Numerical Simulation Methods in Geophysics, Lecture 10: 2D/3D EM modelling

1. MGPY+MGIN

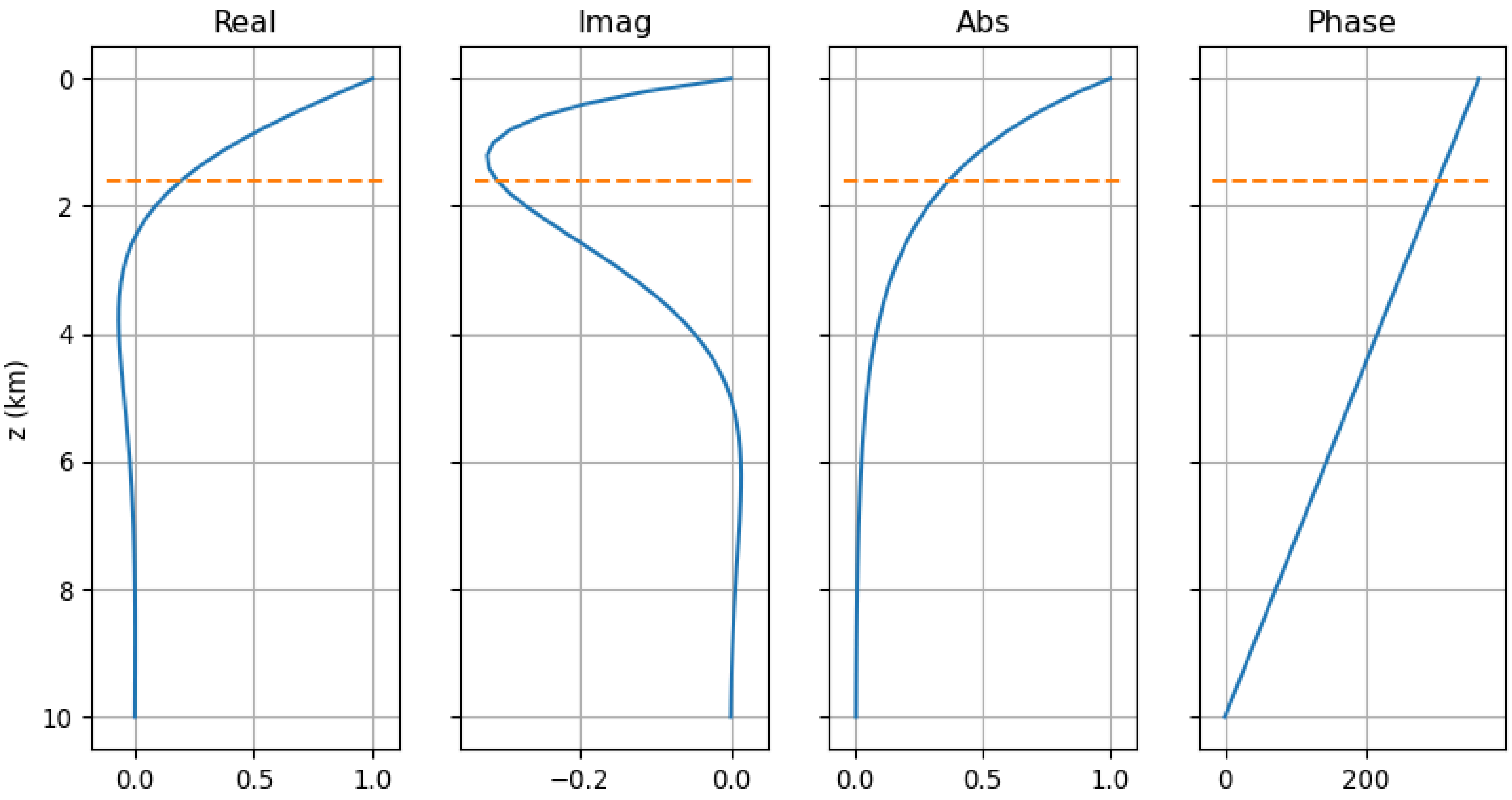
thomas.guenther@geophysik.tu-freiberg.de

Recap

- Maxwells equations (inductive) in time- and frequency domain
- single frequency approach (decomposition) so far 1D
- Helmholtz term ($\frac{\partial^2 u}{\partial x^2}$ and u term)
- u term needs mass matrix (like in time-stepping)
- complex-valued problem: complex or double real matrix
- secondary field approach: σ anomalies as sources

remaining dates: 22.1. (L11), 29.1. (L12), 5.2. (L13 online), 12.2. (TAP)

