Numerical Simulation Methods in Geophysics, Part 10: 2D Helmholtz equation

1. MGPY+MGIN

thomas.guenther@geophysik.tu-freiberg.de

TUBAF
Die Ressourcenuniversität.
Seit 1765.

Recap

- Maxwell equations in time domain
- harmonic approach (or decomposition)
  $\Rightarrow$ (complex-valued) Helmholtz equation for E and B
- solve 1D Helmholtz equation complex-values
  - compare with analytic solution

Next lectures and exercises

- LV10: 15.01. exercise on 16.01.
- LV11: 22.01. exercise on 23.01.
- LV12: 29.01. exercise on 30.01.
- VL13: 05.02., exercise on 06.02.
- report on 2D Helmholtz equations

Todo

- solve 2D Helmholtz equation
  - use secondary field approach
- use wide range of frequencies
  - combine E and H to yield MT sounding curves
- excurse on 3D vectorial Maxwell solvers
- overview on equation solvers and high-performance computing
- outlook to computational fluid dynamics

Electromagnetic fields in the Earth

Maxwell equations lead to diffusion equation

$$\frac{\partial^2 \mathbf{B}}{\partial z^2} = \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t}$$

A periodic excitation ($B_0 e^{\imath\omega t}$) leads to (cf. temperature problem)

$$B = B_0 e^{-z/d} \cos(\omega t - z/d)$$

with the skin depth $d = \sqrt{2/(\mu_0 \sigma \omega)} \approx 503 \sqrt{\rho/f}$

Electromagnetic fields in the Earth

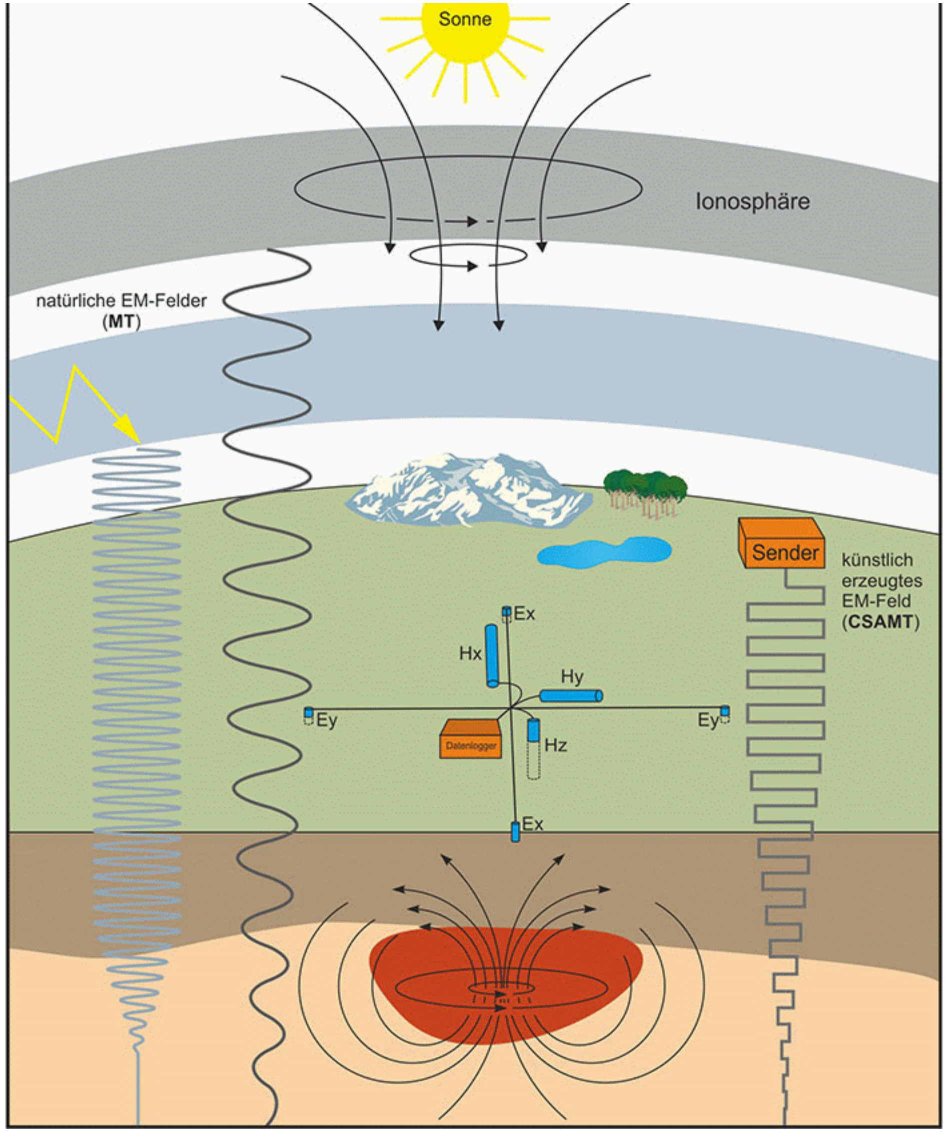
Assume E field in x direction

$$E_x = -\frac{1}{\mu_0 \sigma} \frac{\partial B_y}{\partial z} = \frac{B_0}{\mu_0 \sigma d} e^{-z/d} \sqrt{2} \cos(\omega t - z/d + \pi/4)$$

