MECH366 : Modeling of Mechatronic Systems

L12 : Modeling of thermal systems

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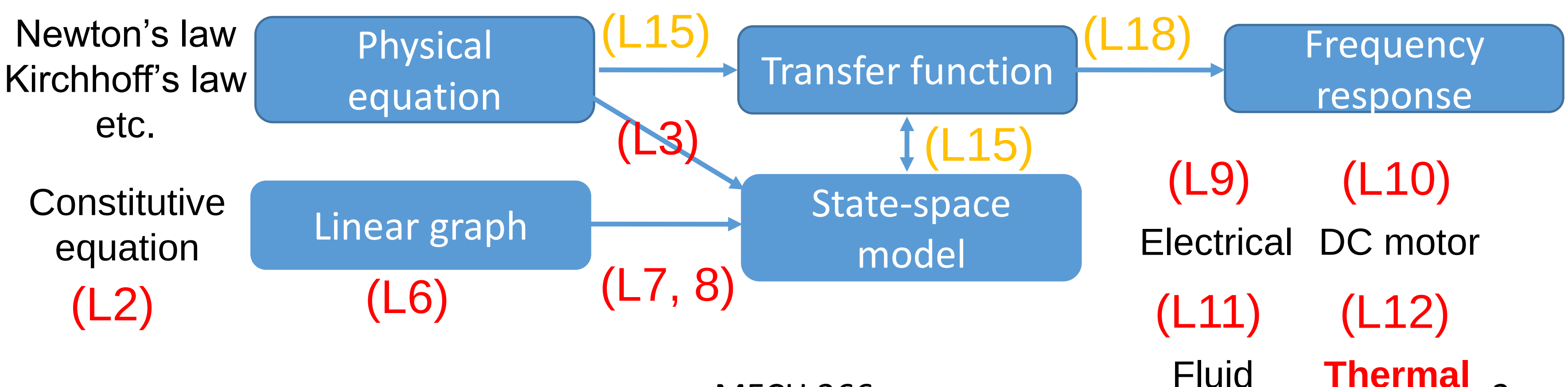
Review and today's topic

- Up to now, we have studied for mechanical, electrical and fluid systems:
  - How to draw linear graphs & derive state-space models
- Today, we will study modeling of thermal systems.
- Various models and their relations



__ : State variable

	Constitutive relation for			
	Energy storage element		Energy dissipating element	
System type	A-Type	T-Type	D-Type	power
Mechanical	Mass	Spring	Viscous Damper	
(translational)				
v: velocity across var.	$m\underline{\dot{v}} = f$	$\underline{\dot{f}} = kv$	$f = bv$	$\mathcal{P} = fv$
f: force through var.	m: mass	k: stiffness	b: damping const.	
Electrical	Capacitor	Inductor	Resistor	
v: voltage across var.	$C\underline{\dot{v}} = i$	$L\underline{\dot{i}} = v$	$v = Ri$	$\mathcal{P} = iv$
i: current through	C: capacitance	L: inductance	R: resistance	
Thermal	Thermal capacitor	None	Thermal resistor	
T: temperature $[K]$ across	$C_t\underline{\dot{T}} = Q$	“Thermal inductor” does not exist!	$T = R_t Q$	$\mathcal{P} = Q$
Q: heat transfer rate $[J/s]$, $[W]$ through var.	C_t: thermal capacitance		R_t: thermal resistance	
Fluid	Fluid capacitor	Fluid inertor	Fluid resistor	
P: pressure difference $[N/m^2]$ across var.	$C_f\underline{\dot{P}} = Q$	$I_f\underline{\dot{Q}} = P$	$P = R_f Q$	$\mathcal{P} = QP$
Q: volume flow rate $[m^3/s]$ through var.	C_f: fluid capacitance	I_f: fluid inertance	R_f: fluid resistance	

