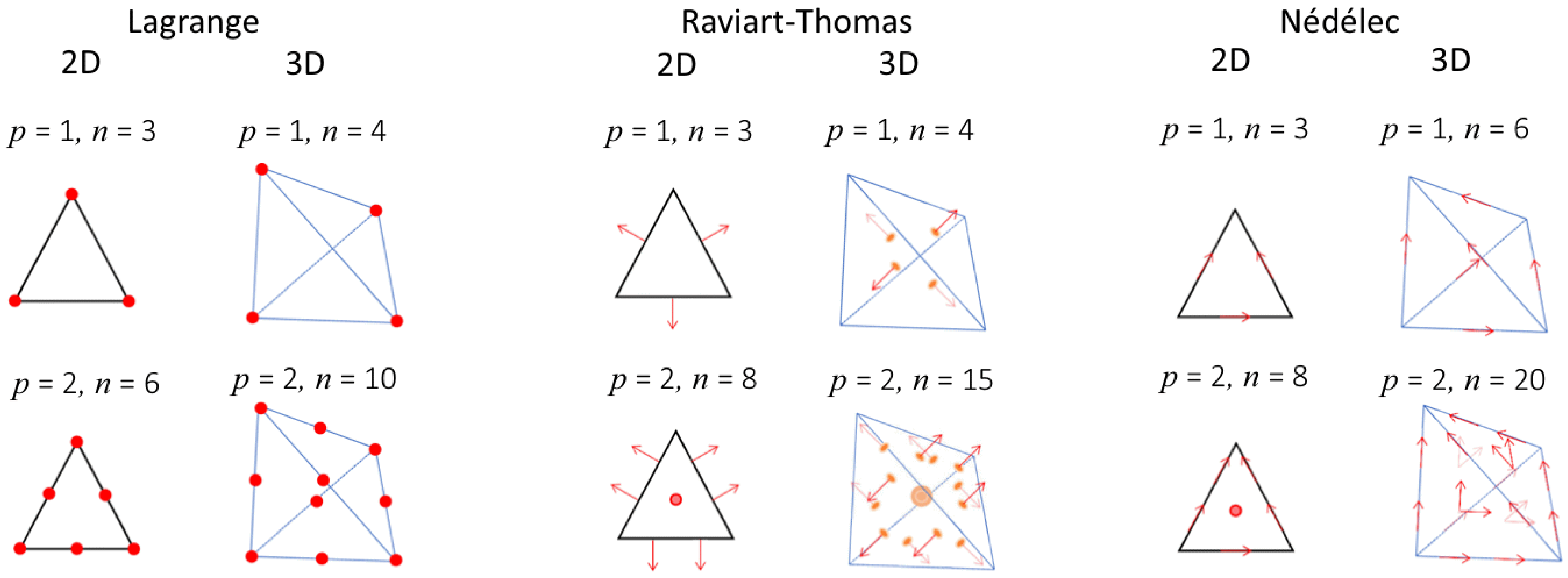
Numerical Simulation Methods in Geophysics, Exercise 0

1. MGPY+MGIN, 3. MDRS+MGEX-CMG

Thomas Günther, thomas.guenther@geophysik.tu-freiberg.de

TU BERGAKADEMIE FREIBERG TUBAF Die Ressourcenuniversität. Seit 1765.

Introduction



$\mathcal{H}^1(\Omega) \;:=\; \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^3\}$ $\quad$ $\mathcal{H}(\mathrm{div};\Omega) \;:=\; \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\}$ $\quad$ $\mathcal{H}(\mathrm{curl};\Omega) \;:=\; \{v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3\}$

Content

0. Introduction
1. Partial differential equations in geophysics
2. Finite Differences
3. The Finite element method
4. Integral equations and Method of Moments
5. Solving linear systems
6. The Finite Volume method
7. High-performance computing

Schedule

Lectures Thursday, 09:45-11:15, MEI-0150

14 slots: 23.10., 30.10., 06.11., 13.11., 20.11., 27.11., 04.12., 11.12., 18.12., 08.01., 22.01., 28.01., 05.02., 06.02. (15.01. dies)

Exercises Wednesday, 09:45-11:15, CIP pool MEI1203a

14 slots: 22.10., 29.10., 05.11., 12.11., 26.11., 03.12., 10.12., 17.12., 07.01., 14.01., 21.01., 28.01., 04.02., 11.02. (19.11. holiday)

