Numerical Simulation Methods in Geophysics, Part 9: 2D Helmholtz equation

1. MGPY+MGIN

thomas.guenther@geophysik.tu-freiberg.de

Recap

Done:

- finite difference method for Poisson (1D) and heat transfer
- wave equation modelling in 1D
- finite element method for Poisson equation
- weak form decreases order of derivatives
- Maxwell equations lead to Helmholtz equation

The next lectures and exercises

- today (LV09): 08.01.25, no exercise on 09.01. (dies academicus)
- LV10: 15.01.25, exercise on 16.01.
- LV11: 22.01.25, exercise on 23.01.
- LV12: 29.01.25, exercise on 30.01.
- last VL13: 05.02.25, exercise on 06.02.
- report on 2D Helmholtz equations

Helmholtz equation for E

Maxwell equation in frequency domain (see also Theory EM)

$$\nabla \times \mathbf{H} = \imath\omega\epsilon\mathbf{E} + \mathbf{j} \qquad (\mathbf{j} = \sigma\mathbf{E} + \mathbf{j}_s)$$

$$\nabla \times \mathbf{E} = -\imath\omega\mu\mathbf{H} \Rightarrow \mathbf{H} = \imath\frac{1}{\mu\omega}\nabla \times \mathbf{E}$$

$$\nabla \times \mathbf{H} = \imath\omega^{-1}\nabla \times \mu^{-1}\nabla \times \mathbf{E} = (\imath\omega\epsilon + \sigma)\mathbf{E} + \mathbf{j}_s$$

$$\nabla \times \mu^{-1}\nabla \times \mathbf{E} + (\imath\sigma\omega - \omega^2\epsilon)\mathbf{E} = -\imath\omega\mathbf{j}_s$$

Quasistatic approximation ($\omega\epsilon \ll \sigma$): neglect ϵ term

Helmholtz equation for H

Maxwell equation in frequency domain (see also Theory EM)

$$\boldsymbol{\nabla} \times \mathbf{E} = -\imath\omega\mu\mathbf{H}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \sigma\mathbf{E} + \mathbf{j}_s \Rightarrow \mathbf{E} = \sigma^{-1}\boldsymbol{\nabla} \times \mathbf{H} - \sigma^{-1}\mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \mathbf{E} = \imath\omega\epsilon\mathbf{E} + \boldsymbol{\nabla} \times \sigma^{-1}\boldsymbol{\nabla} \times \mathbf{H} - \boldsymbol{\nabla} \times \sigma^{-1}\mathbf{j}_s = -\imath\omega\mu\mathbf{H}$$

$$\boldsymbol{\nabla} \times \sigma^{-1}\boldsymbol{\nabla} \times \mathbf{H} + \imath\mu\omega\mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1}\mathbf{j}_s$$

Quasistatic approximation ($\omega\epsilon \ll \sigma$) neglect ϵ term

Helmholtz equations

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\sigma\mathbf{E} - \omega^2 \epsilon \mathbf{E} = -\imath\omega\mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2 \epsilon\mu/\sigma\mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1}\mathbf{j}_s$$

PDEs identical $\mathbf{E}$ and $\mathbf{H}$ through exchanging μ and σ

component perpendicular to modelling frame (E/H polarization)

$$\boldsymbol{\nabla} \times a \boldsymbol{\nabla} \times = -\boldsymbol{\nabla} \cdot a \boldsymbol{\nabla}$$

Finite element discretization

- weak formulation (for E)

$$\int_{\Omega} \mu^{-1} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \imath\omega \int_{\Omega} \sigma v_i v_j \mathrm{d}\Omega = \int_{\Omega} v_i f \mathrm{d}\Omega$$

- stiffness = second derivative $\boldsymbol{\nabla} \cdot \mathbf{v}_i$, expressed by 2 gradients

$$\mathbf{A}_{i,j} = \int_{\Omega} \mu^{-1} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega$$

