University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Homework 5

Due: November 4 (Monday), 2019, 3pm

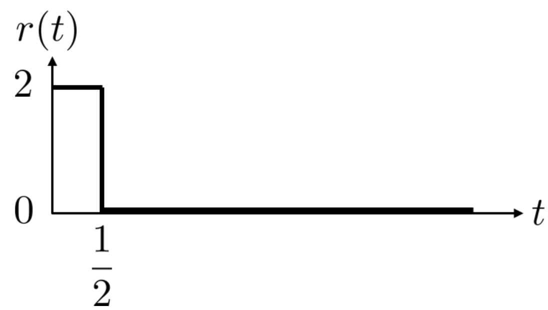
Consider the following ordinary differential equation (ODE):

$$y^{(3)}(t) + 2y^{(2)}(t) + y^{(1)}(t) = r(t),$$

with zero initial conditions $y(0) = y^{(1)}(0) = y^{(2)}(0) = 0$. Here, $y^{(k)}(t)$ denotes the k-th derivative of $y(t)$.

1. Assume that $r(t) = \delta(t)$ (i.e., unit impulse function).
   (a) By using the Laplace transform, solve the ODE (i.e., obtain $y(t)$).
   (b) By using the final value theorem, obtain the final value $\lim_{t\to\infty} y(t)$. (You should verify the applicability of the final value theorem.)

2. Next, assume that $r(t)$ is the function given in the figure below. Note that this is an approximation of the unit impulse function.



By using the final value theorem, obtain the final value $\lim_{t\to\infty} y(t)$. (In this question, you can assume (i.e., you do not need to check) the applicability of the final value theorem.)

<u>**Hint:**</u> You can use the L'Hospital's Rule:

$$\lim_{s\to 0} \frac{f(s)}{g(s)} = \lim_{s\to 0} \frac{f'(s)}{g'(s)} \quad \text{if } f(0) = g(0) = 0.$$

Solution By applying the Laplace transform to the ODE, we get

$$s(s+1)^2 Y(s) = R(s) \;\Rightarrow\; Y(s) = \frac{1}{s(s+1)^2} R(s).$$

1. When $R(s) = 1$,

   (a)

   $$Y(s) = \frac{1}{s(s+1)^2} \cdot 1 = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

   where $A = 1$, $B = -1$ and $C = -1$. So,

   $$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}\right\} \\ &= A + Be^{-t} + Cte^{-t} \\ &= 1 - e^{-t}(1+t) \end{aligned}$$

   (b) Since the poles of $sY(s)$ are $-1, -1$ which are in the open left-half plane, the final value theorem is applicable.

   $$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{(s+1)^2} = 1.$$

2. When $R(s) = 2(1 - e^{-\frac{1}{2}s})/s$,

   $$Y(s) = \frac{1}{s(s+1)^2} \cdot \frac{2(1 - e^{-\frac{1}{2}s})}{s}.$$

   Using the final value theorem and L'Hospital's Rule, we have

   $$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{2(1 - e^{-\frac{1}{2}s})}{s(s+1)^2} = \lim_{s\to 0} \frac{e^{-\frac{1}{2}s}}{3s^2 + 4s + 1} = 1.$$