Inverse Problems in Geophysics
Part 7: Non-linear inversion

2. MGPY+MGIN

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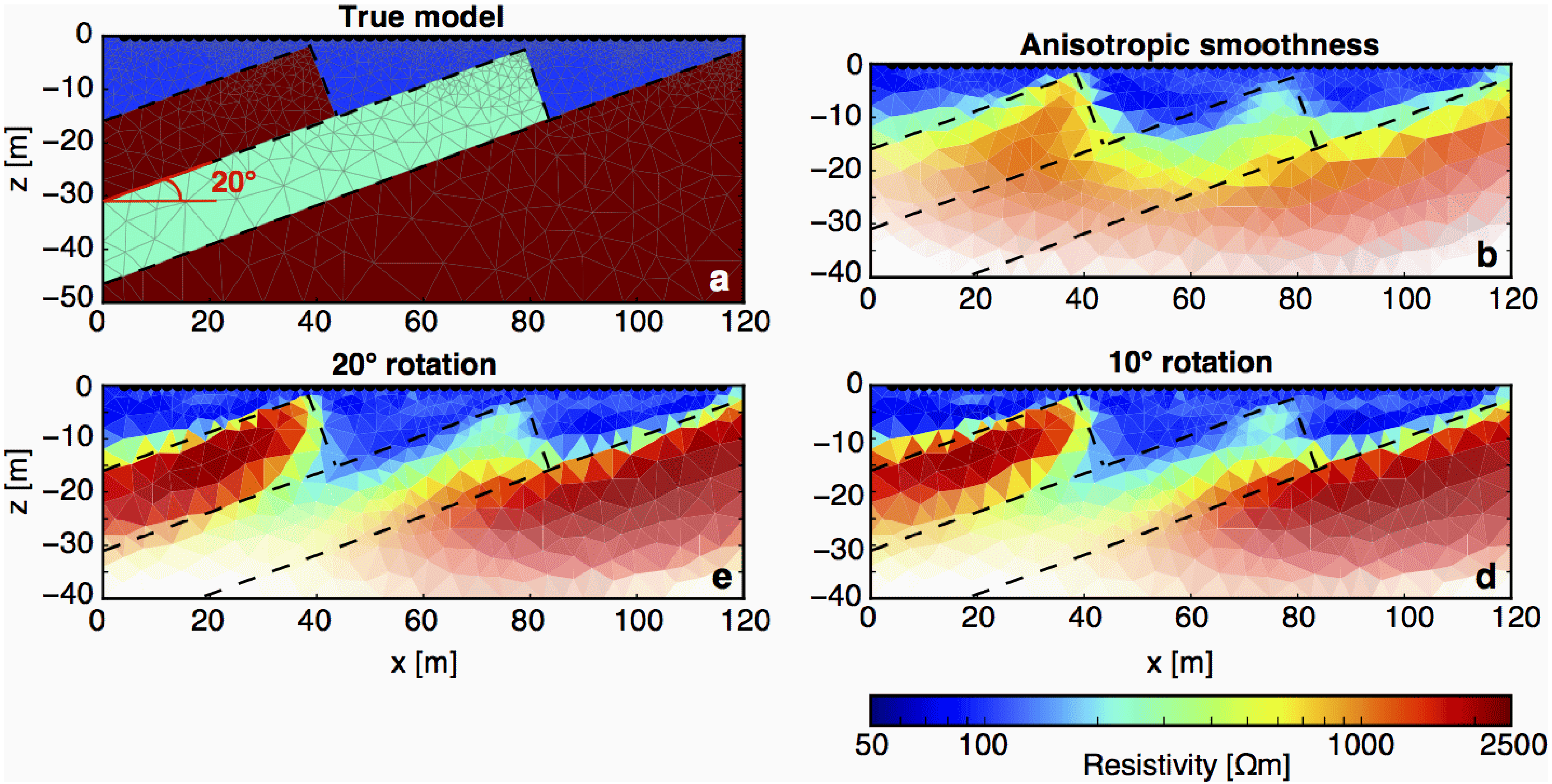
Modern regularization

> ⓘ **Regularization is key to subsurface imaging**

$\Rightarrow$ Add any information about the subsurface:

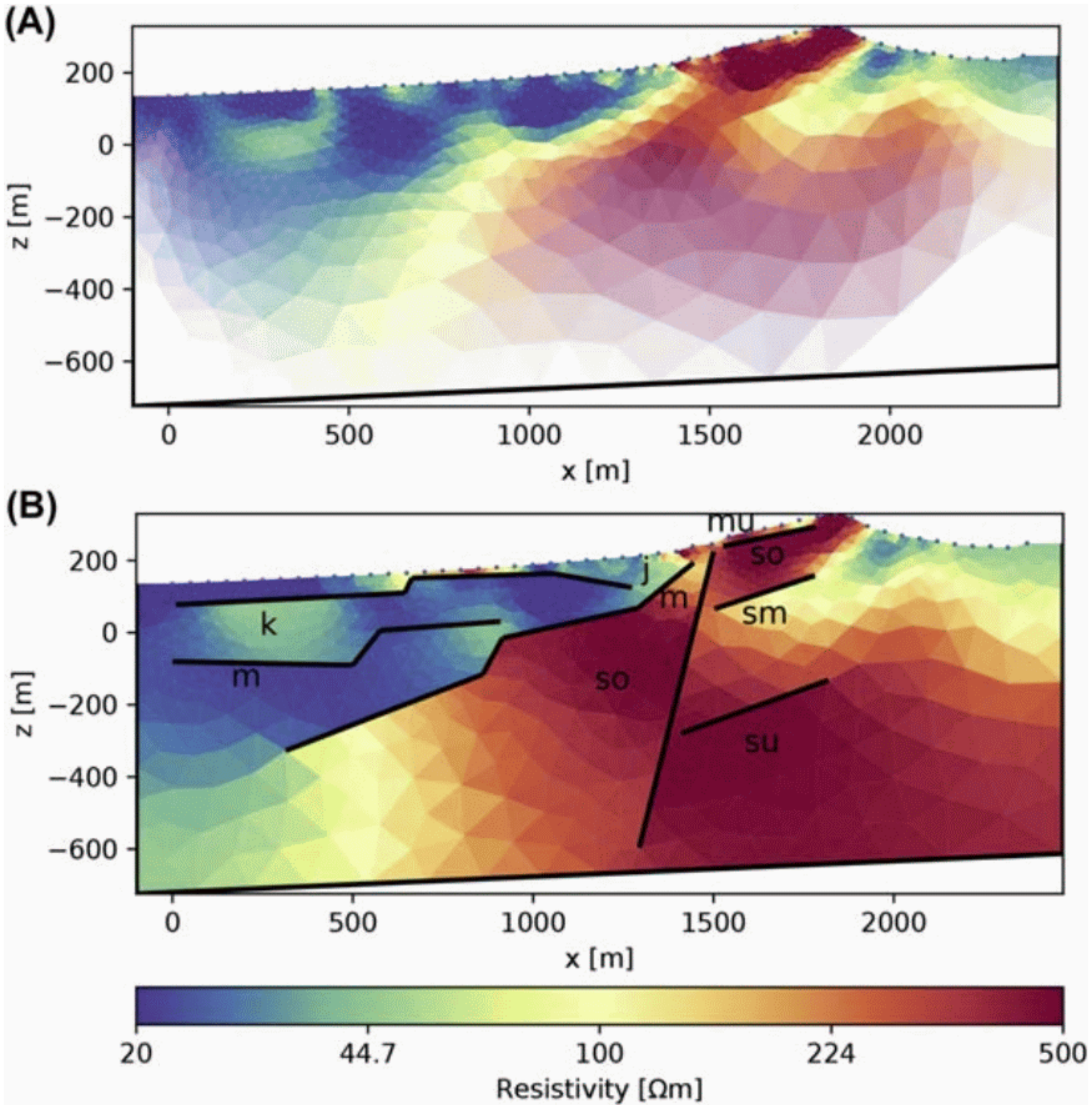
- correlation lengths and angles
- structural boundaries (from boreholes, seismics, GPR)
- point information (samples, borehole logs,)
- limits of the possible parameters (e.g. positivity, natural bounds)