Finite element discretization

- weak formulation (for E)

$$\int_\Omega \mu^{-1} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \imath\omega \int_\Omega \sigma v_i v_j \mathrm{d}\Omega = \int_\Omega v_i f \mathrm{d}\Omega$$

- mass matrix resembles functions $\mathbf{v}_i$

$$\mathbf{M}_{i,j} = \int_\Omega \sigma v_i \cdot v_j \mathrm{d}\Omega$$

Next steps

- solve 1D Helmholtz equation complex-values
  - compare with analytic solution
- solve 2D Helmholtz equation
  - use secondary field approach
- use wide range of frequencies
  - combine E and H to yield MT sounding curves
- excurse on 3D vectorial Maxwell solvers
- overview on equation solvers and high-performance computing
- outlook to computational fluid dynamics

Complex or real-valued?

Either discretize the complex system

$$(\mathbf{A} + \imath\omega\mathbf{M})(\mathbf{u}_r + \imath\mathbf{u}_i) = \mathbf{b}_r + \imath\mathbf{b}_i$$

by complex shape functions OR transfer into real

$$\mathbf{A}\mathbf{u}_r + \imath\mathbf{A}u_i + \imath\omega\mathbf{M}\mathbf{b}_i - \omega\mathbf{M}\mathbf{b}_i = \mathbf{b}_r + \imath\mathbf{b}_i$$

$$\begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{u}_r \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{b}_r \\ \mathbf{b}_i \end{pmatrix}$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$ (or $F_p + F_s$)

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for F_a, because

- F_a is weaker or smoother (e.g. $F_0 \propto 1/r$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

Secondary field Helmholtz equation

The equation $-\boldsymbol{\nabla}^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\boldsymbol{\nabla}^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\boldsymbol{\nabla}^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\boldsymbol{\nabla}^2 F_a - k^2 F_a = \delta k^2 F_0$$

Note

Same operator, source terms at anomalies, weighted by the primary field.

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

leads to the discretized form ($\mathbf{A}$-stiffness, $\mathbf{M}$-mass)

$$\mathbf{A}\mathbf{E}_a + \imath\omega\mathbf{M}_\sigma E_a = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{E}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```

