Numerical Simulation Methods in Geophysics, Lecture 11: 3D EM modelling

1. MGPY+MGIN

thomas.guenther@geophysik.tu-freiberg.de

TUBAF

Die Ressourcenuniversität. Seit 1765.

Recap

- Maxwells equations (inductive) in time- and frequency domain
- FD: u term needs mass matrix (like in time-stepping)
- from staggered grids (curl) to curl-conforming Nédélec elements
- today: finish modern FE methods for Maxwell, HPC

remaining: 29.1. (FV+advection), 5./6.2. (summary, online), 12.2. (TAP)

Governing equations

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} = \boldsymbol{\nabla} \times \sigma^{-1} \mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\sigma\mathbf{E} = -\imath\omega\mathbf{j}_s$$

- air (σ small): H linear (or constant)

TE and TM polarizations: scalar problems

$$-\boldsymbol{\nabla} \cdot \sigma^{-1} \boldsymbol{\nabla} H_x(y, z) + \imath\omega\mu H_x(y, z) = 0$$

$$-\boldsymbol{\nabla} \cdot \mu^{-1} \boldsymbol{\nabla} E_x(y, z) + \imath\omega\sigma E_x(y, z) = 0$$

Original source for Nédélec

A New Family of Mixed Finite Elements in $\mathbb{R}^3$

J.C. Nédélec

Mathématiques Appliqueées, Ecole Polytechnique, F-91120 Plaiseau, France

Summary. We introduce two families of mixed finite element on conforming in $H(\mathrm{div})$ and one conforming in $H(\mathrm{curl})$. These finite elements can be used to approximate the Stokes' system.

Designed for streaming problems (p-pressure, $\mathbf{u}$-velocity)

$$-v\nabla^2\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

Approach: $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u} = -\nabla^2\phi \Rightarrow \mathbf{u} = \boldsymbol{\nabla} \times \phi \qquad \boldsymbol{\nabla} \cdot \phi = 0$

EM vector modelling

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} + \imath\omega\sigma\mathbf{E} - \omega^2 \epsilon \mathbf{E} = -\imath\omega\mathbf{j}_s$$

$$\nabla \times \sigma^{-1} \nabla \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2 \mu\epsilon \mathbf{H} = \nabla \times \sigma^{-1} \mathbf{j}_s$$

- air (σ small): H linear (or constant)

The weak formulation

$$\int_\Omega \mathbf{w}\boldsymbol{\nabla}\times\mu^{-1}\boldsymbol{\nabla}\times\mathbf{E}\mathrm{d}\Omega - \imath\omega\int \mathbf{w}\sigma\mathbf{E}\mathrm{d}\Omega = 0$$

Integration, Greens identity $\int a\nabla^2 b = -\int \boldsymbol{\nabla} a\cdot\boldsymbol{\nabla} b - \int a\,\partial b/\partial n$

$$\int_\Omega \mathbf{w}\ldots = \int_\Omega \boldsymbol{\nabla}\times\mathbf{w}\cdot(\mu^{-1}\boldsymbol{\nabla}\times\mathbf{E})\mathrm{d}\Omega + \int \mathbf{n}\times(\mu^{-1}\boldsymbol{\nabla}\times\mathbf{E})\mathrm{d}\Gamma$$

choose basis that latter term vanishes

$$\int_\Omega \boldsymbol{\nabla}\times\mathbf{w}\cdot(\mu^{-1}\boldsymbol{\nabla}\times\mathbf{E})\mathrm{d}\Omega - \imath\omega\int \mathbf{w}\sigma\mathbf{E}\mathrm{d}\Omega = 0$$

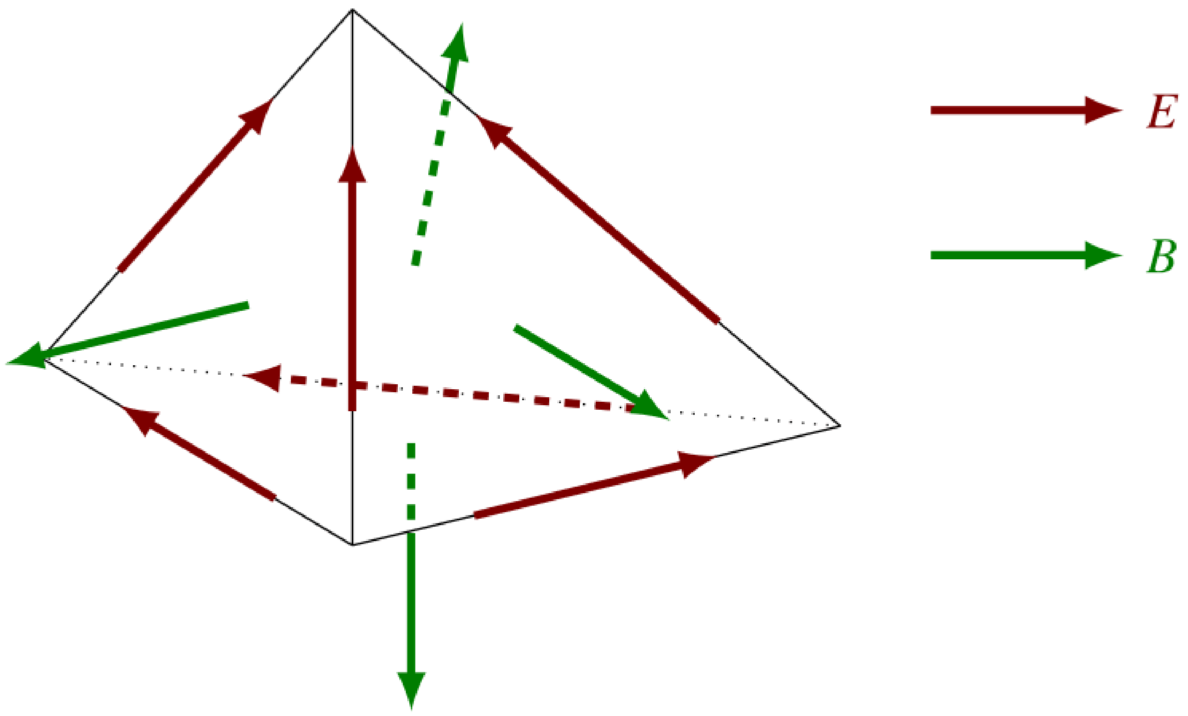
Galerkins method

$$\mathbf{E} = \sum_i u_i \mathbf{u} \quad \text{with} \quad \mathbf{w} \in \mathbf{u}_i$$

$$\left\langle \boldsymbol{\nabla} \times \mathbf{w} \middle| \mu^{-1} \boldsymbol{\nabla} \times \mathbf{u}_i \right\rangle$$

$$\mathbf{Au} = \mathbf{b} \quad \text{with} \quad A_{i,j} = \int_\Omega (\boldsymbol{\nabla} \times u_i) \cdot (\mu^{-1} \boldsymbol{\nabla} \times u_j) \mathrm{d}\Omega$$