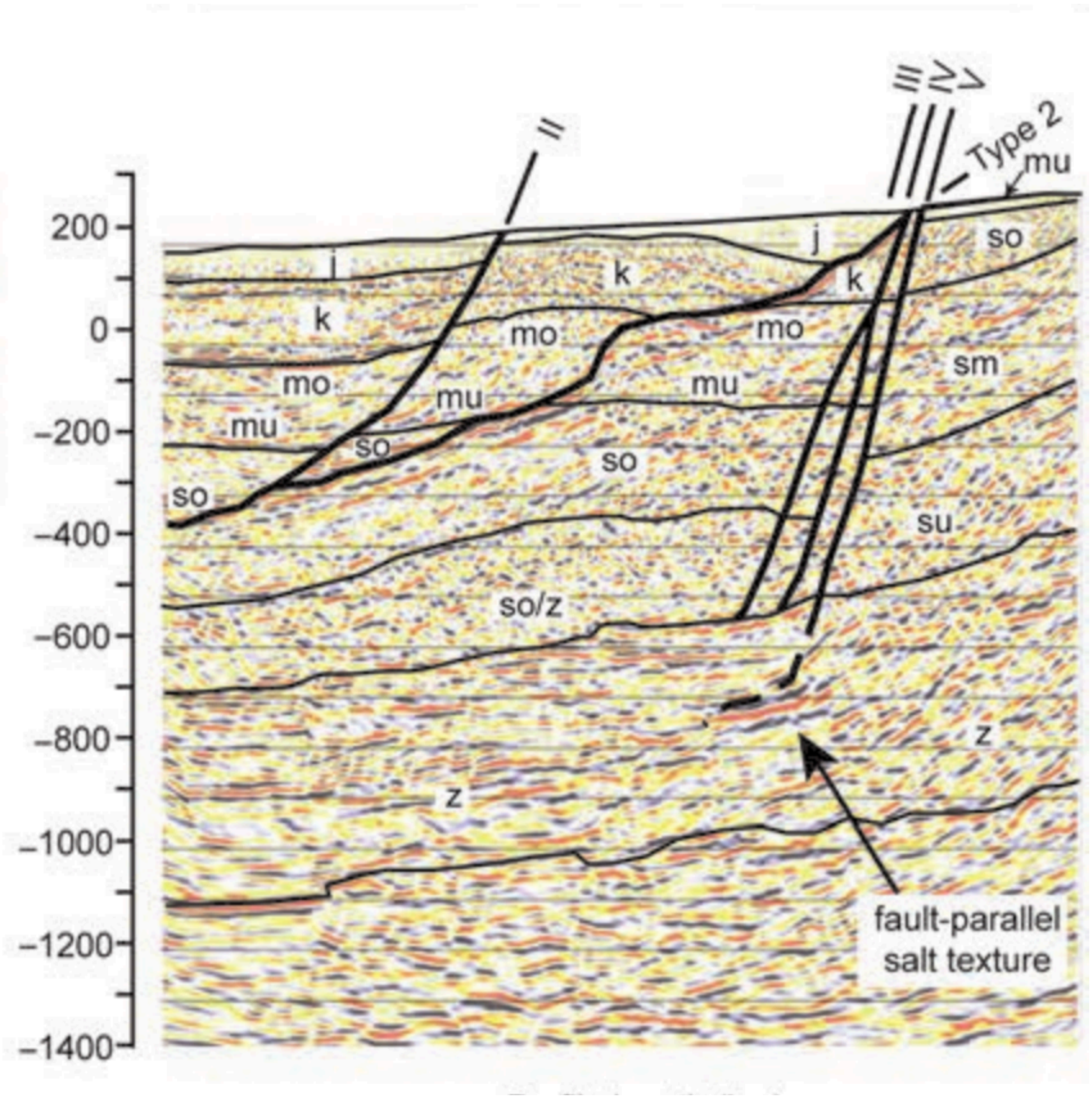
Geostatistical regularization



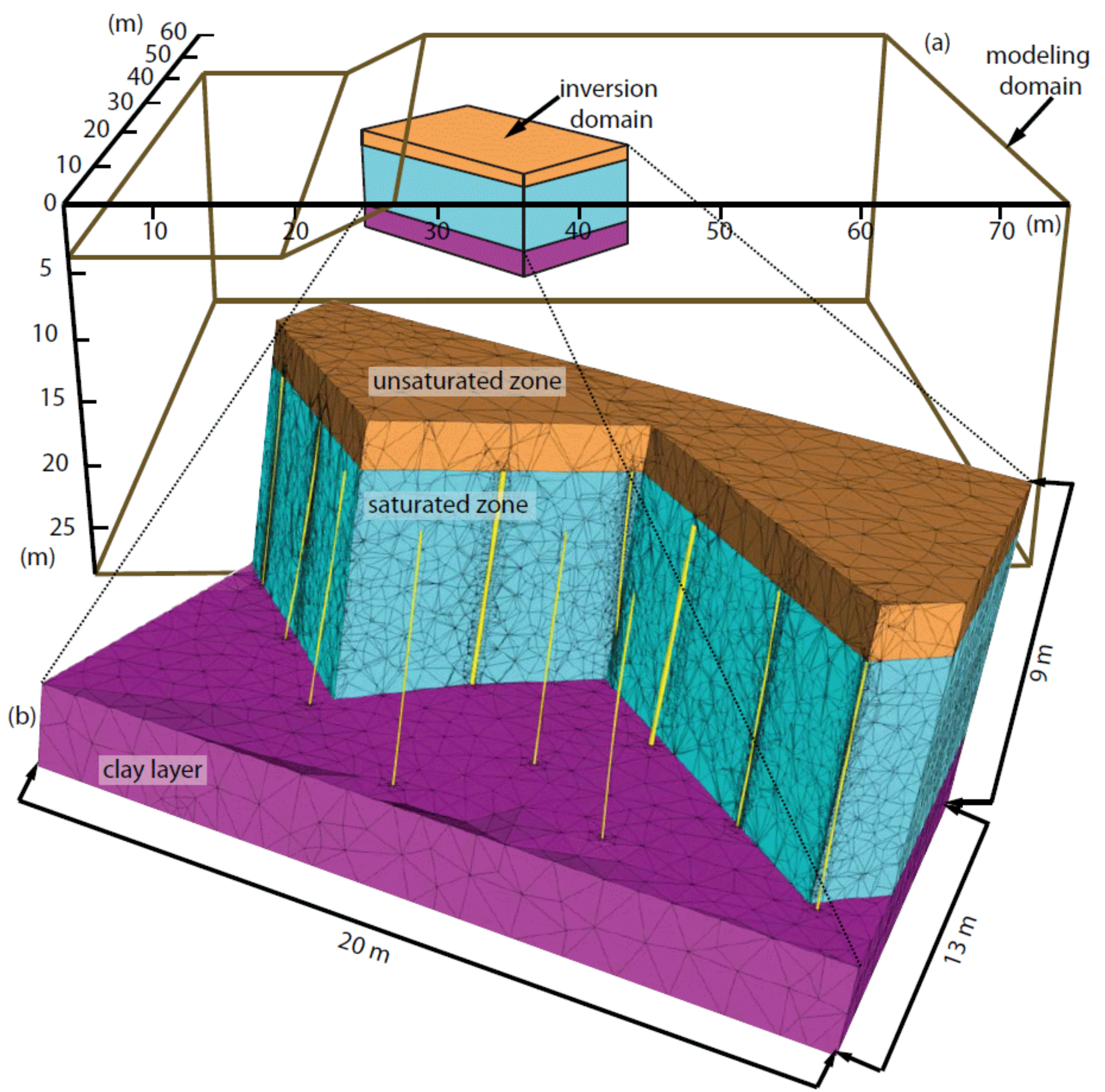
Jordi et al. (2018)

