a place of mind

MECH366 : Modeling of Mechatronic Systems

L8 : Derivation of state-space models from linear graphs: Examples

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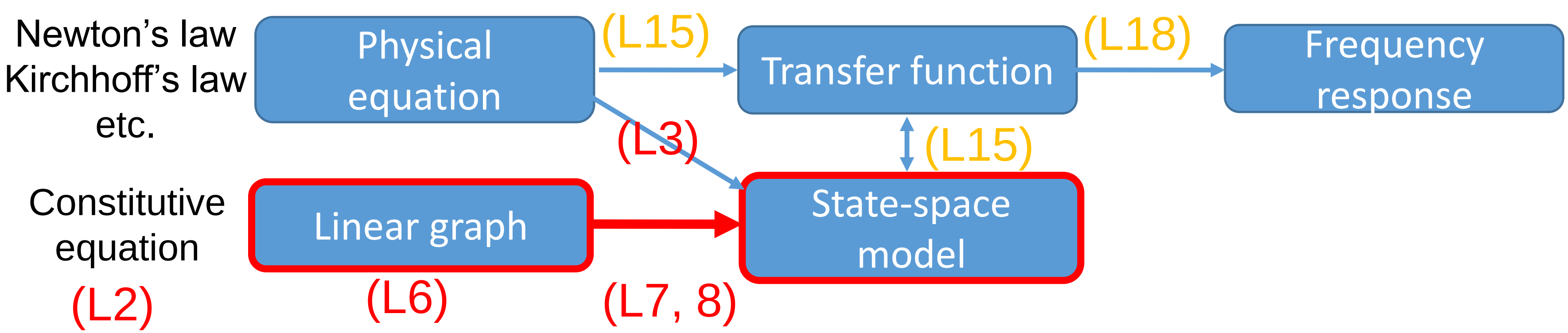
Review and today's topic

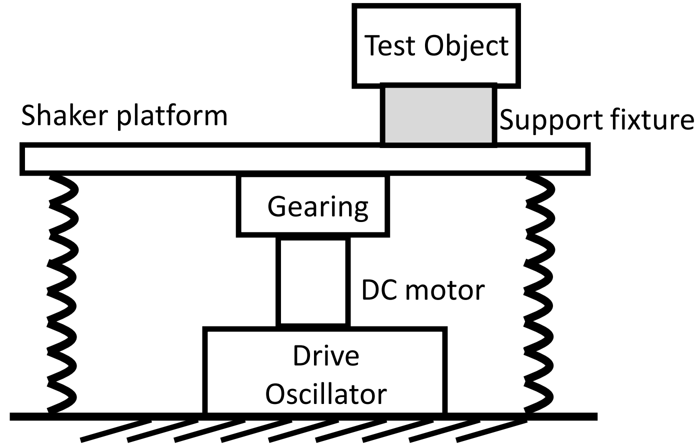
- Up to now, we have studied
  - How to draw linear graphs
  - How to derive state-space models from linear graphs
- Today, we give two examples (in de Silva's book).
- Various models and their relations



How to derive state-space models from linear graphs (review)

- Key steps
  1. Draw a linear graph.
  2. Define state variables.
  3. Write a constitutive equation for each element.
  4. Write loop equations and node equations.
     - Loop equations are similar to Kirchhoff voltage law.
     - Node equations are similar to Kirchhoff current law.
  5. Eliminate redundant variables.