Energy expressions based on across and through variables

	A-type element	T-type element
Mechanical v : Across variable f : Through variable	Kinetic energy $\frac{1}{2}mv^2$	Potential energy $\left(\frac{1}{2}kx^2 =\right) \frac{1}{2}\frac{f^2}{k}$
Electrical v : Across variable i : Through variable	Electrostatic energy $\frac{1}{2}Cv^2$	Electromagnetic energy $\frac{1}{2}Li^2$
Thermal T : Across variable Q : Through variable	Thermal energy $\int Q = C_t T$	N/A
Fluid P : Across variable Q : Through variable	Potential energy $\frac{1}{2}C_f P^2$	Kinetic energy $\frac{1}{2}I_f Q^2$

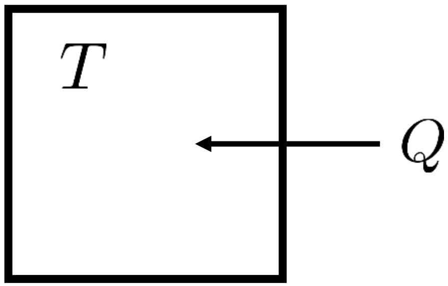
Main differences of thermal systems from other systems

- No thermal inductor
  - Only one energy storage element
  - No natural oscillatory behaviour
    - Mechanical examples: Pendulum, mass spring system exchanges kinetic and potential energies.
    - Electrical example: LC circuit exchanges electrostatic (in C) and electromagnetic (in L) energies.
- Power is defined by only one variable.
- Resistor does not dissipate energy, but impedes heat flow.

Linear graph representation

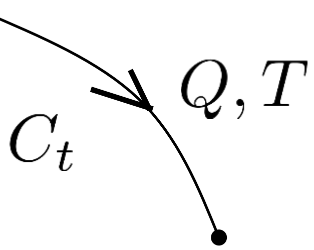
- Single-port elements
    - Energy storage elements
    - Energy (dissipation) elements
    - Energy sources

Energy storage element
Thermal capacitor



- Constitutive equation

$$C_t \frac{dT}{dt} = Q \qquad C_t = \rho V c$$



$$\begin{aligned} C_t \ [J/K] &: \text{thermal capacitance} \\ \rho \ [kg/m^3] &: \text{mass density of the object} \\ V \ [m^3] &: \text{volume of the object} \\ c \ [J/(K \cdot kg)] &: \text{specific heat of the object} \end{aligned}$$

Amount of heat per unit mass required
to raise temperature by one degree

- Thermal energy stored $\mathcal{E} = C_t T$

Linear graph representation

- Single-port elements
  - Energy storage elements
  - Energy (dissipation) elements
  - Energy sources

Energy dissipation element Thermal resistor

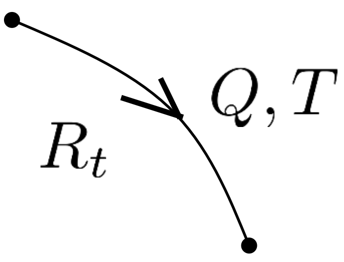
- Constitutive equation

$$R_t Q = T$$

$$R_t \; [Ks/J] \; ([K/W]) : \text{thermal resistance}$$

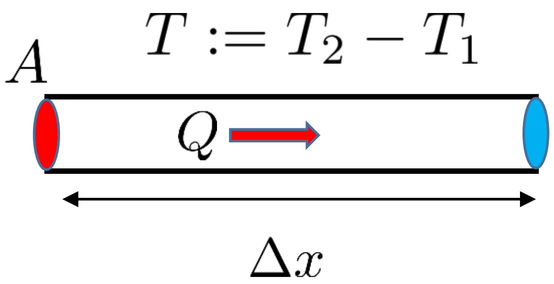
- No energy dissipation
  - Thermal resistance simply acts to impede heat flow.
- Three heat transfer mechanisms
  - Conduction
  - Convection
  - Radiation (nonlinear) $R_t Q = T_1^4 - T_2^4$

Linear graph



Energy dissipation element
Thermal resistor: Conduction

- Heat transfer through a material (e.g. metal block)
  - For a uniform material below