Structural constraints

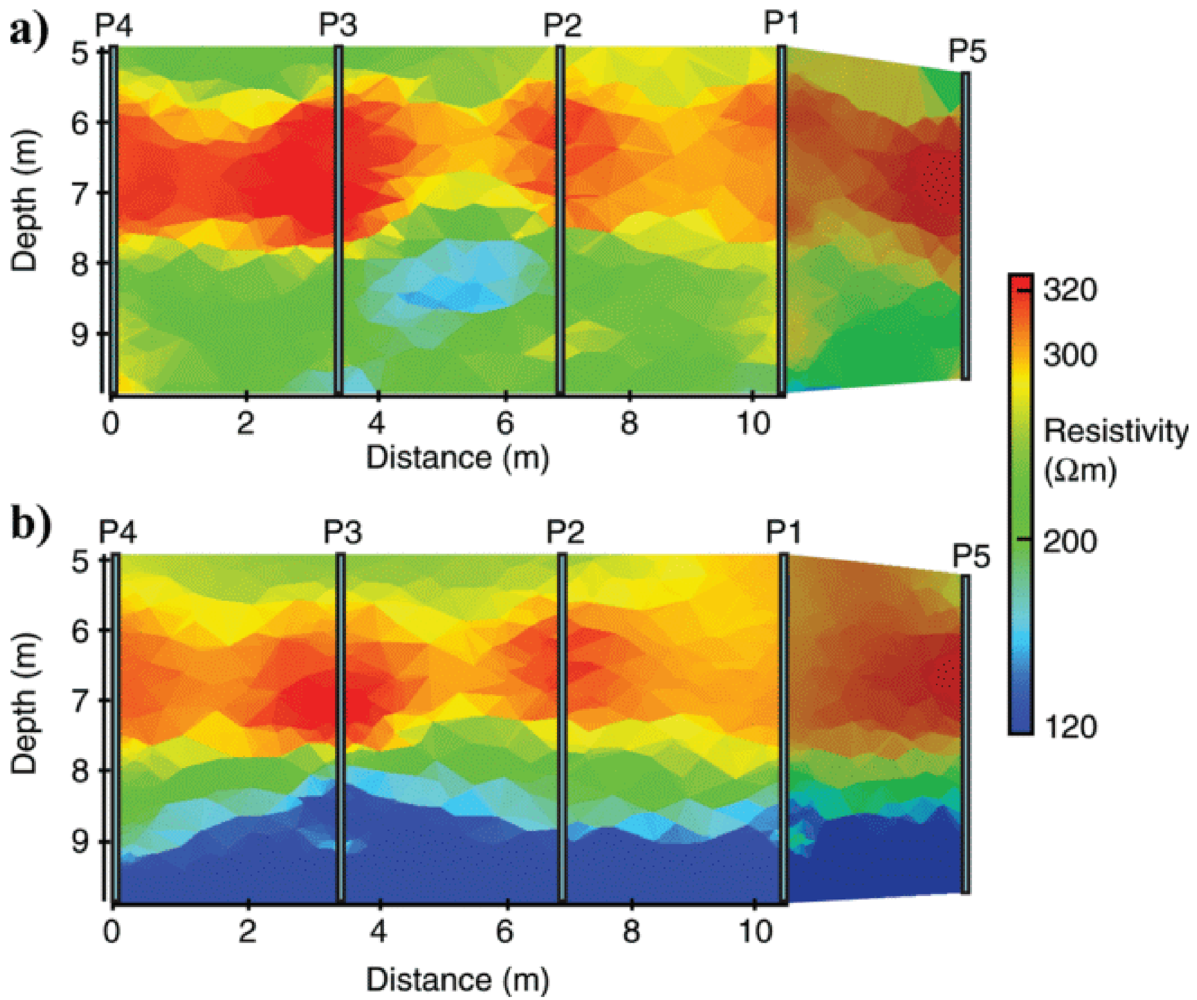

(A)
z [m]
x [m]
(B)
k
m
j
mu
so
sm
su
Resistivity [Ωm]
20
44.7
100
224
500


Type 2
mu
so
j
k
mo
mu
sm
su
so/z
z
fault-parallel
salt texture

Region-wise settings

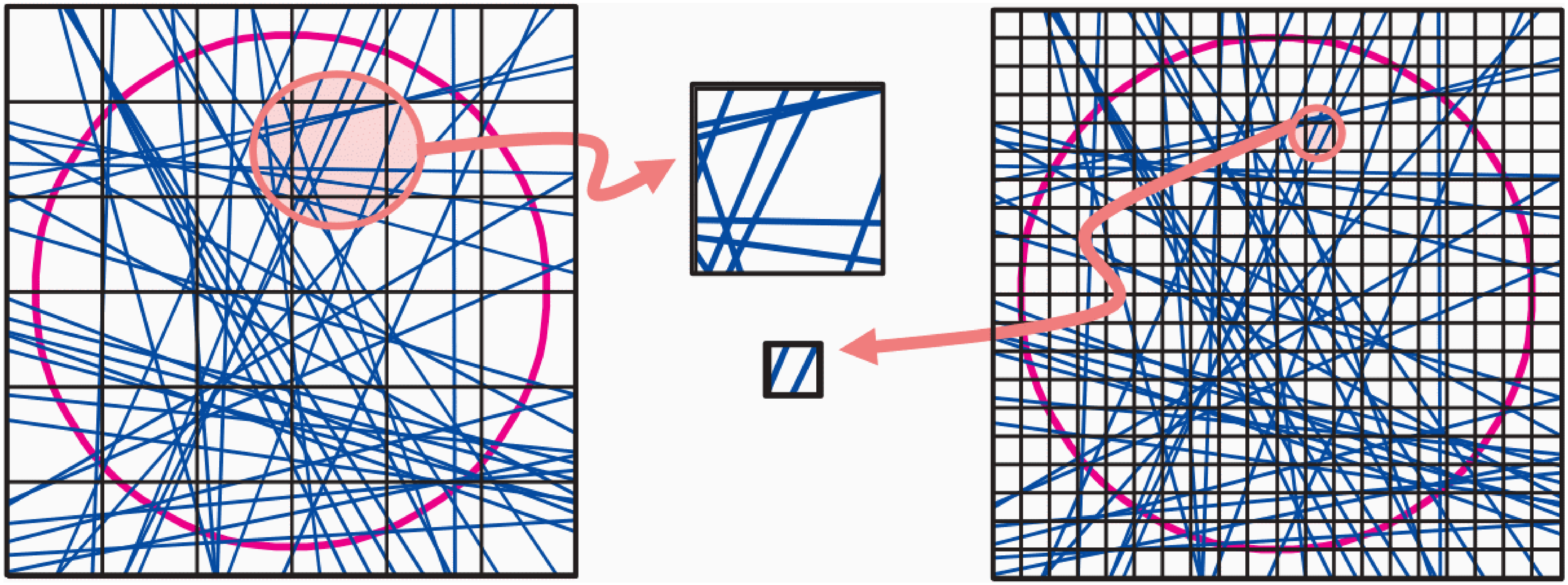


(a)
modeling domain
inversion domain
unsaturated zone
saturated zone
clay layer
(b)
20 m
13 m
9 m



a)
b)
P4
P3
P2
P1
P5
Depth (m)
Distance (m)
Resistivity (Ωm)
320
300
200
120

Discretization (Menke, 2012)