Analytical solution

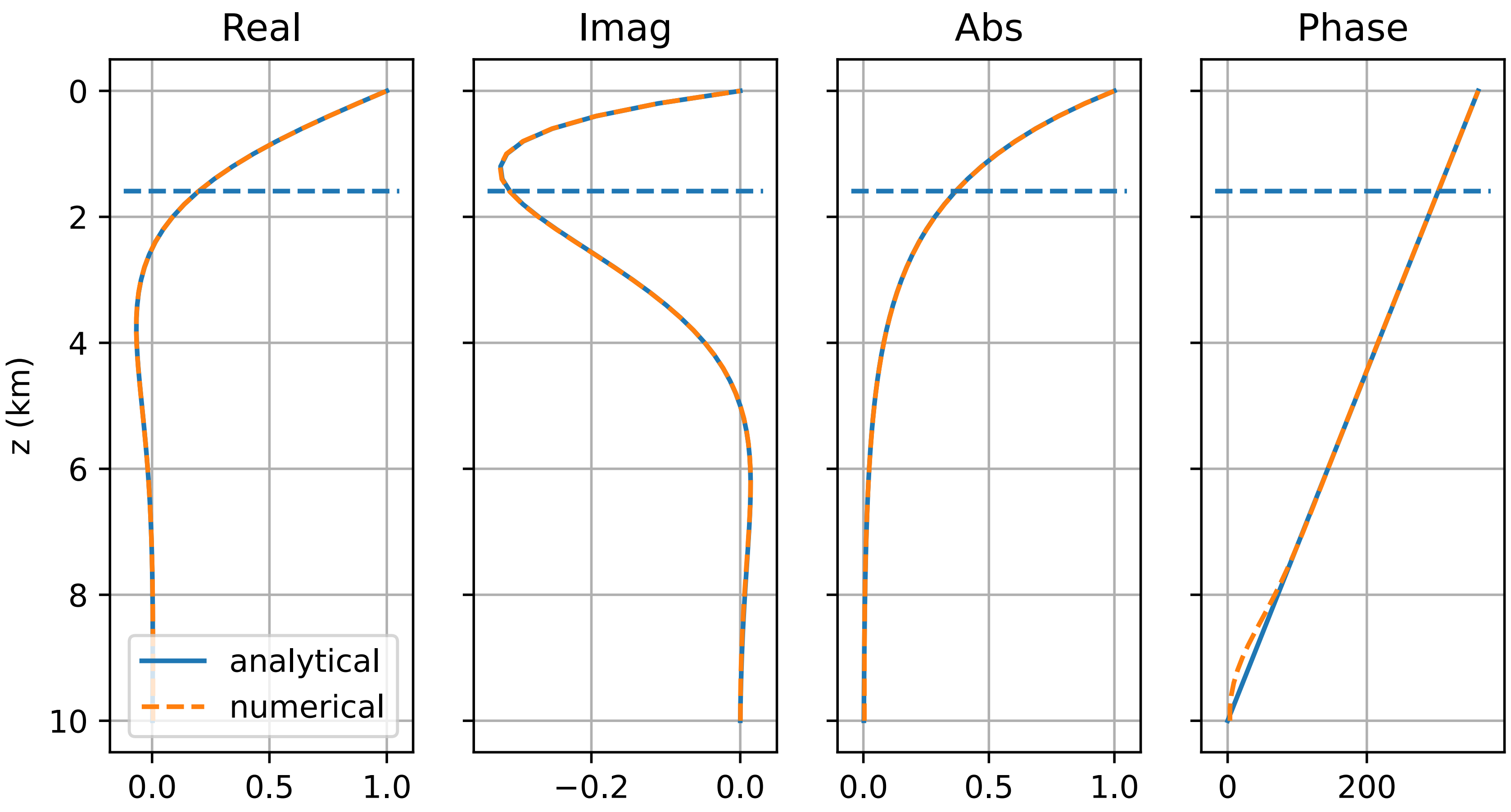


Complex-to-real in pyGIMLi

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

Complex in pyGIMLi

```
C = pg.matrix.CSparseMatrix(A.vecColPtr(), A.vecRowIdx(),
      pg.core.toComplex(A.vecVals(), M.vecVals()*w))
c = pg.core.toComplex(b, b*0)
u = ps.linSolve(C, c).array()  # important (bug in pyGIMLi!)
```



Secondary field for EM

Let E_0 be the solution to the equation for $\sigma = \sigma_0$

$$-\boldsymbol{\nabla}^2 E_0 + \imath\omega\mu\sigma_0 E_0 = 0$$

and $E = E_0 + E_a$ for $\sigma = \sigma_0 + \delta\sigma$

$$-\boldsymbol{\nabla}^2 (E_0 + E_a) + \imath\omega\mu(\sigma_0 + \delta\sigma)(E_0 + E_a) = 0$$

subtracting leads to

$$-\boldsymbol{\nabla}^2 E_a + \imath\omega\mu(\sigma_0 + \delta\sigma)E_a = -\imath\omega\mu\delta\sigma E_0$$

Secondary field for EM

$$-\boldsymbol{\nabla}^2 E_a + \imath\omega\mu\sigma E_a = -\imath\omega\mu\delta\sigma E_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

is solved for $E_0(\mathbf{u})$ using $\mathbf{A}$-stiffness, $\mathbf{M}$-mass and $E_0(\mathbf{u}_0)$

$$(\mathbf{A} + \imath\omega\mathbf{M}_\sigma)\mathbf{u} = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{u}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```

2D scalar EM - TM polarization

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1} \mathbf{j}_s$$

Transverse magnetic (TM) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e. $\mathbf{H} = [H_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar H_x

$$-\boldsymbol{\nabla} \cdot \sigma^{-1} \boldsymbol{\nabla} H_x(y, z) + \imath\omega\mu H_x(y, z) = 0$$

2D scalar EM - TE polarization

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\sigma\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{j}_s$$

Transverse electric (TE) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{E} = [E_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar E_x

$$-\boldsymbol{\nabla} \cdot \mu^{-1} \boldsymbol{\nabla} E_x(y, z) + \imath\omega\sigma E_x(y, z) = 0$$

EM vector modelling

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\sigma\mathbf{E} - \omega^2 \epsilon \mathbf{E} = -\imath\omega\mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2 \mu\epsilon \mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1} \mathbf{j}_s$$

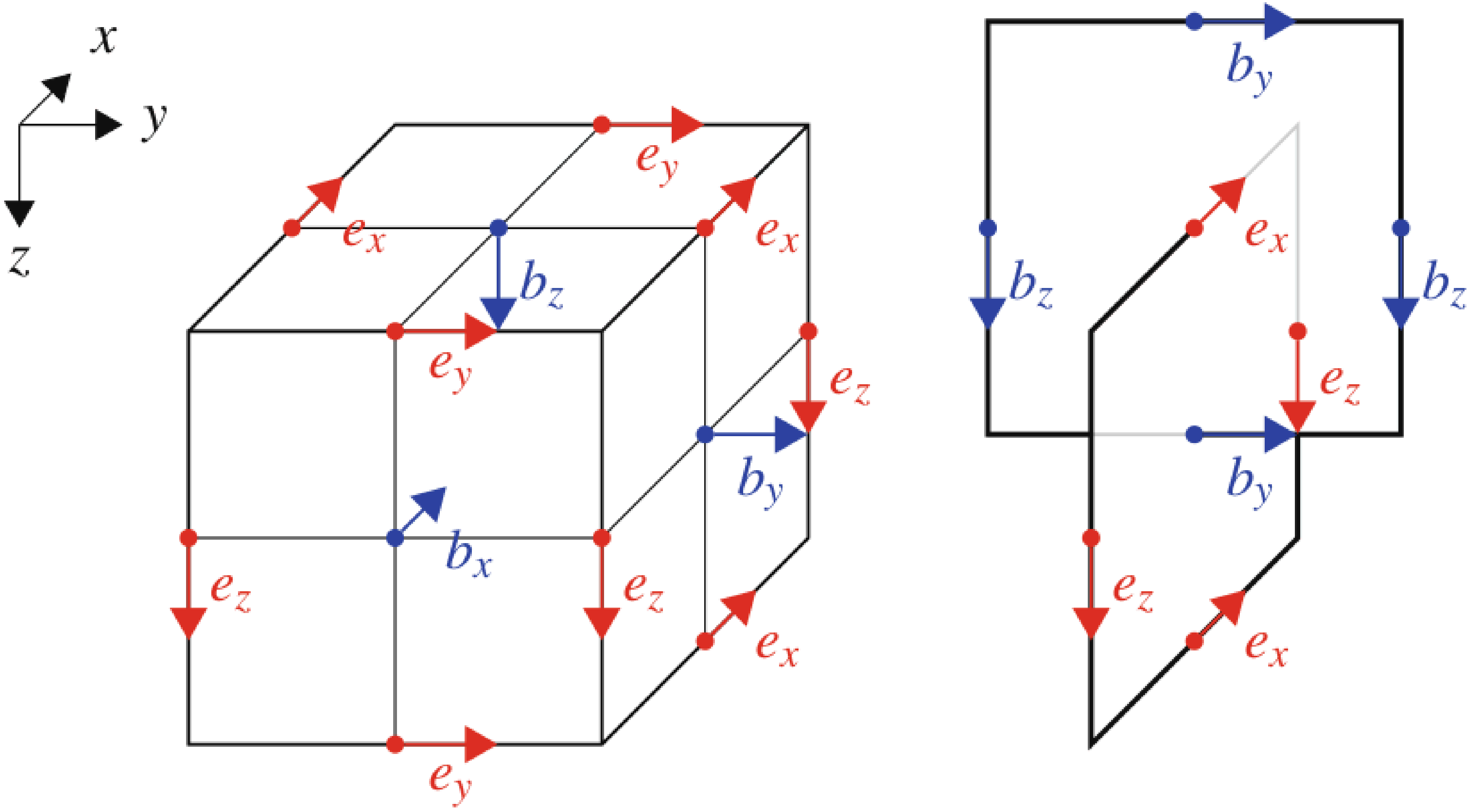
History

James Clerk Maxwell, Treatise on Electricity & Magnetism, 1891

“Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.”

