

# Compositional Reasoning for Generator-Based Shrinking (Extended Version)

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Property-based testing (PBT) is a powerful tool for identifying software bugs. Users provide *generators*, which compute random test cases, and *shrinkers*, which compute a set of “smaller” test cases from an existing test case. Under *generator-based shrinking*, shrinkers are not specified separately, but derived from annotated generator specifications. Generator behaviour is not compositional in many frameworks that support generator-based shrinking (such as Hedgehog [21], falsify [7], RapidCheck [8], and jqwik [13]). In this paper, we address the problem of designing a language with generator-based shrinking that supports compositional reasoning. We propose an eDSL, halcheck, where users must manually annotate random choices with unique labels. In return, halcheck generators can be reasoned about compositionally. We justify our design by (1) showing that existing language designs do not — and cannot — support compositional reasoning and (2) the labelling requirement has negligible overhead in practice.

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## 1 Introduction

*Property-based testing* (PBT) [3] is a powerful tool for identifying software bugs. Users provide an executable specification — or *property* — to a PBT framework, which automatically checks the specification against a system under test (SUT) over a wide range of randomly generated inputs (or *test cases*). If the framework finds a *counterexample* — a test case that causes the SUT to violate the specification — then the framework computes and reports a minimal counterexample. A complete overview is given in Figure 1.

A property consists of three components: (1) *generators*, which compute random test cases, (2) an *oracle*, which decides if a test case is a counterexample, and (3) *shrinkers*, which compute a set of “smaller” test cases from an existing test case. These components are integral to the operation of a PBT framework, which consists of two stages: *testing* and *shrinking*. The testing stage is responsible for identifying a counterexample, and operates using the generators and oracle of the property. The shrinking stage performs counterexample minimization, and operates using the shrinkers and oracle.

The testing stage proceeds by constructing a random test case using the generators, then checking if the test case is a counterexample using the oracle. If the test case is not a counterexample, the testing process restarts with a different test case. Otherwise, the counterexample is passed along to the shrinking stage.

The shrinking stage is essential for the usability of property-based testing. Without it, counterexamples can be arbitrarily large, which complicates debugging. Shrinking operates as follows: (1) starting from the previously discovered counterexample, the shrinkers compute a set of “smaller”

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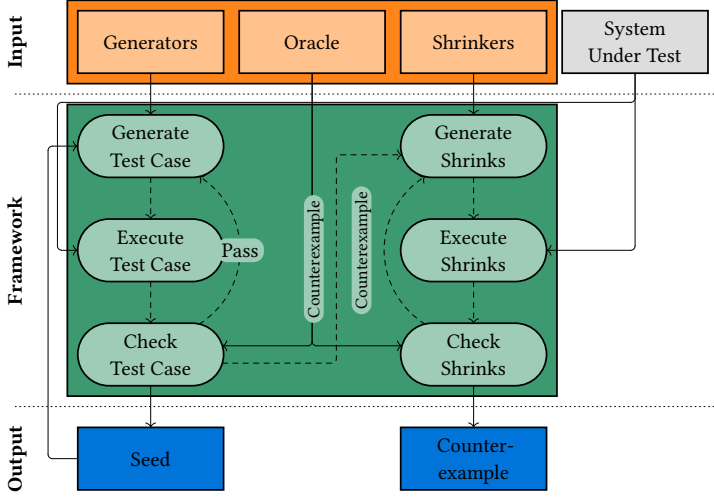


Fig. 1. Property-Based Testing Overview.

```

1 coin :: Gen Bool
2 coin = do x ← prune bool
3         y ← prune bool
4         return (x == y)

```

Fig. 2. Generator Example.

test cases, called *shrinks*, then (2) the oracle determines whether any of the shrinks are counterexamples. If so, the shrinking process restarts using the smaller counterexample. Otherwise, the current counterexample is deemed minimal and reported to the user.

Generators are specified using an embedded domain-specific language (eDSL) provided by the PBT framework. These languages are essentially probabilistic programming languages and hence users view generators as probability distributions. For example, consider the program given in Figure 2, written using the Hedgehog framework [21] for Haskell. Concretely, the `coin` generator computes two random booleans, `x` and `y` (lines 3 and 4), by invoking a subgenerator `prune bool` which performs a fair coin flip. The `coin` generator then computes whether `x` and `y` are equal (line 5). Abstractly, `coin` represents a probability distribution that assigns 0.5 to both `True` and `False`.

Shrinkers are also specified using an eDSL provided by the PBT framework. The QuickCheck [3] framework which popularized PBT employs *manual shrinking*, where shrinkers and generators are specified separately. Shrinkers are surprisingly difficult to write manually, thus subsequent work such as Hedgehog [21] employs *generator-based shrinking*, where shrinkers are derived from annotated generator specifications. Under generator-based shrinking, generators represent distributions of labelled trees, where labels lower in the tree are considered “smaller”. While most languages with generator-based shrinking allow users to influence shrinker behaviour [7, 21], others — such as hypothesis [14] — rely solely upon built-in heuristics. This lack of *user-controlled shrinking* saves users from having to think about how to shrink test cases, but inhibits correct and efficient shrinking since the correct notion of “smaller” test case is application-dependent. Furthermore, the heuristics used by these frameworks are unpredictable and can change when the framework is updated [7]. Thus this work focuses on user-controlled generator-based shrinking.

Frameworks	QuickCheck [3]	falsify [7], Hedgehog [21], RapidCheck [8], jqwik [13]	hypothesis [14]
Generator-Based Shrinking	✗	✓	✓
User-Controlled Shrinking	✓	✓	✗
Compositional Reasoning	✓	✗	✓

Table 1. Existing Frameworks and Supported Features.

```

1 context :: Gen Bool → Gen Bool
2 context m = do
3   x ← shrink (\b → if b then [False] else []) (return True)
4   if x then coin else m

```

Fig. 3. Non-Compositionality Example.

Since generators can be arbitrarily complex [11, 17], it is desirable to construct and reason about them in a *compositional* manner. In particular, a generator’s distribution should be a pure function of its subgenerator’s distributions. For example, we can determine `coin`’s distribution simply by knowing `prune bool`’s distribution. Compositional reasoning holds in frameworks without generator-based shrinking (such as QuickCheck [3]) and frameworks without user-controlled shrinking (such as hypothesis<sup>1</sup> [14]). However, compositional reasoning does not hold in any framework that supports both, such as Hedgehog [21], falsify [7], RapidCheck [8], or jqwik [13]. We summarize existing frameworks and their support for generator-based shrinking, user-controlled shrinking, and compositional reasoning in Table 1.

In this paper, we address the problem of designing a language featuring user-controlled generator-based shrinking and compositional reasoning. We propose a new eDSL, `halcheck`, where users must manually annotate random choices with unique labels. In exchange for this modest overhead, `halcheck` generators can be constructed and reasoned about compositionally.

*Motivating Example.* To illustrate how compositional reasoning fails in a language with user-controlled generator-based shrinking, consider the function `context` given in Figure 3. The `context` function takes in a generator (`m`) and returns another generator. The returned generator first generates a boolean `x` (lines 3 to 5). This boolean is always `True` during the testing stage, but can change to `False` during the shrinking stage. If `x` is `True`, then `coin` is invoked, and otherwise `m` is invoked.

Compositionality implies that `context coin` and `context (prune bool)` should represent the same distribution (since `coin` and `prune bool` represent the same distribution). This is not the case in practice: `context coin` produces the trees given in Figure 4a, while `context (prune bool)` produces the trees given in both Figures 4a and 4b. This difference indicates that `context` can somehow differentiate between `context coin` and `prune bool`, despite them representing identical distributions.

*Contributions.* The contributions of this work are as follows:

<sup>1</sup>While hypothesis’ heuristics make it impossible to predict or control the distribution of generators, hypothesis generators can be reasoned about compositionally if viewed as nondeterministic programs instead of probabilistic ones.



Fig. 4. Trees produced by context (see example in Figure 3).

- We define a new language, *halcheck*, which features an additional *labelling* operator. We show that *halcheck* supports compositional reasoning modulo some minor constraints on the surrounding context. We also provide a complete set of rules for determining whether two *halcheck* generators have identical distributions.
- To justify our additional constraints, we show that unconstrained compositional reasoning is not possible in a language with user-controlled generator-based shrinking.
- To justify the labelling requirement, we provide a translation from generators defined in an existing language to *halcheck* generators. We prove that the translation is sound and results in only a linear increase in program size.
- We explore how incorrect labelling interacts with other types of errors, and describe how to detect, locate, and fix labelling errors.
- We implement *halcheck* as a simple modification of the Hedgehog library for Haskell. We evaluate the effects of *halcheck*'s design on generator construction and performance using a data set drawn from the literature and industry.

*Organization.* The structure of this paper is as follows. In Section 2, we introduce background material, which includes fixing basic notation and reviewing the QuickCheck and Hedgehog languages and their (non-)properties. In Section 3, we present our proof that unconstrained compositional reasoning is not possible in a language with user-controlled generator-based shrinking. In Section 4, we present the language *halcheck*. In Section 5, we discuss labelling errors. In Section 6, we describe our implementation and evaluation of *halcheck*. In Section 7, we describe and compare related work. In Section 8, we conclude and describe future work.

## 2 Background

### 2.1 Notation

*Sets.* The class of all sets is denoted by *Set*. The set of natural numbers is denoted by  $\mathbb{N}$ , the set of rational numbers is denoted by  $\mathbb{Q}$ , and the set of real numbers is denoted by  $\mathbb{R}$ . The set of booleans is defined as  $\mathbb{B} = \{0, 1\}$ . The set of rational weights is defined as  $\mathbb{W} = \mathbb{Q} \cap \mathbb{W}$ . The cartesian product of sets  $A$  and  $B$  is written as  $A \times B$ . For any pair  $(a, b)$  we define  $\text{fst}(a, b) = a$  and  $\text{snd}(a, b) = b$ . The set of all subsets of  $A$  is denoted by  $2^A$ .

*Functions.* We use juxtaposition (e.g.,  $f a$ ) to denote function application. The type of dependent functions from  $A$  to  $B$  (where a variable  $a \in A$  may appear in  $B$ ) is denoted by  $\prod_{a \in A} B$ . The composition of two functions  $f : B \rightarrow C$  and  $g : A \rightarrow B$  is denoted by  $f \circ g$ . The notation  $a \mapsto \dots$  denotes an unnamed function with parameter  $a$ . We also write functions using blanks for arguments (e.g.,  $(a + -) = (b \mapsto a + b)$ ). The pre-image of a function  $f : A \rightarrow B$  is given by  $f^{-1} : 2^B \rightarrow 2^A$ .

Given  $f : A \rightarrow B$ ,  $a \in A$ , and  $b \in B$ , we define  $f[a \mapsto b]$  as

$$f[a \mapsto b]c = \begin{cases} b & \text{if } a = c \\ f c & \text{otherwise.} \end{cases}$$

*Words.* The set of finite words over a set  $A$  is written as  $A^*$ . A list with elements  $a_0, \dots, a_{n-1}$  is written as  $[a_0, \dots, a_{n-1}]$ . The concatenation of two words  $as$  and  $bs$  is denoted by  $as \mathbin{++} bs$ . The length of a word  $as$  is denoted by  $|as|$ . The  $i$ th element of  $as$  is written as  $as_i$ , and  $as_A$  denotes the substring of  $as$  that only contains the elements indexed by elements of  $A \subseteq \mathbb{N}$  (e.g.,  $as_{[0..5]}$  denotes the prefix containing the first five elements of  $as$ .) The expression  $a :: as$  denotes the word  $[a, as_0, \dots, as_{|as|-1}]$ . We define  $\text{map} : (A \rightarrow B) \rightarrow A^* \rightarrow B^*$  as  $\text{map } f [a_0, \dots, a_n] = [f a_0, \dots, f a_n]$ .

## 2.2 Probability

In this section, we review the basics of probability and measure theory.

*Measurable Sets and Functions.* A  $\sigma$ -algebra on a set  $A$  is a set of sets  $\Sigma \subseteq 2^A$  that includes  $A$  and is closed under complementation and countable union. Note that  $2^A$  is a  $\sigma$ -algebra, known as the discrete  $\sigma$ -algebra. We assign each set  $A$  a canonical  $\sigma$ -algebra  $\Sigma_A$  which, unless otherwise specified, is the discrete  $\sigma$ -algebra. A set  $X \subseteq A$  is measurable iff  $X \in \Sigma_A$ , and a function  $f : A \rightarrow B$  is measurable iff  $f^{-1} X$  is measurable for all for all measurable  $X \subseteq B$ .

*Probability Measures and Random Variables.* A probability measure on a set  $A$  is a function  $\mathbb{P}_A : \Sigma_A \rightarrow \mathbb{W}$  satisfying  $\mathbb{P}_A A = 1$  and  $\mathbb{P}_A (\bigcup \mathbf{A}) = \sum_{X \in \mathbf{A}} \mathbb{P}_A X$  for all countable collections of pairwise disjoint measurable sets  $\mathbf{A} \subseteq \Sigma_A$ . A probability space is any set  $A$  equipped with a probability measure  $\mathbb{P}_A$ . If  $A$  is a probability space, then a measurable function  $f : A \rightarrow B$  is called a random variable. For a random variable  $f : A \rightarrow B$ , we define the following shorthands within the context of a call to  $\mathbb{P}_A$ :

$$\begin{aligned} (f \in X) &= f^{-1} X & \text{where } X \subseteq B \\ (f = a) &= f^{-1} \{a\} & \text{where } a \in B \end{aligned}$$

In general, we lift arbitrary operations to random variables in a pointwise manner within the context of a call to  $\mathbb{P}_A$ , e.g.

$$\mathbb{P}_A (f + g \in X) = \mathbb{P}_A ((a \mapsto f a + g a) \in X) \quad \text{where } f, g : A \rightarrow \mathbb{R}.$$

Two sets  $X, Y \subseteq A$  are independent (written  $X \perp\!\!\!\perp Y$ ) iff  $\mathbb{P}_A (X \cap Y) = \mathbb{P}_A (X) \cdot \mathbb{P}_A (Y)$ , and two random variables  $f, g : A \rightarrow B$  are independent if  $f^{-1} X \perp\!\!\!\perp g^{-1} Y$  for all measurable  $X, Y \subseteq B$ .

*Seeds.* Following [5, 22, 24, 25], we fix a probability space *Seed* (called an *entropy space*) of seeds with a non-discrete canonical  $\sigma$ -algebra. From this point on we are only concerned with the measure  $\mathbb{P}_{\text{Seed}}$  therefore omit the subscript. The set *Seed* comes equipped with a family of measurable sampling functions  $\psi_p : \text{Seed} \rightarrow \mathbb{B}$  for all  $p \in \mathbb{W}$  and two measurable projection functions  $\pi_L, \pi_R : \text{Seed} \rightarrow \text{Seed}$  such that (1) the projection functions are measure-preserving, i.e.  $\mathbb{P}(\pi_L^{-1} A) = \mathbb{P}(\pi_R^{-1} A) = \mathbb{P} A$  for all  $A \in \Sigma_{\text{Seed}}$ , (2) the projection functions are independent, i.e.  $\pi_L \perp\!\!\!\perp \pi_R$ , (3) different samples are independent, i.e.  $\psi_p \perp\!\!\!\perp \psi_q$  for all  $p \neq q$ , and (4)  $\mathbb{P}(\psi_p = 1) = p$  for all  $p \in \mathbb{W}$ . We assume every countable set  $A$  is equipped with a canonical injection  $\#- : A \rightarrow \mathbb{N}$ . Then for every countable  $A$  and  $a \in A$  we define  $\pi_a : \text{Seed} \rightarrow \text{Seed}$  as

$$\pi_a = \pi_L \circ \overbrace{\pi_R \circ \dots \circ \pi_R}^{\#a \text{ times}}.$$

Note that any pair of projection functions on different values are independent, i.e.

$$\pi_a \perp\!\!\!\perp \pi_b \iff a \neq b.$$

### 2.3 Metalinguage

In this section we define a simple metalinguage which serves as a general framework for studying languages with generic effects [18–20] such as probabilistic choice and shrinking. Instances of the language are constructed by providing a *signature* (which specifies a set of language-specific operations, see Definition 2.2) and an *interpretation* (which specifies the meaning of language-specific operations, see Definition 2.5.)

