# Numerical Simulation Methods in Geophysics, Part 3: FD Implementation

1. MGPY+MGIN, 3. MDRS+MGEX-CMG

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# Recap and exercise

# Task: solve Poisson equation

$$\nabla \cdot (a \nabla u) = f$$

(stationary) potential field, e.g., temperature, flux, current

simplest method: Finite differences

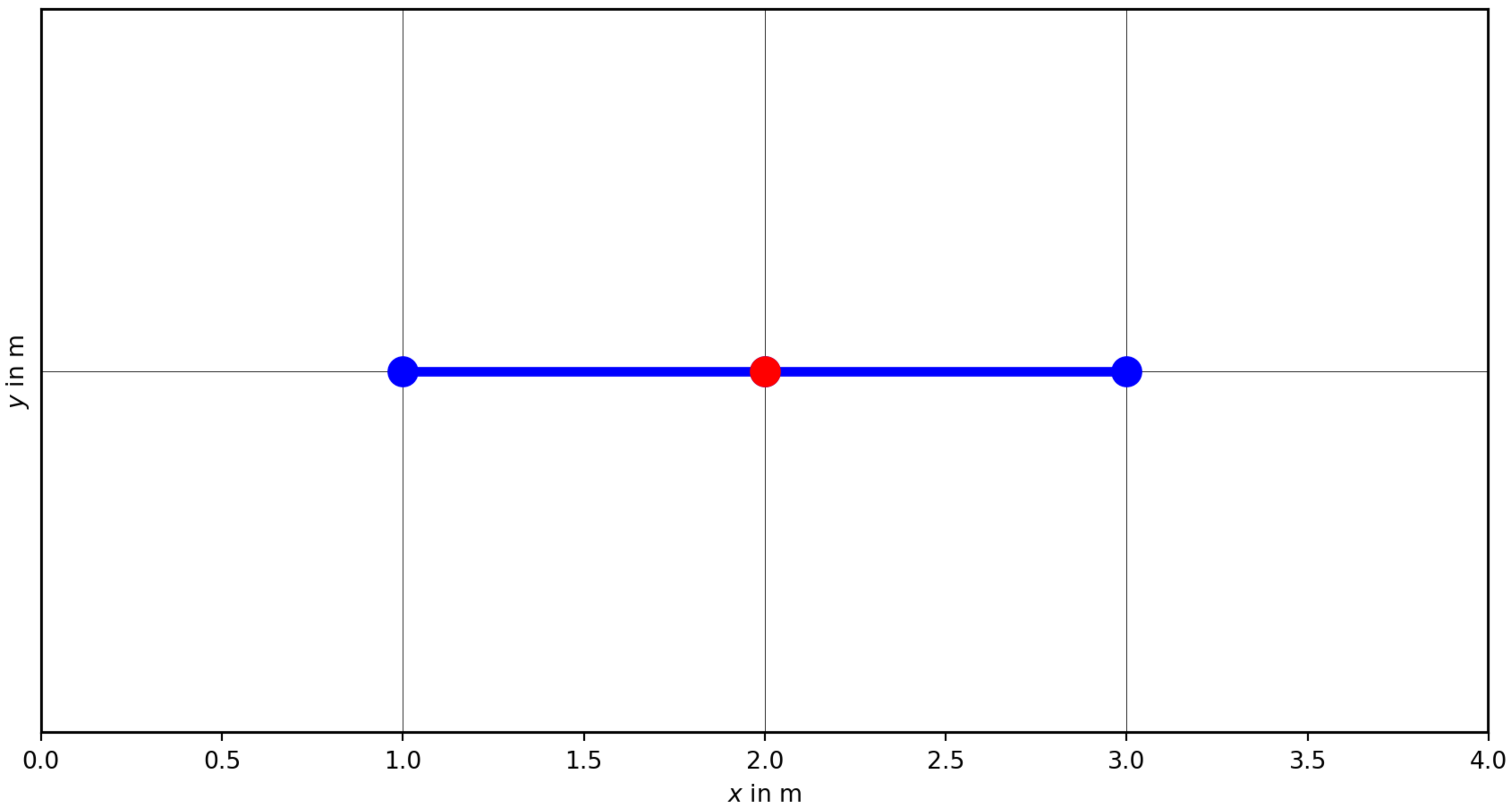
FD: Approximate derivative operators by differences using finite values $u_i$ at points $x_i$, e.g. a

