Numerical Simulation Methods in Geophysics, Part 4: The heat equation

1. MGPY+MGIN

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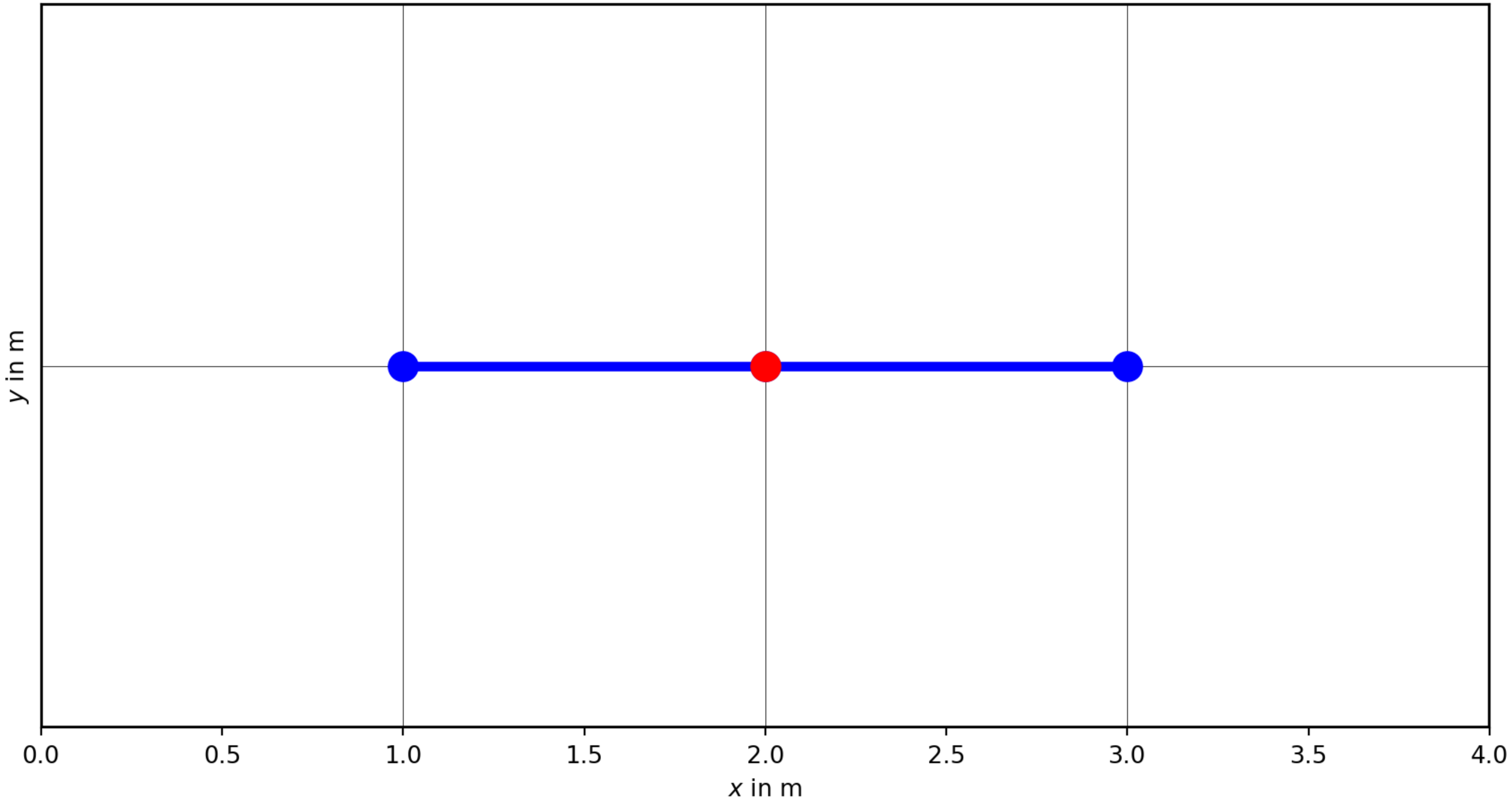
Recap last lessons & exercises

Poisson equation

$$\boldsymbol{\nabla} \cdot (a \boldsymbol{\nabla} u) = f$$

(stationary) potential field, e.g., temperature, flux, current



compute each value (red) with the help of its neighbors (blue)

Boundary conditions (Dirichlet, Neumann)

