Numerical Simulation Methods in Geophysics, Part 7: 2D Poisson+Helmholtz

1. MGPY+MGIN

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Recap

- finite differences approximate partial derivatives
- finite elements approximate solution $\Rightarrow$ preferred
- spatial discretization determines accuracy of solution
- implicit & mixed time-stepping schemes accurate & stable
- basic elements in 1D/2D/3D & higher order shape functions
- numerical integration , e.g. by Gaussian quadrature

Inner vs. outer nodes

distinguish dofs into inner and outer $[\mathbf{u}_i, \mathbf{u}_o]^T$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{ii} & \mathbf{A}_{io} \\ \mathbf{A}_{oi} & \mathbf{A}_{oo} \end{pmatrix}$$

$$\mathbf{A} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_o \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_o \end{bmatrix}$$

$$\Rightarrow \mathbf{A}_{ii}\mathbf{u}_i = \mathbf{f}_i - \mathbf{A}_{oi}\mathbf{u}_o$$

Green's functions

The Green's function G is the solution for a Dirac source δ

$$\mathfrak{L}G = \delta(\mathbf{r})$$

The solution can then be obtained by convolution

$$u(\mathbf{r}) = G(\mathbf{r} - \mathbf{r}') * f(\mathbf{r}') =$$

Secondary field approach

- singular or highly nonlinear source field
  - i.e. $1/r$ behaviour of potential fields from point sources
- boundary conditions require *far-away* boundaries
- well-known field (*primary field*) for background

Solution of Poisson's equation in 1D/2D/3D

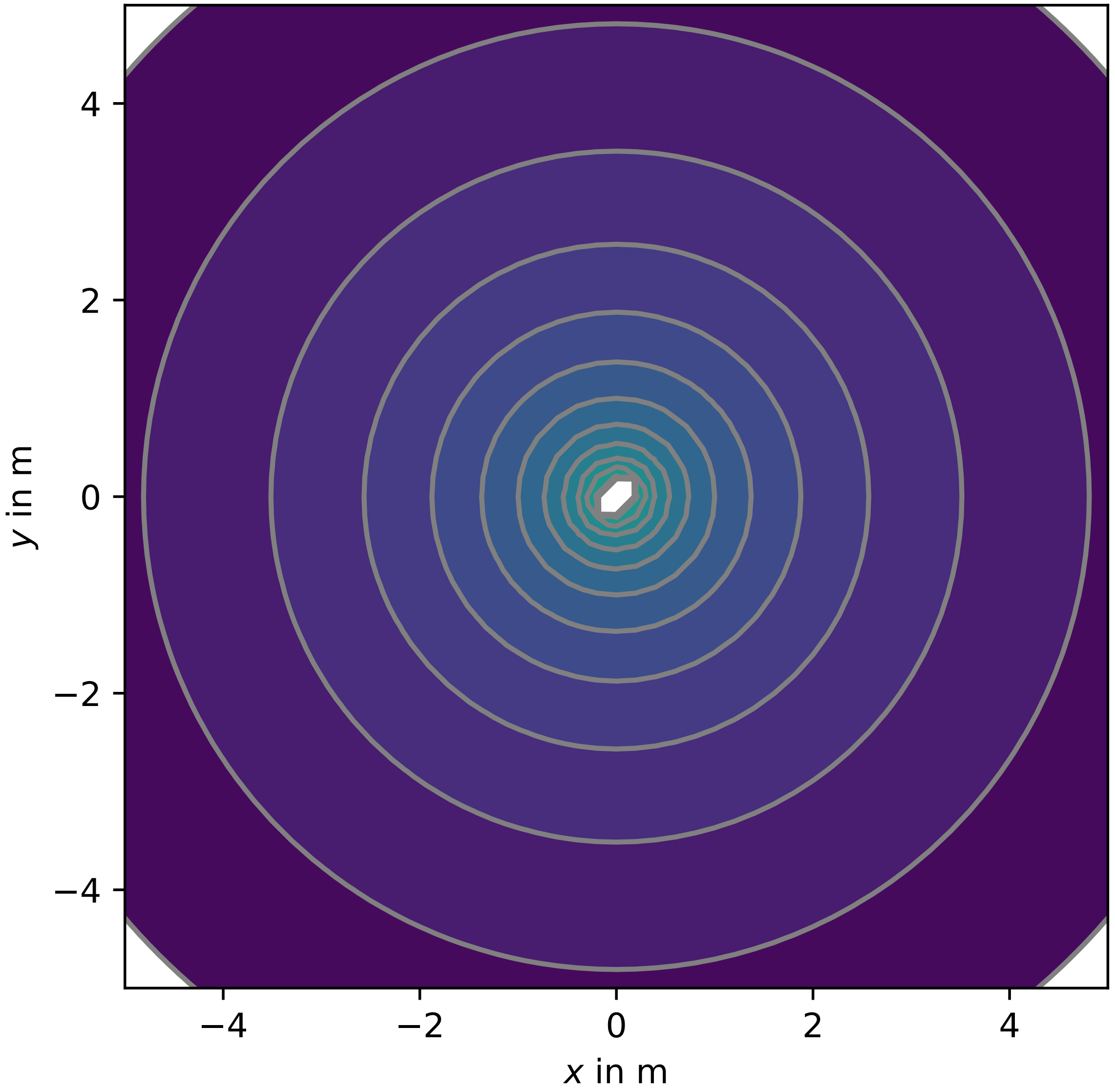
- point source (fundamental solution = Greens function)
- 1D: flow is constant (cannot spread)
- 2D: flow distributes on circle circumference (constant a)

$$q(r) = \frac{Q}{2\pi r} \quad \Rightarrow \quad u = -\frac{Q\rho}{2\pi} \ln r$$

