Numerical Simulation Methods in Geophysics, Part 5: Timestepping

1. MGPY+MGIN

thomas.guenther@geophysik.tu-freiberg.de

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Die Ressourcenuniversität. Seit 1765.

Recap Poisson equation

Recap

- ☑ solve the Poisson equation for arbitrary x and a
- ☑ sources and a contrasts cause curvature in u
  - positive source or a increase $\Rightarrow$ negative u” $\Rightarrow$ maximum
  - single f $\Rightarrow$ piecewise linear, full f $\Rightarrow$ parabola
- ☑ Dirichlet BC determine shift (& slope if double)
- ☑ Neumann BC determine slope of u
- ☑ accuracy (compare analytical) depends on discretization
- ☐ now go for instationary (parabolic) problem by time stepping
  - curvature in u causes negative change of u

Time stepping

Time stepping - explicit method

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial z^2} = 0$$

Finite-difference approximation

$$\frac{\partial T}{\partial t}^n \approx \frac{T^{n+1} - T^n}{\Delta t} = a\frac{\partial^2 T^n}{\partial z^2}$$

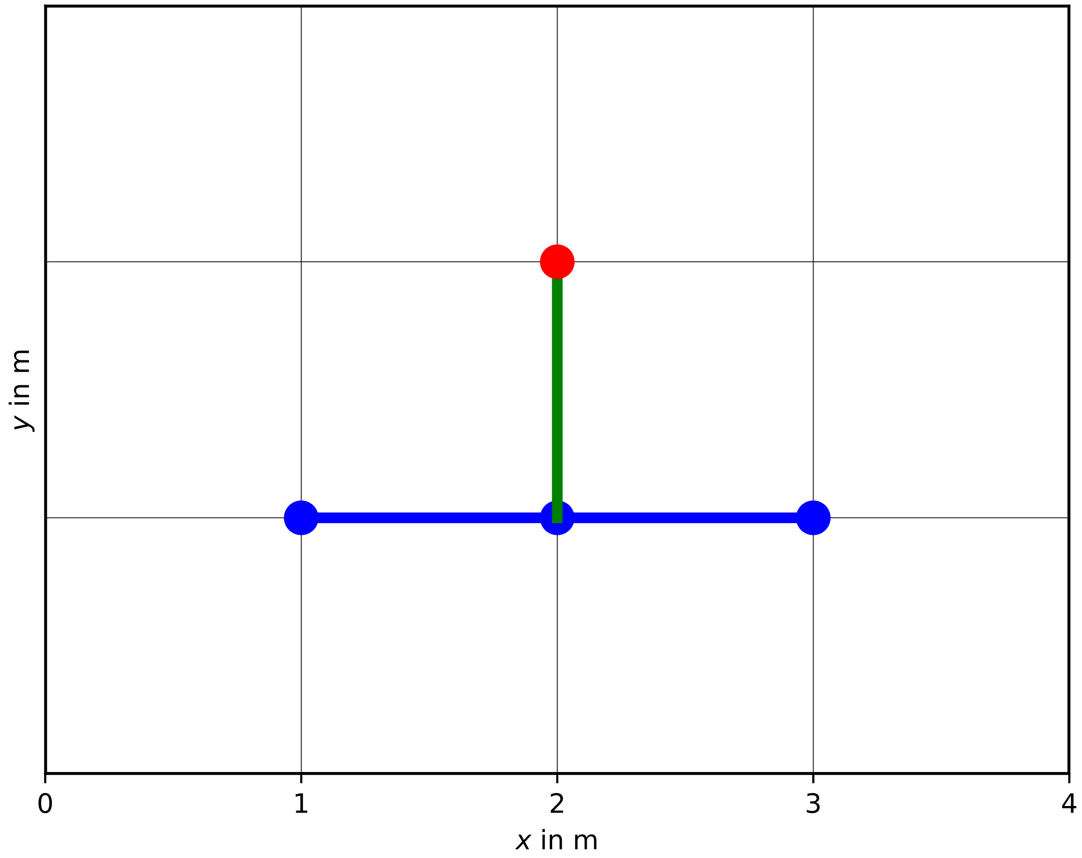
Explicit

Start T^0 with initial condition (e.g. 0)

