a place of mind

MECH366 : Modeling of Mechatronic Systems

L3 : State-space models

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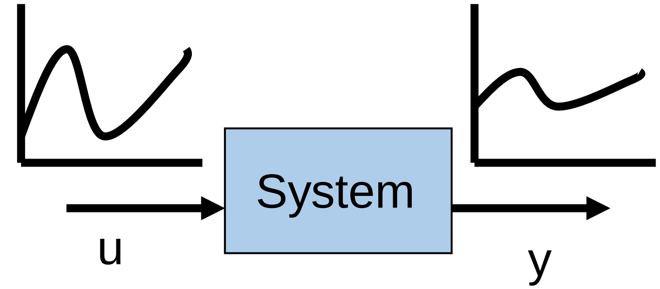
Review and today's topic

- Last lecture was about:
  - Dynamic model
  - Modeling procedure (next slide)
  - Analogies among different domains (incomplete)
- Today, we will study the state-space model.
  - Basic elements are used to generate a linear graph (in later classes, not today), from which we derive the state-space model.
  - State-space models are used for prediction, simulation, and controller design.

Modeling procedure (Review)

1. Identify a physical system to be modeled, and associated input and output variables.
2. Simplify the physical system with basic elements, based on your assumptions.
3. By applying physical laws (Newton's second law, Kirchhoff's law etc.) to the basic elements, obtain differential equations.
4. Identify (estimate) parameter values in the differential equations.
5. Validate the obtained model (differential equations with estimated parameter values) experimentally. If the model turns out to be invalid (the model is invalidated), go back to Step 2 with modifications.

State-space model (Linear)



- Matrix-vector form

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

x : state vector
u : input vector
y : output vector

- Element-wise representation

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \end{cases}$$

Remarks

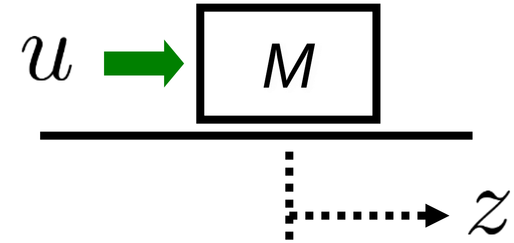
- The first equation, called *state equation*, is a first-order ordinary differential.
- The second equation, called *output equation*, is an algebraic equation.
- Two equations are called *state-space model*.
- Pay attention to sizes of matrices and vectors. They must be always compatible!
- “How to select the state vector?” is explained through examples.

Examples

- Mass with a driving force
- Mass-spring-damper system
  - 1-DOF (degree-of-freedom)
  - 2-DOF
- RLC circuit

Mass with a driving force

- By Newton's second law, we have



$$M\ddot{z}(t) = u(t)$$

where input: force *u*, output: position z

- Define state variables $x_1 := z, \;\; x_2 := \dot{z}$
- Then,

$$\left\{ \begin{array}{l} \dot{x}_1 = \dot{z} = x_2 \\ \dot{x}_2 = \ddot{z} = \frac{1}{M}u \\ y = x_1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u(t) \\ y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} u(t) \end{array} \right.$$

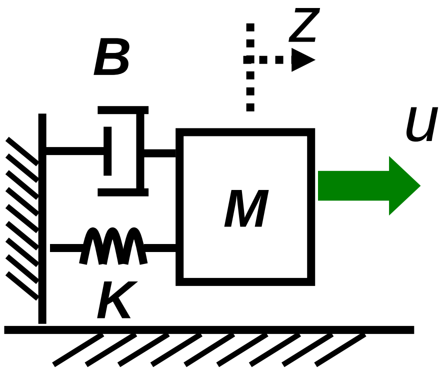
Mass with a driving force
Another SS model

- Define state variables $x_1 := \dot{z},\ x_2 := z$
- Then, $$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/M \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$
  - Derivation 1: Directly calculate.
  - Derivation 2: Using the SS model in the previous slide, exchange rows/columns of *A*, *B*, *C* matrices.
- *SS model is not unique; it depends on the selection of states.*

Mass-spring-damper system (1-DOF)