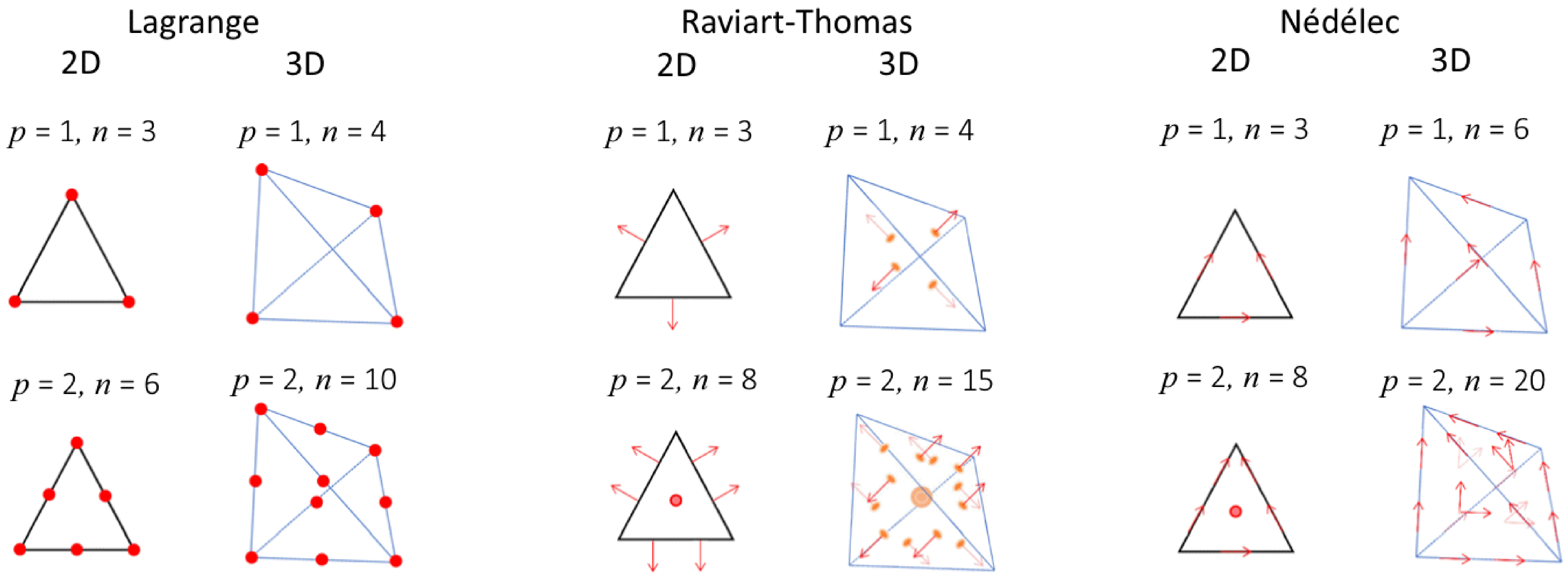
Vectorial solution for EM fields



Schwarzbach & Haber (2013)

- $\mathbf{E} \in H^{\mathrm{curl}}(\Omega)$
- $\mathbf{B} \in H^{\mathrm{div}}(\Omega)$
- natural BC: Dirichlet $\mathbf{n} \times \mathbf{E} = 0$
- Neumann BC:
  $\mathbf{n} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} = 0$

Elements used in EM modelling



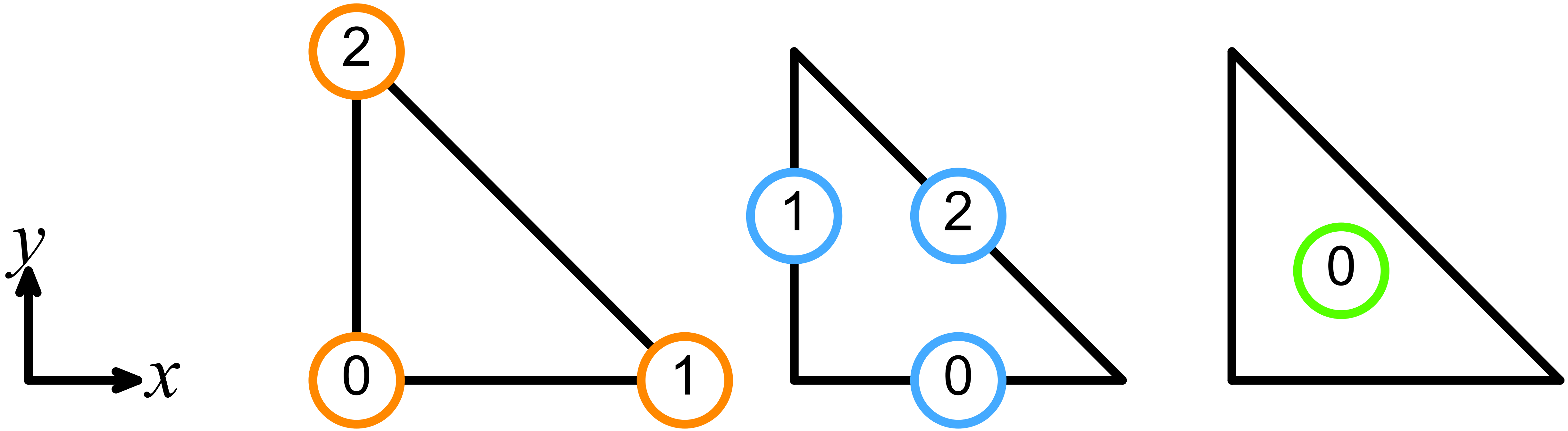
$$\mathcal{H}^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^3\}$$

$$\mathcal{H}(\mathrm{div};\Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\}$$

$$\mathcal{H}(\mathrm{curl};\Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3\}$$

Types of elements relevant for EM problems (Spitzer, 2024)