  - Thermal resistance is

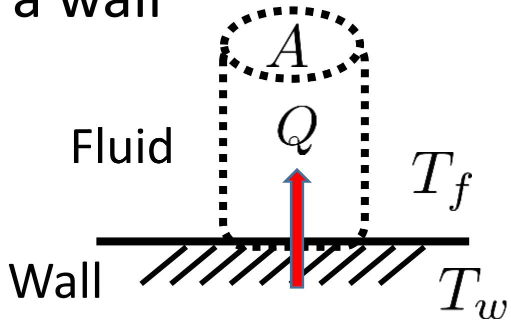
$$R_t = \frac{\Delta x}{kA} \quad k\ [W/K] : \text{thermal conductivity}$$

High *k*-value: Thermal conductor (e.g. metal)

Low *k*-value: Insulator (e.g. wood, plastic)

Energy dissipation element Thermal resistor: Convection

- Heat transfer by moving fluids (e.g. air)
  - Ex: Fluid flowing against a wall



$$T := T_w - T_f$$

  - Thermal resistance is

$$R_t = \frac{1}{h_c A} \qquad \begin{array}{l} h_c \ [W/m^2K] : \text{convection heat transfer coefficient} \\ A \ [m^2] : \text{section area of heat transfer} \end{array}$$

  - Natural convection (e.g. hot air rises)
  - Forced convection (e.g. fluid moved by fan or pump)

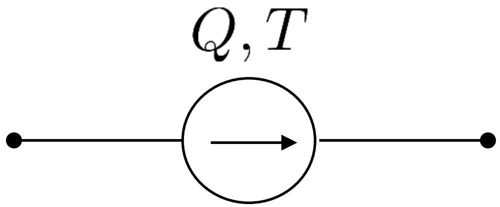
Linear graph representation

- Single-port elements
  - Energy storage elements
  - Energy dissipation elements
  - Energy sources

