Numerical Simulation Methods in Geophysics, Exercise 7: Timestepping with FE

1. MGPY+MGIN

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Recap Poisson and heat equations

Recap

- ☑ solve the Poisson equation for arbitrary x and a
- ☑ sources and a contrasts cause curvature in u
  - positive source or a increase $\Rightarrow$ negative u” $\Rightarrow$ maximum
  - single $f \Rightarrow$ piecewise linear, full $f \Rightarrow$ parabola
- ☑ Dirichlet BC determine shift (& slope if double)
- ☑ Neumann BC determine slope of u
- ☑ accuracy (compare analytical) depends on discretization
- ☑ now go for instationary (parabolic) problem by time stepping
  - curvature in u causes negative change of u

Time stepping

Time stepping - explicit method

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial z^2} = 0$$

Finite-difference approximation

$$\frac{\partial T}{\partial t}^{n} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = a\frac{\partial^2 T^{n}}{\partial z^2}$$

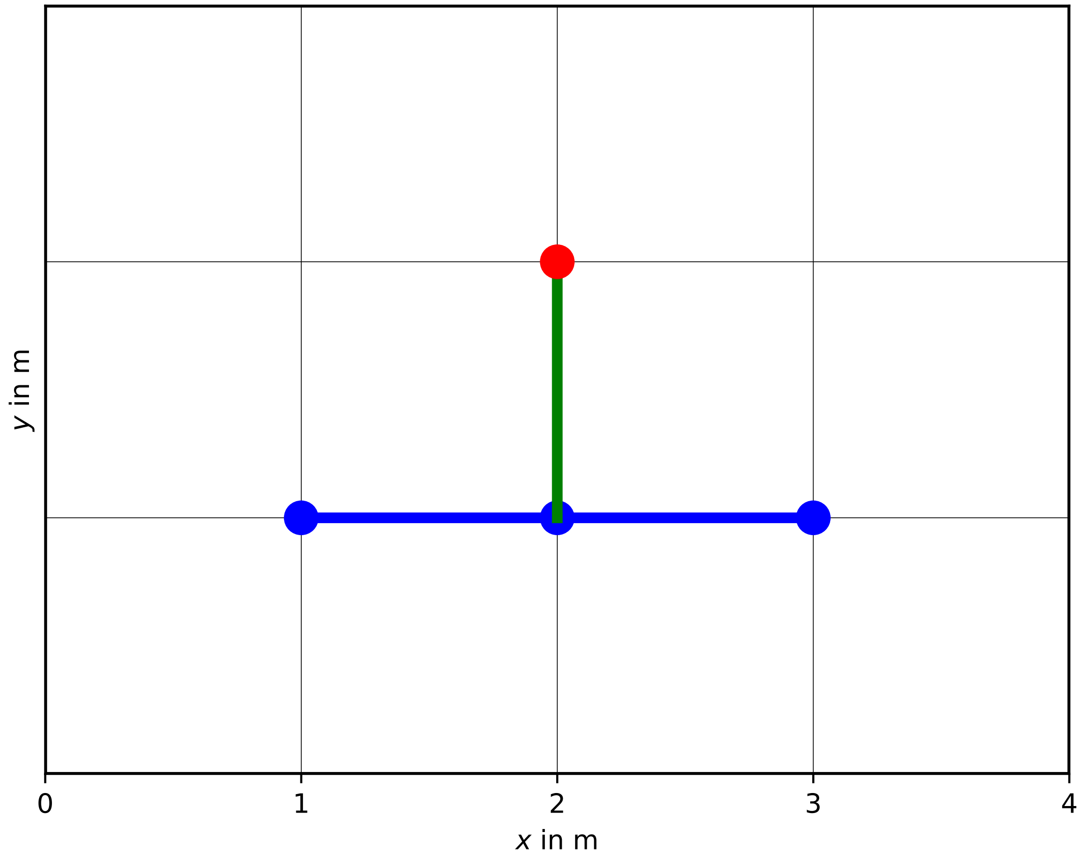
Explicit

Start T^0 with initial condition (e.g. 0)