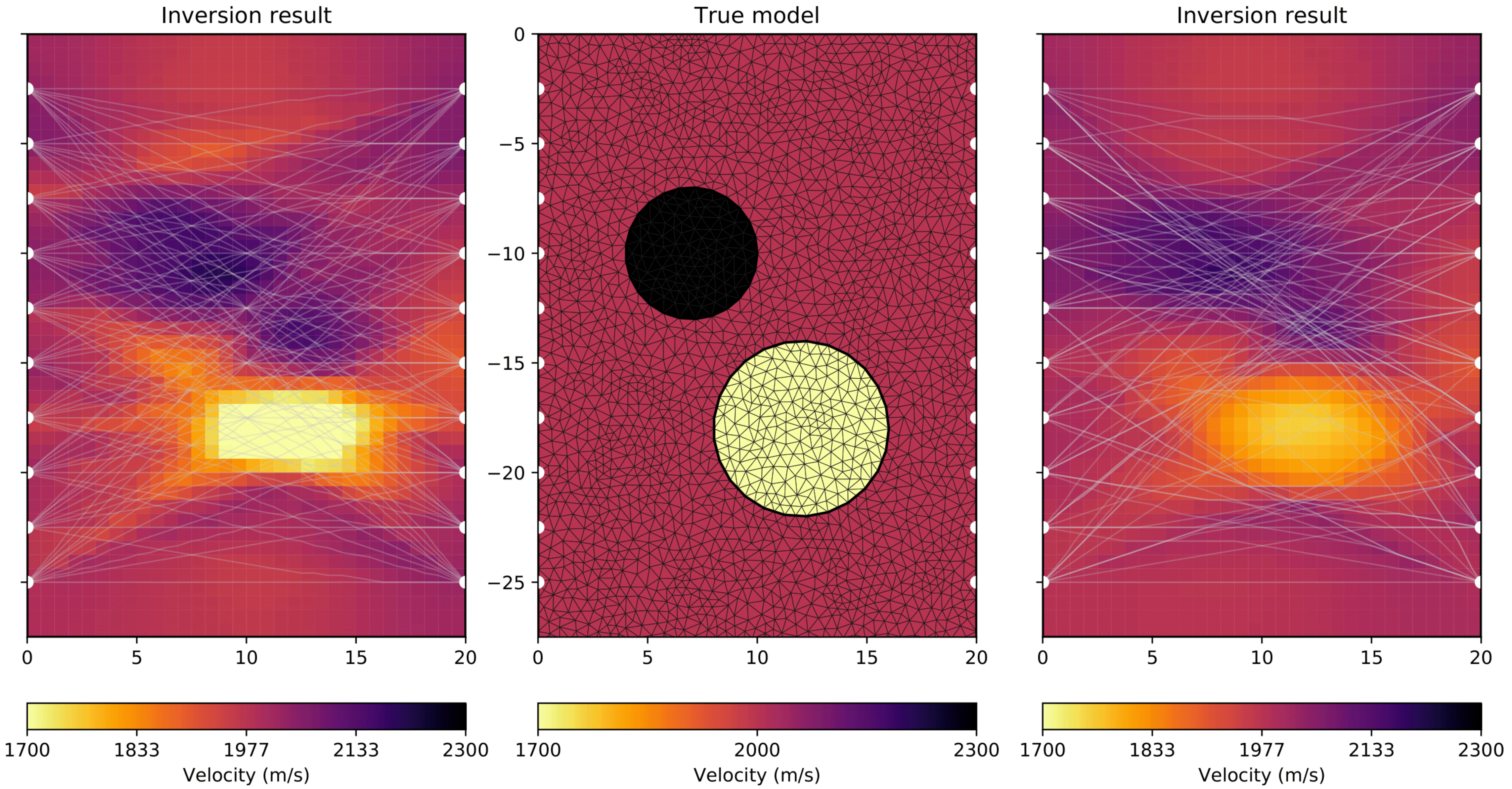
Non-linear inversion

Up to now: $\mathbf{f}(\mathbf{m}) = \mathbf{G}\mathbf{m}$ (with constant $\mathbf{G}$)

Now: either there is no $\mathbf{G}$ or $\mathbf{G}$ depends on $\mathbf{m}$

Example: traveltime tomography with curved rays

Comparison straight vs. curvey rays



Linearization of non-linears

Non-linear problem $y = y_0 e^{\beta x}$ with $\mathbf{m} = (y_0, \tau)^T$

Data transformation: $\hat{y} = \log y = \log y_0 + \beta x$

$\Rightarrow$ Application of linear inverse theory

Logarithmic model transform (positivity constraint)

$\hat{m}_i = \log m_i \Rightarrow$ non-linear problem for $\hat{\mathbf{m}}$ even if linear for $\mathbf{m}$

Linearization using Taylor series

One-dimensional function $f(x)$ of single parameter x:

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + f''(x_0)/2 \cdot (x - x_0)^2 + \ldots$$

Multi-dimensional function of model parameters $m_1, m_2, \ldots$:

$$f(\mathbf{m}) = f(\mathbf{m}_0 + \mathbf{\Delta m}) \approx f(\mathbf{m}_0) + \sum_i^M \frac{\partial f(\mathbf{m}_0)}{\partial m_j} \Delta m_j$$

$$\mathbf{f}(\mathbf{m}_0 + \mathbf{\Delta m}) = \mathbf{f}(\mathbf{m}_0) + \sum_i^M \frac{\partial \mathbf{f}(\mathbf{m}_0)}{\partial m_j} \Delta m_j = \mathbf{S \Delta m}$$

Linearization

$$\mathbf{f}(\mathbf{m}_0 + \boldsymbol{\Delta}\mathbf{m}) = \mathbf{f}(\mathbf{m}_0) + \sum_i^M \frac{\partial \mathbf{f}(\mathbf{m}_0)}{\partial m_j} \Delta m_j = \mathbf{f}(\mathbf{m}_0) + \mathbf{S}\boldsymbol{\Delta}\mathbf{m}$$

We hope the change in the model fits our data: $\mathbf{f}(\mathbf{m}_0 + \boldsymbol{\Delta}\mathbf{m}) \approx \mathbf{d}$

