Solutions - Problem Set # 2

Problem 1:

Assumptions: 1-D Steady-state radial heat conduction with Source term **(S≠constant)**

a) q_{gen}=?

$$q_{gen} = \int_0^{r_o} S dV$$

$$q_{gen} = \int_0^{r_o} S_o \left[1 - \left(\frac{r}{r_o}\right)^2\right] \underbrace{4\pi r^2 dr}_{dV}$$

$$q_{gen} = 4\pi S_o \left[\frac{1}{3}r^3 - \frac{1}{5}\frac{r^5}{r_o^{\,2}}\right]_0^{r_o}$$

$$q_{gen} = \frac{8}{15}\pi S_o r_o^{\,3}$$

b) T_s =?

$$q_{gen} - q_{\substack{conv.loss \\ total}} = 0$$

$$\frac{8}{15}\pi S_o r_o^{\,3} - \left(4\pi r_o^{\,2}\right) h \left(T_s - T_\infty\right) = 0$$

$$T_s = T_\infty + \frac{2}{15}\frac{S_o r_o}{h}$$

Problem 2:

Assumptions: 1-D Steady-state heat conduction with Source term **(S≠constant)**

Given: $\dfrac{(T - T_1)}{(T_2 - T_1)} = C_1 + C_2 x^2 + C_3 x^3$ BCs.: $(i)\quad x = 0 \Rightarrow T = T_1$; $(ii)\quad x = L \Rightarrow T = T_2$

Additional condition $(iii)\quad x = 0 \Rightarrow S = S_o$

Governing Eq.

$$\frac{d^2T}{dx^2} + \frac{S}{k} = 0 \Rightarrow S = -k\frac{d^2T}{dx^2}$$

Applying BCs.

$$\left.\begin{aligned}&(i)\quad x=0\quad T=T_1 \Rightarrow C_1=0\\ &(ii)\quad x=L\quad T=T_2 \Rightarrow C_2=\frac{1}{L^2}-C_3L\end{aligned}\right\}\Rightarrow \frac{T-T_1}{T_2-T_1}=\left(\frac{1}{L^2}-C_3L\right)x^2+C_3x^3$$

Put the temperature profile in the governing Equation

$$\frac{(T-T_1)}{(T_2-T_1)}=C_1+C_2x^2+C_3x^3$$

$$\frac{dT}{dx}=(T_2-T_1)\left[2C_2x+3C_3x^2\right]$$

$$\frac{d^2T}{dx^2}=(T_2-T_1)\left[2C_2+6C_3x\right]\Rightarrow \frac{d^2T}{dx^2}=(T_2-T_1)\left[2\left(\frac{1}{L^2}-C_3L\right)+6C_3x\right]$$

or $$\frac{d^2T}{dx^2}=(T_2-T_1)\left[\frac{2}{L^2}-C_3L^3\left(\frac{2}{L^2}-\frac{6x}{L^3}\right)\right]$$

Thus, $S=-k\dfrac{d^2T}{dx^2}=k(T_1-T_2)\left[\dfrac{2}{L^2}-C_3L^3\left(\dfrac{2}{L^2}-\dfrac{6x}{L^3}\right)\right]$

$$(iii)\quad x=0\quad S=S_o \Rightarrow S_o=k(T_1-T_2)\left[\frac{2}{L^2}-C_3L^3\left(\frac{2}{L^2}-\frac{6\times 0}{L^3}\right)\right]$$

thus, $C_3=\left(\dfrac{-S_o}{k(T_1-T_2)}+\dfrac{2}{L^2}\right)/2L=\dfrac{1}{L^3}-\dfrac{S_o}{2kL(T_1-T_2)}$

$$\rightarrow S=k(T_1-T_2)\left[\frac{2}{L^2}-\left(\frac{1}{L^3}-\frac{S_o}{2kL(T_1-T_2)}\right)L^3\left(\frac{2}{L^2}-\frac{6x}{L^3}\right)\right]$$

$$\rightarrow S=\frac{2k(T_1-T_2)}{L^2}-\left(\frac{k(T_1-T_2)}{L^2}-\frac{S_o}{2}\right)\left(2-6\frac{x}{L}\right)\ \text{or}\ S=S_o-\left(\frac{3S_o}{L}-\frac{6k(T_1-T_2)}{L^3}\right)x$$