Example 4
Electrodynamic shaker

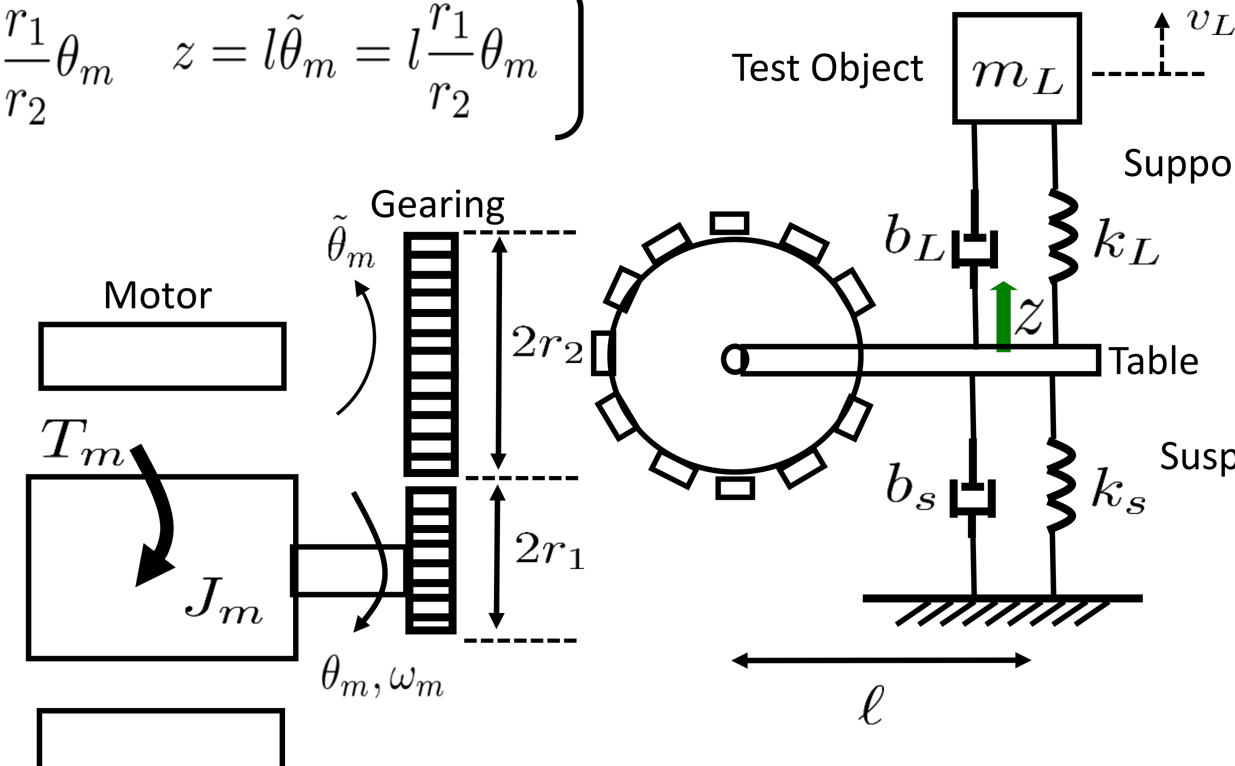


Motion ratio $r := \dfrac{z}{\theta_m} = l\dfrac{r_1}{r_2}$

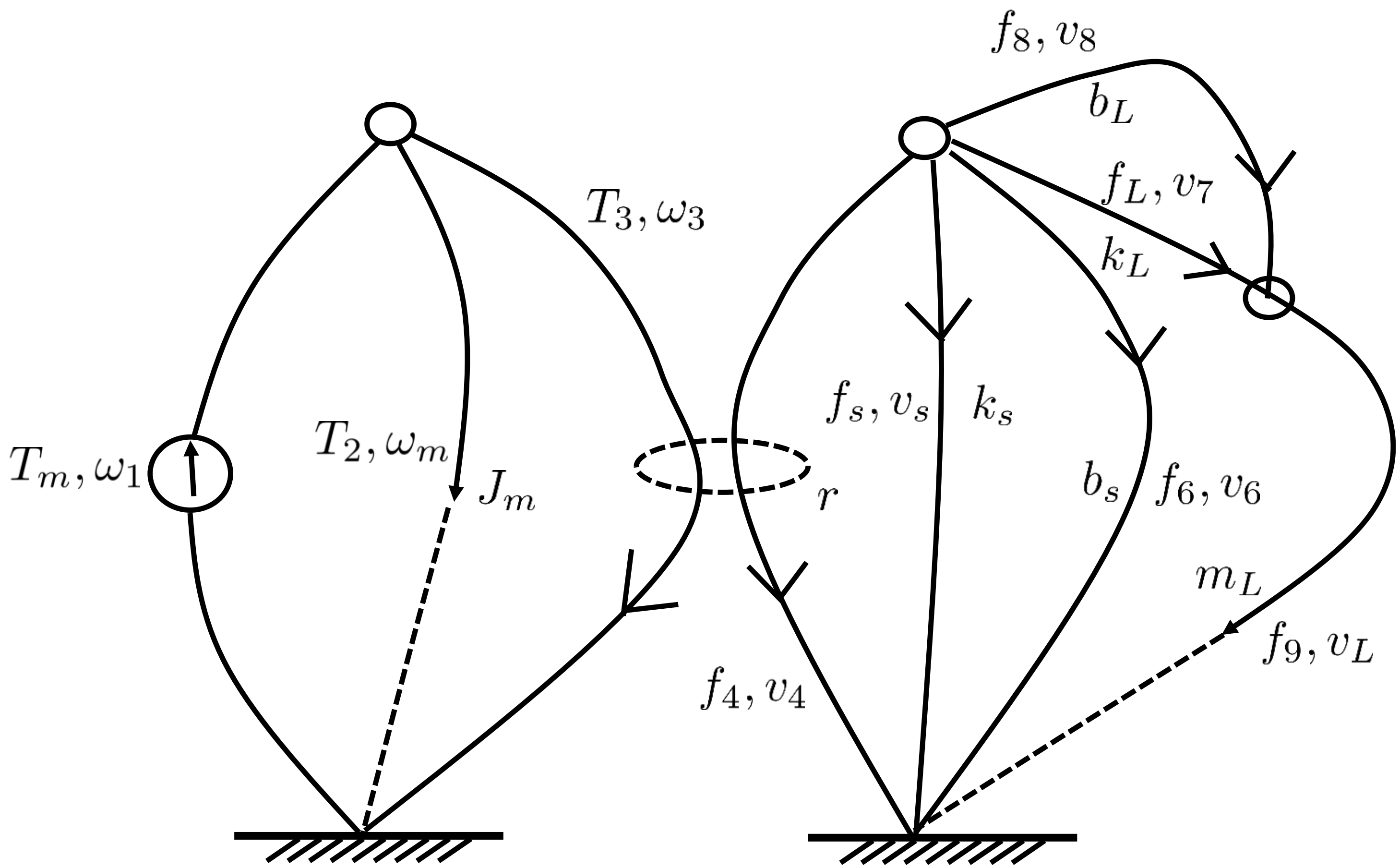
$$\left(\tilde{\theta}_m = \frac{r_1}{r_2}\theta_m \quad z = l\tilde{\theta}_m = l\frac{r_1}{r_2}\theta_m \right)$$

