model equations are appended), and for h_o = 1000 W/m^2·K, $T_{\infty,o}$ = 1700 K, h_i = 200 W/m^2·K and $T_{\infty,i}$ = 400 K, the following temperature field was obtained, where coordinate (x,y) locations are in mm and temperatures are in °C.

y\x	0	1	2	3	4	5
0	1536	1535	1534	1533	1533	1532
0.5	1473	1472	1471	1469	1468	1468
0.5	1456	1456	1454	1452	1451	1451
1.5	1450	1450	1447	1446	1444	1444
2.5	1446	1445	1441	1438	1437	1436
3.5	1445	1443	1438	0	0	0

Note the significant reduction in the turbine blade temperature, as, for example, from a surface temperature of T_1 = 1526 K without the TBC to T_{1i} = 1456 K with the coating. Hence, the coating is serving its intended purpose.

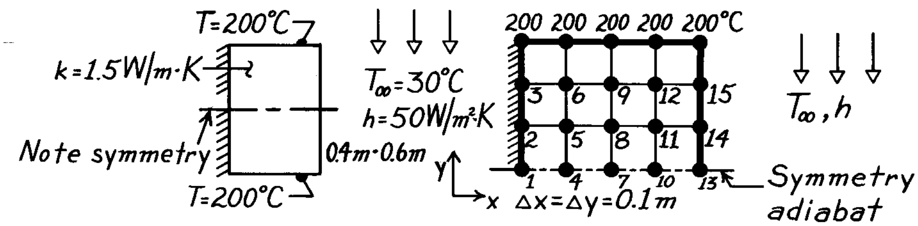
COMMENTS: (1) Significant additional benefits may still be realized by increasing h_i. (2) The foregoing solution may be used to determine the temperature field without the TBC by setting $k_c \to \infty$ and $R''_{t,c} \to 0$.

PROBLEM 4.60

KNOWN: Bar of rectangular cross-section subjected to prescribed boundary conditions.

FIND: Using a numerical technique with a grid spacing of 0.1m, determine the temperature distribution and the heat transfer rate from the bar to the fluid.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: The nodal network has $\Delta x = \Delta y = 0.1\text{m}$. Note the adiabat corresponding to system symmetry. The finite-difference equations for each node can be written using either Eq. 4.33, for interior nodes, or Eq. 4.46, for a plane surface with convection. In the case of adiabatic surfaces, Eq. 4.46 is used with h = 0. Note that

$$\frac{h\Delta x}{k} = \frac{50\text{W/m}^2 \cdot \text{K} \times 0.1\text{m}}{1.5\ \text{W/m} \cdot \text{K}} = 3.333.$$

Node	*Finite-Difference Equations*
1	$-4T_1 + 2T_2 + 2T_4 = 0$
2	$-4T_2 + T_1 + T_3 + 2T_5 = 0$
3	$-4T_3 + 200 + 2T_6 + T_2 = 0$
4	$-4T_4 + T_1 + 2T_5 + T_7 = 0$
5	$-4T_5 + T_2 + T_6 + T_8 + T_4 = 0$
6	$-4T_6 + T_5 + T_3 + 200 + T_9 = 0$
7	$-4T_7 + T_4 + 2T_8 + T_{10} = 0$
8	$-4T_8 + T_7 + T_5 + T_9 + T_{11} = 0$
9	$-4T_9 + T_8 + T_6 + 200 + T_{12} = 0$
10	$-4T_{10} + T_7 + 2T_{11} + T_{13} = 0$
11	$-4T_{11} + T_{10} + T_8 + T_{12} + T_{14} = 0$
12	$-4T_{12} + T_{11} + T_9 + 200 + T_{15} = 0$
13	$2T_{10} + T_{14} + 6.666\times30 - 10.666\ T_{13} = 0$
14	$2T_{11} + T_{13} + T_{15} + 6.666\times30 - 2(3.333+2)T_{14} = 0$
15	$2T_{12} + T_{14} + 200 + 6.666\times30 - 2(3.333+2)\ T_{15} = 0$

Using the matrix inversion method, Section 4.5.2, the above equations can be written in the form [A] [T] = [C] where [A] and [C] are shown on the next page. Using a stock matrix inversion routine, the temperatures [T] are determined.

Continued …..

$$
[A]=\begin{bmatrix}
-4 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -4 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -4 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -10.66 & 2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & -10.66 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & -10.66
\end{bmatrix}
$$

$$
[C]=\begin{bmatrix} 0\\ 0\\ -200\\ 0\\ 0\\ -200\\ 0\\ 0\\ -200\\ 0\\ 0\\ -200\\ -200\\ -200\\ -400 \end{bmatrix}
\qquad
[T]=\begin{bmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12}\\ T_{13}\\ T_{14}\\ T_{15} \end{bmatrix}
=\begin{bmatrix} 153.9\\ 159.7\\ 176.4\\ 148.0\\ 154.4\\ 172.9\\ 129.4\\ 137.0\\ 160.7\\ 95.6\\ 103.5\\ 132.8\\ 45.8\\ 48.7\\ 67.0 \end{bmatrix}\left({}^\circ C\right)
$$

Considering symmetry, the heat transfer rate to the fluid is twice the convection rate from the surfaces of the control volumes exposed to the fluid. Using Newton's law of cooling, considering a unit thickness of the bar, find

$$q_{conv} = 2\left[h\cdot\frac{\Delta y}{2}\cdot\left(T_{13}-T_\infty\right)+h\cdot\Delta y\cdot\left(T_{14}-T_\infty\right)+h\cdot\Delta y\left(T_{15}-T_\infty\right)+h\cdot\frac{\Delta y}{2}\left(200-T_\infty\right)\right]$$

$$q_{conv} = 2h\cdot\Delta y\left[\frac{1}{2}\left(T_{13}-T_\infty\right)+\left(T_{14}-T_\infty\right)+\left(T_{15}-T_\infty\right)+\frac{1}{2}\left(200-T_\infty\right)\right]$$

$$q_{conv} = 2\times 50\frac{W}{m^2\cdot K}\times 0.1m\left[\frac{1}{2}\left(45.8-30\right)+\left(48.7-30\right)+\left(67.0-30\right)+\frac{1}{2}\left(200-30\right)\right]$$

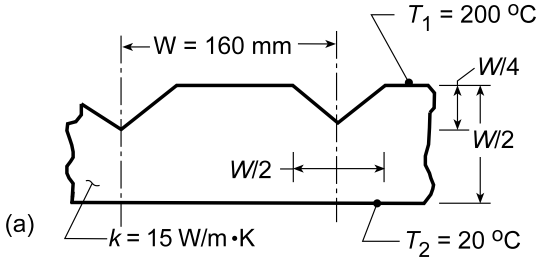
$$q_{conv} = 1487\ W/m. \qquad <$$

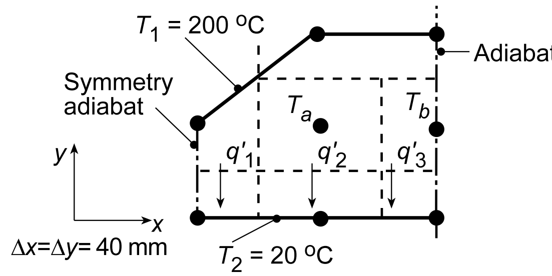
PROBLEM 4.61

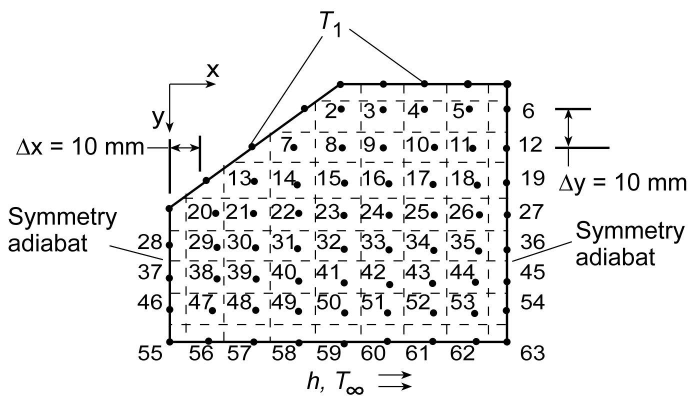
KNOWN: Upper surface and grooves of a plate are maintained at a uniform temperature T_1, while the lower surface is maintained at T_2 or is exposed to a fluid at T_∞.

FIND: (a) Heat rate per width of groove spacing (w) for isothermal top and bottom surfaces using a finite-difference method with Δx = 40 mm, (b) Effect of grid spacing and convection at bottom surface.

SCHEMATIC:







ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) Using a space increment of Δx = 40 mm, the symmetrical section shown in schematic (a) corresponds to one-half the groove spacing. There exist only two interior nodes for which finite-difference equations must be written.

Node a:
$$4T_a - (T_1 + T_b + T_2 + T_1) = 0$$
$$4T_a - (200 + T_b + 20 + 200) = 0 \qquad \text{or} \qquad 4T_a - T_b = 420 \qquad (1)$$

Node b:
$$4T_b - (T_1 + T_a + T_2 + T_a) = 0$$
$$4T_b - (200 + 2T_a + 20) = 0 \qquad \text{or} \qquad -2T_a + 4T_b = 220 \qquad (2)$$

Multiply Eq. (2) by 2 and subtract from Eq. (1) to obtain

$7T_b = 860$ or $T_b = 122.9°C$

From Eq. (1),

$4T_a - 122.9 = 420$ or $T_a = (420 + 122.9)/4 = 135.7°C$.

The heat transfer through the symmetrical section is equal to the sum of heat flows through control volumes adjacent to the lower surface. From the schematic,

$$q' = q'_1 + q'_2 + q'_3 = k\left(\frac{\Delta x}{2}\right)\frac{T_1 - T_2}{\Delta y} + k(\Delta x)\frac{T_a - T_2}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_b - T_2}{\Delta y}.$$

Continued...

Noting that $\Delta x = \Delta y$, regrouping and substituting numerical values, find

$$q' = k\left[\frac{1}{2}(T_1 - T_2) + (T_a - T_2) + \frac{1}{2}(T_b - T_2)\right]$$

$$q' = 15\,\mathrm{W/m\cdot K}\left[\frac{1}{2}(200-20) + (135.7-20) + \frac{1}{2}(122.9-20)\right] = 3.86\,\mathrm{kW/m}.$$

For the full groove spacing, $q'_{total} = 2 \times 3.86$ kW/m = 7.72 kW/m. <

(b) Using the *Finite-Difference Equations* option from the *Tools* portion of the IHT menu, the following two-dimensional temperature field was computed for the grid shown in schematic (b), where x and y are in mm and the nodal temperatures are in °C. Nodes 2-54 are interior nodes, with those along the symmetry adiabats characterized by $T_{m-1,n} = T_{m+1,n}$, while nodes 55-63 lie on a plane surface.

y\x	0	10	20	30	40	50	60	70	80
0					200	200	200	200	200
10				200	191	186.6	184.3	183.1	182.8
20			200	186.7	177.2	171.2	167.5	165.5	164.8
30		200	182.4	169.5	160.1	153.4	149.0	146.4	145.5
40	200	175.4	160.3	148.9	140.1	133.5	128.7	125.7	124.4
50	141.4	134.3	125.7	118.0	111.6	106.7	103.1	100.9	100.1
60	97.09	94.62	90.27	85.73	81.73	78.51	76.17	74.73	74.24
70	57.69	56.83	55.01	52.95	51.04	49.46	48.31	47.60	47.36
80	20	20	20	20	20	20	20	20	20

The foregoing results were computed for $h = 10^7$ W/m^2·K ($h \to \infty$) and $T_\infty = 20$°C, which is tantamount to prescribing an isothermal bottom surface at 20°C. Agreement between corresponding results for the coarse and fine grids is surprisingly good ($T_a = 135.7$°C $\leftrightarrow$ $T_{23} = 140.1$°C; $T_b = 122.9$°C $\leftrightarrow$ $T_{27} = 124.4$°C). The heat rate is

$$q' = 2\times k\left[(T_{46} - T_{55})/2 + (T_{47} - T_{56}) + (T_{48} - T_{57}) + (T_{49} - T_{58}) + (T_{50} - T_{59})\right.$$
$$\left. + (T_{51} - T_{60}) + (T_{52} - T_{61}) + (T_{53} - T_{62}) + (T_{54} - T_{63})/2\right]$$

$$q' = 2\times 15\,\mathrm{W/m\cdot K}\left[18.84 + 36.82 + 35.00 + 32.95 + 31.04 + 29.46\right.$$
$$\left. + 28.31 + 27.6 + 13.68\right]^\circ\mathrm{C} = 7.61\,\mathrm{kW/m}$$ <

The agreement with q' = 7.72 kW/m from the coarse grid of part (a) is excellent and a fortuitous consequence of compensating errors. With reductions in the convection coefficient from $h \to \infty$ to h = 1000, 200 and 5 W/m^2·K, the corresponding increase in the thermal resistance reduces the heat rate to values of 6.03, 3.28 and 0.14 kW/m, respectively. With decreasing h, there is an overall increase in nodal temperatures, as, for example, from 191°C to 199.8°C for T_2 and from 20°C to 196.9°C for T_{55}.

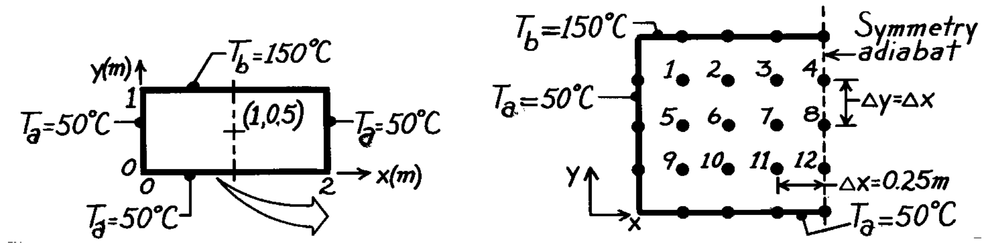
NOTE TO INSTRUCTOR: To reduce computational effort, while achieving the same educational objectives, the problem statement has been changed to allow for convection at the bottom, rather than the top, surface.

PROBLEM 4.62

KNOWN: Rectangular plate subjected to uniform temperature boundaries.

FIND: Temperature at the midpoint using a finite-difference method with space increment of 0.25m

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: For the nodal network above, 12 finite-difference equations must be written. It follows that node 8 represents the midpoint of the rectangle. Since all nodes are interior nodes, Eq. 4.33 is appropriate and is written in the form

$$4T_m - \sum T_{neighbors} = 0.$$

For nodes on the symmetry adiabat, the neighboring nodes include two symmetrical nodes. Hence, for Node 4, the neighbors are T_b, T_8 and $2T_3$. Because of the simplicity of the finite-difference equations, we may proceed directly to the matrices [A] and [C] – see Eq. 4.52 – and matrix inversion can be used to find the nodal temperatures T_m.

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -4 \end{bmatrix} \quad C = \begin{bmatrix} -200 \\ -150 \\ -150 \\ -150 \\ -50 \\ 0 \\ 0 \\ 0 \\ -100 \\ -50 \\ -50 \\ -50 \end{bmatrix} \quad T = \begin{bmatrix} 96.5 \\ 112.9 \\ 118.9 \\ 120.4 \\ 73.2 \\ 86.2 \\ 92.3 \\ 94.0 \\ 59.9 \\ 65.5 \\ 69.9 \\ 71.0 \end{bmatrix}$$

The temperature at the midpoint (Node 8) is

$$T(1,0.5) = T_8 = 94.0^\circ C. \qquad <$$

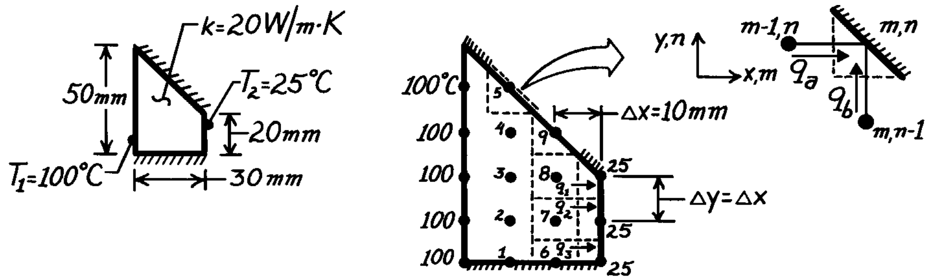
COMMENTS: Using the exact analytical, solution – see Eq. 4.19 and Problem 4.2 – the midpoint temperature is found to be 94.5°C. To improve the accuracy of the finite-difference method, it would be necessary to decrease the nodal mesh size.

PROBLEM 4.63

KNOWN: Long bar with trapezoidal shape, uniform temperatures on two surfaces, and two insulated surfaces.

FIND: Heat transfer rate per unit length using finite-difference method with space increment of 10mm.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: The heat rate can be found after the temperature distribution has been determined. Using the nodal network shown above with Δx = 10mm, nine finite-difference equations must be written. Nodes 1-4 and 6-8 are interior nodes and their finite-difference equations can be written directly from Eq. 4.33. For these nodes

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0 \qquad m = 1-4,\ 6-8. \qquad (1)$$

For nodes 5 and 9 located on the diagonal, insulated boundary, the appropriate finite-difference equation follows from an energy balance on the control volume shown above (upper-right corner of schematic),

$\dot{E}_{in} - \dot{E}_{out} = q_a + q_b = 0$

$$k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} = 0.$$

Since $\Delta x = \Delta y$, the finite-difference equation for nodes 5 and 9 is of the form

$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} = 0 \qquad m = 5,9. \qquad (2)$$

The system of 9 finite-difference equations is first written in the form of Eqs. (1) or (2) and then written in explicit form for use with the Gauss-Seidel iteration method of solution; see Section 4.5.2.

Node	*Finite-difference equation*	*Gauss-Seidel form*
1	$T_2+T_2+T_6+100-4T_1 = 0$	$T_1 = 0.5T_2+0.25T_6+25$
2	$T_3+T_1+T_7+100-4T_2 = 0$	$T_2 = 0.25(T_1+T_3+T_7)+25$
3	$T_4+T_2+T_8+100-4T_3 = 0$	$T_3 = 0.25(T_2+T_4+T_8)+25$
4	$T_5+T_3+T_9+100-4T_4 = 0$	$T_4 = 0.25(T_3+T_5+T_9)+25$
5	$100+T_4-2T_5 = 0$	$T_5 = 0.5T_4+50$
6	$T_7+T_7+25+T_1-4T_6 = 0$	$T_6 = 0.25T_1+0.5T_7+6.25$
7	$T_8+T_6+25+T_2-4T_7 = 0$	$T_7 = 0.25(T_2+T_6+T_8)+6.25$
8	$T_9+T_7+25+T_3-4T_8 = 0$	$T_8 = 0.25(T_3+T_7+T_9)+6.25$
9	$T_4+T_8-2T_9 = 0$	$T_9 = 0.5(T_4+T_8)$

Continued …..

The iteration process begins after an initial guess (k = 0) is made. The calculations are shown in the table below.

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9(°C)
0	75	75	80	85	90	50	50	60	75
1	75.0	76.3	80.0	86.3	92.5	50.0	52.5	57.5	72.5
2	75.7	76.9	80.0	86.3	93.2	51.3	52.2	57.5	71.9
3	76.3	77.0	80.2	86.3	93.2	51.3	52.7	57.3	71.9
4	76.3	77.3	80.2	86.3	93.2	51.7	52.7	57.5	71.8
5	76.6	77.3	80.3	86.3	93.2	51.7	52.9	57.4	71.9
6	76.6	77.5	80.3	86.4	93.2	51.9	52.9	57.5	71.9

Note that by the sixth iteration the change is less than 0.3°C; hence, we assume the temperature distribution is approximated by the last row of the table.

The heat rate per unit length can be determined by evaluating the heat rates in the x-direction for the control volumes about nodes 6, 7, and 8. From the schematic, find that

$$q' = q_1' + q_2' + q_3'$$

$$q' = k\Delta y \frac{T_8 - 25}{\Delta x} + k\Delta y \frac{T_7 - 25}{\Delta x} + k\frac{\Delta y}{2}\frac{T_6 - 25}{\Delta x}$$

Recognizing that $\Delta x = \Delta y$ and substituting numerical values, find

$$q' = 20\frac{\text{W}}{\text{m}\cdot\text{K}}\left[(57.5 - 25) + (52.9 - 25) + \frac{1}{2}(51.9 - 25)\right]\text{K}$$

$$q' = 1477 \text{ W/m.}$$ <

COMMENTS: (1) Recognize that, while the temperature distribution may have been determined to a reasonable approximation, the uncertainty in the heat rate could be substantial. This follows since the heat rate is based upon a gradient and hence on temperature differences.

(2) Note that the initial guesses (k = 0) for the iteration are within 5°C of the final distribution. The geometry is simple enough that the guess can be very close. In some instances, a flux plot may be helpful and save labor in the calculation.