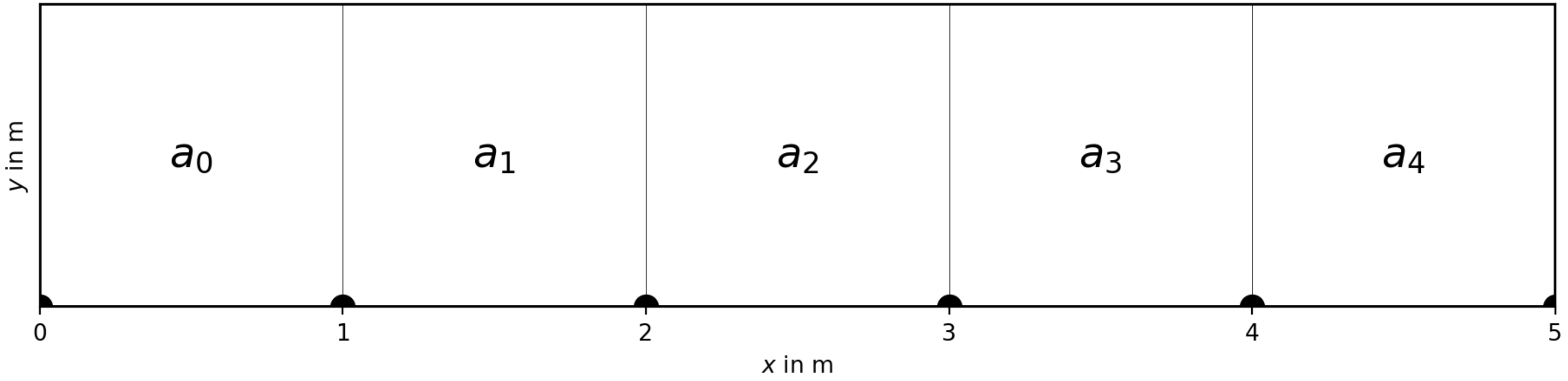
- remove outside value and move to RHS
- adjust self-coupling coefficient

Tasks (1) - done

1. Create a stiffness matrix for unit quantities
2. Implement Dirichlet BC on one and Neumann on other side
3. Solve system for different right-hand sides:
    - no source at all
    - single source in the middle or at the boundary
    - several sources with different strengths (& signs)
    - source on part of the domain
4. Alwas plot the solution and its Laplacian

The general case

$$\Delta x \neq 1 \;\&\; a \neq 1 \Rightarrow a\frac{\partial u}{\partial x} \approx a_i \frac{u_{i+1}-u_i}{x_{i+1}-x_i}$$



$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\partial u}{\partial x}\right) \approx (a_i \frac{u_{i+1}-u_i}{x_{i+1}-x_i} - a_{i-1}\frac{u_i-u_{i-1}}{x_i-x_{i-1}})/(x_{i+1}-x_{i-1}) \cdot 2$$

$$A_{i,i-1} = a_{i-1}/(x_i - x_{i-1})/(x_{i+1}-x_{i-1}) \cdot 2$$

The coupling coefficients

$$C_{left} = a_{i-1}/(x_i - x_{i-1})/(x_{i+1} - x_{i-1}) \cdot 2$$

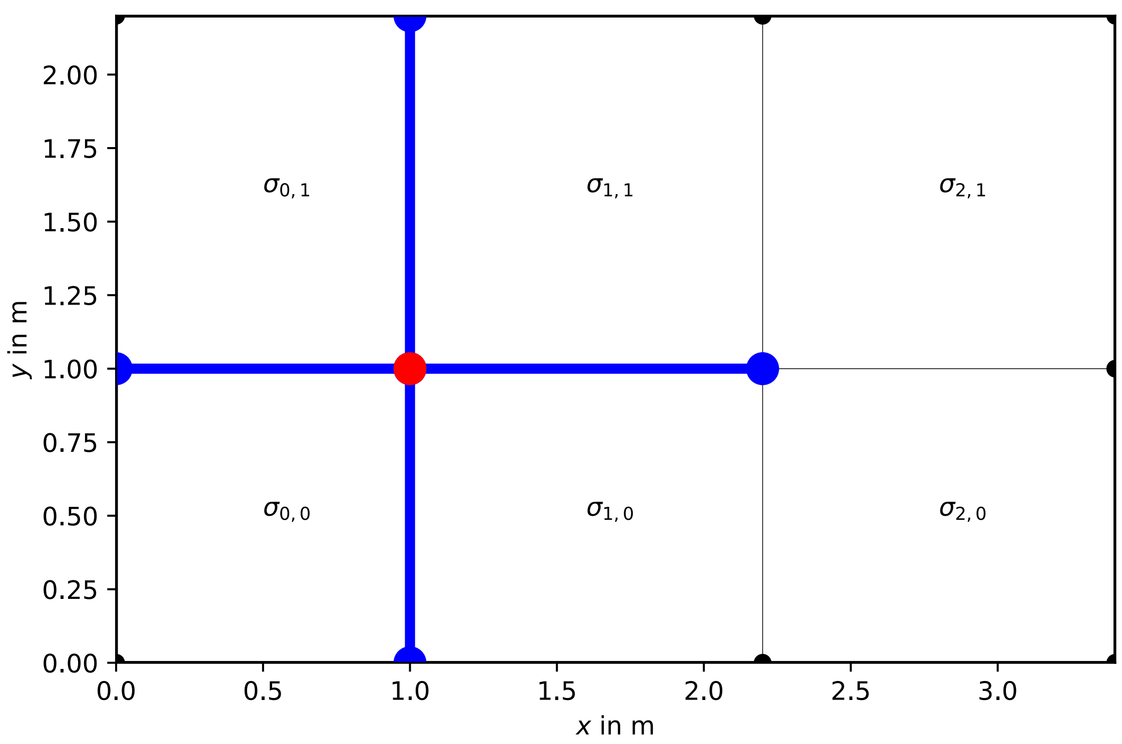
$$C_{right} = a_{i+1}/(x_{i+1} - x_i)/(x_{i+1} - x_{i-1}) \cdot 2$$

