Numerical Simulation Methods in Geophysics, Exercise 12: I open at the close

1. MGPY+MGIN

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Recap time-stepping in FD

There have been problems using the simple time-stepping schemes known from FD. We need to have an FE eye onto the problem at hand.

Explicit

Start T^0 with initial condition

Update field by

$$T^{n+1} = T^n + a\frac{\partial^2 T^n}{\partial z^2} \cdot \Delta t$$

E.g. by using the matrix $\mathbf{A}$ using

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T[1:] += A.dot(T)[1:]
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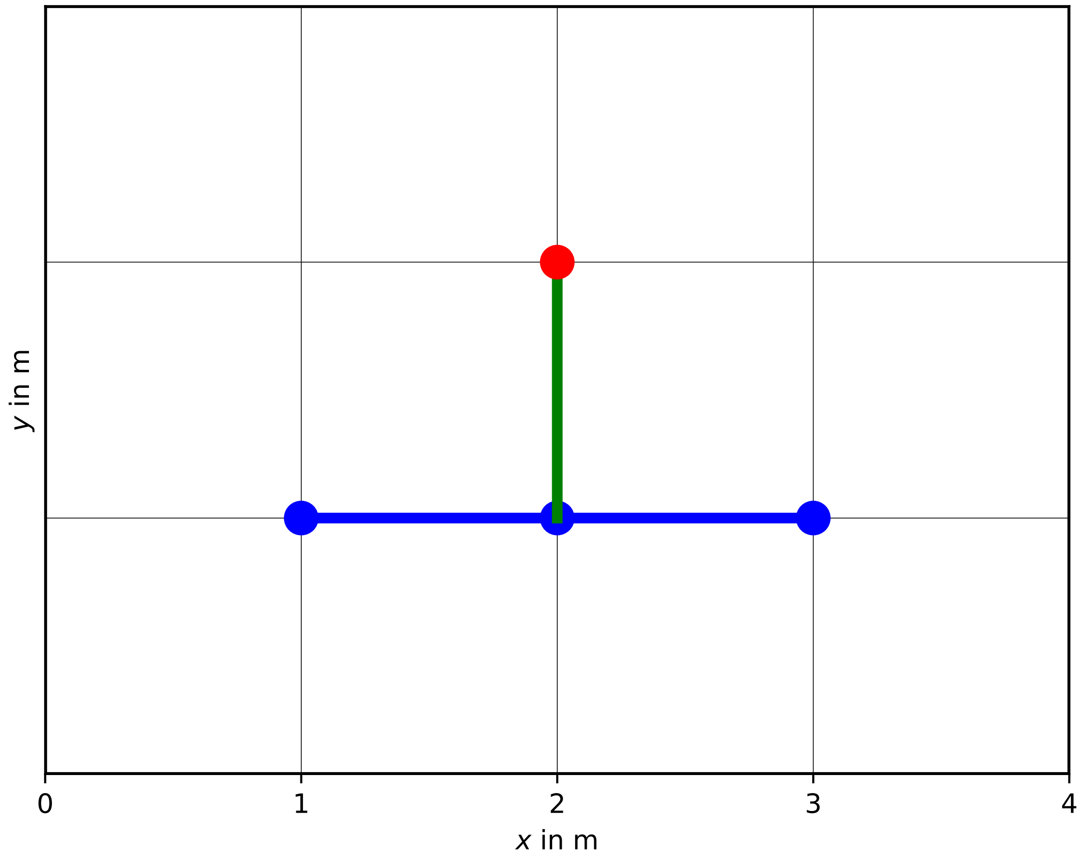
Care for upper boundary!