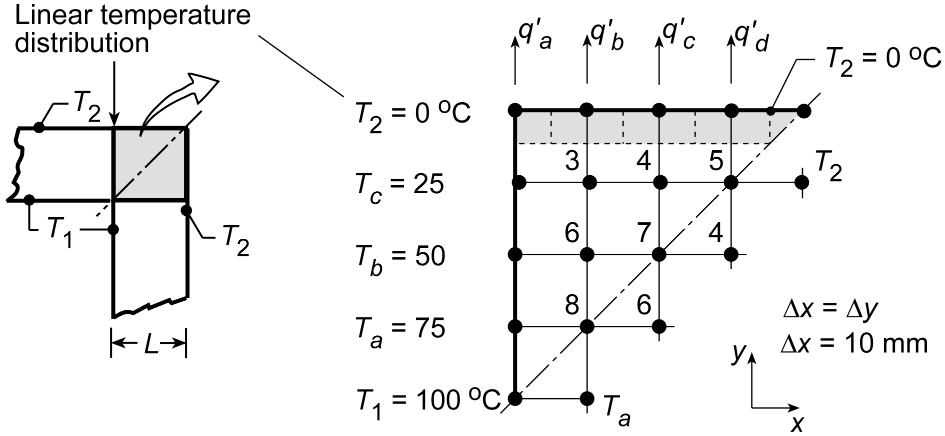
(3) In writing the FDEs, the iteration index (superscript k) was not included to simplify expression of the equations. However, the most recent value of $T_{m,n}$ is always used in the computations. Note that this system of FDEs is diagonally dominant and no rearrangement is required.

PROBLEM 4.64

KNOWN: Edge of adjoining walls (k = 1 W/m·K) represented by symmetrical element bounded by the diagonal symmetry adiabat and a section of the wall thickness over which the temperature distribution is assumed to be linear.

FIND: (a) Temperature distribution, heat rate and shape factor for the edge using the nodal network with = Δx = Δy = 10 mm; compare shape factor result with that from Table 4.1; (b) Assess the validity of assuming linear temperature distributions across sections at various distances from the edge.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties, and (3) Linear temperature distribution at specified locations across the section.

ANALYSIS: (a) Taking advantage of symmetry along the adiabat diagonal, all the nodes may be treated as interior nodes. Across the left-hand boundary, the temperature distribution is specified as linear. The finite-difference equations required to determine the temperature distribution, and hence the heat rate, can be written by inspection.

$$T_3 = 0.25(T_2 + T_4 + T_6 + T_c)$$

$$T_4 = 0.25(T_2 + T_5 + T_7 + T_3)$$

$$T_5 = 0.25(T_2 + T_2 + T_4 + T_4)$$

$$T_6 = 0.25(T_3 + T_7 + T_8 + T_b)$$

$$T_7 = 0.25(T_4 + T_4 + T_6 + T_6)$$

$$T_8 = 0.25(T_6 + T_6 + T_a + T_a)$$

The heat rate for both surfaces of the edge is

$$q'_{tot} = 2[q'_a + q'_b + q'_c + q'_d]$$

$$q'_{tot} = 2[k(\Delta x/2)(T_c - T_2)/\Delta y + k\Delta x(T_3 - T_2)/\Delta y + k\Delta x(T_4 - T_2)/\Delta y + k\Delta x(T_5 - T_2)/\Delta x]$$

The shape factor for the full edge is defined as

$$q'_{tot} = kS'(T_1 - T_2)$$

Solving the above equation set in IHT, the temperature (°C) distribution is

Continued...

0	0	0	0	0
25	18.75	12.5	6.25	
50	37.5	25.0		
75	56.25			
00				

<

and the heat rate and shape factor are

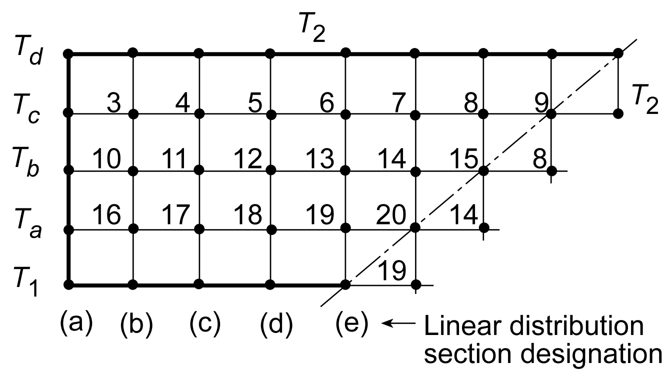
$$q'_{tot} = 100\,\mathrm{W/m} \qquad\qquad S = 1$$ <

From Table 4.1, the edge shape factor is 0.54, considerably below our estimate from this coarse grid analysis.

(b) The effect of the linear temperature distribution on the shape factor estimate can be explored using a more extensive grid as shown below. The FDE analysis was performed with the linear distribution imposed as the different sections a, b, c, d, e. Following the same approach as above, find

Location of linear distribution	(a)	(b)	(c)	(d)	(e)
Shape factor, S	0.797	0.799	0.809	0.857	1.00

The shape factor estimate decreases as the imposed linear temperature distribution section is located further from the edge. We conclude that assuming the temperature distribution across the section directly at the edge is a poor-one.



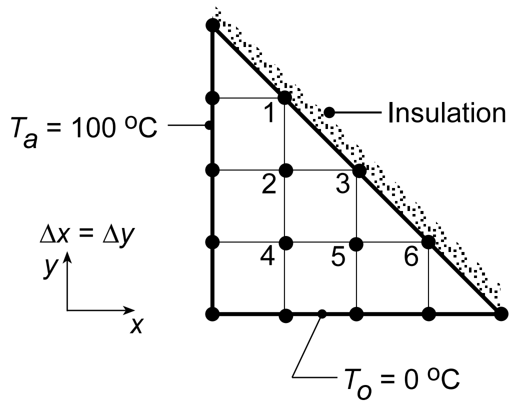
COMMENTS: The grid spacing for this analysis is quite coarse making the estimates in poor agreement with the Table 4.1 result. However, the analysis does show the effect of positioning the linear temperature distribution condition.

PROBLEM 4.65

KNOWN: Long triangular bar insulated on the diagonal while sides are maintained at uniform temperatures T_a and T_b.

FIND: (a) Using a nodal network with five nodes to the side, and beginning with properly defined control volumes, derive the finite-difference equations for the interior and diagonal nodes and obtain the temperature distribution; sketch the 25, 50 and 75°C isotherms and (b) Recognizing that the insulated diagonal surface can be treated as a symmetry line, show that the diagonal nodes can be treated as interior nodes, and write the finite-difference equations by inspection.

SCHEMATIC:

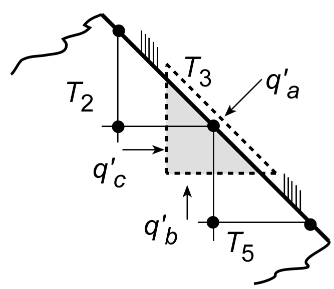


ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional heat transfer, and (3) Constant properties.

ANALYSIS: (a) For the nodal network shown above, nodes 2, 4, 5, 7, 8 and 9 are interior nodes and, since Δx = Δy, the corresponding finite-difference equations are of the form, Eq. 4.33,

$$T_j = 1/4 \sum T_{neighbors} \qquad (1)$$

For a node on the adiabatic, diagonal surface, an energy balance, $\dot{E}_{in} - \dot{E}_{out} = 0$, yields



$$q'_a + q'_b + q'_c = 0$$

$$0 + k\Delta x \frac{T_5 - T_3}{\Delta y} + k\Delta y \frac{T_2 - T_3}{\Delta x} = 0$$

$$T_3 = 1/2\left(T_2 + T_5\right) \qquad (2)$$

That is, for the diagonal nodes, m,

$$T_m = 1/2 \sum T_{neighbors} \qquad (3)$$

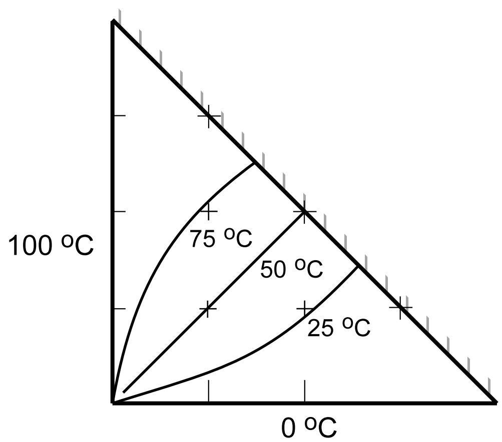
To obtain the temperature distributions, enter Eqs. (1, 2, 3) into the IHT workspace and solve for the nodal temperatures (°C), tabulated according to the nodal arrangement:

Continued...

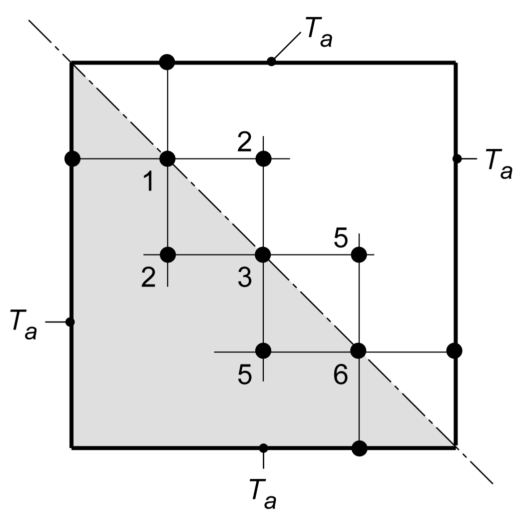
–

00	85.71			
00	71.43	50.00		
00	50.00	28.57	14.29	
–	0	0	0	–

The 25, 50 and 75°C isotherms are sketched below, using an interpolation scheme to scale the isotherms on the triangular bar.



(b) If we consider the insulated surface as a symmetry plane, the nodal network appears as shown. As such, the diagonal nodes can be treated as interior nodes, as Eq. (1) above applies. Recognize the form is the same as that of Eq. (2) or (3).



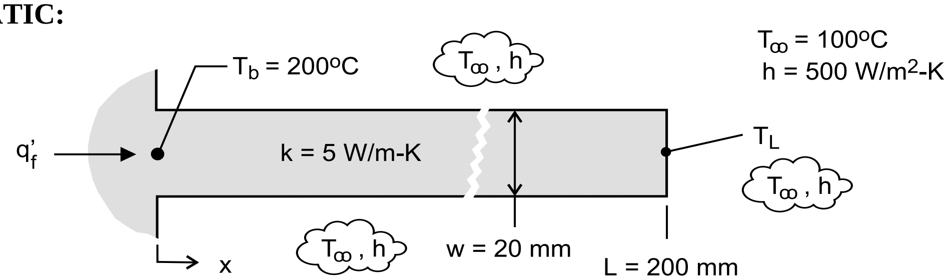
COMMENTS: Always look for symmetry conditions which can greatly simplify the writing of nodal equations. In this situation, the adiabatic surface can be treated as a symmetry plane such that the nodes can be treated as interior nodes, and the finite-difference equations can be written by inspection.

PROBLEM 4.66

KNOWN: Straight fin of uniform cross section with prescribed thermal conditions and geometry; tip condition allows for convection.

FIND: (a) Calculate the fin heat rate, q'_f, and tip temperature, T_L, assuming one-dimensional heat transfer in the fin; calculate the Biot number to determine whether the one-dimensional assumption is valid, (b) Using the finite-element software FEHT, perform a two-dimensional analysis to determine the fin heat rate and the tip temperature; display the isotherms; describe the temperature field and the heat flow pattern inferred from the display, and (c) Validate your FEHT code against the 1-D analytical solution for a fin using a thermal conductivity of 50 and 500 W/m·K.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conduction with constant properties, (2) Negligible radiation exchange, (3) Uniform convection coefficient.

ANALYSIS: (a) Assuming one-dimensional conduction, q'_L and T_L can be determined using Eqs. 3.72 and 3.70, respectively, from Table 3.4, Case A. Alternatively, use the IHT *Model | Extended Surfaces | Temperature Distribution and Heat Rate | Straight Fin | Rectangular*. These results are tabulated below and labeled as "1-D." The Biot number for the fin is

$$Bi = \frac{h(t/2)}{k} = \frac{500\ W/m^2 \cdot K\ (0.020\ m/2)}{5\ W/m \cdot K} = 1$$

(b, c) The fin can be drawn as a two-dimensional outline in FEHT with convection boundary conditions on the exposed surfaces, and with a uniform temperature on the base. Using a fine mesh (at least 1280 elements), solve for the temperature distribution and use the *View | Temperature Contours* command to view the isotherms and the *Heat Flow* command to determine the heat rate into the fin base. The results of the analysis are summarized in the table below.

k (W/m·K)	Bi	Tip temperature, T_L (°C)		Fin heat rate, q'_f (W/m)		Difference* (%)
		1-D	2-D	1-D	2-D	
5	1	100	100	1010	805	20
50	0.1	100.3	100	3194	2990	6.4
500	0.01	123.8	124	9812	9563	2.5

* Difference = $\left(q'_{f,1D} - q'_{f,2D}\right) \times 100 / q'_{f,1D}$

COMMENTS: (1) From part (a), since $Bi = 1 > 0.1$, the internal conduction resistance is not negligible. Therefore significant transverse temperature gradients exist, and the one-dimensional conduction assumption in the fin is a poor one.

Continued …..

PROBLEM 4.66 (Cont.)

(2) From the table, with k = 5 W/m·K (Bi = 1), the 2-D fin heat rate obtained from the FEA analysis is 20% lower than that for the 1-D analytical analysis. This is as expected since the 2-D model accounts for transverse thermal resistance to heat flow. Note, however, that analyses predict the same tip temperature, a consequence of the fin approximating an infinitely long fin (mL = 20.2 >> 2.56; see Ex. 3.8 Comments).

(3) For the k = 5 W/m·K case, the FEHT isotherms show considerable curvature in the region near the fin base. For example, at x = 10 and 20 mm, the difference between the centerline and surface temperatures are 15 and 7°C.

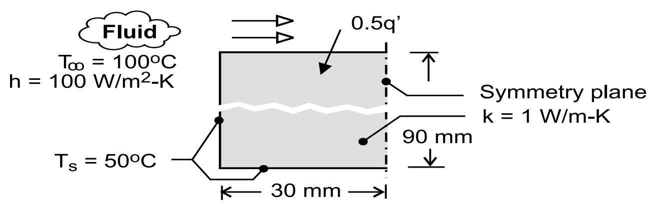
(4) From the table, with increasing thermal conductivity, note that Bi decreases, and the one-dimensional heat transfer assumption becomes more appropriate. The difference for the case when k = 500 W/m·K is mostly due to the approximate manner in which the heat rate is calculated in the FEA software.

PROBLEM 4.67

KNOWN: Long rectangular bar having one boundary exposed to a convection process (T_∞, h) while the other boundaries are maintained at constant temperature T_s.

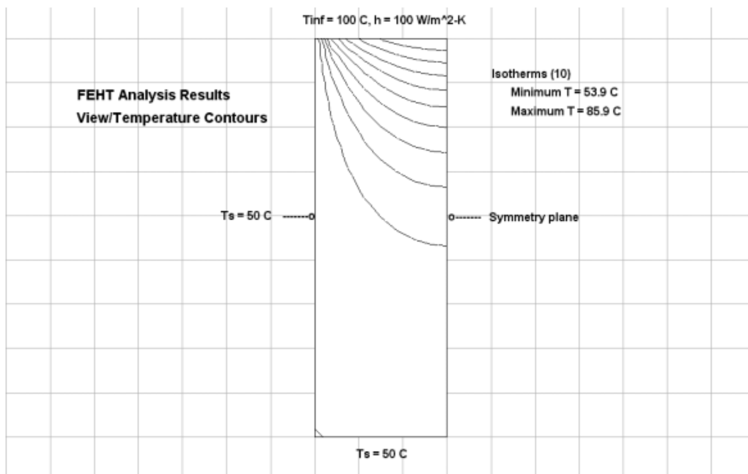
FIND: Using the finite-element method of FEHT, (a) Determine the temperature distribution, plot the isotherms, and identify significant features of the distribution, (b) Calculate the heat rate per unit length (W/m) into the bar from the air stream, and (c) Explore the effect on the heat rate of increasing the convection coefficient by factors of two and three; explain why the change in the heat rate is not proportional to the change in the convection coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two dimensional conduction, (2) Constant properties.

ANALYSIS: (a) The symmetrical section shown in the schematic is drawn in FEHT with the specified boundary conditions and material property. The *View | Temperature Contours* command is used to represent ten isotherms (isopotentials) that have minimum and maximum values of 53.9°C and 85.9°C, respectively.



Because of the symmetry boundary condition, the isotherms are normal to the center-plane indicating an adiabatic surface. Note that the temperature change along the upper surface of the bar is substantial (≈ 40°C), whereas the lower half of the bar has less than a 3°C change. That is, the lower half of the bar is largely unaffected by the heat transfer conditions at the upper surface.

(b, c) Using the *View | Heat Flows* command considering the upper surface boundary with selected convection coefficients, the heat rates into the bar from the air stream were calculated.

$h\left(W/m^2\cdot K\right)$	100	200	300
$q'\left(W/m\right)$	128	175	206

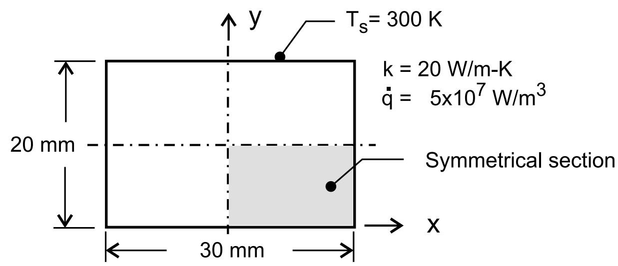
Increasing the convection coefficient by factors of 2 and 3, increases the heat rate by 37% and 61%, respectively. The heat rate from the bar to the air stream is controlled by the thermal resistances of the bar (conduction) and the convection process. Since the conduction resistance is significant, we should not expect the heat rate to change proportionally to the change in convection resistance.

PROBLEM 4.68

KNOWN: Log rod of rectangular cross-section of Problem 4.55 that experiences uniform heat generation while its surfaces are maintained at a fixed temperature. Use the finite-element software FEHT as your analysis tool.

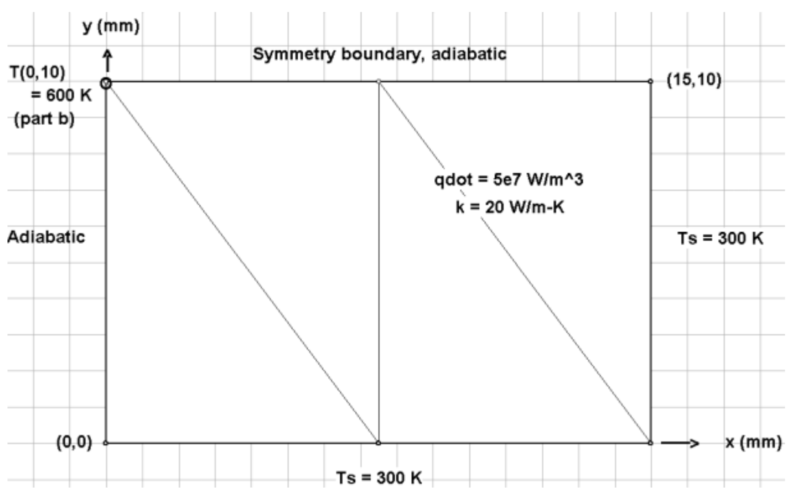
FIND: (a) Represent the temperature distribution with representative isotherms; identify significant features; and (b) Determine what heat generation rate will cause the midpoint to reach 600 K with unchanged boundary conditions.

SCHEMATIC:



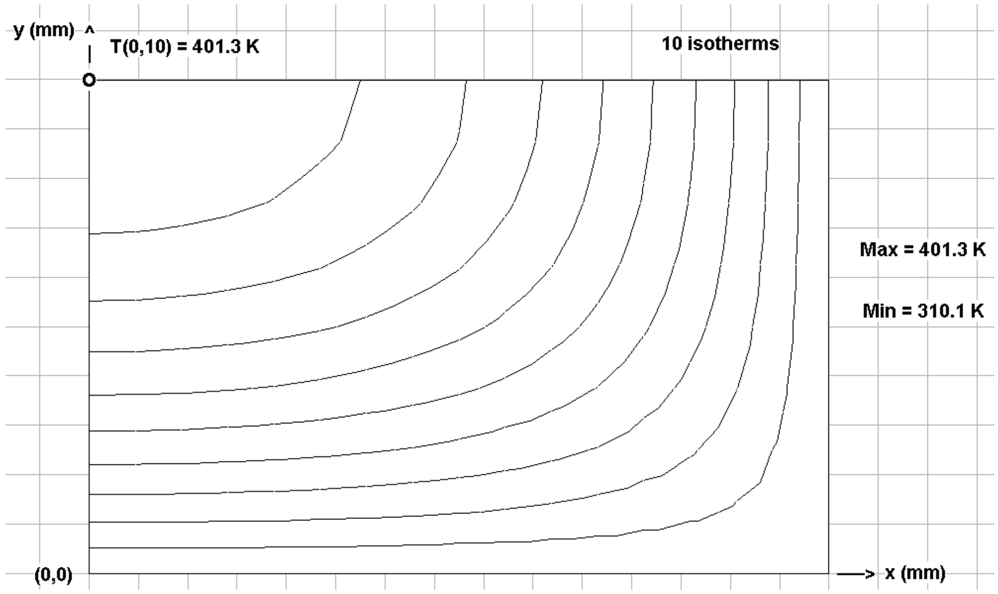
ASSUMPTIONS: (1) Steady-state conditions, and (2) Two-dimensional conduction with constant properties.

ANALYSIS: (a) Using *FEHT*, do the following: in *Setup*, enter an appropriate scale; *Draw* the outline of the symmetrical section shown in the above schematic; *Specify* the *Boundary Conditions* (zero heat flux or adiabatic along the symmetrical lines, and isothermal on the edges). Also *Specify* the *Material Properties* and *Generation* rate. *Draw* three *Element Lines* as shown on the annotated version of the *FEHT* screen below. To reduce the mesh, hit *Draw/Reduce Mesh* until the desired fineness is achieved (256 elements is a good choice).