Linear graph representation
Fluid energy sources

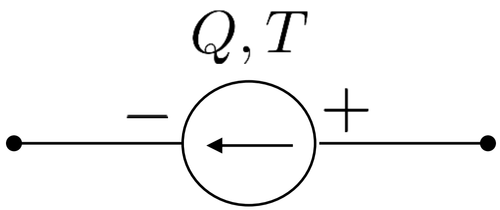
- Heat source
  - Heater

Linear graph

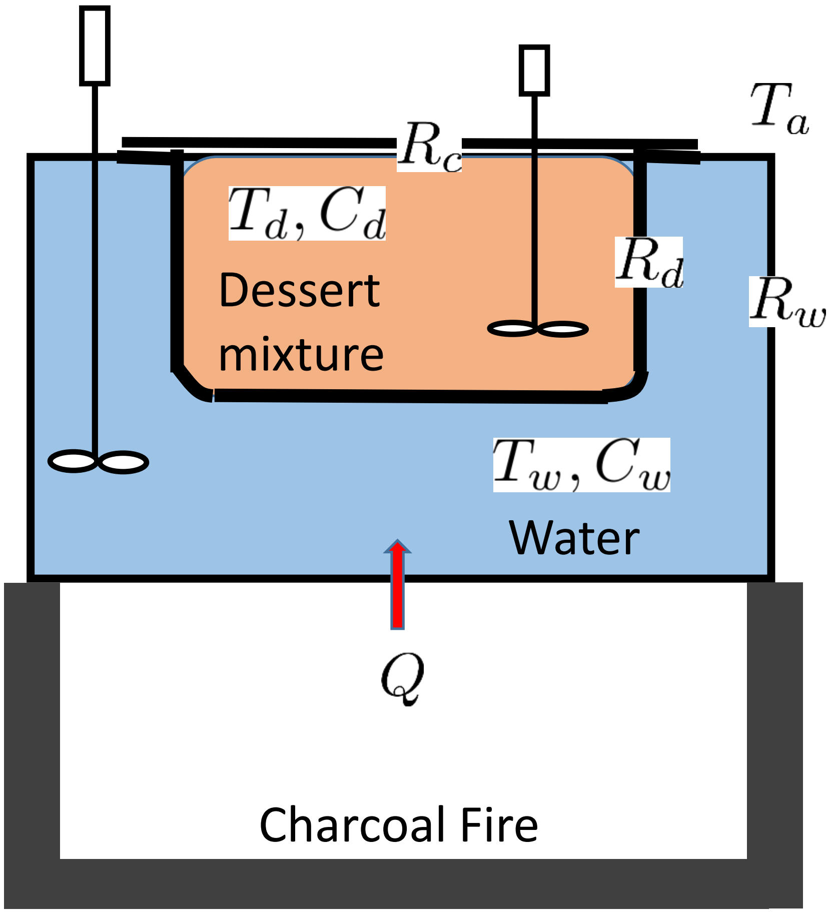


- Temperature source
  - Heater
  - Large reservoir

Linear graph



Example: “watalappam” making (taken from de Silva’s book)



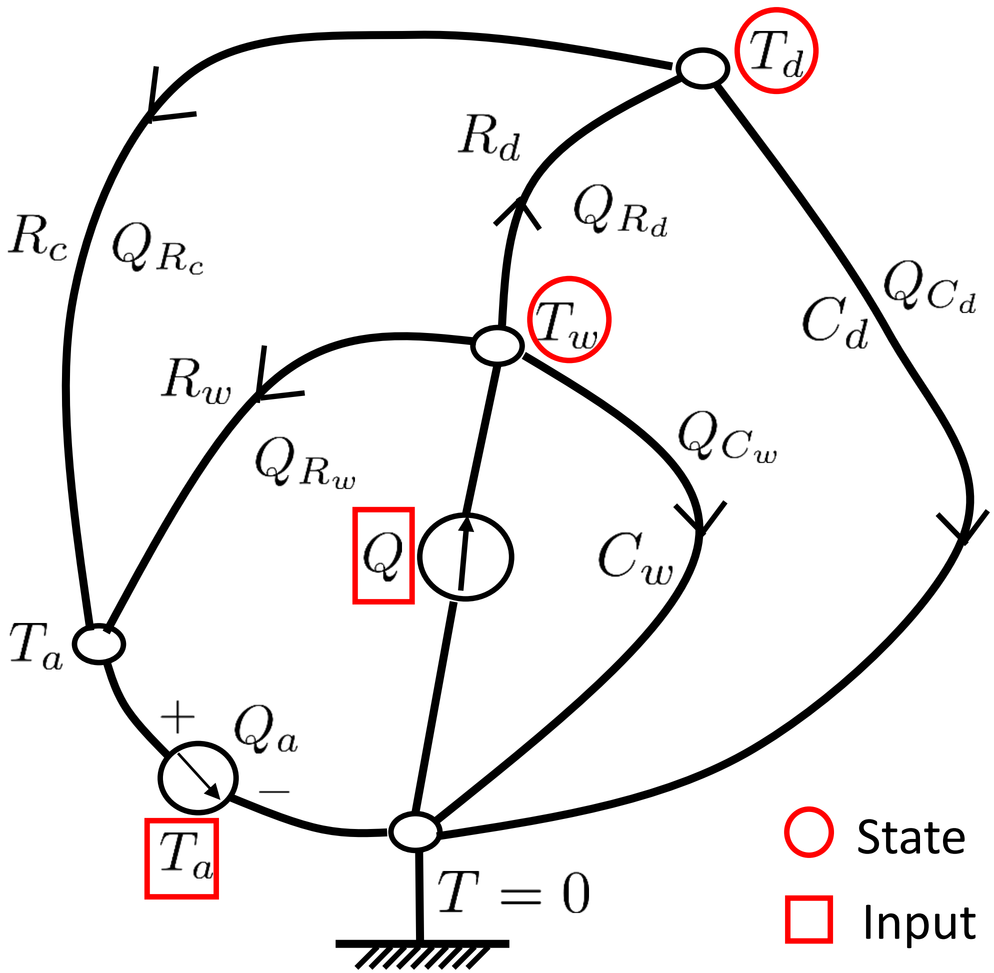
C : thermal capacitance
R : thermal resistance
T : temperature
Q : heat transfer rate

Input: heat transfer rate Q

Disturbance input: ambient temperature T_a

Output: temperature T_d

Example
Linear graph drawing



- Constitutive eq.