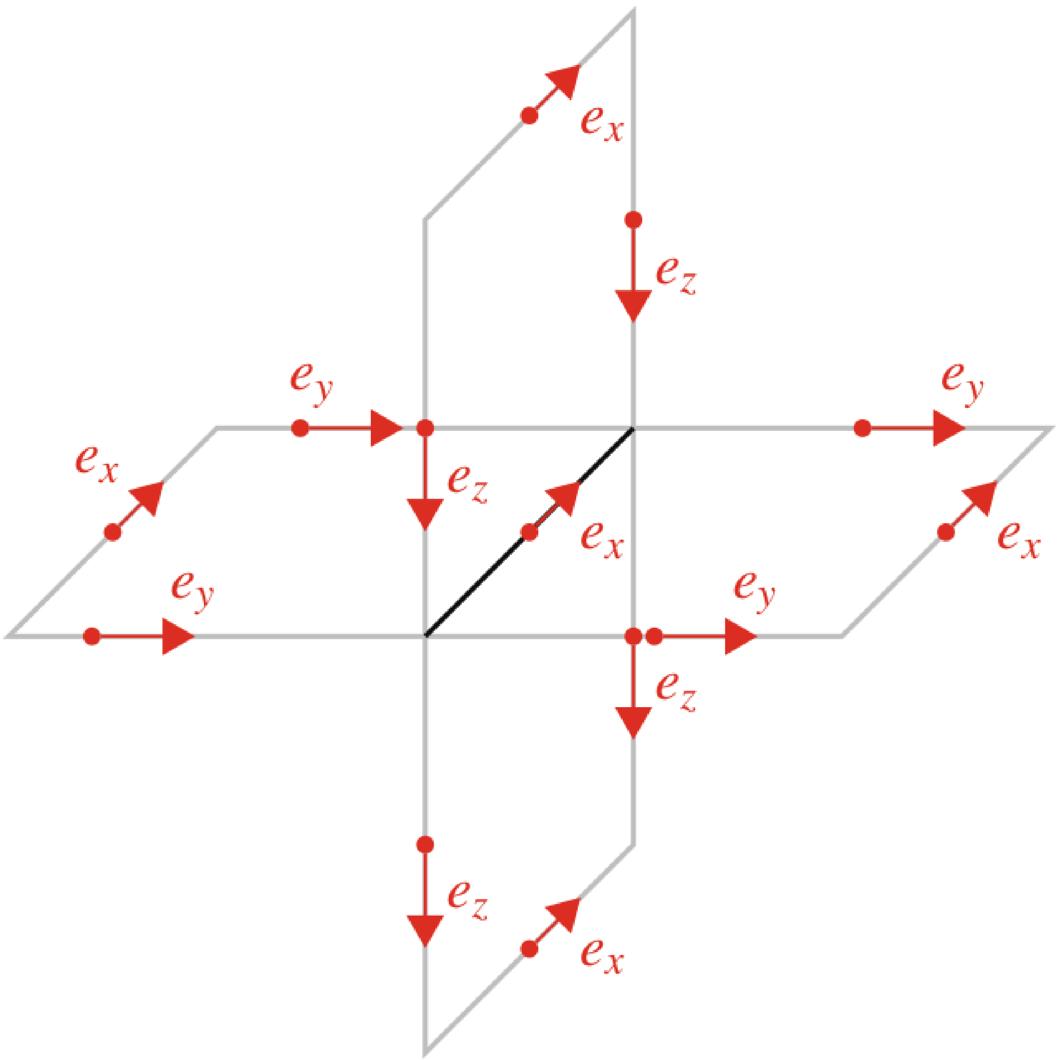
- integral equations (IE), finite differences (FD) and elements (FE)
- nodal and vector basis functions, secondary field
- decompose 3D source (2D inverse) in wavenumber domain
- improvement of equation solvers and preconditioners
- parallel solvers, high-performance computing

Solving EM problems with staggered grid



Staggered grid cell after Yee (1966) (Börner, 2010)

Solving EM problems with staggered grid



Electric field components from one row of curl-curl (Börner, 2010)

The way from FD to FE

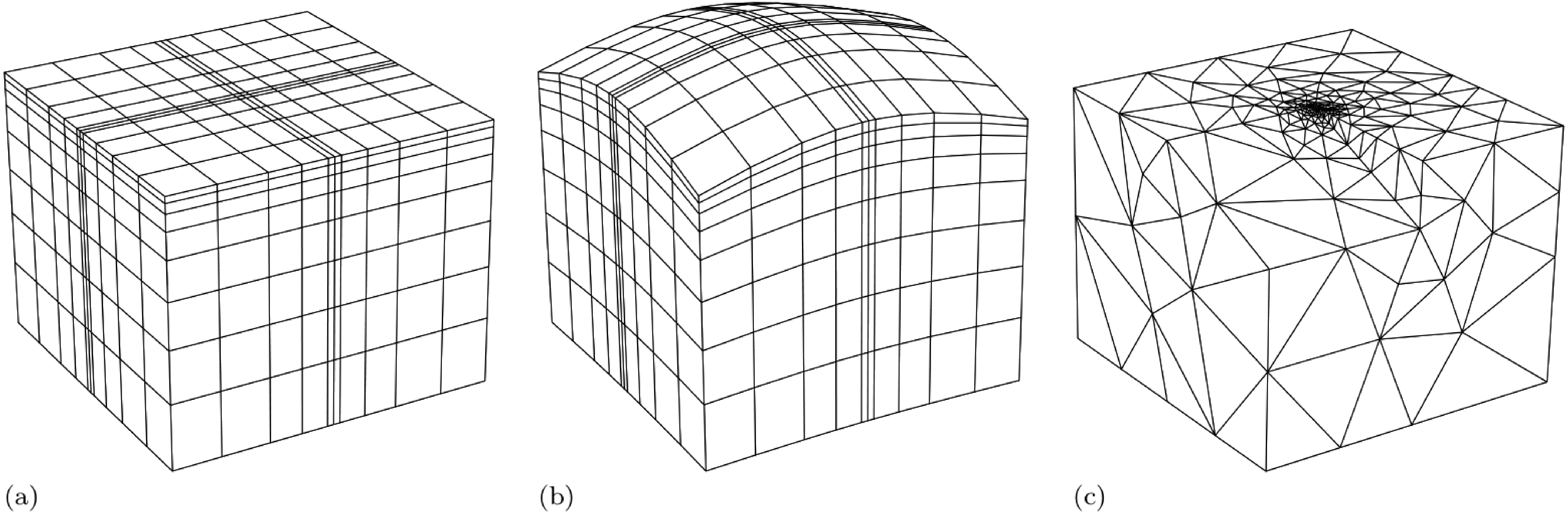


Figure 1. Different grid types: Orthogonal cuboid (a), non-orthogonal hexahedral (b) and unstructured tetrahedral (c) grid.