# Unit solution

2nd derivative $[+1, -2, +1] \Rightarrow \mathbf{A} \cdot \mathbf{u} = \mathbf{f}$ with the stiffness matrix

$$\mathbf{A} = \begin{bmatrix} +1 & -2 & +1 & 0 & \dots & & \\ 0 & +1 & -2 & +1 & 0 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ \dots & \dots & 0 & +1 & -2 & +1 & \end{bmatrix}$$

# Finite difference stencil




compute each value (red) with the help of its neighbors (blue)

# Dirichlet boundary conditions

$$u_B - 2u_1 + u_3 = f_1$$

$$\begin{bmatrix} -2 & +1 & 0 & \dots & \\ +1 & -2 & +1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \dots & 0 & +1 & -2 & +1 \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} f_1 - u_B \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

no change in coefficients, $u_B$ on rhs act as outer source

# Neumann boundary conditions

$$u_0 - 2u_1 + u_2 = f_1 \qquad u_1 - u_0 = g_B \Rightarrow u_2 - u_1 = f_1 + g_B$$

$$\begin{bmatrix} -1 & +1 & 0 & \dots & \\ +1 & -2 & +1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \dots & 0 & +1 & -2 & +1 \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} f_1 + g_B \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

change in self-coupling, $g_B$ on rhs adds to source

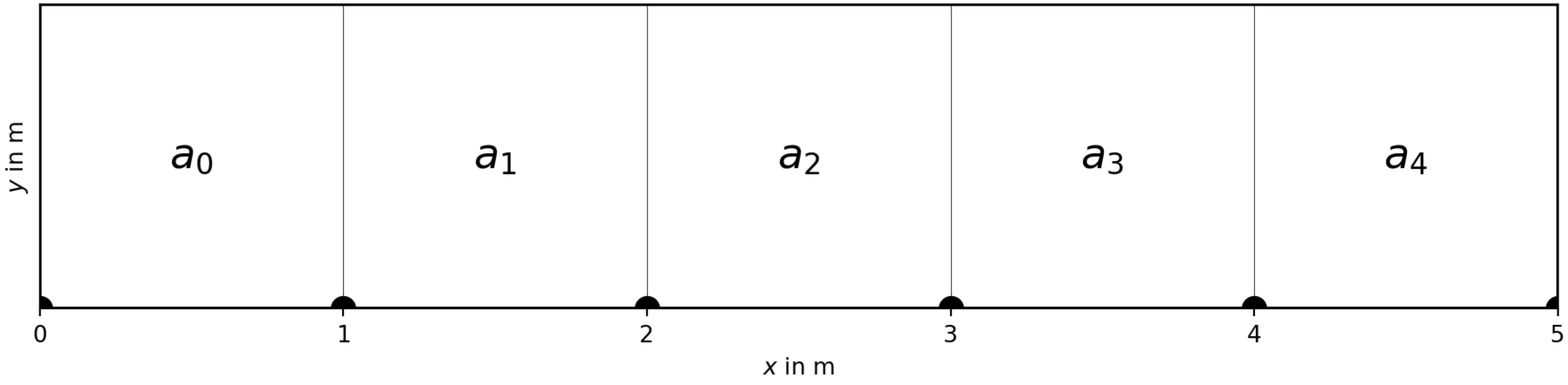
# Tasks

1. Create a stiffness matrix for unit quantities
2. Implement Dirichlet BC on one and Neumann on other side
3. Solve system for different right-hand sides:
    - no source at all
    - single source in the middle or at the boundary
    - several sources with different strengths (& signs)
    - source on part of the domain
4. Alwas plot the solution and itsLaplacian

# The general case

$$\Delta x \neq 1 \;\&\; a \neq 1$$

$$a\frac{\partial u}{\partial x}$$



# Tasks

1. Derive the coefficiencs for the general case
2. Write a function implementing the general case
3. Divide the “subsurface” in regions with different $a$
4. Compute the solution for different source fields
5. Use a non-equidistant discretisation
6. Always plot solution along with source and Laplacian