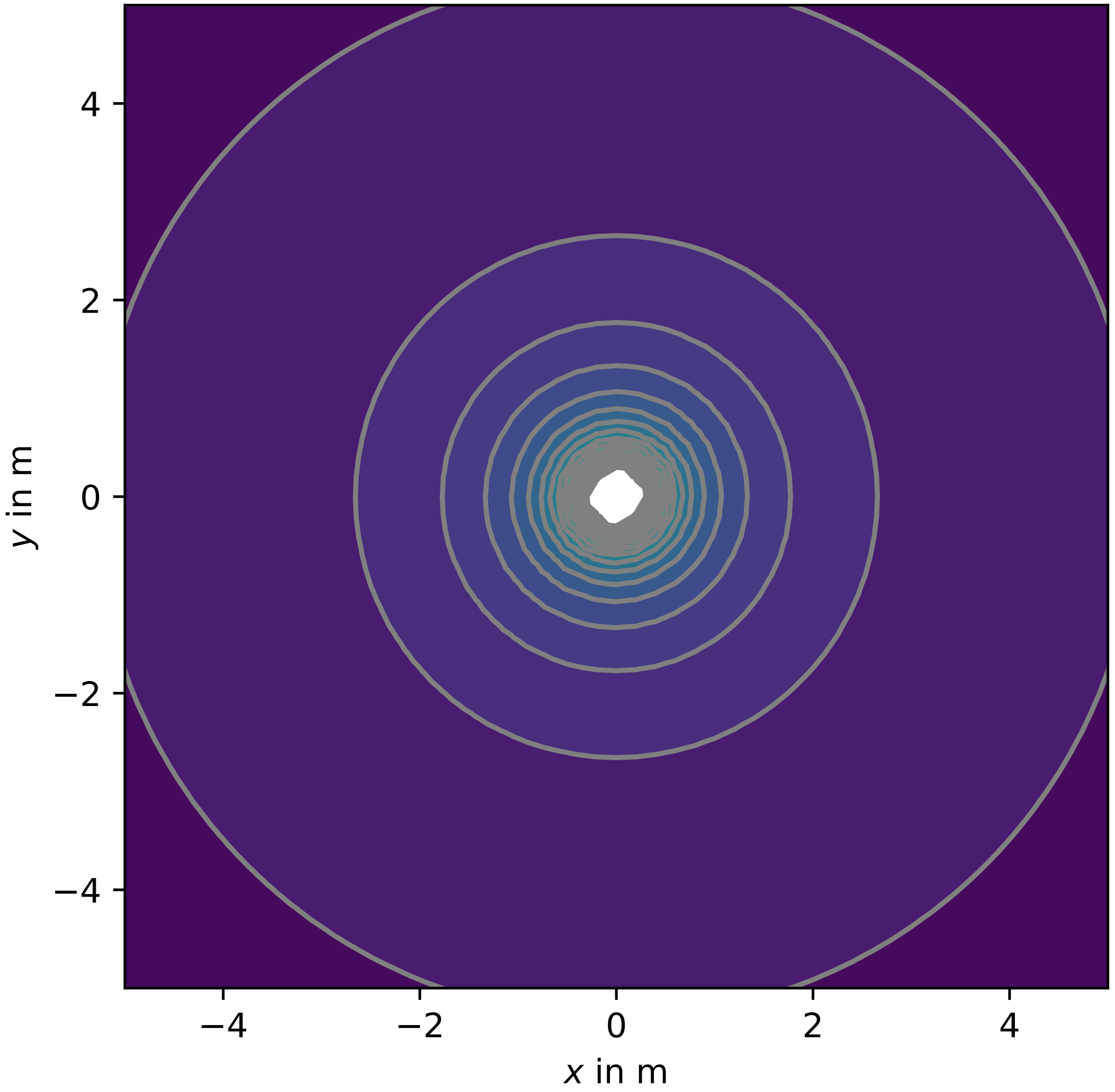
- 3D: flow distributes on sphere surface