Output velocity

Input torque



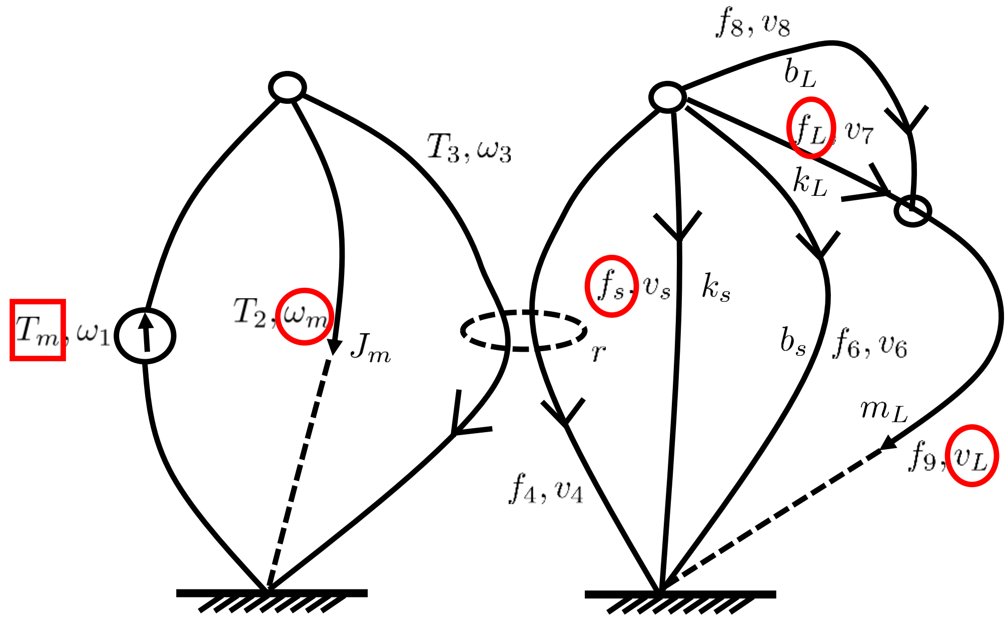
Example 4
Linear graph drawing



Example 4
State-variable selection

- Select the following as state variables:
  - Across variable (v & ω) for A-type element (m & J)
  - Through variable (f & T) for T-type element (k)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} \omega_m \\ f_s \\ f_L \\ v_L \end{bmatrix}$$



