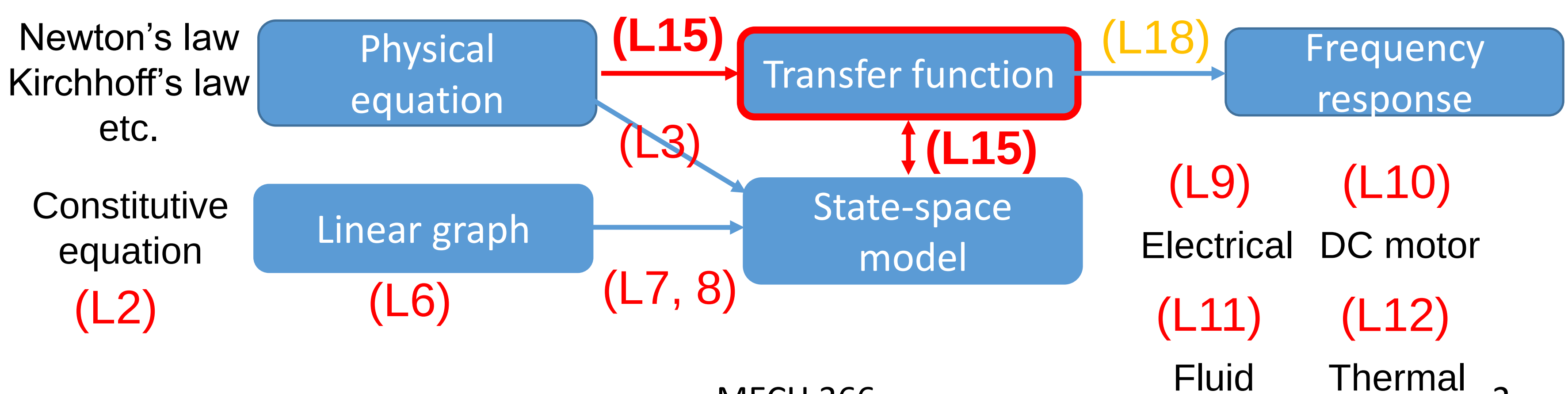
# MECH366 : Modeling of Mechatronic Systems

