Numerical Simulation Methods in Geophysics, Exercise 6: Timestepping with FE

1. MGPY+MGIN

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Recap Poisson & heat PDE

$$\frac{\partial T}{\partial t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} T = \boldsymbol{\nabla} \cdot q_s$$

Recap

- ☑ solve the Poisson equation for arbitrary x and a
- ☑ sources and a contrasts cause curvature in u
  - positive source or a increase $\Rightarrow$ negative u” $\Rightarrow$ maximum
  - single $f \Rightarrow$ piecewise linear, full $f \Rightarrow$ parabola
- ☑ BC: Dirichlet determine shift, Neumann BC determine slope of u
- ☑ accuracy (compare analytical) depends on discretization
- ☑ now go for instationary (parabolic) problem by time stepping
  - curvature in u causes negative change of u (diffusion)
- ☑ FE integrates curvature (stiffness) and solution (mass)

The stiffness matrix

Matrix integrating gradients of base functions for neighbors with a

$$\mathbf{A}_{i,i+1} = -\frac{a_i}{\Delta x_i^2} \cdot \Delta x_i = -\frac{a_i}{\Delta x_i}$$

$$A_{i,i} = \int_\Omega a \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_i \mathrm{d}\Omega = -A_{i,i+1} - A_{i+1,i}$$

$\Rightarrow$ matrix-vector equation $\mathbf{Au} = \mathbf{b}$ with bending&shear stiffness in $\mathbf{A}$

The mass matrix

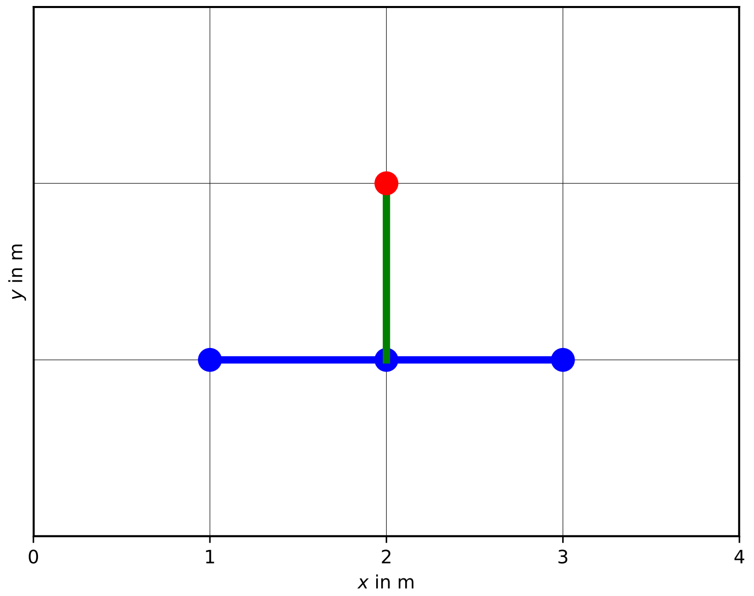
Matrix integrating base functions for neighbors

$$\mathbf{M}_{i,j} = \int_\Omega v_i \cdot v_j \mathrm{d}\Omega \quad \Rightarrow \quad \mathbf{M}_{i,i+1} = \int_{x_i}^{x_{i+1}} v_i v_{i+1}$$

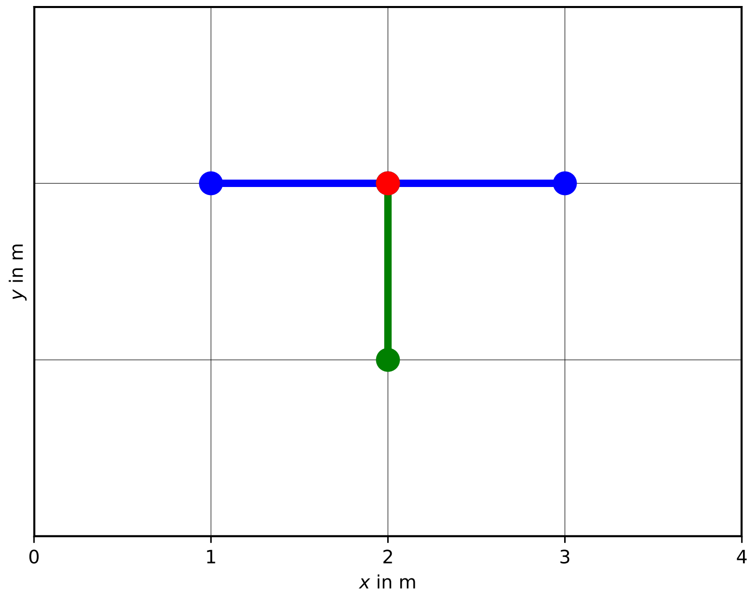
$$\mathbf{M}_{i,i+1} = \Delta x_i \int_0^1 (\xi - \xi^2) = \Delta x_i \left| \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3 \right|_0^1 = \frac{\Delta x_i}{6}$$

$$\mathbf{M}_{i,i} = \frac{\Delta x_{i-1}}{3} + \frac{\Delta x_i}{3} = 2(\mathbf{M}_{i-1,i} + \mathbf{M}_{i,i+1})$$

