# Numerical Simulation Methods in Geophysics, Part 8: 2D Electromagnetics

1. MGPY+MGIN

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# Recap & Motivation

- use *pyGIMLi* for generating meshes, matrices & applying BC
- spatial discretization determines accuracy of solution
- source fields ($1/r$ etc.) require fine mesh or higher order elements
- boundaries for decaying fields need to be far away
- secondary field approach addresses both problems

# Remaining lectures & topics

8. 11.12. 2D electromagnetics
9. 18.12. 2D/3D EM and other element types
10. 08.01. 3D vectorial electromagnetics (dies academicus on 15.1.)
11. 22.01.
12. 29.01. Advection problems and Finite Volumes
13. 05.02. (online) special simulation methods
14. 12.02. (not available) shift to later

- 7 excercises

# Electromagnetics in 2D

- move from Poisson to Helmhotlz type of PDE
- real-valued to complex-valued problem
- move from 1D to 2D (and eventually 3D)
- scalar to vectorial solution
- secondary field approach

# Maxwells equations

- Faraday's law: currents & varying electric fields $\Rightarrow$ magnetic field

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$$

- Ampere's law: time-varying magnetic fields induce electric field

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- $\boldsymbol{\nabla} \cdot \mathbf{D} = \varrho$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\mathbf{D} = \epsilon \mathbf{E}$ & $\mathbf{B} = \mu \mathbf{H}$

# Maxwell in frequency domain

Assumption: AC fields with angular frequency $\omega$ (decomposition)

$$\mathbf{E} = \mathbf{E}_0 e^{\imath\omega t} \quad \text{or} \quad \mathbf{H} = \mathbf{H}_0 e^{\imath\omega t}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \imath\omega\epsilon\mathbf{E} + \sigma\mathbf{E} = (\sigma + \imath\omega\epsilon)\mathbf{E}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\imath\omega\mu\mathbf{H}$$

take curl of one of the equations and insert in the other

# Helmholtz equation

see also Theory EM

take curl of one of the equations and insert in the other

$$\boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} + \imath\omega\sigma\mathbf{E} - \omega^2 \epsilon \mathbf{E} = \boldsymbol{\nabla} \times \mathbf{j}_s$$

$$\boldsymbol{\nabla} \times \sigma^{-1} \boldsymbol{\nabla} \times \mathbf{H} + \imath\omega\mu\mathbf{H} - \omega^2 \mu\epsilon \mathbf{H} = 0$$

- Helmholtz type of PDE, vectorial

# Quasi-static approximation

Assume: $\omega^2\mu\epsilon < \omega\mu\sigma$, no sources ($\boldsymbol{\nabla}\cdot\mathbf{j}_s = 0$), + vector identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{F} - \boldsymbol{\nabla}^2\mathbf{F}$$

leads with $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 = \boldsymbol{\nabla}\cdot\mathbf{B}$ to the vector Helmholtz PDE

$$-\nabla^2\mathbf{E} + \imath\omega\mu\sigma\mathbf{E} = 0$$

$$-\boldsymbol{\nabla}\cdot\sigma^{-1}\nabla\mathbf{H} + \imath\omega\mu\mathbf{H} = 0$$

Helmholtz equation $-\nabla^2\mathbf{H} - k^2\mathbf{H} = 0$ with $k^2 = -\imath\omega\mu\sigma$

# TM polarization

**Transverse magnetic (TM) mode**

Assume the source field is oscillating perpendicular to the modelling plane, i.e.

$\mathbf{H} = [H_x, 0, 0]^T e^{\imath\omega t}$.

Then the PDE holds for the scalar $H_x$ (now only $H$)

$$-\boldsymbol{\nabla} \cdot \sigma^{-1} \boldsymbol{\nabla} H_x(y, z) + \imath\omega\mu H_x(y, z) = 0$$

# Halfspace solution

$$\frac{\partial^2 H}{\partial z^2} + k^2 H = 0 \quad \text{with} \quad k^2 = -\imath\omega\mu\sigma$$