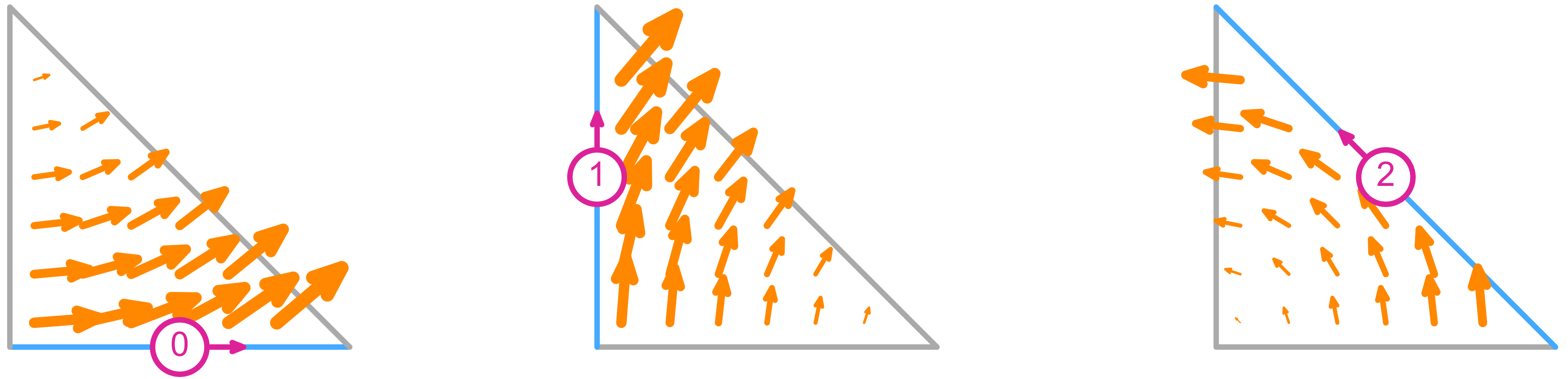
Mesh entities on triangle



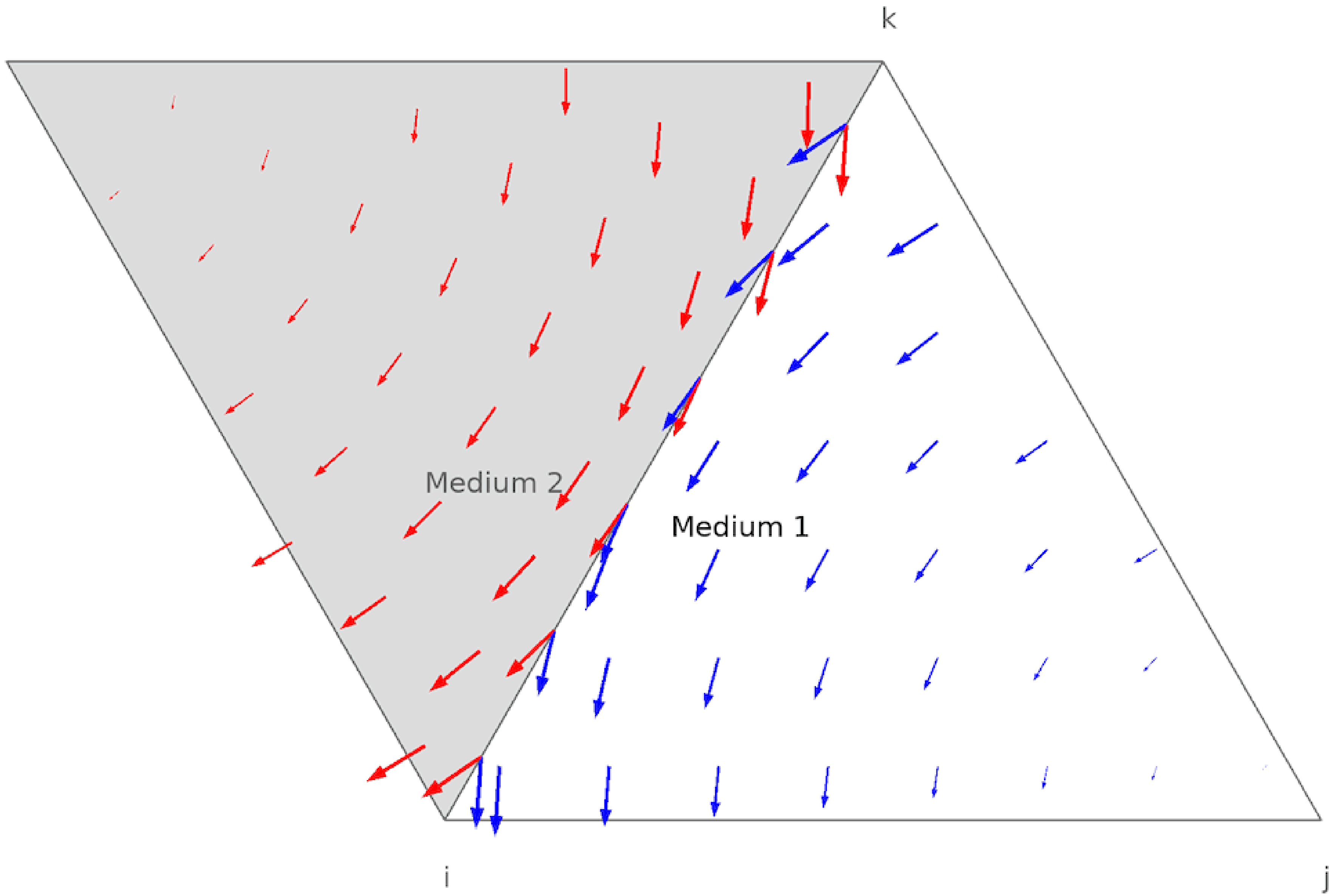
Nodes (yellow), edges (blue) and cells (green) of a triangle

Documentation of DefElements.org

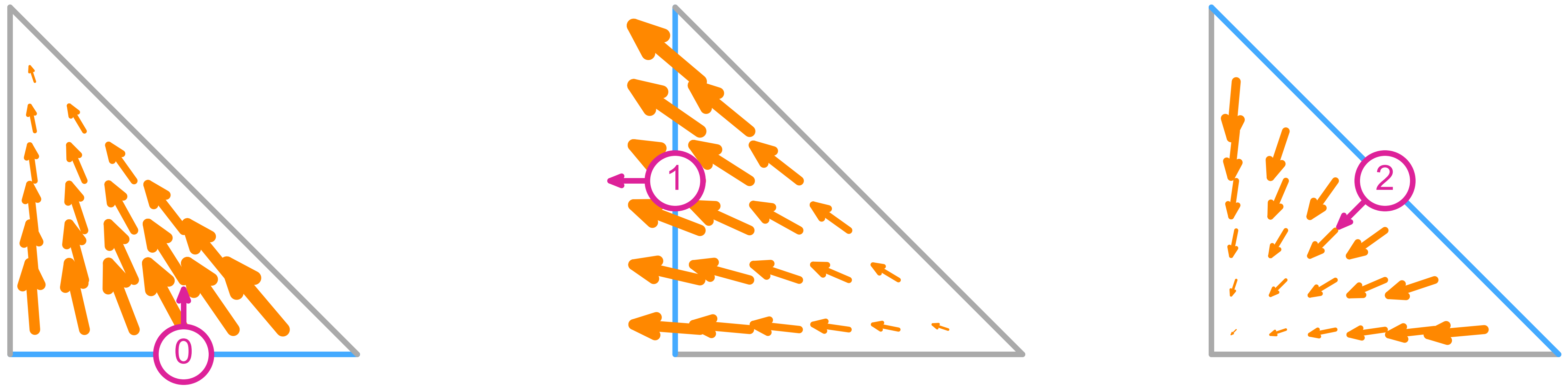
Nédélec elements (first kind) on triangle