Orthogonal, non-orthogonal and irregular grids (Rücker et al., 2006)

The weak formulation

$$\int_\Omega \mathbf{w} \boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} \mathrm{d}\Omega - \imath\omega \int \mathbf{w} \sigma \mathbf{E} \mathrm{d}\Omega = 0$$

Integration, Greens identity $\int a \nabla^2 b = -\int \boldsymbol{\nabla} a \cdot \boldsymbol{\nabla} b - \int a \, \partial b / \partial n$

$$\int_\Omega \mathbf{w} \ldots = \int_\Omega \boldsymbol{\nabla} \times \mathbf{w} \cdot (\mu^{-1} \boldsymbol{\nabla} \times \mathbf{E}) \mathrm{d}\Omega + \int \mathbf{n} \times (\mu^{-1} \boldsymbol{\nabla} \times \mathbf{E}) \mathrm{d}\Gamma$$

choose basis that latter term vanishes

$$\int_\Omega \boldsymbol{\nabla} \times \mathbf{w} \cdot (\mu^{-1} \boldsymbol{\nabla} \times \mathbf{E}) \mathrm{d}\Omega - \imath\omega \int \mathbf{w} \sigma \mathbf{E} \mathrm{d}\Omega = 0$$

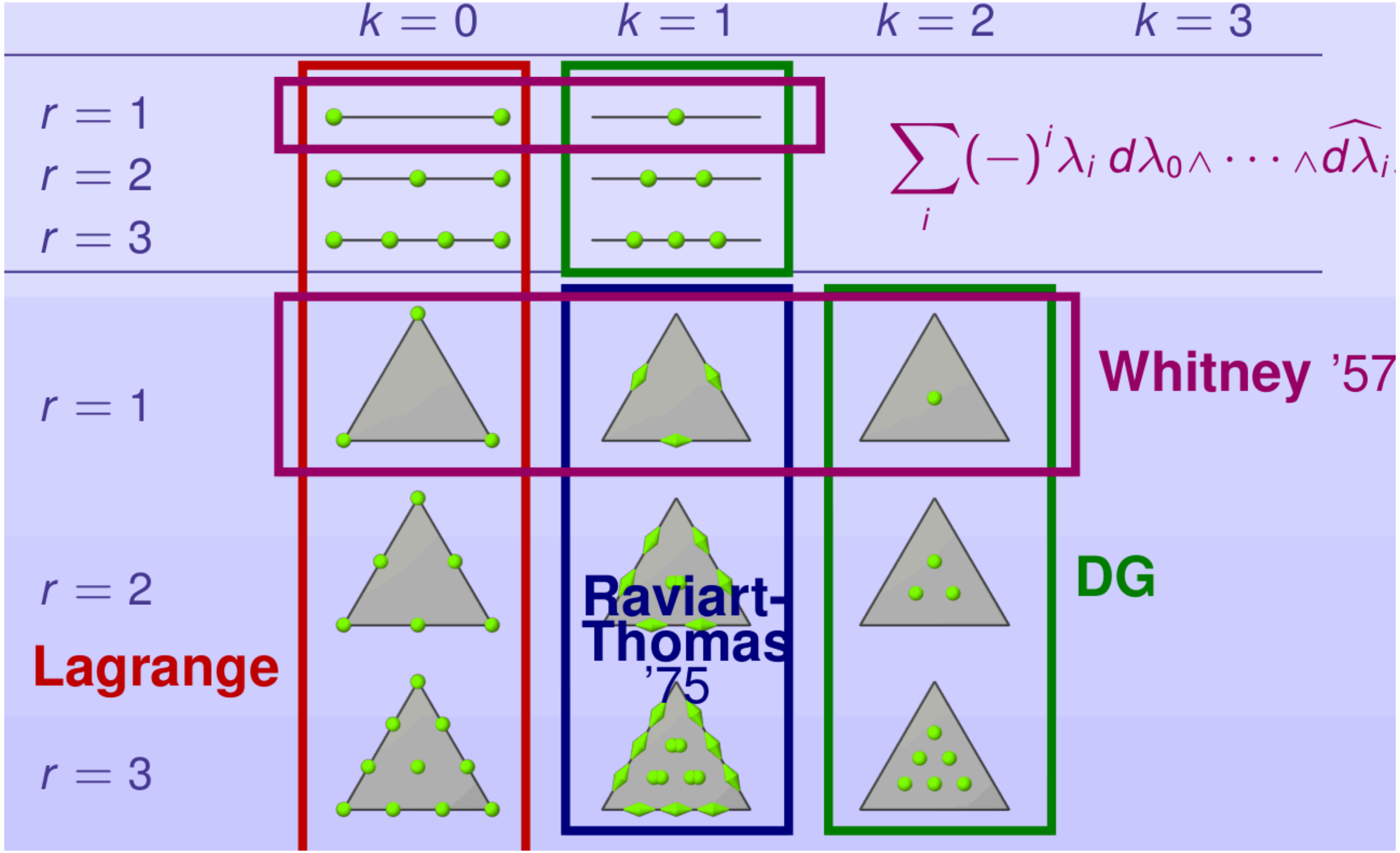
Galerkins method

$$\mathbf{E} = \sum_i u_i \mathbf{u} \quad \text{with} \quad \mathbf{w} \in \mathbf{u}_i$$

$$\left\langle \boldsymbol{\nabla} \times \mathbf{w} \middle| \mu^{-1} \boldsymbol{\nabla} \times \mathbf{u}_i \right\rangle$$

$$\mathbf{Au} = \mathbf{b} \quad \text{with} \quad A_{i,j} = \int_\Omega (\boldsymbol{\nabla} \times u_i) \cdot (\mu^{-1} \boldsymbol{\nabla} \times u_j) \mathrm{d}\Omega$$

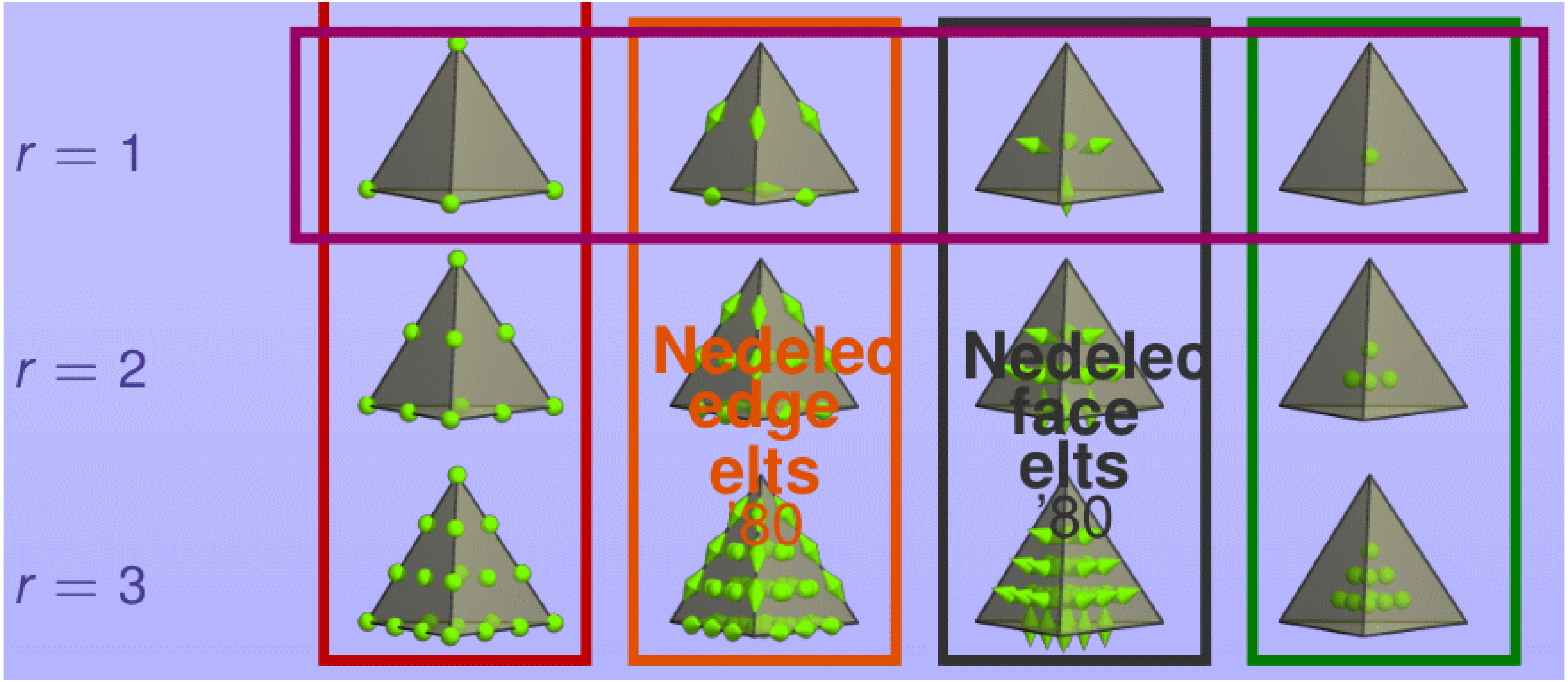
The Finite Element zoo (1D & 2D)



