Numerical Simulation Methods in Geophysics, Part 8: 2D Helmholtz equation

1. MGPY+MGIN

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Recap

- finite differences approximate partial derivatives
- finite elements approximate solution $\Rightarrow$ preferred
- spatial discretization determines accuracy of solution
- different time-stepping approaches, implicit and mixed schemes most accurate and stable
- tasks for report on 1D instationary heat equation with periodic boundary conditions
- basic elements and higher order shape functions
- shape functions on the triangle
- numerical integration , e.g. by Gaussian quadrature

Helmholtz equation in 2D

- move to another type of PDE
- move from 1D to 2D (and eventually 3D)
- complex-valued system
- secondary field approach

Maxwells equations

- Faraday's law: currents & varying electric fields $\Rightarrow$ magnetic field

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$$

- Ampere's law: time-varying magnetic fields induce electric field

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\boldsymbol{\nabla} \cdot \mathbf{D} = \varrho$ (charge $\Rightarrow$), $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (no magnetic charge)
- material laws $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

Maxwell in frequency domain

$$\mathbf{E} = \mathbf{E}_0 e^{\imath\omega t} \quad \text{or} \quad \mathbf{H} = \mathbf{H}_0 e^{\imath\omega t}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \imath\omega\epsilon\mathbf{E} + \sigma\mathbf{E}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\imath\omega\mu\mathbf{H}$$

Helmholtz equation

see also Theory EM

take curl of one of the equations and insert in the other

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\mu\sigma\mathbf{E} - \omega^2\mu\epsilon\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \rho\boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2\mu\epsilon\rho\mathbf{H} = 0$$

Quasi-static approximation

Assume: $\omega^2\mu\epsilon < \omega\mu\sigma$, no sources ($\boldsymbol{\nabla}\cdot\mathbf{j}_s = 0$), + vector identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{F} - \boldsymbol{\nabla}^2\mathbf{F}$$

leads with $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = \boldsymbol{\nabla}\cdot\mathbf{B}$ to the vector Helmholtz PDE

$$-\boldsymbol{\nabla}^2\mathbf{E} + \imath\omega\mu\sigma\mathbf{E} = 0$$

$$-\boldsymbol{\nabla}\cdot\rho\mathbf{H} + \imath\omega\mu\mathbf{H} = 0$$

Variational form

$$-\boldsymbol{\nabla}^2 u + \imath\omega\mu\sigma u = f$$

$$-\int_\Omega w\boldsymbol{\nabla}^2 u \mathrm{d}\Omega + \int_\Omega w\imath\omega\mu\sigma u \mathrm{d}\Omega = \int_\Omega wf \mathrm{d}\Omega$$

Gauss's integral law

$$\int_\Omega \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega + \imath\omega \int_\Omega \mu\sigma w u \mathrm{d}\Omega = \int_\Omega wf \mathrm{d}\Omega$$

Weak formulation

$u = \sum_i u_i \mathbf{v}_i$ and $w_i \in v_i$ leads to

$$\int_\Omega \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \imath\omega \int_\Omega \mu\sigma v_i v_j \mathrm{d}\Omega = \int_\Omega v_i f \mathrm{d}\Omega$$

$$\langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle + \imath\omega \langle v_i | \mu\sigma v_j \rangle = \langle v_i | f \rangle \quad \text{inner products}$$

representation by matrix-vector product $(\mathbf{A} + \imath\omega\mathbf{M})\mathbf{u} = \mathbf{b}$

$A_{ij} = \langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle$ and $b_i = \langle v_i | f \rangle$