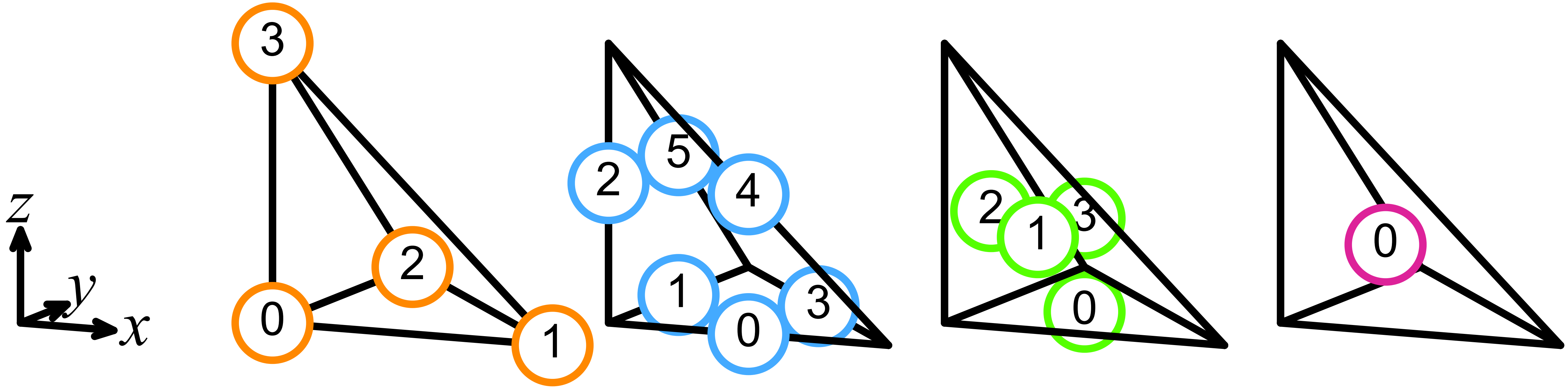
Nedelec shape functions



Raviart-Thomas on the triangle



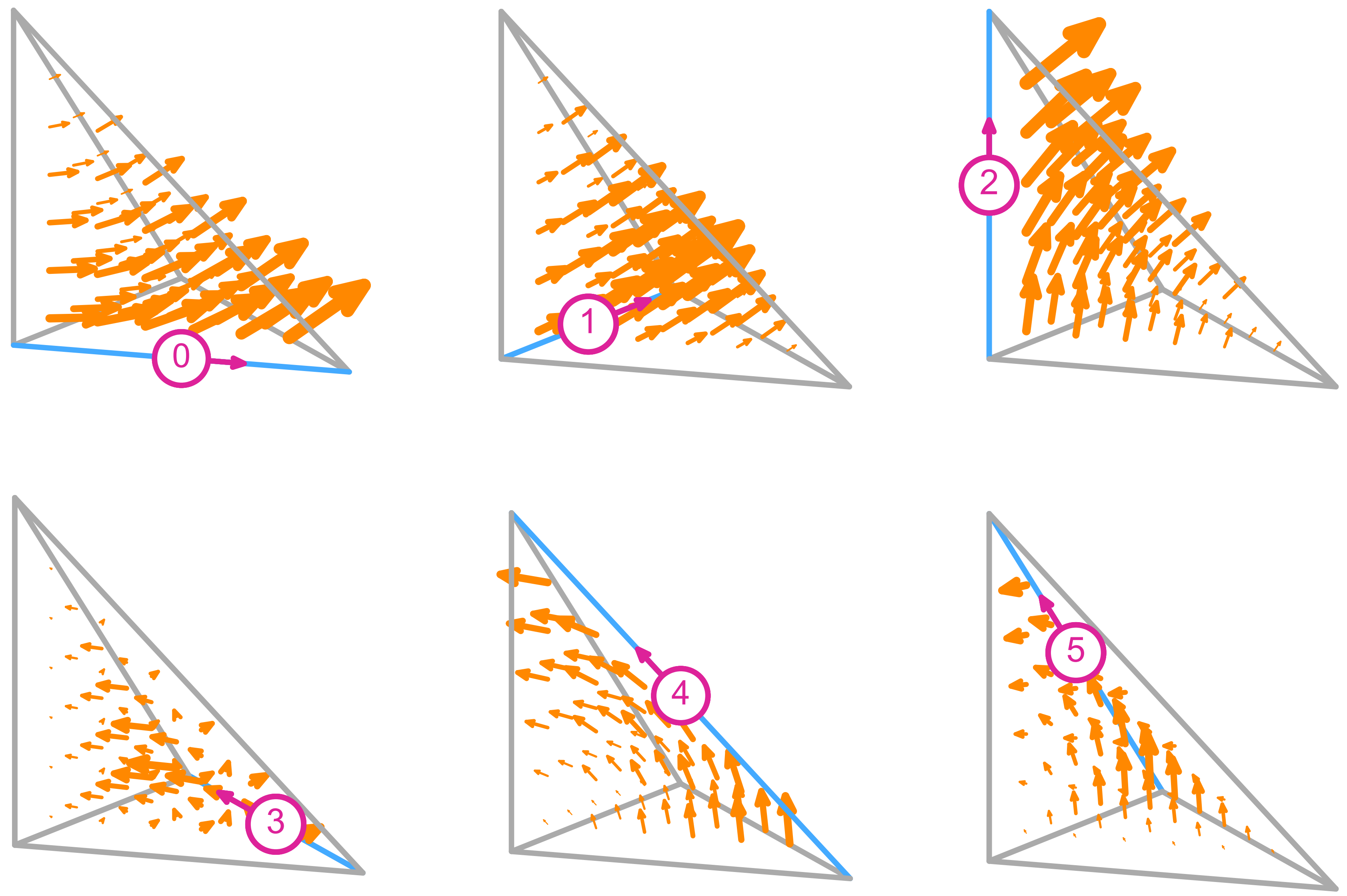
Mesh entities tetrahedron



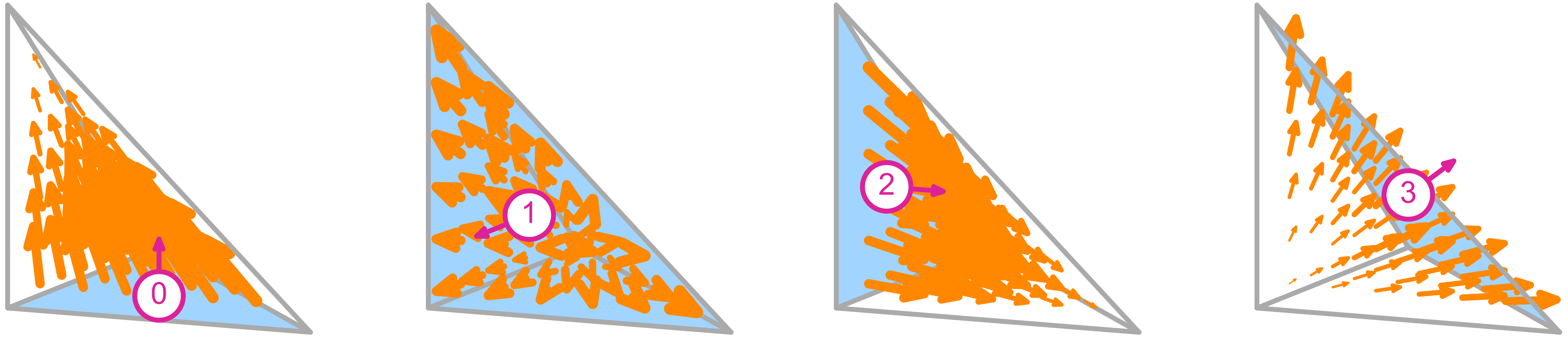
Nodes (orange), edges (blue), faces (green) & cells (purple)

Documentation of DefElements.org

Nédélec elements (first kind) on tetrahedron


0
1
2
3
4
5

Raviart-Thomas elements on tetrahedron



Packages

Mesh generation: TetGen (3D), GMsh (2D/3D)

FE packages: FEniCS, NETGEN/NGsolve

Equation solvers: Suitesparse, MUMPS, SciPy

Computational frameworks: PetSc, MPI

EM modelling (and inversion) packages: Mare2DEM, emg3d, GoFEM, PETGEM, custEM, SimPEG, ModEM, FEMTIC

FEniCS for pressure & flow

$$-v\nabla^2\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

```
import dolfinx as dfx  # API to FEniCS
import ufl  # Unified form language

lagrange = element("P", mesh.basix_cell(), p) # shape=(3,)
nedelec = element("N1curl", mesh.basix_cell(), p)

u = ufl.TestFunctions(nedelev)