$\Rightarrow$ phase shift of 45° $(\pi/4)$ between E_x und B_y

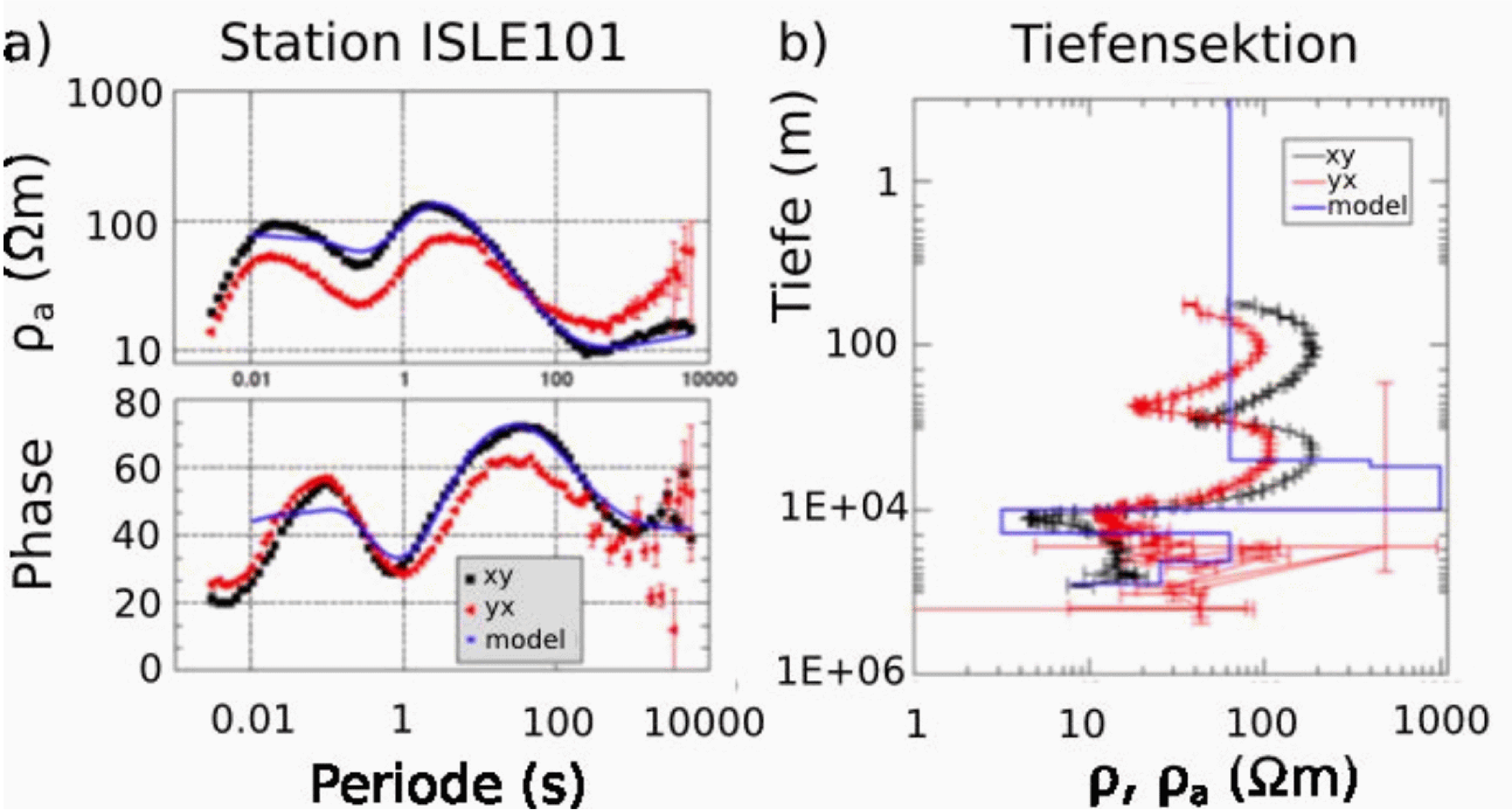
$$\frac{|E_x|}{|B_y|} = \frac{\sqrt{2}}{\mu_0 \sigma d} \Rightarrow \rho = \frac{\mu_0}{\omega} \left| \frac{E_x}{B_y} \right|^2$$