Example 4
Constitutive equations

- Basic elements

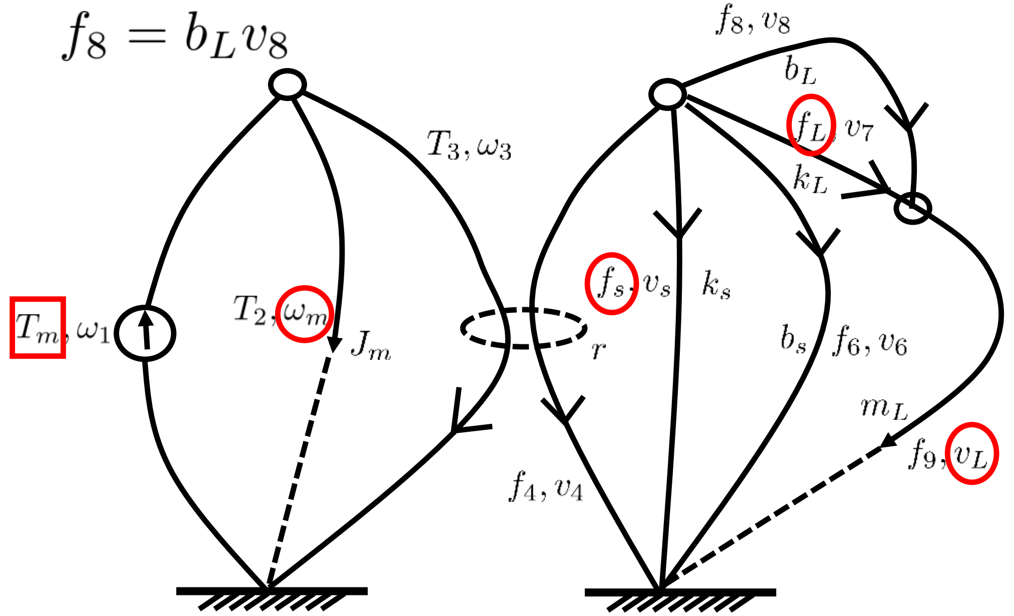
$$\dot{\omega}_m = \frac{1}{J_m} T_2 \qquad \dot{f}_s = k_s v_5 \qquad f_6 = b_s v_6$$

$$\dot{v}_L = \frac{1}{m_L} f_9 \qquad \dot{f}_L = k_L v_7 \qquad f_8 = b_L v_8$$

- Transformer

$$v_4 = r\omega_3$$

$$f_4 = -\frac{1}{r} T_3$$



Example 4
Loop and node equations

States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} \omega_m \\ f_s \\ f_L \\ v_L \end{bmatrix}$$

- From loop equations:

$$\begin{cases} \omega_1 = \omega_m = \omega_3 \\ v_4 = v_5 = v_6 \\ v_8 = v_7 \\ v_6 = v_7 + v_L \end{cases}$$

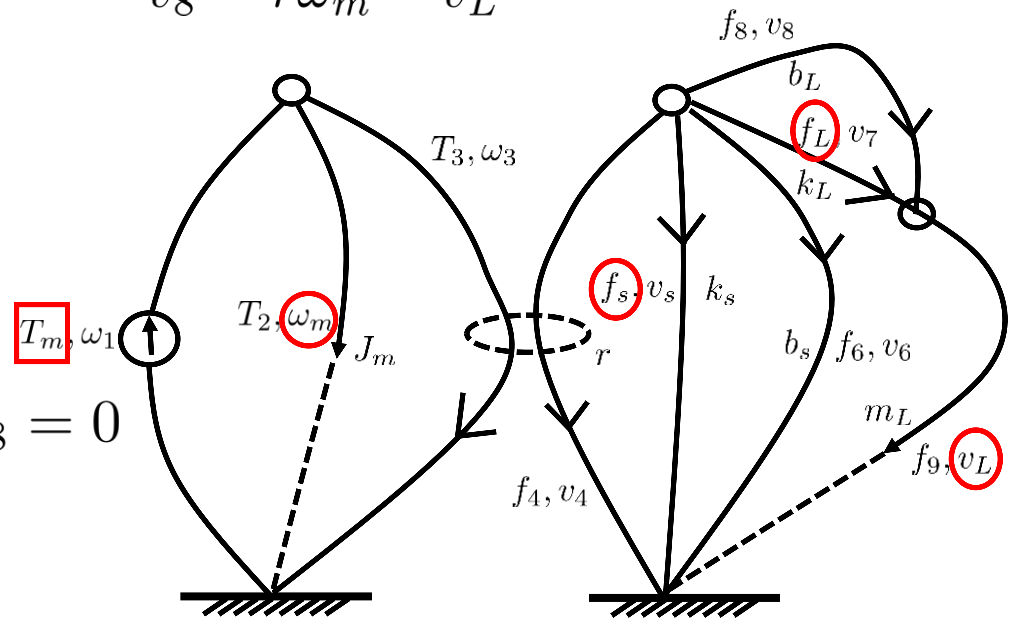
$$v_4 = r\omega_3$$

$$\longrightarrow \quad v_6 = r\omega_m$$

$$v_8 = r\omega_m - v_L$$

- From node equations:

$$\begin{cases} T_m = T_2 + T_3 \\ f_4 + f_s + f_6 + f_L + f_8 = 0 \\ f_L + f_8 = f_9 \end{cases}$$