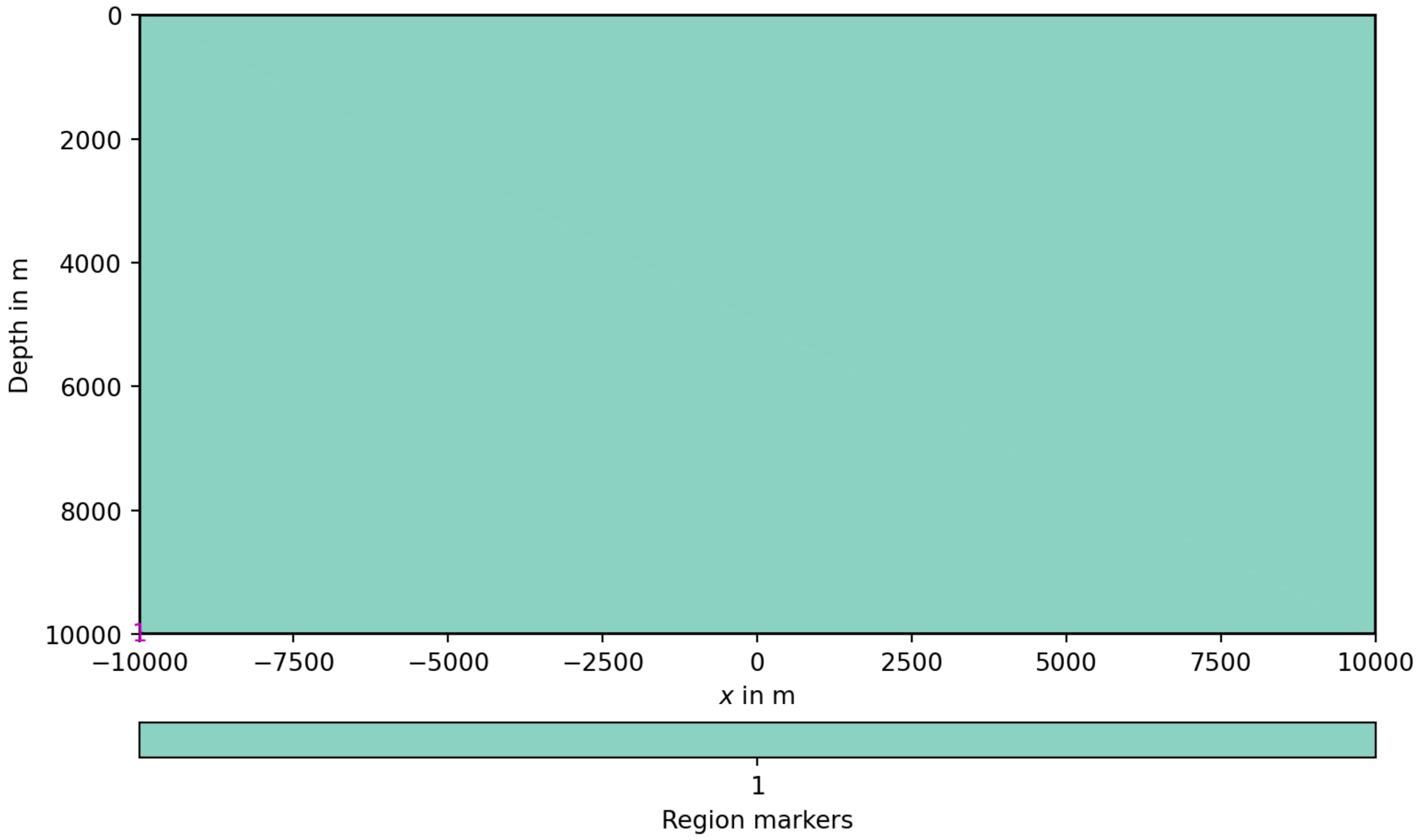
2D problems

Make use of pyGIMLi

See documentation on pyGIMLi.org

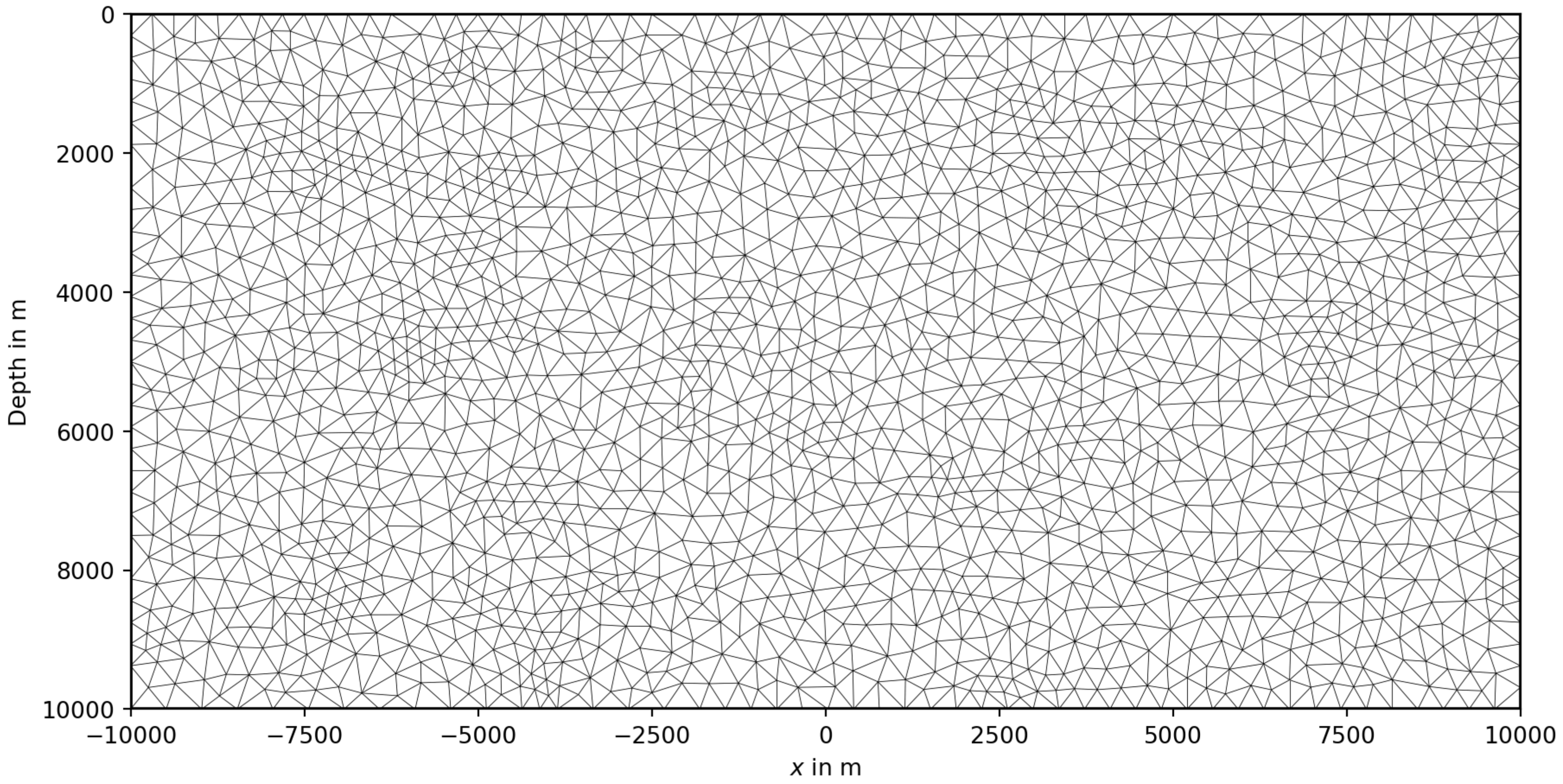
The meshtools module

```
import pygimli as pg
import pygimli.meshtools as mt
world = mt.createWorld(start=[-10000, -10000], end=[10000, 0])
pg.show(world)
```