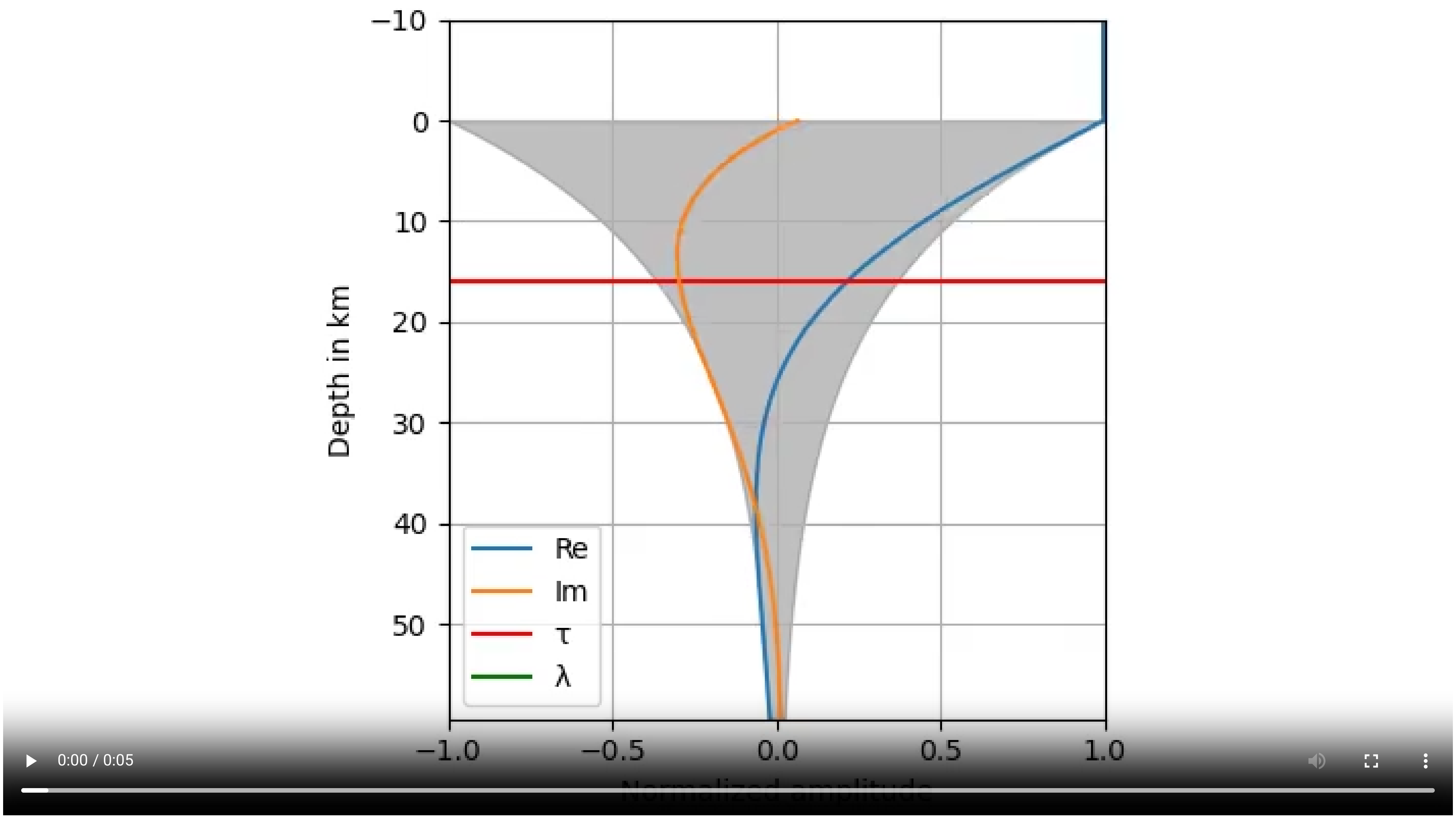
In the Earth, the solution is $H(z) = H_0 e^{-\imath k z}$ $\quad \partial_{zz} H = -k^2 H$

$$k = \sqrt{-\imath\omega\mu\sigma} = \sqrt{\omega\mu\sigma/2}(1-\imath) = 1/d(1-\imath)$$

with skin depth (1/e decay) $d = \sqrt{2/\omega\mu\sigma} = \sqrt{T/\pi\mu\sigma}$ (~500$\sqrt{\rho/f}$)

$$-\imath k = -(1+i)/d \Rightarrow H = H_0 e^{-z/d} e^{-\imath z/d} \qquad \equiv \cos(\omega t - z/d)$$

# Fields in homogeneous half-space



−10
0
10
20
30
40
50
Depth in km
Re
Im
τ
λ
−1.0
−0.5
0.0
0.5
1.0
Normalized amplitude
0:00 / 0:05