Update field by

$$T^{n+1} = T^n + a\frac{\partial^2 T^n}{\partial z^2} \cdot \Delta t$$

E.g. by using the matrix A



Forward stencil

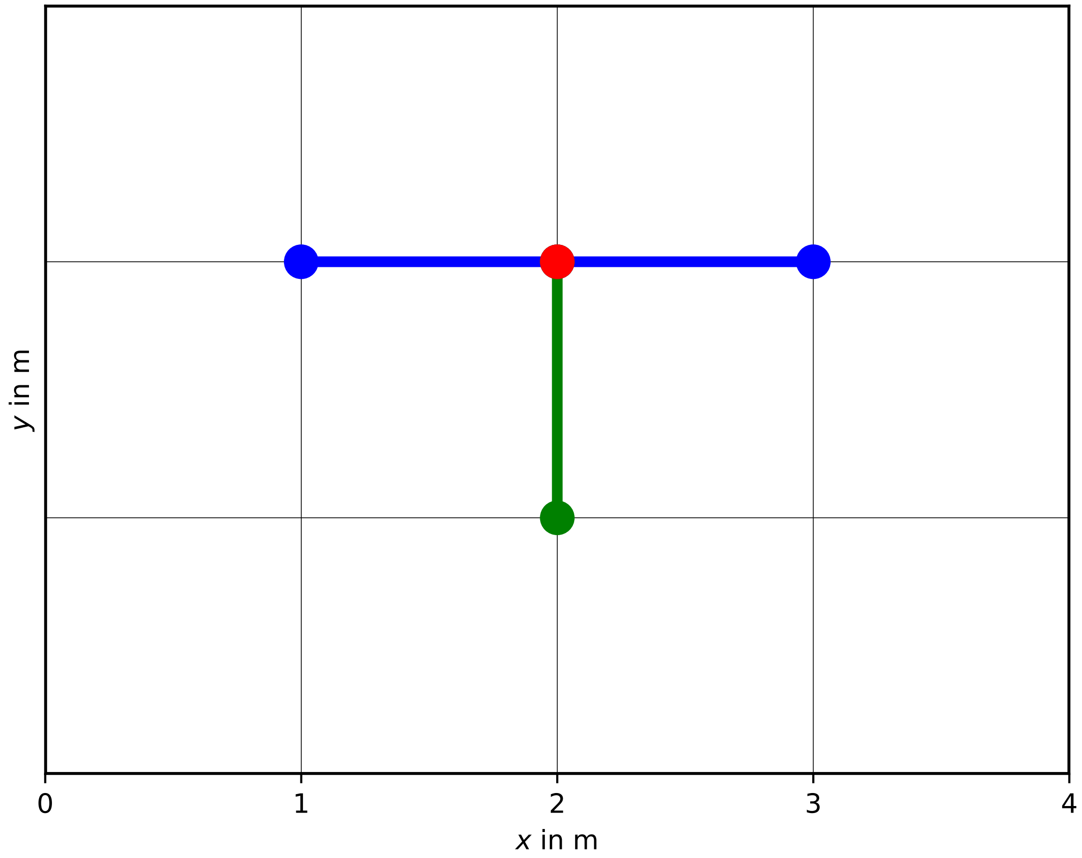
Implicit methods

$$\frac{\partial T}{\partial t}^{n+1} \approx \frac{T^{n+1} - T^n}{\Delta t} = a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{1}{\Delta t} T^{n+1} - a \frac{\partial^2 T}{\partial z^2}^{n+1} = \frac{1}{\Delta t} T^n$$