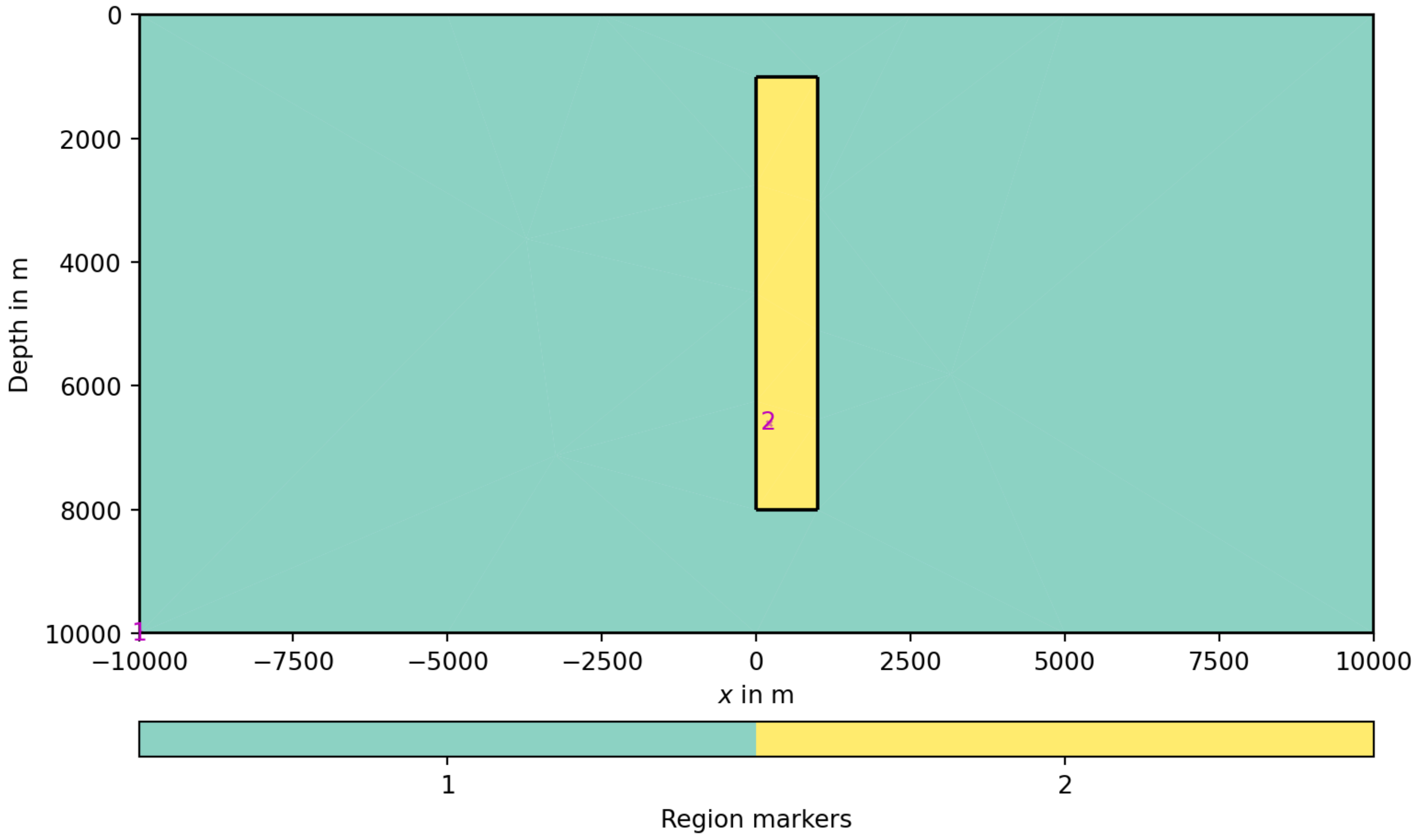
The meshtools module

```
mesh = mt.createMesh(world, quality=34, area=1e5)
pg.show(mesh)
```