# Electromagnetic fields in the Earth

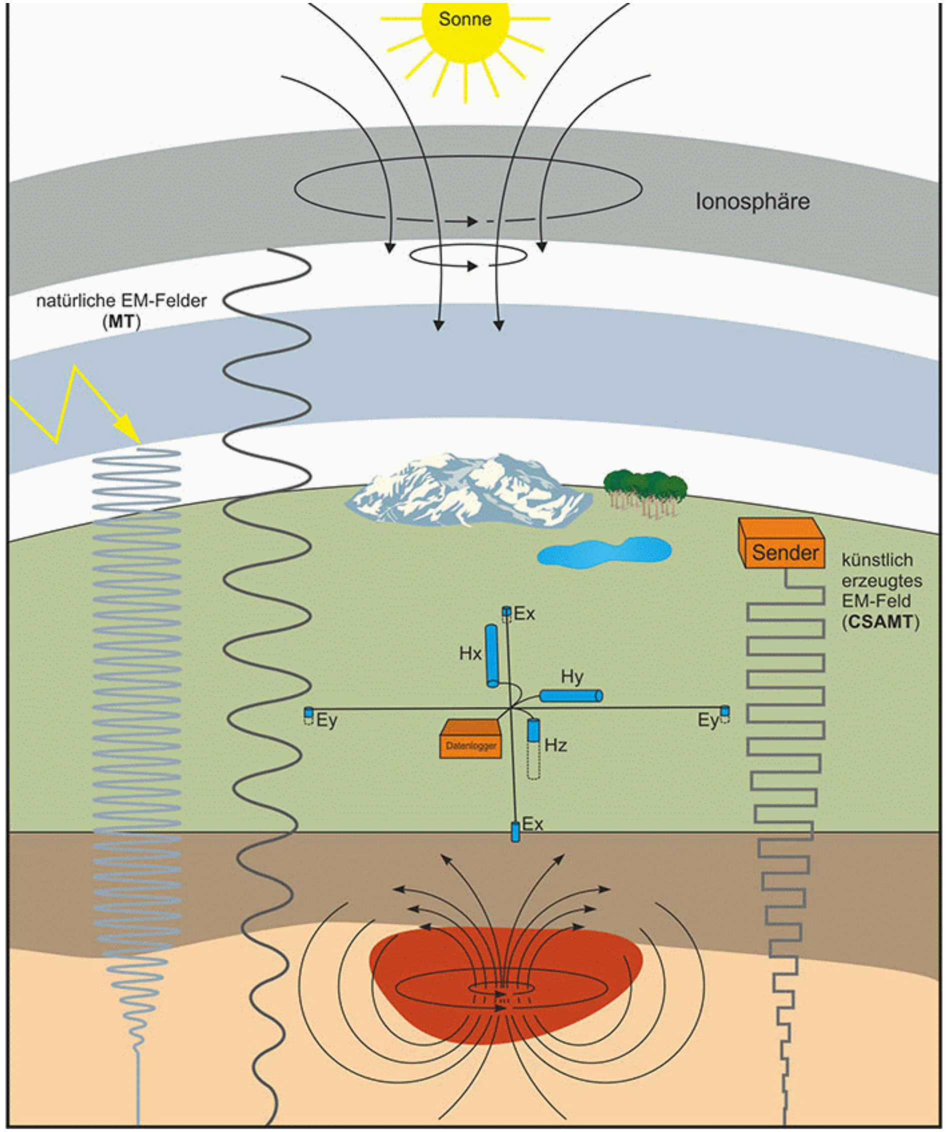
Assume E field in x direction

$$E_x = -\frac{1}{\mu_0 \sigma}\frac{\partial B_y}{\partial z} = \frac{B_0}{\mu_0 \sigma d} e^{-z/d}\sqrt{2}\cos(\omega t - z/d + \pi/4)$$