The magnetotelluric (MT) method

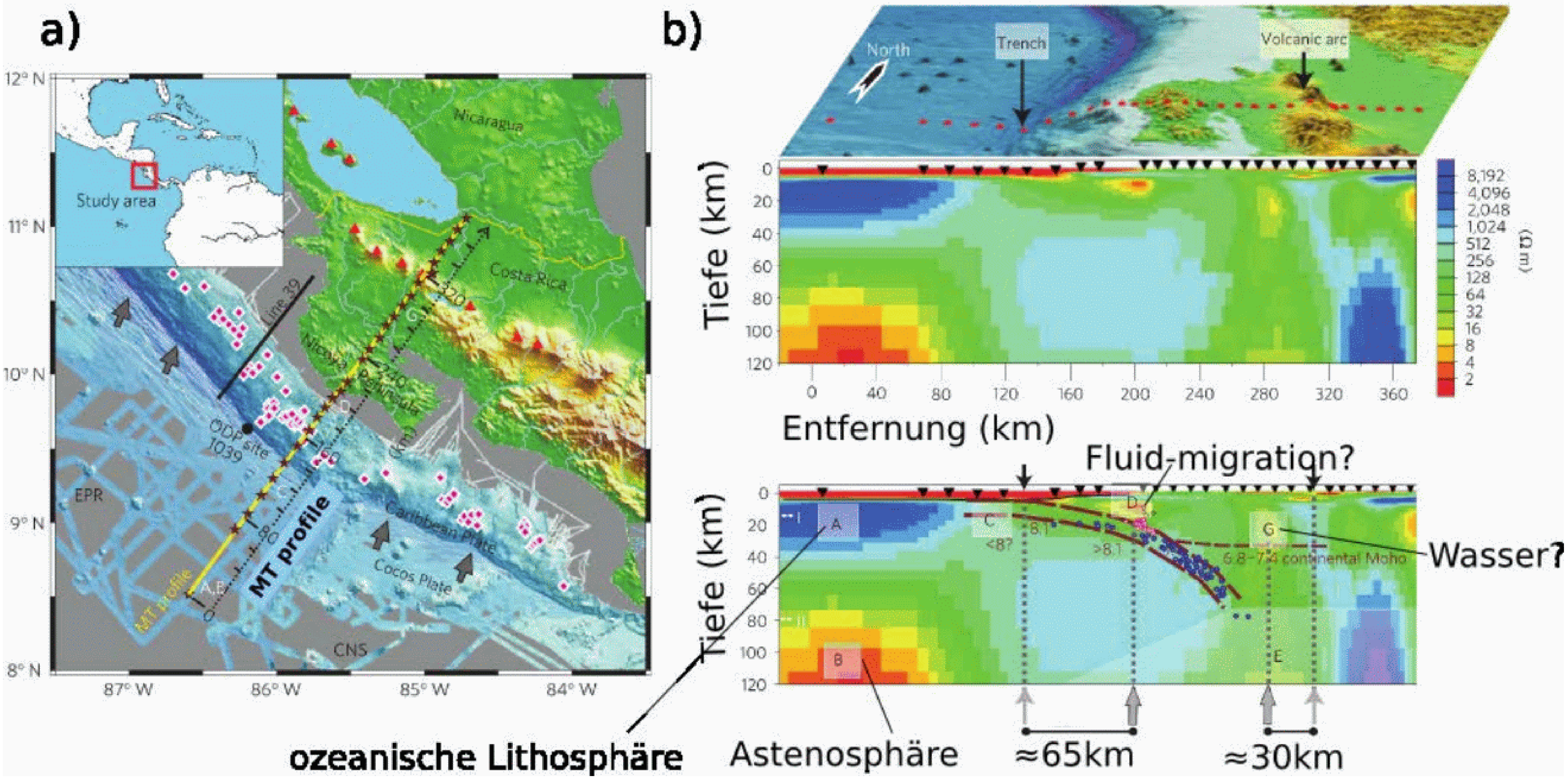


- inductive electromagnetics with f=0.001-100,Hz (T=0.01-1000s)
- source in ionosphere (natural source) or on ground (controlled source)
- measure magnetic and electric fields
- analyse (complex) ratio in frequency domain
- depth sounding (ρ_a & ϕ) with T
- MT course of Anna Marti (U Barcelona)

Magnetotelluric depth sounding



2D/3D Magnetotellurics



Imaging of a subduction zone (Worszewski et al., 2011)

Helmholtz equations

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} + \imath\omega\sigma\mathbf{E} - \omega^2 \epsilon \mathbf{E} = -\imath\omega\mathbf{j}_s$$

$$\nabla \times \sigma^{-1} \nabla \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2 \epsilon\mu/\sigma\mathbf{H} = \nabla \times \sigma^{-1}\mathbf{j}_s$$

PDEs identical $\mathbf{E}$ and $\mathbf{H}$ through exchanging μ and σ

component perpendicular to modelling frame (E/H polarization)

$$\nabla \times a \nabla \times = -\nabla \cdot a \nabla$$

Finite element discretization

- weak formulation (for E)

$$\int_\Omega \mu^{-1} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \imath\omega \int_\Omega \sigma v_i v_j \mathrm{d}\Omega = \int_\Omega v_i f \mathrm{d}\Omega$$

- stiffness = second derivative $\boldsymbol{\nabla} \cdot \mathbf{v}_i$, expressed by 2 gradients

$$\mathbf{A}_{i,j} = \int_\Omega \mu^{-1} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega$$

Finite element discretization

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- mass matrix resembles functions $\mathbf{v}_i$

$$\mathbf{M}_{i,j} = \int_\Omega \sigma v_i \cdot v_j \mathrm{d}\Omega$$

Complex or real-valued?

Either discretize the complex system

$$(\mathbf{A} + \imath\omega\mathbf{M})(\mathbf{u}_r + \imath\mathbf{u}_i) = \mathbf{b}_r + \imath\mathbf{b}_i$$

by complex shape functions OR transfer into real

$$\mathbf{A}\mathbf{u}_r + \imath\mathbf{A}\mathbf{u}_i + \imath\omega\mathbf{M}\mathbf{u}_i - \omega\mathbf{M}\mathbf{u}_i = \mathbf{b}_r + \imath\mathbf{b}_i$$

$$\begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{b}_r \\ \mathbf{b}_i \end{pmatrix}$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$ (or $F_p + F_s$)

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for F_a, because

- F_a is weaker or smoother (e.g. $F_0 \propto 1/r^n$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

Example DC resistivity

$$-\boldsymbol{\nabla} \cdot (\sigma \boldsymbol{\nabla} u) = \boldsymbol{\nabla} \cdot j_s = I\delta(\mathbf{r} - \mathbf{r}_s)$$

Problem: point source leads to infinite potential

$$u(\mathbf{r}) = \frac{I}{2\pi\sigma} \frac{1}{|\mathbf{r} - \mathbf{r}_s|}$$

Approach: $u = u_p(\sigma_0) + u_s(\sigma - \sigma_0)$:

$$-\boldsymbol{\nabla} \cdot (\sigma \boldsymbol{\nabla} u_s) = \boldsymbol{\nabla} \cdot ((\sigma - \sigma_0) \boldsymbol{\nabla} u_p)$$

Example DC resistivity

$$-\boldsymbol{\nabla} \cdot (\sigma \boldsymbol{\nabla} u_s) = \boldsymbol{\nabla} \cdot ((\sigma - \sigma_0) \boldsymbol{\nabla} u_p)$$

discrete form using unit conductivity $\sigma_1 = 1$S/m

$$\mathbf{A}^{\sigma} \mathbf{u}_s = \mathbf{A}^{\delta\sigma} \mathbf{u}_p = \mathbf{A}^{\sigma} \mathbf{u}_p - \sigma_0 \mathbf{A}^{1} \mathbf{u}_p$$

Secondary field Helmholtz equation

The equation $-\boldsymbol{\nabla}^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\boldsymbol{\nabla}^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\boldsymbol{\nabla}^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\boldsymbol{\nabla}^2 F_a - k^2 F_a = \delta k^2 F_0$$

Note

Same operator, source terms at anomalies, weighted by the primary field.

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

leads to the discretized form ($\mathbf{A}$-stiffness, $\mathbf{M}$-mass)

$$\mathbf{A}\mathbf{E}_a + \imath\omega\mathbf{M}_\sigma E_a = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{E}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```