Problem 3:

Given:

ρ = 4500 kg/m^3, k = 22 W/m-K, c = 510 J/kg-K, L =10 cm

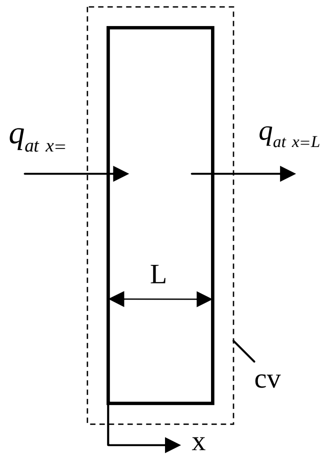
Inside wall **instantaneous** temperature profile: $T = 500 - 2500x + 6000x^2$ [T in K and x in m]

Assumptions: 1-D Unsteady-state heat conduction with Source term

(i): at some time t, the temperature profile is given and S =0 W/m^3. The rate of heat at boundaries of the wall can be estimated:

$$\frac{q_{x=0}}{A_{c.s.}} = q''_{x=0} = -k\left.\frac{dT}{dx}\right|_{x=0} = -22(-2500) = 55000 \text{ W/m}^2$$

$$\frac{q_{x=L}}{A_{c.s.}} = q''_{x=L} = -k\left.\frac{dT}{dx}\right|_{x=L} = -22(-2500 + 12000\times 0.1) = 28600 \text{ W/m}^2$$



Both $q''_{x=0}$ and $q''_{x=L}$ are positive (i.e., heat transfers in positive x direction), thus, heat is added to the wall at x=0, and lost at x=L (see Figure).

As a result:

$$\begin{bmatrix}\text{Net rate of heat transfer to} \\ \text{the plane wall by conduction} \\ \text{across its boundaries} \\ \text{per unit area}\end{bmatrix} = \frac{q_{x=0} - q_{x=L}}{A_{c.s.}} = q''_{x=0} - q''_{x=L}$$

$$= 55000 - 28600 = 26400 \text{ W/m}^2$$

(ii): Governing unsteady 1-D heat conduction in plane wall:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \frac{\partial T}{\partial t}\rho c$$

$\rho, c,$ and k are constant:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2}$$

$$T = 500 - 2500x + 6000x^2 \Rightarrow \frac{\partial^2 T}{\partial x^2} = 12000 \left[\text{K/m}^2\right]$$

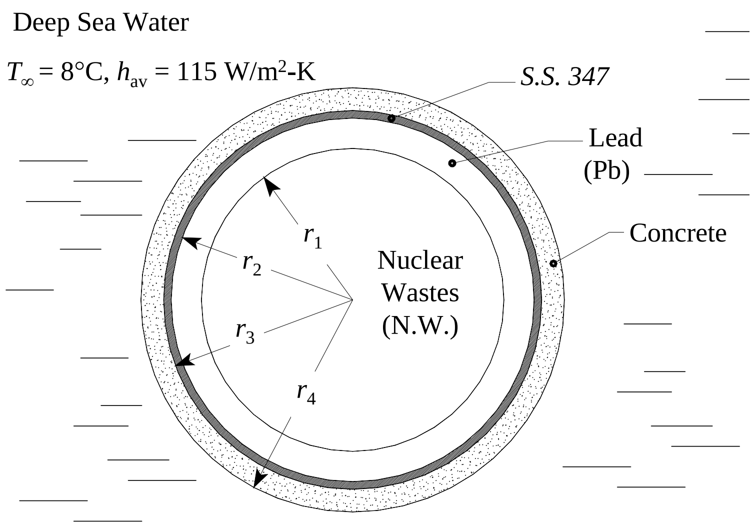
Thus, for a given time, t

$$\frac{\partial^2 T}{\partial x^2} \text{ is constant} \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2} = \frac{22}{4500\times 510}12000 = 0.115\left[\text{K/s}\right]$$

$$\text{Thus, } \frac{\partial T}{\partial t} \text{ is also constant} \Rightarrow \left.\frac{\partial T}{\partial t}\right|_{x=0.05} = 0.115\left[\text{K/s}\right]$$