$$j(r) = \frac{I}{4\pi r^2} \quad \Rightarrow \quad u = \frac{I\rho}{4\pi r}$$

Spatial 2D and 3D solutions

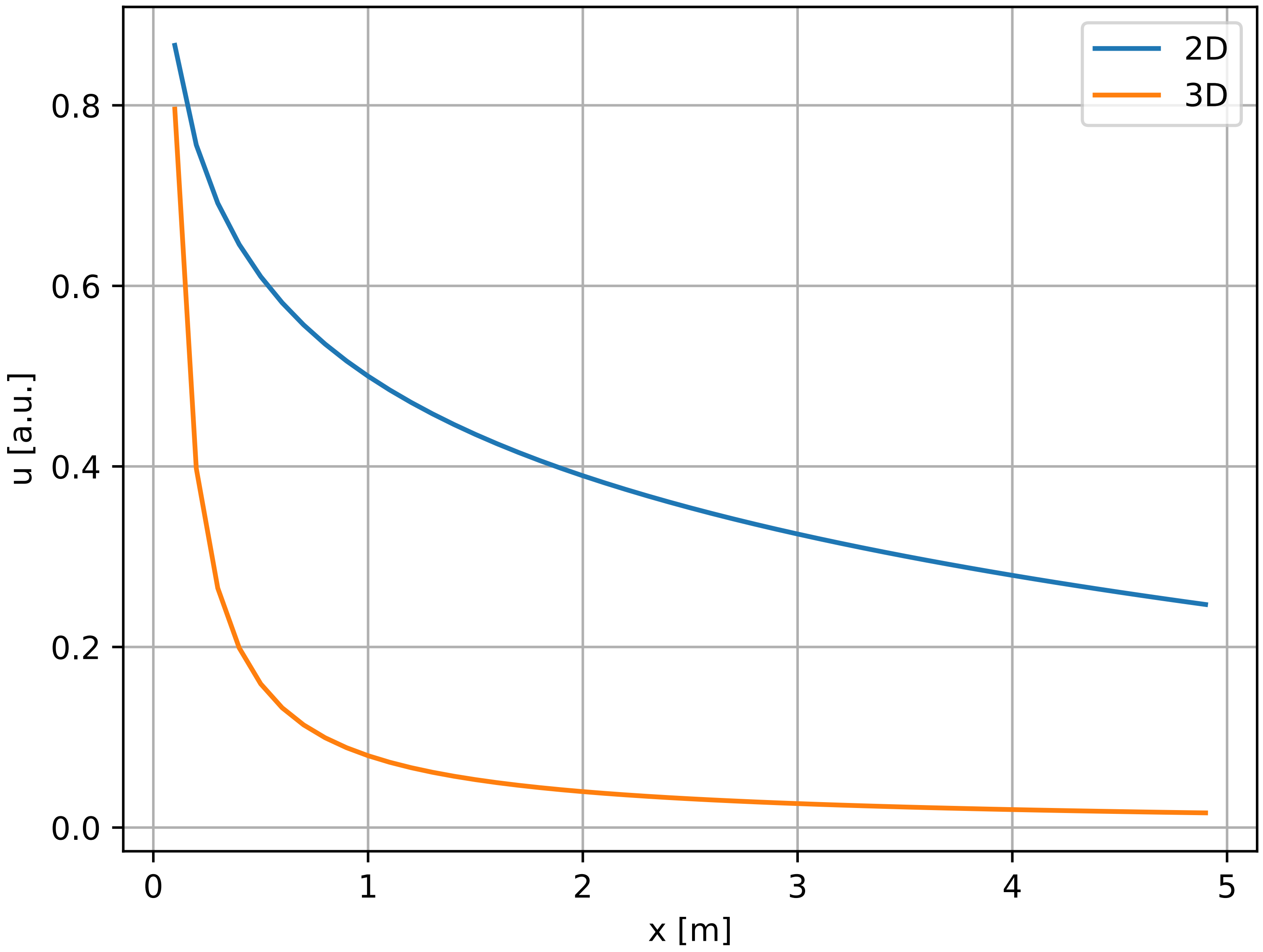


2D solution (-ln r)



3D solution (1/r)

Comparison of 2D and 3D solutions



Direct current resistivity

$$\boldsymbol{\nabla} \cdot \sigma \boldsymbol{\nabla} u = -\boldsymbol{\nabla} \cdot \mathbf{j}_s$$

See notebook for problems

Secondary field approach

$$-\boldsymbol{\nabla}\cdot\sigma\boldsymbol{\nabla}u = \boldsymbol{\nabla}\cdot\mathbf{j}_s$$

u for σ=const: $u_0 = u(\sigma_0) = \frac{I}{2\pi\sigma_0}$ fulfils eq.

Idea: split σ solution in $u = u_0 + u_a$

$$-\boldsymbol{\nabla}\cdot\sigma\boldsymbol{\nabla}u_0 - \boldsymbol{\nabla}\cdot\sigma\boldsymbol{\nabla}u_a = \boldsymbol{\nabla}\cdot\mathbf{j}_s = -\boldsymbol{\nabla}\cdot\sigma_0\boldsymbol{\nabla}u_0$$

$$\Rightarrow -\boldsymbol{\nabla}\cdot\sigma\boldsymbol{\nabla}u_a = \boldsymbol{\nabla}\cdot(\sigma-\sigma_0)\boldsymbol{\nabla}u_0$$

Anomalies *secondary* sources: $\mathbf{A}^{\sigma}\mathbf{u}_a = -\mathbf{A}^{\delta\sigma}\mathbf{u}_0 = (\mathbf{A}^{\sigma_0} - \mathbf{A}^{\sigma})\mathbf{u}_0$

Helmholtz equation in 2D

- move to another type of PDE
- move from 1D to 2D (and eventually 3D)
- scalar to vectorial solution
- complex-valued system
- secondary field approach

Maxwells equations

- Faraday's law: currents & varying electric fields $\Rightarrow$ magnetic field

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$$

- Ampere's law: time-varying magnetic fields induce electric field

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\boldsymbol{\nabla} \cdot \mathbf{D} = \varrho$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\mathbf{D} = \epsilon \mathbf{E}$ & $\mathbf{B} = \mu \mathbf{H}$

Maxwell in frequency domain

$$\mathbf{E} = \mathbf{E}_0 e^{\jmath\omega t} \quad \text{or} \quad \mathbf{H} = \mathbf{H}_0 e^{\jmath\omega t}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \jmath\omega\epsilon\mathbf{E} + \sigma\mathbf{E} = (\sigma + \jmath\omega\epsilon)\mathbf{E}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\jmath\omega\mu\mathbf{H}$$

take curl of one of the equations and insert in the other

Helmholtz equation

see also Theory EM

take curl of one of the equations and insert in the other

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \jmath\omega\sigma\mathbf{E} - \omega^2\epsilon\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \jmath\omega\mu\mathbf{H} - \omega^2\mu\epsilon\mathbf{H} = 0$$

Quasi-static approximation

Assume: $\omega^2\mu\epsilon < \omega\mu\sigma$, no sources ($\boldsymbol{\nabla}\cdot\mathbf{j}_s = 0$), + vector identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{F} - \boldsymbol{\nabla}^2\mathbf{F}$$

leads with $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = \boldsymbol{\nabla}\cdot\mathbf{B}$ to the vector Helmholtz PDE

$$-\nabla^2\mathbf{E} + \jmath\omega\mu\sigma\mathbf{E} = 0$$

$$-\boldsymbol{\nabla}\cdot\sigma^{-1}\nabla\mathbf{H} + \jmath\omega\mu\mathbf{H} = 0$$