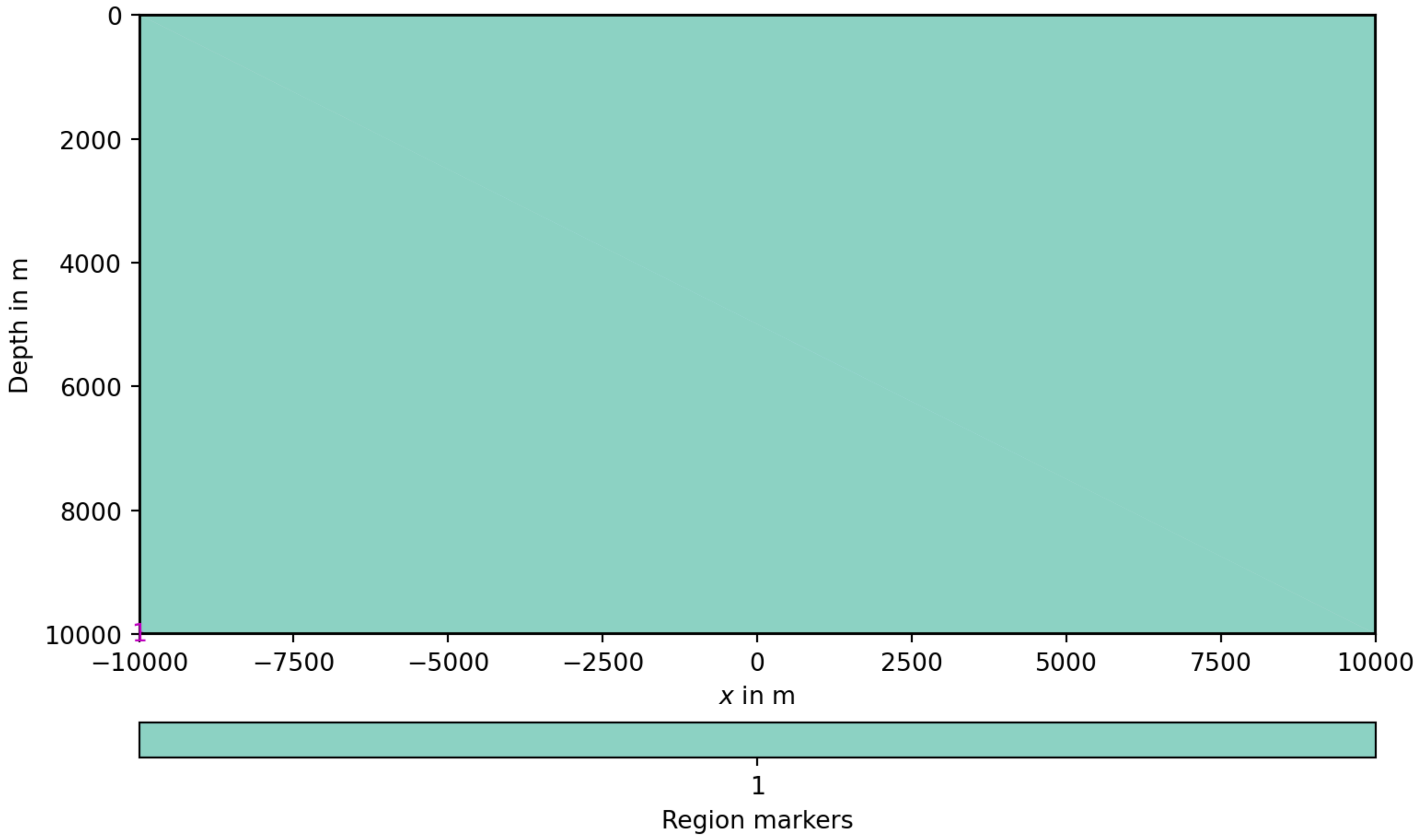
2D problems

Make use of pyGIMLi

See documentation on pyGIMLi.org

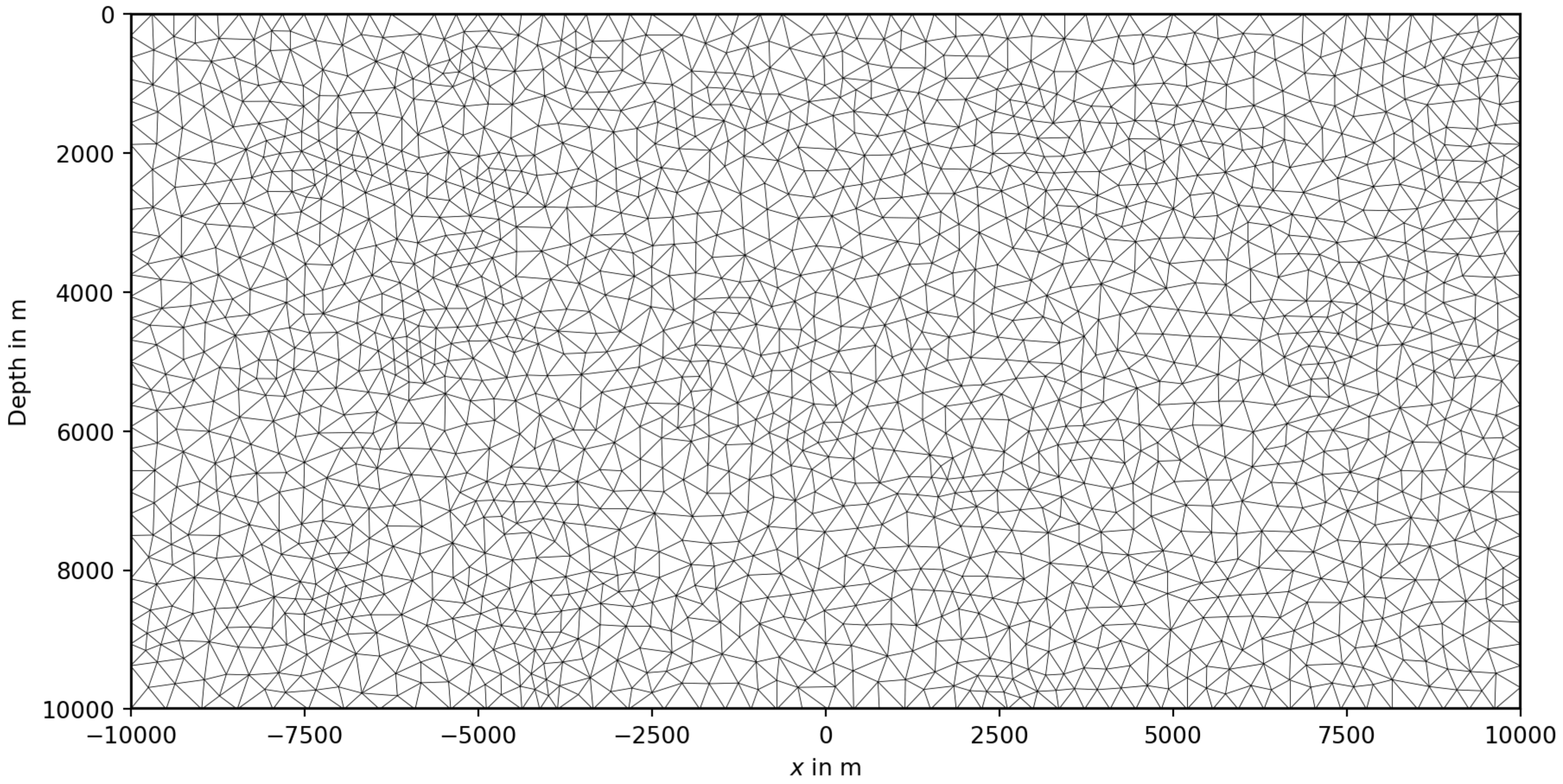
The meshtools module

```
import pygimli as pg
import pygimli.meshtools as mt
world = mt.createWorld(start=[-10000, -10000], end=[10000, 0])
pg.show(world)
```