## L15 : Transfer function

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

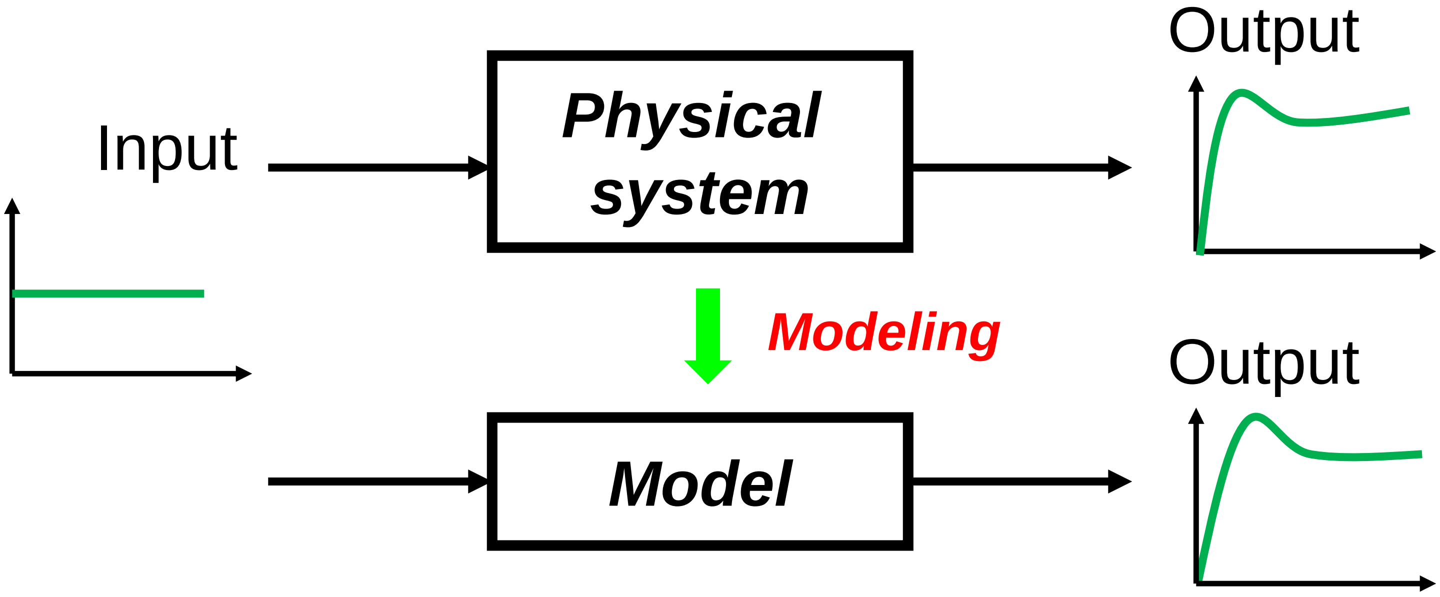
# Review and today's topic

- Up to now, we have studied state-space modeling based on linear graphs, and Laplace transform.
- Today, we will learn another type of models, i.e. transfer functions.
- Various models and their relations



# Model and modeling (Review)

- Model: Representation of input-output (signal) relationship of a system
- Modeling: Process to derive models



# Outline

- Definition of the transfer function
- An advantage of using the transfer function
- Transformation from the state-space model into the transfer function
- Some examples

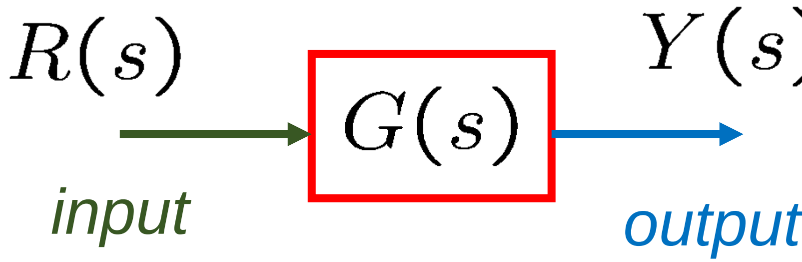
# Transfer function: Definition

- A transfer function is defined by

$$G(s) := \frac{Y(s)}{R(s)}$$

*Laplace transform of system output*

*Laplace transform of system input*



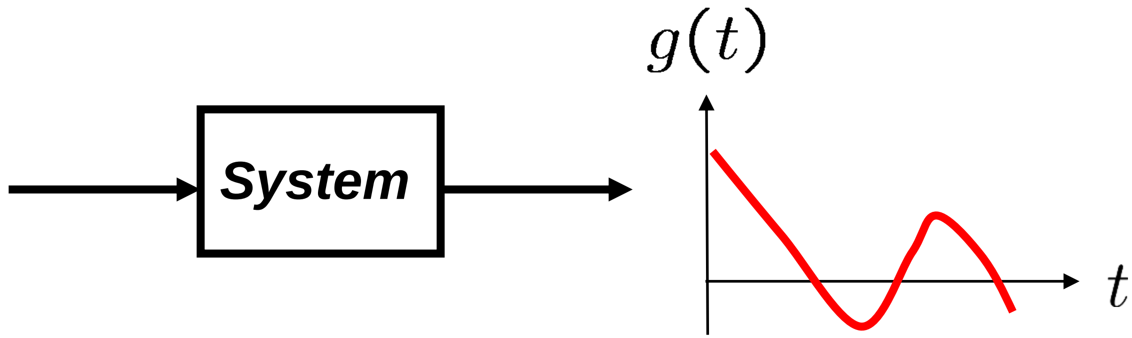

$$Y(s) = G(s)R(s)$$

- A system is assumed to be at rest. (zero initial condition)
- Transfer function is a generalization of "gain" concept.