$$\begin{bmatrix} +1 & 0 & 0 & \dots & & \\ C_1^L & -(C_1^L + C_1^R) & C_1^R & 0 & \dots & \\ \vdots & \vdots & \ddots & \vdots & & \\ \dots & \dots & 0 & C_N^L & -(C_N^L + C_N^R) & C_N^R \\ \dots & \dots & & 0 & -1 & +1 \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} u_B \\ f_1 \\ \vdots \\ f_N \\ g_B \end{bmatrix}$$

Tasks (2)

1. Derive the coefficients for the general case
2. Write a function implementing the general case
3. Divide the “subsurface” in regions with different a
4. Compute the solution for different source fields
5. Use a non-equidistant discretisation
6. Always plot solution along with source and Laplacian

Next spatial dimension



Simple 2D conductivity grid with FD stencil

Parabolic PDEs

Heat flow in 1D

Heat sources

E.g. radioactive elements, heat elements or sinks

$$\boldsymbol{\nabla} \cdot (\mathbf{q} + \mathbf{q}_s) = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{q} = -\boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} T = -\boldsymbol{\nabla} \cdot \mathbf{q}_s$$

Heat conduction equation

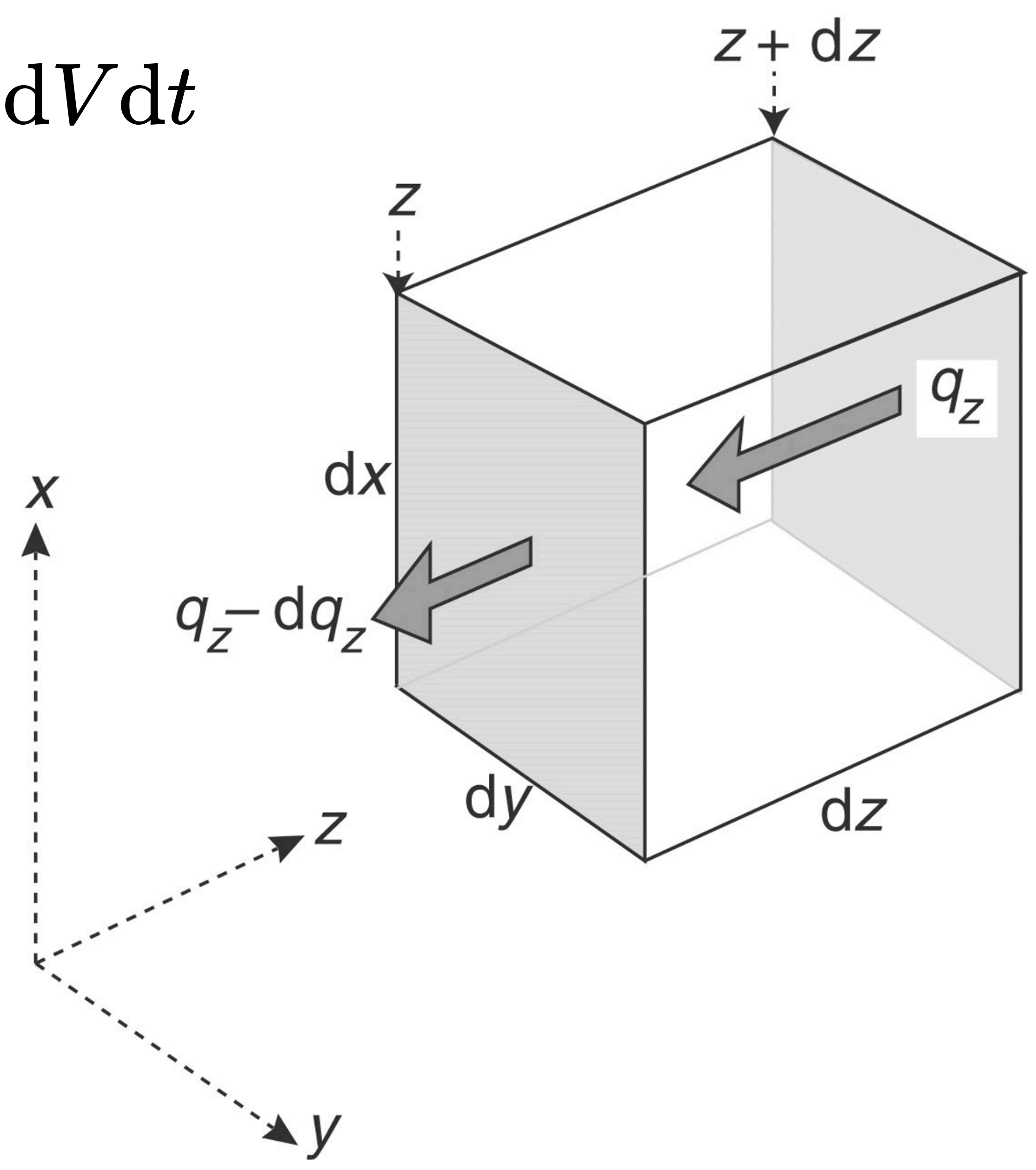
$$\frac{\mathrm{d}Q_z}{\mathrm{d}z}\mathrm{d}z = \frac{\mathrm{d}Q_z}{\mathrm{d}z}\mathrm{d}z\mathrm{d}x\mathrm{d}y\mathrm{d}t = k\frac{\mathrm{d}^2T}{\mathrm{d}z^2}\mathrm{d}V\mathrm{d}t$$