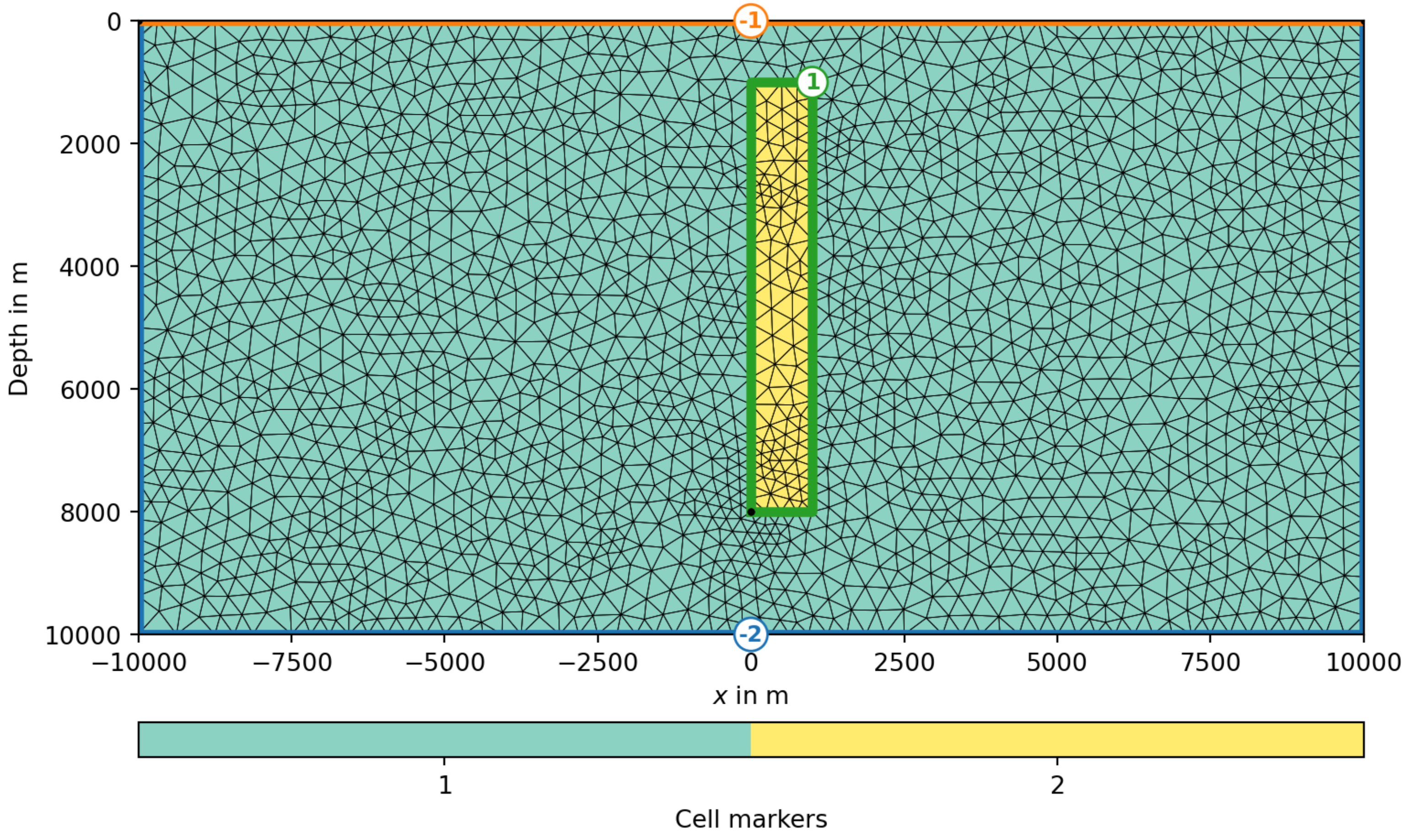
Creating a 2D geometry

```
anomaly = mt.createRectangle(start=[0, -8000], end=[1000, -1000], marker=2)
pg.show(world+anomaly)
```