$$R_d Q_{R_d} = T_w - T_d$$
$$R_c Q_{R_c} = T_d - T_a$$
$$R_w Q_{R_w} = T_w - T_a$$
$$C_w \dot{T}_w = Q_{C_w} \quad C_d \dot{T}_d = Q_{C_d}$$

- Node eq.

$$Q_{R_d} = Q_{R_c} + Q_{C_d}$$
$$Q_{R_d} + Q_{R_w} + Q_{C_w} = Q$$
$$Q_{R_c} + Q_{R_w} + Q_a = 0$$

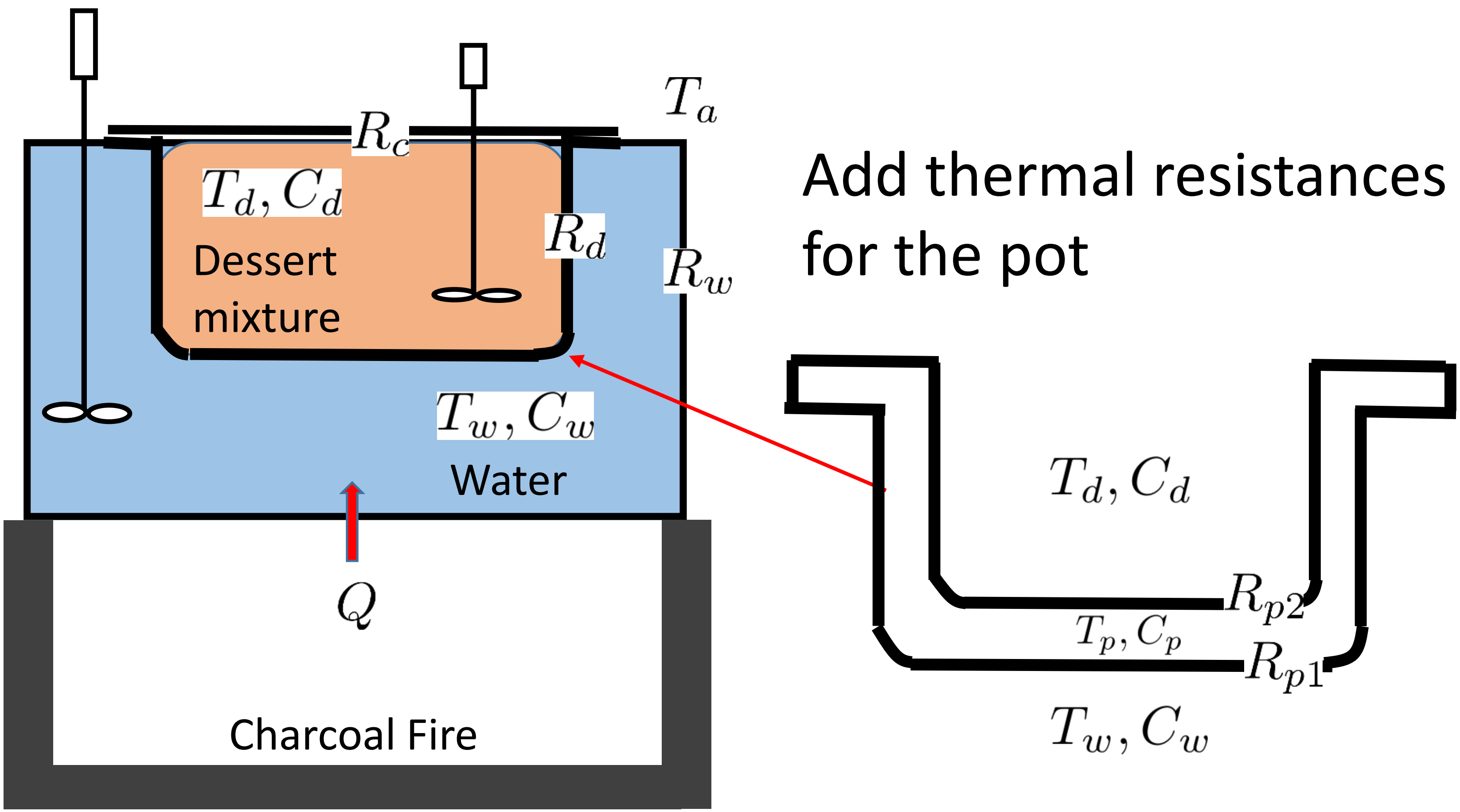
Example
State-space model derivation

$$\dot{T}_w = \frac{1}{C_w} Q_{C_w} = \frac{1}{C_w}\{Q - \underbrace{\frac{1}{R_w}(T_w - T_a))}_{Q_{R_w}} - \underbrace{\frac{1}{R_d}(T_w - T_d)}_{Q_{R_d}}\}$$

$$\dot{T}_d = \frac{1}{C_d} Q_{C_d} = \frac{1}{C_d}\{\underbrace{\frac{1}{R_d}(T_w - T_d)}_{Q_{R_d}} - \underbrace{\frac{1}{R_c}(T_d - T_a)}_{Q_{R_c}}\}$$