with the heat capacity c_p [W/kg/K]

$$c_p m \mathrm{d}T = c_p \rho \mathrm{d}V \mathrm{d}T$$

$\Rightarrow$ heat conduction equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{k}{\rho c_p}\frac{\mathrm{d}^2T}{\mathrm{d}z^2} = -Q_s$$



Instationary heat flow in 3D

$$\frac{\partial T}{\partial t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} T = \boldsymbol{\nabla} \cdot q_s$$

- $a = \frac{k}{\rho c_p}$ [m²/s] thermal diffusivity - measure of heat transfer
- k [W/m/K] thermal conductivity - measure of temperature transfer
- c_p [J/kg/K] - heat capacity - measure of heat storage per mass
- ρ (kg/m³) density

Water k=0.6 W/m/K, ρ=1000 kg/m³, c=4180 J/kg/K $\Rightarrow$ a=1.43e-7 m²/s

Periodic boundary conditions

Separation of variables

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial z^2}$$

$$T(t, z)/\Delta T + T_0 = \theta(t)Z(z)$$

$$Z\frac{\partial \theta}{\partial t} = a\theta\frac{\partial^2 Z}{\partial z^2}$$

$$\frac{1}{\theta}\frac{\partial \theta}{\partial t} = C = a\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

Solution

regarding the BC $e^{\imath\omega t}$ leads to $C = \imath\omega$ and thus $\theta = \theta_0 e^{\imath\omega t}$

$$\frac{\partial^2 Z}{\partial z^2} - \imath\frac{\omega}{a} Z = \frac{\partial^2 Z}{\partial z^2} + n^2 Z = 0$$

Helmholtz equation with solution $Z = Z_0 e^{\imath n z}$ $(n^2 = \imath\omega/a)$

$$Z = Z_0 e^{\imath n z} = Z_0 e^{\sqrt{\imath\omega/a}z} = Z_0 e^{\sqrt{\omega/2a}(1+\imath)z}$$

$$T(t,z)/\Delta T + T_0 = Z(z)\theta(t) = Z_0\theta_0 e^{-\sqrt{\omega/2a}z} e^{i(\omega t - \sqrt{\omega/2a}z)}$$