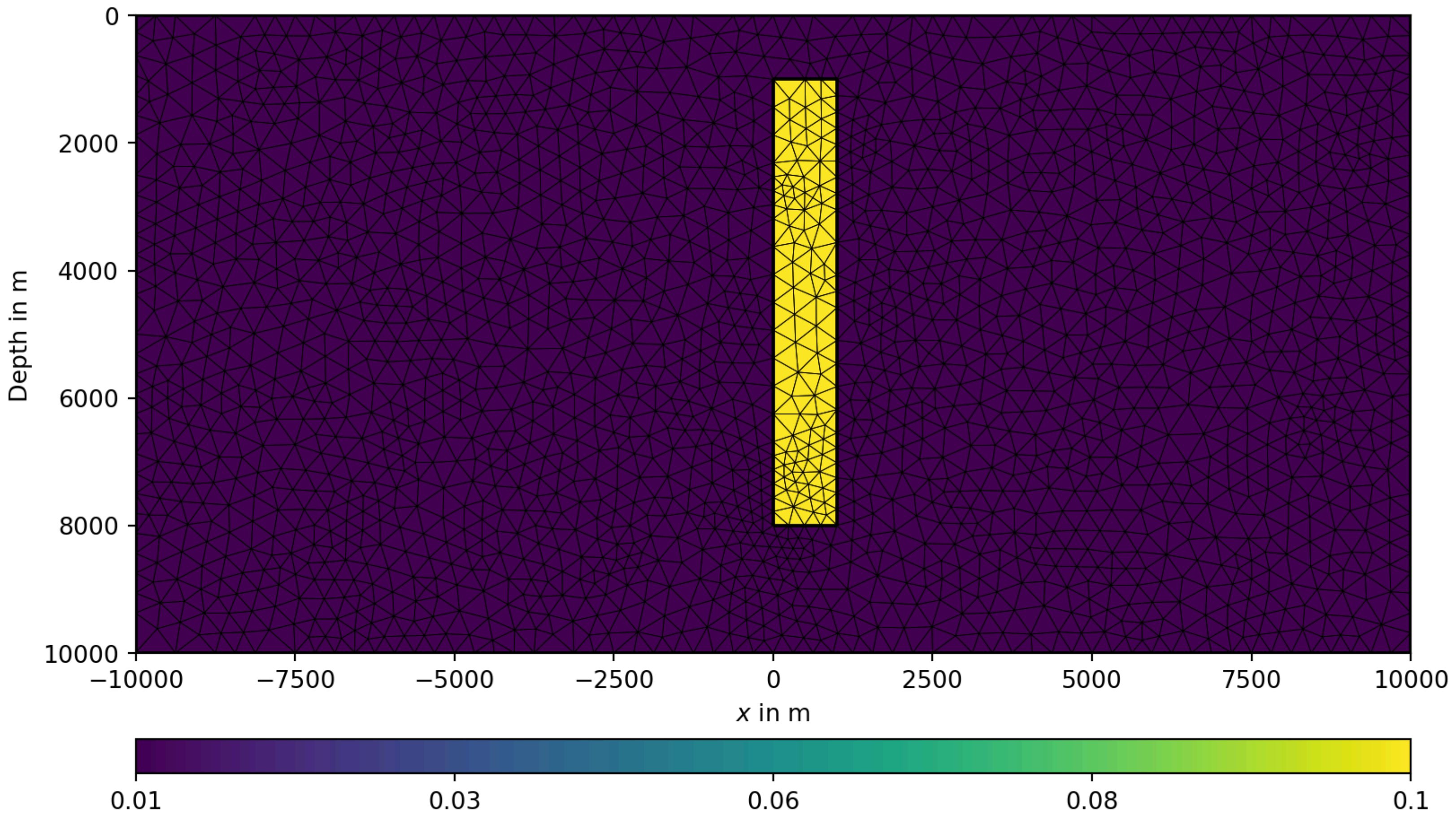
Creating a 2D mesh

```python
mesh = mt.createMesh(world+anomaly, quality=34, smooth=True, area=1e5)
pg.show(mesh, markers=True, showMesh=True);
```