Variational form

$$-\boldsymbol{\nabla}^2 u + \jmath\omega\mu\sigma u = f$$

$$-\int_\Omega w\boldsymbol{\nabla}^2 u \mathrm{d}\Omega + \int_\Omega w\jmath\omega\mu\sigma u \mathrm{d}\Omega = \int_\Omega wf \mathrm{d}\Omega$$

Gauss's integral law

$$\int_\Omega \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega + \imath\omega \int_\Omega \mu\sigma w u \mathrm{d}\Omega = \int_\Omega wf \mathrm{d}\Omega$$

Weak formulation

$u = \sum_i u_i \mathbf{v}_i$ and $w_i \in \{v_i\}$ leads to

$$\int_\Omega \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \jmath\omega\mu \int_\Omega \sigma v_i v_j \mathrm{d}\Omega = \int_\Omega v_i f \mathrm{d}\Omega$$

$$\langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle + \jmath\omega\mu \langle v_i | \sigma v_j \rangle = \langle v_i | f \rangle \quad \text{inner products}$$

representation by matrix-vector product $(\mathbf{A} + \jmath\omega\mathbf{M}^\sigma)\mathbf{u} = \mathbf{b}$

with $A_{ij} = \langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle$, $M_{ij}^\sigma = \langle v_i | v_j \rangle$ and $b_i = \langle v_i | f \rangle$

The finite element mass matrix

The mass matrix

$$M_{i,j} = \int_{\Omega} \mu \sigma v_i v_j \mathrm{d}\Omega = \sum_c \int_{\Omega_c} \mu_c \sigma_c v_i v_j \mathrm{d}\Omega$$

can be written for element-wise conductivity and permittivity

$$M_{i,j} = \sum_c \mu_c \sigma_c \int_{\Omega_c} v_i v_j$$

Complex or real-valued?

The complex valued system

$$(\mathbf{A} + \imath\omega\mathbf{M})\mathbf{u} = (\mathbf{A} + \imath\omega\mathbf{M})(\mathbf{u}_r + \imath\mathbf{u}_i) = \mathbf{b}_r + \imath\mathbf{b}_i$$

can be transferred into a doubled real-valued system

$$\mathbf{A}\mathbf{u}_r + \imath\mathbf{A}u_i + \imath\omega\mathbf{M}\mathbf{u}_r - \omega\mathbf{M}\mathbf{u}_i = \mathbf{b}_r + \imath\mathbf{b}_i$$

$$\begin{pmatrix} A & -\omega M \\ \omega M & A \end{pmatrix} \begin{pmatrix} u_r \\ u_i \end{pmatrix} = \begin{pmatrix} b_r \\ b_i \end{pmatrix}$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for F_a, because

- F_a is weaker or smoother (e.g. $F_0 \propto 1/$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

Secondary field Helmholtz equation

The equation $-\boldsymbol{\nabla}^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\boldsymbol{\nabla}^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\boldsymbol{\nabla}^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\boldsymbol{\nabla}^2 F_a - k^2 F_a = \delta k^2 F_0$$

Note

Source terms only arise at anomalous terms, weighted by the primary field.

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

$$-\nabla^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

leads to the discretized form ($\mathbf{A}$-stiffness, $\mathbf{M}$-mass)

$$\mathbf{A}\mathbf{E}_a + \imath\omega\mathbf{M}_\sigma E_a = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{E}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```