Forward stencil

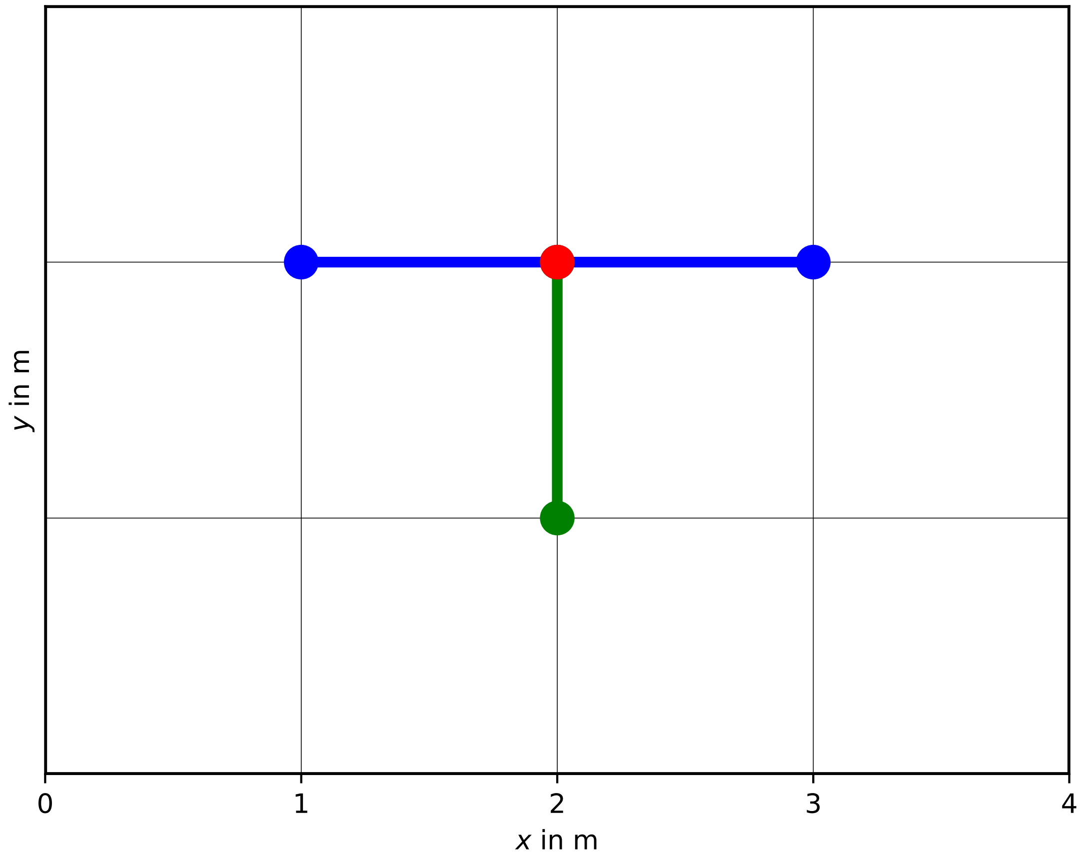
Implicit methods

$$\frac{\partial T}{\partial t}^{n+1} \approx \frac{T^{n+1} - T^n}{\Delta t} = a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{1}{\Delta t} T^{n+1} - a \frac{\partial^2 T}{\partial z^2}^{n+1} = \frac{1}{\Delta t} T^n$$