v = ufl.TrialFunctions(nedelec)

var_form = dfx.fem.form((ufl.curl(ur), ufl.curl(vr)) * ufl.dx))

mixed_ele = mixed_element([lagrange, nedelec]) # real imag
MixedSpace = dfx.fem.functionspace(mesh, mixed_ele)

A = dfx.fem.petsc.assemble_matrix(var_form, bcs=self.bcs)
A.assemble()
```

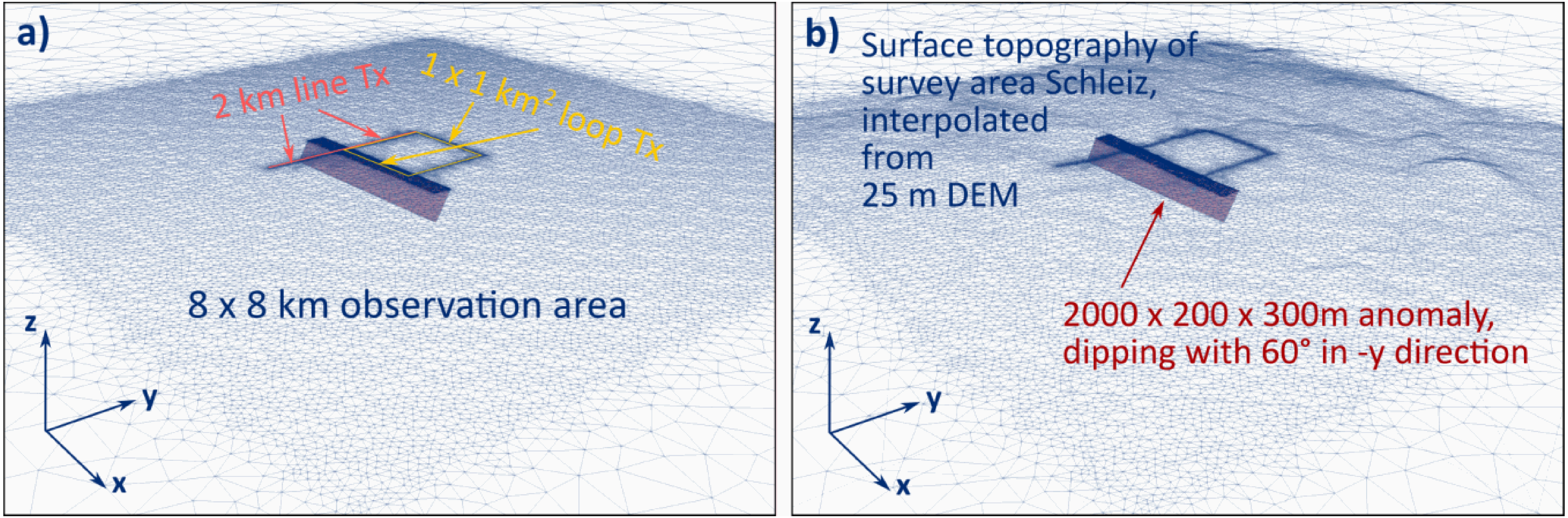
FEniCS for EM

```
# E lives on edges
ele = element("N1curl", mesh.basix_cell(), p)
# real and imaginary part of E combined
mixed_ele = mixed_element([ele, ele])
MixedSpace = dfx.fem.functionspace(mesh, mixed_ele)
# Sigma lives on Cells
dg_space = dfx.fem.functionspace(mesh, ("DG", 0)) # sigma
para = dfx.fem.Function(dg_space)
para.x.array[:] = sigma_vector
# Trial and test function
vr, vi = ufl.TrialFunctions(MixedSpace)
ur, ui = ufl.TestFunctions(MixedSpace)
# variational form
var_form = dfx.fem.form((
    ufl.inner(mu1 * ufl.curl(ur), ufl.curl(vr)) * ufl.dx -
    omega * ufl.inner(sig * vi, ur) * ufl.dx - omega * ufl.inner(sig * vr, ui) * ufl.dx
```