$\Rightarrow$ phase shift of 45° $(\pi/4)$ between $E_x$ und $B_y$

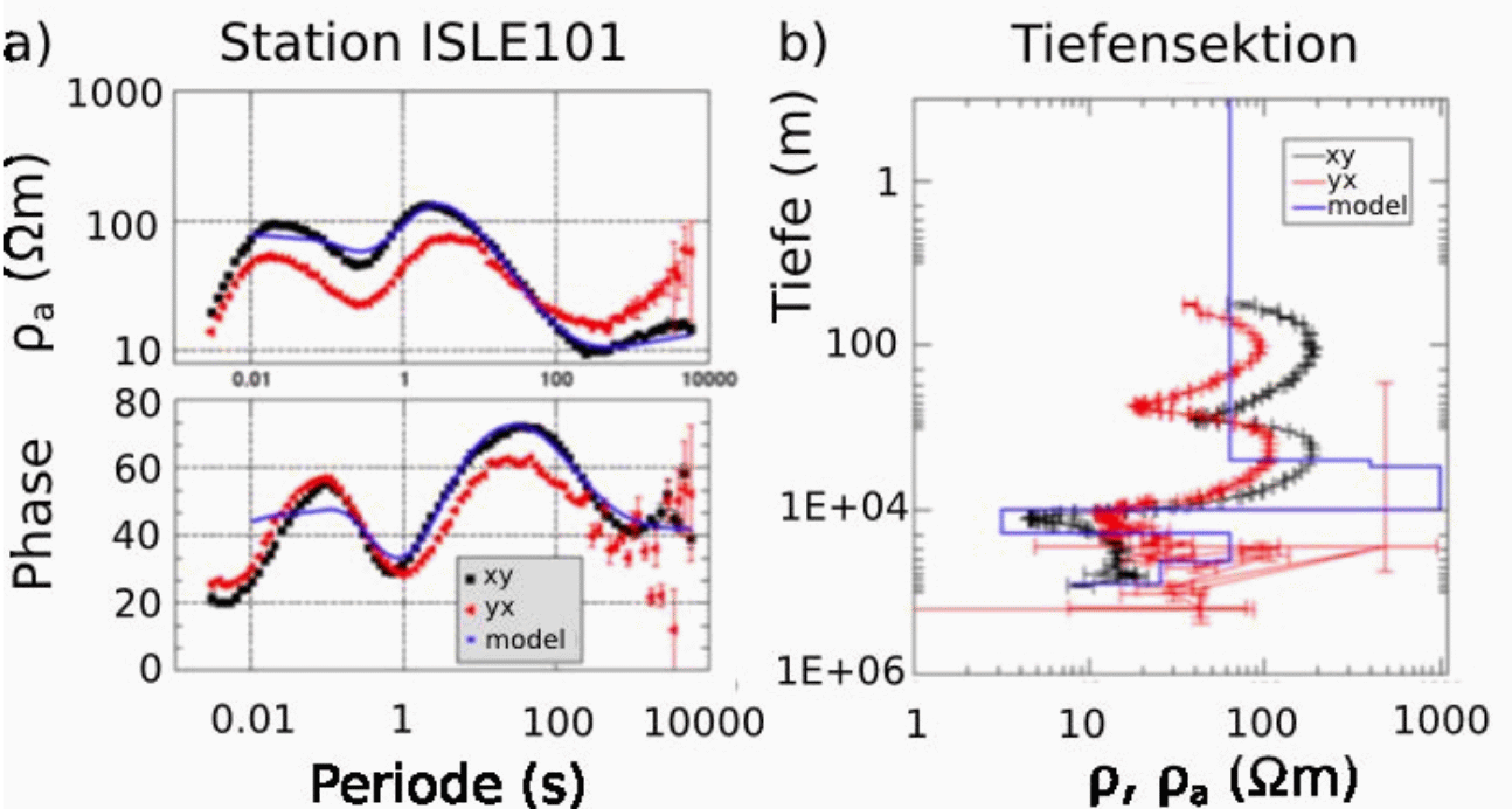
$$\frac{|E_x|}{|B_y|} = \frac{\sqrt{2}}{\mu_0 \sigma d} \Rightarrow \rho = \frac{\mu_0}{\omega}\left|\frac{E_x}{B_y}\right|^2$$