# Transfer function via unit impulse response

- Suppose that *r(t)* is the unit impulse function and system is at rest.

$$r(t) = \delta(t)$$
$$R(s) = 1$$




- The output *g(t)* for the unit impulse input is called *unit impulse response*.
- Since *R(s)=1*, the transfer function can also be defined as the Laplace transform of unit impulse response:

$$G(s) := \mathcal{L}\{g(t)\}$$

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# Advantages of *s*-domain (transfer function)

- We can transform an ordinary differential equation into an algebraic equation which is easy to solve.

  (Today's class)

- It is easy to analyze and design interconnected (series, parallel, feedback etc.) systems.

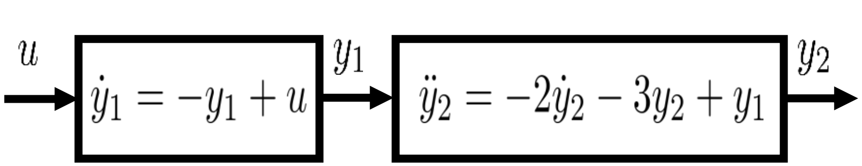
  (In classical control such as MECH467, next slide)

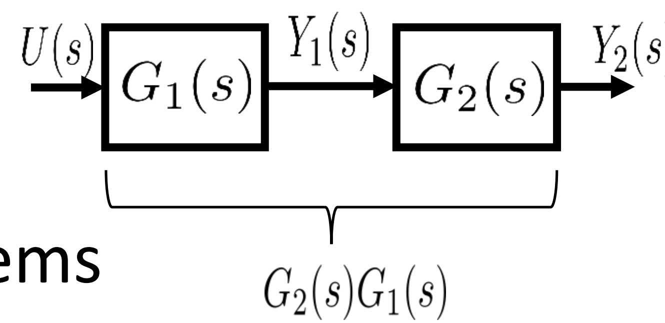
- Frequency domain information of signals can be dealt with.

  (Frequency responses)

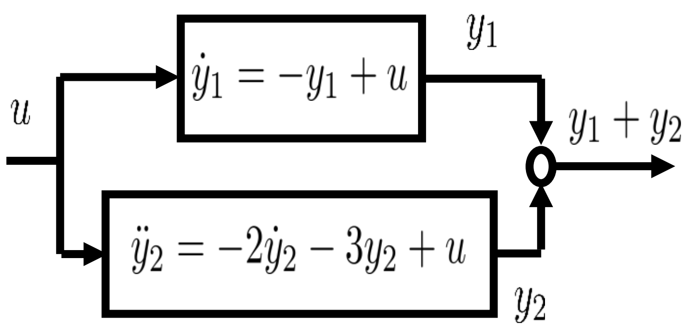
# Examples of interconnected systems

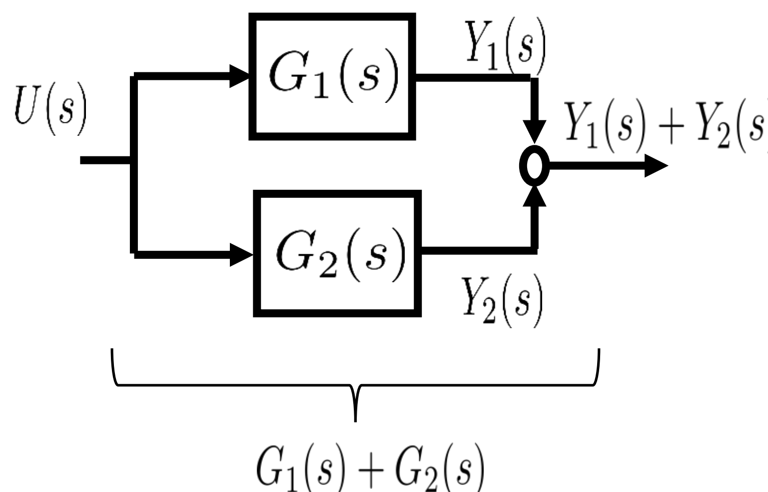
- Series connection of two systems




- Parallel connection of two systems



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# State-space model to transfer function (Matlab: tf(ss(A,B,C,D))