Interpretation

replacing the term $\sqrt{2a/\omega} = \sqrt{at_P/\pi} = d$ leads to

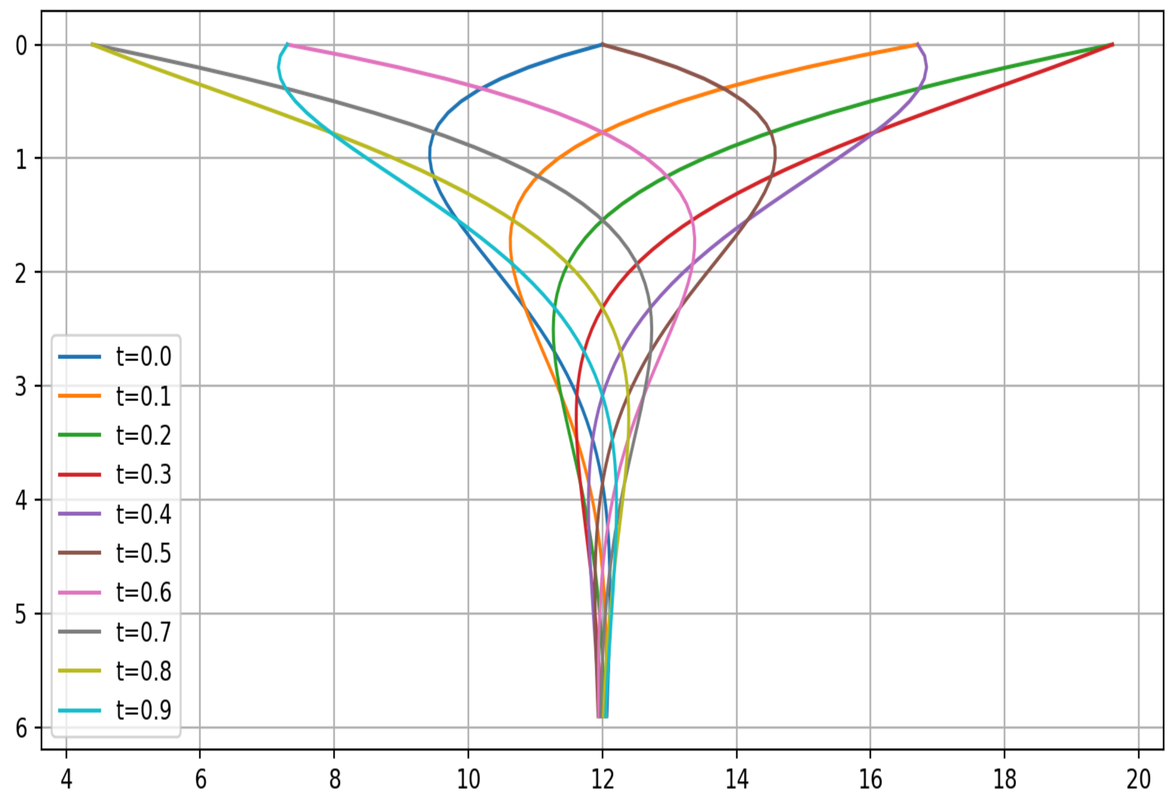
$$T(z,t) = T_0 + \Delta T e^{-z/d} \sin(\omega t - z/d)$$

- exponential damping of the temperature variation with decay depth d
- phase lag z/d increases with depth, $z_\pi = \sqrt{2a/\omega}\pi = \sqrt{at_P\pi}$

1. Daily cycle: decay depth d=6cm, minimum depth=20cm
2. Yearly cycle: decay depth d=1.2m, minimum depth=4m

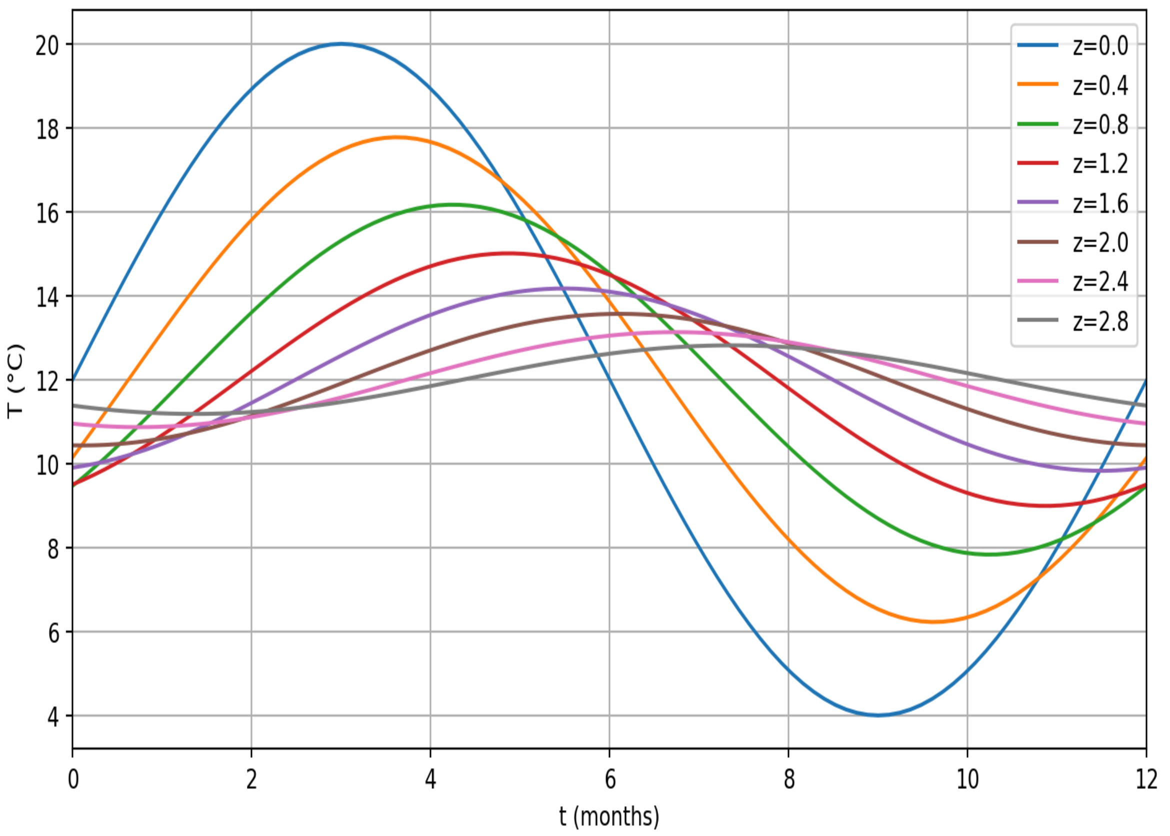
Depth profiles

```
a = 1.5e-7
year = day*365
d = sqrt(a*year/pi)
t = np.arange(0, 1, 0.1) * year
z = np.arange(0, 6, 0.1)
fig, ax = plt.subplots()
for ti in t:
    Tz = np.exp(-z/d)*np.sin(ti*2*pi/year-
                              z/d) * dT + T0
    ax.plot(Tz, z, label="t={:.1f}".format(
        ti/year))

ax.invert_yaxis()
ax.legend()
ax.grid()
```

