Numerical Simulation Methods in Geophysics, Part 3: Time-stepping

1. MGPY+MGIN, 3. MDRS+MGEX-CMG

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Recap

- Poisson's equation, solved with FD in 1D
- boundary conditions determine shift (Dirichlet) and flow (Neumann)
- no sources: linear potential (constant flow)
- sources: positive curvature (max), continuous: parabola
- slope changes with conductivity (a contrasts act like source)

Parabolic PDEs

- describe diffusion problems (of potential fields)
- second spatial and first temporal derivative, e.g. in 1D

$$\frac{\partial T}{\partial t} - a \frac{\partial^2 T}{\partial z^2} = \boldsymbol{\nabla} \cdot \mathbf{q_s}$$

Time-stepping with FD

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial z^2} = 0$$

Finite-difference approximation (^{n} means time step n)

$$\frac{\partial u}{\partial t}^n \approx \frac{u^{n+1} - u^n}{\Delta t} = a\frac{\partial^2 u^n}{\partial z^2}$$

Explicit (forward) Euler method

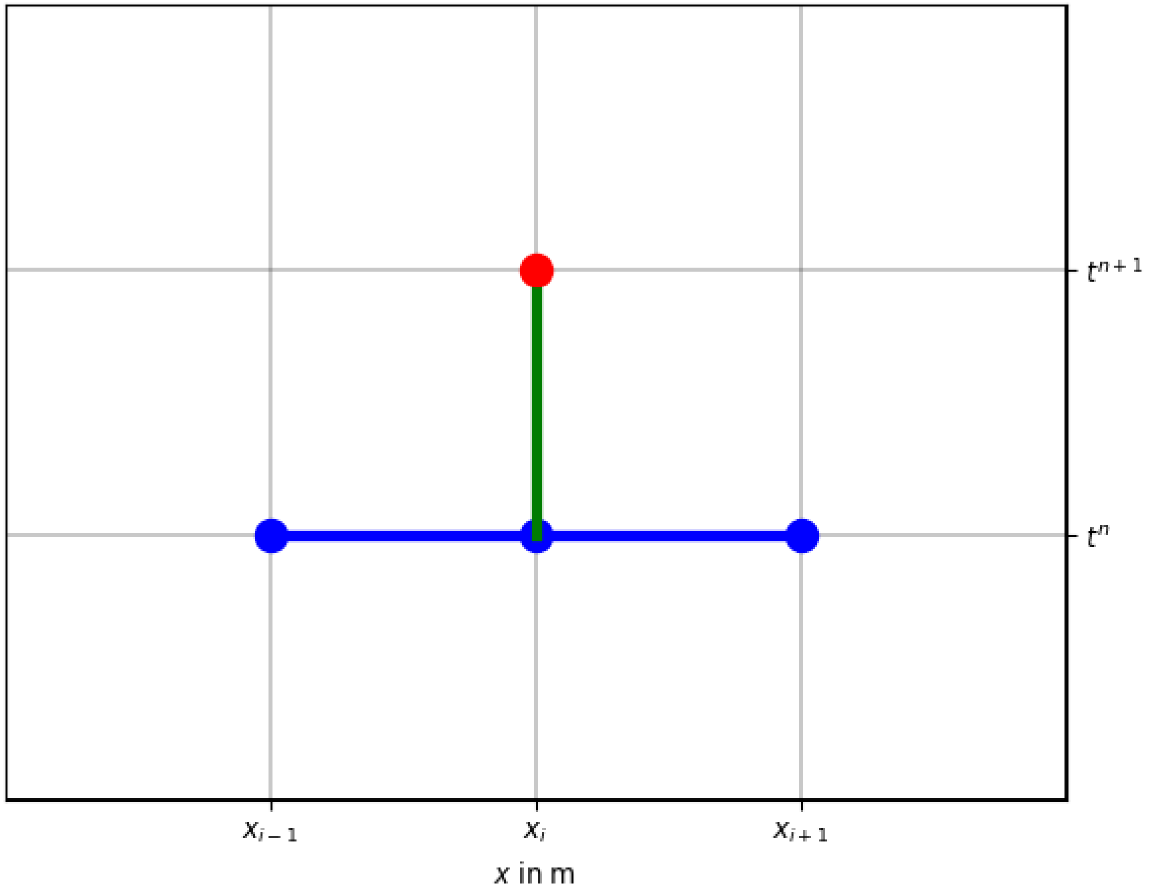
Start T^0 with initial condition

Update field by

$$u^{n+1} = u^n + a\frac{\partial^2 u^n}{\partial z^2} \cdot \Delta t$$

E.g. by using the matrix $\mathbf{A}$:

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t \mathbf{A} \mathbf{u}^n = (\mathbf{I} - \Delta t \mathbf{A})\mathbf{u}^n$$