$$(\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$$

$\mathbf{M}$ - mass matrix



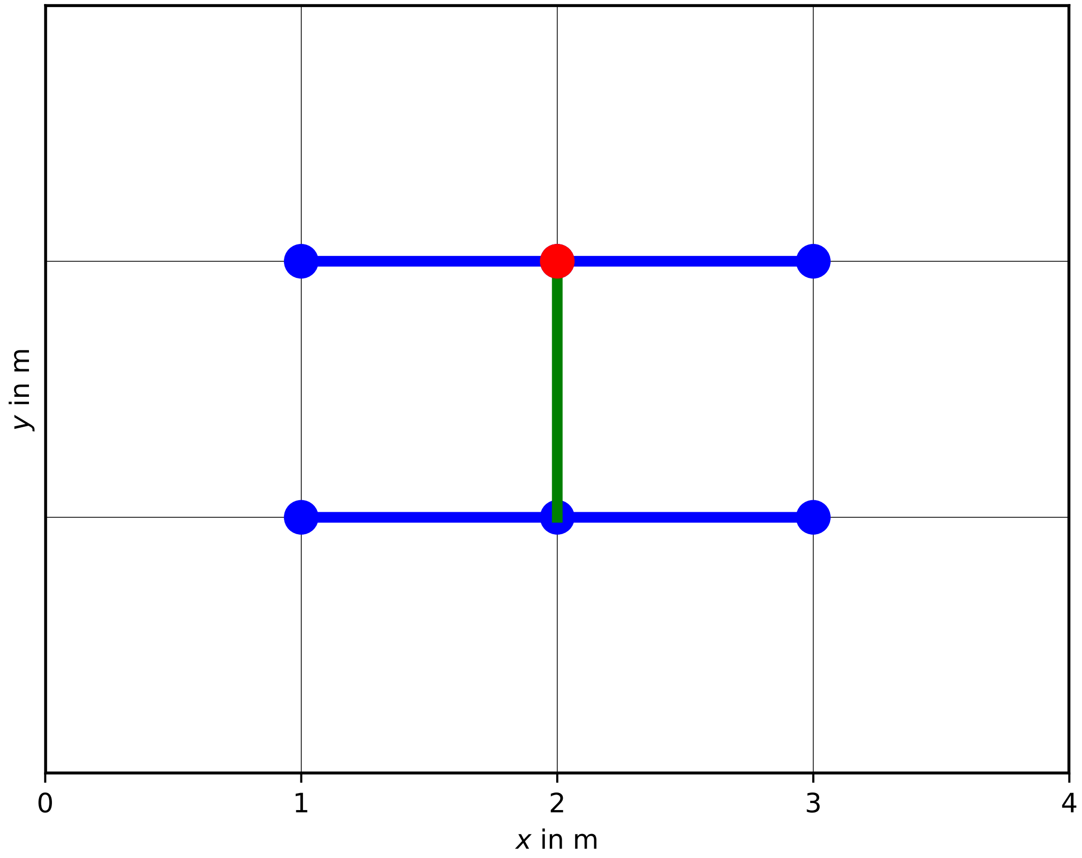
Backward stencil

Mixed - Crank-Nicholson method

$$\frac{\partial T}{\partial t}^{n+1/2} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{2}{\Delta t}T^{n+1} - a\frac{\partial^2 T^{n+1}}{\partial z^2} = \frac{2}{\Delta t}T^n + a$$

$$(2\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = (2\mathbf{M} + \mathbf{A})\mathbf{u}^n$$



Mixed forward/backward stencil

Stability of time-stepping schemes

Consider Newton cooling problem

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{T}{\tau}$$

with solution

$$T(t) = T_0 \exp(-t/\tau)$$

Explicit

$$T_{i+1} = T_i - \frac{dt}{\tau} T_i = T_i(1 - \frac{dt}{\tau}) = T_0(1 - \frac{dt}{\tau})^{i+1}$$

$T_{i+1} < T_i$ requires $0 \leq 1 - dt/\tau < 1$

$$\Rightarrow 0 < dt \leq \tau$$

Implicit

$$\frac{T_{i+1} - T_i}{dt} = -\frac{T_{i+1}}{\tau}$$

$$T_{i+1} = T_i \frac{1}{1 + dt/\tau}$$

$$0 \leq \frac{1}{1 + dt/\tau} < 1$$

unconditionally stable (but maybe still inaccurate)

Mixed

$$\frac{T_{i+1} - T_i}{dt} = -\frac{T_{i+1} + T_i}{2\tau}$$

$$T_{i+1}(1 + dt/2\tau) = T_i(1 - dt/2\tau)$$

$$T_{i+1} = T_i \frac{1 - dt/2\tau}{1 + dt/2\tau}$$

always stable and decreasing for $dt < 2\tau$

Tasks instationary heat equation

- setup a discretization and compute the stiffness matrix A for some a
- choose an initial condition (e.g. homogeneous)
- choose a time step Δt and perform the explicit method using the surface temperature
- change the spatial/temporal discretization and observe the solution
- setup mass matrix and implement the implicit method for diff. Δt
- implement the Crank-Nicholson method and compare all three
- compare the solutions with the analytical solution