Problem 4:

Assumptions: 1-D Radial Steady-state Heat transfer; * Constant properties



Given:

$r_1 = 0.2$ m $\quad k_{NW} = 16$ W/m – K

$r_2 = 0.24$ m $\quad k_{PB} = 32.7$ W/m – K

$r_3 = 0.25$ m $\quad k_{SS} = 17.5$ W/m – K

$r_4 = 0.28$ m $\quad k_{Conc.} = 1.1$ W/m – K

$T_{\substack{melting\\Pb}} = 601\text{ K}$

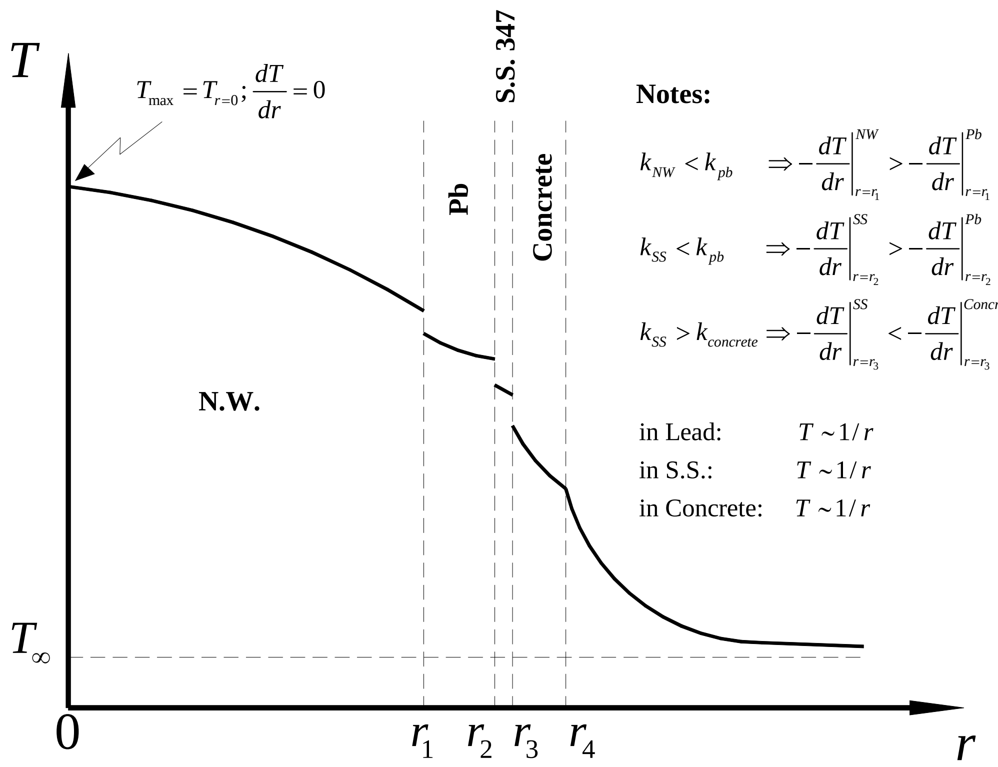
$h_{c,\text{ nuclear wastes-lead}} = h_{c,\text{ lead-ss}} = 2000\text{ W/m}^2\text{-K}$

$h_{c,\text{ ss-concrete}} = 1000\text{ W/m}^2\text{-K}.$

Source Term in N.W.:

$$S = S_o\left[1-\left(\frac{r}{r_1}\right)^2\right] \quad \left[\text{W/m}^3\right]; S_o > 0 \ (S_o \text{ is const.})$$

Design Consideration: Maximum steady-state temp. in Lead must not exceed (601-100) =501K→ $T_{\text{max, in Pb}} \le 227.85$°C



Notes:

$k_{NW} < k_{pb} \quad \Rightarrow -\left.\dfrac{dT}{dr}\right|_{r=r_1}^{NW} > -\left.\dfrac{dT}{dr}\right|_{r=r_1}^{Pb}$