Example 4
State equation

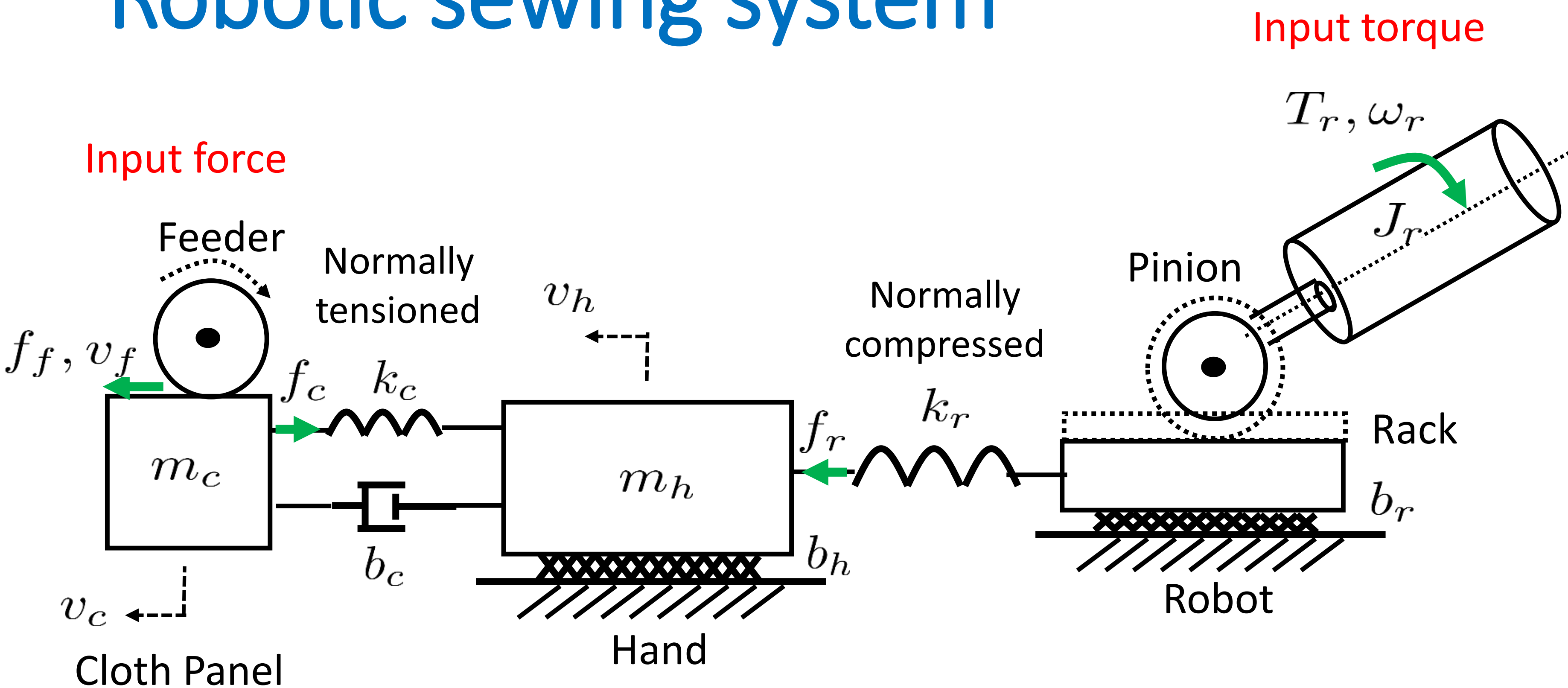
States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} \omega_m \\ f_s \\ f_L \\ v_L \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -r^2(b_s + b_L)/J_m & -r/J_m & -r/J_m & -rb_L/J_m \\ rk_s & 0 & 0 & 0 \\ rk_L & 0 & 0 & -k_L \\ rb_L/m_L & 0 & 1/m_L & -b_L/m_L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1/J_m \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