Continued …

After hitting *Run*, C*heck* and then *Calculate*, use the *View/Temperature Contours* and select the 10-isopotential option to display the isotherms as shown in an annotated copy of the *FEHT* screen below.



The isotherms are normal to the symmetrical lines as expected since those surfaces are adiabatic. The isotherms, especially near the center, have an elliptical shape. Along the x = 0 axis and the y = 10 mm axis, the temperature gradient is nearly linear. The hottest point is of course the center for which the temperature is

$$(T(0, 10 \text{ mm}) = 401.3 \text{ K}. \qquad <$$

The temperature of this point can be read using the *View/Temperatures* or *View*|*Tabular Output* command.

(b) To determine the required generation rate so that T(0, 10 mm) = 600 K, it is necessary to re-run the model with several guessed values of $\dot{q}$. After a few trials, find

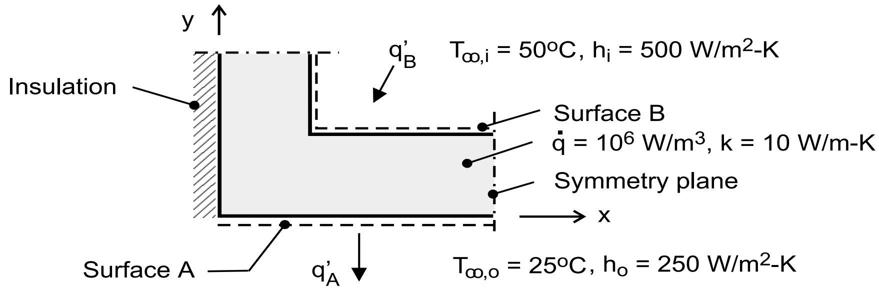
$$\dot{q} = 1.48 \times 10^8 \text{ W/m}^3 \qquad <$$

PROBLEM 4.69

KNOWN: Symmetrical section of a flow channel with prescribed values of $\dot{q}$ and k, as well as the surface convection conditions. See Problem 4.5(S).

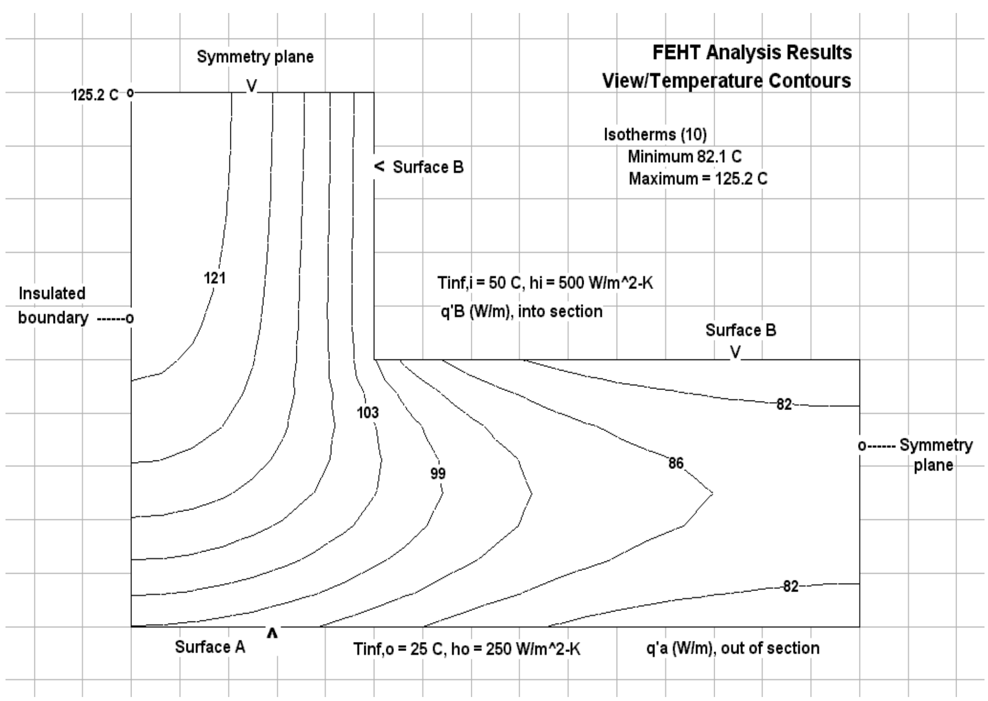
FIND: Using the finite-element method of FEHT, (a) Determine the temperature distribution and plot the isotherms; identify the coolest and hottest regions, and the region with steepest gradients; describe the heat flow field, (b) Calculate the heat rate per unit length (W/m) from the outer surface A to the adjacent fluid, (c) Calculate the heat rate per unit length (W/m) to surface B from the inner fluid, and (d) Verify that the results are consistent with an overall energy balance on the section.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) The symmetrical section shown in the schematic is drawn in FEHT with the specified boundary conditions, material property and generation. The *View | Temperature Contours* command is used to represent ten isotherms (isopotentials) that have minimum and maximum values of 82.1°C and 125.2°C.



The hottest region of the section is the upper vertical leg (left-hand corner). The coolest region is in the lower horizontal leg at the far right-hand boundary. The maximum and minimum section temperatures (125°C and 77°C), respectively, are higher than either adjoining fluid. Remembering that heat flow lines are normal to the isotherms, heat flows from the hottest corner directly to the inner fluid and downward into the lower leg and then flows out surface A and the lower portion of surface B.

Continued …..

PROBLEM 4.69 (Cont.)

(b, c) Using the *View* | *Heat Flows* command considering the boundaries for surfaces A and B, the heat rates are:

$$q'_S = 1135 \text{ W/m} \qquad q'_B = -1365 \text{ W/m.} \qquad <$$

From an energy balance on the section, we note that the results are consistent since conservation of energy is satisfied.

$$\dot{E}'_{in} - \dot{E}'_{out} + \dot{E}_g = 0$$

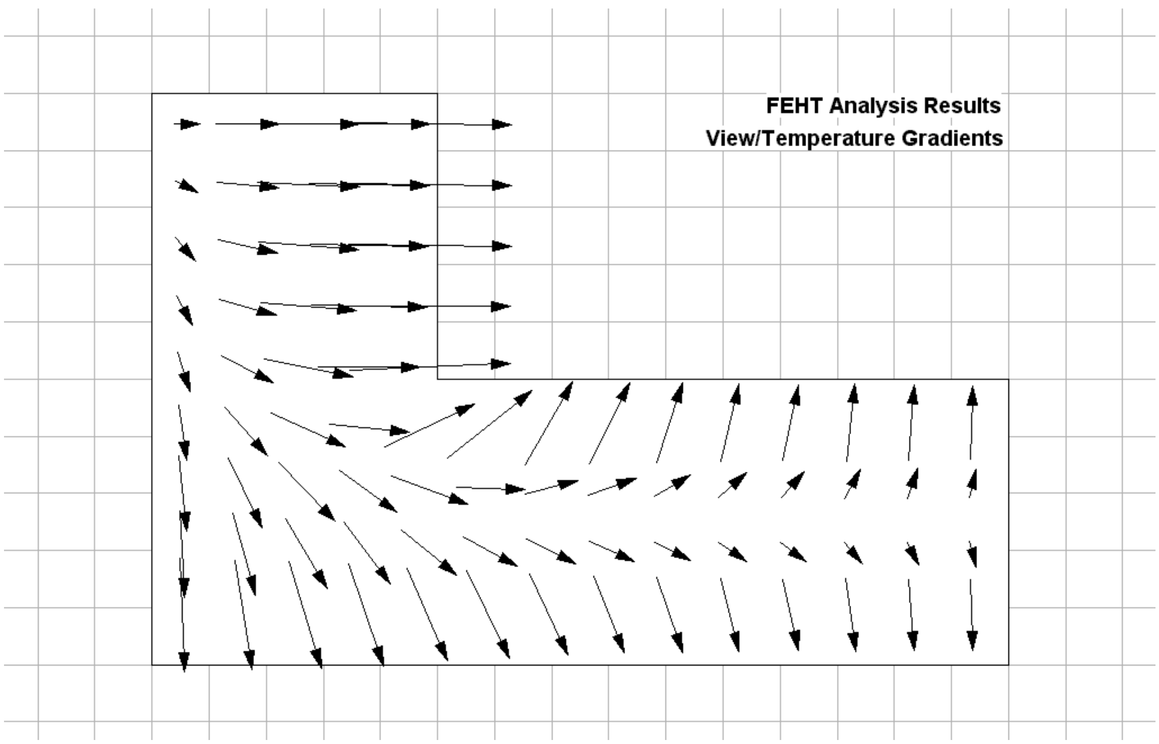
$$-q'_A + q'_B + \dot{q}\forall' = 0$$

$$-1135 \text{ W/m} + (-1365 \text{ W/m}) + 2500 \text{ W/m} = 0 \qquad <$$

where $\dot{q}\forall' = 1\times10^6 \text{ W/m}^3 \times [25\times50 + 25\times50]\times10^{-6} \text{m}^2 = 2500 \text{ W/m.}$

COMMENTS: (1) For background on setting up this problem in FEHT, see the tutorial example of the User's Manual. While the boundary conditions are different, and the internal generation term is to be included, the procedure for performing the analysis is the same.

(2) The heat flow distribution can be visualized using the *View* | *Temperature Gradients* command. The direction and magnitude of the heat flow is represented by the directions and lengths of arrows. Compare the heat flow distribution to the isotherms shown above.

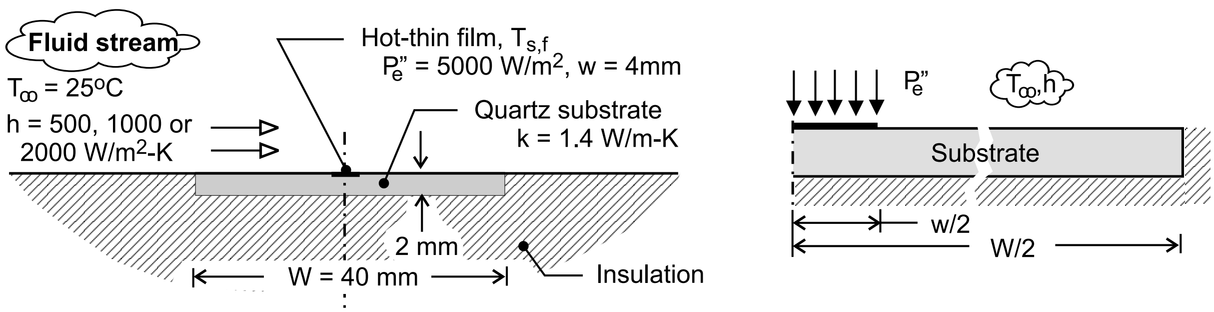


PROBLEM 4.70

KNOWN: Hot-film flux gage for determining the convection coefficient of an adjoining fluid stream by measuring the dissipated electric power, P_e, and the average surface temperature, $T_{s,f}$.

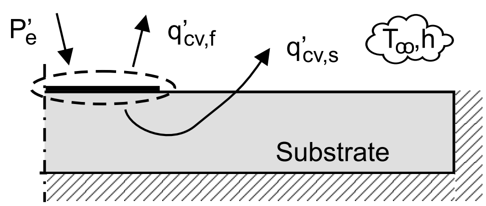
FIND: Using the finite-element method of *FEHT*, determine the fraction of the power dissipation that is conducted into the quartz substrate considering three cases corresponding to convection coefficients of 500, 1000 and 2000 $W/m^2 \cdot K$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant substrate properties, (3) Uniform convection coefficient over the hot-film and substrate surfaces, (4) Uniform power dissipation over hot film.

ANALYSIS: The symmetrical section shown in the schematic above (right) is drawn into *FEHT* specifying the substrate material property. On the upper surface, a convection boundary condition (T_∞,h) is specified over the full width W/2. Additionally, an applied uniform flux $\left(P_e'',\ W/m^2\right)$ boundary condition is specified for the hot-film region (w/2). The remaining surfaces of the two-dimensional system are specified as adiabatic. In the schematic below, the electrical power dissipation P_e' (W/m) in the hot film is transferred by convection from the film surface, $q'_{cv,f}$, and from the adjacent substrate surface, $q'_{cv,s}$.



The analysis evaluates the fraction, F, of the dissipated electrical power that is conducted into the substrate and convected to the fluid stream,

$$F = q'_{cv,s} / P_e' = 1 - q'_{cv,f} / P_e'$$

where $\quad P_e' = P_e''(w/2) = 5000\ W/m^2 \times (0.002\ m) = 10\ W/m.$

After solving for the temperature distribution, the *View|Heat Flow* command is used to evaluate $q'_{cv,f}$ for the three values of the convection coefficient.

Continued …..

Case	$h(W/m^2 \cdot K)$	$q'_{cv,f}\ (W/m)$	F(%)	$T_{s,f}$ (°C)
1	500	5.64	43.6	30.9
2	1000	6.74	32.6	28.6
3	2000	7.70	23.3	27.0

COMMENTS: (1) For the ideal hot-film flux gage, there is negligible heat transfer to the substrate, and the convection coefficient of the air stream is calculated from the measured electrical power, P''_e, the average film temperature (by a thin-film thermocouple), $T_{s,f}$, and the fluid stream temperature, T_∞, as $h = P''_e / (T_{s,f} - T_\infty)$. The purpose in performing the present analysis is to estimate a correction factor to account for heat transfer to the substrate.

(2) As anticipated, the fraction of the dissipated electrical power conducted into the substrate, F, decreases with increasing convection coefficient. For the case of the largest convection coefficient, F amounts to 25%, making it necessary to develop a reliable, accurate heat transfer model to estimate the applied correction. Further, this condition limits the usefulness of this gage design to flows with high convection coefficients.

(3) A reduction in F, and hence the effect of an applied correction, could be achieved with a substrate material having a lower thermal conductivity than quartz. However, quartz is a common substrate material for fabrication of thin-film heat-flux gages and thermocouples. By what other means could you reduce F?

(4) In addition to the tutorial example in the *FEHT* User's Manual, the solved models for Examples 4.3 and 4.4 are useful for developing skills helpful in solving this problem.

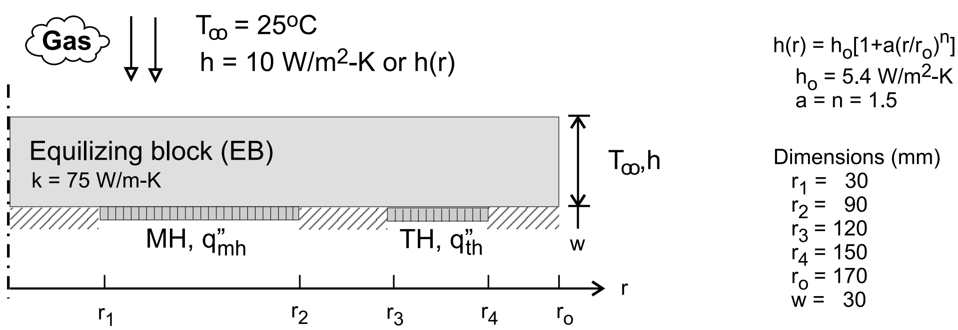
PROBLEM 4.71

KNOWN: Hot-plate tool for micro-lithography processing of 300-mm silicon wafer consisting of an aluminum alloy equalizing block (EB) heated by ring-shaped main and trim electrical heaters (MH and TH) providing two-zone control.

FIND: The assignment is to size the heaters, MH and TH, by specifying their applied heat fluxes, q''_{mh} and q''_{th}, and their radial extents, Δr_{mh} and Δr_{th}, to maintain an operating temperature of 140°C with a uniformity of 0.1°C. Consider these steps in the analysis: (a) Perform an energy balance on the EB to obtain an initial estimate for the heater fluxes with $q''_{mh} = q''_{th}$ extending over the full radial limits; using *FEHT*, determine the upper surface temperature distribution and comment on whether the desired uniformity has been achieved; (b) Re-run your *FEHT* code with different values of the heater fluxes to obtain the best uniformity possible and plot the surface temperature distribution; (c) Re-run your *FEHT* code for the best arrangement found in part (b) using the representative distribution of the convection coefficient (see schematic for h(r) for downward flowing gas across the upper surface of the EB; adjust the heat flux of TH to obtain improved uniformity; and (d) Suggest changes to the design for improving temperature uniformity.

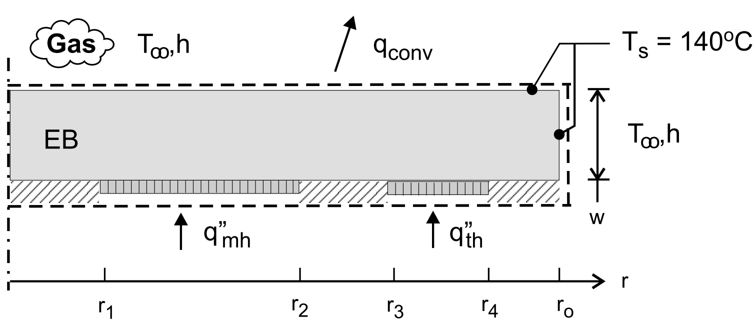
SCHEMATIC:

<



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction with uniform and constant properties in EB, (3) Lower surface of EB perfectly insulated, (4) Uniform convection coefficient over upper EB surface, unless otherwise specified and (5) negligible radiation exchange between the EB surfaces and the surroundings.

ANALYSIS: (a) To obtain initial estimates for the MH and TH fluxes, perform an overall energy balance on the EB as illustrated in the schematic below.



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q''_{mh}\pi\left(r_2^2 - r_1^2\right) + q''_{th}\pi\left(r_4^2 - r_3^2\right) - h\left[\pi r_o^2 + 2\pi r_o w\right]\left(T_s - T_\infty\right) = 0$$

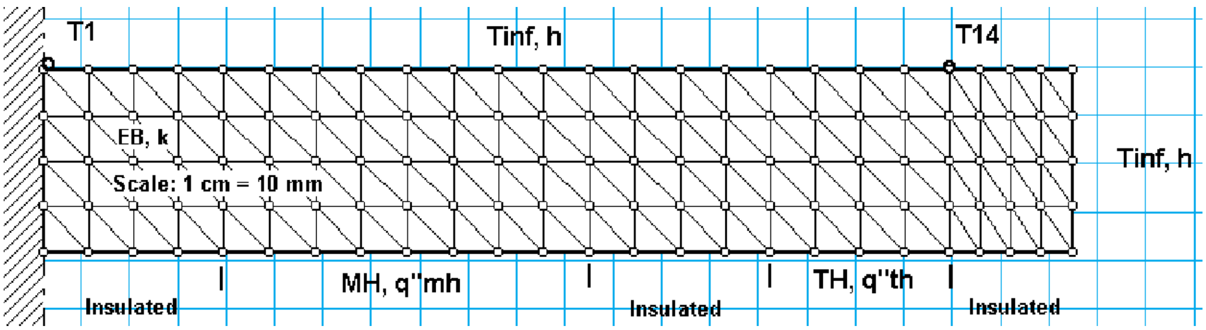
Continued …..

PROBLEM 4.71 (Cont.)

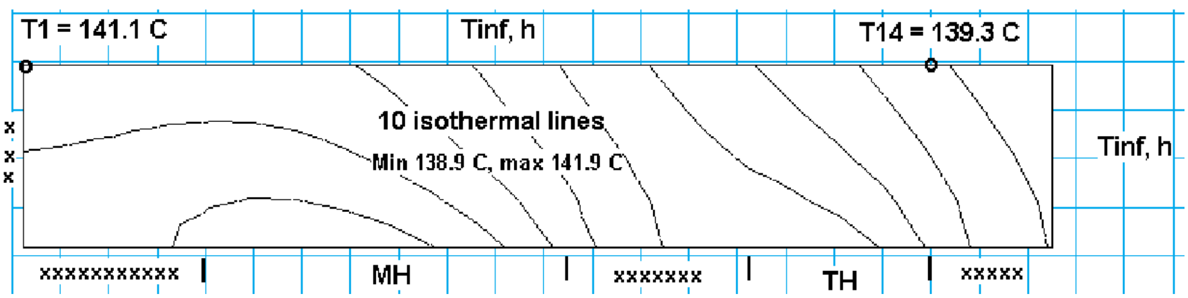
Substituting numerical values and letting $q''_{mh} = q''_{th}$, find

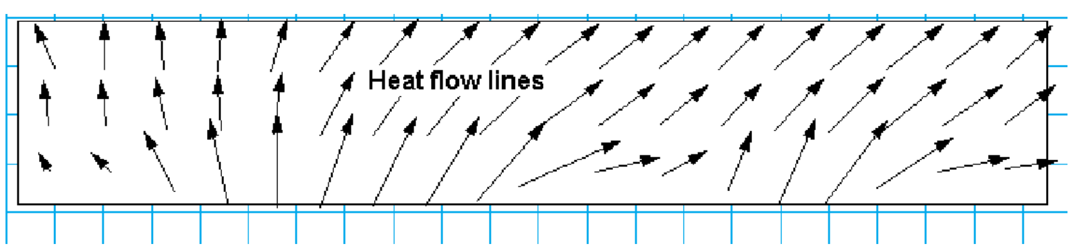
$$q''_{mh} = q''_{th} = 2939\ \mathrm{W/m^2}$$ <

Using *FEHT*, the analysis is performed on an axisymmetric section of the EB with the nodal arrangement as shown below.