$k_{SS} < k_{pb} \quad \Rightarrow -\left.\dfrac{dT}{dr}\right|_{r=r_2}^{SS} > -\left.\dfrac{dT}{dr}\right|_{r=r_2}^{Pb}$

$k_{SS} > k_{concrete} \Rightarrow -\left.\dfrac{dT}{dr}\right|_{r=r_3}^{SS} < -\left.\dfrac{dT}{dr}\right|_{r=r_3}^{Concrete}$

in Lead: $T \sim 1/r$

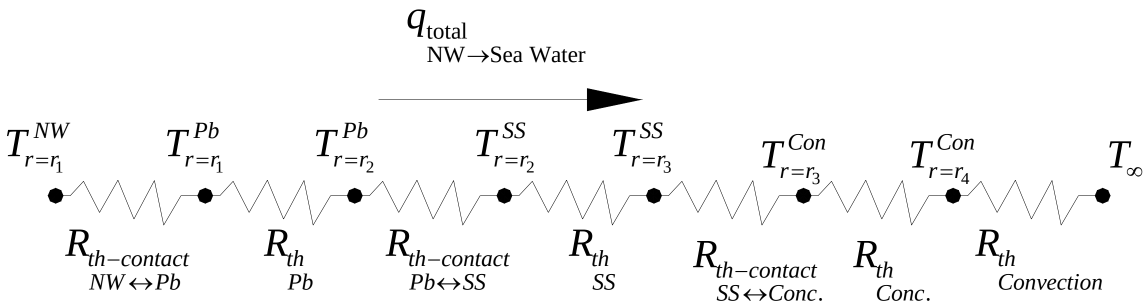
in S.S.: $T \sim 1/r$

in Concrete: $T \sim 1/r$

Part (i)

Part (ii) $S_{o,max}$=?

We have temperature limitation in the Lead (Pb). The highest temperature in Pb occurs at $r = r_1$. Referring to the plot in part (i), we can present the resistance analogy for element outside of the NW (where it applies, as S is zero):



$$q_{\substack{\text{total}\\ \text{NW}\to\text{Sea Water}}} = \frac{T^{NW}_{r=r_1} - T^{Pb}_{r=r_1}}{R_{\substack{th\\ \text{contact}\\ \text{NW}\leftrightarrow\text{Pb}}}} = \frac{T^{Pb}_{r=r_1} - T_\infty}{\sum_{\substack{Pb\to\\ \text{Sea Water}}} R_{th}}$$

$$\sum_{\substack{Pb\to\\ \text{Sea Water}}} R_{th} = R_{\substack{th\\ \text{Pb}}} + R_{\substack{th\\ \text{contact}\\ \text{Pb}\leftrightarrow SS}} + R_{\substack{th\\ \text{SS}}} + R_{\substack{th\\ \text{contact}\\ \text{SS}\leftrightarrow\text{Conc.}}} + R_{\substack{th\\ \text{Conc.}}} + R_{\substack{th\\ \text{Convection}}}$$

$$R_{\substack{th\\ \text{contact}\\ \text{NW}\leftrightarrow Pb}} = \frac{1}{4\pi(0.2)^2\,2000} = 9.947\times10^{-4}\ \text{°C/W}$$

$$R_{\substack{th\\ \text{Pb}}} = \frac{1}{4\pi 32.7}\left[\frac{1}{0.2} - \frac{1}{0.24}\right]$$

$$R_{\substack{th\\ \text{Pb}}} = 2.028\times10^{-3}\ \text{°C/W}$$

$$R_{\substack{th\\ \text{contact}\\ \text{SS}\leftrightarrow\text{Conc.}}} = \frac{1}{4\pi(0.25)^2\,1000}$$

$$R_{\substack{th\\ \text{contact}\\ \text{SS}\leftrightarrow\text{Conc.}}} = 1.273\times10^{-3}\ \text{°C/W}$$

$$R_{\substack{th\\ \text{contact}\\ \text{Pb}\leftrightarrow SS}} = \frac{1}{4\pi(0.24)^2\,2000}$$

$$R_{\substack{th\\ \text{contact}\\ \text{Pb}\leftrightarrow SS}} = 6.908\times10^{-4}\ \text{°C/W}$$