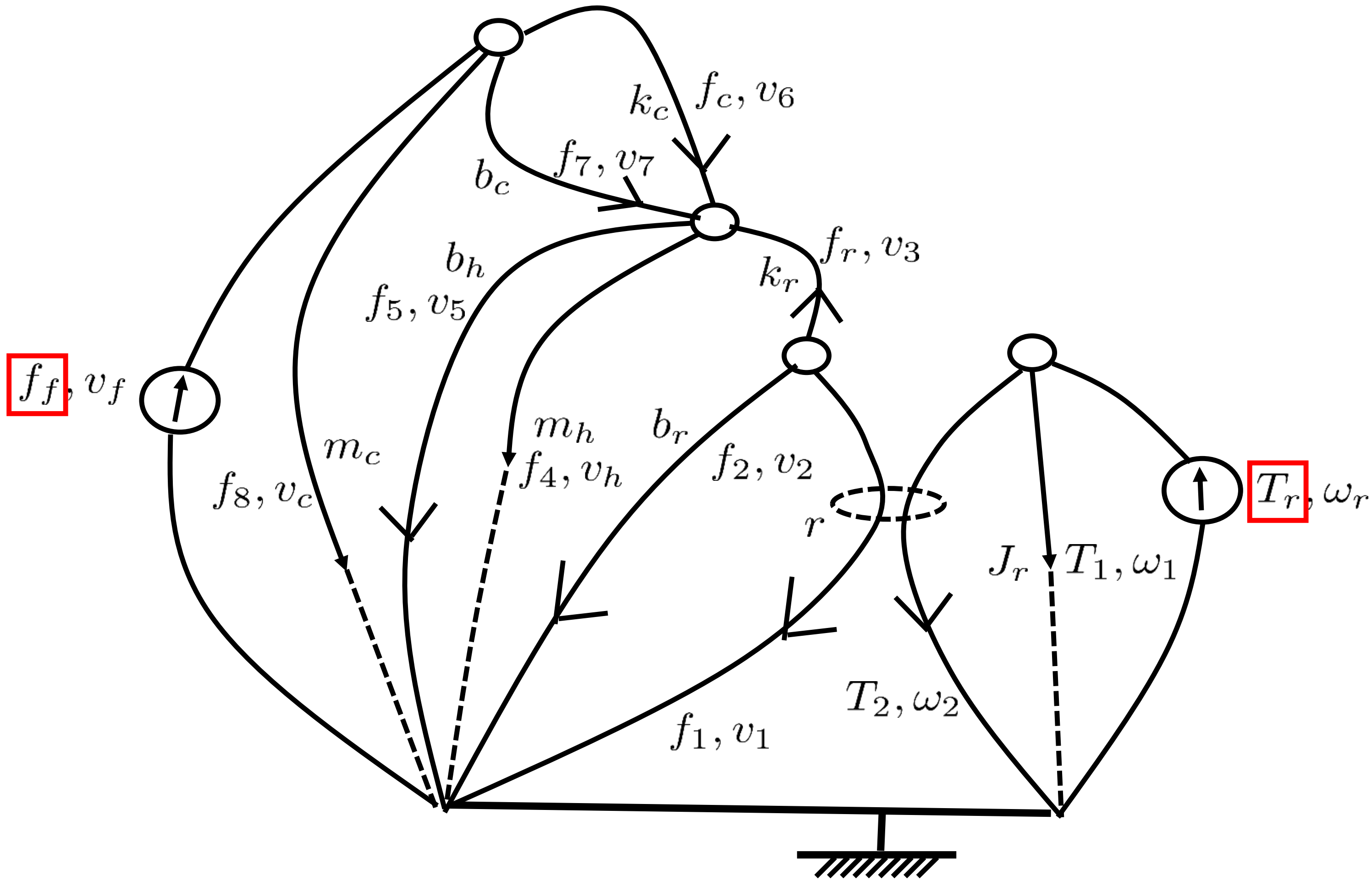
$$\left(\begin{aligned}
\dot{\omega}_m &= \frac{1}{J_m} T_2 = \frac{1}{J_m}(T_m - T_3) = \frac{1}{J_m}\Big(T_m + r\,\underbrace{\Big(\underbrace{-f_s - b_s \underbrace{r\omega_m}_{v_6}}_{f_6} - f_L - \underbrace{b_L\,\underbrace{(r\omega_m - v_L)}_{v_8}}_{f_8}\Big)}_{f_4}\Big) \\
\dot{f}_s &= k_s v_5 = k_s r \omega_m \\
\dot{f}_L &= k_L v_7 = k_L (r\omega_m - v_L) \\
\dot{v}_L &= \frac{1}{m_L} f_9 = \frac{1}{m_L}(f_L + f_8) = \frac{1}{m_L}(f_L + b_L v_8) = \frac{1}{m_L}(f_L + b_L(r\omega_m - v_L))
\end{aligned} \right)$$

