Numerical Simulation Methods in Geophysics, Exercise 5: Finite Elements

1. MGPY+MGIN

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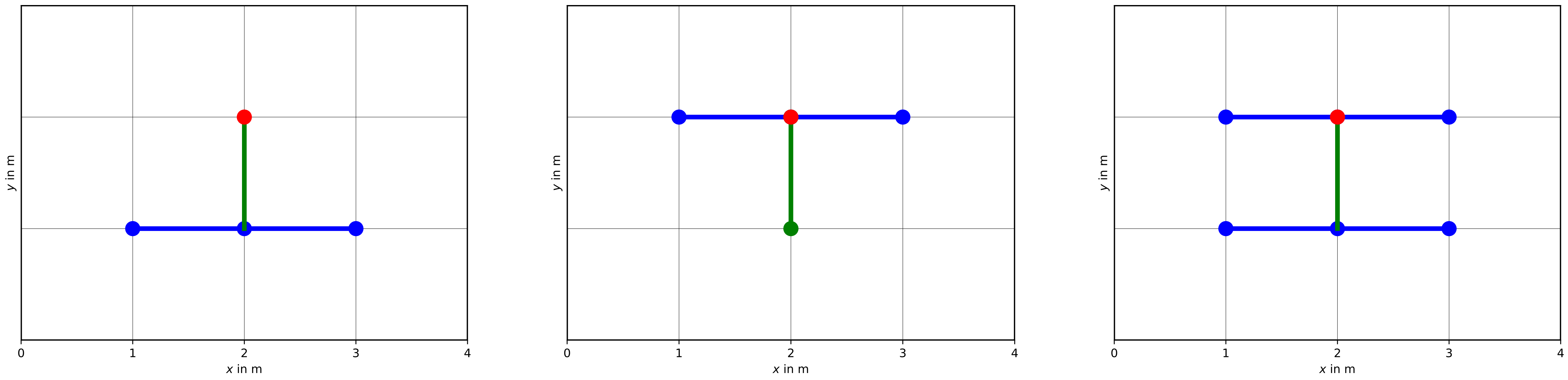
Recap time-stepping in FD

Explicit:	$\mathbf{u}^{n+1} = (\mathbf{I} - \Delta t \mathbf{A})\ \mathbf{u}^n$
Implicit:	$(\mathbf{I} + \Delta t \mathbf{A})\ \mathbf{u}^{n+1} = \mathbf{u}^n$
Mixed:	$(2\mathbf{I} + \Delta t \mathbf{A})\ \mathbf{u}^{n+1} = (2\mathbf{I} - \Delta t \mathbf{A})\ \mathbf{u}^n$



Explicit

Implicit

Mixed

The FE stiffness matrix

Matrix integrating gradients of base functions for neighbors with a

$$\mathbf{A}_{i,i+1} = -\frac{a_i}{\Delta x_i^2} \cdot \Delta x_i = -\frac{a_i}{\Delta x_i}$$

$$A_{i,i} = \int_\Omega a \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_i \mathrm{d}\Omega = -A_{i,i+1} - A_{i+1,i}$$

$\Rightarrow$ matrix-vector equation $\mathbf{Au} = \mathbf{b}$ with bending&shear stiffness in $\mathbf{A}$

Right-hand side vector

The right-hand-side vector $b = \int v_i f \mathrm{d}\Omega$ also scales with Δx

e.g. $f = \boldsymbol{\nabla} \cdot \mathbf{j}_s \Rightarrow b = \int v_i \boldsymbol{\nabla} \cdot \mathbf{j}_s \mathrm{d}\Omega = \int_\Gamma v_i \mathbf{j}_S \cdot \mathbf{n}$

RHS = integrated source function (includes Δx)

(both $\mathbf{A}$ and $\mathbf{b}$ identical to FD for Δx=1)

Difference of FE to FD

Any source function $f(x)$ can be integrated on the whole space!

Tasks

1. Write a function computing the FE stiffness matrix for 1D discretization
2. Test it by solving the Poisson equation with $f = 1$
3. Compare with analytical solution
4. Compute FD solution and compare with FE
5. Change discretization and check again
6. Change conductivity & compare FD and FE

Analytical solution for f=1

$$u(x) = -\frac{1}{2}x^2 + C_1 x + C_0 \quad \Rightarrow \quad u'(x) = -x + C_1$$

BC x=0	BC $x = X$	C_0	C_1
Dirichlet	Dirichlet	u_0	$X/2 + (u_X - u_0)/X$
Dirichlet	Neumann	u_0	$u'_X + X$
Neumann	Dirichlet	u'_0	$u_X - u'_0 X + X^2/2$