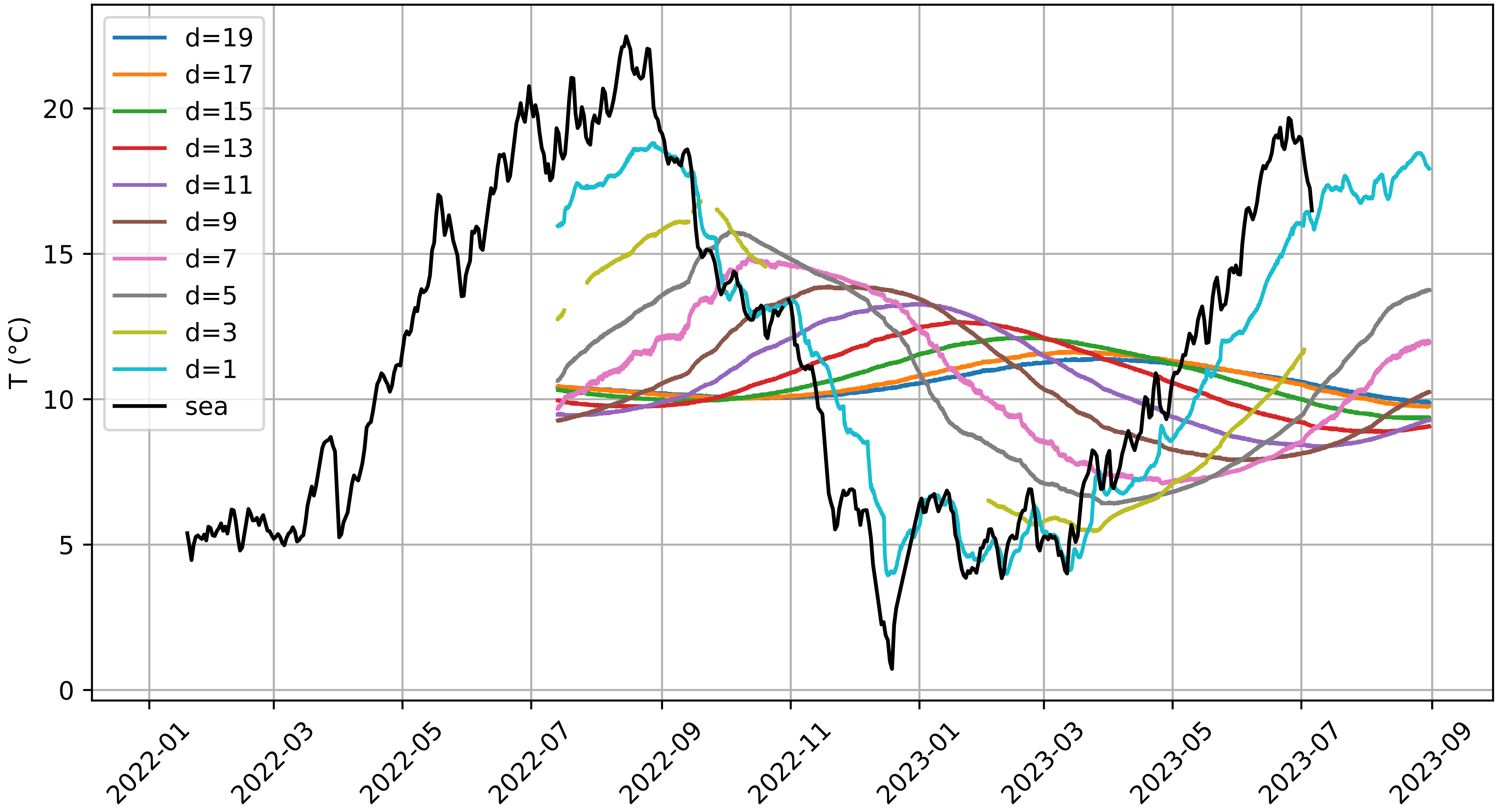


Temporal behaviour

```
t = np.arange(0, 1.01, 0.01) * year
z = np.arange(0, 3, 0.4)
fig, ax = plt.subplots()
for zi in z:
    Tt = np.exp(-zi/d)*np.sin(t*2*pi/year-
                              zi/d) * dT + T0
    ax.plot(t/year*12, Tt, label=f"z={zi:.1f}"

ax.set_xlim(0, 12)
ax.set_xlabel("t (months)")
ax.set_ylabel("T (°C)")
ax.legend()
ax.grid()
```



Experimental data from North Sea beach



Explicit methods

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial z^2} = 0$$

Finite-difference approximation

$$\frac{\partial T}{\partial t}^{n} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = a\frac{\partial^2 T^{n}}{\partial z^2}$$

