Numerical Simulation Methods in Geophysics, Exercise 9: 1D/2D EM

1. MGPY+MGIN

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2D Poisson problem with singular source

$$-\boldsymbol{\nabla} \cdot \sigma \boldsymbol{\nabla} u = I\delta(\mathbf{r} - \mathbf{r}_s)$$

- write function for analytical solution $u = -\frac{I}{2\pi\sigma} \ln r$
- generate a model with two electrodes at the surface
- start with homogeneous σ, create stiffness matrix & load vector
- solve the matrix-vector equation, plot solution and compare with analytical solution
- use inhomogeneous conductivity and try secondary field approach

1D/2D (FD) EM modelling

The induction equation for perpendicular (E or H) fields

$$-\boldsymbol{\nabla}^2 u + \imath\omega\mu\sigma u = f$$

is discretized by stiffness matrix $\mathbf{A}$ and mass matrix $\mathbf{M}$

$$(\mathbf{A} + \imath\omega\mathbf{M})\mathbf{u} = \mathbf{f}$$

TM polarization

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1} \mathbf{j}_s$$

Transverse magnetic (TM) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{H} = [H_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar H_x (now only H)

$$-\boldsymbol{\nabla} \cdot \sigma^{-1} \boldsymbol{\nabla} H_x(y, z) + \imath\omega\mu H_x(y, z) = 0$$

TE polarization

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} + \imath\omega\sigma\mathbf{E} = \nabla \times \mathbf{j}_s$$

Transverse electric (TE) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{E} = [E_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar E_x (now only E)

$$-\nabla \cdot \mu^{-1} \nabla E_x(y, z) + \imath\omega\sigma E_x(y, z) = 0$$

Complex or real-valued?

The complex-valued system

$$(\mathbf{A} + \imath\omega\mathbf{M})\mathbf{u} = (\mathbf{A} + \imath\omega\mathbf{M})(\mathbf{u}_r + \imath\mathbf{u}_i) = \mathbf{b}_r + \imath\mathbf{b}_i$$

can be transferred into a doubled real-valued system

$$\mathbf{A}\mathbf{u}_r + \imath\mathbf{A}u_i + \imath\omega\mathbf{M}\mathbf{u}_r - \omega\mathbf{M}\mathbf{u}_i = \mathbf{b}_r + \imath\mathbf{b}_i$$

$$\begin{pmatrix} A & -\omega M \\ \omega M & A \end{pmatrix} \begin{pmatrix} u_r \\ u_i \end{pmatrix} = \begin{pmatrix} b_r \\ b_i \end{pmatrix}$$

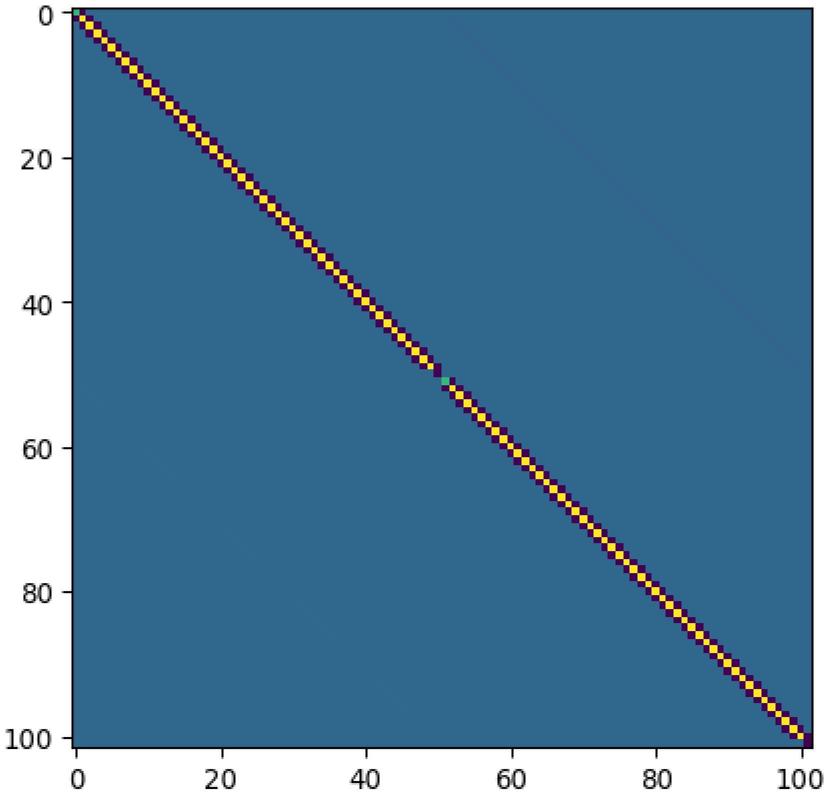
The problem in 1D

```
import numpy as np
import matplotlib.pyplot as plt
from poisson1d import stiffnessMatrix1DFE, massMatrix1DFE
T = 0.1  # 0.1
w = 2 * np.pi / T
sigma0 = 1/100  # 1/100
mu = np.pi * 4e-7
z = np.arange(-10000, 0.1, 200)
A, b = stiffnessMatrix1DFE(x=z, uR=1)
M = massMatrix1DFE(x=z, a=mu*sigma0)
AM = A + M * 1j * w
u = np.linalg.solve(AM, b)
```

Complex-to-real conversion

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

```
D = np.vstack([np.hstack([A, -M*w]),
               np.hstack([M*w, A])])
plt.imshow(D)
d = np.hstack([b, b*0])
uri = np.linalg.solve(D, d)
u = uri[:len(z)] + uri[len(z):] * 1j
```



Complex-to-real conversion

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for F_a, because

- F_a is weaker or smoother (e.g. $F_0 \propto 1/$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

Secondary field Helmholtz equation

The equation $-\boldsymbol{\nabla}^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\boldsymbol{\nabla}^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\boldsymbol{\nabla}^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\boldsymbol{\nabla}^2 F_a - k^2 F_a = \delta k^2 F_0$$

Note

Source terms only arise at anomalous terms, weighted by the primary field.

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field

Secondary field for EM

$$-\nabla^2 \mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

leads to the discretized form ($\mathbf{A}$-stiffness, $\mathbf{M}$-mass)

$$\mathbf{A}\mathbf{u}_a + \imath\omega\mathbf{M}_\sigma\mathbf{u}_a = (\mathbf{A} + \imath\omega\mathbf{M}_\sigma)\mathbf{u}_a = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{u}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```