*Definition 2.1 (Types).* The sets of *value types*  $\text{Type}_v$  and *computation types*  $\text{Type}_c$  are generated by

$$\tau_v ::= \text{Unit} \mid \text{Bool} \mid \text{Nat} \qquad \tau_c ::= \text{Gen } \tau_v.$$

We define the set of all types as  $\text{Type} = \text{Type}_v \cup \text{Type}_c$ .

The set of value types consists of the singleton type (Unit), the type of booleans (Bool) and the type of natural numbers (Nat). The computation type  $\text{Gen } \tau$  represents generators which produce elements of the value type  $\tau$ . Value and computation types are stratified so that computation types may not be nested.

*Definition 2.2 (Signature).* A signature is a set of function symbols of the form  $f : \tau_1 \times \dots \times \tau_n \Rightarrow \tau$ , for  $\tau_1, \dots, \tau_n, \tau \in \text{Type}_v$ .

A signature describes a language-specific set of effectful operations. For example, the signature

$$\Sigma_{\text{St}} = \{\text{get} : \Rightarrow \text{Nat}, \text{put} : \text{Nat} \Rightarrow \text{Unit}\}$$

describes operations on some mutable Nat-valued state, where (1) get denotes an operation which takes no arguments and returns the current value of the state, and (2) put denotes an operation which overwrites the current value of the state with its argument. A function symbol may only include value types, i.e. we exclude higher-order generators.

*Definition 2.3 (Terms).* Given a signature  $\Sigma$ , the set of  $\Sigma$ -terms (denoted by  $\text{Term } \Sigma$ ) is generated by

$$\begin{aligned} t ::= & x \mid \text{return } x \mid \text{let } x \leftarrow t_1 \text{ in } t_2 \mid f \ t_1 \dots t_n \mid () \mid \text{true} \\ & \mid \text{false} \mid \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid n \mid t_1 + t_2 \mid t_1 \cdot t_2 \mid t_1 = t_2 \end{aligned}$$

where variables (e.g.  $x$ ) are sourced from a countably infinite set of names  $\text{Var}$ .

Definition 2.3 defines constructs common to all languages we study in this work. Such constructs include variables ( $x$ ), side-effect-free computations (return), sequential composition (let), the singleton value ( $()$ ), natural numbers ( $n$ ), booleans (true and false), as well as standard operations on these types. For simplicity, we omit general recursion constructs. In addition to the standard constructs, the set of terms also includes function symbol invocations of the form  $f \ t_1 \dots t_n$ . For example, the set of terms generated by  $\Sigma_{\text{St}}$  includes terms such as ‘get’ and ‘put (3 + 2)’. We define shorthands for some common operations:

$$t_1 \wedge t_2 = \text{if } t_1 \text{ then } t_2 \text{ else false} \qquad t_1 \vee t_2 = \text{if } t_1 \text{ then true else } t_2 \qquad \neg t = \text{if } t \text{ then true else false}$$

*Definition 2.4 (Environments and Typing).* The set of environments is defined as the set  $\text{Env} = \text{Var} \rightarrow \text{Type}_v$  of all functions assigning variables to value types. The typing relation

$$- \vdash - : - \subseteq \text{Env} \times \text{Term } \Sigma \times \text{Type}$$

is the smallest relation satisfying the following rules:

$$\begin{array}{c}
\frac{}{\Gamma \vdash x : \Gamma x} \quad \frac{\Gamma \vdash t : \tau}{\Gamma \vdash \text{return } t : \text{Gen } \tau} \quad \frac{\Gamma \vdash t_1 : \text{Gen } \tau_1 \quad \Gamma[x \mapsto \tau_1] \vdash t_2 : \text{Gen } \tau_2}{\Gamma \vdash \text{let } x \leftarrow t_1 \text{ in } t_2 : \text{Gen } \tau_2} \\
\frac{(f : \tau_1 \times \dots \times \tau_n \Rightarrow \tau) \in \Sigma \quad \Gamma \vdash t_1 : \tau_1 \quad \dots \quad \Gamma \vdash t_n : \tau_n}{\Gamma \vdash f t_1 \dots t_n : \text{Gen } \tau} \quad \frac{}{\Gamma \vdash () : \text{Unit}} \\
\frac{}{\Gamma \vdash \text{true} : \text{Bool}} \quad \frac{}{\Gamma \vdash \text{false} : \text{Bool}} \quad \frac{\Gamma \vdash t_1 : \text{Bool} \quad \Gamma \vdash t_2 : \tau \quad \Gamma \vdash t_3 : \tau}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : \tau} \quad \frac{n \in \mathbb{N}}{\Gamma \vdash n : \text{Nat}} \\
\frac{\Gamma \vdash t_1 : \text{Nat} \quad \Gamma \vdash t_2 : \text{Nat}}{\Gamma \vdash t_1 + t_2 : \text{Nat}} \quad \frac{\Gamma \vdash t_1 : \text{Nat} \quad \Gamma \vdash t_2 : \text{Nat}}{\Gamma \vdash t_1 \cdot t_2 : \text{Nat}} \\
\frac{\Gamma \vdash t_1 : \tau \quad \Gamma \vdash t_2 : \tau \quad \tau \in \text{Type}_v}{\Gamma \vdash t_1 = t_2 : \text{Bool}}
\end{array}$$

If  $t$  has no free variables, we write  $t : \tau$  in place of  $\forall \Gamma, \Gamma \vdash t : \tau$ , since the environment has no bearing on whether  $t$  is typed.

Not every term is sensible (e.g., consider  $() + \text{true}$ ). The typing relation  $- \vdash - : -$  describes exactly which terms are sensible within a given environment. Note that the type argument is unique, i.e.,  $\Gamma \vdash t : \tau_1$  and  $\Gamma \vdash t : \tau_2$  implies  $\tau_1 = \tau_2$ .

*Definition 2.5 (Interpretations and Semantics).* The semantics of value types and environments are given by the function  $\llbracket - \rrbracket : \text{Type}_v \cup \text{Env} \rightarrow \text{Set}$ , defined as follows:

$$\llbracket \text{Unit} \rrbracket = \{0\} \quad \llbracket \text{Bool} \rrbracket = \mathbb{B} \quad \llbracket \text{Nat} \rrbracket = \mathbb{N} \quad \llbracket \Gamma \rrbracket = \prod_{x \in \text{Var}} \llbracket \Gamma x \rrbracket$$

An interpretation of a signature  $\Sigma$  (or  $\Sigma$ -interpretation) is a tuple  $F = (\text{Gen}^F, \text{return}^F, \text{let}^F, (f^F)_{f \in \Sigma})$  satisfying

$$\begin{array}{ll}
\text{Gen}^F : \text{Set} \rightarrow \text{Set} & \text{return}^F : A \rightarrow \text{Gen}^F A \\
\text{let}^F : (A \rightarrow \text{Gen}^F B) \rightarrow \text{Gen}^F A \rightarrow \text{Gen}^F B & f^F : \llbracket \tau_1 \rrbracket \rightarrow \dots \rightarrow \llbracket \tau_n \rrbracket \rightarrow \text{Gen} \llbracket \tau \rrbracket
\end{array}$$

for each  $A, B \in \text{Set}$  and function symbol  $(f : \tau_1 \times \dots \times \tau_n \Rightarrow \tau) \in \Sigma$ . The interpretation  $F$  gives rise to a type semantics  $\llbracket - \rrbracket^F : \text{Type} \rightarrow \text{Set}$ , defined as

$$\llbracket \text{Gen } \tau \rrbracket^F = \text{Gen}^F \llbracket \tau \rrbracket \quad \llbracket \tau \rrbracket^F = \llbracket \tau \rrbracket \quad (\text{where } \tau \in \text{Type}_v),$$

and a term semantics  $\llbracket - \vdash - \rrbracket^F$  satisfying  $\llbracket \Gamma \vdash t \rrbracket^F : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$  for all  $\Gamma \vdash t : \tau$ , defined as follows:

$$\begin{array}{ll}
\llbracket \Gamma \vdash x \rrbracket^F \rho = \rho x & \llbracket \Gamma \vdash \text{return } t \rrbracket^F \rho = \text{return}^F (\llbracket \Gamma \vdash t \rrbracket^F \rho) \\
\llbracket \Gamma \vdash \text{let } x \leftarrow t_1 \text{ in } t_2 \rrbracket^F \rho = \text{let}^F (a \mapsto \llbracket \Gamma[x \mapsto a] \vdash t_2 \rrbracket^F \rho[x \mapsto a]) (\llbracket \Gamma \vdash t_1 \rrbracket^F \rho) \\
\llbracket \Gamma \vdash f t_1 \dots t_n \rrbracket^F \rho = f^F (\llbracket \Gamma \vdash t_1 \rrbracket^F \rho) \dots (\llbracket \Gamma \vdash t_n \rrbracket^F \rho) & \llbracket \Gamma \vdash () \rrbracket^F \rho = 0 \quad \llbracket \Gamma \vdash \text{true} \rrbracket^F \rho = 1 \\
\llbracket \Gamma \vdash \text{false} \rrbracket^F \rho = 0 & \llbracket \Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rrbracket^F \rho = \begin{cases} \llbracket \Gamma \vdash t_2 \rrbracket^F \rho & \text{if } \llbracket \Gamma \vdash t_1 \rrbracket^F \rho = 1 \\ \llbracket \Gamma \vdash t_3 \rrbracket^F \rho & \text{otherwise} \end{cases} \\
\llbracket \Gamma \vdash n \rrbracket^F \rho = n & \llbracket \Gamma \vdash t_1 + t_2 \rrbracket^F \rho = \llbracket \Gamma \vdash t_1 \rrbracket^F \rho + \llbracket \Gamma \vdash t_2 \rrbracket^F \rho \\
\llbracket \Gamma \vdash t_1 \cdot t_2 \rrbracket^F \rho = \llbracket \Gamma \vdash t_1 \rrbracket^F \rho \cdot \llbracket \Gamma \vdash t_2 \rrbracket^F \rho & \llbracket \Gamma \vdash t_1 = t_2 \rrbracket^F \rho = \begin{cases} 1 & \text{if } \llbracket \Gamma \vdash t_1 \rrbracket^F \rho = \llbracket \Gamma \vdash t_2 \rrbracket^F \rho \\ 0 & \text{otherwise} \end{cases}
\end{array}$$



If  $t : \tau$  has no free variables, then we define  $\llbracket t \rrbracket^F = \llbracket \Gamma \vdash t \rrbracket^F \rho$  for an arbitrarily chosen  $\Gamma \in \text{Env}$  and  $\rho \in \llbracket \Gamma \rrbracket$ , as neither  $\Gamma$  nor  $\rho$  influence the semantics of  $t$ .

An interpretation assigns language-specific semantics to the type of generators ( $\text{Gen}^F$ ), the return operator ( $\text{return}^F$ ), sequential composition ( $\text{let}^F$ ), and each function symbol ( $f^F$ ). The remaining (non-Gen) types and terms are assigned standard semantics as functions from variable assignments to suitable domains. As an example, we give the standard interpretation  $\text{St}$  for  $\Sigma_{\text{St}}$ , which interprets computations as state transition functions:

$$\begin{aligned} \text{Gen}^{\text{St}} A &= \mathbb{N} \rightarrow A \times \mathbb{N} & \text{return}^{\text{St}} x &= n \mapsto (x, n) \\ \text{let}^{\text{St}} f \, m &= n \mapsto f \, a \, n' \text{ where } (a, n') = m \, n & \text{get}^{\text{St}} &= n \mapsto (n, n) \\ & & \text{put}^{\text{St}} n &= n' \mapsto (0, n) \end{aligned}$$

It is standard to assume that  $(\text{Gen}^F, \text{return}^F, \text{let}^F)$  satisfies certain conditions (e.g., forming a monad [16]). Definition 2.5 does not require any such conditions so that we may consider a wider range of possible semantics.

**Definition 2.6 (Semantic Value Equivalence).** Two terms  $\Gamma \vdash t_1 : \tau$  and  $\Gamma \vdash t_2 : \tau$  (for  $\tau \in \text{Type}_v$ ) are semantically equivalent (written  $\Gamma \vdash t_1 = t_2$ ) iff  $\llbracket \Gamma \vdash t_1 \rrbracket = \llbracket \Gamma \vdash t_2 \rrbracket$ .

Two value-typed terms are semantically equivalent if they denote the same value, regardless of their syntactic representation. For example,  $\Gamma \vdash x + 0 = x$  holds for all environments  $\Gamma \in \text{Env}$  satisfying  $\Gamma x = \text{Nat}$  — even though  $x + 0$  and  $x$  are not syntactically identical — since  $x + 0$  and  $x$  denote the same value.

**Definition 2.7 (Semantic Computation Equivalence).** Let  $F$  be a  $\Sigma$ -interpretation. A semantic equivalence relation on  $F$  is a family of equivalence relations  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$  (indexed on  $A, B \in \text{Set}$ ) which satisfies the following rules:

$$\begin{array}{c} \text{RETURN-CONG} \\ \frac{\Gamma \vdash t = t'}{\Gamma \vdash \text{return } t \approx \text{return } t'} \\ \\ \text{LET-CONG} \\ \frac{\Gamma \vdash t_1 \approx t'_1 \quad \Gamma[x \mapsto \tau] \vdash t_2 \approx t'_2}{\Gamma \vdash (\text{let } x \leftarrow t_1 \text{ in } t_2) \approx (\text{let } x \leftarrow t'_1 \text{ in } t'_2)} \\ \\ \text{IF-CONG} \\ \frac{\Gamma \vdash t_1 = t'_1 \quad \Gamma \vdash t_2 \approx t'_2 \quad \Gamma \vdash t_3 \approx t'_3}{\Gamma \vdash (\text{if } t_1 \text{ then } t_2 \text{ else } t_3) \approx (\text{if } t'_1 \text{ then } t'_2 \text{ else } t'_3)} \end{array}$$

Two terms  $\Gamma \vdash t_1 : \text{Gen } \tau$  and  $\Gamma \vdash t_2 : \text{Gen } \tau$  are semantically equivalent with respect to  $\approx$  (written  $\Gamma \vdash t_1 \approx^F t_2$ ) iff  $\llbracket \Gamma \vdash t_1 \rrbracket^F \approx \llbracket \Gamma \vdash t_2 \rrbracket^F$ . We omit the superscript when  $F$  can be unambiguously determined from the equivalence relation  $\approx$ . We also write  $t_1 \approx^F t_2$  in place of  $\forall \Gamma. \Gamma \vdash t_1 \approx t_2$ , where  $\Gamma$  ranges over all environments satisfying  $\Gamma \vdash t_1 : \text{Gen } \tau$  and  $\Gamma \vdash t_2 : \text{Gen } \tau$  for some  $\tau \in \text{Type}_v$ .

A semantic equivalence relation defines which computation-typed terms are considered to have the same behaviour, regardless of their syntactic representation. Rules **RETURN-CONG** to **IF-CONG** are called the *congruence rules*. The congruence rules imply that the semantic equivalence two terms holds independent of the context they appear, and hence these rules are essential for compositional reasoning. For example, the equivalence  $(\text{let } x \leftarrow \text{get in put } x) =^{\text{St}} \text{return } ()$  holds since, intuitively, assigning some mutable state to its own value is a no-op.

## 2.4 QuickCheck

In this section, we review the QuickCheck language for generator specification [3]. We frame the QuickCheck language within our metalanguage framework by providing an appropriate signature



and interpretation. At a high level, QuickCheck is a probabilistic programming language and generators are specified using probabilistic operators.