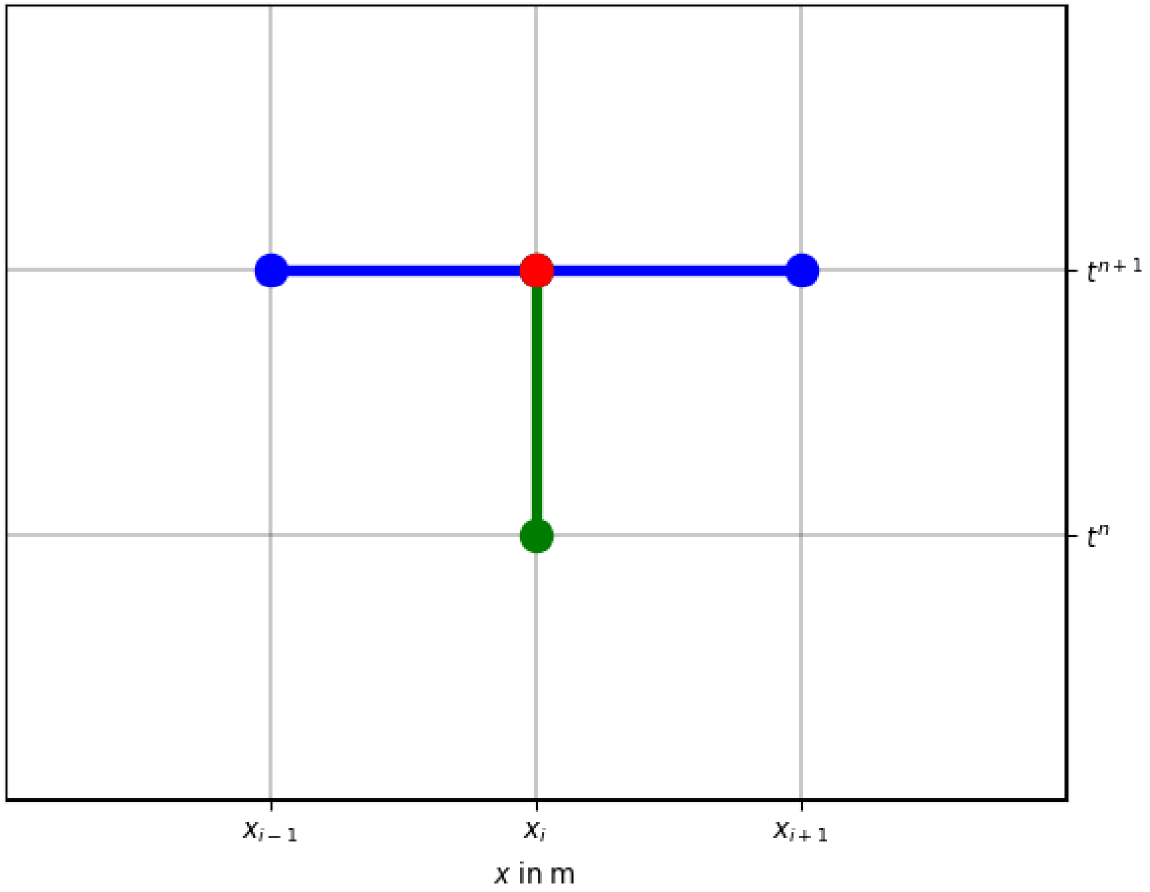


Implicit (backward) Euler method

$$\left.\frac{\partial u}{\partial t}\right|^{n+1} \approx \frac{u^{n+1} - u^n}{\Delta t} = a \left.\frac{\partial^2 u}{\partial z^2}\right|^{n+1}$$

$$\frac{1}{\Delta t} u^{n+1} - a \left.\frac{\partial^2 u}{\partial z^2}\right|^{n+1} = \frac{1}{\Delta t} u^n$$

$$(\mathbf{I} + \Delta t \mathbf{A})\mathbf{u}^{n+1} = \mathbf{u}^n$$



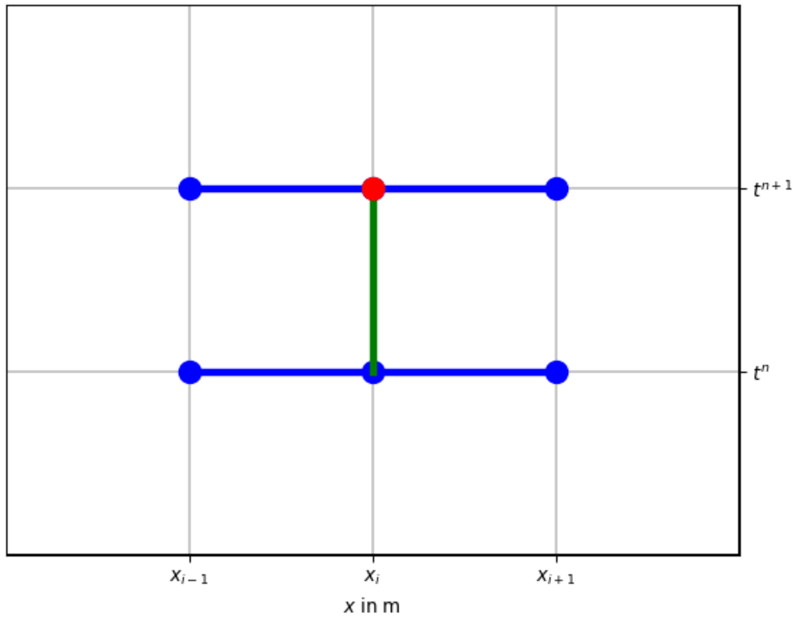
Solving an equation system

Mixed (Crank-Nicholson) method

$$\frac{\partial u}{\partial t}^{n+½} \approx \frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2} a \frac{\partial^2 u}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 u}{\partial z^2}^{n+1}$$

$$\frac{2}{\Delta t} u^{n+1} - a \frac{\partial^2 u^{n+1}}{\partial z^2} = \frac{2}{\Delta t} u^n + a \frac{\partial^2 u^n}{\partial z^2}$$

$$(2\mathbf{I} + \Delta t \mathbf{A})\mathbf{u}^{n+1} = (2\mathbf{I} - \Delta t \mathbf{A})\mathbf{u}^n$$



Dive into time-stepping

Consider *Newtonian cooling* (a 0D toy problem)

$$\frac{\partial T}{\partial t} = -\frac{T}{\tau} \approx \frac{\mathrm{d}T}{\mathrm{d}t}$$

with solution

$$T(t) = T_0 \exp(-t/\tau)$$

Explicit and Implicit Euler methods

Explicit

$$T_{i+1} = T_i - \frac{dt}{\tau}T_i = T_i(1 - \frac{dt}{\tau}) = T_0(1 - \frac{dt}{\tau})^{i+1}$$