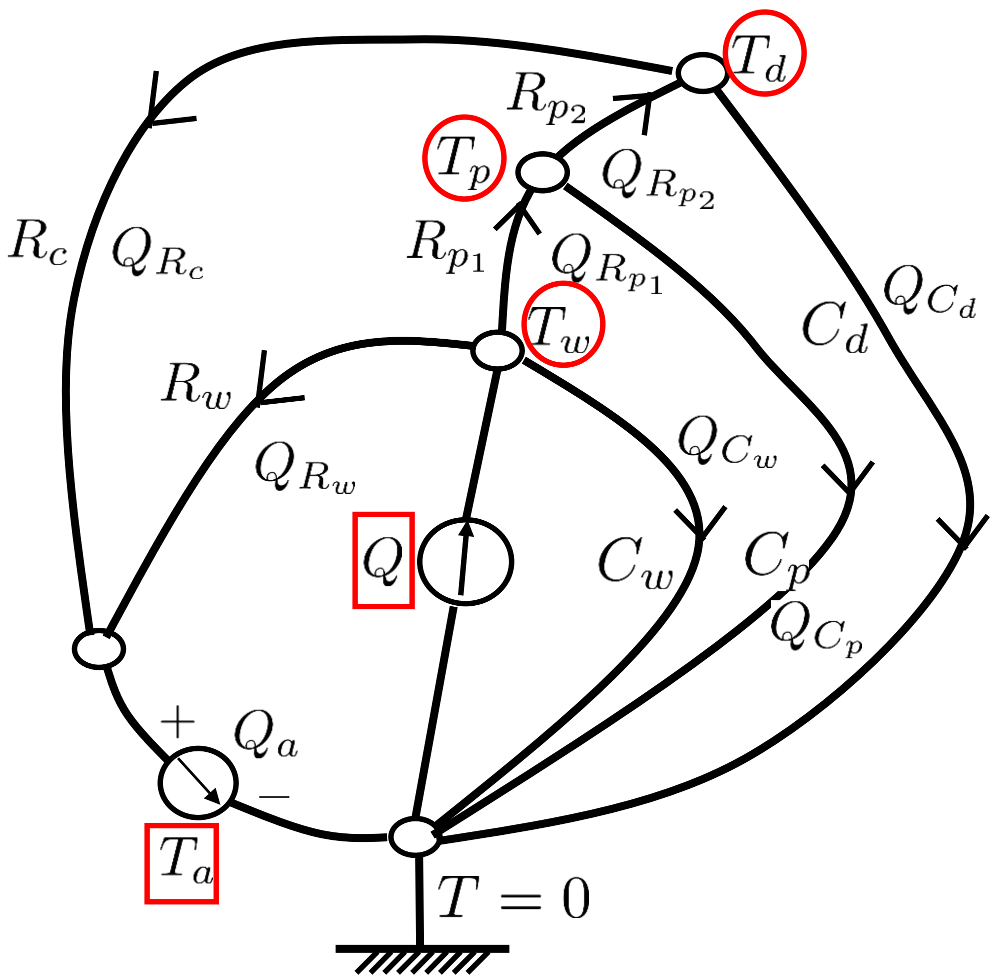
$$\left\{\begin{array}{lcl} \begin{bmatrix} \dot{T}_w \\ \dot{T}_d \end{bmatrix} & = & \begin{bmatrix} -\frac{1}{C_w}\left(\frac{1}{R_w} + \frac{1}{R_d}\right) & \frac{1}{C_w R_d} \\ \frac{1}{C_d R_d} & -\frac{1}{C_d}\left(\frac{1}{R_d} + \frac{1}{R_c}\right) \end{bmatrix} \begin{bmatrix} T_w \\ T_d \end{bmatrix} + \begin{bmatrix} \frac{1}{C_w} & \frac{1}{C_w R_w} \\ 0 & \frac{1}{C_d R_c} \end{bmatrix} \begin{bmatrix} Q \\ T_a \end{bmatrix} \\ T_d & = & \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} T_w \\ T_d \end{bmatrix} \end{array}\right.$$