Arnold (2013): Periodic table of elements

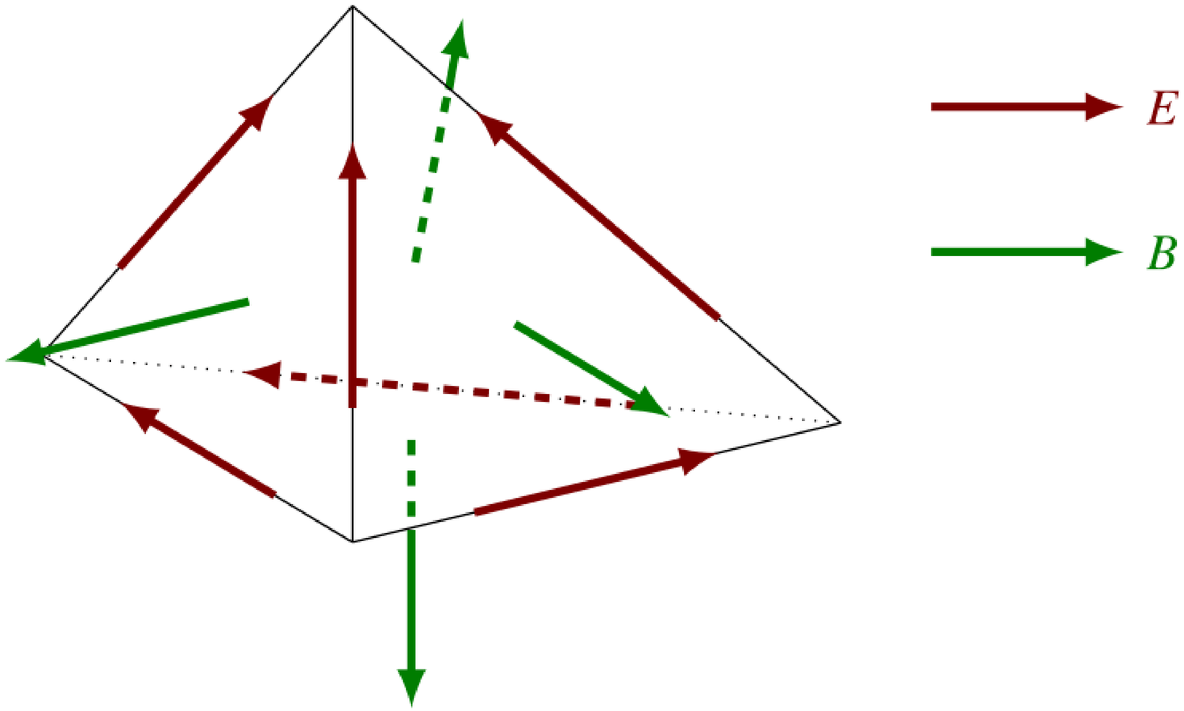
- k=0 Nodal elements
- k=1 edge elements
- k=2 face elements (Discontinuous Galerkin)
- r/p higher order

The Finite Element zoo (3D)



Arnold (2013): Periodic table of elements

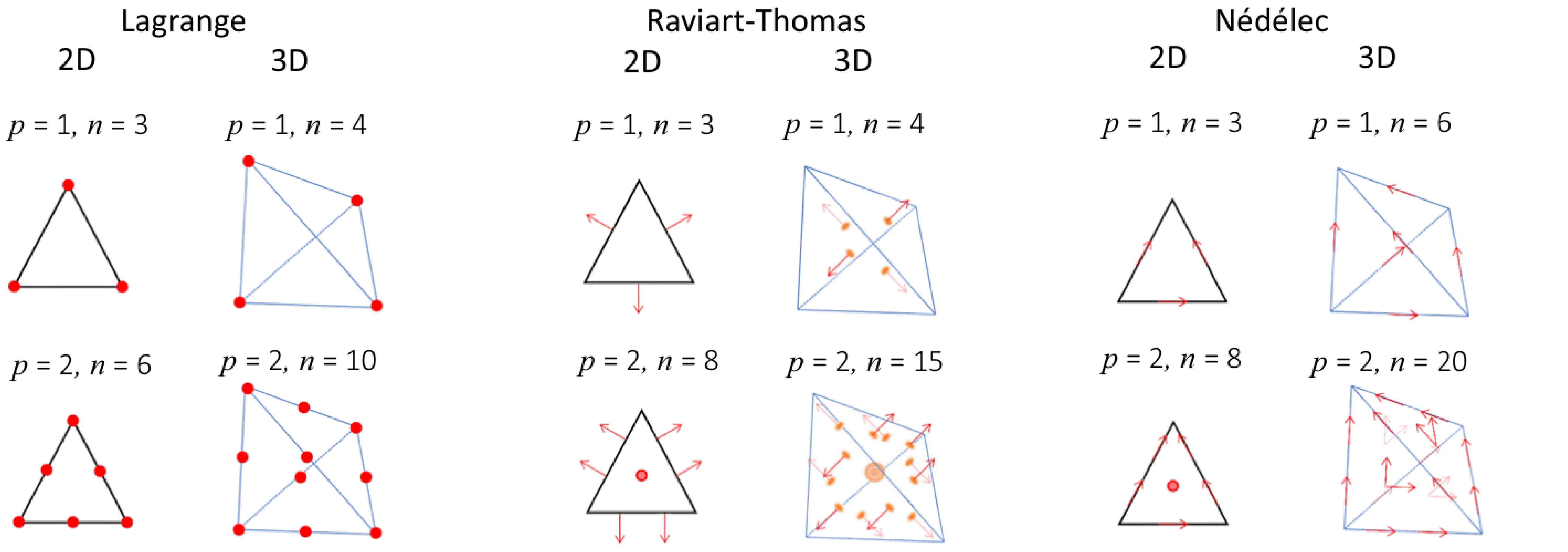
Vectorial solution for EM fields



Schwarzbach & Haber (2013)