Implicit

$$\frac{T_{i+1} - T_i}{dt} = -\frac{T_{i+1}}{\tau}$$

$$T_{i+1} = T_i \frac{1}{1 + dt/\tau}$$

Mixed (Crank-Nicholson) method

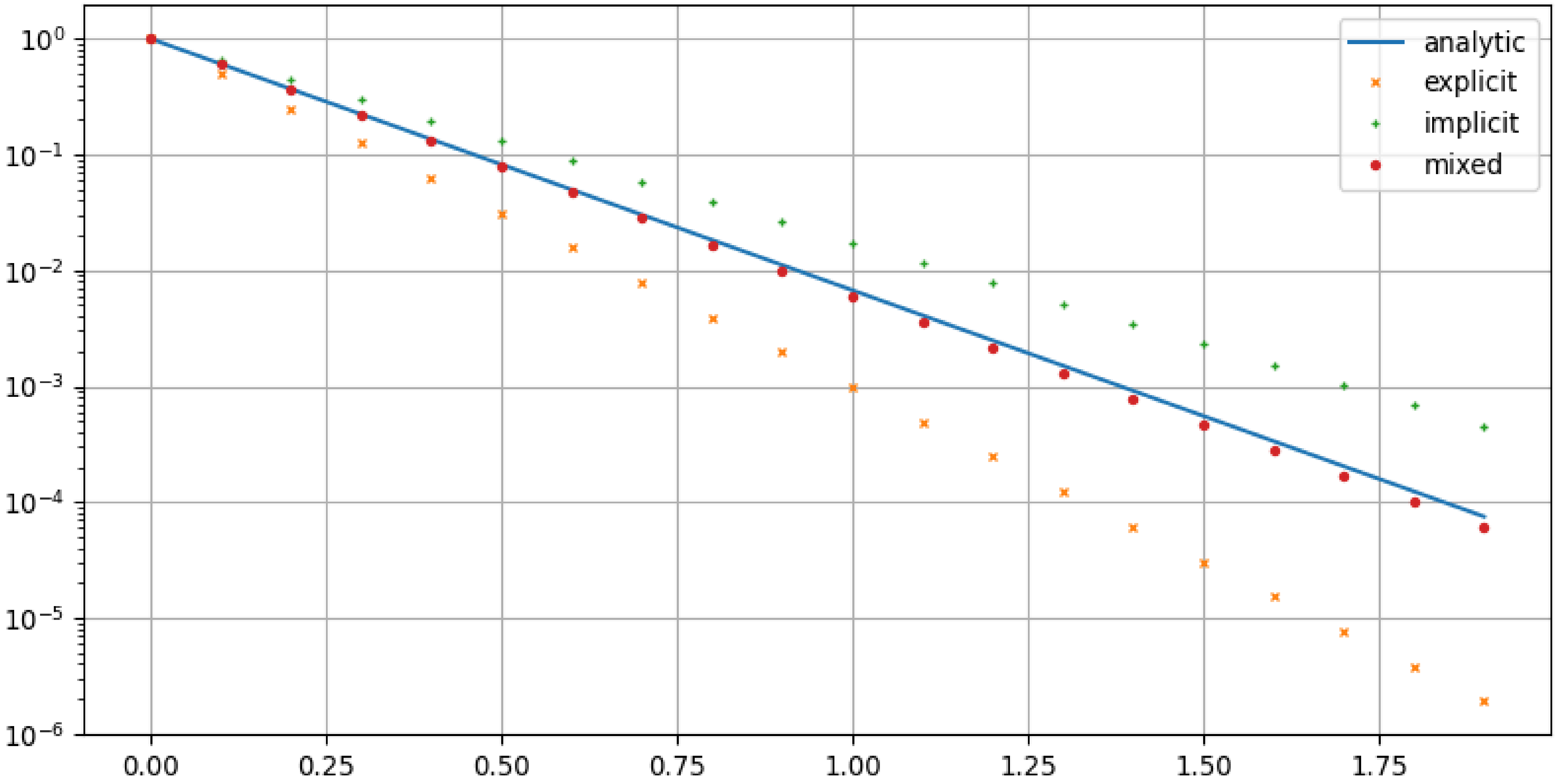
$$\frac{T_{i+1} - T_i}{dt} = -\frac{T_{i+1} + T_i}{2\tau}$$