Example
A detailed model



Add thermal resistances for the pot

Example
Linear graph drawing



- Constitutive eq.

$$R_{p1} Q_{R_{p1}} = T_w - T_p$$
$$R_{p2} Q_{R_{p2}} = T_p - T_d$$
$$R_c Q_{R_c} = T_d - T_a$$
$$R_w Q_{R_w} = T_w - T_a$$
$$C_w \dot{T}_w = Q_{C_w} \quad C_d \dot{T}_d = Q_{C_d}$$
$$C_p \dot{T}_p = Q_{C_p}$$

- Node eq.

$$Q_{R_{p2}} = Q_{R_c} + Q_{C_d}$$
$$Q_{R_{p1}} = Q_{R_{p2}} + Q_{C_p}$$
$$Q_{R_{p1}} + Q_{R_w} + Q_{C_w} = Q$$
$$Q_{R_c} + Q_{R_w} + Q_a = 0$$

Example
State-space model derivation

$$\dot{T}_w = \frac{1}{C_w} Q_{C_w} = \frac{1}{C_w}\{Q - \underbrace{\frac{1}{R_w}(T_w - T_a))}_{Q_{R_w}} - \underbrace{\frac{1}{R_{p1}}(T_w - T_p)\}}_{Q_{R_{p1}}}$$

$$\dot{T}_d = \frac{1}{C_d} Q_{C_d} = \frac{1}{C_d}\{\underbrace{\frac{1}{R_{p2}}(T_p - T_d)}_{Q_{R_{p2}}} - \underbrace{\frac{1}{R_c}(T_d - T_a)\}}_{Q_{R_c}}$$

$$\dot{T}_p = \frac{1}{C_p} Q_{C_p} = \frac{1}{C_p}\{\underbrace{\frac{1}{R_{p1}}(T_w - T_p)}_{Q_{R_{p1}}} - \underbrace{\frac{1}{R_{p2}}(T_p - T_d)\}}_{Q_{R_{p2}}}$$

$$\Rightarrow \left\{ \begin{array}{lcl} \begin{bmatrix} \dot{T}_w \\ \dot{T}_d \\ \dot{T}_p \end{bmatrix} & = & A \begin{bmatrix} T_w \\ T_d \\ T_p \end{bmatrix} + B \begin{bmatrix} Q \\ T_a \end{bmatrix} \\ T_d & = & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_w \\ T_d \\ T_p \end{bmatrix} \end{array} \right.$$

Simscape in Matlab/Simulink

- Physical system modeling tool in Matlab/Simulink
- Software which utilizes linear graph concept
- Read/watch
  - www.mathworks.com/products/simscape.html
  - www.mathworks.com/help/physmod/simscape/getting-started-with-simscape.html
    - Essential Steps for Constructing a Physical Model
    - Evaluating Performance of a DC Motor

Summary

- Linear graph for thermal systems
  - Single-port elements
    - Energy storage elements
    - Energy (dissipation) elements
    - Energy sources
- We are done with linear graph!
- Next, Laplace transform and transfer function
- Homework 4: Due Oct 28 (Monday), 3pm
- Lab 3 report: Due Nov 1 (Friday), 6pm