$$(\mathbf{M} - \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$$

$\mathbf{M}$ - mass matrix



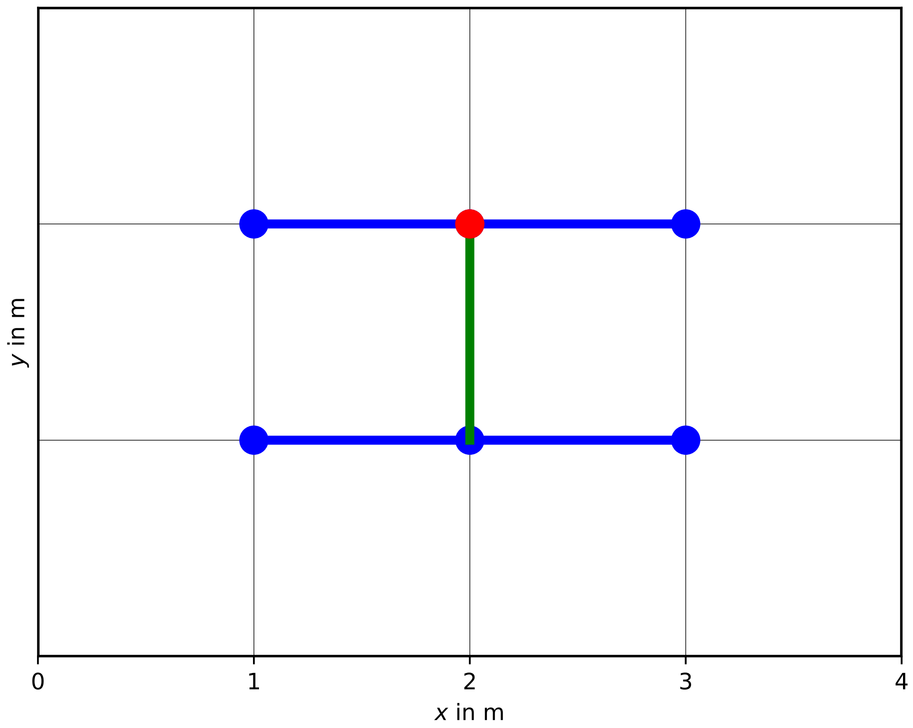
Backward stencil

Mixed - Crank-Nicholson method

$$\frac{\partial T}{\partial t}^{n+1/2} \approx \frac{T^{n+1} - T^{n}}{\Delta t} = \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n} + \frac{1}{2} a \frac{\partial^2 T}{\partial z^2}^{n+1}$$

$$\frac{2}{\Delta t}T^{n+1} - a\frac{\partial^2 T^{n+1}}{\partial z^2} = \frac{2}{\Delta t}T^{n} + a\frac{\partial^2 T^{n}}{\partial z^2}$$

$$(\mathbf{M} - \mathbf{A}/2)\mathbf{u}^{n+1} = (\mathbf{M} + \mathbf{A}/2)\mathbf{u}^{n}$$



Mixed forward/backward stencil

Time-stepping in FE

Variational formulation of Diffusion equation

$$\frac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Finite Difference in Time (NOT in space)

$$\frac{u^{n+1} - u^n}{\Delta t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Variational formulation

$$\frac{u^{n+1} - u^n}{\Delta t} - \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u = f$$

Multiplication with test function w and integration $\Rightarrow$ weak form

$$1/\Delta t\left(\int_\Omega w u^{n+1} \mathrm{d}\Omega - \int_\Omega w u^n \mathrm{d}\Omega\right) - \int_\Omega w \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

$$1/\Delta t\left(\int_\Omega w u^{n+1} \mathrm{d}\Omega - \int_\Omega w u^n \mathrm{d}\Omega\right) - \int_\Omega a \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

Variational formulation of Diffusion equation

u is constructed of shape functions $\mathbf{v}_i$ that are identical to w

The integral over the Poisson term $\int_\Omega a\boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} u \mathrm{d}\Omega$ is represented using the stiffness matrix $\mathbf{A}\mathbf{v}$

$$\mathbf{A}_{i,j} = \int_\Omega \sigma \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega$$

Variational formulation of Poisson equation

Weighted integrals over both u are represented by the mass matrix $\mathbf{Mv}$

$$\mathbf{M}_{i,j} = \int_\Omega v_i \cdot v_j \mathrm{d}\Omega$$

explicit method: $\mathbf{M}\mathbf{u}^{n+1} = (\mathbf{M} - \mathbf{A})\mathbf{u}^n$

implicit method: $(\mathbf{M} + \mathbf{A})\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n$

mixed method: $(\mathbf{M} + \mathbf{A}/2)\mathbf{u}^{n+1} = (\mathbf{M} - \mathbf{A}/2)\mathbf{u}^n$

same as in FD but with FE mass matrix