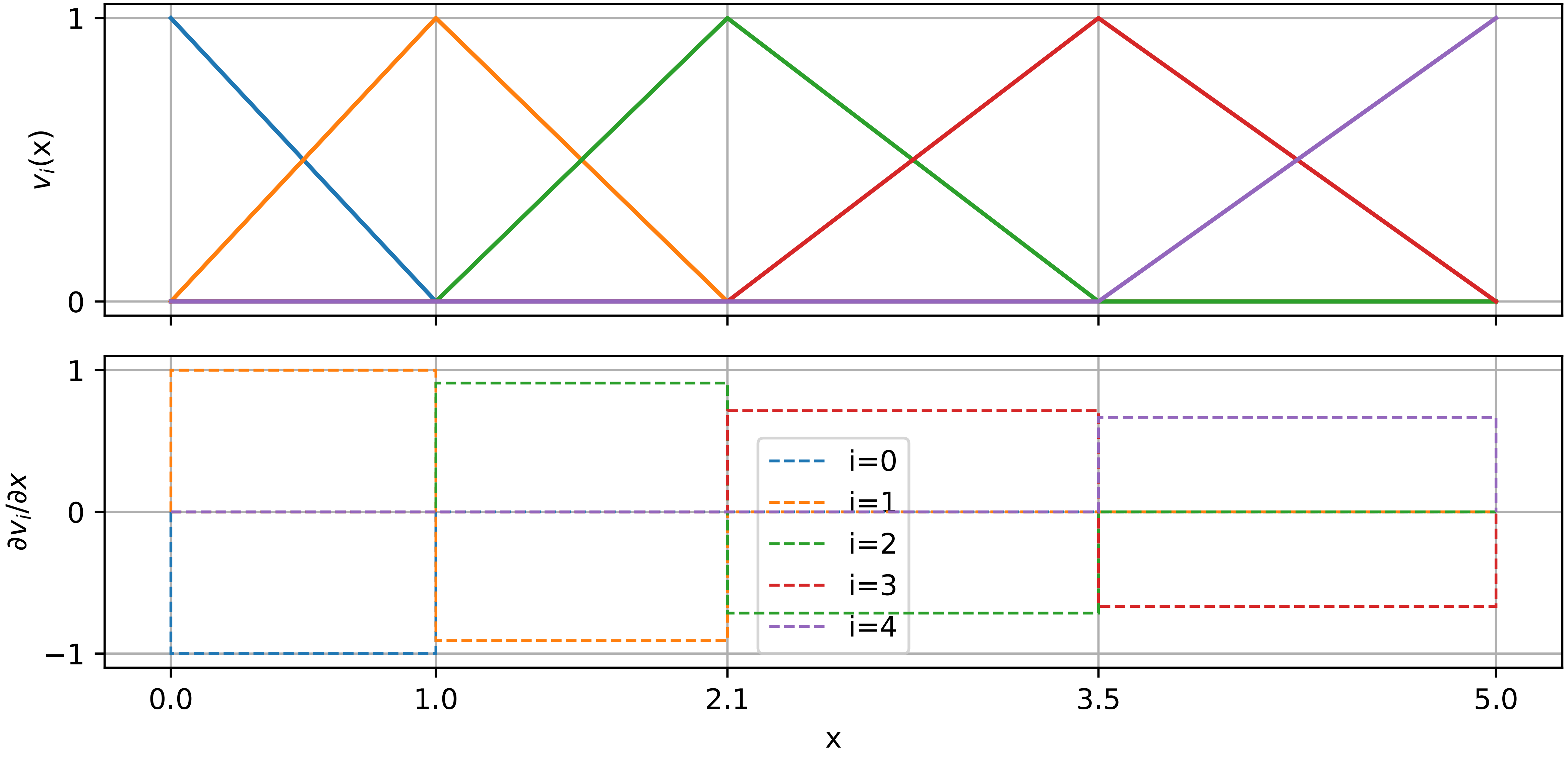
The finite element mass matrix

$$M_{i,j} = \int_{\Omega} \mu \sigma v_i v_j \mathrm{d}\Omega = \sum_c \int_{\Omega_c} \mu_c \sigma_c v_i v_j \mathrm{d}\Omega$$

$$M_{i,j} = \sum_c \mu_c \sigma_c \int_{\Omega_c} v_i v_j$$

in 1D: $v_i = (x - x_i)/\Delta x_i$, $v_j = (x_{i+1} - x)/\Delta x_i$

Hat functions



Complex or real-valued?

The complex valued system

$$\mathbf{A} + \imath\mathbf{M} = \mathbf{u} = b$$

can be transferred into a doubled real-valued system

$$\begin{pmatrix} A & -M \\ M & A \end{pmatrix} \begin{pmatrix} u_r \\ u_i \end{pmatrix} = \begin{pmatrix} b_r \\ b_i \end{pmatrix}$$

Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$

solution for F_0 known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for F_a, because

- F_a is weaker or smoother (e.g. $F_0 \propto 1/$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

Secondary field Helmholtz equation

The equation $-\nabla^2 F - k^2 F = 0$ is solved by the primary field for k_0:

$-\nabla^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\nabla^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\nabla^2 F_a - k^2 F_a = \delta k^2 F_0$$

Note

Source terms only arise at anomalous terms

Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\nabla^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\nabla^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

Note

Source terms only arise at anomalous conductivities and increase with primary field