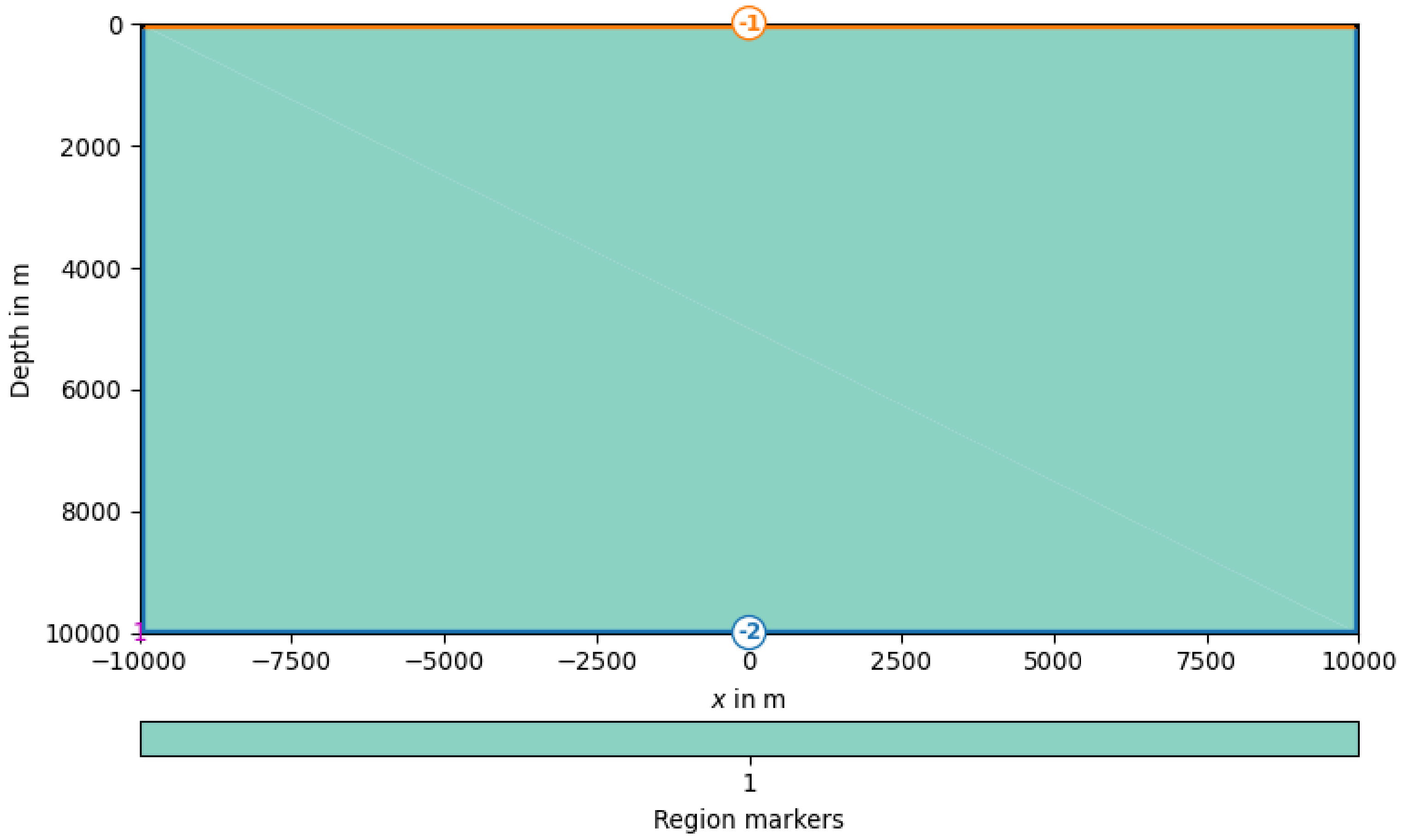
2D/3D problems

Make use of pyGIMLi

See documentation on pyGIMLi.org

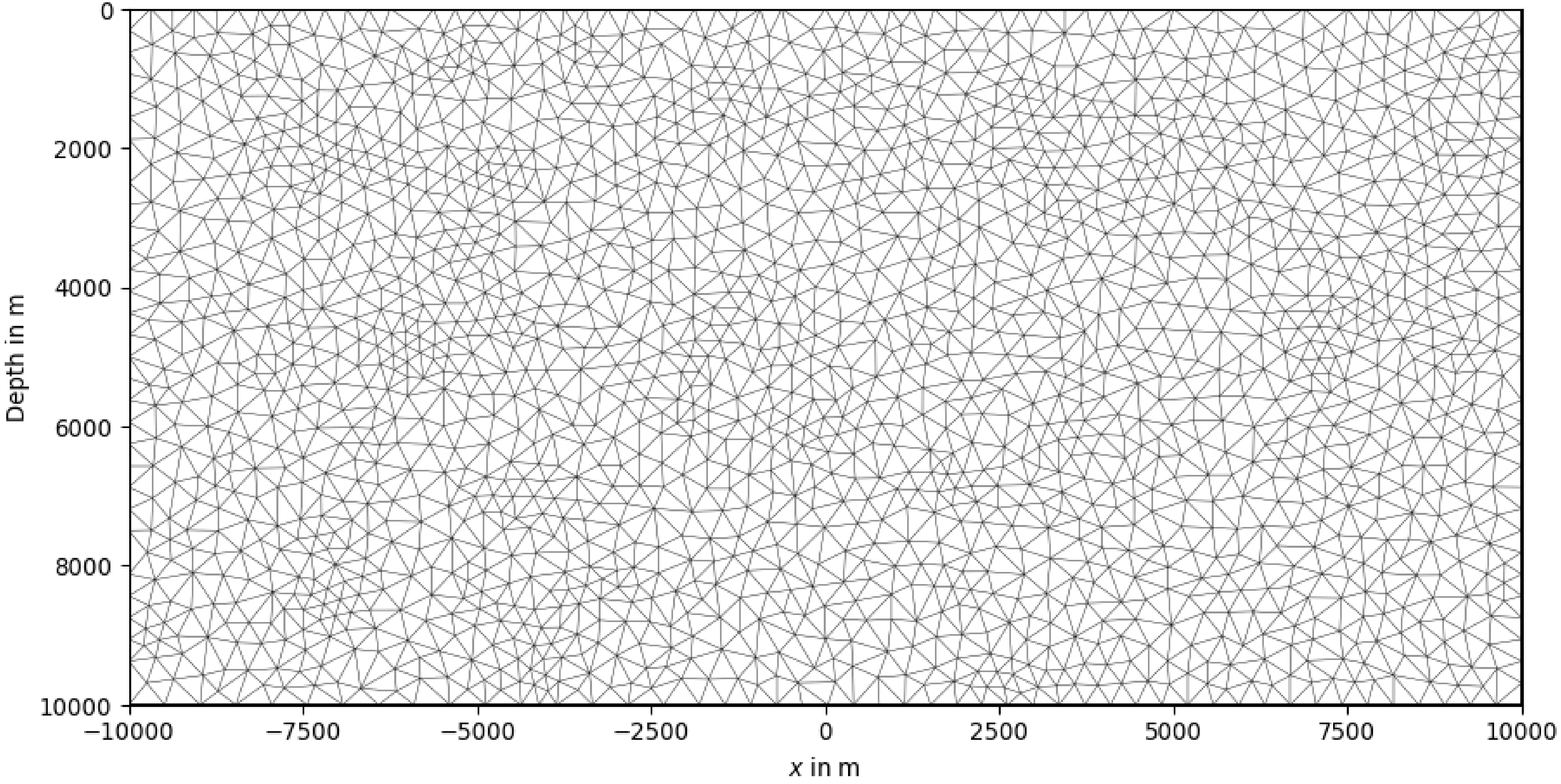
The meshtools module

```
import pygimli as pg
import pygimli.meshtools as mt
world = mt.createWorld(start=[-10000, -10000], end=[10000, 0])
pg.show(world, boundaryMarkers=True)
```