- $\mathbf{E} \in H^{\mathrm{curl}}(\Omega)$
- $\mathbf{B} \in H^{\mathrm{div}}(\Omega)$
- natural BC: Dirichlet $\mathbf{n} \times \mathbf{E} = 0$
- Neumann BC:
  $\mathbf{n} \times \mu^{-1}\boldsymbol{\nabla} \times \mathbf{E} = 0$

Elements used in EM modelling



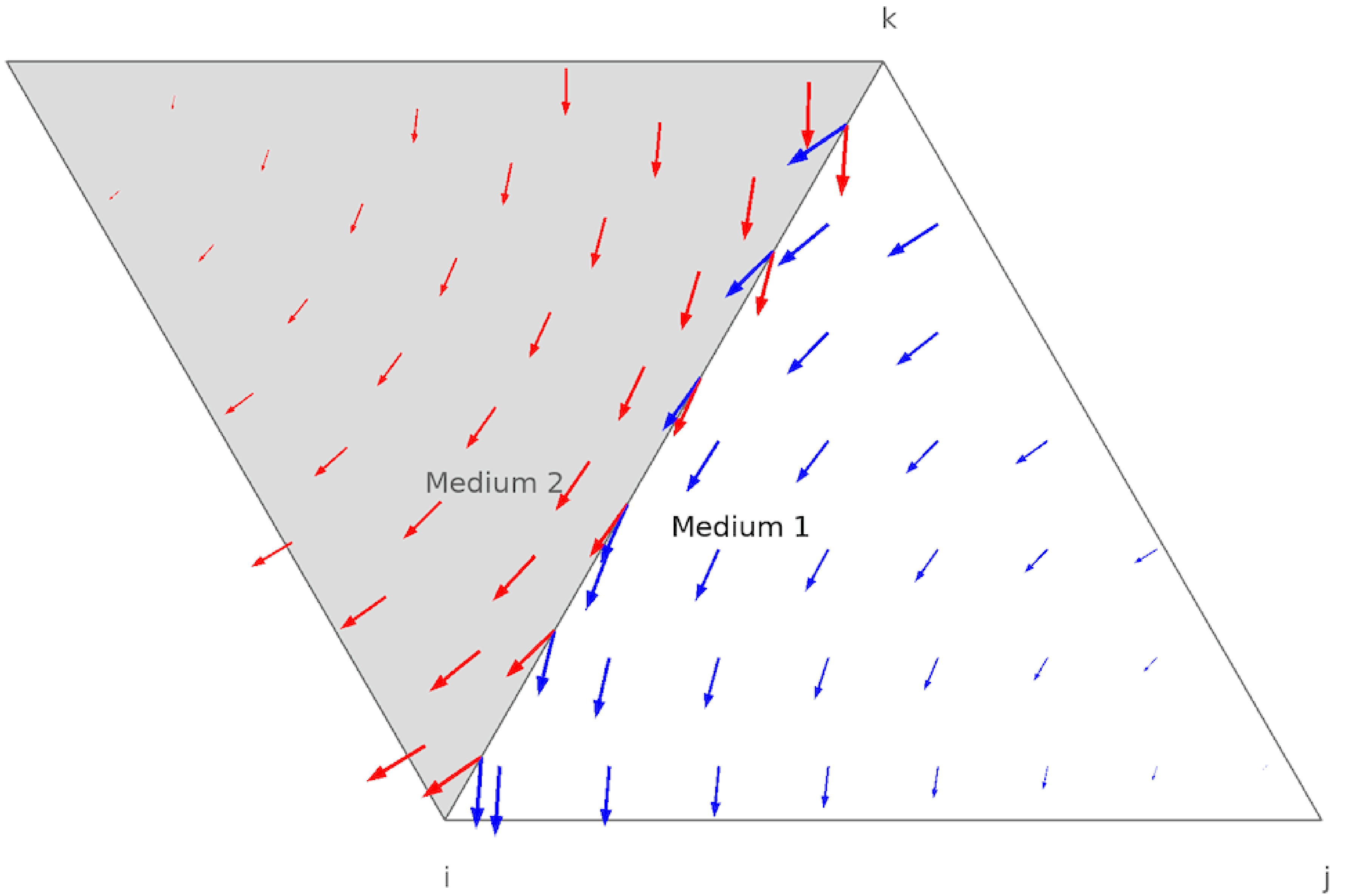
$$\mathcal{H}^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^3\}$$

$$\mathcal{H}(\mathrm{div};\Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\}$$

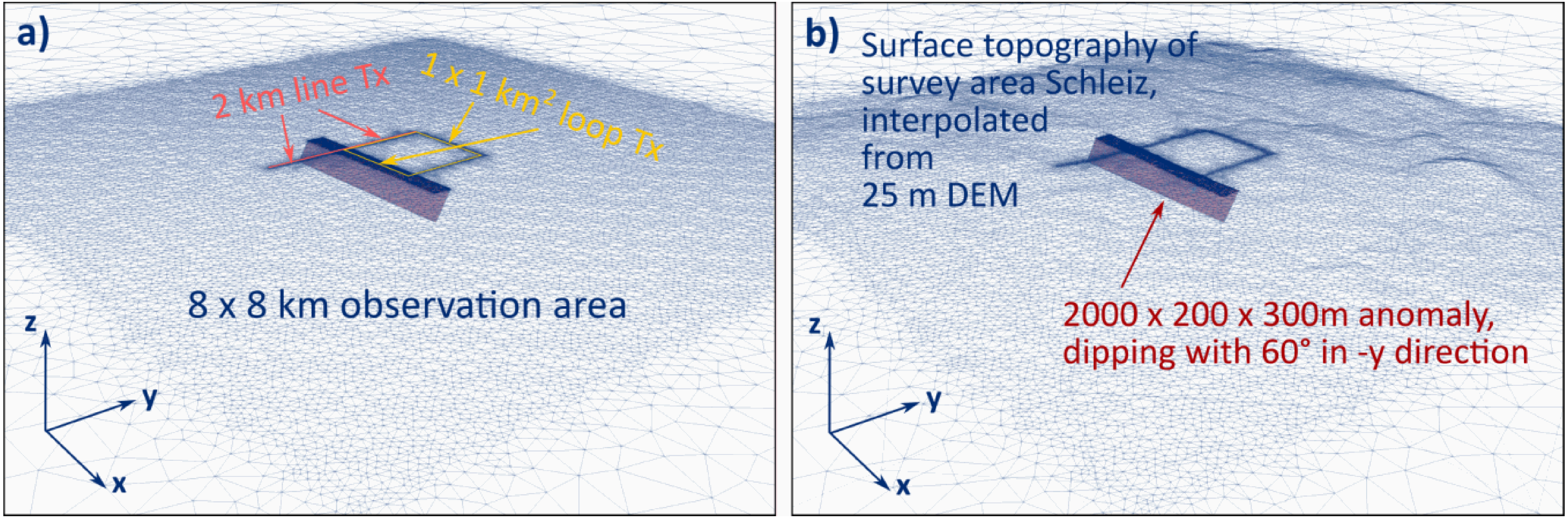
$$\mathcal{H}(\mathrm{curl};\Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3\}$$