Example 5
Robotic sewing system



$$\frac{\text{rack translational motion}}{\text{pinion rotatioal motion}} = r$$

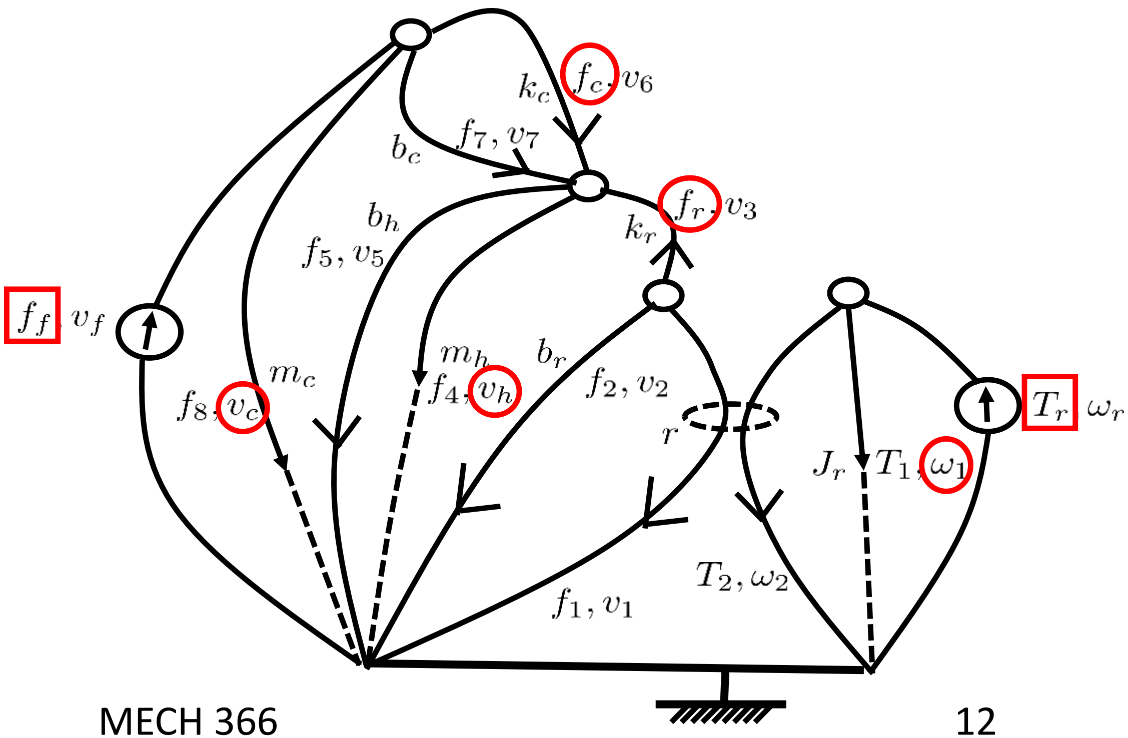
Example 5
Linear graph drawing



Example 5
State-variable selection

- Select the following as state variables:
  - Across variable (*v* & ω) for A-type element (*m* & *J*)
  - Through variable (*f*) for T-type element (*k*)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} := \begin{bmatrix} \omega_1 \\ f_r \\ v_h \\ f_c \\ v_c \end{bmatrix}$$



Example 5
Constitutive equations

- Basic elements

$$\dot{\omega}_1 = \frac{1}{J_r} T_1 \qquad \dot{f}_r = k_r v_3 \qquad f_2 = b_r v_2$$

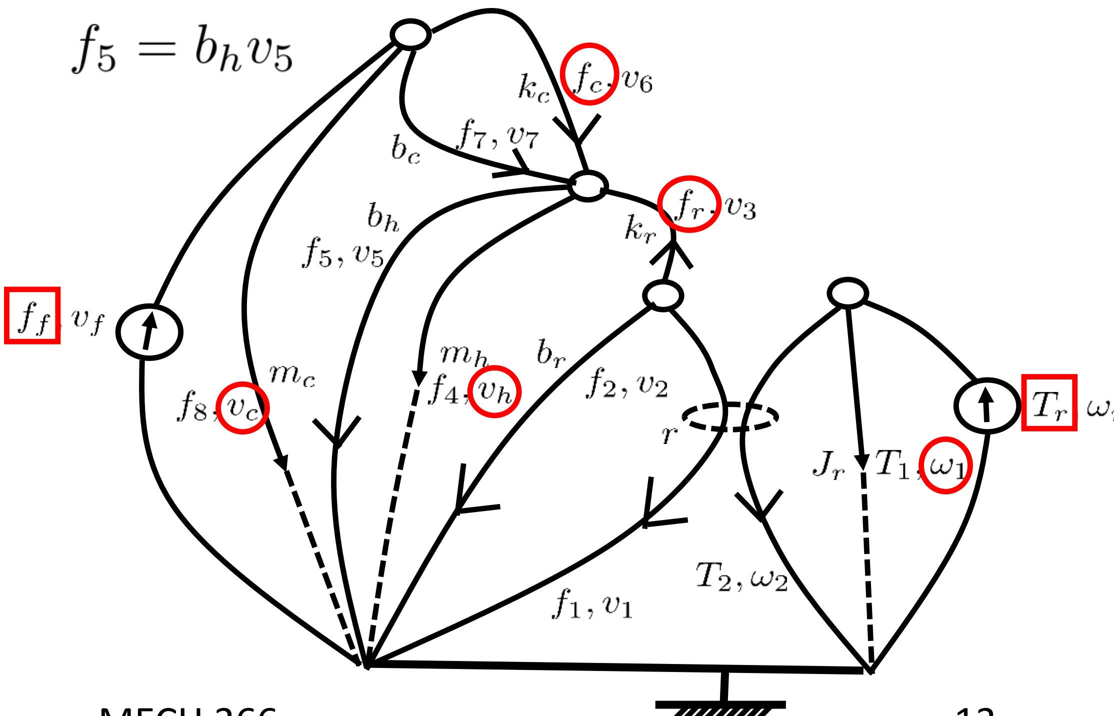
$$\dot{v}_h = \frac{1}{m_h} f_4 \qquad \dot{f}_c = k_c v_6 \qquad f_5 = b_h v_5$$