This leads to a linearized problem

$$\mathbf{S}\boldsymbol{\Delta}\mathbf{m} = \mathbf{d} - \mathbf{f}(\mathbf{m}_0) = \boldsymbol{\Delta}\mathbf{d}$$

$\mathbf{S}(\mathbf{m}_0)$: sensitivity matrix of partial derivatives $S_{ij} = \frac{\partial f_i(\mathbf{m}_0)}{\partial m_j}$

Iterative solution of non-linear problems

Starting with a model $\mathbf{m}_0$, we iteratively $(n = 0, 1, \ldots)$ improve the model by

$$\mathbf{m}^{n+1} = \mathbf{m}^n + \Delta\mathbf{m}^n$$

by solving the linear sub-problem

$$\mathbf{S}^n \Delta\mathbf{m}^n = \Delta\mathbf{d}^n = \mathbf{d} - \mathbf{f}(\mathbf{m}^n)$$

The objective function

$$\Phi_d = \|\mathbf{d} - \mathbf{f}(\mathbf{m})\|^2 = (\mathbf{d} - \mathbf{f}(\mathbf{m}))^T \cdot \mathbf{d} - \mathbf{f}(\mathbf{m})$$

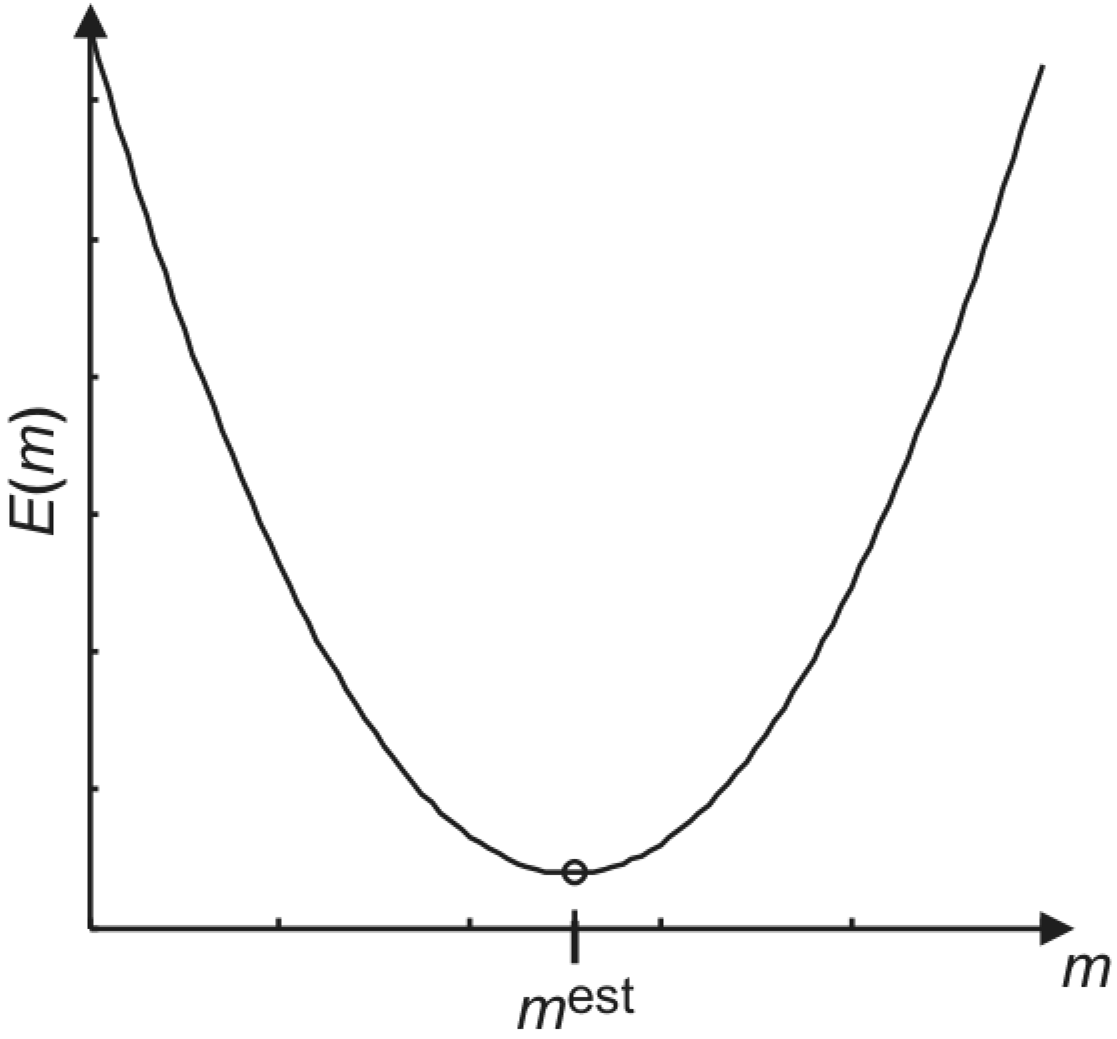
Topography of Φ_d determines non-linearity (s. Notebook on VES)