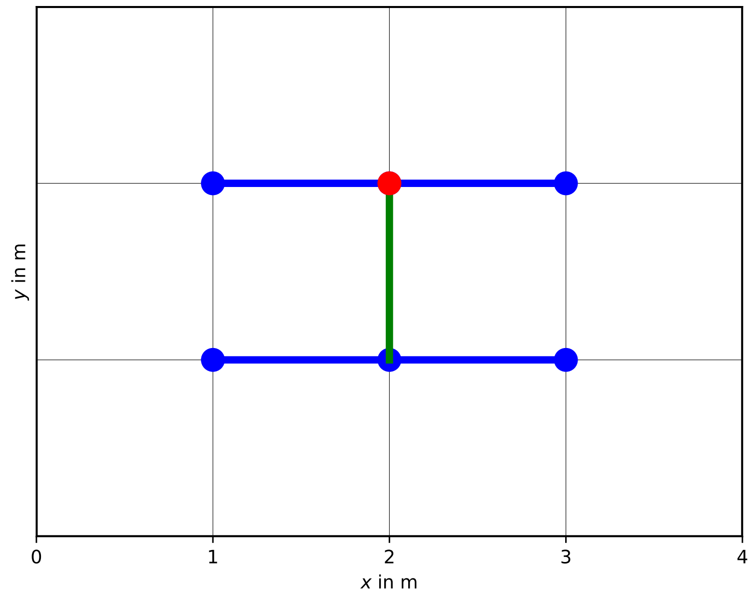
Time stepping



Explicit



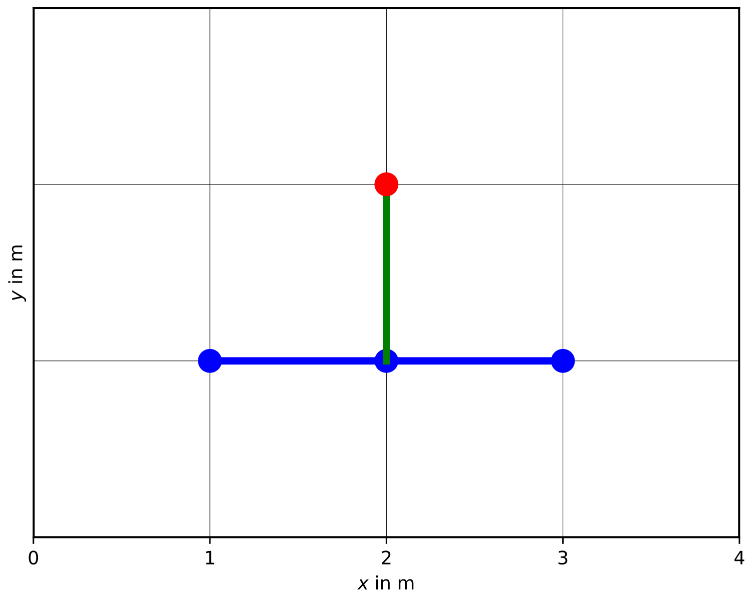
Implicit



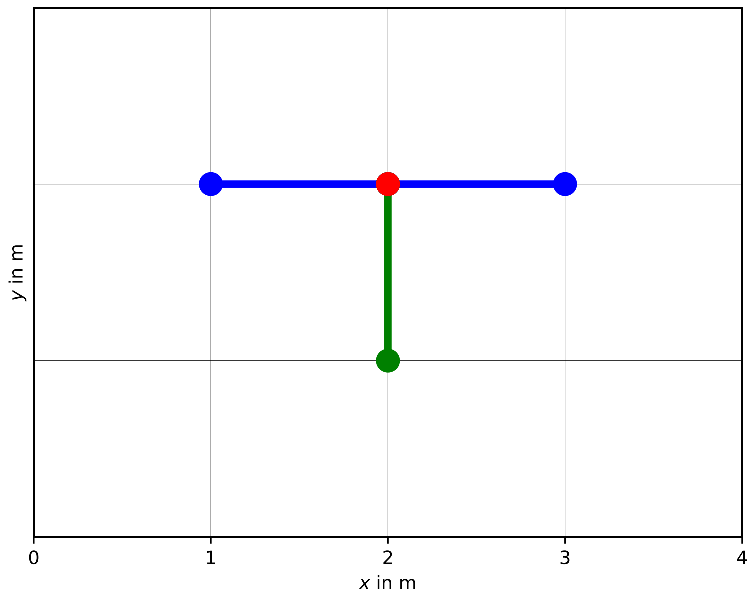
Mixed

Time-stepping in FE

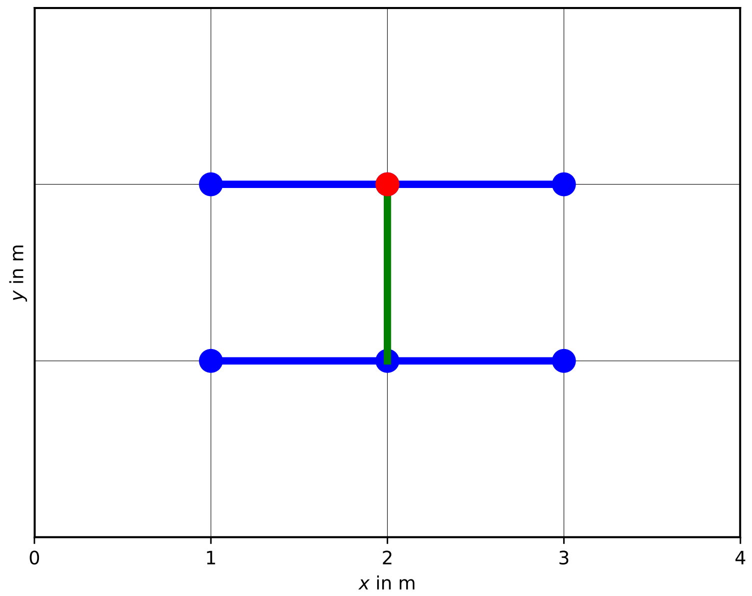
$$
\begin{array}{lrcr}
\textbf{Explicit:} & \mathbf{M} \quad \mathbf{u}^{n+1} & = & (\mathbf{M} - \Delta t \mathbf{A}) \quad \mathbf{u}^{n} \\
\hline
\textbf{Implicit:} & (\mathbf{M} + \Delta t \mathbf{A}) \quad \mathbf{u}^{n+1} & = & \mathbf{M} \quad \mathbf{u}^{n} \\
\hline
\textbf{Mixed:} & (2\mathbf{M} + \Delta t \mathbf{A}) \quad \mathbf{u}^{n+1} & = & (2\mathbf{M} - \Delta t \mathbf{A}) \quad \mathbf{u}^{n}
\end{array}
$$



Explicit



Implicit



Mixed

Tasks for today

1. Code functions stiffness matrix, mass matrix & RHS vector for FE
2. Make sure it corresponds with analytical solution (f=1) for static case
3. check whether it works with different discretizations
4. Start time-stepping with different methods ($\mathbf{B}\mathbf{u}^{n+1} = \mathbf{C}\mathbf{u}^{n}$)

- make sure the boundary conditions are on the total matrix B
- start with homogeneous initial conditions
- use different but constant potential at upper (left) boundary

1. test different time steps and observe behaviour

Report 1

Objective: Use Finite Elements to solve heat transfer into the Earth caused by periodic surface temperature (upper BC)

$$T(z = 0) = T_0 + \Delta T \sin \omega t$$

initial condition: $T(z) = T_0$ (ignoring the previous heat!)

Task

1. Implement functions stiffness matrix, mass matrix and right-hand-side vector using FE
2. Use a non-equidistant discretization of the Earth with increasing layer thicknesses (choose and substantiate).
3. Solve instationary heat equation with periodic boundary condition (yearly cycle) for the Earth using a constant but meaningfull thermal diffusivity.
4. Compare the solutions using explicit, implicit and mixed timestepping methods with the analytical solution.

Questions

- Interpret the results in terms of physical behaviour. How does a change in the diffusivity affect the result?
- Is there a difference between FD and FE discretizations? Why (not)?
- Make a statement about the stability and accuracy of the methods.
- After which time approaches the numeric solution the analytical one?
- How can you evaluate the numerical accuracy if there is no analytical solution?

Deliverables

Format can be Jupyter Notebook and/or PDF

Complete codes to run the results