*Definition 2.8.* The QuickCheck signature is defined as

$$\Sigma_Q = \{\text{flip}_p : \Rightarrow \text{Bool} \mid p \in \mathbb{W}\}$$

and the QuickCheck interpretation  $Q$  is defined as follows:

$$\begin{aligned} \text{Gen}^Q A = \text{Seed} &\rightarrow A & \text{return}^Q a = \sigma &\mapsto a \\ \text{let}^Q f \ m = \sigma &\mapsto f \ (m \ (\pi_L \ \sigma)) \ (\pi_R \ \sigma) & \text{flip}_p^Q &= \psi_p \end{aligned}$$

In addition to the common return and let operators, the QuickCheck language features a biased coin flip operator  $\text{flip}_p$ . Intuitively, the  $\text{flip}_p$  operator produces true with probability  $p$  and false with probability  $1 - p$ . Semantically, generators are represented as random variables (i.e. functions from seeds to values), mirroring the actual implementation of QuickCheck. The core operators are implemented as follows:

- $\text{flip}_p$  samples the seed it is given to produce a random boolean.
- $\text{return } a$  ignores its input and always returns  $a$ .
- $\text{let } x \leftarrow t_1 \text{ in } t_2$  assigns  $x$  to a value produced by  $t_1$ , then returns the value produced by  $t_2$ . The input seed is split between  $t_1$  and  $t_2$  using  $\pi_L$  and  $\pi_R$ , which effectively ensures that the results of  $t_1$  and  $t_2$  are uncorrelated.

As a shorthand, we define a binary probabilistic choice operator  $- \oplus_p -$  as

$$t_1 \oplus_p t_2 = \text{let } b \leftarrow \text{flip}_p \text{ in if } b \text{ then } t_1 \text{ else } t_2.$$

Intuitively,  $t_1 \oplus_p t_2$  behaves like  $t_1$  with probability  $p$  and behaves like  $t_2$  with probability  $1 - p$ .

*Example 2.9.* To demonstrate the QuickCheck language, we define a family of generators  $\text{nat}^i \in \text{Term } \Sigma_Q$  recursively as

$$\begin{aligned} \text{nat}^0 &= \text{return } 0 & \text{nat}^{i+1} &= \text{let } n \leftarrow \text{nat}^i \text{ in} \\ & & & \text{let } b \leftarrow \text{flip}_{1/2} \text{ in} \\ & & & \text{return } (2 \cdot n + \text{if } b \text{ then } 1 \text{ else } 0) \end{aligned}$$

Intuitively,  $\text{nat}^i$  computes a random integer in the range  $[0..2^i)$  by combining  $i$  random bits.

## 2.5 Hedgehog

In this section, we review the Hedgehog language for generator-based shrinking specification [21]. As with QuickCheck, we provide an appropriate signature and interpretation. At a high level, the Hedgehog language extends the QuickCheck language with an extra operation for specifying shrinking behaviour.

*Definition 2.10 (Shrink Tree).* The set  $\text{Tree } A$  of *shrink trees* is the smallest set satisfying

$$\text{Tree } A = A \times (\text{Tree } A)^*$$

and  $\text{let}^\top$ , the sequential composition operator for trees, is defined recursively as

$$\begin{aligned} \text{let}^\top : (A \rightarrow \text{Tree } B) &\rightarrow \text{Tree } A \rightarrow \text{Tree } B \\ \text{let}^\top f \ (a, us) &= (b, vs \text{ ++ } \text{map} \ (\text{let}^\top f) \ us) \\ \text{where } (b, vs) &= f \ a. \end{aligned}$$

For a tree  $u = (a, us) \in \text{Tree } A$ , the value  $a \in A$  is called the *label* of  $u$ , and  $us \in (\text{Tree } A)^*$  denotes the *children* of  $u$ . The set of all labels in a tree  $u$  is denoted by  $\text{labels } u$ .

A shrink tree represents a value and its shrinks, its shrink's shrinks, etc. The topmost value is produced during the testing stage, while the lower values are produced during the shrinking stage. In practice, shrink trees may be infinite or contain partial values. For simplicity we assume shrink trees are finite and contain no partial values.

*Definition 2.11.* The Hedgehog signature is defined as

$$\Sigma_{\mathcal{H}} = \Sigma_Q \cup \{\text{shrinkNat} : \text{Nat} \Rightarrow \text{Nat}\}$$

and the Hedgehog interpretation  $\mathcal{H}$  is given as follows:

$$\begin{aligned} \text{Gen}^{\mathcal{H}} A = \text{Seed} &\rightarrow \text{Tree } A & \text{return}^{\mathcal{H}} a = \sigma &\mapsto (a, []) \\ \text{let}^{\mathcal{H}} f \, m = \sigma &\mapsto \text{let}^{\top} (a \mapsto f \, a \, (\pi_L \sigma)) \, (m \, (\pi_R \sigma)) & \text{flip}_p^{\mathcal{H}} = \sigma &\mapsto (\psi_p \sigma, []) \\ \text{shrinkNat}^{\mathcal{H}} n = \sigma &\mapsto (0, [(1, []), \dots, (n, [])]) \end{aligned}$$

Syntactically, the Hedgehog language extends the QuickCheck language with an additional shrinkNat operator, which is used to express shrinking behaviour. Semantically, the primary difference between Hedgehog and QuickCheck is that Hedgehog interprets generators as random variables over shrink trees instead of singular values. Hedgehog's core operators are implemented as follows:

- shrinkNat  $n$  always returns 0 during testing and specifies the values  $[1..n]$  as shrinks.
- flip <sub>$p$</sub>  returns 1 with probability  $p$  and returns 0 with probability  $1-p$  (just like QuickCheck), and specifies no shrinks.
- return  $a$  represents a generator which returns  $a$  with probability 1, and specifies no shrinks.
- let  $x \leftarrow t_1$  in  $t_2$  behaves as it does under the QuickCheck interpretation during testing, but evaluates  $t_2$  for every value in the tree produced by  $t_1$ . This gives a tree of trees, which is flattened for the final result. In practice, this process is performed lazily so that the entire tree is not computed and stored in memory at once.

As a shorthand, we define an  $n$ -ary shrinking operator shrink  $t_0 \, t_1 \dots t_n$  (for  $n \in \mathbb{N}$ ) as

$$\begin{aligned} \text{shrink } t_0 \, t_1 \dots t_n &= \text{let } i \leftarrow \text{shrinkNat } n \text{ in if } i = n \text{ then } t_n \text{ else} \\ &\quad \dots \\ &\quad \text{if } i = 1 \text{ then } t_1 \text{ else} \\ &\quad t_0 \end{aligned}$$

Intuitively, shrink  $t_0 \, t_1 \dots t_n$  represents a generator that behaves like  $t_0$  during the testing stage, but may behave like any of  $t_i$  during the shrinking stage.

*Example 2.12.* To demonstrate the Hedgehog language, we define a family of generators  $\text{sNat}^i \in \text{Term } \Sigma_{\mathcal{H}}$  recursively in Figure 5a. The term  $\text{sNat}^i$  behaves similarly to  $\text{Nat}^i$ : they both produce a random natural number in the range  $[0..2^i)$ . However,  $\text{sNat}^i$  also specifies some shrinking behaviour. Every result produced by  $\text{sflip}_p$  may shrink to false, which effectively changes one bit  $\text{sNat}$ 's output to zero. For example, if  $\text{sNat}^4$  produces 13, then one possible sequence of shrinks is  $13 \, (1101) \rightarrow 5 \, (0101) \rightarrow 4 \, (0100)$ . Note that no further shrinks are possible because, in Hedgehog, shrinking does not backtrack: once a later bit is flipped, earlier ones may no longer be flipped. The full shrink tree is given in Figure 5b.

## 2.6 Monads

In this section, we review the notion of monadic interpretations (in the sense of Moggi [16]), which characterizes the expected behaviour of sequential composition. We also review some additional properties which characterize the expected behaviour of probabilistic choice and shrinking.

```

491  snat0 = return 0
492
493  snati+1 = let n ← snati in
494             let b ← sflip1/2 in
495             return (2 · n + if b then 1 else 0)
496
497  sflipp = shrink (return true) (return false)
498             ⊕p return false.

```

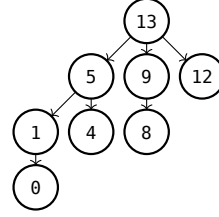
(a) Definition of snat<sup>i</sup>.(b) A shrink tree produced by snat<sup>4</sup>.

Fig. 5. Hedgehog example.

**Definition 2.13 (Monadic Interpretation).** An interpretation  $F$  is monadic up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$  iff  $\approx$  is a semantic equivalence on  $F$  and satisfies the equivalences

<p>LET-<math>\beta</math></p> <p>let <math>x \leftarrow \text{return } t_1</math> in <math>t_2 \approx t_2[t_1/x]</math></p> <p>LET-ASSOC</p> <p>let <math>x \leftarrow (\text{let } y \leftarrow t_1 \text{ in } t_2)</math> in <math>t_3 \approx \text{let } x \leftarrow t_1</math> in let <math>y \leftarrow t_2</math> in <math>t_3</math></p>	<p>LET-<math>\eta</math></p> <p>let <math>x \leftarrow t</math> in return <math>x \approx t</math></p>
---	--

where  $t_1[t_2/x]$  denotes the substitution of  $t_2$  for  $x$  in  $t_1$ .

Intuitively, a monadic interpretation ensures that sequential composition behaves as expected, i.e. sequential composition is associative (rule LET-ASSOC) and return behaves like a no-op (rules LET- $\beta$  and LET- $\eta$ .) It is particularly useful to have an interpretation which is monadic up to standard equality ( $=$ ): sequential composition satisfies Definition 2.13 in most languages (such as C++), so a monadic interpretation is a necessary (but not sufficient) requirement if the language implementor wishes to reuse the host language's sequential composition constructs.

When reasoning about generator behaviour, the key property of interest is a generator's distribution, i.e. the probability a generator produces a particular value. It is known that the QuickCheck interpretation  $Q$  is *not* monadic, although Claessen and Hughes [3] claim that this is “morally” acceptable since rules LET- $\beta$  to LET-ASSOC hold up to some notion of distribution equivalence. We formally investigate this claim in Section 3.

**Definition 2.14 (Probabilistic Interpretation).** A  $\Sigma$ -interpretation  $F$  (for  $\Sigma \supseteq \Sigma_Q$ ) is probabilistic up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$  iff  $F$  is monadic up to  $\approx$  and satisfies the following rules:

<p>⊕-1</p> <p><math>t_1 \oplus_1 t_2 \approx t_1</math></p>	<p>⊕-0</p> <p><math>t_1 \oplus_0 t_2 \approx t_2</math></p>	<p>⊕-IDEM</p> <p><math>t \oplus_p t \approx t</math></p>	<p>⊕-COMM</p> <p><math>t_1 \oplus_p t_2 \approx t_2 \oplus_{1-p} t_1</math></p>
<p>⊕-Assoc</p> $\frac{0 < p < 1 \quad 0 < q < 1}{(t_1 \oplus_p t_2) \oplus_q t_3 \approx t_1 \oplus_{p \cdot q} (t_2 \oplus_{(q \cdot (1-p))/(1-p \cdot q)} t_3)}$			
<p>⊕-<math>\eta</math></p> <p><math>(\text{let } x \leftarrow t_1 \text{ in } t_2 \oplus_p t_3) \approx (\text{let } x \leftarrow t_1 \text{ in } t_2) \oplus_p (\text{let } x \leftarrow t_1 \text{ in } t_3)</math></p>			

Definition 2.14, adapted from Gibbons and Hinze [9], characterizes the expected behaviour of the probabilistic choice operator  $\oplus_p$  (and by extension the operator  $\text{flip}_p$ ). We omit the right distributivity rule from Gibbons and Hinze [9] as it holds for all monadic interpretations in our formulation (Proposition 2.15).

PROPOSITION 2.15. *The following rule holds for all  $\Sigma$ -interpretations  $F$  (for  $\Sigma \supseteq \Sigma_Q$ ) that are monadic up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$ :*

$$\oplus\text{-}\beta \\ (\text{let } x \leftarrow t_1 \oplus_p t_2 \text{ in } t_3) \approx (\text{let } x \leftarrow t_1 \text{ in } t_3) \oplus_p (\text{let } x \leftarrow t_2 \text{ in } t_3)$$

Example 2.16. Probabilistic interpretations allow us to give algebraic proofs of generator properties. For example,  $\text{nat}^1$  intuitively generates a 0 or 1 with probability 1/2. We can prove this formally for any interpretation  $F$  which is probabilistic up to some equivalence relation  $\approx$ :

$$\begin{aligned} \text{nat}^1 &\approx \text{let } n \leftarrow \text{nat}^0 \text{ in} && \text{(by definition)} \\ &\quad \text{let } b \leftarrow \text{flip}_{1/2} \text{ in} \\ &\quad \text{return } (2 \cdot n + \text{if } b \text{ then } 1 \text{ else } 0) \\ &\approx \text{let } n \leftarrow \text{nat}^0 \text{ in} && \text{(since } \text{flip}_p \approx \text{return true } \oplus_p \text{return false)} \\ &\quad \text{let } b \leftarrow \text{return true } \oplus_{1/2} \text{return false in} \\ &\quad \text{return } (2 \cdot n + \text{if } b \text{ then } 1 \text{ else } 0) \\ &\approx \text{let } n \leftarrow \text{nat}^0 \text{ in return } (2 \cdot n + 1) \oplus_{1/2} \text{return } (2 \cdot n) && \text{(by } \oplus\text{-}\beta) \\ &\approx \text{let } n \leftarrow \text{return } 0 \text{ in return } (2 \cdot n + 1) \oplus_{1/2} \text{return } (2 \cdot n) && \text{(by definition)} \\ &\approx \text{return } 1 \oplus_{1/2} \text{return } 0 && \text{(by LET-}\beta) \end{aligned}$$

Definition 2.17. A  $\Sigma$ -interpretation  $F$  (for  $\Sigma \supseteq \Sigma_{\mathcal{H}} \setminus \Sigma_Q$ ) is a shrinking interpretation up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$  iff  $F$  is monadic up to  $\approx$  and satisfies the following rules:

$$\begin{array}{ll} \text{SHRINK-ID} & \text{SHRINK-JOIN} \\ \text{shrink } m \approx m & \text{shrink } (\text{shrink } m_0 m_1 \dots m_n) m'_1 \dots m'_k \approx \text{shrink } m_0 m_1 \dots m_n m'_1 \dots m'_k \end{array}$$

Definition 2.17 characterizes the expected behaviour of the shrink operator. As with the probabilistic choice operator  $\oplus_p$ , the shrink operator distributes over let.

PROPOSITION 2.18. *The following rule holds for all  $\Sigma$ -interpretations  $F$  (for  $\Sigma \supseteq \Sigma_{\mathcal{H}} \setminus \Sigma_Q$ ) that are monadic up to a semantic equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$ :*

$$\text{SHRINK-}\beta \\ \text{let } x \leftarrow \text{shrink } m_0 \dots m_n \text{ in } m' \approx \text{shrink } (\text{let } x \leftarrow m_0 \text{ in } m') \dots (\text{let } x \leftarrow m_n \text{ in } m')$$

Example 2.19. As in Example 2.16, we formally show that  $\text{sNat}^1$  returns 0 or 1 with probability 1/2. Additionally, our proof shows that, when  $\text{sNat}^1$  returns 1, it may shrink to 0.

$$\begin{aligned} \text{sNat}^1 &\approx \text{let } n \leftarrow \text{sNat}^0 \text{ in} && \text{(by definition)} \\ &\quad \text{let } b \leftarrow \text{sflip}_{1/2} \text{ in} \\ &\quad \text{return } (2 \cdot n + \text{if } b \text{ then } 1 \text{ else } 0) \\ &\approx \text{let } n \leftarrow \text{sNat}^0 \text{ in} && \text{(by definition)} \\ &\quad \text{let } b \leftarrow \text{shrink } (\text{return true}) (\text{return false}) \oplus_p \text{return false in} \\ &\quad \text{return } (2 \cdot n + \text{if } b \text{ then } 1 \text{ else } 0) \\ &\approx \text{let } n \leftarrow \text{sNat}^0 \text{ in} && \text{(by } \oplus\text{-}\beta \text{ and SHRINK-}\beta) \\ &\quad \text{shrink } (\text{return } (2 \cdot n + 1)) (\text{return } (2 \cdot n)) \oplus_{1/2} \text{return } (2 \cdot n) \\ &\approx \text{let } n \leftarrow \text{return } 0 \text{ in} && \text{(by definition)} \\ &\quad \text{shrink } (\text{return } (2 \cdot n + 1)) (\text{return } (2 \cdot n)) \oplus_{1/2} \text{return } (2 \cdot n) \\ &\approx \text{shrink } (\text{return } 1) (\text{return } 0) \oplus_{1/2} \text{return } 0 && \text{(by LET-}\beta) \end{aligned}$$

### 3 Analysis

In this section, we formally investigate the QuickCheck and Hedgehog languages and their properties. We first define distribution equivalence and show (1) the QuickCheck interpretation is monadic up to distribution equivalence, but (2) the same is not true of Hedgehog's interpretation. Finally, we show that this is not a flaw in Hedgehog, but rather an inherent problem with the combination of probabilistic and shrinking effects.