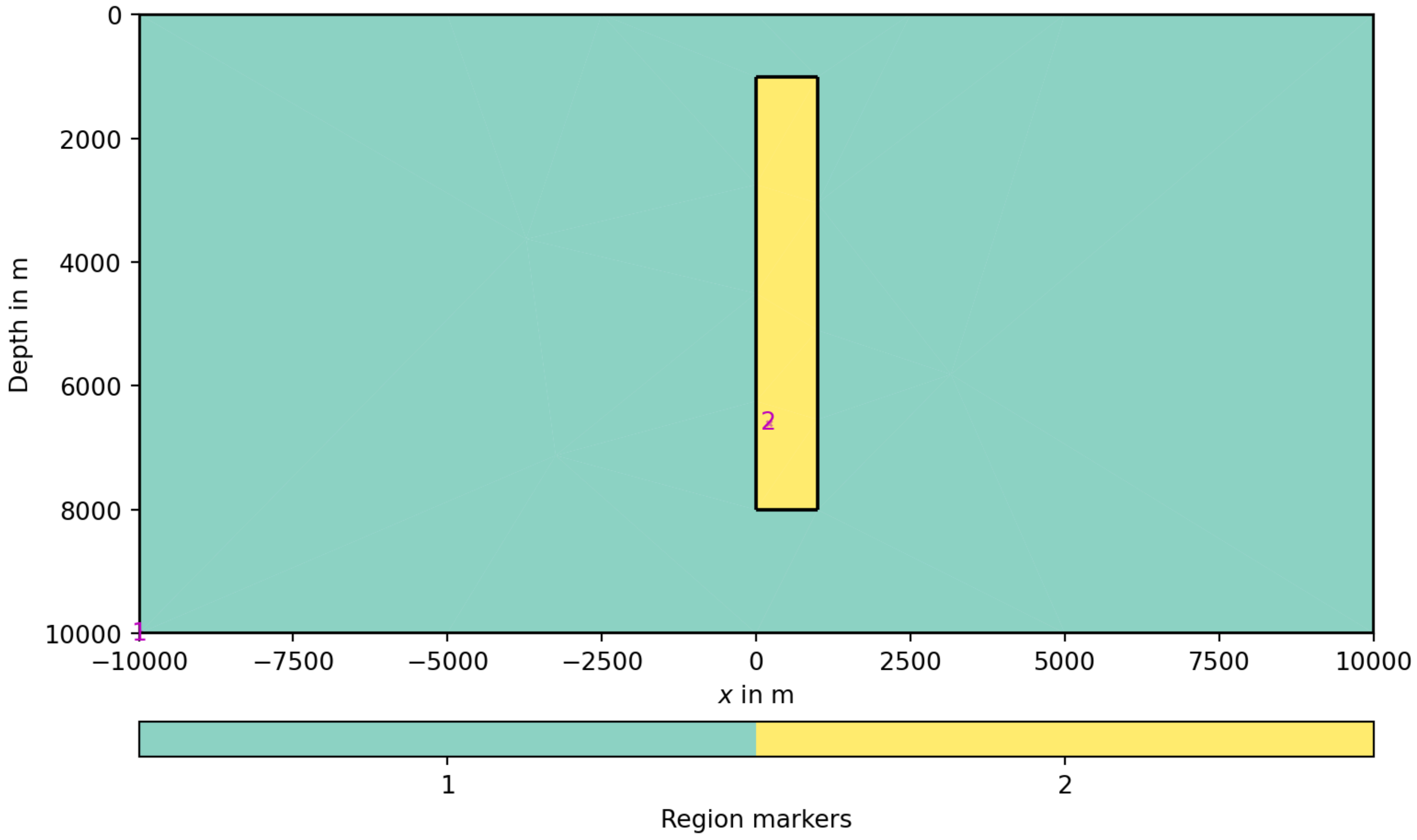
The meshtools module

```python
mesh = mt.createMesh(world, quality=34, area=1e5)
pg.show(mesh)
```