$$R_{\substack{th\\ \text{Conc.}}}\ \frac{1}{4\pi 1.1}\left[\frac{1}{0.25} - \frac{1}{0.28}\right]$$

$$R_{\substack{th\\ \text{Pb}}} = 0.031\ \text{°C/W}$$

$$R_{\substack{th\\ \text{SS}}}\ \frac{1}{4\pi 17.5}\left[\frac{1}{0.24} - \frac{1}{0.25}\right]$$

$$R_{\substack{th\\ \text{SS}}} = 7.579\times10^{-4}\ \text{°C/W}$$

$$R_{\substack{th\\ \text{Convection}}} = \frac{1}{4\pi(0.28)^2\,115}$$

$$R_{\substack{th\\ \text{Convection}}} = 8.826\times10^{-3}\ \text{°C/W}$$

$$\sum_{\substack{Pb\to\\ \text{Sea Water}}} R_{th} = 0.044576\quad \text{°C/W}$$

E-Balance on the NW:

$$q_{\substack{\text{gen}\\ \text{inside}\\ \text{NW}}} = q_{\substack{\text{total}\\ \text{NW}\to\text{Sea Water}}}$$

$$\int_0^{r_1} S_o\left[1-\left(\frac{r}{r_1}\right)^2\right]4\pi r^2 dr = \frac{T^{Pb}_{r=r_1} - T_\infty}{\sum\limits_{\substack{Pb\to\\ \text{Sea Water}}} R_{th}}$$

$$\frac{8}{15}\pi r_1^3 \left(S_o\right)_{Max} = \frac{\left(T^{Pb}_{r=r_1}\right)_{Max} - T_\infty}{\sum\limits_{\substack{Pb\to\\ \text{Sea Water}}} R_{th}}$$

$$\to 0.0134\left(S_o\right)_{Max} = \frac{227.85-8}{0.044576} = 4932.03\text{W} \;= q_{\substack{\text{gen}\\ \text{inside}\\ \text{NW}}} = q_{\substack{\text{total}\\ \text{NW}\to\text{Sea Water}}}$$

$$\to \left(S_o\right)_{Max} = 368061.94 \quad \text{W/m}^3$$

Part (iii) When S_o=$S_{o,max}$, T^{NW}_{Max} =?

When $\mathbf{S_o}$=$\mathbf{S_{o,max}}$: $$q_{\substack{\text{gen}\\ \text{inside}\\ \text{NW}}} = q_{\substack{\text{total}\\ \text{NW}\to\text{Sea Water}}} = \frac{T^{NW}_{r=r_1} - T^{Pb}_{r=r_1}}{R_{\substack{th\\ \text{contact}\\ \text{NW}\leftrightarrow Pb}}}$$

$$\to T^{NW}_{r=r_1} = 4932.03\times 9.947\times 10^{-4} + 227.85 = 232.76°\text{C}$$

To obtain T^{NW}_{Max} which is equal to $T^{NW}_{r=0}$ we need to get the temperature profile inside the NW.

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 k_{NW}\frac{dT}{dr}\right] + S = 0$$

$BC(i) \quad r=0 \quad T \quad$ is Finite

$BC(ii) \quad r=r_1 \quad T = T^{NW}_{r_1} = 232.76°\text{C}$

$$\frac{d}{dr}\left[r^2\frac{dT}{dr}\right] = -\frac{S_o}{k_{NW}}\left[r^2 - \frac{r^4}{r_1^2}\right]$$

Intg. W.R.T. r: $\dfrac{dT}{dr} = \dfrac{-S_o}{k_{NW}}\left[\dfrac{1}{3}r - \dfrac{1}{5}\dfrac{r^3}{r_1^2}\right] + \dfrac{C_1}{r^2}$

Intg. Again W.R.T. r: $T = \dfrac{-S_o}{k_{NW}}\left[\dfrac{1}{6}r^2 - \dfrac{1}{20}\dfrac{r^4}{r_1^2}\right] - \dfrac{C_1}{r} + C_2$

Applying BCs: C_1=0; C_2 =340.11°C

$\to T^{NW}_{Max} = T^{NW}_{r=0} = C_2 = 340.11°\text{C}$