$$T_{i+1}(1 + dt/2\tau) = T_i(1 - dt/2\tau)$$

$$T_{i+1} = T_i \frac{1 - dt/2\tau}{1 + dt/2\tau}$$

Task: simulate Newtonian cooling using all three methods

Numerical comparison



Periodic BC

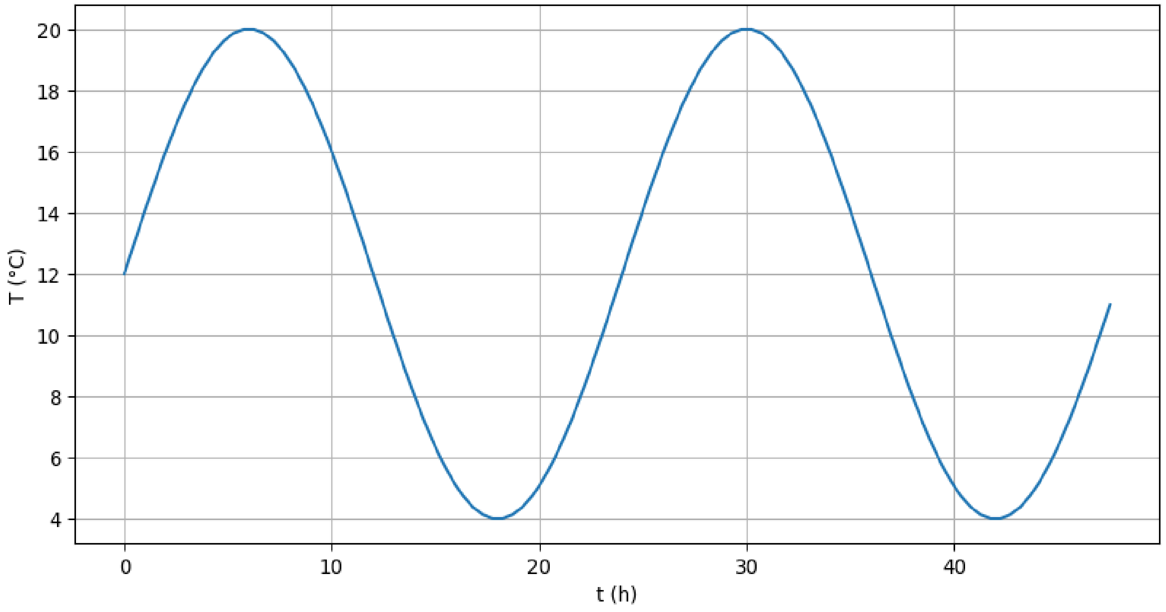
Upper boundary: daily/yearly variation

$$T(z=0) = T_0 + \Delta T \sin \omega t$$

T_0 mean temperature (e.g. 12°C), ΔT variation, e.g. 8°C

$\omega = 2\pi/\tau$ daily (τ_d=3600*24s) or yearly ($\tau_y = 365\tau_d$) cycle

```
day = 3600 * 24
T0, dT = 12, 8
t = np.arange(100) / 50 * day
T = T0+dT*np.sin(t/day*2*np.pi)
plt.plot(t/day*24, T)
plt.xlabel("t (h)")
plt.ylabel("T (°C)")
plt.grid()
```



Tasks for today

1. Complete the stationary tasks from last exercise
2. Keep using the FD matrix for the time-stepping
3. Start with zero temperature distribution
4. Set upper temperature to higher value (source)
5. Choose Δt and step up in time with explicit method
6. Change Δt and observe solution
7. Repeat last steps with implicit and mixed methods
8. Compare with analytical solution