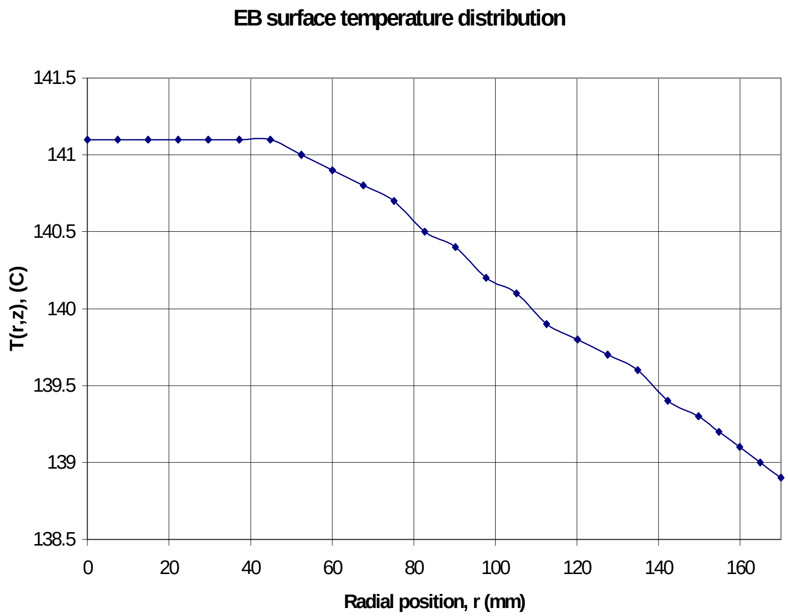
The *Temperature Contour* view command is used to create the temperature distribution shown below. The temperatures at the center (T_1) and the outer edge of the wafer (r = 150 mm, T_{14}) are read from the *Tabular Output* page. The *Temperature Gradients* view command is used to obtain the heat flow distribution when the line length is proportional to the magnitude of the heat rate.





From the analysis results, for this base case design $(q''_{mh} = q''_{th})$, the temperature difference across the radius of the wafer is 1.7°C, much larger than the design goal of 0.1°C. The upper surface temperature distribution is shown in the graph below.

Continued …..



(b) From examination of the results above, we conclude that if q''_{mh} is reduced and q''_{th} increased, the EB surface temperature uniformity could improve. The results of three trials compared to the base case are tabulated below.

Trial	q''_{mh} (W/m^2)	q''_{th} (W/m^2)	T_1 $(°C)$	T_{14} $(°C)$	$T_1 - T_{14}$ $(°C)$
Base	2939	2939	141.1	139.3	1.8
1	2880 (-2%)	2997 (+2%)	141.1	139.4	1.7
2	2880 (-2%)	3027 (+3%)	141.7	140.0	1.7
3	2910 (-1%)	2997 (+2%)	141.7	139.9	1.8
Part (c)	2939	2939	141.7	139.1	2.6
Part (d) k=150 W/m·K	2939	2939	140.4	139.5	0.9
Part (d) k=300 W/m·K	2939	2939	140.0	139.6	0.4

The strategy of changing the heater fluxes (trials 1-3) has not resulted in significant improvements in the EB surface temperature uniformity.

Continued …..

(c) Using the same *FEHT* code as with part (b), base case, the boundary conditions on the upper surface of the EB were specified by the function h(r) shown in the schematic. The value of h(r) ranged from 5.4 to 13.5 $W/m^2 \cdot K$ between the centerline and EB edge. The result of the analysis is tabulated above, labeled as part (c). Note that the temperature uniformity has become significantly poorer.

(d) There are at least two options that should be considered in the re-design to improve temperature uniformity. *Higher thermal conductivity material for the EB.* Aluminum alloy is the material most widely used in practice for reasons of low cost, ease of machining, and durability of the heated surface. The results of analyses for thermal conductivity values of 150 and 300 W/m·K are tabulated above, labeled as part (d). Using pure or oxygen-free copper could improve the temperature uniformity to better than 0.5°C.

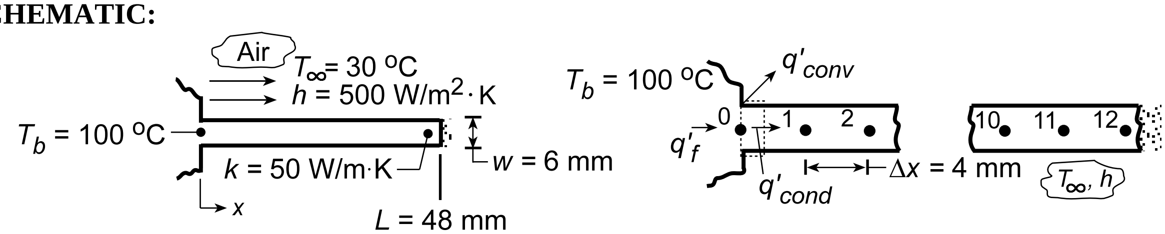
Distributed heater elements. The initial option might be to determine whether temperature uniformity could be improved using two elements, but located differently. Another option is a single element heater spirally embedded in the lower portion of the EB. By appropriately positioning the element as a function of the EB radius, improved uniformity can be achieved. This practice is widely used where precise and uniform temperature control is needed.

PROBLEM 4.72

KNOWN: Straight fin of uniform cross section with insulated end.

FIND: (a) Temperature distribution using finite-difference method and validity of assuming one-dimensional heat transfer, (b) Fin heat transfer rate and comparison with analytical solution, Eq. 3.76, (c) Effect of convection coefficient on fin temperature distribution and heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in fin, (3) Constant properties, (4) Uniform film coefficient.

ANALYSIS: (a) From the analysis of Problem 4.45, the finite-difference equations for the nodal arrangement can be directly written. For the nodal spacing Δx = 4 mm, there will be 12 nodes. With ℓ >> w representing the distance normal to the page,

$$\frac{hP}{kA_c}\cdot\Delta x^2 \approx \frac{h\cdot 2\ell}{k\cdot\ell\cdot w}\Delta x^2 = \frac{h\cdot 2}{kw}\Delta x^2 = \frac{500\,\mathrm{W/m^2\cdot K}\times 2}{50\,\mathrm{W/m\cdot K}\times 6\times 10^{-3}\,\mathrm{m}}\left(4\times 10^{-3}\mathrm{mm}\right) = 0.0533$$

Node 1: $100 + T_2 + 0.0533\times 30 - (2+0.0533)T_1 = 0$ or $-2.053T_1 + T_2 = -101.6$

Node n: $T_{n+1} + T_{n-1} + 1.60 - 2.0533T_n = 0$ or $T_{n-1} - 2.053T_n + T_{n-1} = -1.60$

Node 12: $T_{11} + (0.0533/2)30 - (0.0533/2+1)T_{12} = 0$ or $T_{11} - 1.0267T_{12} = -0.800$

Using matrix notation, Eq. 4.52, where [A] [T] = [C], the A-matrix is tridiagonal and only the non-zero terms are shown below. A matrix inversion routine was used to obtain [T].

Tridiagonal Matrix A						*Column Matrices*		
Nonzero Terms			Values			Node	C	T
	$a_{1,1}$	$a_{1,2}$		-2.053	1	1	-101.6	85.8
$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	1	-2.053	1	2	-1.6	74.5
$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	1	-2.053	1	3	-1.6	65.6
$a_{4,3}$	$a_{4,4}$	$a_{4,5}$	1	-2.053	1	4	-1.6	58.6
$a_{5,4}$	$a_{5,5}$	$a_{5,6}$	1	-2.053	1	5	-1.6	53.1
$a_{6,5}$	$a_{6,6}$	$a_{6,7}$	1	-2.053	1	6	-1.6	48.8
$a_{7,6}$	$a_{7,7}$	$a_{7,8}$	1	-2.053	1	7	-1.6	45.5
$a_{8,7}$	$a_{8,8}$	$a_{8,9}$	1	-2.053	1	8	-1.6	43.0
$a_{9,8}$	$a_{9,9}$	$a_{9,10}$	1	-2.053	1	9	-1.6	41.2
$a_{10,9}$	$a_{10,10}$	$a_{10,11}$	1	-2.053	1	10	-1.6	39.9
$a_{11,10}$	$a_{11,11}$	$a_{11,12}$	1	-2.053	1	11	-1.6	39.2
$a_{12,11}$	$a_{12,12}$	$a_{12,13}$	1	-1.027	1	12	-0.8	38.9

The assumption of one-dimensional heat conduction is justified when Bi ≡ h(w/2)/k < 0.1. Hence, with Bi = 500 W/m²·K(3 × 10^{-3} m)/50 W/m·K = 0.03, the assumption is reasonable.

Continued...

(b) The fin heat rate can be most easily found from an energy balance on the control volume about Node 0,

$$q'_f = q'_1 + q'_{conv} = k \cdot w \frac{T_0 - T_1}{\Delta x} + h\left(2\frac{\Delta x}{2}\right)(T_0 - T_\infty)$$

$$q'_f = 50\,W/m \cdot K\left(6\times10^{-3}m\right)\frac{(100-85.8)^\circ C}{4\times10^{-3}m} + 500\,W/m^2 \cdot K\left(2\cdot\frac{4\times10^{-3}m}{2}\right)(100-30)^\circ C$$

$$q'_f = (1065+140)\,W/m = 1205\,W/m\,.$$ <

From Eq. 3.76, the fin heat rate is

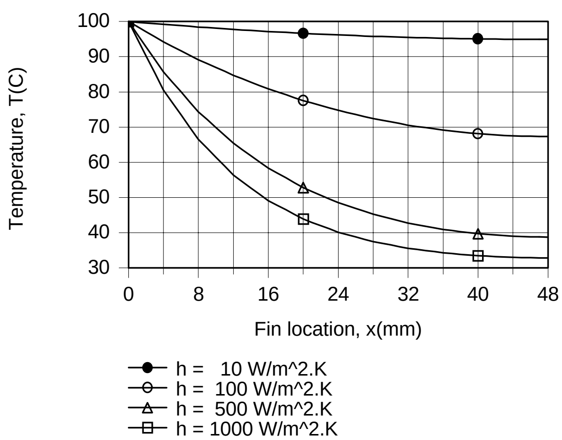
$$q = \left(hPkA_c\right)^{1/2} \cdot \theta_b \cdot \tanh mL\,.$$

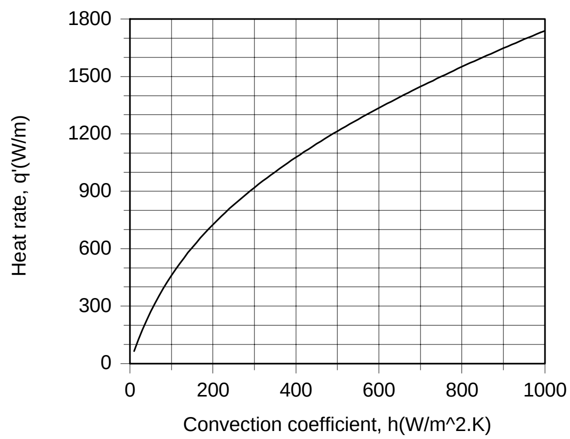
Substituting numerical values with $P = 2(w + \ell) \approx 2\ell$ and $A_c = w\cdot\ell$, $m = (hP/kA_c)^{1/2} = 57.74\ m^{-1}$ and $M = (hPkA_c)^{1/2} = 17.32\,\ell$ W/K. Hence, with $\theta_b = 70°C$,

$$q' = 17.32\,W/K \times 70\,K \times \tanh(57.44\times0.048) = 1203\,W/m$$

and the finite-difference result agrees very well with the exact (analytical) solution.

(c) Using the IHT *Finite-Difference Equations Tool Pad* for *1D, SS* conditions, the fin temperature distribution and heat rate were computed for h = 10, 100, 500 and 1000 $W/m^2\cdot K$. Results are plotted as follows.





The temperature distributions were obtained by first creating a *Lookup Table* consisting of 4 rows of nodal temperatures corresponding to the 4 values of h and then using the *LOOKUPVAL2* interpolating function with the *Explore* feature of the IHT menu. Specifically, the function T_EVAL = LOOKUPVAL2(t0467, h, x) was entered into the workspace, where t0467 is the file name given to the Lookup Table. For each value of h, *Explore* was used to compute T(x), thereby generating 4 data sets which were placed in the *Browser* and used to generate the plots. The variation of q' with h was simply generated by using the *Explore* feature to solve the finite-difference model equations for values of h incremented by 10 from 10 to 1000 $W/m^2\cdot K$.

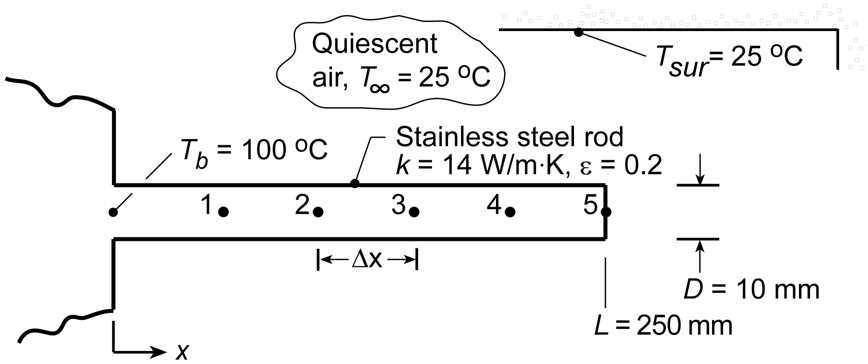
Although q'_f increases with increasing h, the effect of changes in h becomes less pronounced. This trend is a consequence of the reduction in fin temperatures, and hence the fin efficiency, with increasing h. For $10 \le h \le 1000\ W/m^2\cdot K$, $0.95 \ge \eta_f \ge 0.24$. Note the nearly isothermal fin for h = 10 $W/m^2\cdot K$ and the pronounced temperature decay for h = 1000 $W/m^2\cdot K$.

PROBLEM 4.73

KNOWN: Pin fin of 10 mm diameter and length 250 mm with base temperature of 100°C experiencing radiation exchange with the surroundings and free convection with ambient air.

FIND: Temperature distribution using finite-difference method with five nodes. Fin heat rate and relative contributions by convection and radiation.

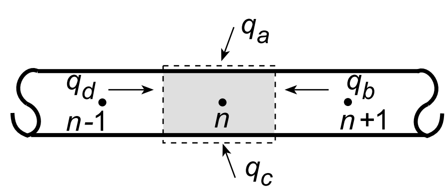
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in fin, (3) Constant properties, (4) Fin approximates small object in large enclosure, (5) Fin tip experiences convection and radiation, (6) $h_{fc} = 2.89[0.6 + 0.624(T - T_\infty)^{1/6}]^2$.

ANALYSIS: To apply the finite-difference method, define the 5-node system shown above where $\Delta x = L/5$. Perform energy balances on the nodes to obtain the finite-difference equations for the nodal temperatures.

Interior node, n = 1, 2, 3 or 4



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q_a + q_b + q_c + q_d = 0 \qquad (1)$$

$$h_{r,n} P\Delta x\left(T_{sur} - T_n\right) + kA_c \frac{T_{n+1} - T_n}{\Delta x} + h_{fc,n} P\Delta x\left(T_\infty - T_n\right) + kA_c \frac{T_{n-1} - T_n}{\Delta x} = 0 \qquad (2)$$

where the free convection coefficient is

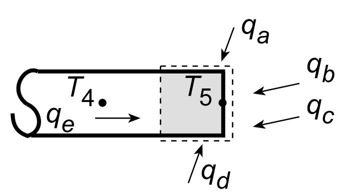
$$h_{fc,n} = 2.89\left[0.6 + 0.624\left(T_n - T_\infty\right)^{1/6}\right]^2 \qquad (3)$$

and the linearized radiation coefficient is

$$h_{r,n} = \varepsilon\sigma\left(T_n + T_{sur}\right)\left(T_n^2 + T_{sur}^2\right) \qquad (4)$$

with $P = \pi D$ and $A_c = \pi D^2/4$. (5,6)

Tip node, n = 5



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q_a + q_b + q_c + q_d + q_e = 0$$

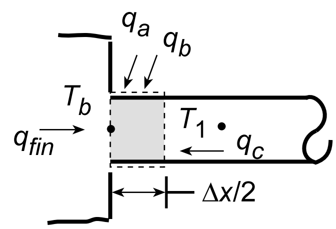
$$\begin{aligned} &h_{r,5}\left(P\Delta x/2\right)\left(T_{sur} - T_5\right) + h_{r,5}A_c\left(T_{sur} - T_5\right) + h_{fc,5}A_c\left(T_\infty - T_5\right) \\ &\quad + h_{fc,5}\left(P\Delta x/2\right)\left(T_\infty - T_5\right) + kA_c \frac{T_4 - T_5}{\Delta x} = 0 \end{aligned} \qquad (7)$$

Continued...

PROBLEM 4.73 (Cont.)

Knowing the nodal temperatures, the heat rates are evaluated as:

Fin Heat Rate: Perform an energy balance around Node b.



$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q_a + q_b + q_c + q_{fin} = 0$$

$$h_{r,b}(P\Delta x/2)(T_{sur} - T_b) + h_{fc,b}(P\Delta x/2)(T_\infty - T_b) + kA_c \frac{(T_1 - T_b)}{\Delta x} + q_{fin} = 0 \qquad (8)$$

where $h_{r,b}$ and $h_{fc,b}$ are evaluated at T_b.

Convection Heat Rate: To determine the portion of the heat rate by convection from the fin surface, we need to sum contributions from each node. Using the convection heat rate terms from the foregoing energy balances, for, respectively, node b, nodes 1, 2, 3, 4 and node 5.

$$q_{cv} = -q_b)_b - \sum q_c)_{1-4} - (q_c + q_d)_5 \qquad (9)$$

Radiation Heat Rate: In the same manner,

$$q_{rad} = -q_a)_b - \sum q_b)_{1-4} - (q_a + q_b)_5$$

The above equations were entered into the IHT workspace and the set of equations solved for the nodal temperatures and the heat rates. Summary of key results including the temperature distribution and heat rates is shown below.

Node	b	1	2	3	4	5	Fin
T_j (°C)	100	58.5	40.9	33.1	29.8	28.8	-
q_{cv} (W)	0.603	0.451	0.183	0.081	0.043	0.015	1.375
q_{fin} (W)	-	-	-	-	-	-	1.604
q_{rad} (W)	-	-	-	-	-	-	0.229
h_{cv} ($W/m^2 \cdot K$)	10.1	8.6	7.3	6.4	5.7	5.5	-
h_{rod} ($W/m^2 \cdot K$)	1.5	1.4	1.3	1.3	1.2	1.2	-

<

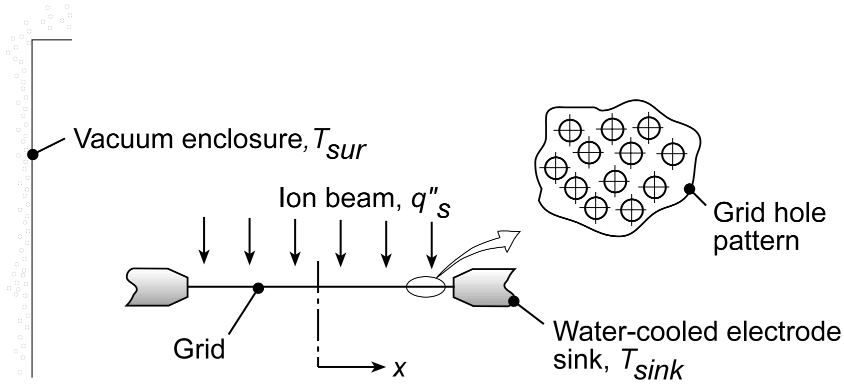
COMMENTS: From the tabulated results, it is evident that free convection is the dominant node. Note that the free convection coefficient varies almost by a factor of two over the length of the fin.

PROBLEM 4.74

KNOWN: Thin metallic foil of thickness, t, whose edges are thermally coupled to a sink at temperature T_{sink} is exposed on the top surface to an ion beam heat flux, q_s'', and experiences radiation exchange with the vacuum enclosure walls at T_{sur}.

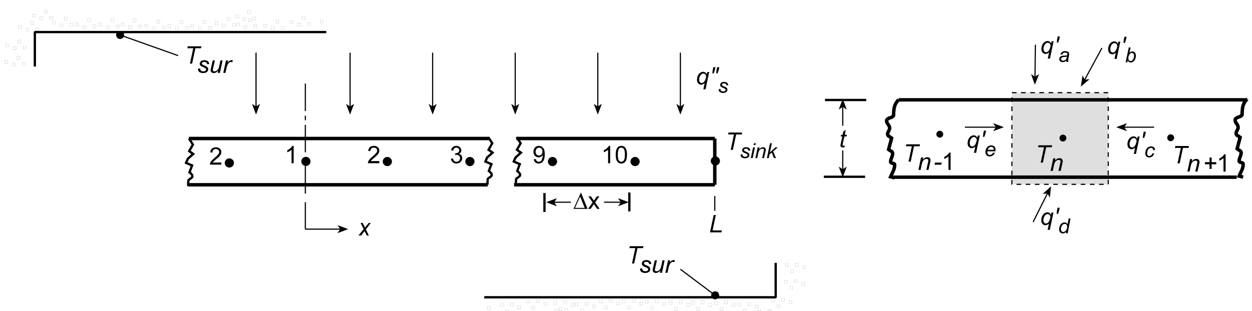
FIND: Temperature distribution across the foil.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in the foil, (2) Constant properties, (3) Upper and lower surfaces of foil experience radiation exchange, (4) Foil is of unit length normal to the page.

ANALYSIS: The 10-node network representing the foil is shown below.