#### 3.1 Distribution Equivalence

**Definition 3.1 (Distribution Equivalence).** Two families of random variables  $f_1, f_2 : A \rightarrow \text{Seed} \rightarrow B$  have pointwise equivalent distributions (written  $f_1 \doteq f_2$ ) iff  $\mathbb{P}(f_1 a \in X) = \mathbb{P}(f_2 a \in X)$  for all  $a \in A$  and  $X \subseteq B$ .

Two random variables have equivalent distributions if they produce each possible set of values with the same likelihood. Definition 3.1 lifts this idea to families of random variables. Note that Definition 3.1 also applies to Hedgehog's tree-valued random variable generator representation.

**LEMMA 3.2.** Suppose  $\Gamma \vdash t : \tau, \Gamma \vdash t_1 : \text{Gen } \tau, \Gamma[x \mapsto \tau] \vdash t_2 : \text{Gen } \tau', p \in \mathbb{W}, X \subseteq \llbracket \tau \rrbracket$ , and  $Y \subseteq \llbracket \tau' \rrbracket$ . The following rules hold:

$$\begin{aligned} \mathbb{P}(\llbracket \Gamma \vdash \text{return } t \rrbracket^Q \rho \in X) &= \begin{cases} 1 & \text{if } \llbracket \Gamma \vdash t \rrbracket^Q \rho \in X \\ 0 & \text{otherwise.} \end{cases} \\ \mathbb{P}(\llbracket \Gamma \vdash \text{let } x \leftarrow t_1 \text{ in } t_2 \rrbracket^Q \rho \in Y) &= \sum_{a \in \llbracket \tau \rrbracket} \mathbb{P}(\llbracket \Gamma \vdash t_1 \rrbracket^Q \rho = a) \cdot \mathbb{P}(\llbracket \Gamma[x \mapsto \tau] \vdash t_2 \rrbracket^Q \rho[x \mapsto a] \in Y) \\ \mathbb{P}(\llbracket \Gamma \vdash \text{flip}_p \rrbracket^Q \rho = 1) &= p \end{aligned}$$

**PROOF.** Note that, for any  $t \in \text{Term } \Sigma_Q$  satisfying  $\Gamma \vdash t : \text{Gen } \tau$  and  $\rho \in \llbracket \Gamma \rrbracket$ , the random variable  $\llbracket \Gamma \vdash t \rrbracket^Q \rho$  has a countable domain. Hence, it suffices to prove the following three goals:

*Goal 1.* Suppose  $a \in \llbracket \tau \rrbracket$ . Then

$$\mathbb{P}((\sigma \mapsto a) \in X) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise.} \end{cases}$$

**PROOF.** Trivial. ■

*Goal 2.* Suppose  $A, B \in \text{Set}$  are assigned the discrete  $\sigma$ -algebra,  $m : \text{Seed} \rightarrow A$  is a random variable with a countable domain,  $f : A \rightarrow \text{Seed} \rightarrow B$  is a family of random variables with countable domains, and  $Y \subseteq B$ . Then

$$\mathbb{P}(\text{let}^Q f m \in Y) = \sum_{a \in \llbracket \tau \rrbracket} \mathbb{P}(m = a) \cdot \mathbb{P}(f a \in Y).$$

**PROOF.** Let  $A'$  be the countable subset of  $A$  for which  $(m \circ \pi_L)^{-1} A' \neq \emptyset$ . Then

$$\begin{aligned} \mathbb{P}(\text{let}^Q f m \in Y) &= \mathbb{P}\{\sigma \mid f(m(\pi_L \sigma))(\pi_R \sigma) \in Y\} && \text{(by definition)} \\ &= \mathbb{P}\{\sigma \mid \exists a \in A. m(\pi_L \sigma) = a \wedge f a(\pi_R \sigma) \in Y\} && \text{(property of } \exists) \\ &= \mathbb{P}\left(\bigcup_{a \in A} \{\sigma \mid m(\pi_L \sigma) = a \wedge f a(\pi_R \sigma) \in Y\}\right) && \text{(by definition)} \\ &= \mathbb{P}\left(\bigcup_{a \in A'} \{\sigma \mid m(\pi_L \sigma) = a \wedge f a(\pi_R \sigma) \in Y\}\right) && \text{(property of } A') \end{aligned}$$

$$\begin{aligned}
&= \sum_{a \in A'} \mathbb{P} \{ \sigma \mid m(\pi_L \sigma) = a \wedge f a(\pi_R \sigma) \in Y \} && \text{(property of } \mathbb{P} \text{)} \\
&= \sum_{a \in A} \mathbb{P} \{ \sigma \mid m(\pi_L \sigma) = a \wedge f a(\pi_R \sigma) \in Y \} && \text{(property of } A' \text{)} \\
&= \sum_{a \in A} \mathbb{P} \{ \sigma \mid m(\pi_L \sigma) = a \} \cdot \mathbb{P} \{ \sigma \mid f a(\pi_R \sigma) \in Y \} && \text{(indep. of } \pi_L \text{ and } \pi_R \text{)} \\
&= \sum_{a \in A} \mathbb{P} \{ \sigma \mid m \sigma = a \} \cdot \mathbb{P} \{ \sigma \mid f a \sigma \in Y \} && \text{(measure preservation of } \pi_L \text{ and } \pi_R \text{)} \\
&= \sum_{a \in A} \mathbb{P} (m = a) \cdot \mathbb{P} (f a \in Y). && \text{(by definition)}
\end{aligned}$$

■

Goal 3. We have

$$\mathbb{P}(\psi_p = 1) = p.$$

PROOF. Trivial.

■

□

Lemma 3.2 characterizes the distribution of each of QuickCheck's operators. This lemma allows us to formally prove the claim by Claessen and Hughes [3] that the QuickCheck interpretation is monadic up to distribution equivalence. In fact, we prove a stronger property: we show that the QuickCheck interpretation is probabilistic.

PROPOSITION 3.3. *The QuickCheck interpretation  $Q$  is probabilistic up to distribution equivalence ( $\doteq$ ).*

PROOF. Follows by simple calculational proof using Lemma 3.2.

□

Can Proposition 3.3 be extended to Hedgehog? This turns out not to be the case: as distribution equivalence is not even a semantic equivalence on  $\mathcal{H}$  as the rule

$$\frac{\text{LET-CONG} \quad \Gamma \vdash t_1 \doteq^{\mathcal{H}} t'_1 \quad \Gamma[x \mapsto \tau] \vdash t_2 \doteq^{\mathcal{H}} t'_2}{\Gamma \vdash (\text{let } x \leftarrow t_1 \text{ in } t_2) \doteq^{\mathcal{H}} (\text{let } x \leftarrow t'_1 \text{ in } t'_2)}.$$

does not hold. For a counterexample, consider the generators  $\text{coin}_1, \text{coin}_2 \in \text{Term } \Sigma_{\mathcal{H}}$  defined as

$$\text{coin}_1 = \text{return } 1 \oplus_{1/2} \text{return } 0 \quad \text{coin}_2 = \text{return } 0 \oplus_{1/2} \text{return } 1.$$

The  $\text{coin}_2$  generator interprets its seed in the exact opposite way from  $\text{coin}_1$ , i.e.  $\text{coin}_1 \sigma = 1 - \text{coin}_2 \sigma$  for any  $\sigma \in \text{Seed}$ . However, we clearly have  $\text{coin}_1 \doteq^{\mathcal{H}} \text{coin}_2$  (since both generators return 0 or 1 with probability 0.5) and therefore, for any  $x \in \text{Var}$ ,

$$\text{coin}_1 \doteq^{\mathcal{H}} \text{if } x \text{ then } \text{coin}_1 \text{ else } \text{coin}_2. \quad (1)$$

Now define

$$\text{step} = \text{shrink}(\text{return } 1) (\text{return } 0).$$

The step generator always returns 1 during testing, but returns 0 during shrinking. If rule LET-CONG holds then by equation 1 we have

$$\text{let } x \leftarrow \text{step} \text{ in } \text{coin}_1 \doteq^{\mathcal{H}} \text{let } x \leftarrow \text{step} \text{ in if } x \text{ then } \text{coin}_1 \text{ else } \text{coin}_2. \quad (2)$$

However, the left program returns the same value during the testing and shrinking stages, as the value  $x$  obtained from step is ignored. Conversely, the right program may change its value between

stages, since (1)  $\text{coin}_1$  is evaluated during testing, (2)  $\text{coin}_2$  is evaluated during shrinking, and (3)  $\text{coin}_1$  and  $\text{coin}_2$  return opposite values. Hence equation 2 is a contradiction.

PROPOSITION 3.4. *Distribution equivalence ( $\doteq$ ) is not a semantic equivalence on the Hedgehog interpretation  $\mathcal{H}$ .*

### 3.2 Generator-Based Shrinking Monads

Can we somehow “fix” Hedgehog to make it probabilistic with respect to distribution equivalence? Alternatively, can we use a different equivalence relation? We show that neither of these possibilities work.

Definition 3.5. An interpretation  $F$  is a generator-based shrinking interpretation up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$  iff  $F$  is a probabilistic interpretation and a shrinking interpretation up to  $F$ .

Definition 3.5 characterizes the minimal set of properties we expect from a language for generator-based shrinking.

Now consider an arbitrary generator-based shrinking interpretation  $F$  on  $\approx$  and an arbitrary weight  $p \in \mathbb{W}$ . We define  $\text{amb} : \text{Gen Bool}$  as

$$\text{amb} = \text{shrink}(\text{return } 1)(\text{return } 1) \oplus_p \text{shrink}(\text{return } 0)(\text{return } 0).$$

Intuitively,  $\text{amb}$  denotes a generator which returns 1 or 0 during testing, and returns the exact same value during shrinking. Consider also the generator  $\text{amb}' : \text{Gen Bool}$  defined as

$$\begin{aligned} \text{amb}' = & (\text{shrink}(\text{return } 1)(\text{return } 1) \oplus_p \text{shrink}(\text{return } 0)(\text{return } 1)) \oplus_p \\ & (\text{shrink}(\text{return } 1)(\text{return } 0) \oplus_p \text{shrink}(\text{return } 0)(\text{return } 0)). \end{aligned}$$

Intuitively,  $\text{amb}'$  behaves similarly to  $\text{amb}$  but includes the possibility of changing values during shrinking. Clearly these programs should not be equivalent, but the properties of generator-based shrinking interpretations imply that they are.

THEOREM 3.6. *The equivalence  $\text{amb} \approx^F \text{amb}'$  holds for any interpretation  $F$  which is a generator-based shrinking interpretation up to an equivalence relation  $\approx \subseteq (A \rightarrow \text{Gen}^F B)^2$ .*

PROOF. Consider the following lemma:

LEMMA 3.6.1. *The following rule holds:*

$$\begin{array}{c} \oplus\text{-SHRINK-DISTRIB} \\ \text{shrink } t_0 \dots (t_i \oplus_p t'_i) \dots t_n \approx \text{shrink } t_0 \dots t_i \dots t_n \oplus_p \text{shrink } t_0 \dots t'_i \dots t_n \end{array}$$

PROOF. Define

$$\begin{aligned} \text{select } x \ t_0 \dots t_n &= \text{if } x = n \text{ then } t_n \text{ else} \\ &\dots \\ &\text{if } x = 1 \text{ then } t_1 \text{ else} \\ &t_0. \end{aligned}$$

Then

$$\begin{aligned} & \text{shrink } t_0 \dots (t_i \oplus_p t'_i) \dots t_n \\ \approx & \text{let } x \leftarrow \text{shrink } 0 \dots n && \text{(by SHRINK-}\beta\text{)} \\ & \text{in select } x \ t_0 \dots (t_i \oplus_p t'_i) \dots t_n \\ \approx & \text{let } x \leftarrow \text{shrink } 0 \dots n && \text{(by } \oplus\text{-IDEM)} \\ & \text{in select } x \ (t_0 \oplus_p t_0) \dots (t_i \oplus_p t'_i) \dots (t_n \oplus_p t_n) \end{aligned}$$



$$\begin{aligned}
&\approx \text{let } x \leftarrow \text{shrink } 0 \dots n && \text{(property of if (select))} \\
&\quad \text{in select } x \ t_0 \dots t_i \dots t_n \oplus_p \text{select } x \ t'_0 \dots t'_i \dots t_n \\
&\approx (\text{let } x \leftarrow \text{shrink } 0 \dots n \text{ in select } x \ t_0 \dots t_i \dots t_n) && \text{(by } \oplus\text{-}\eta\text{)} \\
&\quad \oplus_p (\text{let } x \leftarrow \text{shrink } 0 \dots n \text{ in select } x \ t'_0 \dots t'_i \dots t_n) \\
&\approx \text{shrink } t_0 \dots t_i \dots t_n \oplus_p \text{shrink } t'_0 \dots t'_i \dots t_n. && \text{(by } \oplus\text{-}\eta\text{)}
\end{aligned}$$

■

Now consider the term

$$\text{shrink } (\text{return } 1 \oplus_p \text{return } 0) \ (\text{return } 1 \oplus_p \text{return } 0).$$

We have

$$\begin{aligned}
&\text{shrink } (\text{return } 1 \oplus_p \text{return } 0) \ (\text{return } 1 \oplus_p \text{return } 0) \\
&\approx \text{shrink } (\text{return } 1) \ (\text{return } 1 \oplus_p \text{return } 0) && \text{(by } \oplus\text{-SHRINK-DISTRIB)} \\
&\approx \oplus_p \text{shrink } (\text{return } 0) \ (\text{return } 1 \oplus_p \text{return } 0) \\
&\approx (\text{shrink } (\text{return } 1) \ (\text{return } 1) \oplus_p \text{shrink } (\text{return } 1) \ (\text{return } 0)) && \text{(by } \oplus\text{-SHRINK-DISTRIB)} \\
&\quad \oplus_p (\text{shrink } (\text{return } 0) \ (\text{return } 1) \oplus_p \text{shrink } (\text{return } 0) \ (\text{return } 0)) \\
&\approx \text{amb}'. && \text{(by definition)}
\end{aligned}$$

We also have

$$\begin{aligned}
&\text{shrink } (\text{return } 1 \oplus_p \text{return } 0) \ (\text{return } 1 \oplus_p \text{return } 0) \\
&\approx \text{let } x \leftarrow \text{shrink } (\text{return } 1) \ (\text{return } 0) \text{ in return } 1 \oplus_p \text{return } 0 && \text{(by SHRINK-}\beta \text{ and LET-}\beta\text{)} \\
&\approx (\text{let } x \leftarrow \text{shrink } (\text{return } 1) \ (\text{return } 0) \text{ in return } 1) \\
&\quad \oplus_p (\text{let } x \leftarrow \text{shrink } (\text{return } 1) \ (\text{return } 0) \text{ in return } 0) && \text{(by } \oplus\text{-}\eta\text{)} \\
&\approx \text{shrink } (\text{return } 1) \ (\text{return } 1) \oplus_p \text{shrink } (\text{return } 0) \ (\text{return } 0) && \text{(by SHRINK-}\beta \text{ and LET-}\beta\text{)} \\
&\approx \text{amb}. && \text{(by definition)}
\end{aligned}$$

Hence  $\text{amb} \approx \text{amb}'$ . □

## 4 The halcheck Language

In this section, we introduce the halcheck language for generator-based shrinking. In Section 4.1, we specify halcheck and discuss some basic properties. In Section 4.2, we define our primary reasoning tool – contextual joint distribution equivalence – and show that halcheck’s interpretation is probabilistic with respect to this equivalence, modulo some assumptions. In Section 4.3, we introduce a conservative extension of the halcheck language which resolves some issues with compositional generator construction and allows us to define a sound translation from Hedgehog and halcheck.