Objective function for linear problems

$$\Phi_d = (\mathbf{d} - \mathbf{Gm})^T(\mathbf{d} - \mathbf{Gm}) =$$

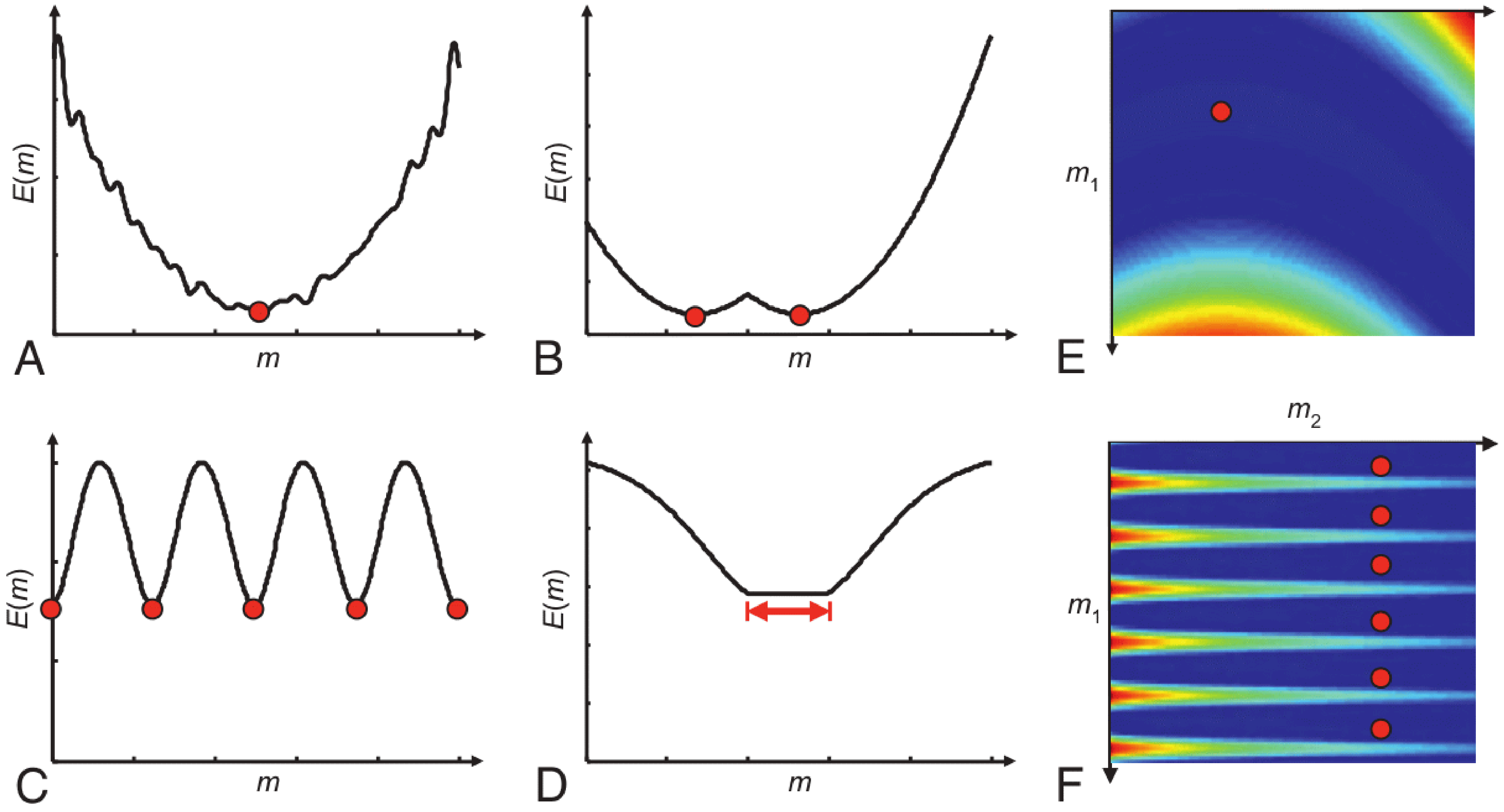
$$\mathbf{d}^T\mathbf{d} + \mathbf{m}^T\mathbf{G}^T\mathbf{Gm} - 2\mathbf{d}^T\mathbf{Gm}$$

Parabola for all m_i



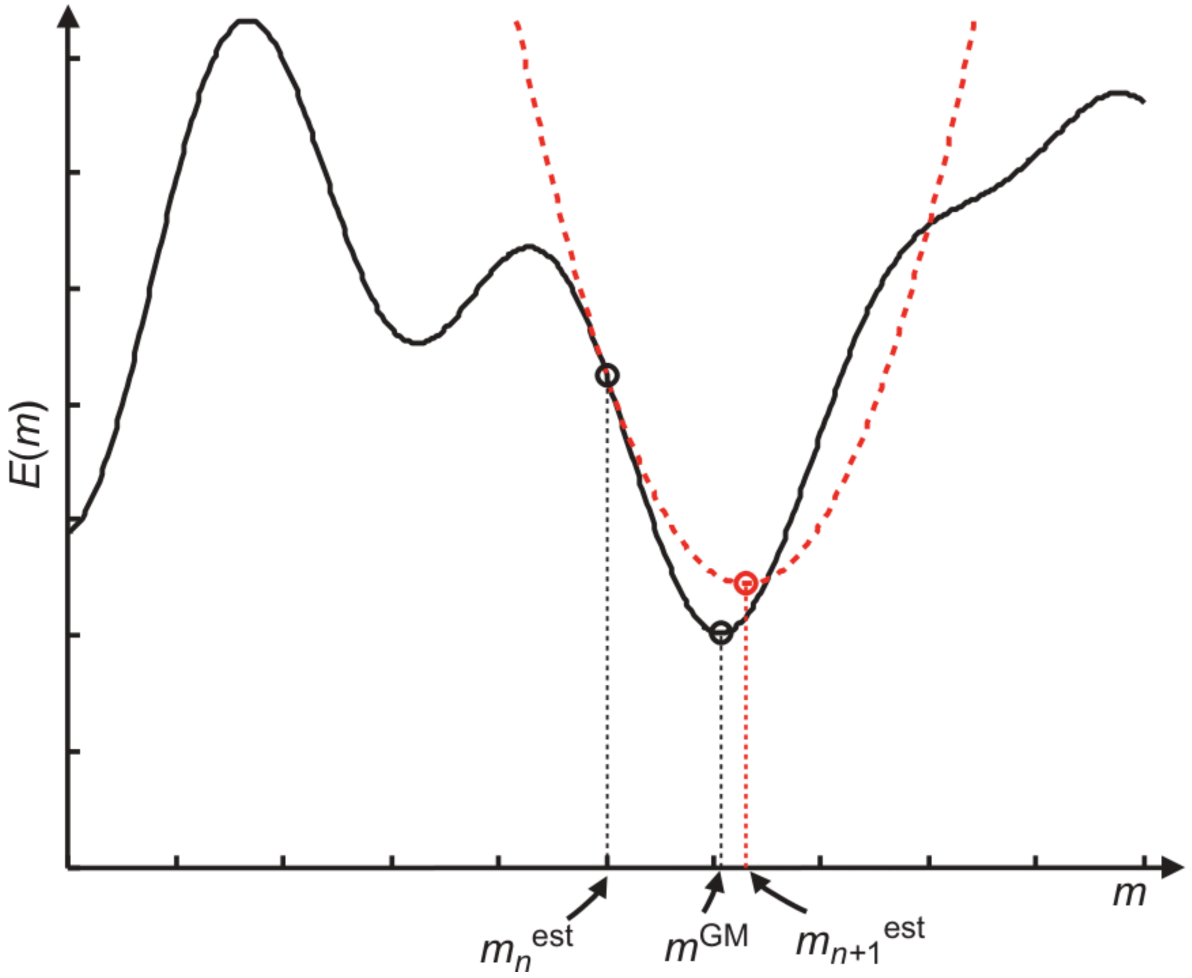
Menke (2012)