Post-processing

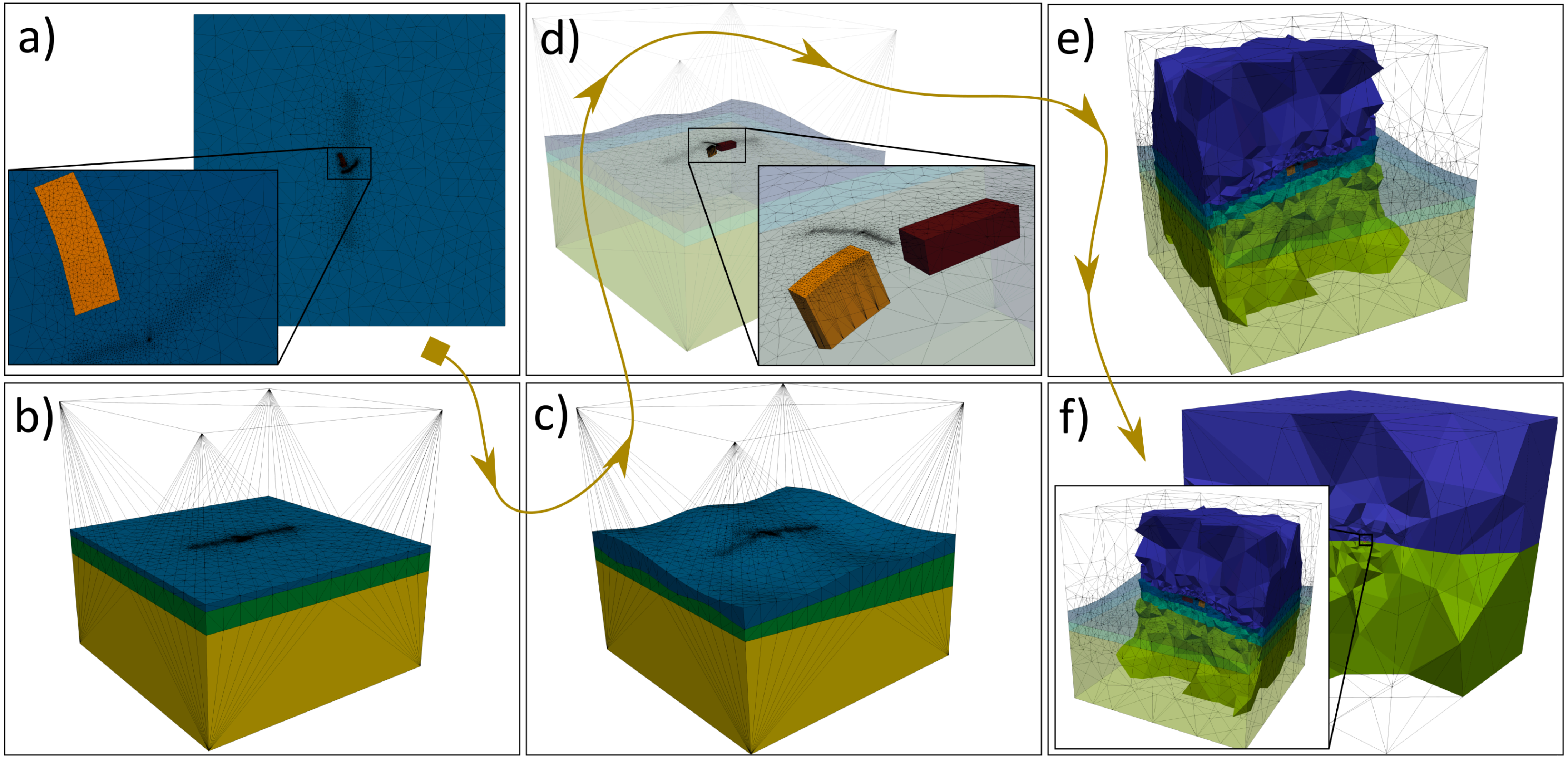
```
raviart_thomas = dfx.fem.functionspace(
    mesh, element("RT", mesh.basix_cell(), p))
```

Modelling topography



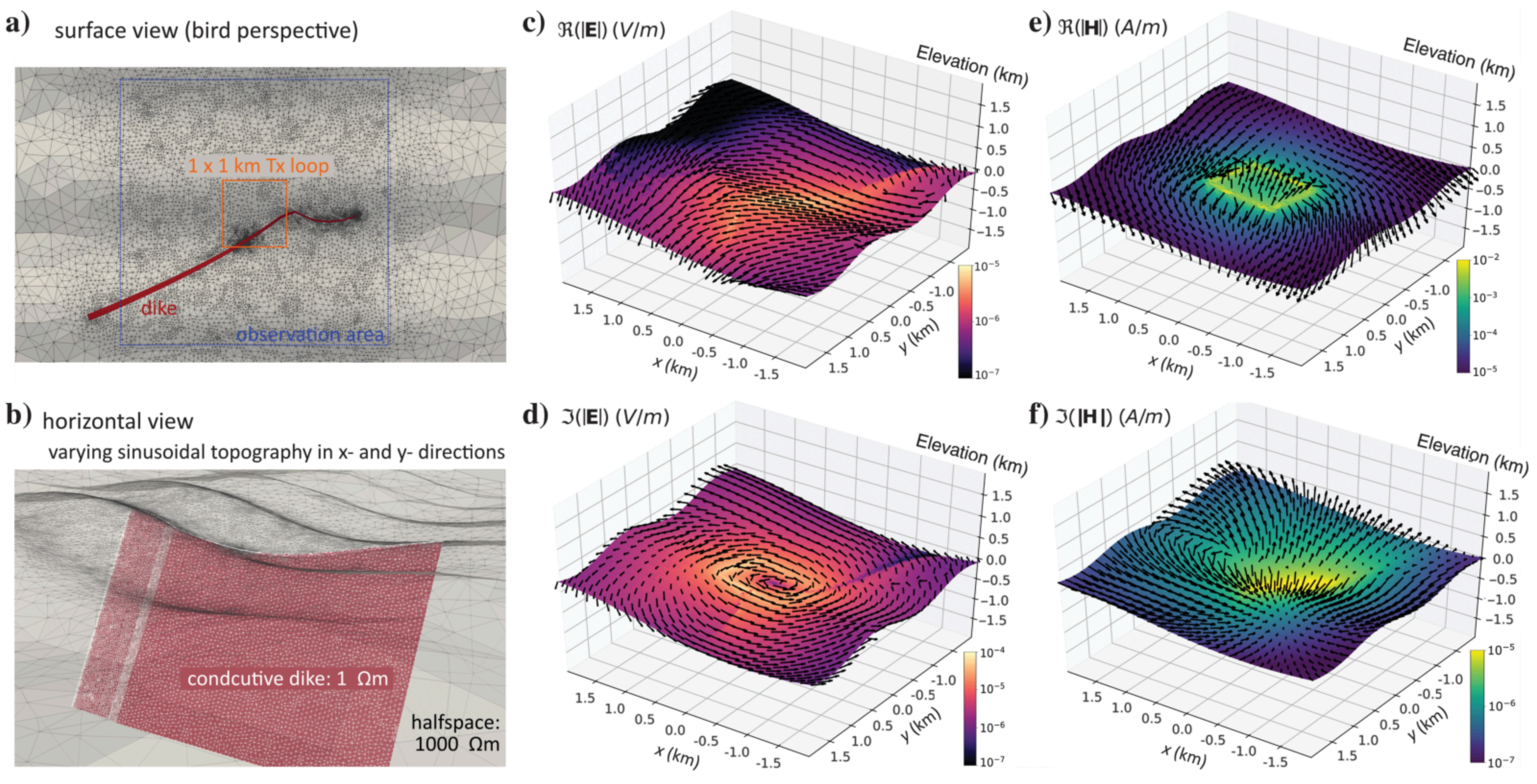
Modelling incorporating topography (Rochlitz, 2019)