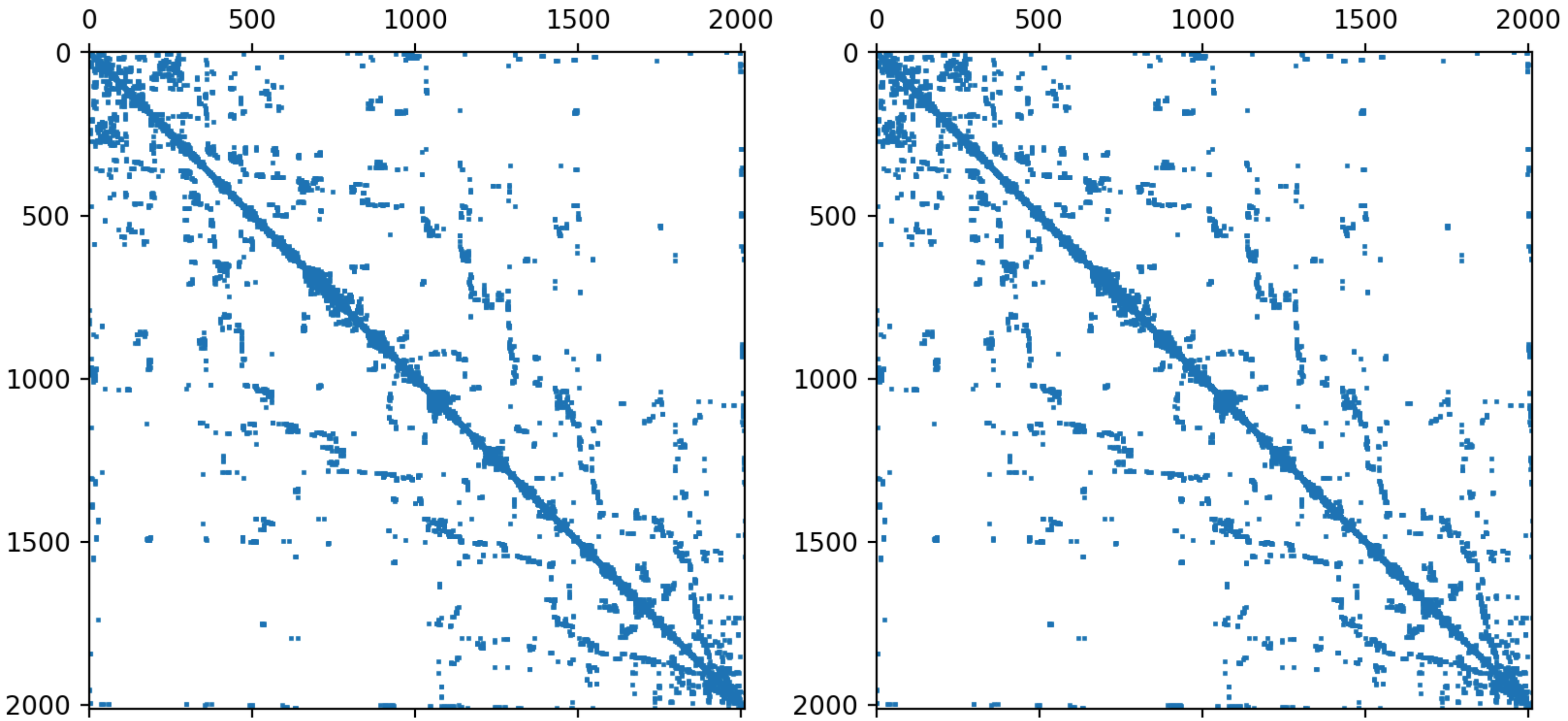
Creating a 2D conductivity model

```
sigma0 = 1 / 100  # 100 Ohmm
sigma = mesh.populate("sigma", {1: sigma0, 2: sigma0*10})
pg.show(mesh, "sigma", showMesh=True);
```



The solver module

```
import pygimli.solver as ps
mesh["my"] = 4 * np.pi * 1e-7
A = ps.createStiffnessMatrix(mesh, a=1/mesh["my"])
M = ps.createMassMatrix(mesh, mesh["sigma"])
fig, ax = plt.subplots(ncols=2)
ax[0].spy(pg.utils.toCSR(A), markersize=1)
ax[1].spy(pg.utils.toCSR(M).todense(), markersize=1)
```



The complex problem matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

```python
w = 0.1
nd = mesh.nodeCount()
B = pg.BlockMatrix()
B.Aid = B.addMatrix(A)
B.Mid = B.addMatrix(M)
B.addMatrixEntry(B.Aid, 0, 0)
B.addMatrixEntry(B.Aid, nd, nd)
B.addMatrixEntry(B.Mid, 0, nd, scale=-w)
B.addMatrixEntry(B.Mid, nd, 0, scale=w)
pg.show(B)
```