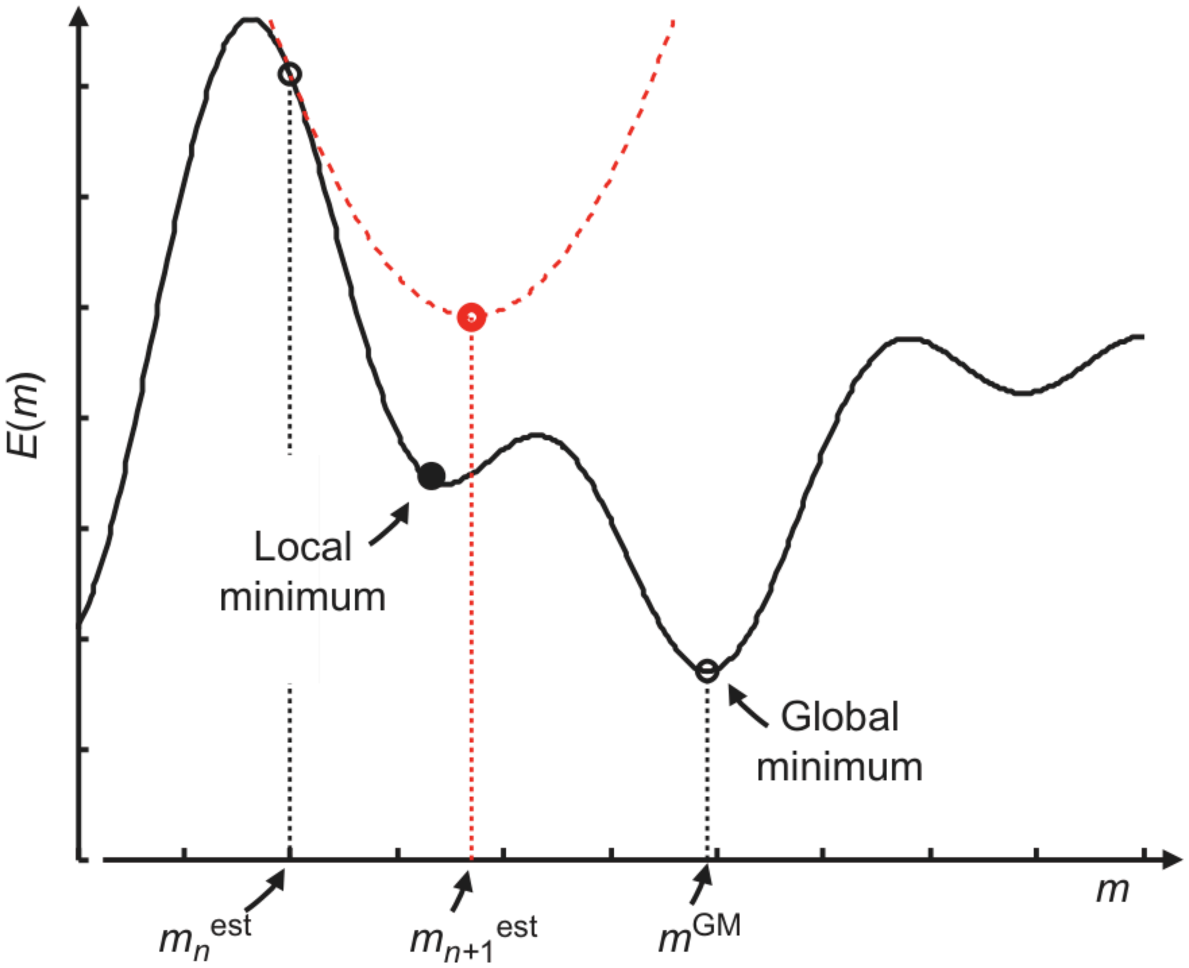
Misfit for non-linear problems (Menke, 2012)



A single minimum
B multiple minima
C periodics
D broad range
E flat valley
F separated valleys

Linearization in the objective (Menke, 2012)





Computation of the sensitivity (Jacobian)

- analytically (derivation of the forward operator)
- transforming the PDE and its numerical solution
- perturbation method (brute force)

$$\frac{\partial f_i(\mathbf{m})}{\partial m_j} \approx \frac{f_i(\mathbf{m} + \delta_j \Delta m) - f_i(\mathbf{m})}{\Delta m}$$

Line search for strong non-linearity

The Taylor approximation might change along $\mathbf{\Delta m}$, so we optimize the step length

$$\mathbf{m}^{n+1} = \mathbf{m}^n + \tau^n \mathbf{\Delta m}^n$$

Summary

- most geophysical problems are non-linear
- linearization around given model
- iterative improvement of the model