Meshing complicated geometries



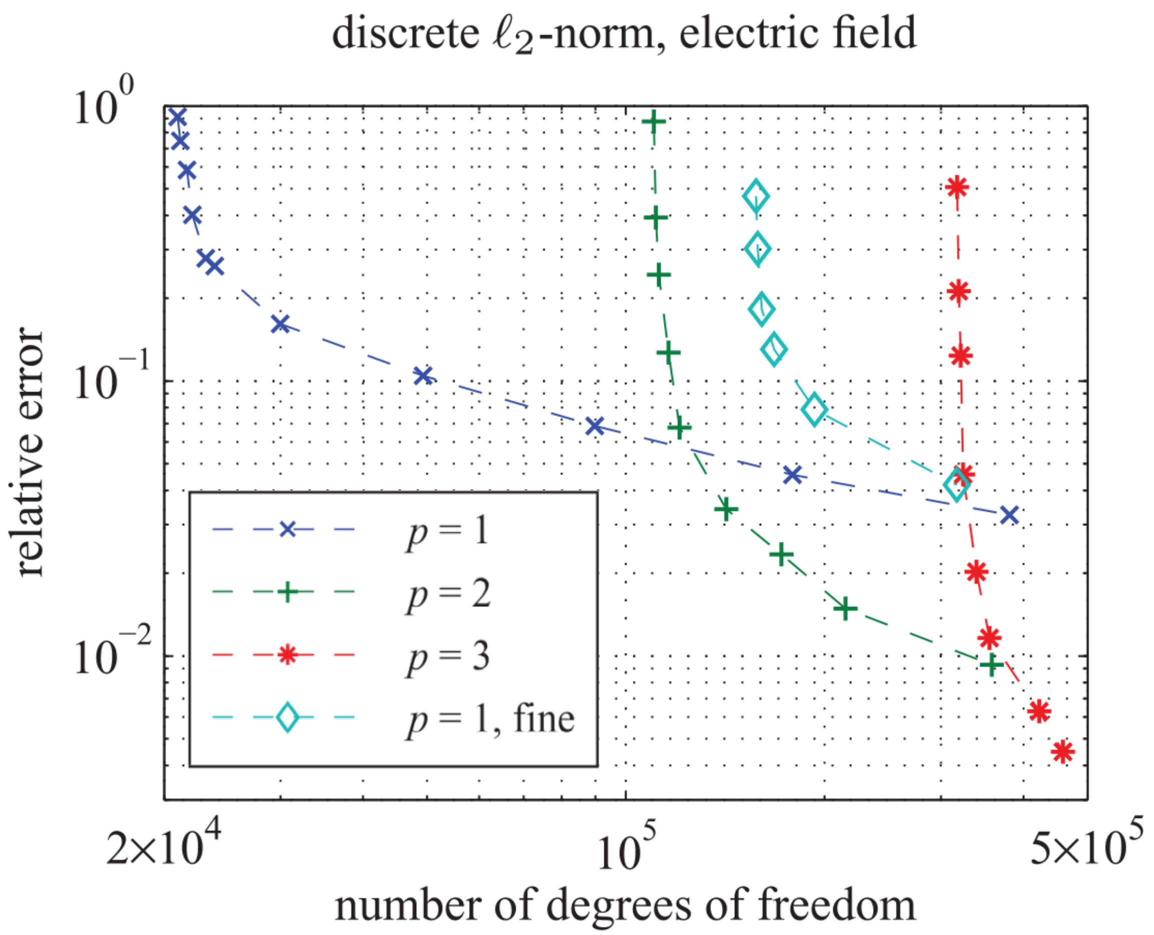
Meshing workflow in custEM (Rochlitz et al., 2019)

Modelling example



Modelling example of a conductive dike (Rochlitz et al., 2019)

Accuracy and refinement



h and p refinement (Schwarzbach et al. 2011)

Boundary conditions

Neumann (implicitly by choice of shape functions)

$$\frac{\partial u}{\partial n} = 0 \Rightarrow \mathbf{n} \times \boldsymbol{\nabla} \times \mathbf{E} = 0 \Rightarrow \mathbf{n} \times \mathbf{B} = 0$$

Dirichlet

$$u = 0 \Rightarrow \mathbf{n} \times \mathbf{E} = 0 \Rightarrow \mathbf{n} \cdot \mathbf{B} = 0$$

Mixed boundary conditions

So far…

- Dirichlet Boundary conditions $u = u_0$
- Neumann Boundary conditions $\frac{\partial u}{\partial n} = g_B$

vectorial problems: $\mathbf{n} \cdot \mathbf{E} = 0$ (Neumann) or $\boldsymbol{\nabla} \times \mathbf{E} = 0$ (Dirichlet)