- Linear time-invariant (LTI) state-space model

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

- Laplace transform with *x(0)=0*

$$\begin{cases} sX(s) - \cancel{x(0)} &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

*Transfer function*

$$\Rightarrow Y(s) = \underbrace{\boxed{\{C(sI - A)^{-1}B + D\}}}_{=:G(s)}U(s)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Transfer function to state-space model

- The transformation from transfer function model to state-space model is called realization, and it is beyond the scope of this course.
- For a given transfer function, there are infinitely many state-space models.
- The realization theory will be covered in MECH468: Modern Control Engineering (elective for MECH undergrads).

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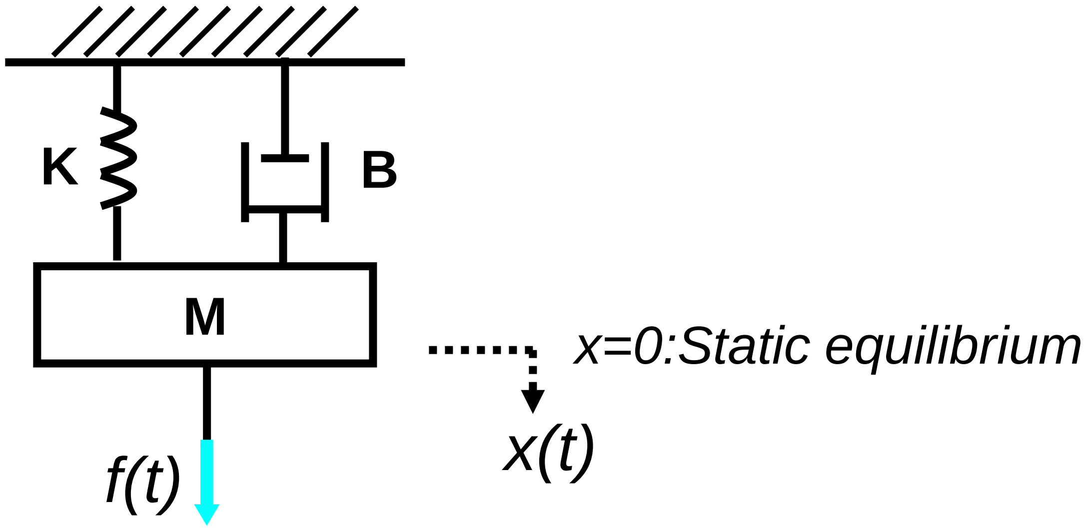
# Ex: Mass-spring-damper system