- By Newton's second law,

$$M\ddot{z} = u - Kz - B\dot{z}$$



- Input: force (*u*), output: position (*z*)
- Define state variables: $x_1 := z,\ x_2 := \dot{z}$

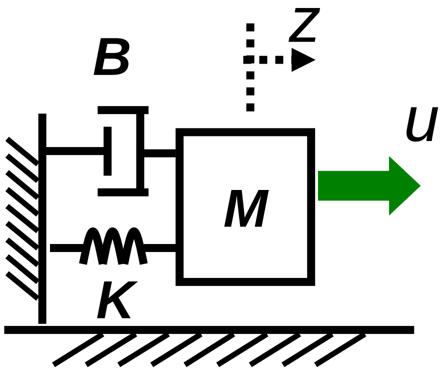
$$\Rightarrow \left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \right.$$

Second order model (because it has 2 states)

Mass-spring-damper system (1-DOF)

- By Newton's second law,

$$M\ddot{z} = u - \underbrace{Kz}_{=:f_K} - B\dot{z}$$

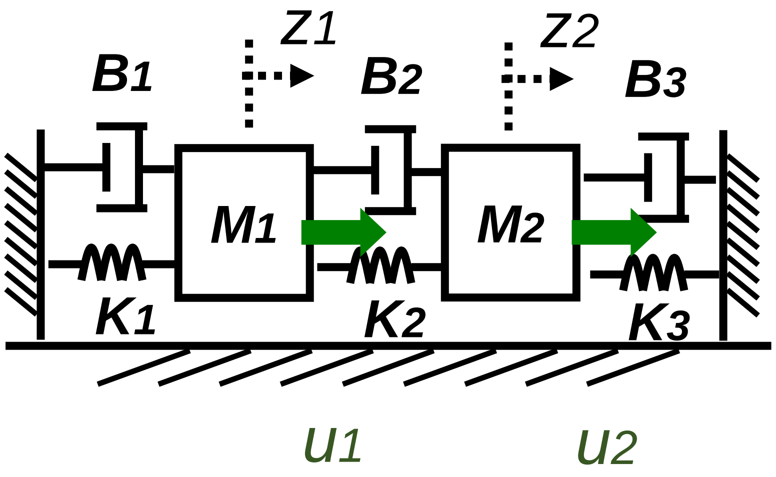


- Define state variables: $x_1 := \dot{z},\ \ x_2 := f_K$
  - Across variable (velocity) for A-type element (Mass)
  - Through variable (force) for T-type element (Spring)

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -B/M & -1/M \\ K & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/M \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1/K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

Mass-spring-damper system (2-DOF)

- Inputs: forces (u_1 & u_2)
- Outputs: position (z_1 & z_2)
- Newton's second law



$$\begin{cases} M_1\ddot{z}_1 & = & u_1 - K_1 z_1 - B_1\dot{z}_1 - K_2(z_1 - z_2) - B_2(\dot{z}_1 - \dot{z}_2) \\ M_2\ddot{z}_2 & = & u_2 - K_2(z_2 - z_1) - B_2(\dot{z}_2 - \dot{z}_1) - K_3 z_2 - B_3\dot{z}_2 \end{cases}$$

Mass-spring-damper system (2-DOF)

$$\begin{cases} M_1\ddot{z}_1 &= u_1 - K_1 z_1 - B_1\dot{z}_1 - K_2(z_1 - z_2) - B_2(\dot{z}_1 - \dot{z}_2) \\ M_2\ddot{z}_2 &= u_2 - K_2(z_2 - z_1) - B_2(\dot{z}_2 - \dot{z}_1) - K_3 z_2 - B_3\dot{z}_2 \end{cases}$$