$$\dot{v}_c = \frac{1}{m_c} f_8 \qquad f_7 = b_c v_7$$

- Transformer (rack & pinion)

$$v_1 = r\omega_2$$

$$f_1 = -\frac{1}{r} T_2$$



Example 5
Loop and node equations

States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} := \begin{bmatrix} \omega_1 \\ f_r \\ v_h \\ f_c \\ v_c \end{bmatrix}$$

- From loop equations:

$$\begin{cases} \omega_r = \omega_1 = \omega_2 \\ v_1 = v_2 = v_3 + v_h \\ v_h = v_5 \\ v_6 = v_7 = v_c - v_h \\ v_f = v_c \end{cases}$$

$$\xrightarrow{v_1 = r\omega_2}$$

$$v_2 = r\omega_2 = r\omega_1$$

$$v_3 = v_2 - v_h = r\omega_1 - v_h$$

- From node equations:

$$\begin{cases} T_r = T_1 + T_2 \\ f_1 + f_2 + f_r = 0 \\ f_5 + f_4 = f_r + f_c + f_7 \\ f_c + f_8 + f_7 = f_f \end{cases}$$

Example 5
State-space model

States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} := \begin{bmatrix} \omega_1 \\ f_r \\ v_h \\ f_c \\ v_c \end{bmatrix}$$

Inputs $u := \begin{bmatrix} T_r \\ f_f \end{bmatrix}$ Outputs $y := \begin{bmatrix} f_c \\ \omega_r \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -r^2 b_r/J_r & -r/J_r & 0 & 0 & 0 \\ r k_r & 0 & -k_r & 0 & 0 \\ 0 & 1/m_h & -(b_c + b_h)/m_h & 1/m_h & b_c/m_h \\ 0 & 0 & -k_c & 0 & k_c \\ 0 & 0 & b_c/m_c & -1/m_c & -b_c/m_c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1/J_r & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/m_c \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Example 5
State equation

States

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} := \begin{bmatrix} \omega_1 \\ f_r \\ v_h \\ f_c \\ v_c \end{bmatrix}$$

Inputs

$$u := \begin{bmatrix} T_r \\ f_f \end{bmatrix}$$

- Derivation

$$\dot{\omega}_1 = \frac{1}{J_r} T_1 = \frac{1}{J_r}(T_r - T_2) = \frac{1}{J_r}(T_r + r f_1) = \frac{1}{J_r}(T_r - r(f_r + \underbrace{b_r v_2}_{f_2})) = \frac{1}{J_r}(T_r - r(f_r + b_r r \omega_1))$$

$$\dot{f}_r = k_r v_3 = k_r(r\omega_1 - v_h)$$

$$\dot{v}_h = \frac{1}{m_h} f_4 = \frac{1}{m_h}(-\underbrace{b_h v_5}_{f_5} + f_r + f_c + \underbrace{b_c v_7}_{f_7}) = \frac{1}{m_h}(-b_h v_h + f_r + f_c + b_c(v_c - v_h))$$

$$\dot{f}_c = k_c v_6 = k_c(v_c - v_h)$$

$$\dot{v}_c = \frac{1}{m_c} f_8 = \frac{1}{m_c}(f_f - f_c - f_7) = \frac{1}{m_c}(f_f - f_c - b_c v_7) = \frac{1}{m_c}(f_f - f_c - b_c(v_c - v_h))$$

Summary

- Today's topic
  - Two mechanical examples of deriving state-space models from linear graphs
  - Examples are taken from:

    "Modeling and Control of Engineering Systems"

    CRC Press, 2009, by C. W. de Silva
- Next, electrical systems
- Homework 3: Due Oct 7 (Monday), 3pm