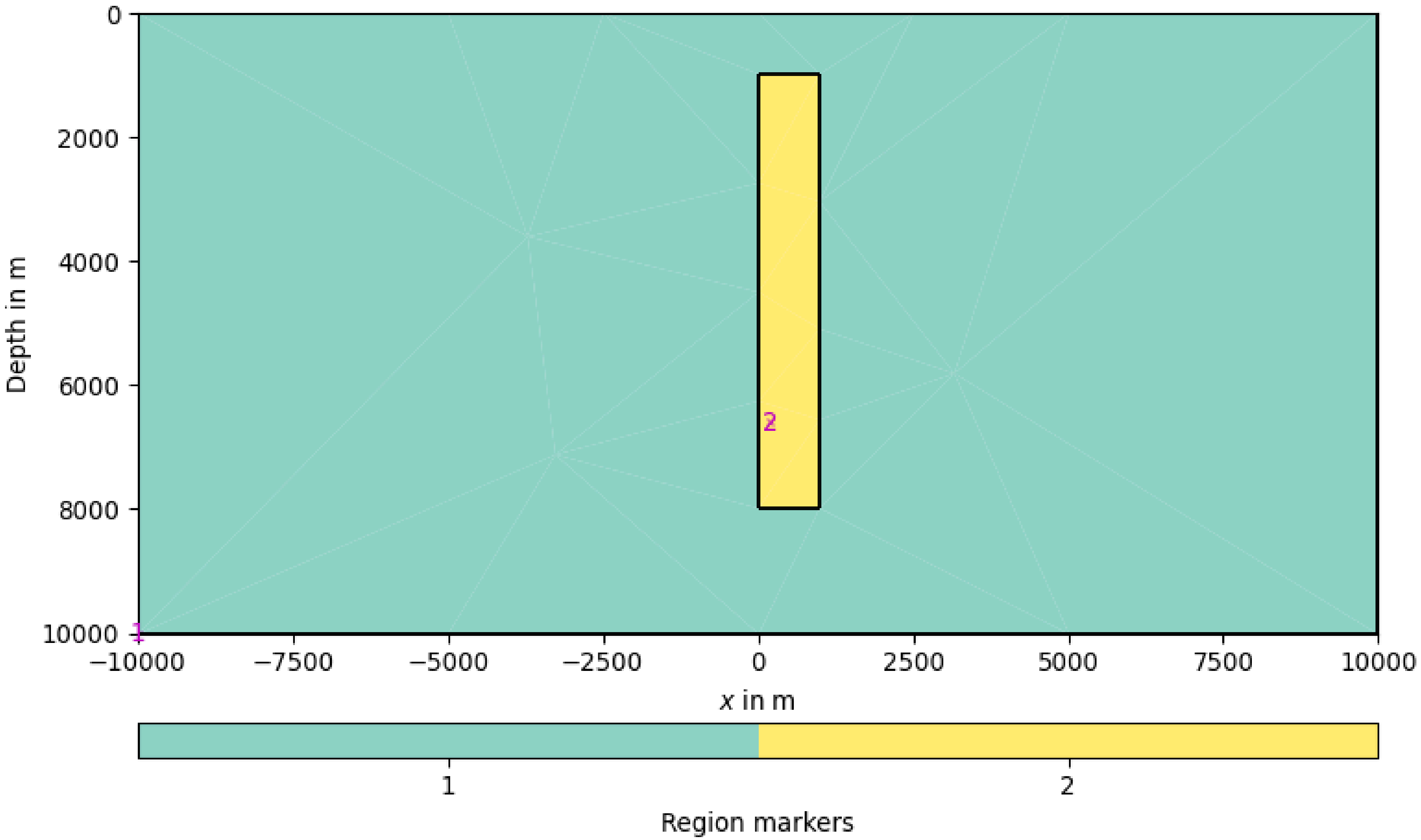
The meshtools module

```
mesh = mt.createMesh(world, quality=34, area=1e5)
pg.show(mesh)
```