Update field by

$$T^{n+1} = T^n + a\frac{\partial^2 T^n}{\partial z^2} \cdot \Delta t$$

E.g. by using the matrix A



Forward stencil

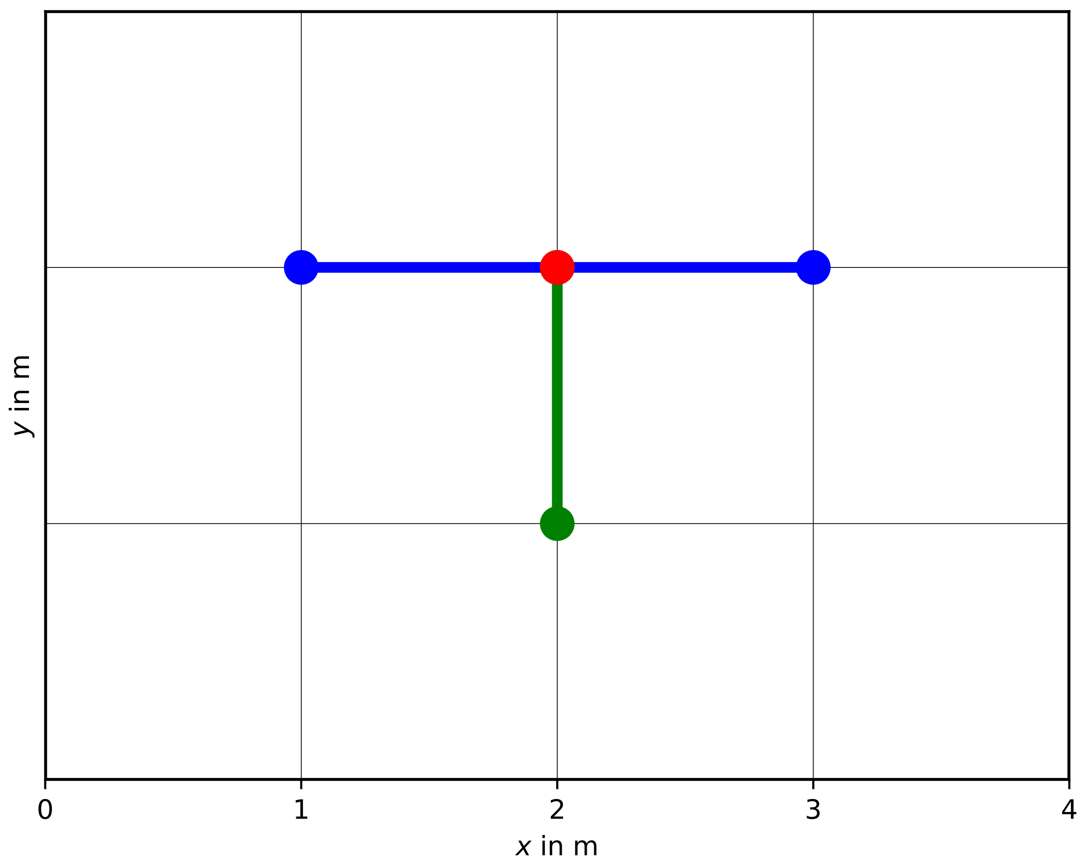
Implicit methods

$$\frac{\partial T}{\partial t}^{n+1} \approx \frac{T^{n+1} - T^n}{\Delta t} = a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{1}{\Delta t} T^{n+1} - a \frac{\partial^2 T}{\partial z^2}^{n+1} = \frac{1}{\Delta t} T^n$$

$$(\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$$

$\mathbf{M}$ - mass matrix



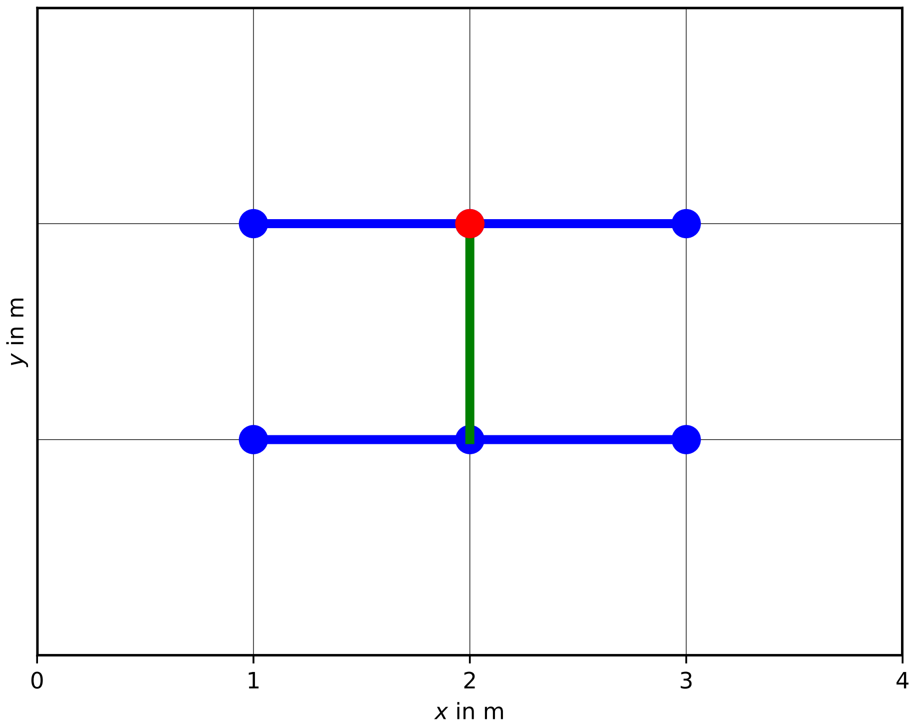
Backward stencil

Mixed - Crank-Nicholson method

$$\frac{\partial T}{\partial t}^{n+1/2} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{2}{\Delta t}T^{n+1} - a\frac{\partial^2 T^{n+1}}{\partial z^2} = \frac{2}{\Delta t}T^{n} + a\frac{\partial^2 T^{n}}{\partial z^2}$$

$$(2\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = (2\mathbf{M} + \mathbf{A})\mathbf{u}^{n}$$



Mixed forward/backward stencil

Report

Task

1. Complete functions delivering both stiffness matrix and right-hand-side vector using FE discretizations
2. Use a non-equidistant discretization of the Earth with increasing layer thicknesses (choose and substantiate).
3. Solve instationary heat equation with periodic boundary condition (yearly cycle) for the Earth using a constant but meaningfull thermal diffusivity.
4. Compare the solutions using explicit, implicit and mixed timestepping methods with the analytical solution.

Questions

- Interpret the results in terms of physical behaviour. How does a change in the diffusivity affect the result.
- Is there a difference between FD and FE discretizations? Why (not)?
- Make a statement about the stability and accuracy of the methods.
- After which time approaches the numeric solution the analytical one?
- How can you evaluate the numerical accuracy if there is not analytical solution?

Deliverables

Format can be Jupyter Notebook and/or PDF

Complete codes to run the results