Digital Image Processing

Image Enhancement

(Filtering in the **Frequency** Domain)

By: Dr. Hafeez

Contents

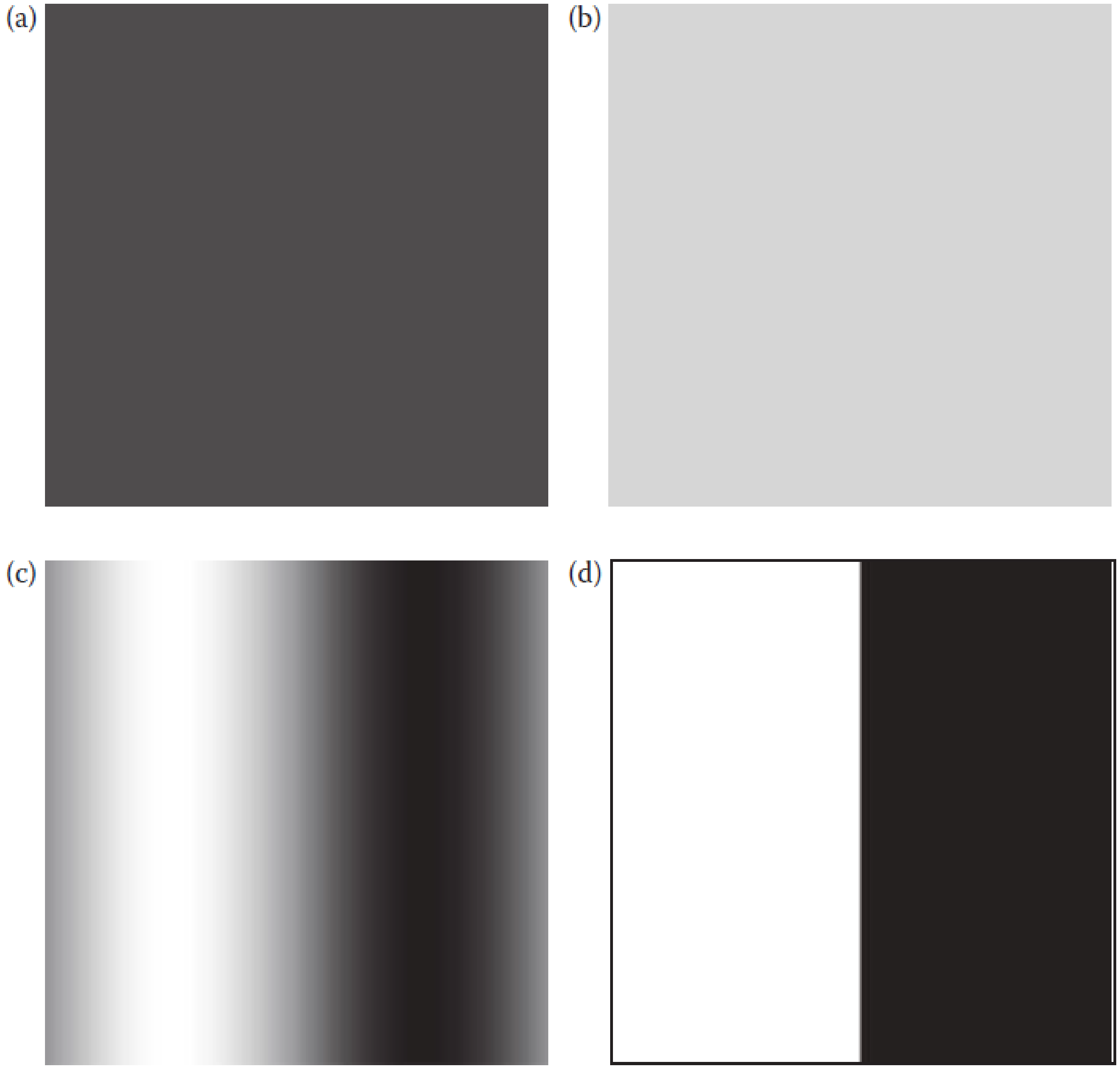
In this lecture we will look at image enhancement in the frequency domain