From an energy balance on node n, $\dot{E}_{in} - \dot{E}_{out} = 0$, for a unit depth,

$$q'_a + q'_b + q'_c + q'_d + q'_e = 0$$

$$q_s''\Delta x + h_{r,n}\Delta x\left(T_{sur} - T_n\right) + k\left(t\right)\left(T_{n+1} - T_n\right)/\Delta x + h_{r,n}\Delta x\left(T_{sur} - T_n\right) + k\left(t\right)\left(T_{n-1} - T_n\right)/\Delta x = 0 \quad (1)$$

where the linearized radiation coefficient for node n is

$$h_{r,n} = \varepsilon\sigma\left(T_{sur} + T_n\right)\left(T_{sur}^2 + T_n^2\right) \quad (2)$$

Solving Eq. (1) for T_n find,

$$T_n = \left[\left(T_{n+1} + T_{n-1}\right) + \left(2h_{r,n}\Delta x^2/kt\right)T_{sur} + \left(\Delta x^2/kt\right)q_s''\right] \Big/ \left[\left(h_{r,n}\Delta x^2/kt\right) + 2\right] \quad (3)$$

which, considering symmetry, applies also to node 1. Using IHT for Eqs. (3) and (2), the set of finite-difference equations was solved for the temperature distribution (K):

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
374.1	374.0	373.5	372.5	370.9	368.2	363.7	356.6	345.3	327.4

Continued...

COMMENTS: (1) If the temperature gradients were excessive across the foil, it would wrinkle; most likely since its edges are constrained, the foil will bow.

(2) The IHT workspace for the finite-difference analysis follows:

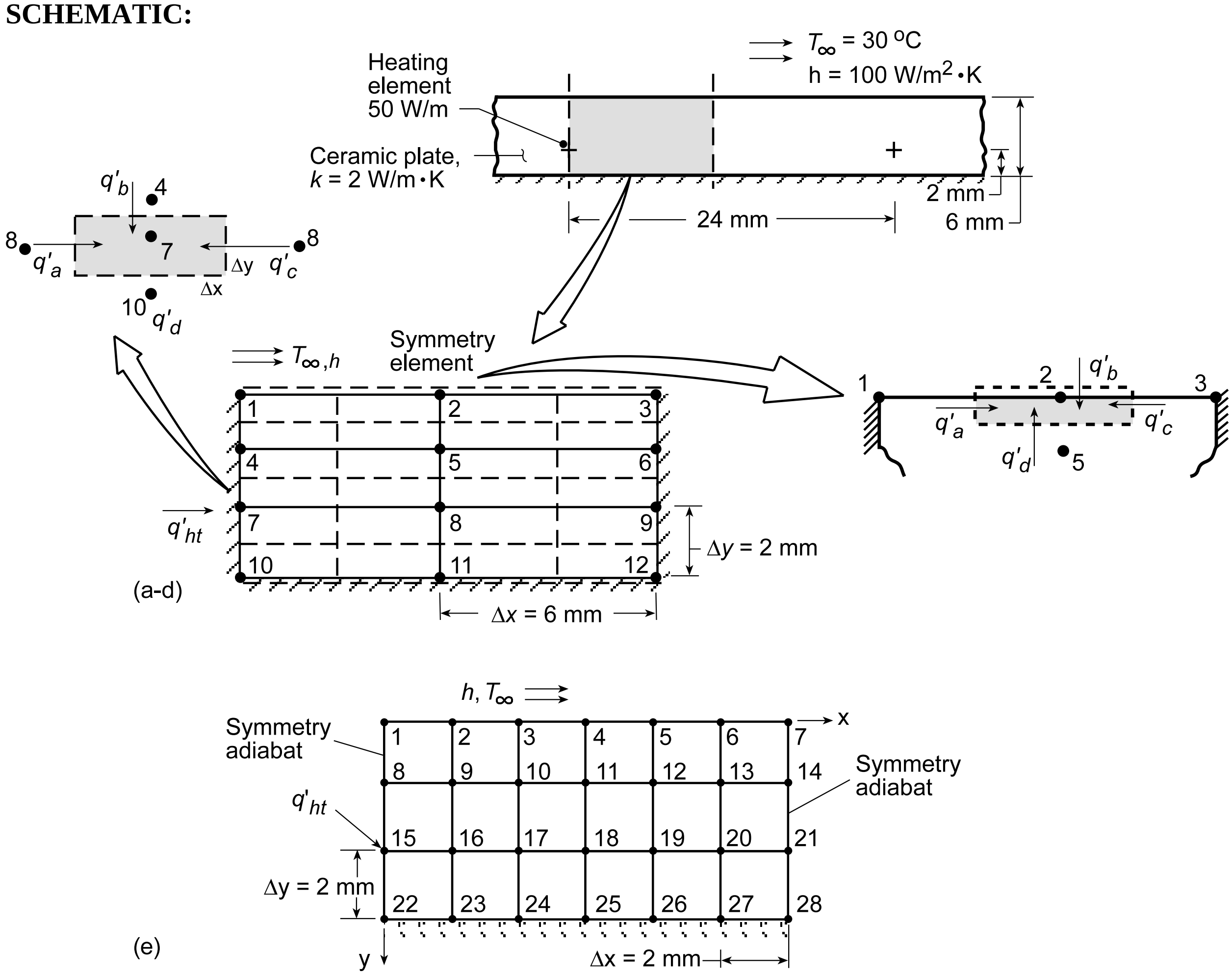
```
// The nodal equations:
T1 = ( (T2 + T2) + A1 * Tsur +  B *q''s ) / ( A1 + 2)
A1= 2 * hr1 * deltax^2 / (k * t)
hr1 = eps * sigma * (Tsur + T1) * (Tsur^2 + T1^2)
sigma = 5.67e-8
B = deltax^2 / (k * t)

T2 = ( (T1 + T3) + A2 * Tsur + B *q''s ) / ( A2 + 2)
A2= 2 * hr2 * deltax^2 / (k * t)
hr2 = eps * sigma * (Tsur + T2) * (Tsur^2 + T2^2)

T3 = ( (T2 + T4) + A3 * Tsur + B *q''s ) / ( A3 + 2)
A3= 2 * hr3 * deltax^2 / (k * t)
hr3 = eps * sigma * (Tsur + T3) * (Tsur^2 + T3^2)

T4 = ( (T3 + T5) + A4 * Tsur + B *q''s ) / ( A4 + 2)
A4= 2 * hr4 * deltax^2 / (k * t)
hr4 = eps * sigma * (Tsur + T4) * (Tsur^2 + T4^2)

T5 = ( (T4 + T6) + A5 * Tsur + B *q''s ) / ( A5 + 2)
A5= 2 * hr5 * deltax^2 / (k * t)
hr5 = eps * sigma * (Tsur + T5) * (Tsur^2 + T5^2)

T6 = ( (T5 + T7) + A6 * Tsur + B *q''s ) / ( A6 + 2)
A6= 2 * hr6 * deltax^2 / (k * t)
hr6 = eps * sigma * (Tsur + T6) * (Tsur^2 + T6^2)

T7 = ( (T6 + T8) + A7 * Tsur + B *q''s ) / ( A7 + 2)
A7= 2 * hr7 * deltax^2 / (k * t)
hr7 = eps * sigma * (Tsur + T7) * (Tsur^2 + T7^2)

T8 = ( (T7 + T9) + A8 * Tsur + B *q''s ) / ( A8 + 2)
A8= 2 * hr8 * deltax^2 / (k * t)
hr8 = eps * sigma * (Tsur + T8) * (Tsur^2 + T8^2)

T9 = ( (T8 + T10) + A9 * Tsur + B *q''s ) / ( A9 + 2)
A9= 2 * hr9 * deltax^2 / (k * t)
hr9 = eps * sigma * (Tsur + T9) * (Tsur^2 + T9^2)

T10 = ( (T9 + Tsink) + A10 * Tsur + B *q''s ) / ( A10 + 2)
A10= 2 * hr10 * deltax^2 / (k * t)
hr10 = eps * sigma * (Tsur + T10) * (Tsur^2 + T10^2)

// Assigned variables
deltax = L / 10          // Spatial increment, m
L = 0.150                // Foil length, m
t = 0.00025              // Foil thickness, m
eps = 0.45               // Emissivity
Tsur = 300               // Surroundings temperature, K
k = 40                   // Foil thermal conductivity, W/m.K
Tsink = 300              // Sink temperature, K
q''s = 600               // Ion beam heat flux, W/m^2

/* Data Browser results: Temperature distribution (K)  and linearized radiation cofficients
(W/m^2.K):

T1      T2      T3      T4      T5      T6      T7      T8      T9      T10
374.1   374     373.5   372.5   370.9   368.2   363.7   356.6   345.3   327.4

hr1     hr2     hr3     hr4     hr5     hr6     hr7     hr8     hr9     hr10
3.956   3.953   3.943   3.926   3.895   3.845   3.765   3.639   3.444   3.157 */
```

PROBLEM 4.75

KNOWN: Electrical heating elements with known dissipation rate embedded in a ceramic plate of known thermal conductivity; lower surface is insulated, while upper surface is exposed to a convection process.

FIND: (a) Temperature distribution within the plate using prescribed grid spacing, (b) Sketch isotherms to illustrate temperature distribution, (c) Heat loss by convection from exposed surface (compare with element dissipation rate), (d) Advantage, if any, in not setting $\Delta x = \Delta y$, (e) Effect of grid size and convection coefficient on the temperature field.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction in ceramic plate, (2) Constant properties, (3) No internal generation, except for Node 7 (or Node 15 for part (e)), (4) Heating element approximates a line source of negligible wire diameter.

ANALYSIS: (a) The prescribed grid for the symmetry element shown above consists of 12 nodal points. Nodes 1-3 are points on a surface experiencing convection; nodes 4-6 and 8-12 are interior nodes. Node 7 is a special case of the interior node having a generation term; because of symmetry, q'_{ht} = 25 W/m. The finite-difference equations are derived as follows:

Continued...

Surface Node 2. From an energy balance on the prescribed control volume with $\Delta x/\Delta y = 3$,

$\dot{E}_{in} - \dot{E}_{out} = q'_a + q'_b + q'_c + q'_d = 0$;

$$k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + h\Delta x\left(T_\infty - T_2\right) + k\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_5 - T_2}{\Delta y} = 0\,.$$

Regrouping, find

$$T_2\left(1 + 2N\frac{\Delta x}{\Delta y} + 1 + 2\left(\frac{\Delta x}{\Delta y}\right)^2\right) = T_1 + T_3 + 2\left(\frac{\Delta x}{\Delta y}\right)^2 T_5 + 2N\frac{\Delta x}{\Delta y}T_\infty$$

where $N = h\Delta x/k = 100\ \mathrm{W/m^2\cdot K} \times 0.006\ \mathrm{m}/2\ \mathrm{W/m\cdot K} = 0.30\ \mathrm{K}$. Hence, with $T_\infty = 30°C$,

$$T_2 = 0.04587T_1 + 0.04587T_3 + 0.82569T_5 + 2.4771 \qquad (1)$$

From this FDE, the forms for nodes 1 and 3 can also be deduced.

Interior Node 7. From an energy balance on the prescribed control volume, with $\Delta x/\Delta y = 3$, $\dot{E}'_{in} - \dot{E}'_g = 0$, where $\dot{E}'_g = 2\,q'_{ht}$ and E'_{in} represents the conduction terms. Hence,

$q'_a + q'_b + q'_c + q'_d + 2q'_{ht} = 0$, or

$$k\Delta y\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_4 - T_7}{\Delta y} + k\Delta y\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_{10} - T_7}{\Delta y} + 2q'_{ht} = 0$$

Regrouping,

$$T_7\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + 1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right] = T_8 + \left(\frac{\Delta x}{\Delta y}\right)^2 T_4 + T_8 + \left(\frac{\Delta x}{\Delta y}\right)^2 T_{10} + \frac{2q'_{ht}}{k}\left(\frac{\Delta x}{\Delta y}\right)$$

Recognizing that $\Delta x/\Delta y = 3$, $q'_{ht} = 25\ \mathrm{W/m}$ and $k = 2\ \mathrm{W/m\cdot K}$, the FDE is

$$T_7 = 0.0500T_8 + 0.4500T_4 + 0.0500T_8 + 0.4500T_{10} + 3.7500 \qquad (2)$$

The FDEs for the remaining nodes may be deduced from this form. Following the procedure described in Section 4.5.2 for the Gauss-Seidel method, the system of FDEs has the form:

$$T_1^k = 0.09174T_2^{k-1} + 0.8257T_4^{k-1} + 2.4771$$
$$T_2^k = 0.04587T_1^k + 0.04587T_3^{k-1} + 0.8257T_5^{k-1} + 2.4771$$
$$T_3^k = 0.09174T_2^k + 0.8257T_6^{k-1} + 2.4771$$
$$T_4^k = 0.4500T_1^k + 0.1000T_5^{k-1} + 0.4500T_7^{k-1}$$
$$T_5^k = 0.4500T_2^k + 0.0500T_4^k + 0.0500T_6^{k-1} + 0.4500T_8^{k-1}$$
$$T_6^k = 0.4500T_3^k + 0.1000T_5^k + 0.4500T_9^{k-1}$$
$$T_7^k = 0.4500T_4^k + 0.1000T_8^{k-1} + 0.4500T_{10}^{k-1} + 3.7500$$
$$T_8^k = 0.4500T_5^k + 0.0500T_7^k + 0.0500T_9^{k-1} + 0.4500T_{11}^{k-1}$$
$$T_9^k = 0.4500T_6^k + 0.1000T_8^k + 0.4500T_{12}^{k-1}$$
$$T_{10}^k = 0.9000T_7^k + 0.1000T_{11}^{k-1}$$
$$T_{11}^k = 0.9000T_8^k + 0.0500T_{10}^{k-1} + 0.0500T_{12}^{k-1}$$
$$T_{12}^k = 0.9000T_9^k + 0.1000T_{11}^k$$

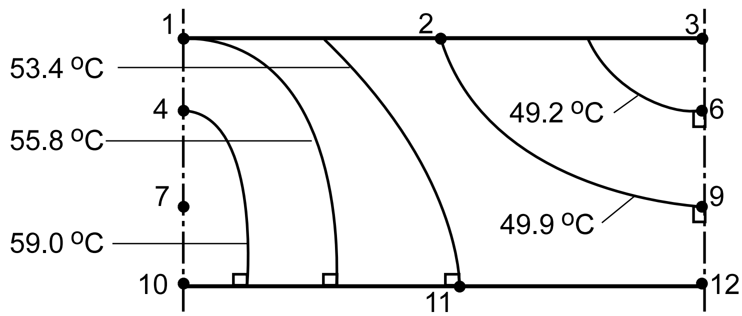
Continued …..

Note the use of the superscript k to denote the level of iteration. Begin the iteration procedure with rational estimates for T_i (k = 0) and prescribe the convergence criterion as $\varepsilon \le 0.1$.

k/T_i	1	2	3	4	5	6	7	8	9	10	11	12
0	55.0	50.0	45.0	61.0	54.0	47.0	65.0	56.0	49.0	60.0	55.0	50.0
1	57.4	51.7	46.0	60.4	53.8	48.1	63.5	54.6	49.6	62.7	54.8	50.1
2	57.1	51.6	46.9	59.7	53.2	48.7	64.3	54.3	49.9	63.4	54.5	50.4
∞	55.80	49.93	47.67	59.03	51.72	49.19	63.89	52.98	50.14	62.84	53.35	50.46

The last row with k = ∞ corresponds to the solution obtained by matrix inversion. It appears that at least 20 iterations would be required to satisfy the convergence criterion using the Gauss-Seidel method.

(b) Selected isotherms are shown in the sketch of the nodal network.



Note that the isotherms are normal to the adiabatic surfaces.

(c) The heat loss by convection can be expressed as

$$q'_{conv} = h\left[\frac{1}{2}\Delta x\left(T_1 - T_\infty\right) + \Delta x\left(T_2 - T_\infty\right) + \frac{1}{2}\Delta x\left(T_3 - T_\infty\right)\right]$$

$$q'_{conv} = 100\,\mathrm{W/m^2 \cdot K} \times 0.006\,\mathrm{m}\left[\frac{1}{2}(55.80-30) + (49.93-30) + \frac{1}{2}(47.67-30)\right] = 25.00\,\mathrm{W/m}. \quad <$$

As expected, the heat loss by convection is equal to the heater element dissipation. This follows from the conservation of energy requirement.

(d) For this situation, choosing Δx = 3Δy was advantageous from the standpoint of precision and effort. If we had chosen Δx = Δy = 2 mm, there would have been 28 nodes, doubling the amount of work, but with improved precision.

(e) Examining the effect of grid size by using the *Finite-Difference Equations* option from the *Tools* portion of the IHT Menu, the following temperature field was computed for Δx = Δy = 2 mm, where x and y are in mm and the temperatures are in °C.

y\x	0	2	4	6	8	10	12
0	55.04	53.88	52.03	50.32	49.02	48.24	47.97
2	58.71	56.61	54.17	52.14	50.67	49.80	49.51
4	66.56	59.70	55.90	53.39	51.73	50.77	50.46
6	63.14	59.71	56.33	53.80	52.09	51.11	50.78

Continued …..

Agreement with the results of part (a) is excellent, except in proximity to the heating element, where T_{15} = 66.6°C for the fine grid exceeds T_7 = 63.9°C for the coarse grid by 2.7°C.

For h = 10 $W/m^2 \cdot K$, the maximum temperature in the ceramic corresponds to T_{15} = 254°C, and the heater could still be operated at the prescribed power. With h = 10 $W/m^2 \cdot K$, the critical temperature of T_{15} = 400°C would be reached with a heater power of approximately 82 W/m.

COMMENTS: (1) The method used to obtain the rational estimates for T_i (k = 0) in part (a) is as follows. Assume 25 W/m is transferred by convection uniformly over the surface; find $\overline{T}_{surf} \approx 50°C$. Set T_2 = 50°C and recognize that T_1 and T_3 will be higher and lower, respectively. Assume 25 W/m is conducted uniformly to the outer nodes; find $T_5 - T_2 \approx 4°C$. For the remaining nodes, use intuition to guess reasonable values. (2) In selecting grid size (and whether $\Delta x = \Delta y$), one should consider the region of largest temperature gradients.

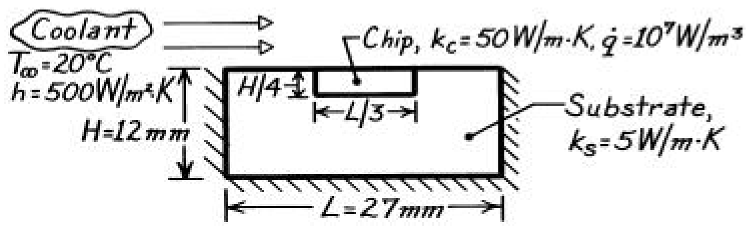
NOTE TO INSTRUCTOR: Although the problem statement calls for calculations with $\Delta x = \Delta y = 1$ mm, the instructional value and benefit-to-effort ratio are small. Hence, consideration of this grid size is not recommended.

PROBLEM 4.76

KNOWN: Silicon chip mounted in a dielectric substrate. One surface of system is convectively cooled while the remaining surfaces are well insulated.

FIND: Whether maximum temperature in chip will exceed 85°C.

SCHEMATIC:



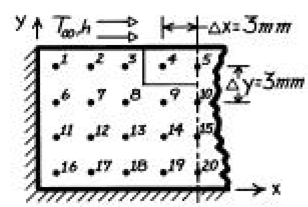
ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Negligible contact resistance between chip and substrate, (4) Upper surface experiences uniform convection coefficient, (5) Other surfaces are perfectly insulated.

ANALYSIS: Performing an energy balance on the chip assuming it is *perfectly insulated from* the substrate, the maximum temperature occurring at the interface with the dielectric substrate will be, according to Eqs. 3.43 and 3.46,

$$T_{max} = \frac{\dot{q}(H/4)^2}{2k_c} + \frac{\dot{q}(H/4)}{h} + T_\infty = \frac{10^7 \text{ W/m}^3 (0.003 \text{ m})^2}{2 \times 50 \text{ W/m} \cdot \text{K}} + \frac{10^7 \text{ W/m}^3 (0.003 \text{ m})}{500 \text{ W/m}^2 \cdot \text{K}} + 20^\circ\text{C} = 80.9^\circ\text{C}.$$

Since $T_{max} < 85°C$ for the assumed situation, for the actual two-dimensional situation with the conducting dielectric substrate, the maximum temperature should be less than 80°C.