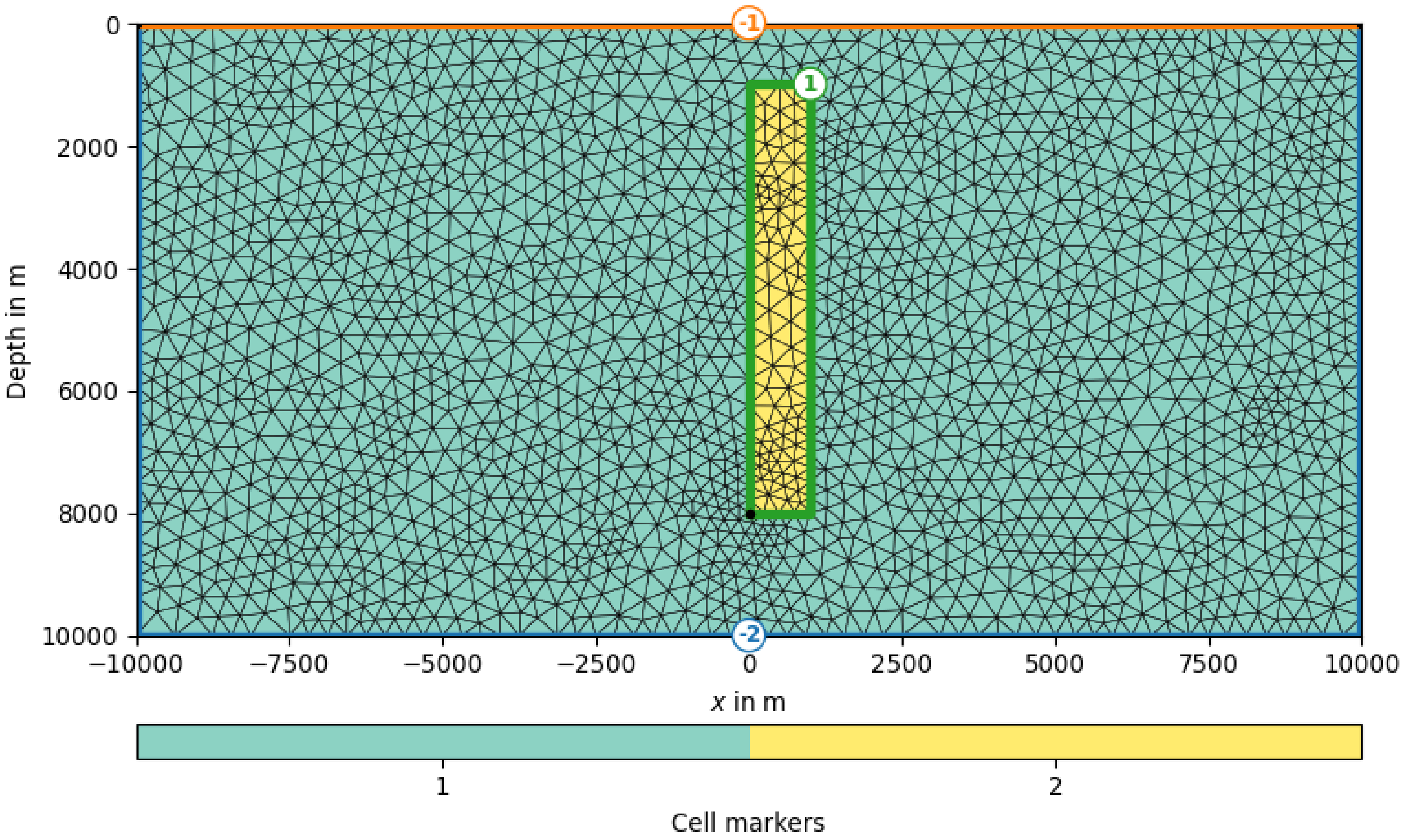
Creating a 2D geometry

```
anomaly = mt.createRectangle(start=[0, -8000], end=[1000, -1000], marker=2)
pg.show(world+anomaly)
```



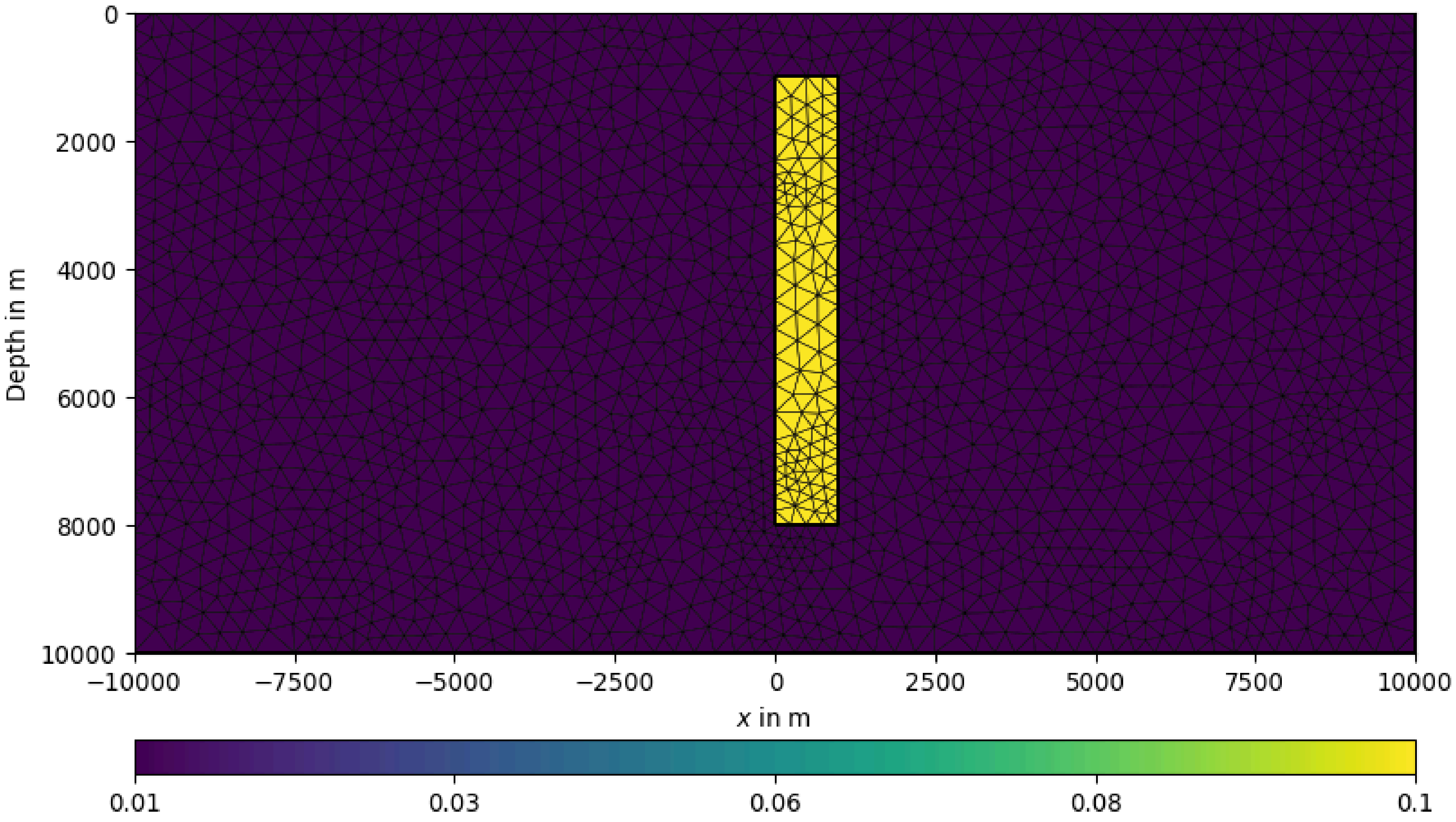
Creating a 2D mesh

```
mesh = mt.createMesh(world+anomaly, quality=34, smooth=True, area=1e5)
pg.show(mesh, markers=True, showMesh=True);
```