# Magnetotellurics (MT)

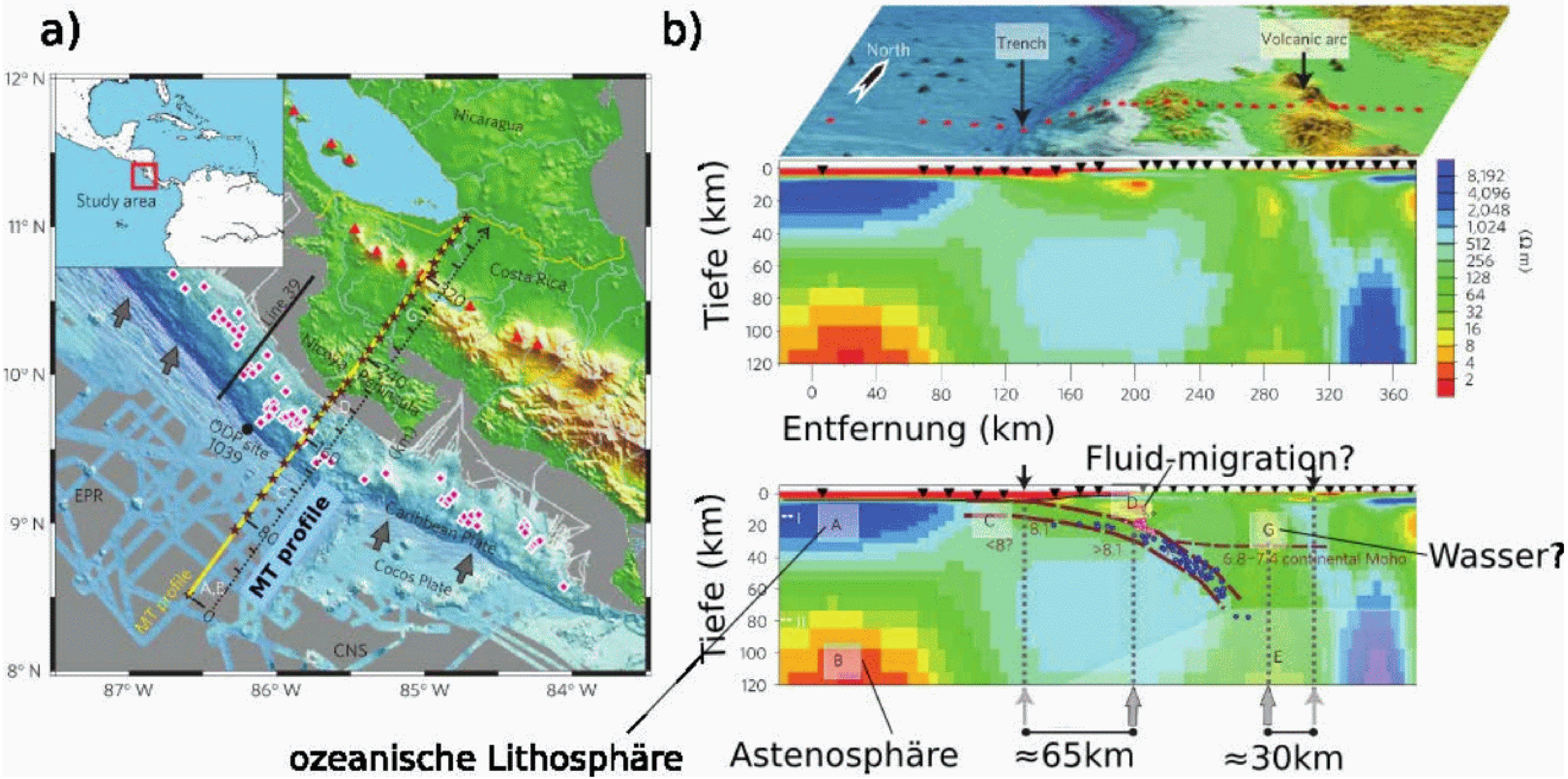



- depth sounding with $T$=0.01-1000s ($f$ =0.001-100,Hz)
- source in ionosphere (natural source) or on ground (controlled source)
- measure magnetic and electric fields
- analyse (complex) ratio in frequ. domain
- depth sounding ($\rho_a$ & $\phi$) with $T$
- MT course of Anna Marti (U Barcelona)

# Magnetotelluric depth sounding



# 2D/3D Magnetotellurics




Imaging of a subduction zone (Worszewski et al., 2011)

# EM with finite elements

$$-\nabla^2 u + \imath\omega\mu\sigma u = f$$

$$-\int_\Omega w\nabla^2 u \mathrm{d}\Omega + \int_\Omega w\imath\omega\mu\sigma u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

Gauss's integral law

$$\int_\Omega \nabla w \cdot \nabla u \mathrm{d}\Omega + \imath\omega \int_\Omega \mu\sigma w u \mathrm{d}\Omega = \int_\Omega w f \mathrm{d}\Omega$$

# Weak formulation

$u = \sum_i u_i \mathbf{v}_i$ and $w_i \in \{v_i\}$ leads to

$$\int_\Omega \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_j \mathrm{d}\Omega + \imath\omega\mu \int_\Omega \sigma v_i v_j \mathrm{d}\Omega = \int_\Omega v_i f \mathrm{d}\Omega$$

$$\langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle + \imath\omega\mu \langle v_i | \sigma v_j \rangle = \langle v_i | f \rangle \quad \text{inner products}$$

representation by matrix-vector product $(\mathbf{A} + \imath\omega\mathbf{M}^\sigma)\mathbf{u} = \mathbf{b}$

with $A_{ij} = \langle \boldsymbol{\nabla} v_i | \boldsymbol{\nabla} v_j \rangle$, $M_{ij}^\sigma = \langle v_i | v_j \rangle$ and $b_i = \langle v_i | f \rangle$

# The finite element mass matrix

The mass matrix

$$M_{i,j} = \int_{\Omega} \mu \sigma v_i v_j \mathrm{d}\Omega = \sum_c \int_{\Omega_c} \mu_c \sigma_c v_i v_j \mathrm{d}\Omega$$

can be written for element-wise conductivity and permittivity

$$M_{i,j} = \sum_c \mu_c \sigma_c \int_{\Omega_c} v_i v_j$$

# Complex or real-valued?

The complex-valued system

$$(\mathbf{A} + \imath\omega\mathbf{M})\mathbf{u} = (\mathbf{A} + \imath\omega\mathbf{M})(\mathbf{u}_r + \imath\mathbf{u}_i) = \mathbf{b}_r + \imath\mathbf{b}_i$$

can be transferred into a doubled real-valued system

$$\mathbf{A}\mathbf{u}_r + \imath\mathbf{A}u_i + \imath\omega\mathbf{M}\mathbf{u}_r - \omega\mathbf{M}\mathbf{u}_i = \mathbf{b}_r + \imath\mathbf{b}_i$$

$$\begin{pmatrix} A & -\omega M \\ \omega M & A \end{pmatrix} \begin{pmatrix} u_r \\ u_i \end{pmatrix} = \begin{pmatrix} b_r \\ b_i \end{pmatrix}$$

# Secondary field approach

Consider the field to consist of a primary (background) and an secondary (anomalous) field $F = F_0 + F_a$

solution for $F_0$ known, e.g. analytically or 1D (semi-analytically)

$\Rightarrow$ form equations for $F_a$, because

- $F_a$ is weaker or smoother (e.g. $F_0 \propto 1/$ at sources)
- boundary conditions easier to set (e.g. homogeneous Dirichlet)

# Secondary field Helmholtz equation

The equation $-\nabla^2 F - k^2 F = 0$ is solved by the primary field for $k_0$:

$-\nabla^2 F_0 - k_0^2 F_0 = 0$ and the total field for $k_0 + \delta k$:

$$-\nabla^2 (F_0 + F_a) - (k_0^2 + \delta k^2)(F_0 + F_a) = 0$$

$$-\nabla^2 F_a - k^2 F_a = \delta k^2 F_0$$

**Note**

Source terms only arise at anomalous terms, weighted by the primary field.

# Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

**Note**

Source terms only arise at anomalous conductivities and increase with primary field

# Secondary field for EM

Maxwells equations $k^2 = -\imath\omega\mu\sigma$

$$-\boldsymbol{\nabla}^2\mathbf{E}_0 + \imath\omega\mu\sigma\mathbf{E}_0 = 0$$

leads to

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

**Note**

Source terms only arise at anomalous conductivities and increase with primary field

# Secondary field for EM

$$-\boldsymbol{\nabla}^2\mathbf{E}_a + \imath\omega\mu\sigma\mathbf{E}_a = -\imath\omega\mu\delta\sigma\mathbf{E}_0$$

leads to the discretized form ($\mathbf{A}$-stiffness, $\mathbf{M}$-mass)

$$\mathbf{A}\mathbf{E}_a + \imath\omega\mathbf{M}_\sigma E_a = -\imath\omega\mathbf{M}_{\delta\sigma}\mathbf{E}_0$$

```
A = stiffnessMatrix1DFE(x=z)
M = massMatrix1DFE(x=z, a=w*mu*sigma)
dM = massMatrix1DFE(x=z, a=w*mu*(sigma-sigma0))
u = uAna + solve(A+M*w*1j, dM@uAna * w*1j)
```

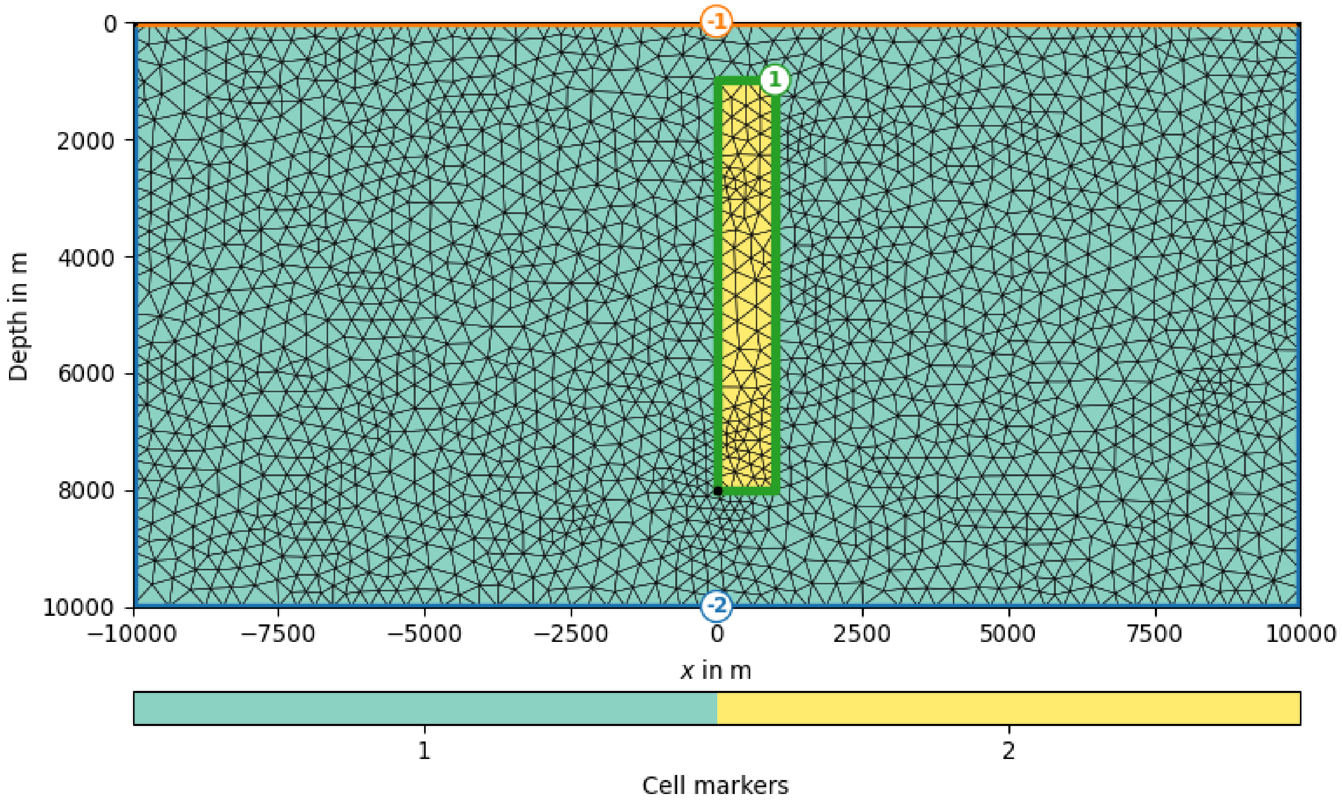
# 2D/3D problems

Make use of pyGIMLi

See documentation on pyGIMLi.org

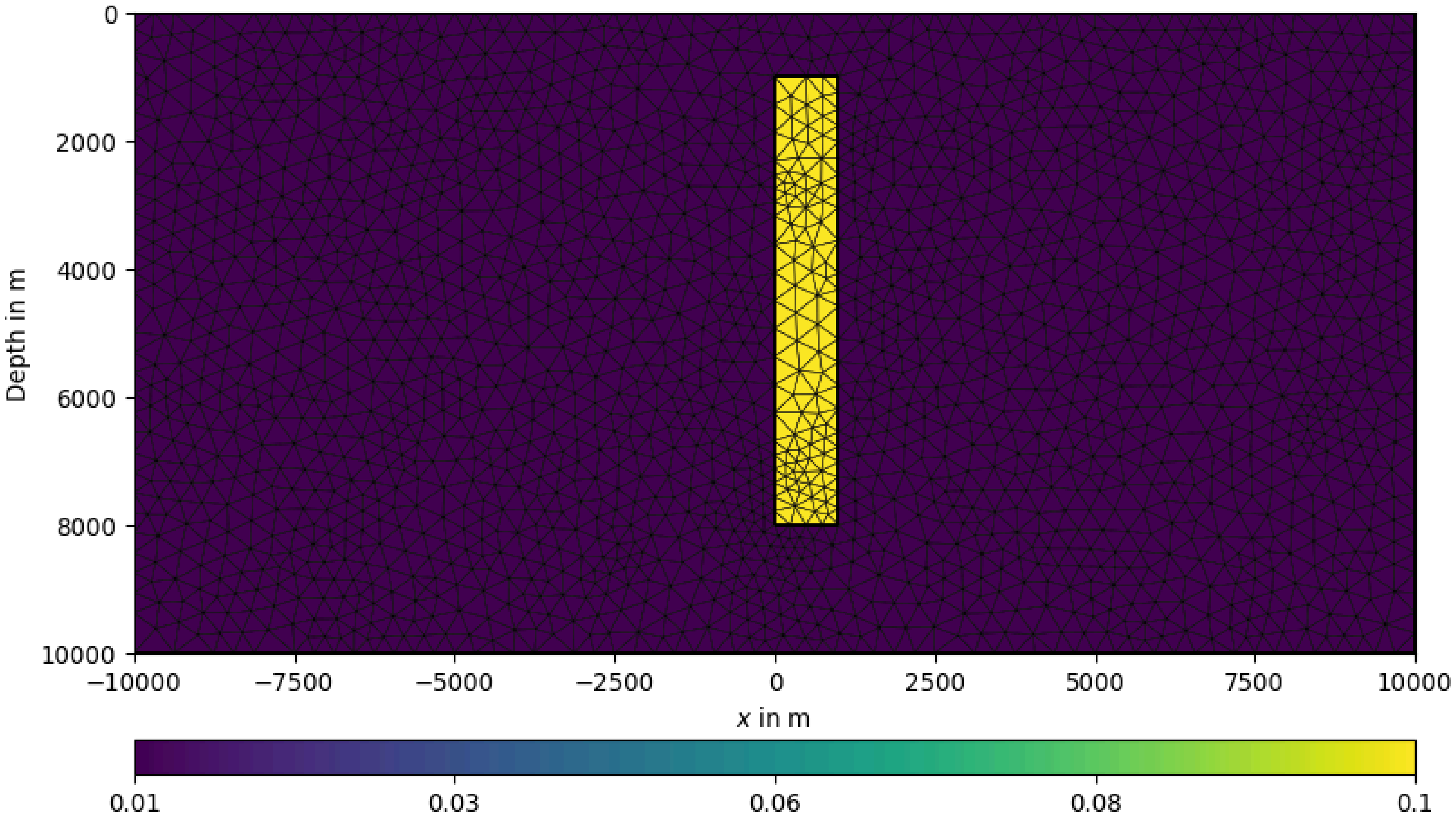
# Generating a 2D model

```python
import pygimli as pg
import pygimli.meshtools as mt
world = mt.createWorld(start=[-10000, -10000], end=[10000, 0])
anomaly = mt.createRectangle(start=[0, -8000], end=[1000, -1000], marker=2)
mesh = mt.createMesh(world+anomaly, quality=34, smooth=True, area=1e5)
pg.show(mesh, markers=True, showMesh=True);
```