Types of elements relevant for EM problems (Spitzer, 2024)

Nedelec shape functions

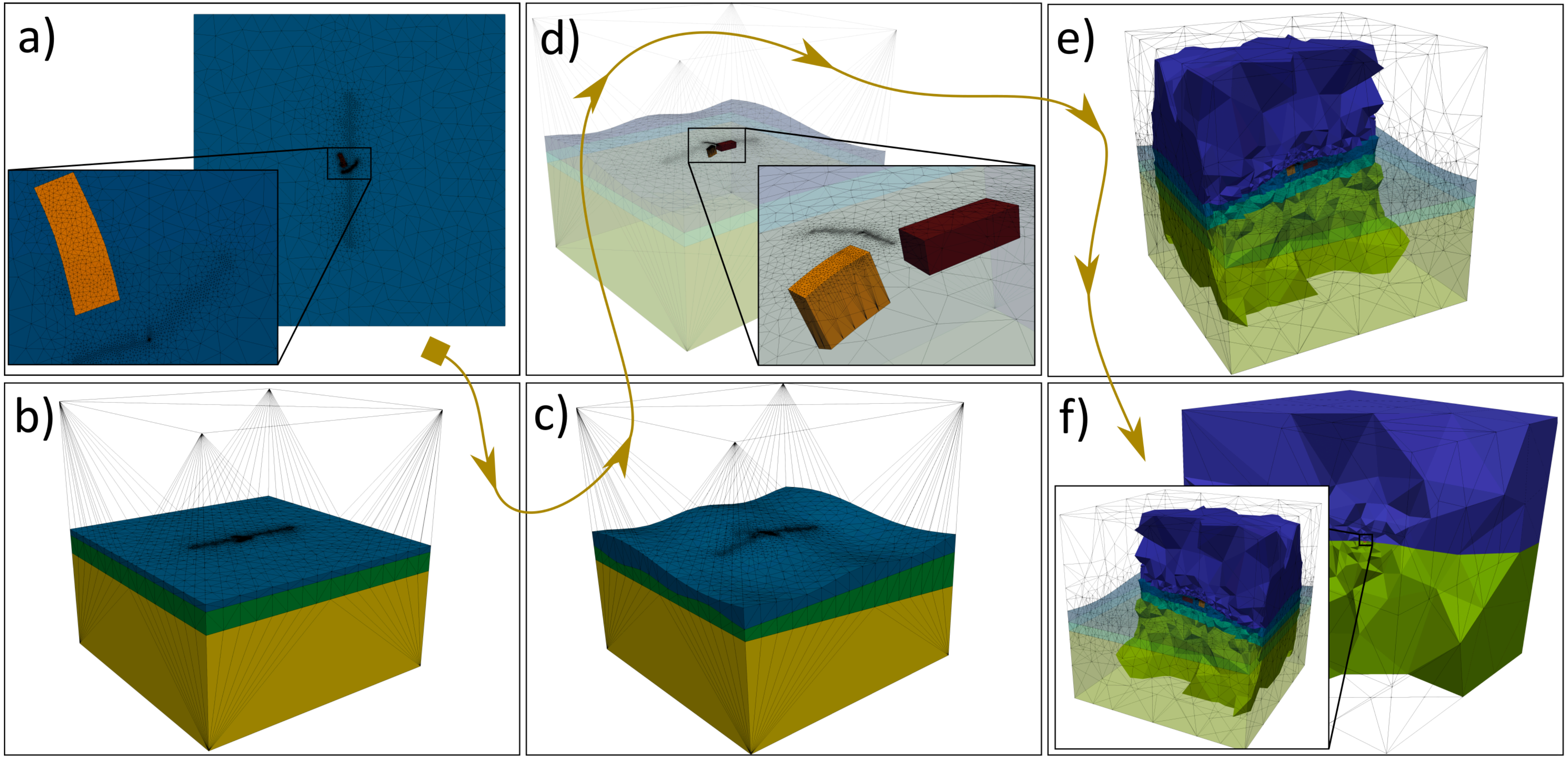


Modelling topography



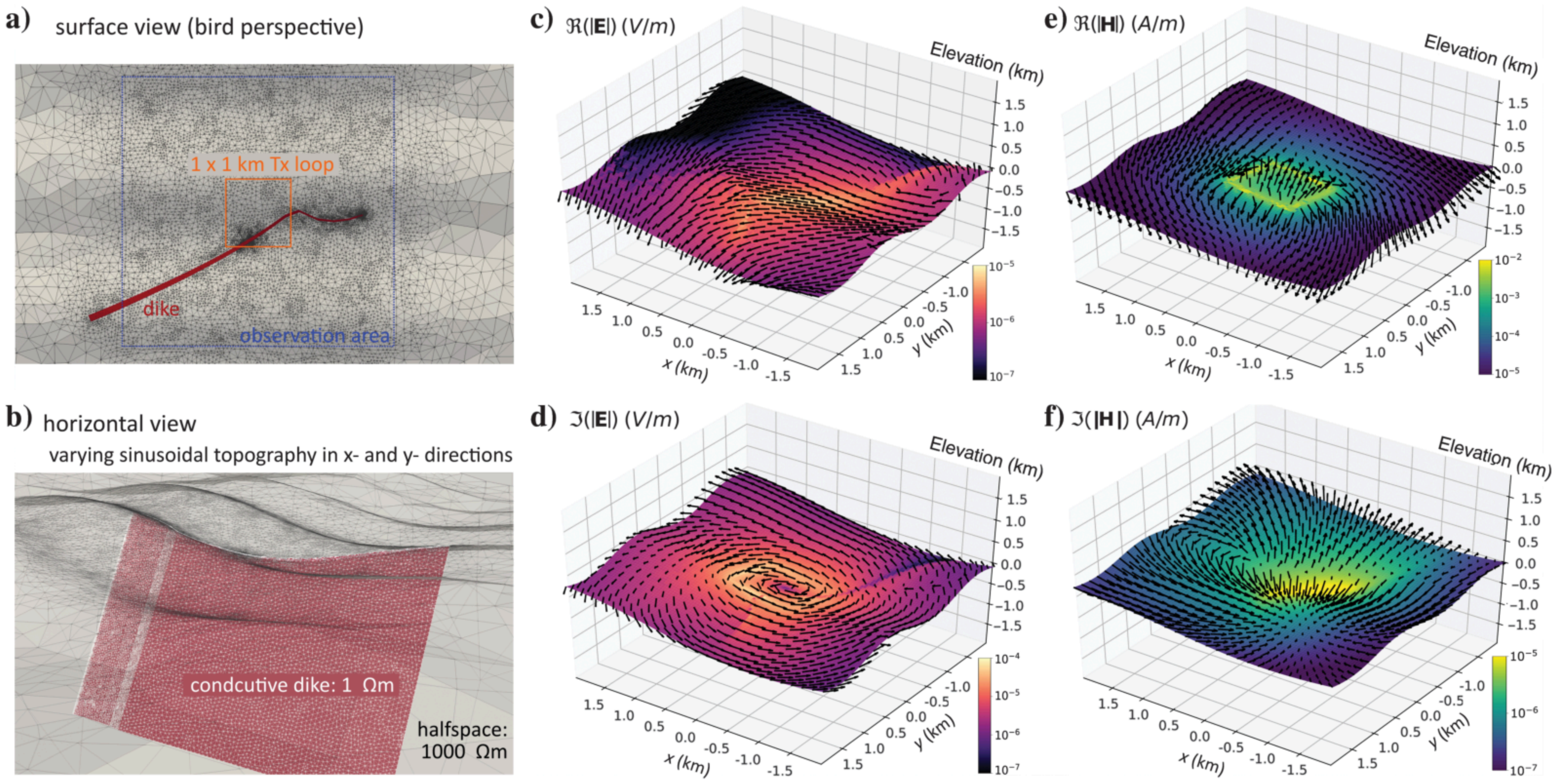
Modelling incorporating topography (Rochlitz, 2019)