### 4.1 Definition and Basic Properties

*Definition 4.1 (halcheck).* The halcheck signature is defined as

$$\Sigma_{\text{hc}} = \{\text{flip}_{\ell s \sim p} : \Rightarrow \text{Bool} \mid \ell s \in \text{Label}^*, p \in \mathbb{W}\} \cup \{\text{shrinkNat} : \text{Nat} \Rightarrow \text{Nat}\}$$

and the halcheck interpretation  $\text{hc}$  is given as follows:

$$\begin{aligned} \text{Gen}^{\text{hc}} A &= \text{Seed} \rightarrow \text{Tree } A & \text{return}^{\text{hc}} a = \sigma &\mapsto (a, []) \\ \text{let}^{\text{hc}} f \ m = \sigma &\mapsto \text{let}^\top (a \mapsto f \ a \ \sigma) \ (m \ \sigma) & \text{shrinkNat}^{\text{hc}} n = \sigma &\mapsto (0, [(1, []), \dots, (n, [])]) \\ \text{flip}_{\ell s \sim p}^{\text{hc}} &= \sigma \mapsto (\psi_p (\pi_{\ell s} \sigma), []) \end{aligned}$$

Syntactically, the halcheck language is nearly identical to the Hedgehog language but substitutes the coin flip operator  $\text{flip}_p$  with a *labelled* coin flip operator  $\text{flip}_{\ell s \sim p}$ . Semantically, halcheck differs from Hedgehog in two important ways:

- let  $x \leftarrow t_1$  in  $t_2$  passes the same seed to both  $t_1$  and  $t_2$  instead of splitting it via  $\pi_L$  and  $\pi_R$ . This means that the results produced by  $t_1$  and  $t_2$  may be correlated even if  $t_2$  does not mention  $x$ .
- $\text{flip}_{\ell s \sim p}$  samples a different part of the input seed depending on the specified label string  $\ell s$ .

As with QuickCheck, we define a labelled probabilistic choice operator  $\oplus_{\ell s \sim p}$  – as

$$t_1 \oplus_{\ell s \sim p} t_2 = \text{let } b \leftarrow \text{flip}_{\ell s \sim p} \text{ in if } b \text{ then } t_1 \text{ else } t_2.$$

*Example 4.2.* To illustrate the difference between halcheck and Hedgehog, consider the Hedgehog generator  $\text{compare}_1$  and its naive halcheck translation  $\text{compare}_2$ :

$$\begin{aligned} \text{compare}_1 &= \text{let } x \leftarrow \text{flip}_{1/2} \text{ in} & \text{compare}_2 &= \text{let } x \leftarrow \text{flip}_{[] \sim 1/2} \text{ in} \\ &\text{let } y \leftarrow \text{flip}_{1/2} \text{ in} & &\text{let } y \leftarrow \text{flip}_{[] \sim 1/2} \text{ in} \\ &\text{return } (x = y) & &\text{return } (x = y) \end{aligned}$$

The generator  $\text{compare}_1$  may evaluate to true or false. For  $\text{compare}_2$ , since  $\text{let}^{\text{hc}}$  passes the same seed to both of its arguments, the two instances of  $\text{flip}$  will always produce the same choice, i.e. either they both return true or they both return false. Hence  $\text{compare}_2$  may only evaluate to true. A truly equivalent port of  $\text{compare}_1$  is written as follows:

$$\begin{aligned} \text{compare}_3 &= \text{let } x \leftarrow \text{flip}_{[\underline{x}] \sim 1/2} \text{ in} \\ &\text{let } y \leftarrow \text{flip}_{[\underline{y}] \sim 1/2} \text{ in} \\ &\text{return } (x = y) \end{aligned}$$

While  $\text{let}^{\text{hc}}$  passes the same seed to both instances of  $\text{flip}$ , the different labels cause different parts of the seed to be used. Thus  $\text{compare}_3$  can evaluate to true or false.

**THEOREM 4.3.** *The halcheck interpretation  $\text{hc}$  is a shrinking interpretation up to standard equivalence and also satisfies the following rules:*

$$\begin{aligned} \oplus\text{-IDEM} & \quad \oplus\text{-CONSISTENT} \\ t \oplus_{\ell s \sim p} t &=^{\text{hc}} t \quad (\text{let } x \leftarrow t_1 \oplus_{\ell s \sim p} t_2 \text{ in } t_3 \oplus_{\ell s \sim p} t_4) =^{\text{hc}} (\text{let } x \leftarrow t_1 \text{ in } t_3) \oplus_{\ell s \sim p} (\text{let } x \leftarrow t_2 \text{ in } t_4) \end{aligned}$$

**PROOF.** Follows trivially by expanding definitions.  $\square$

Unlike QuickCheck and Hedgehog, the halcheck interpretation is monadic. This makes halcheck a better candidate for an eDSL implementation which reuses its host language's sequential composition operator. Additionally, halcheck satisfies some of the rules required by probabilistic interpretations (see Definition 2.14), with the exceptions being rules  $\oplus\text{-1}$ ,  $\oplus\text{-0}$ ,  $\oplus\text{-COMM}$  and  $\oplus\text{-ASSOC}$ . Uniquely, halcheck satisfies the rule  $\oplus\text{-CONSISTENT}$ , which formalizes the intuition that two instances of  $\oplus_{\ell s \sim p}$  always make the same choice. Note that rule  $\oplus\text{-CONSISTENT}$  implies rule  $\oplus\text{-}\eta$  and rule  $\oplus\text{-}\beta$ .

## 4.2 Contextual Joint Distribution Equivalence

*Definition 4.4 (Contextual Joint Distribution Equivalence).* Two families of random variables  $f_1, f_2 : A \rightarrow \text{Gen}^{\text{hc}} B$  have jointwise equivalent distributions in a finite context  $L \subseteq \text{Label}^* \times \mathbb{W}$  (written  $f_1 \stackrel{\circ}{=}_L f_2$ ) iff

$$\mathbb{P} \left( \bigcap_{(ts,p) \notin L} \psi_p \circ \pi_{ts} \in X(ts,p) \cap \bigcap_{a \in A} f_1 a \in Y a \right) = \mathbb{P} \left( \bigcap_{(ts,p) \notin L} \psi_p \circ \pi_{ts} \in X(ts,p) \cap \bigcap_{a \in A} f_2 a \in Y a \right)$$

for all  $X : \bar{L} \rightarrow 2^{\mathbb{B}}$  and  $Y : A \rightarrow 2^B$ , where  $Y$  returns a non-empty set for only finitely many inputs.

Intuitively,  $\Gamma \vdash t_1 \stackrel{\circ}{=} t_2$  means that  $t_1$  and  $t_2$  produce the same distribution of values in all contexts that *do not* depend on any label in  $L$ .

*Example 4.5.* Consider the terms

$$\text{coin}_1 = \text{return } 0 \oplus_{[\underline{x}] \sim 1/2} \text{return } 1 \quad \text{and} \quad \text{coin}_2 = \text{return } 1 \oplus_{[\underline{x}] \sim 1/2} \text{return } 0.$$

These two terms have identical distributions but it is not sound to assume one can be substituted for the other in all contexts. For example, the terms

$$\begin{array}{ll} \text{compare}_{11} = \text{let } x \leftarrow \text{coin}_1 \text{ in} & \text{and} \quad \text{compare}_{12} = \text{let } x \leftarrow \text{coin}_1 \text{ in} \\ \quad \text{let } y \leftarrow \text{coin}_1 \text{ in} & \quad \text{let } y \leftarrow \text{coin}_2 \text{ in} \\ \quad \text{return } (x = y) & \quad \text{return } (x = y) \end{array}$$

are not equal since  $\text{compare}_{11}$  always produces true and  $\text{compare}_{12}$  always produces false. However, if no string with prefix  $[\underline{x}]$  is mentioned anywhere else in the surrounding program then it is safe to replace  $\text{coin}_1$  with  $\text{coin}_2$ . Hence  $\text{coin}_1 \stackrel{\circ}{=} \text{coin}_2$  iff  $[\underline{x}] \in L$ .

*Definition 4.6.* The support of a term is given by  $\text{supp} : \text{Term}_{\Sigma_{\text{hc}}} \rightarrow 2^{(\text{Label}^* \times \mathbb{W})}$ , defined as

$$\begin{aligned} \text{supp}(\text{let } x \leftarrow t_1 \text{ in } t_2) &= \text{supp } t_1 \cup \text{supp } t_2 & \text{supp flip}_{\ell s \sim p} &= \{(\ell s, p)\} \\ \text{supp}(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) &= \text{supp } t_2 \cup \text{supp } t_3 \end{aligned}$$

where  $\text{supp } t = \emptyset$  in all other cases.

The support of a term  $t$  is the set of all pairs of label strings and probabilities that appear in  $t$ . This includes pairs that appear in instances of  $\oplus$ , e.g.

$$\begin{aligned} \text{supp}(\text{return } 0 \oplus_{[\underline{x}] \sim 1/2} \text{return } 1) &= \text{supp}(\text{let } x \leftarrow \text{flip}_{[\underline{x}] \sim 1/2} \text{ in if } x \text{ then return } 0 \text{ else return } 1) \\ &= \{([\underline{x}], 1/2)\}. \end{aligned}$$

Intuitively, two terms with disjoint supports represent independent random variables, and hence their results will be uncorrelated when composed in sequence.

**THEOREM 4.7.** *The following rules hold:*

$$\begin{array}{c} \text{REFL}_1^* \quad \frac{\Gamma \vdash t_1 \stackrel{\text{hc}}{=} t_2}{\Gamma \vdash t_1 \stackrel{\circ}{=} t_2} \quad \text{REFL}_2^* \quad \frac{\Gamma \vdash t_1 \stackrel{\circ}{=} t_2}{\Gamma \vdash t_1 \doteq t_2} \quad \frac{\alpha\text{-RENAME}^* \quad \{(\ell s', p)\} \notin \text{supp } t}{t \stackrel{\circ}{=}_{\{(\ell s, p), (\ell s', p)\}} t[\ell s / \ell s']} \\ \text{MONO}^* \quad \frac{L \subseteq L' \quad \Gamma \vdash t_1 \stackrel{\circ}{=} t_2}{\Gamma \vdash t_1 \stackrel{\circ}{=}_{L'} t_2} \quad \text{RETURN-CONG}^* \quad \frac{\Gamma \vdash t = t'}{\Gamma \vdash \text{return } t \stackrel{\circ}{=} \text{return } t'} \\ \text{LET-CONG}^* \quad \frac{\Gamma \vdash t_1 \stackrel{\circ}{=}_{L_1} t'_1 \quad \text{supp } t_1 \cap L_2 = \emptyset \quad \text{supp } t'_1 \cap L_2 = \emptyset \quad \Gamma[x \mapsto \tau] \vdash t_2 \stackrel{\circ}{=}_{L_2} t'_2 \quad \text{supp } t_2 \cap L_1 = \emptyset \quad \text{supp } t'_2 \cap L_1 = \emptyset}{\Gamma \vdash (\text{let } x \leftarrow t_1 \text{ in } t_2) \stackrel{\circ}{=}_{L_1 \cup L_2} (\text{let } x \leftarrow t'_1 \text{ in } t'_2)} (\Gamma \vdash t_1 : \tau) \end{array}$$

$$\begin{array}{c}
\text{IF-CONG}^* \\
\frac{\Gamma \vdash t_1 \dot{=}_{L_1} t'_1 \quad \Gamma \vdash t_2 \dot{=}_{L_2} t'_2 \quad \text{supp } t_1 \cap L_2 = \emptyset \quad \text{supp } t'_1 \cap L_2 = \emptyset \quad \text{supp } t_2 \cap L_1 = \emptyset \quad \text{supp } t'_2 \cap L_1 = \emptyset}{\Gamma \vdash (\text{if } t_0 \text{ then } t_1 \text{ else } t_2) \dot{=}_{L_1 \cup L_2} (\text{if } t_0 \text{ then } t'_1 \text{ else } t'_2)} \\
\text{⊕-CONG}^* \\
\frac{\Gamma \vdash t_1 \dot{=}_L t'_1 \quad \Gamma \vdash t_2 \dot{=}_L t'_2 \quad (\ell s, p) \notin L}{\Gamma \vdash t_1 \oplus_{\ell s \sim p} t_2 \dot{=}_L t'_1 \oplus_{\ell s \sim p} t'_2} \quad \oplus\text{-1}^* \quad \oplus\text{-0}^* \\
\frac{}{t_1 \oplus_{\ell s \sim 1} t_2 \dot{=}_{\emptyset} t_1} \quad \frac{}{t_1 \oplus_{\ell s \sim 0} t_2 \dot{=}_{\emptyset} t_2} \\
\text{⊕-COMM}^* \\
\frac{(\ell s, p), (\ell s, 1-p) \notin \text{supp } t_1 \cup \text{supp } t_2}{t_1 \oplus_{\ell s \sim p} t_2 \dot{=}_{\{(\ell s, p), (\ell s, 1-p)\}} t_2 \oplus_{\ell s \sim 1-p} t_1} \\
\text{⊕-Assoc}^* \\
\frac{0 < p < 1 \quad 0 < q < 1 \quad (\ell s, p) \neq (\ell s', q) \quad (\ell s, p), (\ell s, p \cdot q), (\ell s', q), (\ell s', q \cdot (1-p)/(1-p \cdot q)) \notin \text{supp } t_1 \cup \text{supp } t_2 \cup \text{supp } t_3 = \emptyset}{(t_1 \oplus_{\ell s \sim p} t_2) \oplus_{\ell s' \sim q} t_3 \dot{=}_{\{(\ell s, p), (\ell s, p \cdot q), (\ell s', q), (\ell s', q \cdot (1-p)/(1-p \cdot q))\}} t_1 \oplus_{\ell s \sim p \cdot q} (t_2 \oplus_{\ell s' \sim q \cdot (1-p)/(1-p \cdot q)} t_3)}
\end{array}$$

PROOF. Our proof relies on the following definitions and lemmas:

*Definition 4.7.1.* Suppose  $\Gamma \vdash t : \tau$ . The function  $\langle \Gamma \vdash t \rangle : \llbracket \Gamma \rrbracket \rightarrow (\text{supp } t \rightarrow \mathbb{B}) \rightarrow \text{Tree } \llbracket \tau \rrbracket$  is defined recursively as follows:

$$\begin{aligned}
\langle \Gamma \vdash \text{return } x \rangle \rho k &= (\llbracket x \rrbracket \rho, []) \\
\langle \Gamma \vdash \text{let } x \leftarrow t_1 \text{ in } t_2 \rangle \rho k &= \text{let}^\top (a \mapsto \langle \Gamma[x \mapsto \tau] \vdash t_2 \rangle \rho[x \mapsto a] k) (\langle \Gamma \vdash t_1 \rangle \rho) \\
\langle \Gamma \vdash \text{flip}_{\ell s \sim p} \rangle \rho k &= k(\ell s, p) \\
\langle \Gamma \vdash \text{shrinkNat } t \rangle \rho k &= (0, [(1, []), \dots, (\llbracket \Gamma \vdash t \rrbracket \rho, [])])
\end{aligned}$$

PROPOSITION 4.7.1. Suppose  $\Gamma \vdash t : \tau$ . Then

$$\langle \Gamma \vdash t \rangle \rho k)^{-1} X = \bigcup_{\substack{k: \text{supp } t \rightarrow \mathbb{B} \\ \langle \Gamma \vdash t \rangle \rho k \in X}} \bigcap_{(\ell s, p) \in \text{supp } t} (\psi_p \circ \pi_{\ell s})^{-1}(k(\ell s, p))$$

for all  $\rho \in \llbracket \Gamma \rrbracket$  and  $k : \text{supp } t \rightarrow \mathbb{B}$ .