Grade: submitting two reports including codes

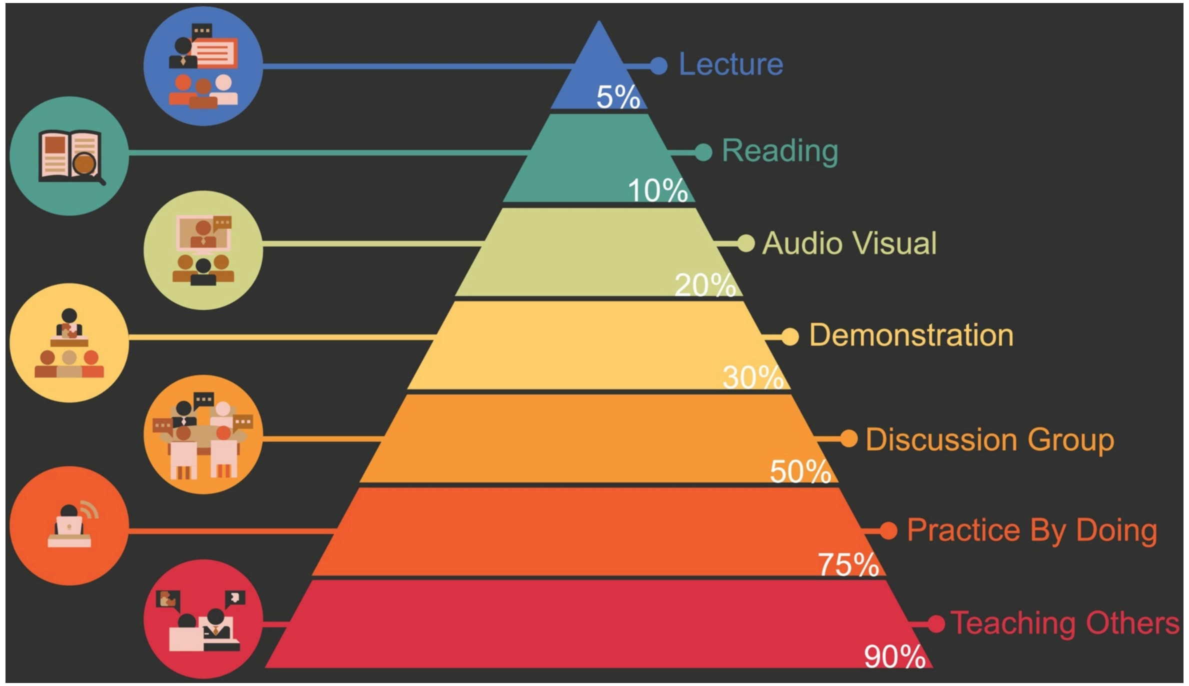
What should you know already?

- Higher mathematics: differential equations, algebra (1.-2. BSc)
- Experimental and theoretical physics: governing equations
- Numerics for engineers (2. BSc)
- Programming (1. BSc), Software development (3. BSc)
- Geophysics: feeling for physical fields & methods
- Electromagnetics (5. BSc), Theory EM,
- now: Scientific programming, HPC, seismic imaging

Topics to be covered

- recap on partial differential equations
- (1D) heat equation: stationary and instationary
- 2D electromagnetic fields in the Earth
- (3D DC modelling - see Spitzer videos)
- 2D ground-penetrating radar (EM) and pressure waves (seismics)
- excurse to hydrodynamic modelling
- modelling the Eikonal equation (the travelling saleman)
- exercises: code FD & FE by hand, use packages to obtain feeling

Learning goal



- basic understanding of the common modelling techniques
- feeling for strengths and weaknesses of numerical solutions
- ability to write your own modelling codes in Python

Literature

- Haber (2015): Computational methods geophysical electromagnetics
- Morra (2018): Pythonic Geodynamics - Implementations for fast computing, frei verfügbar hier, Eintauchen ins Programmieren
- Warnick: Numerical Methods for Engineering : An Introduction Using MATLAB® and Computational Electromagnetics Examples Link
- Igel (2007): Numerical modelling in geophysics, short course
- Logg et al. (2011): Automated Solution of Differential Equations by the Finite Element Method: Link
- Press (2007): Numerical recipes: the art of scientific computing, https://numerical.recipes/book.html

Further links

- Course notes
- pyGIMLi: Python Geophysical Inversion and Modelling Library
  https://pygimli.org
- Homepage of Oskar package
- Geoscience.XYZ
- Fenics handbook
- Theory of electromagnetics

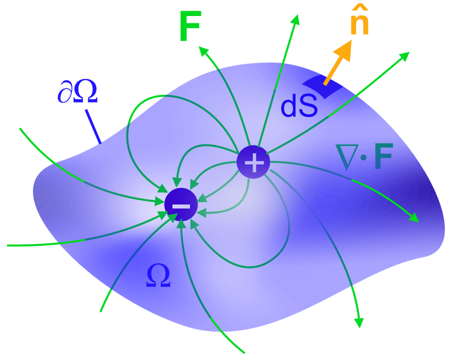
Some mathematical background

Differential operators

- single derivative in space $\frac{\partial}{\partial x}$ or time $\frac{\partial}{\partial t}$
- gradient $\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$
- divergence $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Gauss': *what's in (volume) comes out (surface)*

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$



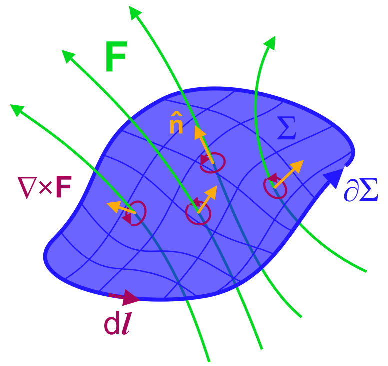
Gauss's theorem in EM

Curl (rotation)