Using the suggested grid spacing of 3 mm, construct the nodal network and write the finite-difference equation for each of the nodes taking advantage of symmetry of the system. Note that we have chosen to *not* locate nodes on the system surfaces for two reasons: (1) fewer total number of nodes, 20 vs. 25, and (2) Node 5 corresponds to center of chip which is likely the point of maximum temperature. Using these numerical values,



$$\frac{h\Delta x}{k_s} = \frac{500 \text{ W/m}^2 \cdot \text{K} \times 0.003 \text{ m}}{5 \text{ W/m} \cdot \text{K}} = 0.30 \qquad a = \frac{2}{(k_s/k_c)+1} = \frac{2}{5/50+1} = 1.818$$

$$\frac{h\Delta x}{k_c} = \frac{500 \text{ W/m}^2 \cdot \text{K} \times 0.003 \text{ m}}{5 \text{ W/m} \cdot \text{K}} = 0.030 \qquad b = \frac{2}{(k_c/k_s)+1} = \frac{2}{50/5+1} = 0.182$$

$$\frac{\dot{q}\Delta x \Delta y}{k_c} = 1.800 \qquad g = \frac{1}{k_c/k_s+1} = 0.0910$$

find the nodal equations:

Node 1 $$k_s\Delta x \frac{T_6 - T_1}{\Delta y} + k_s \Delta y \frac{T_2 - T_1}{\Delta x} + h\Delta x (T_\infty - T_1) = 0$$

Continued …..

$$-\left(2+\frac{h\Delta x}{k_s}\right)T_1 + T_2 + T_6 = -\frac{h\Delta x}{k_s}T_\infty \qquad -2.30T_1 + T_2 + T_6 = -6.00 \qquad (1)$$

Node 2 $\qquad T_1 - 3.3T_2 + T_3 + T_7 = -6.00$

(2)

Node 3

$$k_s\Delta y\frac{T_2 - T_3}{\Delta x} + \frac{T_4 - T_3}{(\Delta x/2)/k_c\Delta y + (\Delta x/2)/k_s\Delta y} + k_s\Delta x\frac{T_8 - T_3}{\Delta y} + h\Delta x(T_\infty - T_3) = 0$$

$$T_2 - \left(2 + a + (h\Delta x/k_s)T_3\right) + aT_4 + T_8 = -(h\Delta x/k)T_\infty$$

$$T_2 - 4.12T_3 + 1.82T_4 + T_8 = -6.00 \qquad (3)$$

Node 4

$$\frac{T_3 - T_4}{(\Delta x/2)/k_s\Delta y + (\Delta x/2)/k_c\Delta y} + k_c\Delta y\frac{T_5 - T_4}{\Delta x} + \frac{T_9 - T_4}{(\Delta y/2)/k_s\Delta x + (\Delta y/2)k_c\Delta x}$$

$$+\dot{q}(\Delta x\Delta y) + h\Delta x(T_\infty - T_4) = 0$$

$$bT_3 - \left(1 + 2b + [h\Delta x/k_c]\right)T_4 + T_5 + bT_9 = -(h\Delta x/k_c)T_\infty - \dot{q}\Delta x\Delta y/k_c$$

$$0.182T_3 - 1.39T_4 + T_5 + 0.182T_9 = -2.40 \qquad (4)$$

Node 5

$$k_c\Delta y\frac{T_4 - T_5}{\Delta x} + \frac{T_{10} - T_5}{(\Delta y/2)/k_s(\Delta x/2) + (\Delta y/2)/k_c(\Delta x/2)} + h(\Delta x/2)(T_\infty - T_5) + \dot{q}\Delta y(\Delta x/2) = 0$$

$$2T_4 - 2.21T_5 + 0.182T_{10} = -2.40 \qquad (5)$$

Nodes 6 and 11

$$k_s\Delta x(T_1 - T_6)/\Delta y + k_s\Delta y(T_7 - T_6)/\Delta x + k_s\Delta x(T_{11} - T_6)/\Delta y = 0$$

$$T_1 - 3T_6 + T_7 + T_{11} = 0 \qquad T_6 - 3T_{11} + T_{12} + T_{16} = 0 \qquad (6,11)$$

Nodes 7, 8, 12, 13, 14 Treat as interior points,

$$T_2 + T_6 - 4T_7 + T_8 + T_{12} = 0 \qquad T_3 + T_7 - 4T_8 + T_9 + T_{13} = 0 \qquad (7,8)$$

$$T_7 + T_{11} - 4T_{12} + T_{13} + T_{17} = 0 \qquad T_8 + T_{12} - 4T_{13} + T_{14} + T_{18} = 0 \qquad (12,13)$$

$$T_9 + T_{13} - 4T_{14} + T_{15} + T_{19} = 0 \qquad (14)$$

Node 9

$$k_s\Delta y\frac{T_8 - T_9}{\Delta x} + \frac{T_4 - T_9}{(\Delta y/2)/k_c\Delta x + (\Delta y/2)/k_s\Delta x} + k_s\Delta y\frac{T_{10} - T_9}{\Delta x} + k_s\Delta x\frac{T_{14} - T_9}{\Delta y} = 0$$

$$1.82T_4 + T_8 - 4.82T_9 + T_{10} + T_{14} = 0 \qquad (9)$$

Node 10 Using the result of Node 9 and considering symmetry,

$$1.82T_5 + 2T_9 - 4.82T_{10} + T_{15} = 0 \qquad (10)$$

Node 15 Interior point considering symmetry $\quad T_{10} + 2T_{14} - 4T_{15} + T_{20} = 0 \qquad (15)$

Node 16 By inspection, $\quad T_{11} - 2T_{16} + T_{17} = 0 \qquad (16)$

Continued …..

Nodes 17, 18, 19, 20

$$T_{12} + T_{16} - 3T_{17} + T_{18} = 0 \qquad T_{13} + T_{17} - 3T_{18} + T_{19} = 0 \qquad (17,18)$$

$$T_{14} + T_{18} - 3T_{19} + T_{20} = 0 \qquad T_{15} + 2T_{19} - 3T_{20} = 0 \qquad (19,20)$$

Using the matrix inversion method, the above system of finite-difference equations is written in matrix notation, Eq. 4.52, [A][T] = [C] where

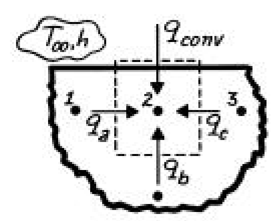
$$[A] = \begin{array}{cccccccccccccccccccc}
-2.3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3.3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4.12 & 1.82 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .182 & -1.39 & 1 & 0 & 0 & 0 & .182 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -2.21 & 0 & 0 & 0 & 0 & .182 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.82 & 0 & 0 & 0 & 1 & -4.82 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.82 & 0 & 0 & 0 & 2 & -4.82 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & -3
\end{array} \qquad [C] = \begin{array}{c} -6 \\ -6 \\ -6 \\ -2.4 \\ -2.4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$$

and the temperature distribution (°C), in geometrical representation, is

34.46	36.13	40.41	45.88	46.23
37.13	38.37	40.85	43.80	44.51
38.56	39.38	40.81	42.72	42.78
39.16	39.77	40.76	41.70	42.06

The maximum temperature is T_5 = 46.23°C which is indeed less than 85°C. <

COMMENTS: (1) The convection process for the energy balances of Nodes 1 through 5 were simplified by assuming the node temperature is also that of the surface. Considering Node 2, the energy balance processes for q_a, q_b and q_c are identical (see Eq. (2)); however,



$$q_{conv} = \frac{T_\infty - T_2}{1/h + \Delta y/2k} \approx h\left(T_\infty - T_2\right)$$

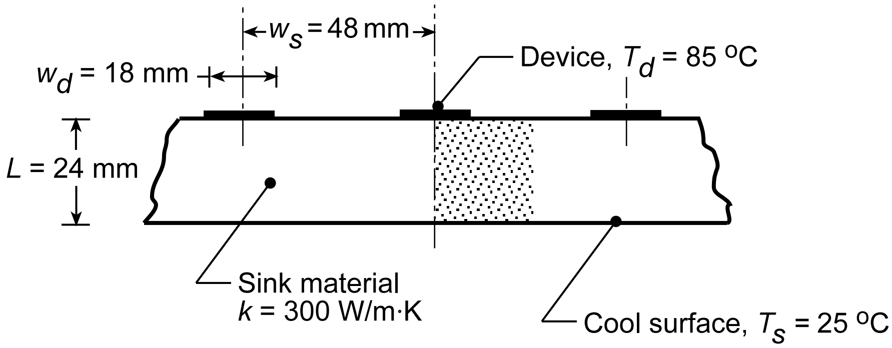
where $h\Delta y/2k = 5\ \text{W/m}^2\cdot\text{K} \times 0.003\ \text{m}/2 \times 50\ \text{W/m}\cdot\text{K} = 1.5\times10^{-4} << 1$. Hence, for this situation, the simplification is justified.

PROBLEM 4.77

KNOWN: Electronic device cooled by conduction to a heat sink.

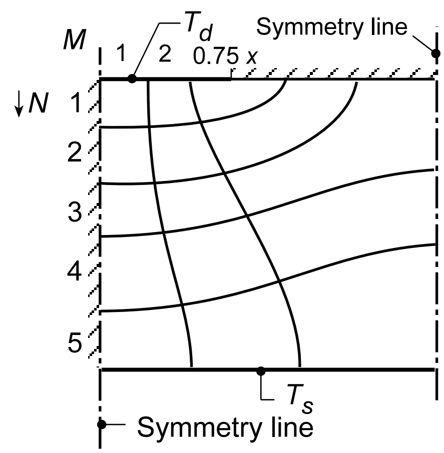
FIND: (a) Beginning with a symmetrical element, find the thermal resistance per unit depth between the device and lower surface of the sink, $R'_{t,d-s}$ (m·K/W) using the flux plot method; compare result with thermal resistance based upon assumption of one-dimensional conduction in rectangular domains of (i) width w_d and length L and (ii) width w_s and length L; (b) Using a coarse (5x5) nodal network, determine $R'_{t,d-s}$; (c) Using nodal networks with finer grid spacings, determine the effect of grid size on the precision of the thermal resistance calculation; (d) Using a fine nodal network, determine the effect of device width on $R'_{t,d-s}$ with w_d/w_s = 0.175, 0.275, 0.375 and 0.475 keeping w_s and L fixed.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, and (3) No internal generation, (4) Top surface not covered by device is insulated.

ANALYSIS: (a) Begin the flux plot for the symmetrical element noting that the temperature drop along the left-hand symmetry line will be almost linear. Choosing to sketch five isotherms and drawing the adiabats, find



N = 5 M = 2.75

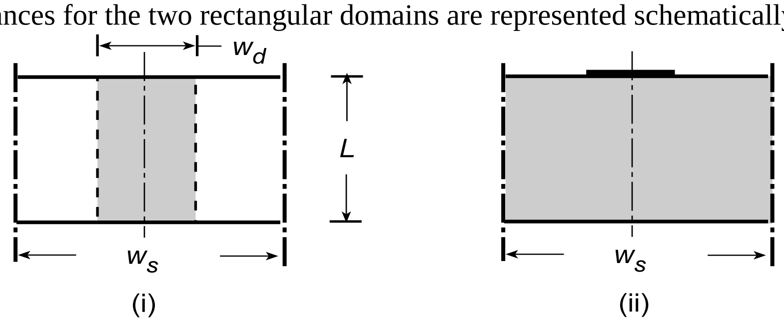
so that the shape factor for the device to the sink considering two symmetrical elements per unit depth is

$$S' = 2S'_o = 2\frac{M}{N} = 1.10$$

and the thermal resistance per unit depth is

$$R'_{t,d-s,fp} = 1/kS' = 1/300\,W/m\cdot K \times 1.10 = 3.03\times10^{-3}\,m\cdot K/W$$ <

The thermal resistances for the two rectangular domains are represented schematically below.



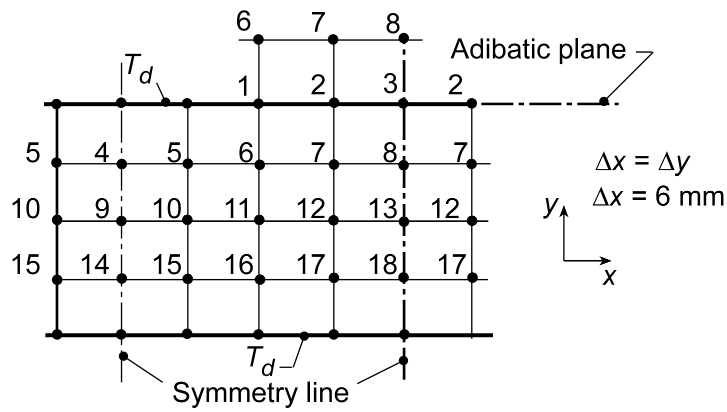
Continued...

$$R'_{t,d-s,i} = \frac{L}{kw_d} = \frac{0.024\,\text{m}}{300\,\text{W/m}\cdot\text{K}\times 0.018\,\text{m}} = 4.44\times10^{-3}\ \text{m}\cdot\text{K/W} \qquad <$$

$$R'_{t,d-s,ii} = \frac{L}{kw_s} = \frac{0.024\,\text{m}}{300\,\text{W/m}\cdot\text{K}\times 0.048\,\text{m}} = 1.67\times10^{-3}\ \text{m}\cdot\text{K/W} \qquad <$$

We expect the flux plot result to be bounded by the results for the rectangular domains. The *spreading* effect can be seen by comparing $R'_{f,d-s,fp}$ with $R'_{t,d-s,i}$.

(b) The coarse 5x5 nodal network is shown in the sketch including the nodes adjacent to the symmetry lines and the adiabatic surface. As such, all the finite-difference equations are interior nodes and can be written by inspection directly onto the IHT workspace. Alternatively, one could use the *IHT Finite-Difference Equations Tool*. The temperature distribution (°C) is tabulated in the same arrangement as the nodal network.

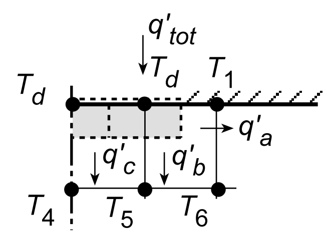


85.00	85.00	62.31	53.26	50.73
65.76	63.85	55.49	50.00	48.20
50.32	49.17	45.80	43.06	42.07
37.18	36.70	35.47	34.37	33.95
25.00	25.00	25.00	25.00	25.00

The thermal resistance between the device and sink per unit depth is

$$R'_{t,s-d} = \frac{T_d - T_s}{2q'_{tot}}$$

Performing an energy balance on the device nodes, find



$$q'_{tot} = q'_a + q'_b + q'_c$$

$$q'_{tot} = k(\Delta y/2)\frac{T_d - T_1}{\Delta x} + k\Delta x\frac{T_d - T_5}{\Delta y} + k(\Delta x/2)\frac{T_d - T_4}{\Delta y}$$

$$q'_{tot} = 300\,\text{W/m}\cdot\text{K}\left[(85-62.31)/2 + (85-63.85) + (85-65.76)/2\right]\text{K} = 1.263\times10^{4}\ \text{W/m}$$

$$R'_{t,s-d} = \frac{(85-25)\,\text{K}}{2\times1.263\times10^{4}\ \text{W/m}} = 2.38\times10^{-3}\,\text{m}\cdot\text{K/W} \qquad <$$

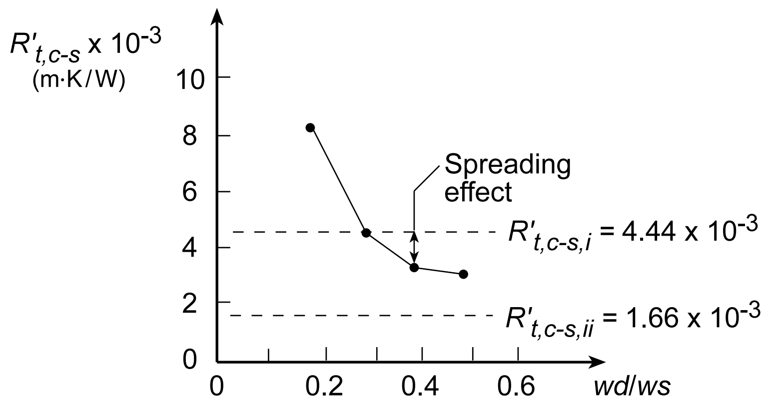
(c) The effect of grid size on the precision of the thermal resistance estimate should be tested by systematically reducing the nodal spacing Δx and Δy. This is a considerable amount of work even with IHT since the equations need to be individually entered. A more generalized, powerful code would be

Continued...

required which allows for automatically selecting the grid size. Using FEHT, a finite-element package, with eight elements across the device, representing a much finer mesh, we found

$$R'_{t,s-d} = 3.64 \times 10^{-3}\ \text{m} \cdot \text{K/W}$$

(d) Using the same tool, with the finest mesh, the thermal resistance was found as a function of w_d/w_s with fixed w_s and L.



As expected, as w_d increases, $R'_{t,d-s}$ decreases, and eventually will approach the value for the rectangular domain (ii). The spreading effect is shown for the base case, $w_d/w_s = 0.375$, where the thermal resistance of the sink is less than that for the rectangular domain (i).

COMMENTS: It is useful to compare the results for estimating the thermal resistance in terms of precision requirements and level of effort,

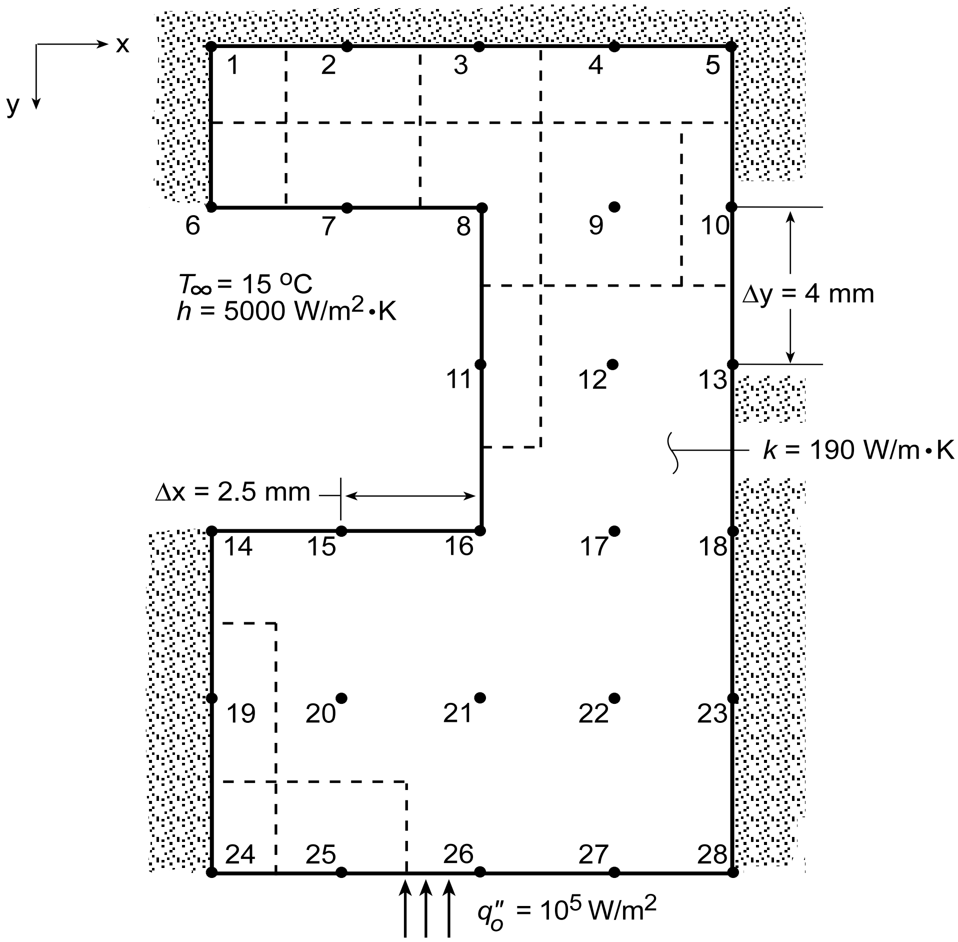
	$R'_{t,d-s} \times 10^3$ (m·K/W)
Rectangular domain (i)	4.44
Flux plot	3.03
Rectangular domain (ii)	1.67
FDE, 5x5 network	2.38
FEA, fine mesh	3.64

PROBLEM 4.78

KNOWN: Nodal network and boundary conditions for a water-cooled cold plate.

FIND: (a) Steady-state temperature distribution for prescribed conditions, (b) Means by which operation may be extended to larger heat fluxes.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: Finite-difference equations must be obtained for each of the 28 nodes. Applying the energy balance method to regions 1 and 5, which are similar, it follows that

Node 1: $\quad (\Delta y/\Delta x)T_2 + (\Delta x/\Delta y)T_6 - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_1 = 0$

Node 5: $\quad (\Delta y/\Delta x)T_4 + (\Delta x/\Delta y)T_{10} - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_5 = 0$

Nodal regions 2, 3 and 4 are similar, and the energy balance method yields a finite-difference equation of the form

Nodes 2,3,4:

$$(\Delta y/\Delta x)\left(T_{m-1,n} + T_{m+1,n}\right) + 2(\Delta x/\Delta y)T_{m,n-1} - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_{m,n} = 0$$

Energy balances applied to the remaining combinations of similar nodes yield the following finite-difference equations.

Continued...

Nodes 6, 14: $(\Delta x/\Delta y)T_1 + (\Delta y/\Delta x)T_7 - [(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)]T_6 = -(h\Delta x/k)T_\infty$

$(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{15} - [(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)]T_{14} = -(h\Delta x/k)T_\infty$

Nodes 7, 15: $(\Delta y/\Delta x)(T_6 + T_8) + 2(\Delta x/\Delta y)T_2 - 2[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)]T_7 = -(2h\Delta x/k)T_\infty$

$(\Delta y/\Delta x)(T_{14} + T_{16}) + 2(\Delta x/\Delta y)T_{20} - 2[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)]T_{15} = -(2h\Delta x/k)T_\infty$

Nodes 8, 16: $(\Delta y/\Delta x)T_7 + 2(\Delta y/\Delta x)T_9 + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_3 - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y)$
$+ (h/k)(\Delta x + \Delta y)]T_8 = -(h/k)(\Delta x + \Delta y)T_\infty$

$(\Delta y/\Delta x)T_{15} + 2(\Delta y/\Delta x)T_{17} + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_{21} - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y)$
$+ (h/k)(\Delta x + \Delta y)]T_{16} = -(h/k)(\Delta x + \Delta y)T_\infty$

Node 11: $(\Delta x/\Delta y)T_8 + (\Delta x/\Delta y)T_{16} + 2(\Delta y/\Delta x)T_{12} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta y/k)]T_{11} = -(2h\Delta y/k)T_\infty$

Nodes 9, 12, 17, 20, 21, 22:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + (\Delta x/\Delta y)T_{m,n+1} + (\Delta x/\Delta y)T_{m,n-1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Nodes 10, 13, 18, 23:

$$(\Delta x/\Delta y)T_{n+1,m} + (\Delta x/\Delta y)T_{n-1,m} + 2(\Delta y/\Delta x)T_{m-1,n} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Node 19: $(\Delta x/\Delta y)T_{14} + (\Delta x/\Delta y)T_{24} + 2(\Delta y/\Delta x)T_{20} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{19} = 0$

Nodes 24, 28: $(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{25} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{24} = -(q_o''\Delta x/k)$

$(\Delta x/\Delta y)T_{23} + (\Delta y/\Delta x)T_{27} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{28} = -(q_o''\Delta x/k)$

Nodes 25, 26, 27:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + 2(\Delta x/\Delta y)T_{m,n+1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = -(2q_o''\Delta x/k)$$

Evaluating the coefficients and solving the equations simultaneously, the steady-state temperature distribution (°C), tabulated according to the node locations, is:

23.77	23.91	24.27	24.61	24.74
23.41	23.62	24.31	24.89	25.07
		25.70	26.18	26.33
28.90	28.76	28.26	28.32	28.35
30.72	30.67	30.57	30.53	30.52
32.77	32.74	32.69	32.66	32.65

Alternatively, the foregoing results may readily be obtained by accessing the IHT *Tools* pat and using the *2-D, SS, Finite-Difference Equations* options (model equations are appended). Maximum and minimum cold plate temperatures are at the bottom (T_{24}) and top center (T_1) locations respectively.