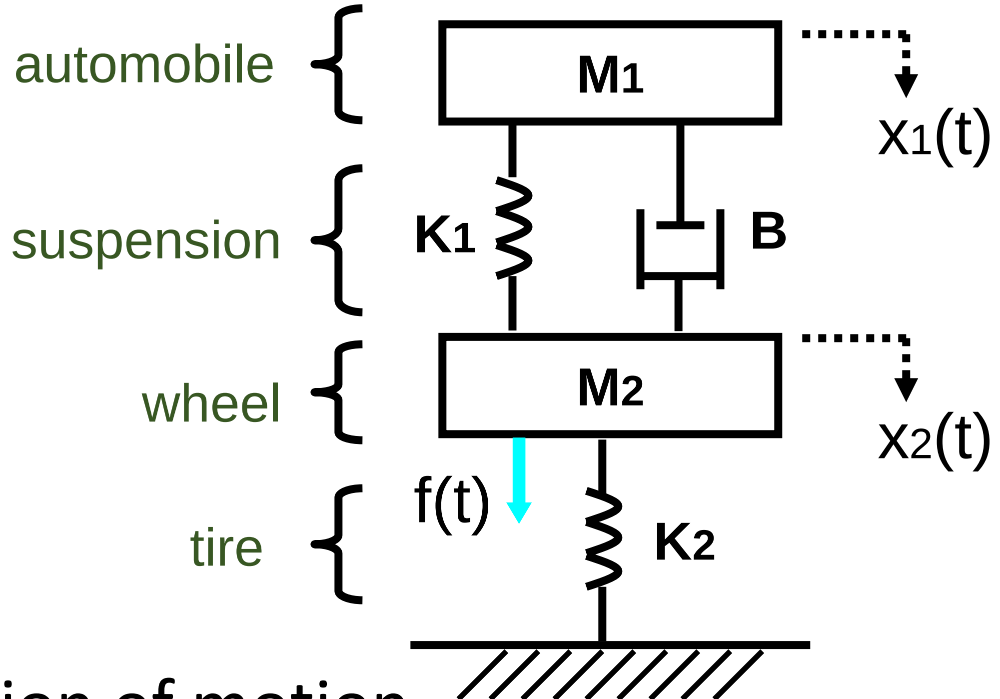

- Equation of motion by Newton's 2nd law

$$Mx''(t) = f(t) - Bx'(t) - Kx(t)$$

- By Laplace transform (with zero initial conditions),

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \quad \textit{(2}^{\textit{nd}}\textit{ order system)}$$

# Ex: Automobile suspension system (Quarter-car model)




- Equation of motion

$$\begin{cases} M_1 x_1''(t) &= -B(x_1'(t) - x_2'(t)) - K_1(x_1(t) - x_2(t)) \\ M_2 x_2''(t) &= f(t) - B(x_2'(t) - x_1'(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t) \end{cases}$$

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Laplace transform with zero ICs

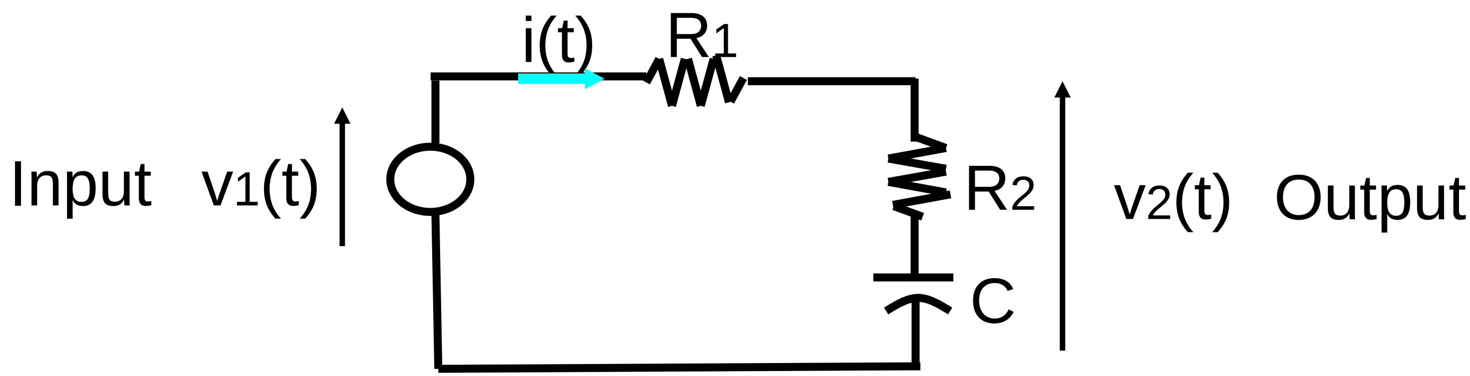
$$\begin{cases} M_1 s^2 X_1(s) &= -B(sX_1(s) - sX_2(s)) - K_1(X_1(s) - X_2(s)) \\ M_2 s^2 X_2(s) &= F(s) - B(sX_2(s) - sX_1(s)) - K_1(X_2(s) - X_1(s)) - K_2 X_2(s) \end{cases}$$