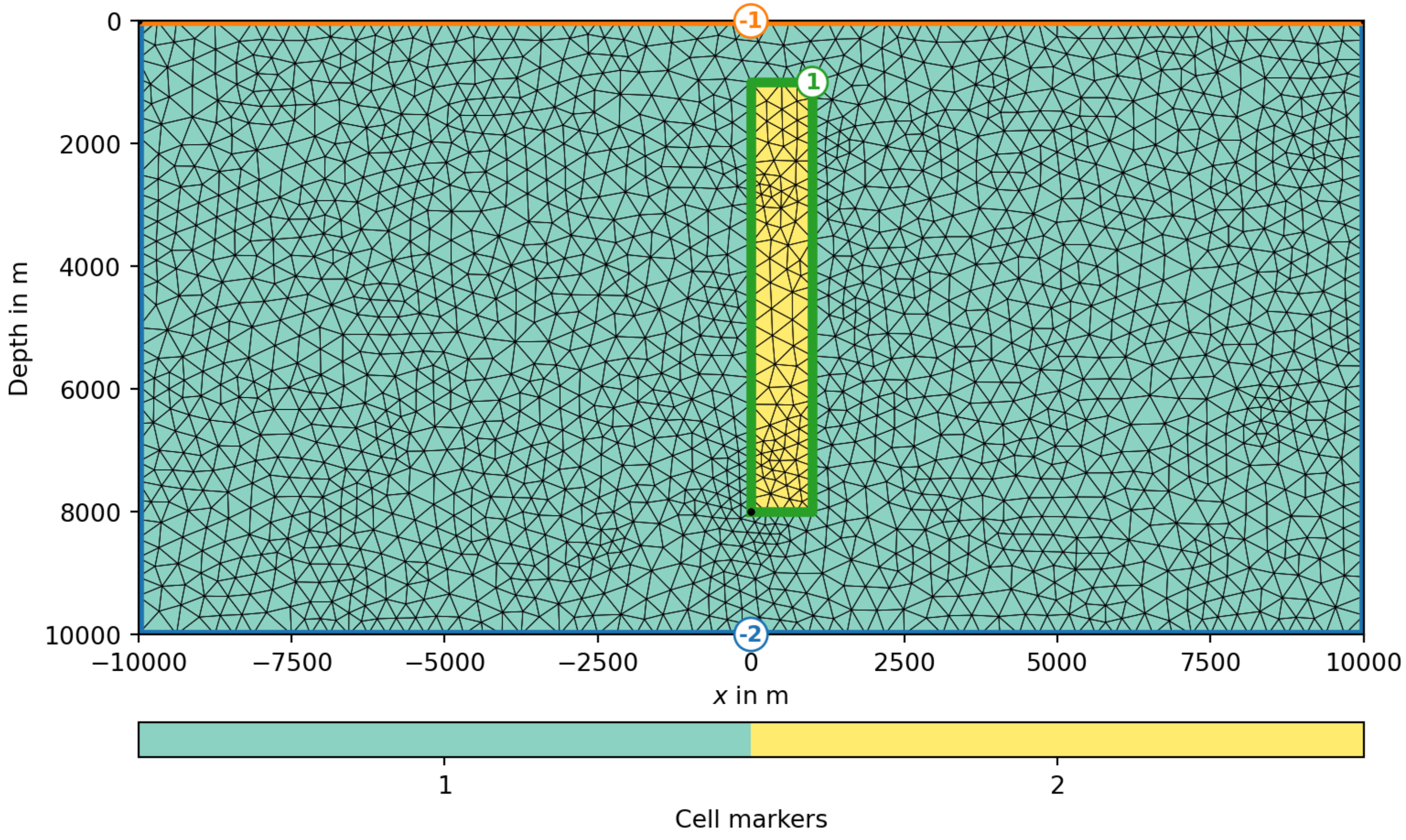
Creating a 2D geometry

```python
anomaly = mt.createRectangle(start=[0, -8000], end=[1000, -1000], marker=2)
pg.show(world+anomaly)
```



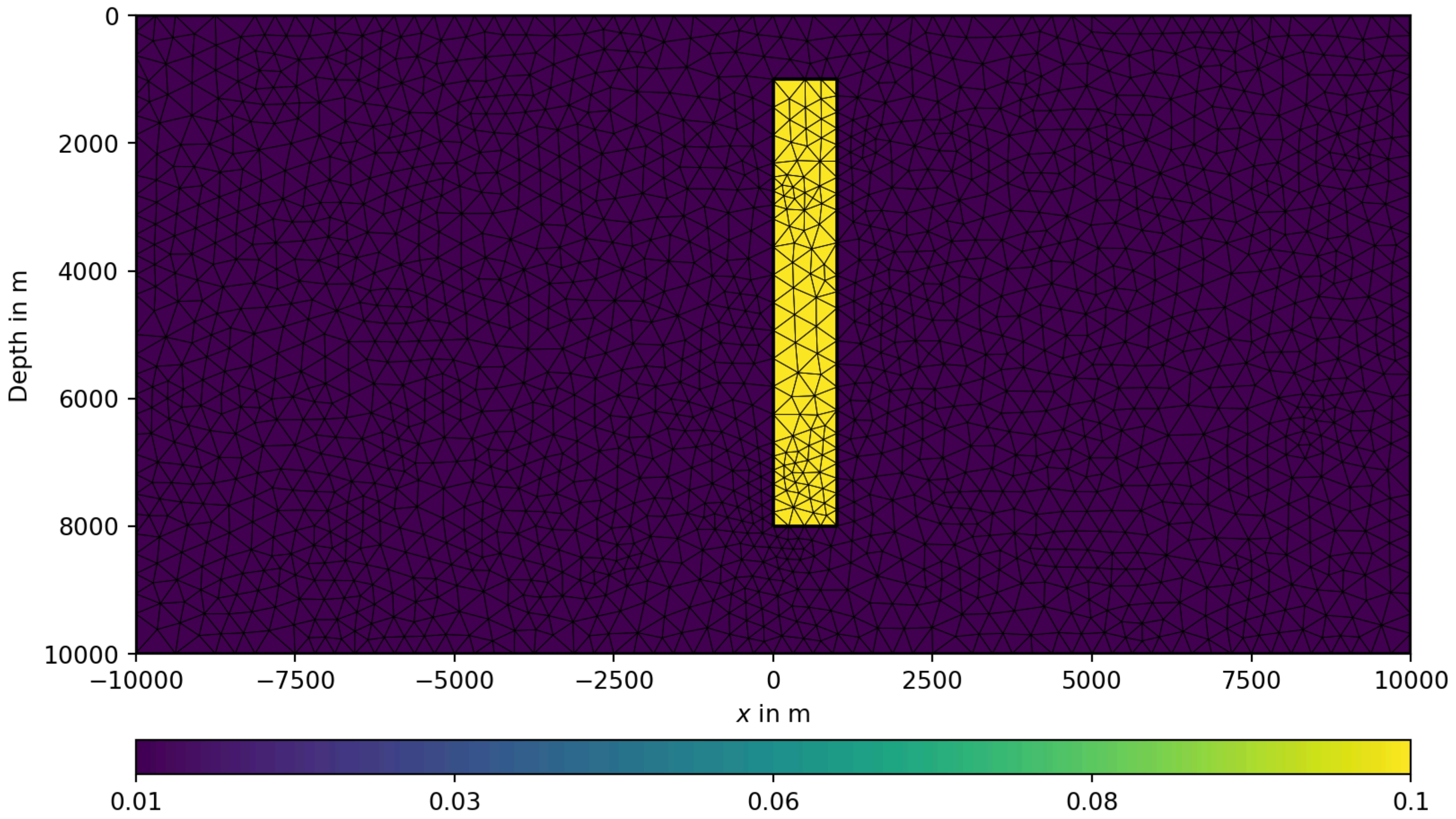
Creating a 2D mesh

```python
mesh = mt.createMesh(world+anomaly, quality=34, smooth=True, area=1e5)
pg.show(mesh, markers=True, showMesh=True);
```