Time-stepping in FE

Variational formulation of Diffusion equation

$$\frac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Finite Difference in Time (NOT in space)

$$\frac{u^{n+1} - u^n}{\Delta t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Variational formulation

$$\frac{u^{n+1} - u^n}{\Delta t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Multiplication with test function w and integration $\Rightarrow$ weak form

$$1/\Delta t\left(\int_\Omega w u^{n+1} \mathrm{d}\Omega - \int_\Omega w u^n \mathrm{d}\Omega\right) - \int_\Omega w \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

$$1/\Delta t\left(\int_\Omega w u^{n+1} \mathrm{d}\Omega - \int_\Omega w u^n \mathrm{d}\Omega\right) - \int_\Omega a \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

Variational formulation of Diffusion equation

u is constructed of shape functions $\mathbf{v}_i$ that are identical to w

The integral over the Poisson term $\int_\Omega a \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega$ is represented using the stiffness matrix $\mathbf{A}\mathbf{v}$

$$\mathbf{A}_{i,j} = \int_\Omega \sigma \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega$$

Variational formulation of Poisson equation

Weighted integrals over both u are represented by the mass matrix $\mathbf{Mv}$

$$\mathbf{M}_{i,j} = \int_\Omega v_i \cdot v_j \mathrm{d}\Omega$$

explicit method: $\mathbf{M}\mathbf{u}^{n+1} = (\mathbf{M} - \mathbf{A})\mathbf{u}^n$

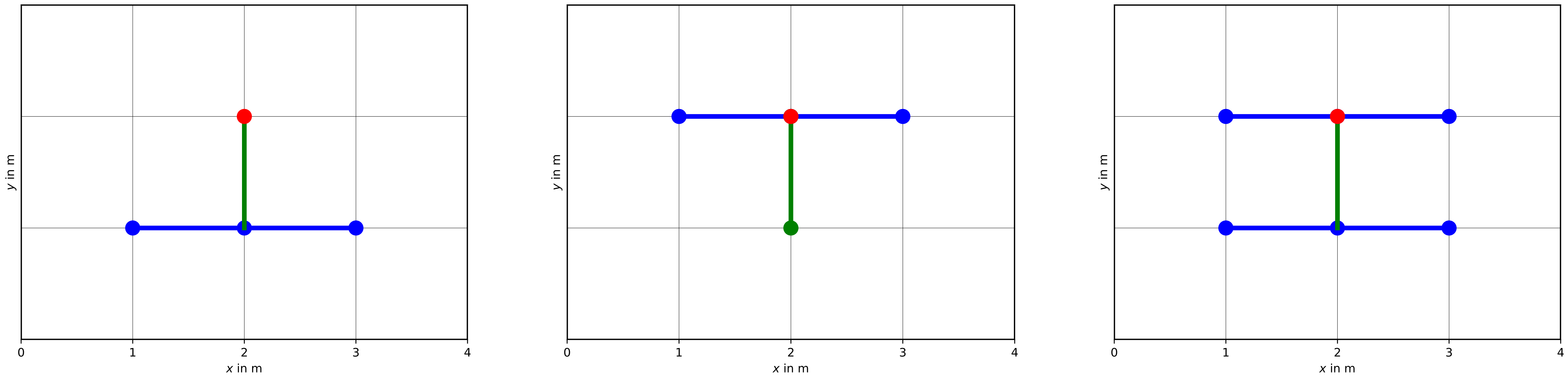
implicit method: $(\mathbf{M} + \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$

mixed method: $(\mathbf{M} + \mathbf{A}/2)\mathbf{u}^{n+1} = (\mathbf{M} - \mathbf{A}/2)\mathbf{u}^n$

same as in FD but with FE mass matrix

Time-stepping in FE

Explicit:	$\mathbf{M}$	$\mathbf{u}^{n+1}$	$=$	$(\mathbf{M} - \Delta t \mathbf{A})$	$\mathbf{u}^{n}$
Implicit:	$(\mathbf{M} + \Delta t \mathbf{A})$	$\mathbf{u}^{n+1}$	$=$	$\mathbf{M}$	$\mathbf{u}^{n}$
Mixed:	$(2\mathbf{M} + \Delta t \mathbf{A})$	$\mathbf{u}^{n+1}$	$=$	$(2\mathbf{M} - \Delta t \mathbf{A})$	$\mathbf{u}^{n}$



Explicit Implicit Mixed

Tasks

1. Write a function computing the FE stiffness matrix for 1D discretization
2. Test it by solving the Poisson equation with $f = 1$ (analytical solution)
3. Compare with analytical and FD solutions
4. Write a function computing the FE mass matrix for 1D discretization
5. Repeat the time-stepping tasks from FD with FE