Meshing complicated geometries



Meshing workflow in custEM (Rochlitz et al., 2019)

Modelling example



Modelling example of a conductive dike (Rochlitz et al., 2019)

Packages

Mesh generation: TetGen (3D), GMsh (2D/3D)

FE packages: FEniCS, NETGEN/NGsolve

Equation solvers: Suitesparse, MUMPS, SciPy

Computational frameworks: PetSc, MPI

EM modelling (and inversion) packages: Mare2DEM, emg3d, GoFEM, PETGEM, custEM, SimPEG, ModEM, FEMTIC

Boundary conditions

Mixed boundary conditions

So far…

- Dirichlet Boundary conditions $u = u_0$
- Neumann Boundary conditions $\frac{\partial u}{\partial n} = g_B$

vectorial problems: $\mathbf{n} \cdot \mathbf{E} = 0$ (Neumann) or $\boldsymbol{\nabla} \times \mathbf{E} = 0$ (Dirichlet)

In general mixed, also called Robin (or impedance convective) BC

$$au + b\frac{\partial u}{\partial n} = c$$

Example DC resistivity with point source

$$\boldsymbol{\nabla} \cdot \sigma \boldsymbol{\nabla} u = \boldsymbol{\nabla} \cdot \mathbf{j} = I\delta(\mathbf{r} - \mathbf{r}_s)$$

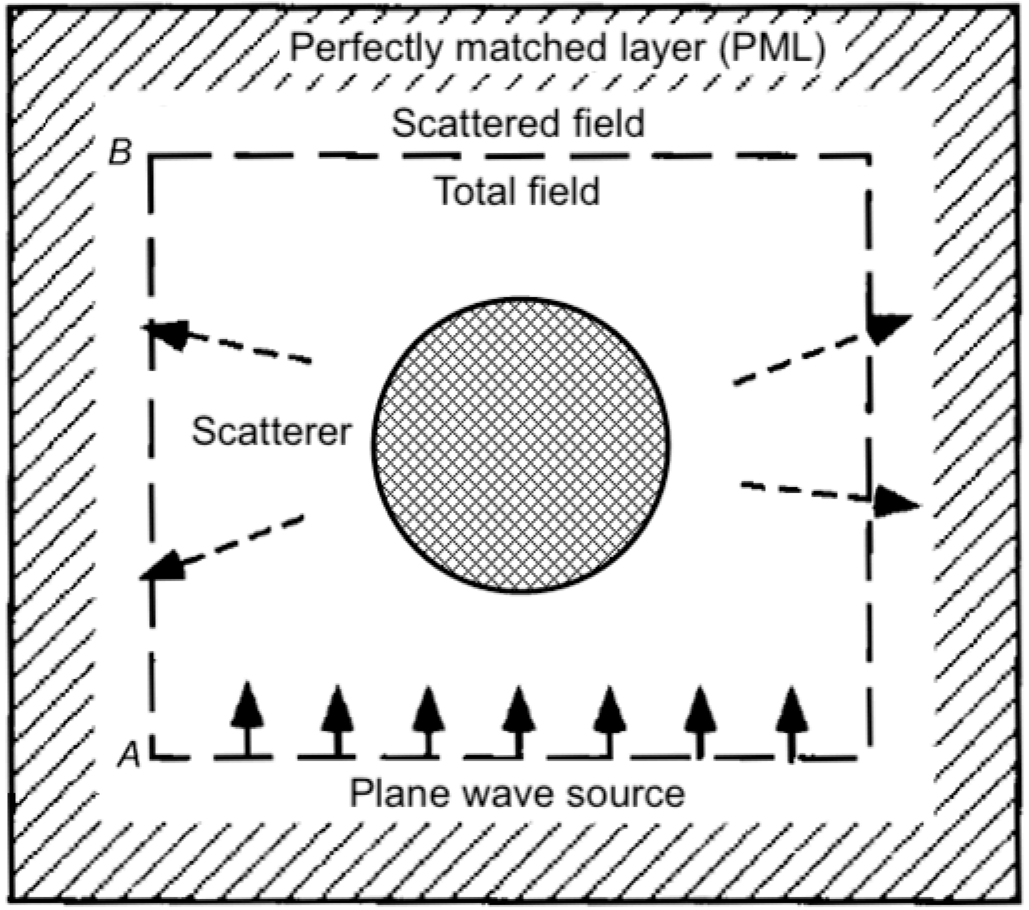
solution for homogeneous σ on surface: $u = \frac{I}{2\pi\sigma} \frac{1}{|\mathbf{r}-\mathbf{r}_s|}$

E-field $\mathbf{E} = -\frac{I}{2\pi\sigma} \frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3}$

normal direction $\mathbf{E} \cdot \mathbf{n} = -\frac{u}{|\mathbf{r}-\mathbf{r}_s|} \cos\phi$ purely geometric

$$\frac{\partial u}{\partial n} + \frac{\cos\phi}{|\mathbf{r} - \mathbf{r}_s|} = 0$$

Perfectly matched layers



$$\frac{\partial}{\partial x} \to \frac{1}{1 + i\sigma/\omega} \frac{\partial}{\partial x}$$

$$x \to x + \frac{i}{\omega} \int^{x} \sigma(x')\mathrm{d}x'$$

Absorbing boundary conditions

wave equation (e.g. in 2D)

$$\frac{\partial^2 u}{\partial t^2} - v^2 \nabla^2 u = 0$$

Fourier transform in t and y (boundary direction) $\Rightarrow \omega, k$

$$\omega^2 \hat{u} - v^2 \frac{\partial^2 \hat{u}}{\partial x^2} + v^2 k^2 \hat{u} = 0$$

ordinary DE with solution $\hat{u} = \sum a_i e^{\lambda x}$ with $\lambda^2 = k^2 - \omega^2 / v^2$

Time-domain EM

- Fourier transform excitation $\Rightarrow$ solve in FD $\Rightarrow$ backtransform
- solve with Time-Stepping: Governing equation

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{e} + \sigma \frac{\partial \mathbf{e}}{\partial t} = - \frac{\partial \mathbf{e}_s}{\partial t}$$

$$\mathbf{K}\mathbf{u} + \mathbf{M} \frac{\partial u}{\partial t} = \mathbf{s}$$

Implicit time stepping

$$(\Delta t\mathbf{K} + \mathbf{M})\mathbf{u}^{n+1} = \mathbf{M}u^{n} + \Delta t\mathbf{s}^{n+1}$$

second-order

$$(2\Delta t\mathbf{K} + 3\mathbf{M})\mathbf{u}^{n+2} = \mathbf{M}(4\mathbf{u}^{n+1} - \mathbf{u}^{n}) - 2\Delta t\mathbf{s}^{n+2}$$