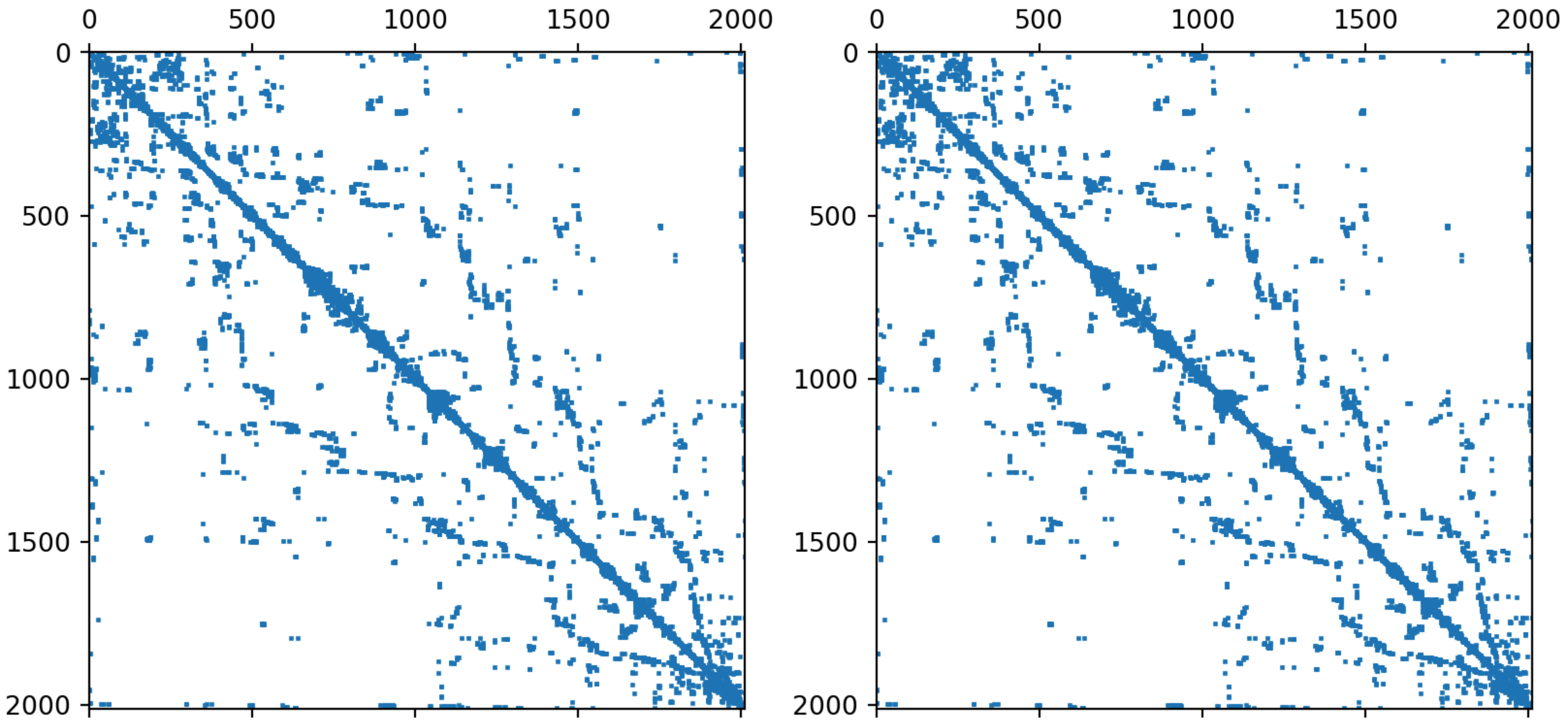
Creating a 2D conductivity model

```
sigma0 = 1 / 100  # 100 Ohmm
sigma = mesh.populate("sigma", {1: sigma0, 2: sigma0*10})
pg.show(mesh, "sigma", showMesh=True);
```



The solver module

```
import pygimli.solver as ps
mesh["my"] = 4 * np.pi * 1e-7
A = ps.createStiffnessMatrix(mesh, a=1/mesh["my"])
M = ps.createMassMatrix(mesh, mesh["sigma"])
fig, ax = plt.subplots(ncols=2)
ax[0].spy(pg.utils.toCSR(A), markersize=1)
ax[1].spy(pg.utils.toCSR(M).todense(), markersize=1)
```