- curl $\boldsymbol{\nabla} \times \mathbf{F} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})^T$

Stoke: *what goes around comes around*

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \mathbf{d}S = \iint_S \mathbf{F} \cdot \mathbf{d}l$$



Stokes' theorem in EM

- curls have no divergence: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$
- potential fields have no curl $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} u) = 0$

Numerical simulation

Mostly: solution of partial differential equations (PDEs) for either scalar (potentials) or vectorial (fields) quantities

PDE Types (u-function, f-source, a/c-parameter):

- elliptic PDE: $\nabla^2 u = f$
- parabolic PDE $\nabla^2 u - a\frac{\partial u}{\partial t} = f$
- hyperbolic $\nabla^2 u - c^2\frac{\partial^2 u}{\partial t^2} = f$ (plus diffusive term)

$$\frac{\partial^2 u}{\partial x^2} - c^2\frac{\partial^2 u}{\partial t^2} = 0$$

- coupled $\nabla \cdot u = f$ & $u = K\nabla p = 0$ (Darcy flow)
- nonlinear $(\nabla u)^2 = s^2$ (Eikonal equation)

Poisson equation

potential field u generates field $\vec{F} = -\nabla u$

causes some flow $\vec{j} = a\vec{F}$

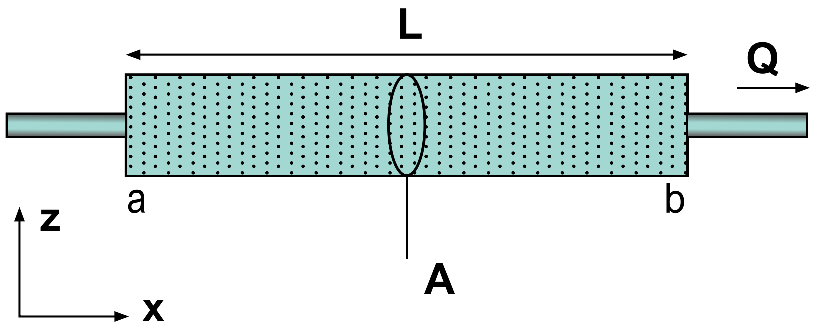
a is some sort of conductivity (electric, hydraulic, thermal)

continuity of flow: divergence of total current $\mathbf{j} + \mathbf{j}_s$ is zero

$$\boldsymbol{\nabla} \cdot (a\nabla u) = -\boldsymbol{\nabla} \cdot \mathbf{j}_s$$

Darcy's law

volumetric flow rate Q caused by gradient of pressure p



Darcy's law

$$Q = \frac{kA}{\mu L}\Delta p$$

$$\mathbf{q} = -\frac{k}{\mu}\nabla p$$

$$\boldsymbol{\nabla}\cdot\mathbf{q} = -\boldsymbol{\nabla}\cdot(k/\mu\boldsymbol{\nabla} p) = 0$$

The heat equation in 1D

sought: Temperature T as a function of space and time

heat flux density $\mathbf{q} = \lambda \boldsymbol{\nabla} T$

q in W/m², λ - heat conductivity/diffusivity in W/(m.K)

Fourier's law: $\frac{\partial T}{\partial t} - a\nabla^2 T = s$ (s - heat source)

temperature conduction $a = \frac{\lambda}{\rho c}$ (ρ - density, c - heat capacity)

Stokes equation

$$\mu \nabla^2 \mathbf{v} - \boldsymbol{\nabla} p + f = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

Navier-Stokes equations

(incompressible, uniform viscosity)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \nu \nabla^2 \mathbf{u} - 1/\rho \boldsymbol{\nabla} p + f$$

Maxwell's equations

- Faraday's law: currents & varying electric fields $\Rightarrow$ magnetic field

$$\mathbf{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$$

- Ampere's law: time-varying magnetic fields induce electric field

$$\mathbf{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\mathbf{\nabla} \cdot \mathbf{D} = \varrho$ (charge $\Rightarrow$), $\mathbf{\nabla} \cdot \mathbf{B} = 0$ (no magnetic charge)
- material laws $\mathbf{D} = \epsilon \mathbf{E}$ and $\vec{B} = \mu \mathbf{H}$

Helmholtz equations

$$\nabla^2 u + k^2 u = f$$

results from wavenumber decomposition of diffusion or wave equations

approach: $\mathbf{F} = \mathbf{F_0} e^{\imath\omega t} \quad \Rightarrow \quad \frac{\partial \mathbf{F}}{\partial t} = \imath\omega\mathbf{F} \quad \Rightarrow \quad \frac{\partial^2 \mathbf{F}}{\partial t^2} = -\omega^2 \mathbf{F}$

$$\nabla^2 \mathbf{F} - a\nabla_t \mathbf{F} - c^2 \nabla_t^2 \mathbf{F} = 0$$

$$\Rightarrow \nabla^2 \mathbf{F} - a\imath\omega\mathbf{F} + c^2\omega^2\mathbf{F} = 0$$

The eikonal equation

Describes first-arrival times t as a function of velocity (v) or slowness (s)

$$|\nabla t| = s = 1/v$$