Numerical Simulation Methods in Geophysics, Part 8: 2D Helmholtz equation

1. MGPY+MGIN

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Recap

- finite differences approximate partial derivatives
- finite elements approximate solution $\Rightarrow$ preferred
- spatial discretization determines accuracy of solution
- different time-stepping approaches, implicit and mixed schemes most accurate and stable
- tasks for report on 1D instationary heat equation with periodic boundary conditions

Helmholtz equation in 2D

Maxwells equations

- Faraday's law: currents & varying electric fields $\Rightarrow$ magnetic field

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$$

- Ampere's law: time-varying magnetic fields induce electric field

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\boldsymbol{\nabla} \cdot \mathbf{D} = \varrho$ (charge $\Rightarrow$), $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (no magnetic charge)
- material laws $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

Maxwell in frequency domain

$$\mathbf{E} = \mathbf{E}_0 e^{\imath\omega t} \quad \text{or} \quad \mathbf{H} = \mathbf{H}_0 e^{\imath\omega t}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \imath\omega\epsilon\mathbf{E} + \sigma E$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\imath\omega\mu\mathbf{H}$$

Helmholtz equation

see also Theory EM

take curl of one of the equations and insert in the other

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\mu\sigma\mathbf{E} - \omega^2\mu\epsilon\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \rho\boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2\mu\epsilon\rho\mathbf{H} = 0$$

Quasi-static approximation

Assume: $\omega^2\mu\epsilon < \omega\mu\sigma$, no sources ($\boldsymbol{\nabla}\cdot\mathbf{j}_s = 0$), + vector identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{F} - \boldsymbol{\nabla}^2\mathbf{F}$$

leads with $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = \boldsymbol{\nabla}\cdot\mathbf{B}$ to the vector Helmholtz PDE

$$-\boldsymbol{\nabla}^2\mathbf{E} + \imath\omega\mu\sigma\mathbf{E} = 0$$

$$-\boldsymbol{\nabla}\cdot\rho\mathbf{H} + \imath\omega\mu\mathbf{H} = 0$$

Boundary value problem

$$-\boldsymbol{\nabla}^2\mathbf{F} + \imath\omega\mu\sigma\mathbf{F} = 0$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

form equations for F_a

Secondary field Helmholtz equation

The equation $-\boldsymbol{\nabla}^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\boldsymbol{\nabla}^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\boldsymbol{\nabla}^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\boldsymbol{\nabla}^2 F_a - k^2 F_a = \delta k^2 F_0$$

for Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2 \mathbf{E}_a + \imath\omega\mu\sigma E_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$