The complex problem matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

Sparse matrices

Up to now: regular (dense) array: save every element including 0

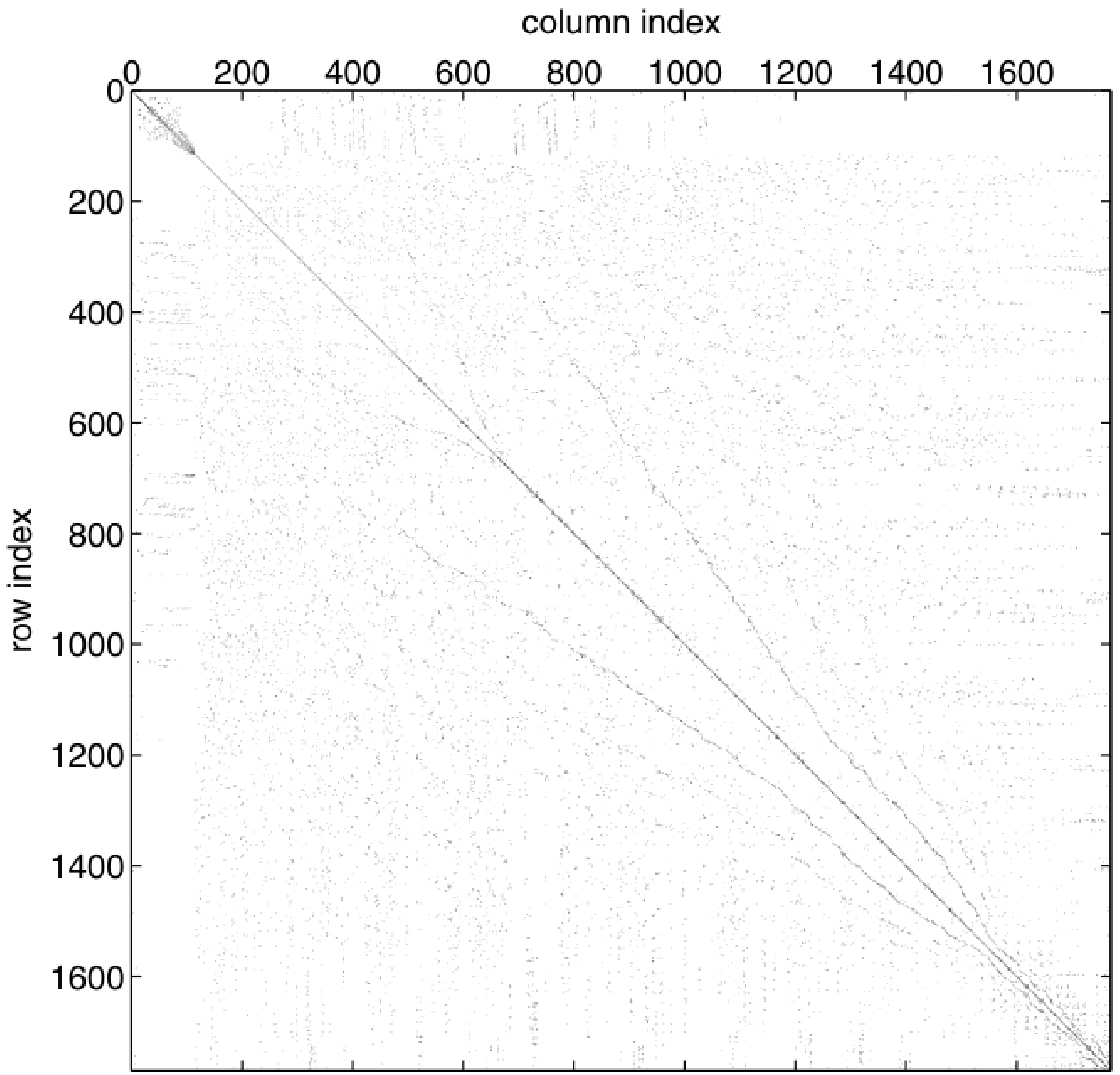
Save only non-zero components (e.g. using `scipy.sparse`)

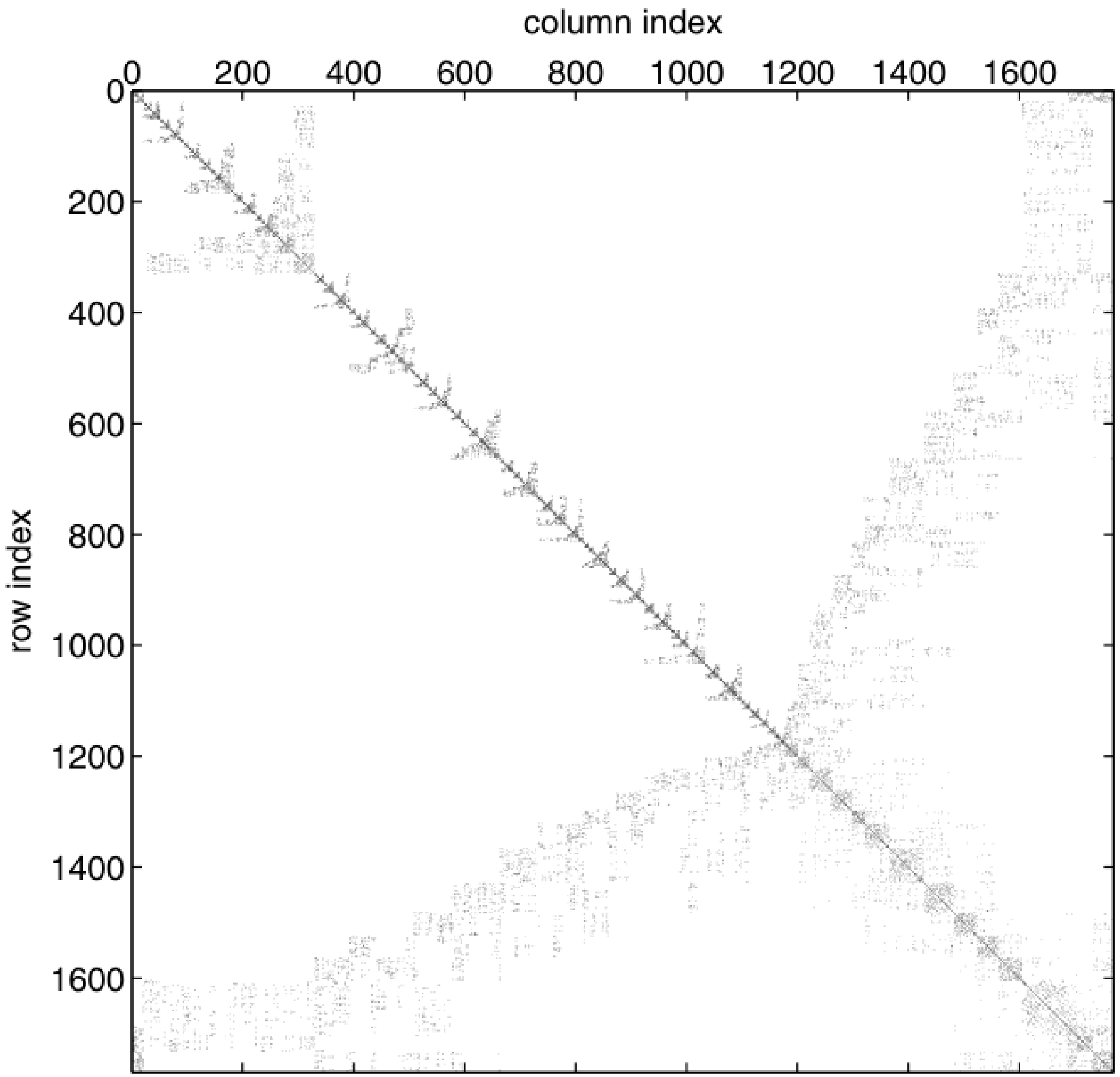
- COO - coordinate format
- CSC/CRS - compressed sparse column/row
- BSR - block sparse row format, …

Solve systems of equations

- Gauss elimination (expensive and dense)
- Cholesky (or ILU) decomposition
- Iterative solvers (conjugate gradients)
- incomplete factorizations or factorization of submatrices

Reordering





original (left) & reordered (right) matrix (Rücker et al. 2006)