(b) For the prescribed conditions, the maximum allowable temperature (T_{24} = 40°C) is reached when q_o'' = 1.407×10^5 W/m^2 (14.07 W/cm^2). Options for extending this limit could include use of a copper cold plate (k ≈ 400 W/m·K) and/or increasing the convection coefficient associated with the coolant. With k = 400 W/m·K, a value of q_o'' = 17.37 W/cm^2 may be maintained. With k = 400 W/m·K and h = 10,000 W/m^2·K (a practical upper limit), q_o'' = 28.65 W/cm^2. Additional, albeit small, improvements may be realized by relocating the coolant channels closer to the base of the cold plate.

COMMENTS: The accuracy of the solution may be confirmed by verifying that the results satisfy the overall energy balance

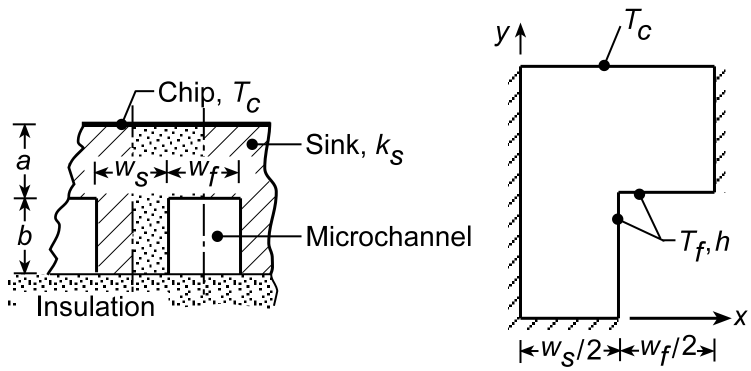
$$q_o''(4\Delta x) = h[(\Delta x/2)(T_6 - T_\infty) + \Delta x(T_7 - T_\infty) + (\Delta x + \Delta y)(T_8 - T_\infty)/2$$
$$+ \Delta y(T_{11} - T_\infty) + (\Delta x + \Delta y)(T_{16} - T_\infty)/2 + \Delta x(T_{15} - T_\infty) + (\Delta x/2)(T_{14} - T_\infty)].$$

PROBLEM 4.79

KNOWN: Heat sink for cooling computer chips fabricated from copper with microchannels passing fluid with prescribed temperature and convection coefficient.

FIND: (a) Using a square nodal network with 100 μm spatial increment, determine the temperature distribution and the heat rate to the coolant per unit channel length for maximum allowable chip temperature $T_{c,max}$ = 75°C; estimate the thermal resistance betweeen the chip surface and the fluid, $R'_{t,c-f}$ (m·K/W); maximum allowable heat dissipation for a chip that measures 10 x 10 mm on a side; (b) The effect of grid spacing by considering spatial increments of 50 and 25 μm; and (c) Consistent with the requirement that a + b = 400 μm, explore altering the sink dimensions to decrease the thermal resistance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, and (3) Convection coefficient is uniform over the microchannel surface and independent of the channel dimensions and shape.

ANALYSIS: (a) The square nodal network with Δx = Δy = 100 μm is shown below. Considering symmetry, the nodes 1, 2, 3, 4, 7, and 9 can be treated as interior nodes and their finite-difference equations representing nodal energy balances can be written by inspection. Using the, *IHT Finite-Difference Equations Tool*, appropriate FDEs for the nodes experiencing surface convection can be obtained. The IHT code including results is included in the Comments. Having the temperature distribution, the heat rate to the coolant per unit channel length for two symmetrical elements can be obtained by applying Newton's law of cooling to the surface nodes,

$$q'_{cv} = 2\left[h\left(\Delta y/2+\Delta x/2\right)\left(T_5-T_\infty\right)+h\left(\Delta x/2\right)\left(T_6-T_\infty\right)+h\left(\Delta y\right)\left(T_8-T_\infty\right)h\left(\Delta y/2\right)\left(T_{10}-T_\infty\right)\right]$$

$$q'_{cv} = 2\times 30,000\,\mathrm{W/m^2\cdot K}\times 100\times 10^{-6}\,\mathrm{m}\left[(74.02-25)+(74.09-25)/2+(73.60-25)+(73.37-25)/2\right]\mathrm{K}$$

$$q'_{cv} = 878\,\mathrm{W/m} \qquad <$$

The thermal resistance between the chip and fluid per unit length for each microchannel is

$$R'_{t,c-f} = \frac{T_c - T_\infty}{q'_{cv}} = \frac{(75-25)^\circ\mathrm{C}}{878\,\mathrm{W/m}} = 5.69\times 10^{-2}\,\mathrm{m\cdot K/W} \qquad <$$

The maximum allowable heat dissipation for a 10 mm × 10 mm chip is

$$P_{chip,max} = q''_c \times A_{chip} = 2.20\times 10^6\,\mathrm{W/m^2}\times(0.01\times 0.01)\,\mathrm{m^2} = 220\,\mathrm{W} \qquad <$$

where A_{chip} = 10 mm × 10 mm and the heat flux on the chip surface ($w_f + w_s$) is

$$q''_c = q'_{cv}/\left(w_f + w_s\right) = 878\,\mathrm{W/m}\Big/(200+200)\times 10^{-6}\,\mathrm{m} = 2.20\times 10^6\,\mathrm{W/m^2}$$

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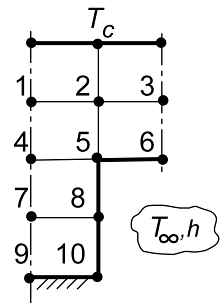
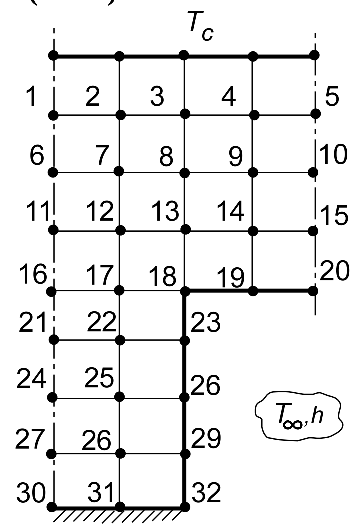




(b) To investigate the effect of grid spacing, the analysis was repreated with a spatial increment of 50 μm (32 nodes as shown above) with the following results

$$q'_{cv} = 881\,\text{W/m} \qquad R'_{t,c-f} = 5.67\times10^{-2}\,\text{m}\cdot\text{K/W} \qquad <$$

Using a finite-element package with a mesh around 25 μm, we found $R'_{t,c-f} = 5.70\times10^{-2}\,\text{m}\cdot\text{K/W}$ which suggests the grid spacing effect is not very significant.

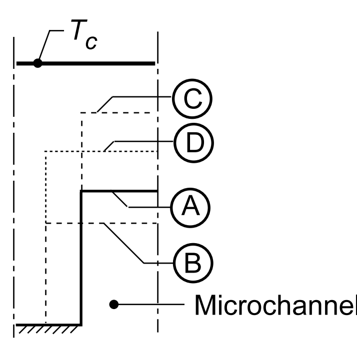
(c) Requring that the overall dimensions of the symmetrical element remain unchanged, we explored what effect changes in the microchannel cross-section would have on the overall thermal resistance, $R'_{t,c-f}$. It is important to recognize that the sink conduction path represents the dominant resistance, since for the convection process

$$R'_{t,cv} = 1/A'_s = 1\Big/\left(30,000\,\text{W/m}^2\cdot\text{K}\times600\times10^{-6}\,\text{m}\right) = 5.55\times10^{-2}\,\text{m}\cdot\text{K/W}$$

where $A'_s = (w_f + 2b) = 600$ μm.

Using a finite-element package, the thermal resistances per unit length for three additional channel cross-sections were determined and results summarized below.

Case	Microchannel (μm) Height	Half-width	$R'_{t,c-s}\times10^2$ (m·K/W)
A	200	100	5.70
B	133	150	6.12
C	300	100	4.29
D	250	150	4.25



Continued...

COMMENTS: (1) The IHT Workspace for the 5x5 coarse node analysis with results follows.

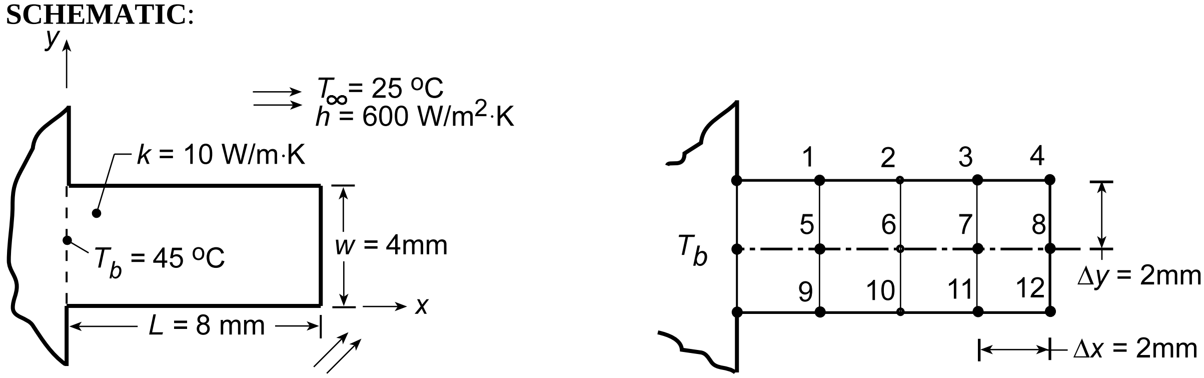
```
// Finite-difference equations - energy balances
// First row - treating as interior nodes considering symmetry
T1 = 0.25 * ( Tc + T2 + T4 + T2 )
T2 = 0.25 * ( Tc + T3 + T5 + T1 )
T3 = 0.25 * ( Tc + T2 + T6 + T2 )

/* Second row - Node 4 treat as interior node; for others, use Tools: Finite-Difference Equations,
Two-Dimensional, Steady-State; be sure to delimit replicated q''a = 0 equations.  */
T4 = 0.25 * ( T1 + T5+ T7 + T5 )
/* Node 5: internal corner node, e-s orientation; e, w, n, s labeled 6, 4, 2, 8. */
0.0 = fd_2d_ic_es(T5,T6,T4,T2,T8,k,qdot,deltax,deltay,Tinf,h,q''a)
q''a = 0                         // Applied heat flux, W/m^2; zero flux shown
/* Node 6: plane surface node, s-orientation; e, w, n labeled 5, 5, 3 . */
0.0 = fd_2d_psur_s(T6,T5,T5,T3,k,qdot,deltax,deltay,Tinf,h,q''a)
//q''a = 0                       // Applied heat flux, W/m^2; zero flux shown

/* Third row - Node 7 treat as interior node; for others, use Tools: Finite-Difference Equations,
Two-Dimensional, Steady-State; be sure to delimit replicated q''a = 0 equations.  */
T7 = 0.25 * (T4 + T8 + T9 + T8)
/* Node 8: plane surface node, e-orientation; w, n, s labeled 7, 5, 10. */
0.0 = fd_2d_psur_e(T8,T7,T5,T10,k,qdot,deltax,deltay,Tinf,h,q''a)
//q''a = 0                       // Applied heat flux, W/m^2; zero flux shown

/* Fourth row - Node 9 treat as interior node; for others, use Tools: Finite-Difference Equations,
Two-Dimensional, Steady-State; be sure to delimit replicated q''a = 0 equations.  */
T9 = 0.25 * (T7 + T10 +T7 + T10)
/* Node 10: plane surface node, e-orientation; w, n, s labeled 9, 8, 8. */
0.0 = fd_2d_psur_e(T10,T9,T8,T8,k,qdot,deltax,deltay,Tinf,h,q''a)
//q''a = 0                       // Applied heat flux, W/m^2; zero flux shown

// Assigned variables
// For the FDE functions,
qdot = 0                         // Volumetric generation, W/m^3
deltax = deltay                  // Spatial increments
deltay = 100e-6                  // Spatial increment, m
Tinf = 25                        // Microchannel fluid temperature, C
h = 30000                               // Convection coefficient, W/m^2.K
// Sink and chip parameters
k = 400                          // Sink thermal conductivity, W/m.K
Tc = 75                          // Maximum chip operating temperature, C
wf = 200e-6                      // Channel width, m
ws = 200e-6                      // Sink width, m

/* Heat rate per unit length, for two symmetrical elements about one microchannel, */
q'cv= 2 * (q'5 + q'6 + q'8 + q'10)
q'5 = h* (deltax / 2 + deltay / 2) * (T5 - Tinf)
q'6 = h * deltax / 2 * (T6 - Tinf)
q'8 = h * deltax  * (T8 - Tinf)
q'10 = h * deltax / 2 * (T10 - Tinf)

/* Thermal resistance between chip and fluid, per unit channel length, */
R'tcf = (Tc - Tinf) / q'cv       // Thermal resistance, m.K/W

// Total power for a chip of 10mm x 10mm, Pchip (W),
q''c = q'cv / (wf + ws)          // Heat flux on chip surface, W/m^2
Pchip = Achip * q''c             // Power, W
Achip = 0.01 * 0.01              // Chip area, m^2

/* Data Browser results: chip power, thermal resistance, heat rates and temperature distribution
Pchip   R'tcf    q''c     q'cv
219.5   0.05694  2.195E6  878.1

T1      T2      T3      T4      T5      T6      T7      T8      T9      T10
74.53   74.52   74.53   74.07   74.02   74.09   73.7    73.6    73.53   73.37 */
```

PROBLEM 4.80

KNOWN: Longitudinal rib (k = 10 W/m·K) with rectangular cross-section with length L= 8 mm and width w = 4 mm. Base temperature T_b and convection conditions, T_∞ and h, are prescribed.

FIND: (a) Temperature distribution and fin base heat rate using a finite-difference method with $\Delta x = \Delta y$ = 2 mm for a total of $5 \times 3 = 15$ nodal points and regions; compare results with those obtained assuming one-dimensional heat transfer in rib; and (b) The effect of grid spacing by reducing nodal spacing to $\Delta x = \Delta y$ = 1 mm for a total of $9 \times 3 = 27$ nodal points and regions considering symmetry of the centerline; and (c) A criterion for which the one-dimensional approximation is reasonable; compare the heat rate for the range $1.5 \le L/w \le 10$, keeping L constant, as predicted by the two-dimensional, finite-difference method and the one-dimensional fin analysis.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, and (3) Convection coefficient uniform over rib surfaces, including tip.

ANALYSIS: (a) The rib is represented by a 5×3 nodal grid as shown above where the symmetry plane is an adiabatic surface. The *IHT Tool, Finite-Difference Equations*, for *Two-Dimensional, Steady-State* conditions is used to formulate the nodal equations (see Comment 2 below) which yields the following nodal temperatures (° C)

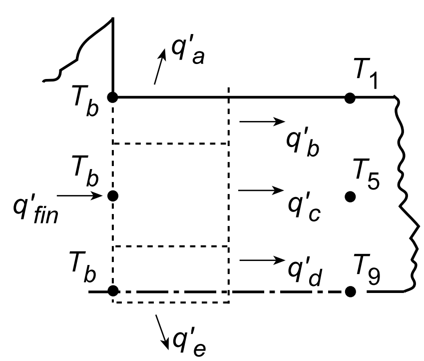
45	39.3	35.7	33.5	32.2
45	40.0	36.4	34.0	32.6
45	39.3	35.7	33.5	32.2

Note that the fin tip temperature is

$$T_{tip} = T_{12} = 32.6^\circ C$$ <

The fin heat rate per unit width normal to the page, q'_{fin}, can be determined from energy balances on the three base nodes as shown in the schematic below.

$$q'_{fin} = q'_a + q'_b + q'_c + q'_d + q'_e$$
$$q'_a = h(\Delta x/2)(T_b - T_\infty)$$
$$q'_b = k(\Delta y/2)(T_b - T_1)/\Delta x$$
$$q'_c = k(\Delta y)(T_b - T_5)/\Delta x$$
$$q'_d = k(\Delta y/2)(T_b - T_9)\Delta x$$
$$q'_3 = h(\Delta x/2)(T_b - T_\infty)$$



Continued...

PROBLEM 4.80 (Cont.)

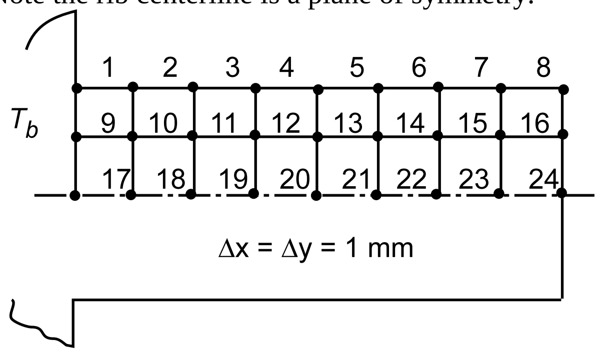
Substituting numerical values, find

$$q'_{fin} = (12.0 + 28.4 + 50.0 + 28.4 + 12.0)\,\text{W/m} = 130.8\,\text{W/m} \qquad <$$

Using the *IHT Model*, *Extended Surfaces*, *Heat Rate* and *Temperature Distributions* for *Rectangular*, *Straight Fins*, with convection tip condition, the one-dimensional fin analysis yields

$$q'_f = 131\,\text{W/m} \qquad T_{tip} = 32.2^\circ\text{C} \qquad <$$

(b) With $\Delta x = L/8 = 1$ mm and $\Delta x = 1$ mm, for a total of $9 \times 3 = 27$ nodal points and regions, the grid appears as shown below. Note the rib centerline is a plane of symmetry.

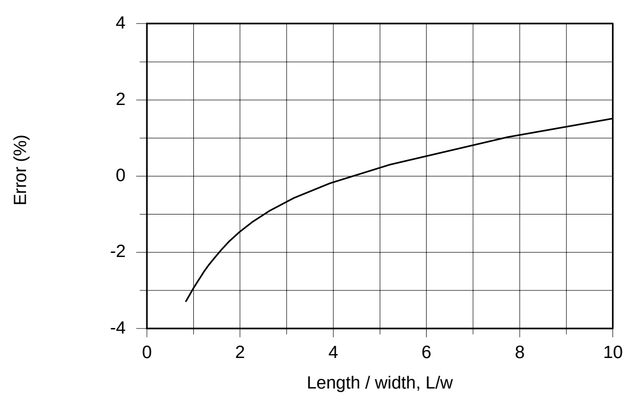


Using the same IHT FDE Tool as above with an appropriate expression for the fin heat rate, Eq. (1), the fin heat rate and tip temperature were determined.

	1-D analysis	2-D analysis (nodes) (5 × 3)	(9 × 3)	
T_{tip} (°C)	32.2	32.6	32.6	<
q'_{fin} (W/m)	131	131	129	<

(c) To determine when the one-dimensional approximation is reasonable, consider a rib of constant length, L = 8 mm, and vary the thickness w for the range $1.5 \le L/w \le 10$. Using the above IHT model for the 27 node grid, the fin heat rates for 1-D, q'_{1d}, and 2-D, q'_{2d}, analysis were determined as a function of w with the error in the approximation evaluated as

$$\text{Error}(\%) = \left(q'_{2d} - q'_{1d}\right) \times 100 / q'_{1d}$$



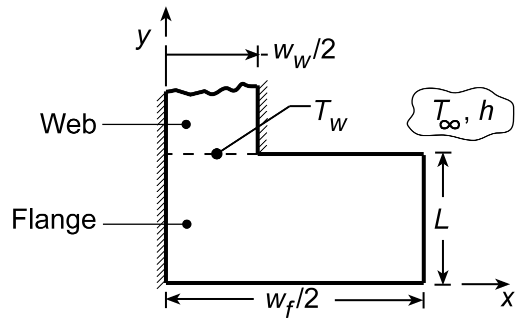
Note that for small L/w, a thick rib, the 1-D approximation is poor. For large L/w, a thin rib which approximates a fin, we would expect the 1-D approximation to become increasingly more satisfactory. The discrepancy at large L/w must be due to discretization error; that is, the grid is too coarse to accurately represent the slender rib.

PROBLEM 4.81

KNOWN: Bottom half of an I-beam exposed to hot furnace gases.