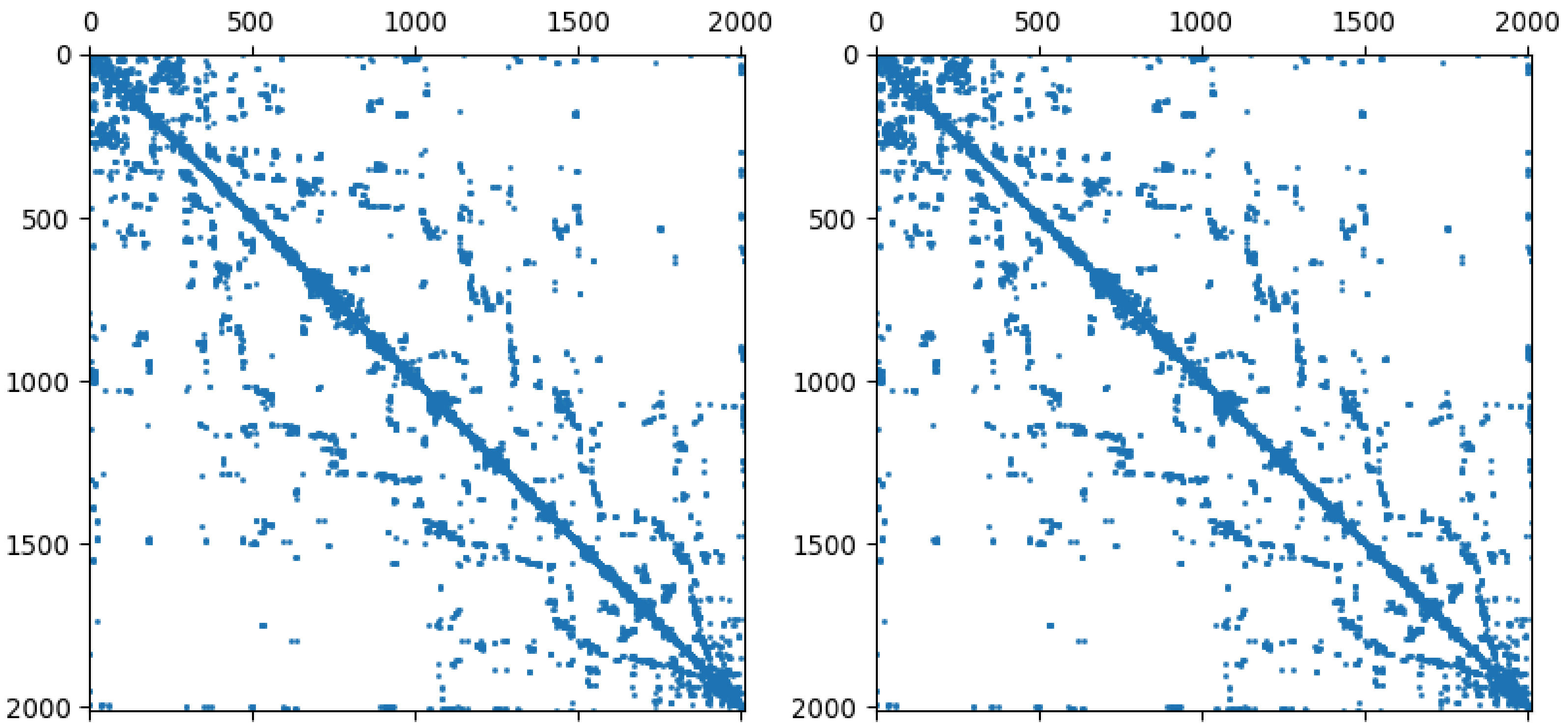
# Creating a 2D conductivity model

```python
sigma0 = 1 / 100  # 100 Ohmm
sigma = mesh.populate("sigma", {1: sigma0, 2: sigma0*10})
pg.show(mesh, "sigma", showMesh=True);
```



# The solver module

```
import pygimli.solver as ps
mesh["my"] = 4 * np.pi * 1e-7
A = ps.createStiffnessMatrix(mesh, a=1/mesh["my"])
M = ps.createMassMatrix(mesh, mesh["sigma"])
fig, ax = plt.subplots(ncols=2)
pg.show(A, ax=ax[0], markersize=1)
pg.show(M, ax=ax[1], markersize=1)
```



# The complex problem matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & -\omega\mathbf{M} \\ \omega\mathbf{M} & \mathbf{A} \end{pmatrix}$$

```python
w = 0.1
nd = mesh.nodeCount()
B = pg.BlockMatrix()
B.Aid = B.addMatrix(A)
B.Mid = B.addMatrix(M)
B.addMatrixEntry(B.Aid, 0, 0)
B.addMatrixEntry(B.Aid, nd, nd)
B.addMatrixEntry(B.Mid, 0, nd, scale=-w)
B.addMatrixEntry(B.Mid, nd, 0, scale=w)
pg.show(B)
```