In general mixed, also called Robin (or impedance convective) BC

$$au + b\frac{\partial u}{\partial n} = c$$

Example DC resistivity with point source

$$\boldsymbol{\nabla} \cdot \sigma \boldsymbol{\nabla} u = \boldsymbol{\nabla} \cdot \mathbf{j} = I\delta(\mathbf{r} - \mathbf{r}_s)$$

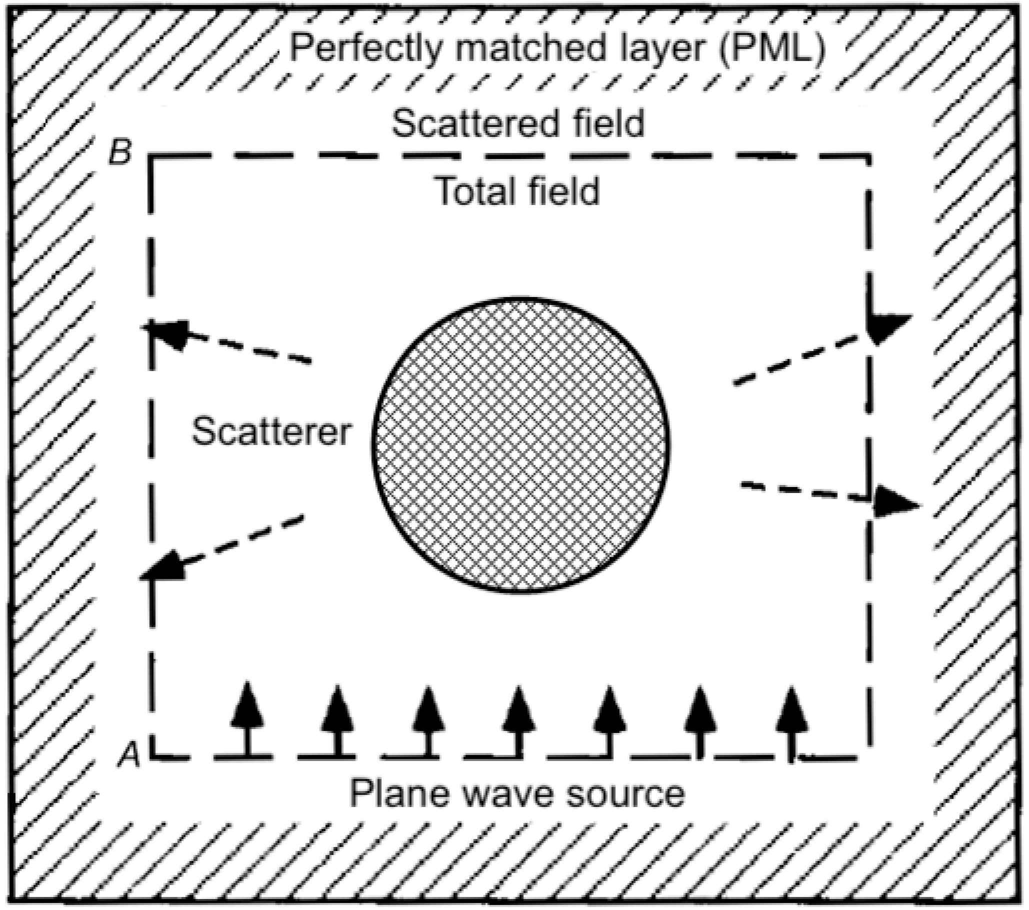
solution for homogeneous σ on surface: $u = \frac{I}{2\pi\sigma} \frac{1}{|\mathbf{r}-\mathbf{r}_s|}$

E-field $\mathbf{E} = -\frac{I}{2\pi\sigma} \frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3}$

normal direction $\mathbf{E} \cdot \mathbf{n} = -\frac{u}{|\mathbf{r}-\mathbf{r}_s|} \cos\phi$ purely geometric

$$\frac{\partial u}{\partial n} + \frac{\cos\phi}{|\mathbf{r} - \mathbf{r}_s|} = 0$$

Perfectly matched layers



$$\frac{\partial}{\partial x} \to \frac{1}{1 + i\sigma/\omega} \frac{\partial}{\partial x}$$

$$x \to x + \frac{i}{\omega} \int^{x} \sigma(x') \mathrm{d}x'$$

Absorbing boundary conditions

wave equation (e.g. in 2D)

$$\frac{\partial^2 u}{\partial t^2} - v^2 \nabla^2 u = 0$$

Fourier transform in t and y (boundary direction) $\Rightarrow \omega, k$

$$\omega^2 \hat{u} - v^2 \frac{\partial^2 \hat{u}}{\partial x^2} + v^2 k^2 \hat{u} = 0$$

ordinary DE with solution $\hat{u} = \sum a_i e^{\lambda x}$ with $\lambda^2 = k^2 - \omega^2/v^2$

Time-domain EM

- Fourier transform excitation $\Rightarrow$ solve in FD $\Rightarrow$ backtransform
- solve with Time-Stepping: Governing equation

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{e} + \sigma \frac{\partial \mathbf{e}}{\partial t} = - \frac{\partial \mathbf{e}_s}{\partial t}$$

$$\mathbf{K} \mathbf{u} + \mathbf{M} \frac{\partial u}{\partial t} = \mathbf{s}$$

Implicit time stepping

$$(\Delta t\mathbf{K} + \mathbf{M})\mathbf{u}^{n+1} = \mathbf{M}u^{n} + \Delta t\mathbf{s}^{n+1}$$

second-order

$$(2\Delta t\mathbf{K} + 3\mathbf{M})\mathbf{u}^{n+2} = \mathbf{M}(4\mathbf{u}^{n+1} - \mathbf{u}^{n}) - 2\Delta t\mathbf{s}^{n+2}$$

Appendix

2D scalar EM - TM polarization

$$\nabla \times \sigma^{-1} \nabla \times \mathbf{H} + \imath\omega\mu\mathbf{H} = \nabla \times \sigma^{-1} \mathbf{j}_s$$

Transverse magnetic (TM) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{H} = [H_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar H_x

$$-\nabla \cdot \sigma^{-1} \nabla H_x(y, z) + \imath\omega\mu H_x(y, z) = 0$$

2D scalar EM - TE polarization

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} + \imath\omega\sigma\mathbf{E} = \nabla \times \mathbf{j}_s$$

Transverse electric (TE) mode

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{E} = [E_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar E_x

$$-\nabla \cdot \mu^{-1} \nabla E_x(y, z) + \imath\omega\sigma E_x(y, z) = 0$$