# Parabolic PDEs

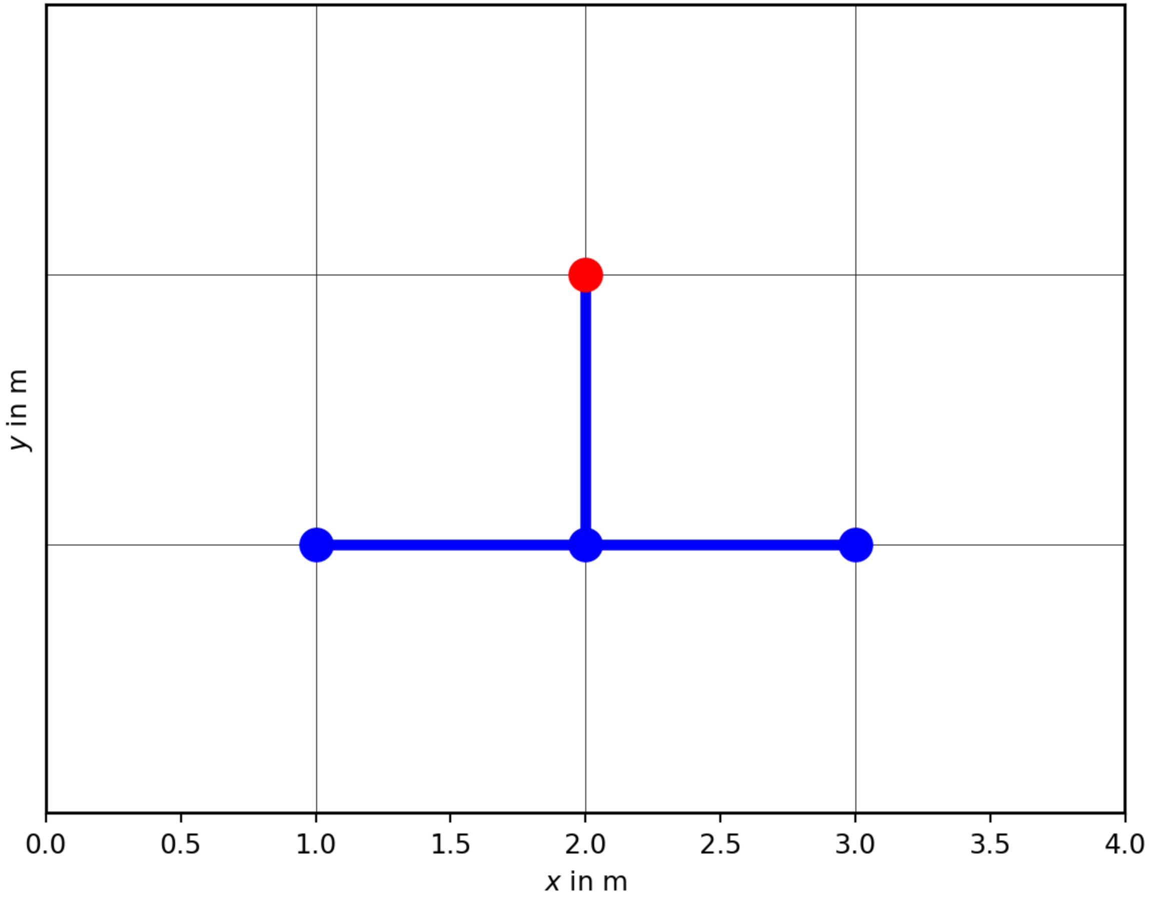
# Heat transfer in 1D

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial z^2} = 0$$

with the periodic boundary conditions: * $T(z=0,t) = T_0 + \Delta T \sin \omega t$ (daily/yearly cycle) * $\frac{\partial T}{\partial z}(z=z_1) = 0$ (no change at depth) and the initial condition $T(z,t=0) = \sin \pi z$ has the analytical solution

$$T(z,t) = \Delta T e^{-\pi^2 t} \sin \pi z$$

# FD stencil



# Explicit methods

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial z^2} = 0$$

Finite-difference approximation

$$\frac{\partial T}{\partial t}^{n} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = a\frac{\partial^2 T}{\partial z^2}^{n}$$

# Explicit

Solve Poisson equation $\boldsymbol{\nabla} \cdot (a\boldsymbol{\nabla} u) = f$

for every time step $i$ (using FDM, FEM, FVM etc.)

Finite-difference step in time: update field by

$$T_{i+1} = T_i + a\frac{\partial^2 u}{\partial z^2} \cdot \Delta t$$

# Implicit methods

$$\frac{\partial T}{\partial t}^{n} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

# Mixed - Crank-Nicholson method

$$\frac{\partial T}{\partial t}^{n} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n+1}$$