PROPOSITION 4.7.2.

$$\begin{aligned}
&(\llbracket \Gamma \vdash \text{let } x \leftarrow t_1 \text{ in } t_2 \rrbracket \rho)^{-1} X \\
&= \bigcup_{b \in \text{range } (\llbracket \Gamma \vdash t_1 \rrbracket \rho)} (\llbracket \Gamma \vdash t_1 \rrbracket \rho)^{-1} \{b\} \cap \bigcup_{\substack{k: \text{labels } b \rightarrow \text{Tree } B \\ \text{let}^\top k b \in X}} \bigcap_{a \in \text{labels } b} (\llbracket \Gamma[x \mapsto \tau] \vdash t_2 \rrbracket \rho[x \mapsto a])^{-1} \{k a\}
\end{aligned}$$

PROPOSITION 4.7.3.

$$\begin{aligned}
&\mathbb{P}(X \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} \llbracket \Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rrbracket \rho \in Y \rho) \\
&= \mathbb{P}(X \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ \llbracket \Gamma \vdash t_1 \rrbracket \rho = 1}} \llbracket \Gamma \vdash t_2 \rrbracket \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ \llbracket \Gamma \vdash t_1 \rrbracket \rho = 0}} \llbracket \Gamma \vdash t_3 \rrbracket \rho \in Y \rho)
\end{aligned}$$

PROPOSITION 4.7.4.

$$\begin{aligned} \mathbb{P}(X \cap \bigcap_{\rho \in [\Gamma]} [\Gamma \vdash t_1 \oplus_{\ell s \sim p} t_2] \mid \rho \in Y \mid \rho) &= \mathbb{P}(X \cap \psi_p \circ \pi_{\ell s \sim p} = 1 \cap \bigcap_{\rho \in [\Gamma]} [\Gamma \vdash t_1] \mid \rho \in Y \mid \rho) \\ &+ \mathbb{P}(X \cap \psi_p \circ \pi_{\ell s \sim p} = 0 \cap \bigcap_{\rho \in [\Gamma]} [\Gamma \vdash t_2] \mid \rho \in Y \mid \rho) \end{aligned}$$

We now prove each rule individually.

Case *REFL<sub>1</sub>*<sup>\*</sup>. Trivial.

Case *REFL<sub>2</sub>*<sup>\*</sup>. It suffices to show that  $f \doteq g$  implies  $f \doteq g$  for all  $f, g : A \rightarrow \text{Seed} \rightarrow \text{Tree } B$ , where  $A, B \in \text{Set}$  have discrete  $\sigma$ -algebras. To show  $f \doteq g$ , we must show

$$\mathbb{P}(f a \in U) = \mathbb{P}(g a \in U)$$

for all  $a \in A$  and  $U \subseteq B$ . By Definition 4.4,

$$\mathbb{P}\left(\bigcap_{(\ell s, p) \notin L} \psi_p \circ \pi_{\ell s} \in X(\ell s, p) \cap \bigcap_{a \in A} f a \in Y a\right) = \mathbb{P}\left(\bigcap_{(\ell s, p) \notin L} \psi_p \circ \pi_{\ell s} \in X(\ell s, p) \cap \bigcap_{a \in A} g a \in Y a\right)$$

for all  $X : \bar{L} \rightarrow 2^{\mathbb{B}}$  and  $Y : A \rightarrow 2^B$  such that  $Y$  returns a non-empty set for only finitely many inputs. Then we get

$$\mathbb{P}(f a \in U) = \mathbb{P}(g a \in U)$$

by instantiating  $X$  and  $Y$  as follows:

$$X = (\ell s, p) \mapsto \mathbb{B} \qquad Y = a' \mapsto \begin{cases} U & \text{if } a' = a \\ \emptyset & \text{otherwise.} \end{cases}$$

Case  *$\alpha$ -RENAME*<sup>\*</sup>. If  $(\ell s, p) \notin \text{supp } t$  then this case holds trivially, so assume  $(\ell s, p) \in \text{supp } t$ . Let  $t' = t[\ell s' / \ell s]$ . By Proposition 4.7.1, it suffices to prove

$$\bigcup_{\substack{k: \text{supp } t \rightarrow \mathbb{B} \\ (\Gamma \vdash t) \mid \rho \mid k \in X}} \bigcap_{(\ell s, p) \in \text{supp } t} (\psi_p \circ \pi_{\ell s})^{-1}(k(\ell s, p)) = \bigcup_{\substack{k: \text{supp } t' \rightarrow \mathbb{B} \\ (\Gamma \vdash t') \mid \rho \mid k \in X}} \bigcap_{(\ell s, p) \in \text{supp } t'} (\psi_p \circ \pi_{\ell s})^{-1}(k(\ell s, p)).$$

We prove

$$\bigcup_{\substack{k: \text{supp } t \rightarrow \mathbb{B} \\ (\Gamma \vdash t) \mid \rho \mid k \in X}} \bigcap_{(\ell s, p) \in \text{supp } t} (\psi_p \circ \pi_{\ell s})^{-1}(k(\ell s, p)) \subseteq \bigcup_{\substack{k: \text{supp } t' \rightarrow \mathbb{B} \\ (\Gamma \vdash t') \mid \rho \mid k \in X}} \bigcap_{(\ell s, p) \in \text{supp } t'} (\psi_p \circ \pi_{\ell s})^{-1}(k(\ell s, p)).$$

The other direction is similar.

First, note that  $(\Gamma \vdash t') \mid \rho \mid k' = (\Gamma \vdash t) \mid \rho \mid k$  for all  $k : \text{supp } t \rightarrow \mathbb{B}$ , where

$$k'(\ell s'', p'') = \begin{cases} k(\ell s, p) & \text{if } (\ell s'', p'') = (\ell s', p) \\ k(\ell s'', p'') & \text{otherwise.} \end{cases}$$

Intuitively, we can get the same result from  $t'$  as  $(\Gamma \vdash t) \mid \rho \mid k$  if we pass an assignment  $k' : \text{supp } t' \rightarrow \mathbb{B}$  that remaps  $(\ell s', p)$  to  $(\ell s, p)$ . Second, note that

$$\text{supp } t' = \text{supp } t \setminus \{(\ell s, p)\} \cup \{(\ell s', p)\}.$$

Now assume 1)  $k : \text{supp } t \rightarrow \mathbb{B}$  is such that  $(\Gamma \vdash t) \mid \rho \mid k \in X$  and 2)  $\sigma \in \text{Seed}$  is such that

$$\psi_p(\pi_{\ell s} \sigma) = k(\ell s, p)$$

for all  $(\ell s, p) \in \text{supp } t$ . It suffices to show 1)  $(\Gamma \vdash t') \rho k \in X$  and 2)

$$\psi_p (\pi_{\ell s} \sigma) = k' (\ell s, p)$$

for all  $(\ell s, p) \in \text{supp } t'$ , which follows immediately from our earlier observations.

*Case **MONO\***.* Trivial. Intuitively, any context that does not depend on  $L'$  also does not depend on  $L$ .

*Case **RETURN-CONG\***.* Trivial.

*Case **LET-CONG\***.* This case is very similar to **IF-CONG\***, but using Proposition 4.7.2 instead of Proposition 4.7.3.

*Case **IF-CONG\***.* Let  $f = \llbracket \Gamma \vdash t_0 \rrbracket$ ,  $g = \llbracket \Gamma \vdash t_1 \rrbracket$ ,  $g' = \llbracket \Gamma \vdash t'_1 \rrbracket$ ,  $h = \llbracket \Gamma \vdash t_2 \rrbracket$ , and  $h' = \llbracket \Gamma \vdash t'_2 \rrbracket$ . By Proposition 4.7.3 it suffices to show

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L_1 \cup L_2} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 1}} g \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 0}} h \rho \in Y \rho \right) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L_1 \cup L_2} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 1}} g' \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 0}} h' \rho \in Y \rho \right) \end{aligned}$$

for all  $X : \overline{L_1 \cup L_2} \rightarrow 2^{\mathbb{B}}$  and  $Y : \llbracket \Gamma \rrbracket \rightarrow 2^{\llbracket \tau \rrbracket}$ , where  $\Gamma \vdash t_1 : \tau$  and  $Y \rho \neq \emptyset$  for only finitely many  $\rho \in \llbracket \Gamma \rrbracket$ . We have

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L_1 \cup L_2} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 1}} g \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 0}} h \rho \in Y \rho \right) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L_1 \cup L_2} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 1}} g \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 0}} h' \rho \in Y \rho \right) \\ & \quad \text{(by Proposition 4.7.1 and assumptions } \Gamma \vdash t_1 \triangleq_{L_2} t'_1 \text{ and } \text{supp } t_1 \cap L_2 = \emptyset) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L_1 \cup L_2} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 1}} g' \rho \in Y \rho \cap \bigcap_{\substack{\rho \in \llbracket \Gamma \rrbracket \\ f \rho = 0}} h' \rho \in Y \rho \right) \\ & \quad \text{(by Proposition 4.7.1 and assumptions } \Gamma \vdash t_2 \triangleq_{L_2} t'_2 \text{ and } \text{supp } t_2 \cap L_1 = \emptyset) \end{aligned}$$

*Case **⊕-CONG\***.* Let  $f_i = \llbracket \Gamma \vdash t_i \rrbracket^{\text{hc}}$  and  $g_i = \llbracket \Gamma \vdash t'_i \rrbracket^{\text{hc}}$ . By Proposition 4.7.4, it suffices to show

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} f_1 a \in Y a \right) \\ &+ \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 0 \cap \bigcap_{a \in A} g_1 a \in Y a \right) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} f_2 a \in Y a \right) \\ &+ \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 0 \cap \bigcap_{a \in A} g_2 a \in Y a \right) \end{aligned}$$

for all  $X : \bar{L} \rightarrow 2^{\mathbb{B}}$  and  $Y : A \rightarrow 2^B$ , where  $Y$  returns a non-empty set for only finitely many inputs, which can be simplified to showing

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} f_1 a \in Y a \right) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} f_2 a \in Y a \right) \end{aligned}$$

(which follows from  $\Gamma \vdash t_1 \stackrel{\circ}{=}_L t'_1$  since  $(p, \ell s) \notin L$ ) and

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} g_1 a \in Y a \right) \\ &= \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} g_2 a \in Y a \right) \end{aligned}$$

(which follows from  $\Gamma \vdash t_2 \stackrel{\circ}{=}_L t'_2$  since  $(p, \ell s) \notin L$ ).

*Case  $\oplus - 1^*$ .* Let  $f = \llbracket \Gamma \vdash t_1 \rrbracket^{\text{hc}}$  and  $g = \llbracket \Gamma \vdash t_2 \rrbracket^{\text{hc}}$ . By Proposition 4.7.4 suffices to show

$$\begin{aligned} & \mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} = 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} f \rho \in Y \rho) + \mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} = 0 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \\ &= \mathbb{P}(X \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} f \rho \in Y \rho). \end{aligned}$$

for any measurable  $X \subseteq \text{Seed}$  and  $Y : \llbracket \Gamma \rrbracket \rightarrow 2^{\llbracket \tau \rrbracket}$  where  $Y$  returns a non-empty set for only finitely many inputs.

By monotonicity of  $\mathbb{P}$ , we have

$$\mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} = 0 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \leq \mathbb{P}(\psi_1 \circ \pi_{\ell s} = 0) = 0 \quad (3)$$

and

$$\mathbb{P}(X \cap \psi_0 \circ \pi_{\ell s} \neq 0 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \leq \mathbb{P}(\psi_0 \circ \pi_{\ell s} \neq 0) = 1 - \mathbb{P}(\psi_0 \circ \pi_{\ell s} = 0) = 0. \quad (4)$$

Furthermore, we have

$$\begin{aligned} & \mathbb{P}(X \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \\ &= \mathbb{P}((X \cap \psi_1 \circ \pi_{\ell s} = 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \cup (X \cap \psi_1 \circ \pi_{\ell s} \neq 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho)) \\ & \quad \text{(since } \psi_1 \circ \pi_{\ell s} = 1 \cup \psi_1 \circ \pi_{\ell s} \neq 1 = \text{Seed}) \\ &= \mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} = 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) + \mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} \neq 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho) \\ & \quad \text{(measure of disjoint union)} \\ &= \mathbb{P}(X \cap \psi_1 \circ \pi_{\ell s} = 1 \cap \bigcap_{\rho \in \llbracket \Gamma \rrbracket} g \rho \in Y \rho). \quad \text{(by (4))} \end{aligned}$$

which, combined with (3), proves this case.

*Case  $\oplus - 0^*$ .* Similar to the case for  $\oplus - 1^*$ .



Case  $\oplus\text{-COMM}^*$ . Let  $f = \llbracket \Gamma \vdash t_1 \rrbracket$  and  $g = \llbracket \Gamma \vdash t_2 \rrbracket$ . By Proposition 4.7.4, it suffices to show

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} f a \in Y a \right) \\ & + \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_p \circ \pi_{\ell s} = 0 \cap \bigcap_{a \in A} g a \in Y a \right) \\ & = \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_{1-p} \circ \pi_{\ell s} = 1 \cap \bigcap_{a \in A} g a \in Y a \right) \\ & + \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \psi_{1-p} \circ \pi_{\ell s} = 0 \cap \bigcap_{a \in A} f a \in Y a \right) \end{aligned}$$

for all  $X : \bar{L} \rightarrow 2^{\mathbb{B}}$  and  $Y : A \rightarrow 2^B$ , where  $Y$  returns a non-empty set for only finitely many inputs. Note that the terms  $\bigcap_{a \in A} f a \in Y a$  and  $\bigcap_{a \in A} g a \in Y a$ , by Proposition 4.7.1, can be expressed as a countable union of a countable finite intersection of sets of the form  $(\psi_{p'} \circ \pi_{\ell s'})^{-1} \in Z$ , where  $(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}$ . Hence the above can be rearranged to

$$\begin{aligned} & \mathbb{P}(\psi_p \circ \pi_{\ell s} = 1) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \\ & + \mathbb{P}(\psi_p \circ \pi_{\ell s} = 0) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\ & = \mathbb{P}(\psi_{1-p} \circ \pi_{\ell s} = 1) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\ & + \mathbb{P}(\psi_{1-p} \circ \pi_{\ell s} = 0) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \end{aligned}$$

which is equivalent to

$$\begin{aligned} & p \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \\ & + (1-p) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\ & = (1-p) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\ & + p \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin \{(\ell s, p), (\ell s, 1-p)\}} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \end{aligned}$$

which holds trivially.