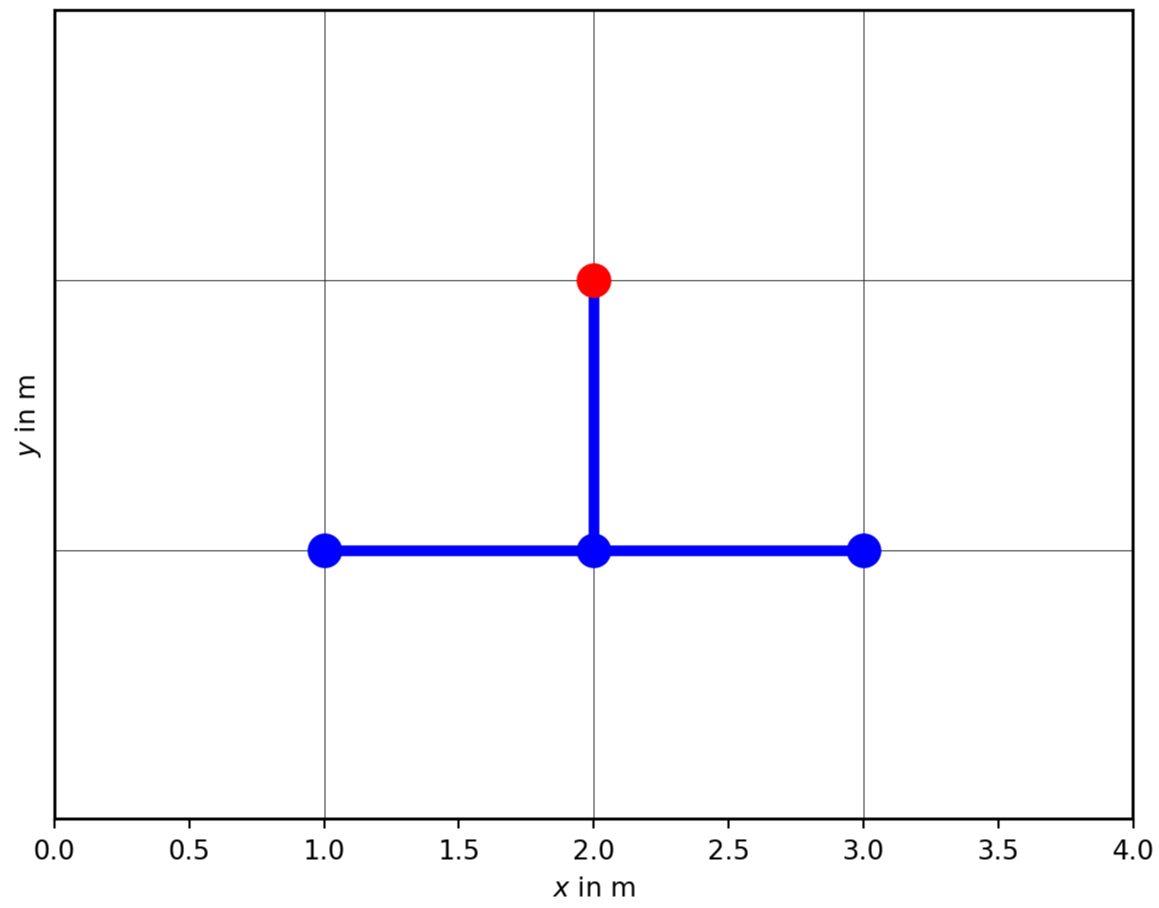
Explicit

Start T^0 with initial condition

Update field by

$$T^{n+1} = T^n + a\frac{\partial^2 T^n}{\partial z^2} \cdot \Delta t$$

E.g. by using matrix A from Poisson solver

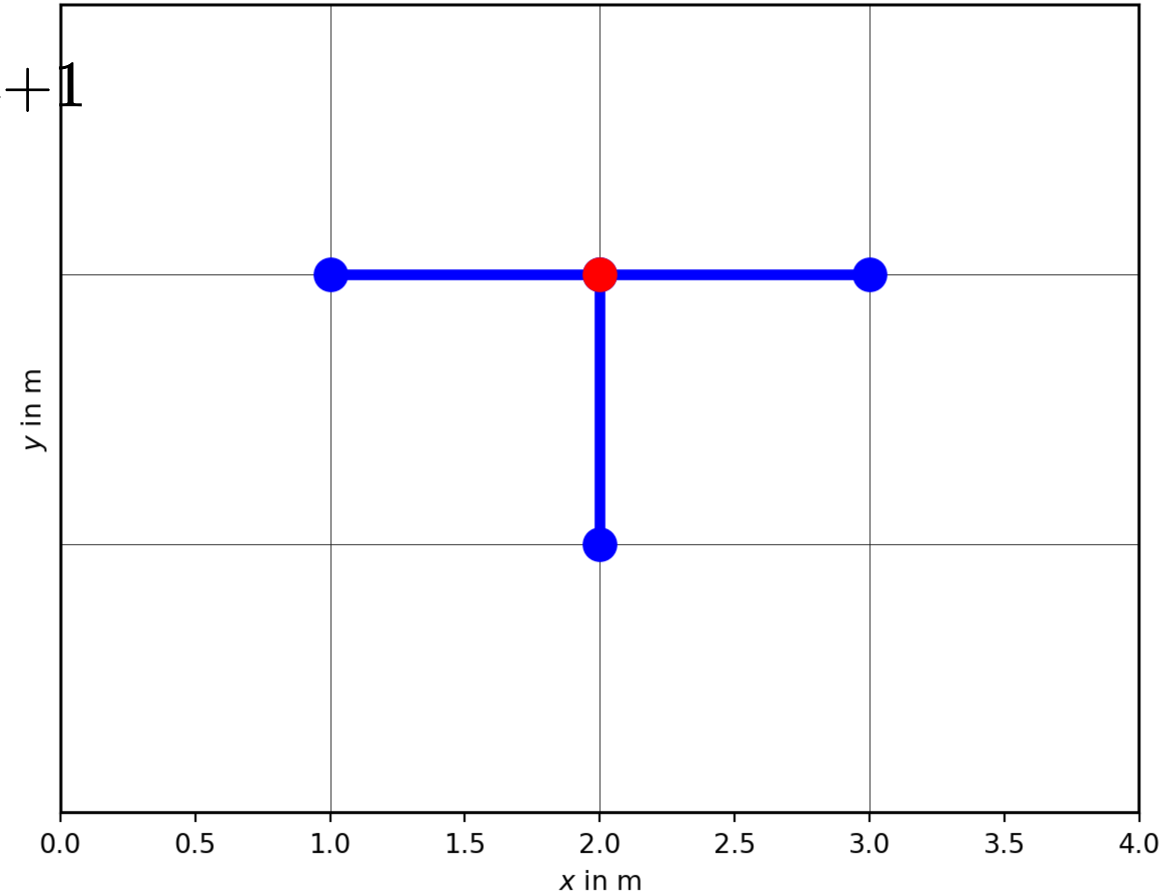


Implicit methods

$$\frac{\partial T}{\partial t}^{n+1} \approx \frac{T^{n+1} - T^n}{\Delta t} = a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{1}{\Delta t} T^{n+1} - a \frac{\partial^2 T}{\partial z^2}^{n+1} = \frac{1}{\Delta t} T^n$$

$$(\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$$



Mixed - Crank-Nicholson method

$$\frac{\partial T}{\partial t}^{n+1/2} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{2}{\Delta t} T^{n+1} - a \frac{\partial^2 T^{n+1}}{\partial z^2} = \frac{2}{\Delta t} T^{n} + a \frac{\partial^2 T^{n}}{\partial z^2}$$

$$(2\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = (2\mathbf{M} + \mathbf{A})\mathbf{u}^{n}$$