University of British Columbia
Department of Mechanical Engineering

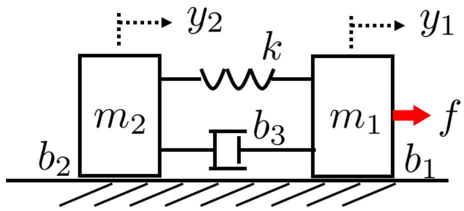
MECH366 Modeling of Mechatronic Systems
Homework 6

Solution

Consider 2-DOF mass-spring-damper system in the figure below, where k [N/m] is the linear spring constant, b_1 [Ns/m] and b_2 [Ns/m] are viscous friction coefficients (between masses and ground), and b_3 [Ns/m] is the damping coefficient.

Obtain the transfer function:

1. from the input force f to the output displacement y_1.
2. from the input force f to the output acceleration $\ddot{y}_2$.
3. from the input force f to the output (displacement difference) $y_1 - y_2$.
4. from the input displacement y_2 to the output displacement y_1.
5. from the input displacement y_1 to the output displacement y_2.



Solutions: The equations of motion are

$$\begin{aligned} m_1\ddot{y}_1 &= f - k(y_1 - y_2) - b_1\dot{y}_1 - b_3(\dot{y}_1 - \dot{y}_2) \\ m_2\ddot{y}_2 &= -k(y_2 - y_1) - b_2\dot{y}_2 - b_3(\dot{y}_2 - \dot{y}_1) \end{aligned}$$

By applying the Laplace transform, we can get

$$\begin{aligned} (m_1s^2 + (b_1 + b_3)s + k)Y_1(s) &= F(s) + (b_3s + k)Y_2(s) \\ (m_2s^2 + (b_2 + b_3)s + k)Y_2(s) &= (b_3s + k)Y_1(s) \end{aligned}$$

This can be rewritten as

$$\begin{aligned} Y_1(s) &= \underbrace{\frac{1}{m_1s^2 + (b_1 + b_3)s + k}}_{=:G_1(s)} F(s) + \underbrace{\frac{b_3s + k}{m_1s^2 + (b_1 + b_3)s + k}}_{=:G_2(s)} Y_2(s) \\ Y_2(s) &= \underbrace{\frac{b_3s + k}{m_2s^2 + (b_2 + b_3)s + k}}_{=:G_3(s)} Y_1(s) \end{aligned}$$

1.

$$Y_1(s) = G_1(s)F(s) + G_2(s)Y_2(s) = G_1(s)F(s) + G_2(s)G_3(s)Y_1(s)$$

$$\Rightarrow \frac{Y_1(s)}{F(s)} = \frac{G_1(s)}{1 - G_2(s)G_3(s)}$$

2.

$$Y_2(s) = G_3(s)Y_1(s) = G_3(s)(G_1(s)F(s) + G_2(s)Y_2(s))$$

$$\Rightarrow Y_2(s) = \frac{G_3(s)G_1(s)}{1 - G_3(s)G_2(s)}F(s) \Rightarrow \frac{s^2Y_2(s)}{F(s)} = \frac{s^2G_3(s)G_1(s)}{1 - G_3(s)G_2(s)}$$

3. Since $Y_1(s) = G_1(s)F(s) + G_2(s)Y_2(s)$, we have the following calculations to derive the transfer function.

$$\begin{aligned}
Y_1(s) - Y_2(s) &= G_1(s)F(s) + (G_2(s) - 1)Y_2(s) \\
&= G_1(s)F(s) + (G_2(s) - 1)\frac{G_3(s)G_1(s)}{1 - G_3(s)G_2(s)}F(s) \\
&= \frac{G_1(s)(1 - G_3(s)G_2(s)) + (G_2(s) - 1)G_3(s)G_1(s)}{1 - G_3(s)G_2(s)}F(s) \\
&= \frac{(1 - G_3(s))G_1(s)}{1 - G_3(s)G_2(s)}F(s) \\
\frac{Y_1(s) - Y_2(s)}{F(s)} &= \frac{(1 - G_3(s))G_1(s)}{1 - G_3(s)G_2(s)}
\end{aligned}$$

4.

$$\frac{Y_1(s)}{Y_2(s)} = G_2(s)$$

5.

$$\frac{Y_2(s)}{Y_1(s)} = G_3(s)$$