Case  $\oplus\text{-ASSOC}^*$ . Let  $f = \llbracket \Gamma \vdash t_1 \rrbracket$ ,  $g = \llbracket \Gamma \vdash t_2 \rrbracket$ ,  $h = \llbracket \Gamma \vdash t_3 \rrbracket$ , and

$$L = \{(\ell s, p), (\ell s, p \cdot q), (\ell s', q), (\ell s', q \cdot (1-p)/(1-p \cdot q))\}.$$

This case proceeds similarly to the case for  $\oplus\text{-COMM}^*$ . Proposition 4.7.1 and Proposition 4.7.4 allows us to reduce this case to

$$\begin{aligned}
& p \cdot q \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \\
& + (1 - p) \cdot q \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\
& + (1 - q) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} h a \in Y a \right) \\
& = p \cdot q \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} f a \in Y a \right) \\
& + (1 - p \cdot q) \cdot q \cdot (1 - p) / (1 - p \cdot q) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} g a \in Y a \right) \\
& + (1 - p \cdot q) \cdot (1 - q) \cdot (1 - p) / (1 - p \cdot q) \cdot \mathbb{P} \left( \bigcap_{(\ell s', p') \notin L} \psi_{p'} \circ \pi_{\ell s'} \in X(\ell s', p') \cap \bigcap_{a \in A} h a \in Y a \right)
\end{aligned}$$

which holds because

$$\begin{aligned}
(1 - p) \cdot q &= q \cdot (1 - p) \\
&= (1 - p \cdot q) \cdot q \cdot (1 - p) / (1 - p \cdot q)
\end{aligned}$$

and

$$\begin{aligned}
1 - q &= 1 - p \cdot q - q + p \cdot q \\
&= 1 - p \cdot q - q \cdot (1 - p) \\
&= 1 - p \cdot q - (1 - p \cdot q) \cdot q \cdot (1 - p) / (1 - p \cdot q) \\
&= (1 - p \cdot q) \cdot (1 - q \cdot (1 - p) / (1 - p \cdot q)).
\end{aligned}$$

□

Theorem 4.7 enumerates the most important properties of contextual joint distribution equivalence. We describe each rule in detail:

- Rule  $\text{REFL}_1^*$  allows any of the equivalences from Theorem 4.3 to be lifted to contextual joint distribution equivalence.
- Rule  $\text{REFL}_2^*$  states that contextual joint distribution equivalence is sound with respect to distribution equivalence.
- Rule  $\text{MONO}^*$  states that it is sound to enlarge the context of an equivalence. One use of this rule is to apply the transitivity rule to equivalences with different contexts, i.e.

$$t_1 \stackrel{\circ}{=}_L t_2 \wedge t_2 \stackrel{\circ}{=}_{L'} t_3 \implies t_1 \stackrel{\circ}{=}_{L \cup L'} t_3.$$

- Rules  $\text{RETURN-CONG}^*$ ,  $\text{LET-CONG}^*$  and  $\oplus\text{-CONG}^*$  make contextual joint distribution equivalence *almost* a semantic equivalence. The rules are almost identical to rules  $\text{RETURN-CONG}$  to  $\text{IF-CONG}$  but include some side conditions which prevent equivalences from being applied in invalid contexts such as those seen in Example 4.5.
- Rules  $\oplus\text{-1}^*$  to  $\oplus\text{-ASSOC}^*$  are almost identical to Rules  $\oplus\text{-1}$ ,  $\oplus\text{-0}$ ,  $\oplus\text{-COMM}$  and  $\oplus\text{-ASSOC}$  of probabilistic interpretations. The primary difference is that rules  $\oplus\text{-COMM}^*$  and  $\oplus\text{-ASSOC}^*$

can only be applied when the labels used for the choice operator are unique within the context they appear. For example, this disallows inferences of the form

$$(t_1 \oplus_{[\underline{x}] \sim 1/2} t_2) \oplus_{[\underline{x}] \sim 1/2} t_3 \stackrel{\circ}{=} t_3 \oplus_{[\underline{x}] \sim 1/2} (t_1 \oplus_{[\underline{x}] \sim 1/2} t_2)$$

since  $\underline{x}$  is mentioned twice.

- Rule  $\alpha$ -RENAME\* states that it is sound to replace a label with another, as long as *all* instances of the label are replaced. For example, the equation

$$(t_1 \oplus_{[\underline{x}] \sim 1/2} t_2) \oplus_{[\underline{x}] \sim 1/2} t_3 \stackrel{\circ}{=}_{\{[\underline{x}], [\underline{y}]\}} (t_1 \oplus_{[\underline{y}] \sim 1/2} t_2) \oplus_{[\underline{y}] \sim 1/2} t_3$$

is allowed, but the following is not:

$$(t_1 \oplus_{[\underline{x}] \sim 1/2} t_2) \oplus_{[\underline{x}] \sim 1/2} t_3 \stackrel{\circ}{=}_{\{[\underline{x}], [\underline{y}]\}} (t_1 \oplus_{[\underline{x}] \sim 1/2} t_2) \oplus_{[\underline{y}] \sim 1/2} t_3$$

**THEOREM 4.8.** *Suppose  $t_1 \doteq t_2$  for some closed  $t_1 : \text{Gen } \tau$  and  $t_2 : \text{Gen } \tau$ . Then  $t_1 \doteq t_2$  is provable solely using the rules given in Theorems 4.3 and 4.7.*

**PROOF.** For any  $t : \tau$ , there must be a finite set of values  $a_0, \dots, a_n \in \text{Tree } \llbracket \tau \rrbracket$  such that  $\mathbb{P}(\llbracket t \rrbracket^{\text{hc}} = a_i) > 0$  and  $1 \leq i \leq n$ . Assume  $a_0, \dots, a_n$  are ordered according to some fixed but arbitrary linear ordering on  $\text{Tree } \llbracket \tau \rrbracket$ . Thus it suffices to show  $t$  is equivalent to a normal form constructed from  $a_0, \dots, a_n$ , i.e.,

$$t \doteq (b_0 \oplus_{p_1 \sim \ell_{s_1}} b_1) \dots \oplus_{p_n \sim \ell_{s_n}} b_n, \quad (5)$$

where each  $\ell_{s_i}$  is unique and

$$b_i = \text{toTreeExpr } a_i$$

$$p_i = \mathbb{P}(\llbracket t \rrbracket^{\text{hc}} = a_i) / p_{i-1} / \dots / p_1,$$

for

$$\text{toTreeExpr} : \text{Tree } \llbracket \tau \rrbracket \rightarrow \text{Term } \Sigma_{\text{hc}}$$

$$\text{toTreeExpr}(\llbracket t \rrbracket^{\text{hc}}, [a_1, \dots, a_n]) = \text{shrink}(\text{return } t) (\text{toTreeExpr } a_1) \dots (\text{toTreeExpr } a_n).$$

The way we construct the proof of (5) can be thought of as a series of transformations. First, we remove all variables from  $t$  using rules **LET- $\beta$** ,  **$\oplus$ - $\beta$**  and **SHRINK- $\beta$** . Then, we push all shrink terms inside all  $\oplus$  terms using rule  **$\oplus$ -CONSISTENT**. Then, we remove duplicate labels using rule  **$\oplus$ -CONSISTENT**. Finally, we sort the shrink terms using rules  **$\oplus$ -COMM** and  **$\oplus$ -ASSOC**.  $\square$

Theorem 4.8 essentially states that the rules given in Theorems 4.3 and 4.7 are complete: no other rules are needed to prove that two generators have equivalent distributions.

### 4.3 Labelling and Translation from Hedgehog

**Definition 4.9.** The labelling operator  $\text{label}_\ell : \text{Term } \Sigma_{\text{hc}} \rightarrow \text{Term } \Sigma_{\text{hc}}$  (for  $\ell \in \text{Label}$ ) is defined by

$$\text{label}_\ell(\text{let } x \leftarrow t_1 \text{ in } t_2) = \text{let } x \leftarrow \text{label}_\ell t_1 \text{ in } \text{label}_\ell t_2$$

$$\text{label}_\ell(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) = \text{if } t_1 \text{ then } \text{label}_\ell t_2 \text{ else } \text{label}_\ell t_3$$

$$\text{label}_\ell(\text{flip}_{\ell s \sim p}) = \text{flip}_{\ell :: \ell s \sim p}$$

where  $\text{label}_\ell t = t$  in all other cases.

The expression  $\text{label}_\ell t$  behaves exactly like  $t$ , except every instance of the coin flip operator  $\text{flip}_{\ell s \sim p}$  is replaced with  $\text{flip}_{\ell :: \ell s \sim p}$ . The main purpose of this operator is to enable compositional construction of generators.

*Example 4.10.* Suppose we have a closed term  $t : \text{Gen Nat}$  and suppose we want to construct a generator  $\text{sum} : \text{Gen Nat}$  that produces the sum of two random values chosen by  $t$ . In other words, we want  $\text{sum}$  to have the distribution

$$\mathbb{P}(\llbracket \text{sum} \rrbracket^{\text{hc}} \in X) = \sum_{x, y \in \mathbb{N}} \mathbb{P}(\llbracket t \rrbracket^{\text{hc}} = x) \cdot \mathbb{P}(\llbracket t \rrbracket^{\text{hc}} = y) \cdot \begin{cases} 1 & \text{if } x + y \in X \\ 0 & \text{otherwise.} \end{cases}$$

Naively, we might write

```
sum = let x ← t in
      let y ← t in
      return (x + y).
```

However this definition of  $\text{sum}$  does not have the right distribution:  $x$  and  $y$  are always assigned to identical values since any occurrence of  $\text{flip}$  within the first  $t$  will also occur in the second  $t$  *with the same label*. To write  $\text{sum}$  correctly we require another version of  $t$  with different labels, which we can achieve with the label operator:

```
sum = let x ← labelx t in
      let y ← labely t in
      return (x + y)
```

This example suggests a general pattern for generator construction: in order to prevent unwanted correlation between generated values, all sub-generator calls should be annotated with unique labels via the label operator.

**PROPOSITION 4.11.** *For any  $t \in \text{Term } \Sigma_{\text{hc}}$ ,  $\text{supp}(\text{label}_\ell t) = \{(\ell :: \ell s, p) \mid (\ell s, p) \in \text{supp } t\}$ .*

**PROOF.** Follows trivially by induction over  $t$ . □

Proposition 4.11 provides formal justification for the correctness of  $\text{sum}$  given in Example 4.10: the two terms  $\text{label}_x t$  and  $\text{label}_y t$  have disjoint supports (since no word can simultaneously begin with both  $x$  and  $y$ ) hence  $\text{label}_x t$  and  $\text{label}_y t$  produce uncorrelated results.

The label operator allows us to demonstrate that  $\text{halcheck}$  loses no expressivity relative to  $\text{Hedgehog}$ . In particular, we provide a translation from  $\text{Hedgehog}$  to  $\text{halcheck}$ .

**Definition 4.12.** The translation  $\langle - \rangle : \text{Term}_{\mathcal{H}} \rightarrow \text{Term}_{\text{hc}}$  from  $\text{Hedgehog}$  to  $\text{halcheck}$  terms is given by

$$\begin{aligned} \langle \text{let } x \leftarrow t_1 \text{ in } t_2 \rangle &= \text{let } x \leftarrow \text{label}_1 \langle t_1 \rangle \text{ in } \text{label}_\square \langle t_2 \rangle & \langle \text{flip}_p \rangle &= \text{flip}_{\square \sim p} \\ \langle \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rangle &= \text{if } t_1 \text{ then } \langle t_2 \rangle \text{ else } \langle t_3 \rangle \end{aligned}$$

where  $\langle t \rangle = t$  in all other cases.

The translation  $\langle t \rangle$  of a  $\text{Hedgehog}$  term  $t$  is obtained by replacing all occurrences of  $\text{let } x \leftarrow t_1 \text{ in } t_2$  with  $\text{let } x \leftarrow \text{label}_1 \langle t_1 \rangle \text{ in } \text{label}_\square \langle t_2 \rangle$ , and all occurrences of  $\text{flip}_p$  with  $\text{flip}_{\square \sim p}$ . To understand why this works, consider the definition of  $\text{let}^{\mathcal{H}}$ :

$$\text{let}^{\mathcal{H}} f m = \sigma \mapsto \text{let}^{\top} (a \mapsto f a (\pi_L \sigma)) (m (\pi_R \sigma))$$

The only difference between the  $\text{Hedgehog}$  version ( $\text{let}^{\mathcal{H}}$ ) and the  $\text{halcheck}$  version ( $\text{let}^{\text{hc}}$ ) is that the  $\text{Hedgehog}$  version splits the input seed ( $\sigma$ ) before forwarding it to its arguments  $f$  and  $m$ . This ensures that the results of  $f$  and  $m$  are uncorrelated, which is exactly what the label operator is for. Note that this translation does not preserve the exact semantics of the  $\text{Hedgehog}$  generator (i.e.  $\llbracket \langle t \rangle \rrbracket^{\text{hc}}$  and  $\llbracket t \rrbracket^{\mathcal{H}}$  denote different random variables) but their distributions are guaranteed to be the same (Theorem 4.13).

THEOREM 4.13. If  $\Gamma \vdash t : \tau$  (where  $t \in \text{Term } \Sigma_{\mathcal{H}}$ ) then  $\llbracket \Gamma \vdash \langle t \rangle \rrbracket^{\text{hc}} \doteq \llbracket \Gamma \vdash t \rrbracket^{\mathcal{H}}$ .

In general, our translation produces programs at most 3 times larger than the original Hedgehog program (we count the size of  $\text{label}_\ell t$  as one plus the size of  $t$ ). In Section 6, we demonstrate that halcheck programs are not much larger than their Hedgehog counterparts in practice.

#### 4.4 Summary

In this section, we specified halcheck by providing a signature and interpretation (Definition 4.1). We defined our primary reasoning tool – contextual joint distribution equivalence (Definition 4.4) – and showed that halcheck’s interpretation is probabilistic with respect to this equivalence, modulo some assumptions (Theorem 4.7). We introduced a conservative extension of the halcheck language called the labelling operator (Definition 4.9) and showed how the labelling operator solves some compositionality issues. Finally, we constructed a translation from Hedgehog to halcheck (Definition 4.12) and proved that the translation is sound (Theorem 4.13).

### 5 Labelling Errors

In this section, we discuss the effects of labelling with respect to generator bugs. We discuss various classes of bugs, how labelling interacts with each class, and how to fix errors which arise from incorrect labelling.

#### 5.1 Error Classes

There are three main classes of errors that can occur when writing generators.

*Soundness.* A soundness error occurs when a generator produces values that it should not. For example, a generator intended to produce sorted lists has a soundness error if it produces unsorted lists.

*Completeness.* A completeness error occurs when a generator is incapable of producing a desired value. For example, a generator intended to produce sorted lists has a completeness error if it never produces the empty list.

*Performance.* A performance error occurs when a generator takes longer than desired to produce a value. We reduce performance errors to soundness errors: given a generator  $G : \text{Gen } \tau$ , we 1) construct another generator  $G' : \text{Gen Nat}$  which behaves like  $G$  but instead returns the number of steps required to produce the output, and then 2) check whether  $G'$  only returns values below some desired threshold.

#### 5.2 Labelling Errors

Labelled probabilistic choice only behaves differently from normal probabilistic choice in the presence of *label collisions*, i.e., when two values are generated from the same string of labels in a single generator execution. Label collisions are problematic because they result in unwanted correlation, which skews generator distributions. For example, suppose we extend our core syntax and semantics with a type of boolean lists  $\text{List}$  with constructor terms  $\text{nil} \in \text{Term } \Sigma$  and  $\text{cons } t_1 t_2 \in \text{Term } \Sigma$  defined for all  $t_1, t_2 \in \text{Term } \Sigma$ , satisfying the usual typing rules. Now consider the family of generators  $\text{list}_n$ , defined as

$$\begin{aligned} \text{list}_0 &= \text{return nil} & \text{list}_{n+1} &= \text{return nil} \oplus_{\underline{x} \sim 1/2} \text{aux}_n \\ \text{aux}_n &= \text{let } x \leftarrow \text{return true} \oplus_{\underline{x} \sim 1/2} \text{return false in} \\ &\quad \text{let } y \leftarrow \text{list}_n \text{ in} \\ &\quad \text{return (cons } x y) \end{aligned}$$

The generator  $\text{list}_n$  is intended to produce an arbitrary list of size  $\leq n$ . However, there are two bugs:

- (1) Since both invocations of  $\oplus$  use the same label, they must always make the same choice, so  $\text{list}_n$  may only ever produce lists containing false.
- (2) Since all (indirect) recursive invocations of  $\text{list}_n$  use the same label,  $\text{list}_n$  will only produce lists of length zero or length  $n$ .