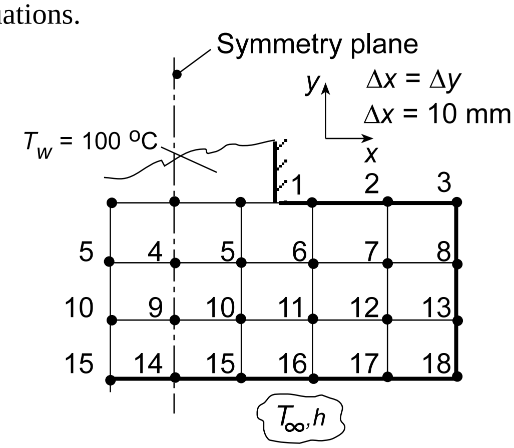
FIND: (a) The heat transfer rate per unit length into the beam using a coarse nodal network (5×4) considering the temperature distribution across the web is uniform and (b) Assess the reasonableness of the uniform web-flange interface temperature assumption.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, and (2) Constant properties.

ANALYSIS: (a) The symmetrical section of the I-beam is shown in the Schematic above indicating the web-flange interface temperature is uniform, $T_w = 100°C$. The nodal arrangement to represent this system is shown below. The nodes on the line of symmetry have been shown for convenience in deriving the nodal finite-difference equations.



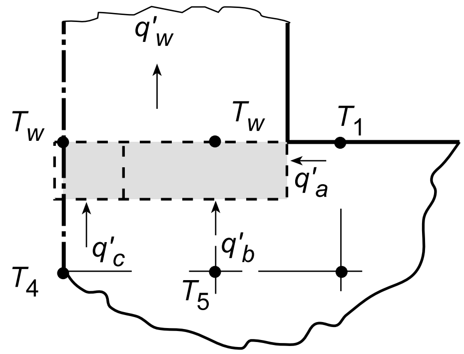
Using the *IHT Finite-Difference Equations Tool*, the set of nodal equations can be readily formulated. The temperature distribution (°C) is tabulated in the same arrangement as the nodal network.

100.00	100.00	215.8	262.9	284.8
166.6	177.1	222.4	255.0	272.0
211.7	219.5	241.9	262.7	274.4
241.4	247.2	262.9	279.3	292.9

The heat rate to the beam can be determined from energy balances about the web-flange interface nodes as shown in the sketch below.

Continued...

PROBLEM 4.81 (Cont.)

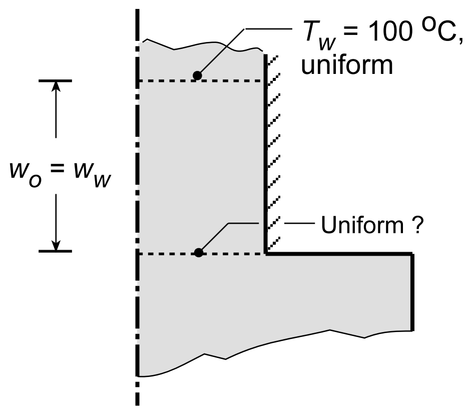


$$q'_w = q'_a + q'_b + q'_c$$

$$q'_w = k(\Delta y/2)\frac{T_1 - T_w}{\Delta x} + k(\Delta x)\frac{T_5 - T_w}{\Delta y} + k(\Delta x/2)\frac{T_4 - T_w}{\Delta y}$$

$$q'_w = 10\,W/m\cdot K\left[(215.8-100)/2 + (177.1-100) + (166.6-100)/2\right]K = 1683\,W/m$$ <

(b) The schematic below poses the question concerning the reasonableness of the uniform temperature assumption at the web-flange interface. From the analysis above, note that T_1 = 215.8°C vs. T_w = 100°C indicating that this assumption is a poor one. This L-shaped section has strong two-dimensional behavior. To illustrate the effect, we performed an analysis with T_w = 100°C located nearly 2 × times further up the web than it is wide. For this situation, the temperature difference at the web-flange interface across the width of the web was nearly 40°C. The steel beam with its low thermal conductivity has substantial internal thermal resistance and given the L-shape, the uniform temperature assumption (T_w) across the web-flange interface is inappropriate.

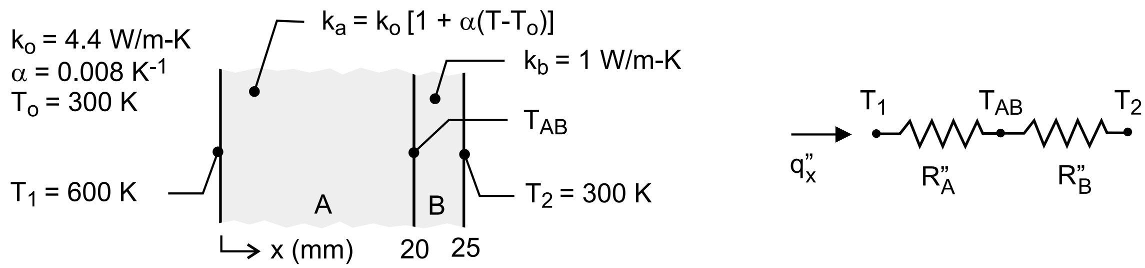


PROBLEM 4.82

KNOWN: Plane composite wall with exposed surfaces maintained at fixed temperatures. Material A has temperature-dependent thermal conductivity.

FIND: Heat flux through the wall (a) assuming a uniform thermal conductivity in material A evaluated at the average temperature of the section, and considering the temperature-dependent thermal conductivity of material A using (b) a finite-difference method of solution in IHT with a space increment of 1 mm and (c) the finite-element method of FEHT.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, one-dimensional conduction, (2) No thermal contact resistance between the materials, and (3) No internal generation.

ANALYSIS: (a) From the thermal circuit in the above schematic, the heat flux is

$$q''_x = \frac{T_1 - T_2}{R''_A + R''_B} = \frac{T_{AB} - T_2}{R''_B} \qquad (1, 2)$$

and the thermal resistances of the two sections are

$$R''_A = L_A / k_A \qquad\qquad R''_B = L_B / k_B \qquad (3, 4)$$

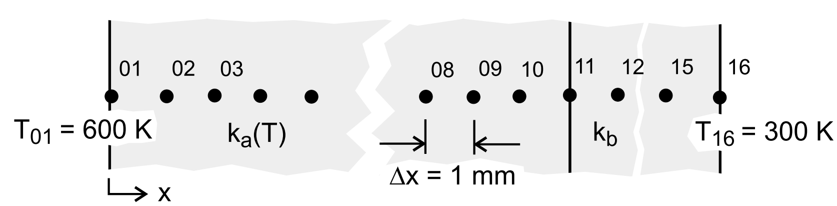
The thermal conductivity of material A is evaluated at the average temperature of the section

$$k_A = k_o \left\{1 + \alpha\left[\left(T_1 + T_{AB}\right)/2 - T_o\right]\right\} \qquad (5)$$

Substituting numerical values and solving the system of equations simultaneously in IHT, find

$$T_{AB} = 563.2 \text{ K} \qquad\qquad q''_x = 52.64 \text{ kW/m}^2 \qquad <$$

(b) The nodal arrangement for the finite-difference method of solution is shown in the schematic below. FDEs must be written for the internal nodes (02 – 10, 12 – 15) and the A-B interface node (11) considering in section A, the temperature-dependent thermal conductivity.



Interior Nodes, Section A (m = 02, 03 … 10)
Referring to the schematic below, the energy balance on node *m* is written in terms of the heat fluxes at the control surfaces using Fourier's law with the thermal conductivity based upon the average temperature of adjacent nodes. The heat fluxes into node *m* are

Continued …..

$$q''_c = k_a\left(m, m+1\right)\frac{T_{m+1} - T_m}{\Delta x} \tag{1}$$

$$q''_d = k_a\left(m-1, m\right)\frac{T_{m-1} - T_m}{\Delta x} \tag{2}$$

and the FDEs are obtained from the energy balance written as

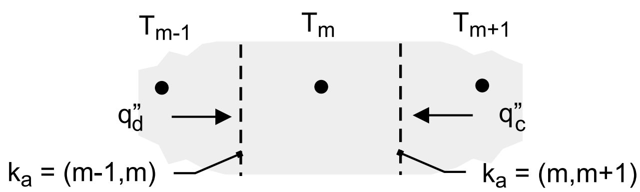
$$q''_c + q''_d = 0 \tag{3}$$

$$k_a\left(m, m+1\right)\frac{T_{m+1} - T_m}{\Delta x} + k_a\left(m-1, m\right)\frac{T_{m-1} - T_m}{\Delta x} = 0 \tag{4}$$

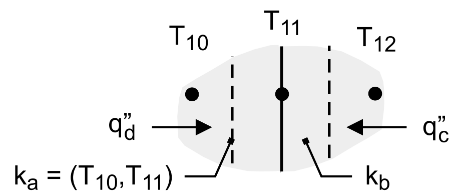
where the thermal conductivities averaged over the path between the nodes are expressed as

$$k_a\left(m-1, m\right) = k_o\left\{1 + \alpha\left[\left(T_{m-1} + T_m\right)/2 - T_o\right]\right\} \tag{5}$$

$$k_a\left(m, m+1\right) = k_o\left\{1 + \alpha\left[\left(T_m + T_{m+1}\right)/2 - T_o\right]\right\} \tag{6}$$



Interior nodes, Section A



A-B interface node

A-B Interface Node 11

Referring to the above schematic, the energy balance on the interface node, $q''_c + q''_d = 0$, has the form

$$k_b\frac{T_{12} - T_{11}}{\Delta x} + k_a\left(10, 11\right)\frac{T_{10} - T_{11}}{\Delta x} = 0 \tag{7}$$

where the thermal conductivity in the section A path is

$$k\left(10, 11\right) = k_o\left\{1 + \left[\left(T_{10} + T_{11}\right)/2 - T_o\right]\right\} \tag{8}$$

Interior Nodes, Section B (n = 12 …15)

Since the thermal conductivity in Section B is uniform, the FDEs have the form

$$T_n = \left(T_{n-1} + T_{n+1}\right)/2 \tag{9}$$

And the heat flux in the x-direction is

$$q''_x = k_b\frac{T_n - T_{n+1}}{\Delta x} \tag{10}$$

Finite-Difference Method of Solution

The foregoing FDE equations for section A nodes (m = 02 to 10), the AB interface node and their respective expressions for the thermal conductivity, k (m, m +1), and for section B nodes are entered into the IHT workspace and solved for the temperature distribution. The heat flux can be evaluated using Eq. (2) or (10). A portion of the IHT code is contained in the Comments, and the results of the analysis are tabulated below.

$$T_{11} = T_{AB} = 563.2\ \text{K} \qquad q''_x = 52.64\ \text{kW/m}^2$$ <

Continued …..

(c) The finite-element method of FEHT can be used readily to obtain the heat flux considering the temperature-dependent thermal conductivity of section A. Draw the composite wall outline with properly scaled section thicknesses in the x-direction with an arbitrary y-direction dimension. In the *Specify* | *Materials Properties* box for the thermal conductivity, specify k_a as 4.4*[1 + 0.008*(T – 300)] having earlier selected *Set* | *Temperatures in K*. The results of the analysis are

$$T_{AB} = 563 \text{ K} \qquad q''_x = 5.26 \text{ kW/m}^2 \qquad <$$

COMMENTS: (1) The results from the three methods of analysis compare very well. Because the thermal conductivity in section A is linear, and moderately dependent on temperature, the simplest method of using an overall section average, part (a), is recommended. This same method is recommended when using tabular data for temperature-dependent properties.

(2) For the finite-difference method of solution, part (b), the heat flux was evaluated at several nodes within section A and in section B with identical results. This is a consequence of the technique for averaging k_a over the path between nodes in computing the heat flux into a node.

(3) To illustrate the use of IHT in solving the finite-difference method of solution, lines of code for representative nodes are shown below.

```
// FDEs – Section A
k01_02 * (T01-T02)/deltax + k02_03 * (T03-T02)/deltax = 0
k01_02 = ko * (1+ alpha * ((T01 + T02)/2 – To))
k02_03 = ko * (1 + alpha * ((T02 + T03)/2 – To))

k02_03 * (T02 – T03)/deltax + k03_04 * (T04 – T03)/deltax = 0
k03_04 = ko * (1 + alpha * ((T03 + T04)/2 – To))

// Interface, node 11
k11 * (T10 –T11)/deltax + kb * (T12 –T11)/deltax =0
k11 = ko * (1 + alpha * ((T10 + T11)/2 – To))

// Section B (using Tools/FDE/One-dimensional/Steady-state)
/* Node 12: interior node; */
0.0 = fd_1d_int(T12, T13, T11, kb, qdot, deltax)
```

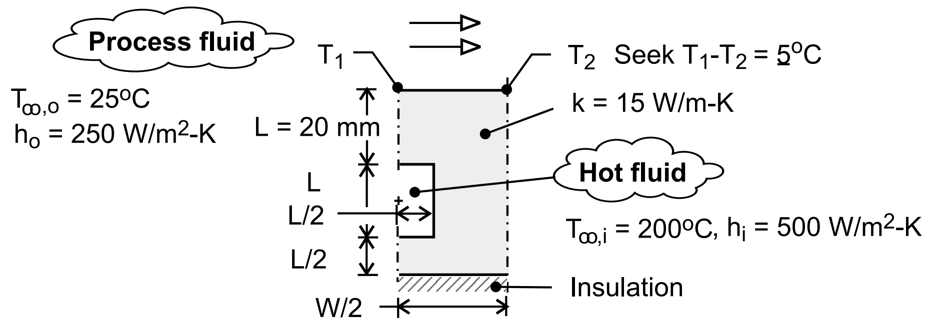
(4) The solved models for Text Examples 4.3 and 4.4, plus the tutorial of the User’s Manual, provide background for developing skills in using FEHT.

PROBLEM 4.83

KNOWN: Upper surface of a platen heated by hot fluid through the flow channels is used to heat a process fluid.

FIND: (a) The maximum allowable spacing, W, between channel centerlines that will provide a uniform temperature requirement of 5°C on the upper surface of the platen, and (b) Heat rate per unit length from the flow channel for this condition.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction with constant properties, and (2) Lower surface of platen is adiabatic.

ANALYSIS: As shown in the schematic above for a symmetrical section of the platen-flow channel arrangement, the temperature uniformity requirement will be met when $T_1 - T_2 = 5°C$. The maximum temperature, T_1, will occur directly over the flow channel centerline, while the minimum surface temperature, T_2, will occur at the mid-span between channel centerlines.

We chose to use FEHT to obtain the temperature distribution and heat rate for guessed values of the channel centerline spacing, W. The following method of solution was used: (1) Make an initial guess value for W; try W = 100 mm, (2) Draw an outline of the symmetrical section, and assign properties and boundary conditions, (3) Make a copy of this file so that in your second trial, you can use the *Draw | Move Node* option to modify the section width, W/2, larger or smaller, (4) Draw element lines within the outline to create triangular elements, (5) Use the *Draw | Reduce Mesh* command to generate a suitably fine mesh, then solve for the temperature distribution, (6) Use the *View | Temperatures* command to determine the temperatures T_1 and T_2, (7) If, $T_1 - T_2 \approx 5°C$, use the *View | Heat Flows* command to find the heat rate, otherwise, change the width of the section outline and repeat the analysis. The results of our three trials are tabulated below.

Trial	W (mm)	T_1 (°C)	T_2 (°C)	$T_1 - T_2$ (°C)	q' (W/m)
1	100	108	98	10	--
2	60	119	118	1	--
3	80	113	108	5	1706

COMMENTS: (1) In addition to the tutorial example in the FEHT User's Manual, the solved models for Examples 4.3 and 4.4 of the Text are useful for developing skills in using this problem-solving tool.

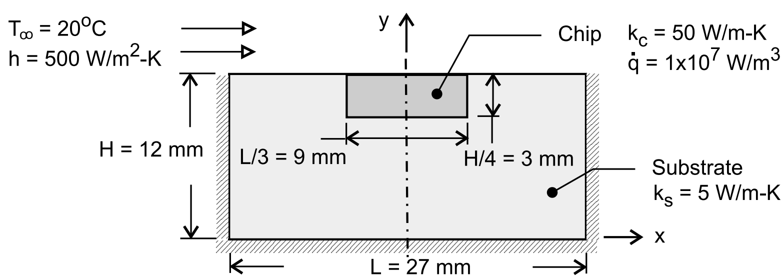
(2) An alternative numerical method of solution would be to create a nodal network, generate the finite-difference equations and solve for the temperature distribution and the heat rate. The FDEs should allow for a non-square grid, $\Delta x \neq \Delta y$, so that different values for W/2 can be accommodated by changing the value of Δx. Even using the IHT tool for building FDEs (*Tools | Finite-Difference Equations | Steady-State*) this method of solution is very labor intensive because of the large number of nodes required for obtaining good estimates.

PROBLEM 4.84

KNOWN: Silicon chip mounted in a dielectric substrate. One surface of system is convectively cooled, while the remaining surfaces are well insulated. See Problem 4.77. Use the finite-element software *FEHT* as your analysis tool.

FIND: (a) The temperature distribution in the substrate-chip system; does the maximum temperature exceed 85°C?; (b) Volumetric heating rate that will result in a maximum temperature of 85°C; and (c) Effect of reducing thickness of substrate from 12 to 6 mm, keeping all other dimensions unchanged with $\dot{q} = 1\times10^7$ W/m^3; maximum temperature in the system for these conditions, and fraction of the power generated within the chip removed by convection directly from the chip surface.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction in system, and (3) Uniform convection coefficient over upper surface.

ANALYSIS: Using *FEHT*, the symmetrical section is represented in the workspace as two connected regions, chip and substrate. *Draw* first the chip outline; *Specify* the material and generation parameters. Now, *Draw* the outline of the substrate, connecting the nodes of the interfacing surfaces; *Specify* the material parameters for this region. Finally, *Assign* the *Boundary Conditions*: zero heat flux for the symmetry and insulated surfaces, and convection for the upper surface. Draw *Element Lines*, making the triangular elements near the chip and surface smaller than near the lower insulated boundary as shown in a copy of the *FEHT* screen on the next page. Use the *Draw*|*Reduce M*esh command and *Run* the model.

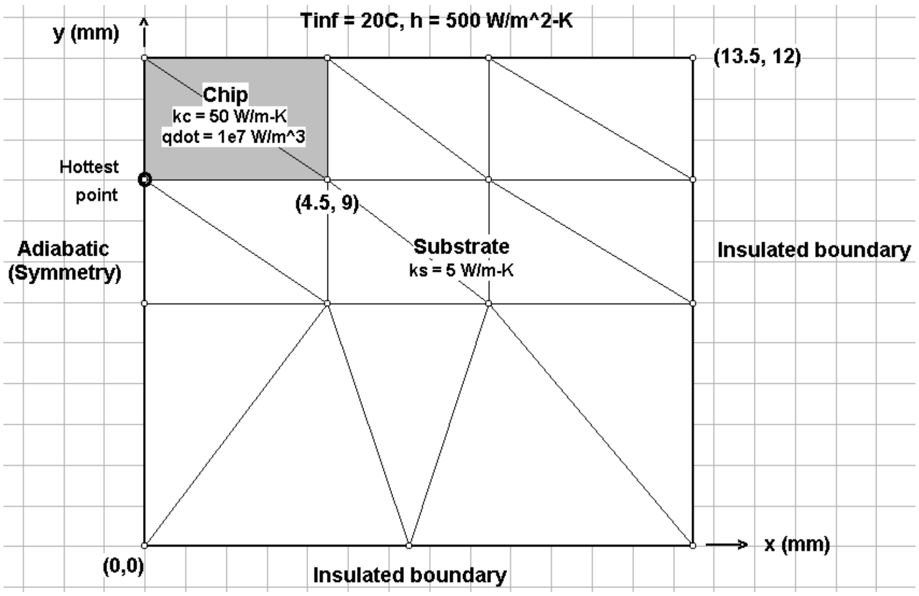
(a) Use the *View*|*Temperature* command to see the nodal temperatures through out the system. As expected, the hottest location is on the centerline of the chip at the bottom surface. At this location, the temperature is

$$T(0,\ 9\ \text{mm}) = 46.7°\text{C}$$ <

(b) Run the model again, with different values of the generation rate until the temperature at this location is T(0, 9 mm) = 85°C, finding

$$\dot{q} = 2.43\times10^7\ \text{W}/\text{m}^3$$ <

Continued …..



(c) Returning to the model code with the conditions of part (a), reposition the nodes on the lower boundary, as well as the intermediate ones, to represent a substrate that is of 6-mm, rather than 12-mm thickness. Find the maximum temperature as

$$T(0, 3\text{ mm}) = 47.5°\text{C} \qquad <$$

Using the *View|Heat Flow* command, click on the adjacent line segments forming the chip surface exposed to the convection process. The heat rate per unit width (normal to the page) is

$$q'_{chip,cv} = 60.26\text{ W/m}$$

The total heat generated within the chip is

$$q'_{tot} = \dot{q}(L/6 \times H/4) = 1\times10^7\text{ W/m}^3 \times (0.0045\times0.003)\text{m}^2 = 135\text{ W/m}$$

so that the fraction of the power dissipated by the chip that is convected directly to the coolant stream is

$$F = q'_{chip,cv}/q'_{tot} = 60.26/135 = 45\% \qquad <$$

COMMENTS: (1) Comparing the maximum temperatures for the system with the 12-mm and 6-mm thickness substrates, note that the effect of halving the substrate thickness is to raise the maximum temperature by less than 1°C. The thicker substrate does not provide significantly improved heat removal capability.

(2) Without running the code for part (b), estimate the magnitude of $\dot{q}$ that would make T(0, 9 mm) = 85°C. Did you get $\dot{q} = 2.43\times10^7$ W/m^3? Why?