$$\begin{cases} X_1(s) &= \underbrace{\frac{Bs + K_1}{M_1 s^2 + Bs + K_1}}_{G_1(s)} X_2(s) \\ X_2(s) &= \underbrace{\frac{1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_2(s)} F(s) + \underbrace{\frac{Bs + K_1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_3(s)} X_1(s) \end{cases}$$

$$\Rightarrow \quad \frac{X_1(s)}{F(s)} = \frac{G_1(s) G_2(s)}{1 - G_1(s) G_3(s)}$$

Always $\deg(\mathrm{den}) \geq \deg(\mathrm{num})$

# Ex: RC circuit



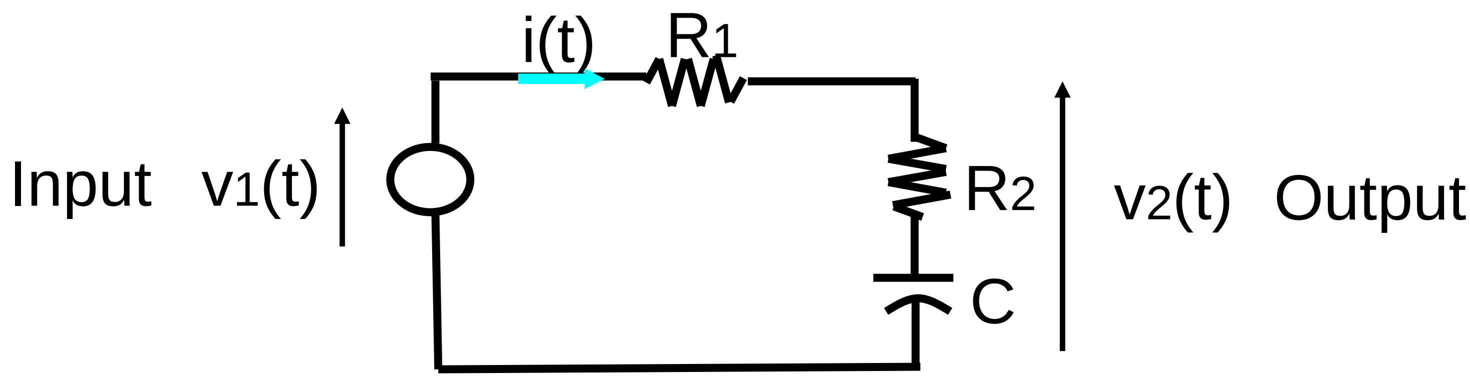

- Kirchhoff voltage law (with zero initial conditions)

$$
\begin{aligned}
v_1(t) &= (R_1 + R_2)i(t) + \tfrac{1}{C}\int_0^t i(\tau)d\tau \\
v_2(t) &= R_2 i(t) + \tfrac{1}{C}\int_0^t i(\tau)d\tau
\end{aligned}
$$

- By Laplace transform,

$$
\begin{aligned}
V_1(s) &= (R_1 + R_2)I(s) + \tfrac{1}{sC}I(s) \\
V_2(s) &= R_2 I(s) + \tfrac{1}{sC}I(s)
\end{aligned}
$$

# Ex: RC circuit




- Transfer function

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{(R_1 + R_2) + \frac{1}{sC}}$$

$$= \frac{R_2 C s + 1}{(R_1 + R_2) C s + 1} \quad \text{(first-order system)}$$

- If output is *i(t)*, then ....

# Summary

- Transfer function
  - Definition
  - State-space model to transfer function
  - Examples in mechanical and electrical systems
- Next,
  - Transfer function for DC motor
  - Block diagram
- Homework 5: Due Nov 4 (Monday), 3pm
- Lab 3 report: Due today, 6pm
- Lab 4: November 8