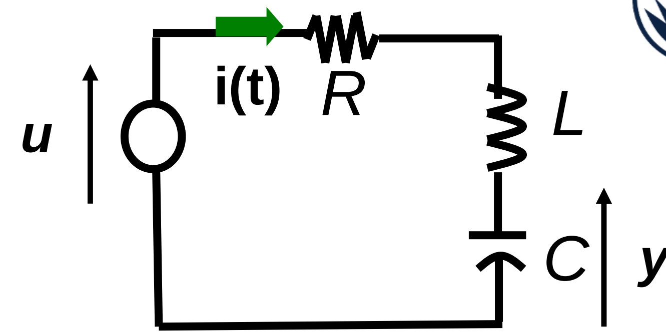
- State variables: $x_1 := z_1,\ x_2 := \dot{z}_1,\ x_3 := z_2,\ x_4 := \dot{z}_2$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1+K_2}{M_1} & -\frac{B_1+B_2}{M_1} & \frac{K_2}{M_1} & \frac{B_2}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_2}{M_2} & \frac{B_2}{M_2} & -\frac{K_2+K_3}{M_2} & -\frac{B_2+B_3}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{M_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases}$$

Fourth order model

(because it has 4 states)

RLC circuit (one-output case)



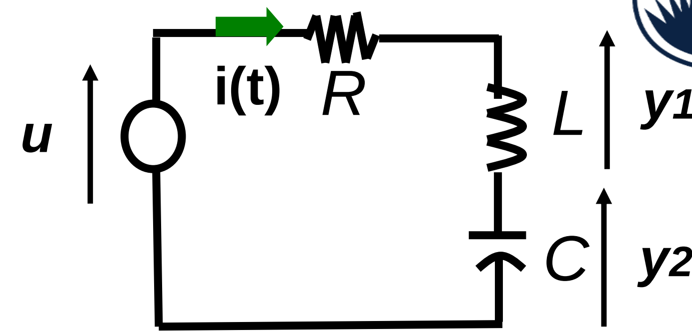
- By Kirchhoff's voltage law

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)d\tau$$

- Define state variables: $x_1(t) := i(t),\ x_2(t) := \frac{1}{C}\int i(\tau)d\tau$
  - Current (through variable) for inductor (T-type element)
  - Voltage (across variable) for capacitor (A-type element)

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

RLC circuit (two-output case)



- By Kirchhoff's voltage law

$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)d\tau$$

- Define state variables: $x_1(t) := i(t), \; x_2(t) := \frac{1}{C}\int i(\tau)d\tau$

$$\Rightarrow \left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -R & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \end{array} \right.$$

Summary

- Today's topics
  - Linear state-space model
  - Examples for deriving state-space models
    - Mechanical system
    - Electrical system
- Next, linearization of nonlinear models
- HW1: given on Monday, and due in one week.

O : State variable

	Constitutive Relation for		
	Energy Storage Elements		Energy Dissipating Elements
System Type	A-type (Across) Element	T-type (Through) Element	D-type (Dissipative) Element
Translatory-Mechanical v = velocity f = force	*Mass* $m\dfrac{dv}{dt}=f$ (Newton's second law) m = mass	*Spring* $\dfrac{df}{dt}=kv$ (Hooke's law) k = stiffness	*Viscous Damper* $f=bv$ b = damping constant
Electrical v = voltage i = current	*Capacitor* $C\dfrac{dv}{dt}=i$ C = capacitance	*Inductor* $L\dfrac{di}{dt}=v$ L = inductance	*Resistor* $Ri=v$ R = resistance
Thermal T = temperature difference Q = heat transfer rate	*Thermal Capacitor* $C_t\dfrac{dT}{dt}=Q$ C_t = thermal capacitance	*None*	*Thermal Resistor* $R_tQ=T$ R_t = thermal resistance
Fluid P = pressure difference Q = volume flow rate	*Fluid Capacitor* $C_f\dfrac{dP}{dt}=Q$ C_f = fluid capacitance	*Fluid Inertor* $I_f\dfrac{dQ}{dt}=P$ I_f = inertance	*Fluid Resistor* $R_fQ=P$ R_f = fluid resistance

Review of linear algebra

- Product of a matrix and a vector

$$
\begin{array}{rcl}
 & & \begin{array}{cc} (n \times n) & (n \times 1) \end{array} \\
Ax & = & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
 & = & \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix} \\
 & & (n \times 1)
\end{array}
$$