Creating a 2D conductivity model

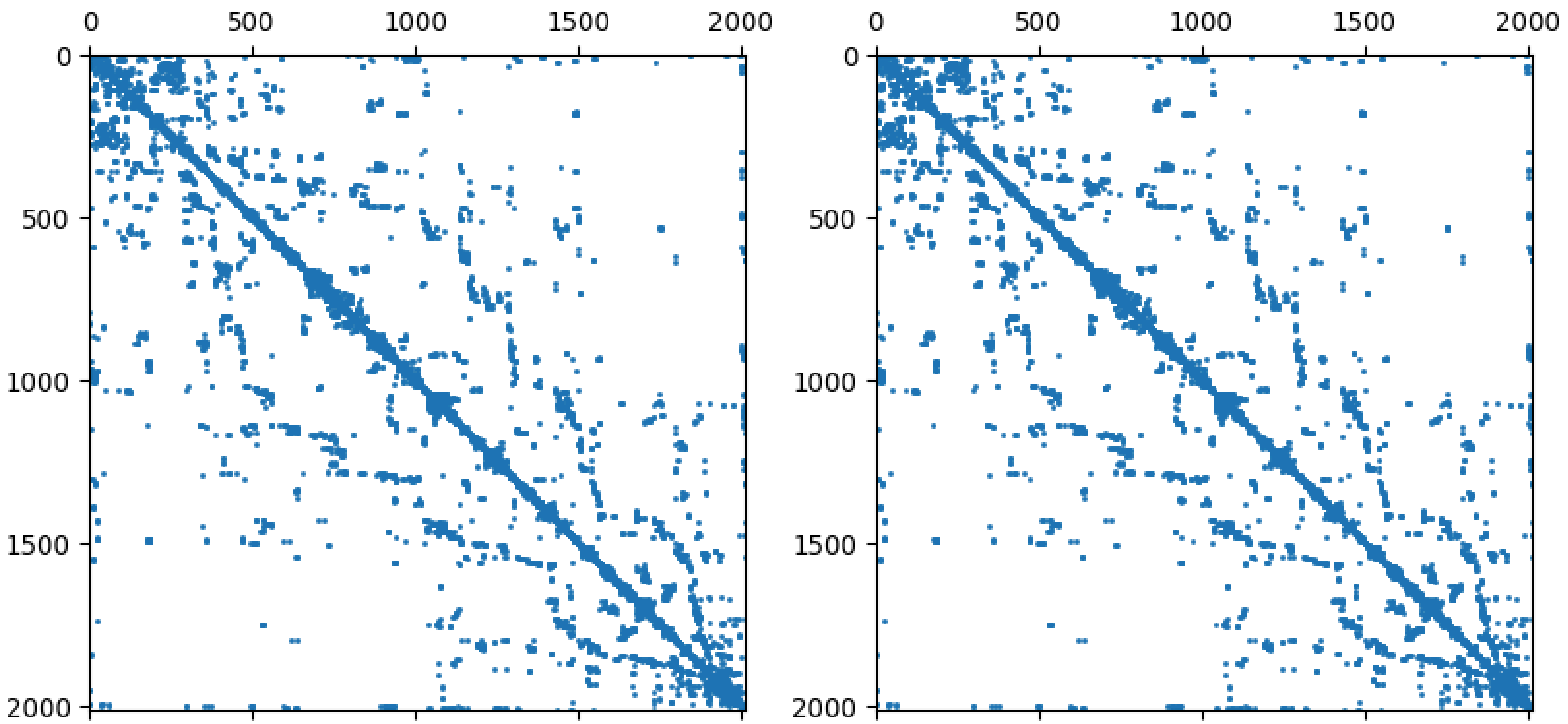
```
sigma0 = 1 / 100  # 100 Ohmm
sigma = mesh.populate("sigma", {1: sigma0, 2: sigma0*10})
pg.show(mesh, "sigma", showMesh=True);
```



The solver module

The solver module

```python
import pygimli.solver as ps
mesh["my"] = 4 * np.pi * 1e-7
A = ps.createStiffnessMatrix(mesh, a=1/mesh["my"])
M = ps.createMassMatrix(mesh, mesh["sigma"])
fig, ax = plt.subplots(ncols=2)
ax[0].spy(pg.utils.toCSR(A), markersize=1)
ax[1].spy(pg.utils.toCSR(M).todense(), markersize=1)
```



The complex problem matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

```
w = 0.1
nd = mesh.nodeCount()
B = pg.BlockMatrix()
B.Aid = B.addMatrix(A)
B.Mid = B.addMatrix(M)
B.addMatrixEntry(B.Aid, 0, 0)
B.addMatrixEntry(B.Aid, nd, nd)
B.addMatrixEntry(B.Mid, 0, nd, scale=-w)
B.addMatrixEntry(B.Mid, nd, 0, scale=w)
pg.show(B)
```