Label collisions are fixed by adding or modifying labels. For the example generator  $\text{list}_n$ , we fix the first problem by replacing exactly one of the labels  $\underline{x}$  with  $\underline{y}$ , and we fix the second problem by wrapping the call to  $\text{aux}_n$  with a call to  $\text{label}_{\underline{z}}$ :

```

list'_0 = return nil                                list'_{n+1} = return nil  $\oplus_{\underline{x} \sim 1/2}$  label_{\underline{z}} aux'_n
aux'_n = let x  $\leftarrow$  return true  $\oplus_{\underline{y} \sim 1/2}$  return false in
        let y  $\leftarrow$  list'_n in
        return (cons x y)

```

We summarize how label collisions cause or interact with generator soundness (and performance) and completeness.

*Soundness.* Label collisions cannot cause soundness errors: fixing label collisions only adds to the possible behaviours of a generator, and soundness errors can only be fixed by restricting behaviour. On the other hand, label collisions may hide soundness errors. For example, suppose we actually intended  $\text{list}_n$  to produce a monotonically increasing list of booleans (ordering  $\text{false} < \text{true}$ ). Clearly, a generic list generator will not produce monotonically increasing lists. However,  $\text{list}_n$  *does* produce monotonically increasing lists, since label collisions restrict  $\text{list}_n$ 's output to lists containing false. On the other hand, the “fixed” version  $\text{list}'_n$  has no label collisions but makes no guarantees of returning monotonically increasing lists.

*Completeness.* Label collisions can result in completeness errors. This is also demonstrated by  $\text{list}_n$ , where incorrect labelling prevents  $\text{list}_n$  from returning any lists containing true.

### 5.3 Eliminating Label Collisions

Given that label collisions interact with all other kinds of errors, it is prudent that they are dealt with as soon as possible in the testing process.

*Detecting Label Collisions.* To detect label collisions, we use a runtime monitor that 1) records all labels as they are used, and 2) reports any labels that are used more than once. Taking an execution of  $\text{list}_1$  as an example, the monitor will record the label  $\underline{x}$  at the first invocation of  $\oplus$ , and then detect a collision when the label  $\underline{x}$  is reused for the second invocation of  $\oplus$ .

The primary downside of runtime monitors is that they can only detect errors that are actually executed. This downside is not problematic in the setting of property-based testing, since failing to trigger a label collision means that the test distribution was unaffected.

*Fixing Label Collisions.* Once a label collision has been detected, it must be located in order to actually fix it. Note that a label collision actually occurs *between two locations*. For example, the label collision in  $\text{list}_1$  occurs at both invocations of  $\oplus$ , and changing the label at either invocation will eliminate the collision.

We identify sources of label collisions by having the runtime monitor record the call stack at each invocation of labelled choice, and then report the longest common prefix of the call stacks associated with the offending invocations. For example, consider the following version of  $\text{list}_n$

which fixes the duplicate label issue but does not label recursive calls:

```

list''0 = return nil
list''n+1 = return nil  $\oplus_{\underline{x} \sim 1/2}$  aux''n
aux''n = let x  $\leftarrow$  return true  $\oplus_{\underline{y} \sim 1/2}$  return false in
    let y  $\leftarrow$  list''n in
    return (cons x y)

```

This version of  $list''$  has a label collision between the first occurrence of  $\oplus$  with itself, and upon executing  $list''_2$ , the monitor 1) records that  $\underline{x}$  is used with call stack  $[list''_2]$ , 2) records that  $\underline{y}$  is used with call stack  $[list''_2, aux''_2]$  (assuming the right choice is taken), 3) records that  $\underline{x}$  is used with call stack  $[list''_2, aux''_2, list''_1]$ , and 4) reports the conflict occurring from steps 1 and 3 with the call stack prefix  $[list''_2]$ . This prompts us to notice that the call to  $aux'_n$  in  $list''_{n+1}$  needs to be wrapped in  $label_{\underline{z}}$ , leading us to the definition of  $list'_n$ .

## 6 Empirical Evaluation

In Section 4, we described a complete, composable, set of rules for reasoning about halcheck generator distributions. Hence, we now evaluate the effects of halcheck's design with respect to other practical considerations such as generator size and performance. Compared to existing solutions for generator-based shrinking [7, 21], halcheck requires that users manually label random choices. Thus, the goal of our evaluation is to answer the following research questions:

**RQ1:** To what extent can we express existing generators?

**RQ2:** What is the performance impact of manual labelling, with and without monitoring?

### 6.1 Setup

The halcheck interpretation (Definition 4.1) is essentially the same as the Hedgehog interpretation (Definition 2.11) extended with labelled choice. Hence, we implemented a version of halcheck by modifying the existing Hedgehog library (v1.5 [21]) for Haskell. Our modifications are extremely minimal: modified Hedgehog consists of 66 inserted lines and 80 deleted lines across 4 out of 27 of vanilla Hedgehog's files. Changes consist solely of (1) the introduction of the label operator and (2) wrapping of all sub-generator calls in existing combinators with the label operator. There are no changes to the core datatypes. We also updated Hedgehog's test suite to conform with the changes to Hedgehog. These updates consist of 25 inserted lines and 33 deleted lines across 6 out of 9 files.

We use a set of example generators drawn from Lampropoulos et al. [12] and from industry. We summarize our set of examples in Table 2. Each example is written using halcheck's operations (i.e., let, return, label, flip, and shrinkNat), and we interpret each operation using the corresponding operations from each (vanilla and modified) library. Under the vanilla interpretation, labelling is ignored (i.e.,  $label_t t$  behaves like  $t$ ), and our examples are carefully constructed to produce the same distribution of results under each interpretation.

Note that labelling primarily affects the semantics of random generation and not shrinking. Other libraries, such as QuickCheck [3] and falsify [7], implement random generation (but not shrinking) in a manner similar to Hedgehog (e.g., compare Definition 2.8 and Definition 2.11). Thus, we also implemented two additional versions of halcheck by modifying the QuickCheck (v2.15) and falsify (v0.2) libraries for Haskell in a similar manner to our Hedgehog modifications. For the vanilla and modified QuickCheck interpretations, shrinkNat always returns 0 (i.e., no shrinking). Studying these other implementations allows us to evaluate our research questions independently of the particular shrinking semantics we gave in Section 4. Additionally, for each modified interpretation, we implemented a monitored interpretation which reports label collisions





Example 4.2 is translated as

$$\langle \text{compare}_1 \rangle = \text{let } x \leftarrow \text{label}_\perp \text{ flip}_{[] \sim 1/2} \text{ in} \\ \text{label}_\perp (\text{let } y \leftarrow \text{label}_\perp \text{ flip}_{[] \sim 1/2} \text{ in} \\ \text{label}_\perp (\text{return } (x = y)))$$

but can be more succinctly expressed as

$$\text{let } x \leftarrow \text{label}_\perp \text{ flip}_{[] \sim 1/2} \text{ in} \\ \text{let } y \leftarrow \text{label}_\perp \text{ flip}_{[] \sim 1/2} \text{ in} \\ \text{return } (x = y).$$

Thus, we investigate the practicality of writing halcheck generators by measuring the increase in program size that comes with manual labelling.

The second column of Table 2 lists the size of each example with and without labelling, along with the size increase as a percentage. Sizes correspond to the abstract syntax tree of the program. We report a single pair of numbers for the instruction sequence generators from Hritcu et al. [11] as the various implementations heavily share code.

Our results show that labelling increases program sizes up to 21% in practice, which is much less significant than the 200% upper bound given at the end of Section 4.3.

**Summary:** We can express all existing generators. Manual labelling incurs a negligible increase in program size.

**RQ2:** *What is the performance impact of manual labelling, with and without monitoring?* Generator performance has a direct impact on how quickly bugs are found by property-based testing. Thus we measure and compare the throughput of each interpretation. Specifically, we measure, for each generator and interpretation, 1) the amount of time it takes to generate 1000 test cases, and 2) the amount of time it takes to compute 1000 shrinks. Measurements are made using the Criterion benchmarking library, which repeats each experiment until a time limit is reached (5 minutes in our case), and estimates the running time via linear regression. All measurements are made on an AMD Ryzen Threadripper 2990WX running at 3GHz, with 128 GB of RAM, on Ubuntu 22.04.5 LTS, compiled with GHC 9.4.8.

Tables 2 and 3 reports the results of our experiments. For each example and interpretation, we report Criterion's estimates for the amount of time it takes to generate and shrink 1000 random values. We also report the difference (as a percentage) between the vanilla, modified, and monitored interpretations of each library. Specifically, we report the results for vanilla vs. modified interpretations in Table 2, and the results for modified vs. monitored interpretations in Table 3. For example, the List generator performs 26% faster under the modified Hedgehog interpretation than the vanilla interpretation (reported as -26% in Table 2), while the monitored interpretation is 247% slower than the modified interpretation (reported in Table 3). We do not report shrink times for QuickCheck since it does not have generator-based shrinking, and hence shrink times are not affected by our modifications.

For Hedgehog and falsify, manual labelling makes generation up to 35% and 50% faster, respectively, and makes shrinking up to 29% and 33% faster, respectively. Modified Hedgehog and falsify exceed the vanilla interpretations' performance in nearly all benchmarks, with a few exceptions due to statistical noise. On the other hand, modified QuickCheck generally performs worse than the unmodified version, up to 27% slower. Finally, monitoring negatively affects performance in all cases, with a maximum 1427% slowdown during generation and a maximum 1382% slowdown during shrinking.

Our results indicate that manual labelling increases generator and shrinker performance for languages featuring generator-based shrinking such as Hedgehog and falsify. However, manual labelling decreases performance of QuickCheck, which does not employ generator-based shrinking. Furthermore, monitoring has a significant impact on generator performance. This suggests that monitoring should not be used during normal test runs. Instead, users can detect labelling collisions with additional trivial test cases which only invoke a generator with monitoring.

**Summary:** Manual labelling has a positive impact on generator performance when generator-based shrinking is employed, and a negative impact otherwise. Monitoring has a significant negative impact on performance, and should only be used on simple tests.

### 6.3 Threats to Validity

Our examples are written entirely in Haskell, which is a lazy and purely functional language. Our results may not generalize to strict, imperative languages. Our example set is fairly small so we cannot guarantee that our analysis extends to a wide variety of generators and shrinkers in practice. We do not formally investigate the semantics of modified QuickCheck and falsify as we are only concerned with their runtime performance. However, a lack of formal semantics means that we do not know for sure that we have made the “correct” modifications to these libraries, and it is possible that an “incorrect” modification has skewed our measurements.

## 7 Related Work

*Property-Based Testing and Shrinking.* Property-based testing was popularized by the QuickCheck [3] framework for Haskell, which features a probabilistic eDSL for specifying generators. In QuickCheck, a shrinker for a type  $a$  is specified separately as a function  $\text{shrinks} :: a \rightarrow [a]$  from values to their immediate shrinks. In contrast, halcheck employs generator-based shrinking, where shrinkers are derived from annotated generator specifications.

Generator-based shrinking approaches were introduced by QuviQ QuickCheck [4] for Erlang. QuviQ QuickCheck implements generator-based shrinking using an approach known as *integrated shrinking*, where generators are represented as tree-valued random variables. Integrated shrinking is used by halcheck as well as many property-based testing frameworks in other languages such as Hedgehog [21] for Haskell and RapidCheck [8] for C++.

The hypothesis [14] library for Python implements a different approach to generator-based shrinking called *internal shrinking*, where generators are represented as random variables drawing from a space of infinite bit sequences, and shrinks are obtained by re-executing generators on modified bit sequences obtained via a set of built-in heuristics. While hypothesis often produces better shrinks than approaches based on integrated shrinking for programs with no user annotations, hypothesis provides no mechanism for user control over shrinking. This is less than ideal when the built-in heuristics are insufficient. The falsify [7] library for Haskell rectifies this downside while retaining the advantages of internal shrinking by allowing users to specify custom shrinking behaviour. Integrating the advantages of internal shrinking into halcheck is left as future work.

Reflective generators [10] extend normal generators with *partial monadic profunctor* operations. Goldstein et al. [10] show how this extension lends itself to a number of features, including the ability to run a generator “backwards” to obtain a random seed from a user-provided test case. Reflective generators also allow labelling random choices, however this labelling is optional and is intended for a different purpose (i.e., enabling “backwards” execution.) Nonetheless, the reflective generator framework does not mandate a particular interpretation, and so reflective generators may be run under a “halcheck-like” interpretation where labels affect a generator’s distribution.

The formal semantics of generator-based shrinking languages have not been explored in detail prior to this work and hence these languages lack sound, formally-specified, compositional rules for reasoning about combined generator and shrinker behaviour.

*Semantics of Probabilistic and Nondeterministic Programming Languages.* Shrinking can be viewed as a form of nondeterminism, and there is a large body of work addressing the combination of probabilistic and nondeterministic effects [1, 2, 9, 15, 23]. Various authors [9, 15, 23] observe that allowing probabilistic choice to distribute over nondeterministic choice, combined with the idempotence of probabilistic choice (rule  $\oplus\text{-IDEM}$ ), leads to unexpected behaviour. These observations bear resemblance to Theorem 3.6, however our distributivity law goes in the other direction (shrinking distributes over probabilistic choice due to rule  $\oplus\text{-}\eta$ ) and shrinking is not idempotent.

Dal Lago et al. [6] propose the Probabilistic Event Lambda-Calculus  $\Lambda_{PE}$  to achieve confluence in a probabilistic  $\lambda$ -calculi without committing to a call-by-value or call-by-name evaluation strategy.

The  $\Lambda_{PE}$  language decomposes probabilistic choice into a labelled choice operator  $t_1 \overset{\ell}{\oplus} t_2$  and a generator term  $\boxed{\ell}.t$ . While labelled choice behaves like  $t_1 \oplus_{[\ell] \sim 1/2} t_2$  in halcheck, the generator term has no direct analogue. Contextual equivalence over a finite number of labels  $t_1 \approx_{\{\ell_1, \dots, \ell_n\}} t_2$  appears expressible as  $\boxed{\ell_1} \dots \boxed{\ell_n}.t_1 \approx \boxed{\ell_1} \dots \boxed{\ell_n}.t_2$ . The semantics of  $\Lambda_{PE}$  is given in an operational style, while halcheck's semantics is denotational. Formally relating the two languages is left for future work.

## 8 Conclusion and Future Work

Existing languages that feature user-controlled generator-based shrinking lack sound compositional rules for reasoning about generator behaviour. In this paper, we showed that this problem is inherent to the combination of generation and shrinking. We then defined a new language, halcheck, which 1) requires users to manually label random choices and 2) features a complete and compositional (modulo some side conditions on labels) set of rules for reasoning about generator behaviour. We also showed that halcheck is able to express all existing Hedgehog programs. We implemented halcheck and measured, using a diverse set of examples, the practical impact of halcheck's labelling mechanism on program size and performance.

Our analysis has some important limitations. First, we do not consider higher-order generators, i.e., generators which produce other generators. Second, labels must be static, i.e., they cannot depend on another generator's behaviour. Lifting these restrictions is left as future work. One non-limitation is that we do not consider programs with general recursion or (pure) higher-order functions. Our results are easily generalized to include such constructs. One of our implementations of halcheck is a modification of falsify. However, this is not a "true" implementation, as falsify's semantics differs significantly with respect to shrinking. In particular, falsify's shrinking is "hierarchical", which essentially means that it allows backtracking (see Example 2.12 for non-backtracking behaviour). One future direction is to formally analyze the combination of hierarchical shrinking (as in falsify) and labelled probabilistic choice (as in halcheck). In particular, we aim to determine how to formally specify "hierarchical shrinking" and how a language with such a combination of effects can support compositional reasoning.

## Data Availability Statement

The source code for our implementation, examples, and experiments is available at <https://github.com/halcheck/halcheck-hs>.

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