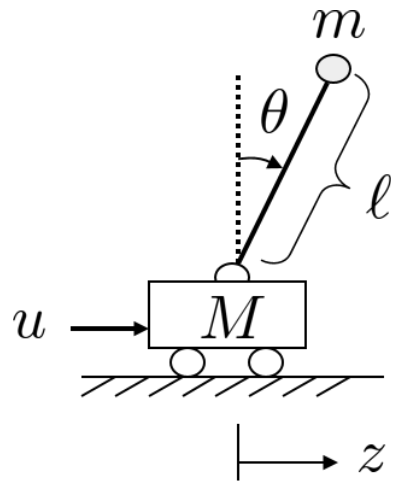
University of British Columbia
Department of Mechanical Engineering

MECH366 Modeling of Mechatronic Systems
Homework 2

Due: September 30 (Monday), 2019, 3pm

Consider the inverted pendulum system below. Here, the input is the force u [N] and the two outputs are the position of the cart z [m] and the pendulum angular position θ [rad]. Other parameters are shown in the figure and below.



ℓ [m]	:	length of the pendulum
m [kg]	:	mass lumped at the top of the pendulum
M [kg]	:	mass of the cart

The equations of motion for this system can be derived as follows:

$$\begin{cases} (M+m)\ddot{z} + (m\ell\cos\theta)\ddot{\theta} = u + m\ell\left(\dot{\theta}\right)^2 \sin\theta \\ (\cos\theta)\ddot{z} + (\ell)\ddot{\theta} = g\sin\theta \end{cases}$$

To answer the following questions, use the equations of motion above. (There is no need to re-derive them. The derivation is given in Appendix.)

1. By defining the states as

$$x_1 := z,\ x_2 = \dot{z},\ x_3 := \theta,\ x_4 := \dot{\theta},$$

obtain the nonlinear state-space model.

2. For an operating point

$$x_0 := \begin{bmatrix} z_0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where z_0 is a constant displacement, derive a linearized state-space model.

Appendix: Derivation of the equations of motion for the ball-and-beam system (not covered in this course)

The Lagrange function is given by

$$L := T - U$$

where kinetic energy T and potential energy U are respectively

$$\begin{aligned}
T &:= \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left\{\frac{d}{dt}(z + \ell\sin\theta)\right\}^2 + \frac{1}{2}m\left\{\frac{d}{dt}(\ell\cos\theta)\right\}^2, \\
&= \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{z} + \ell\cos\theta\cdot\dot{\theta}\right)^2 + \frac{1}{2}m(-\ell\sin\theta\cdot\dot{\theta})^2, \\
&= \frac{1}{2}(M+m)\dot{z}^2 + m\ell\dot{z}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2, \\
U &:= mg\ell\cos\theta.
\end{aligned}$$

The equations of motion can be obtained by using the Euler-Lagrange equation:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= u, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0.
\end{aligned}$$

These equations are calculated as

$$\begin{aligned}
&\frac{d}{dt}\left((M+m)\dot{z} + m\ell\dot{\theta}\cos\theta\right) = u, \\
&\frac{d}{dt}\left(m\ell\dot{z}\cos\theta + m\ell^2\dot{\theta}\right) - \left(-m\ell\dot{z}\dot{\theta}\sin\theta - mg\ell(-\sin\theta)\right) = 0.
\end{aligned}$$

These can be simplified, by dividing the second equation by $m\ell$, as

$$\begin{gathered}
(M+m)\ddot{z} + m\ell\cos\theta\cdot\ddot{\theta} = u + m\ell\left(\dot{\theta}\right)^2\sin\theta, \\
\cos\theta\ddot{z} - \dot{z}\dot{\theta}\sin\theta + \ell\ddot{\theta} + \dot{z}\dot{\theta}\sin\theta - g\sin\theta = 0.
\end{gathered}$$

By simplifying the second equation further, we can obtain the final form of the equations of motion as

$$\begin{aligned}
(M+m)\ddot{z} + m\ell\cos\theta\cdot\ddot{\theta} &= u + m\ell\left(\dot{\theta}\right)^2\sin\theta, \\
(\cos\theta)\ddot{z} + \ell\ddot{\theta} &= g\sin\theta.
\end{aligned}$$