- Jean Baptiste Joseph Fourier
- The Fourier series & the Fourier transform
- **Filtering**: Image Processing in the frequency domain
  - Image smoothing
  - Image sharpening

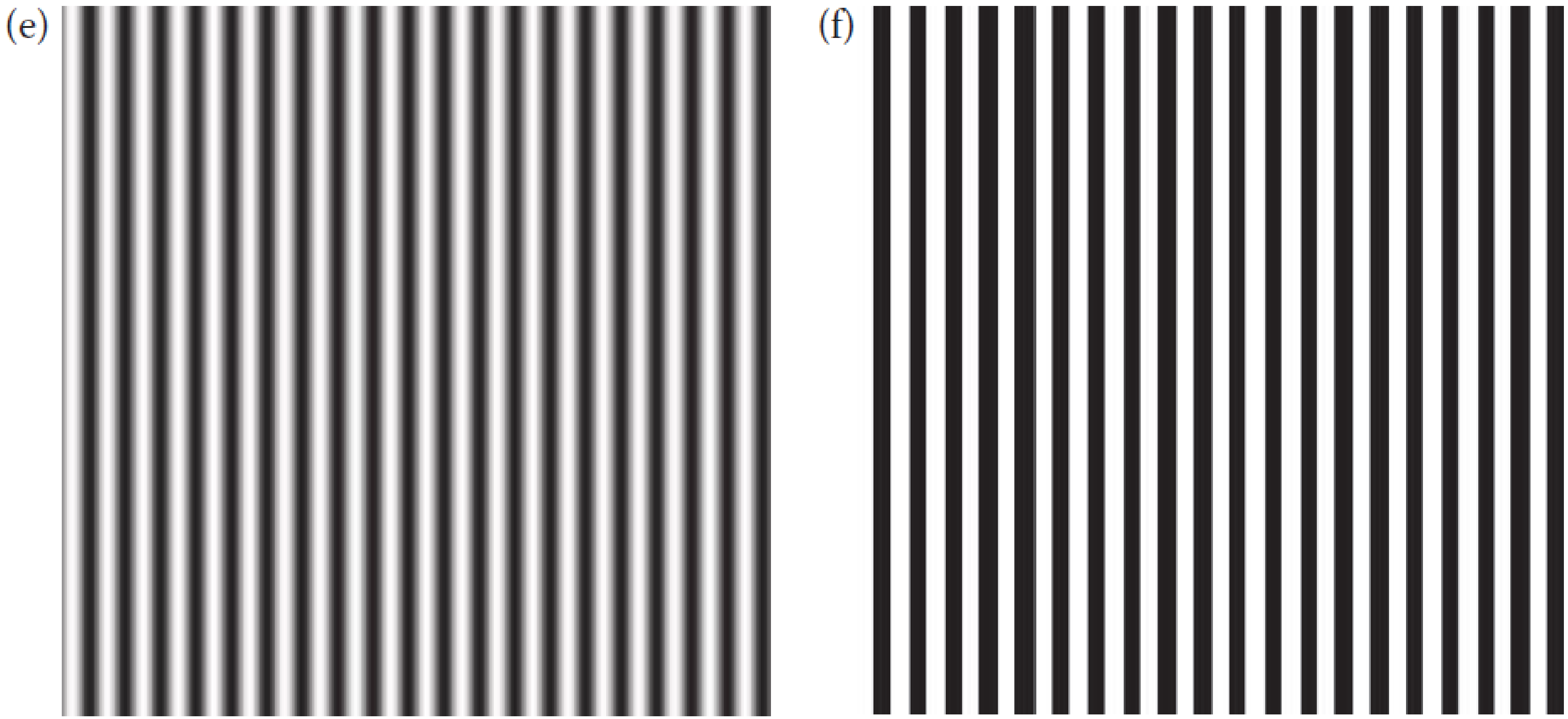
Jean Baptiste Joseph Fourier

- Fourier was born in Auxerre, France in 1768
  - Most famous for his work "*La Théorie Analitique de la Chaleur*" published in 1822
  - Translated into English in 1878: "*The Analytic Theory of Heat*"
- Nobody paid much attention when the work was first published
- One of the most important mathematical theories in modern engineering

Images have frequencies…really?

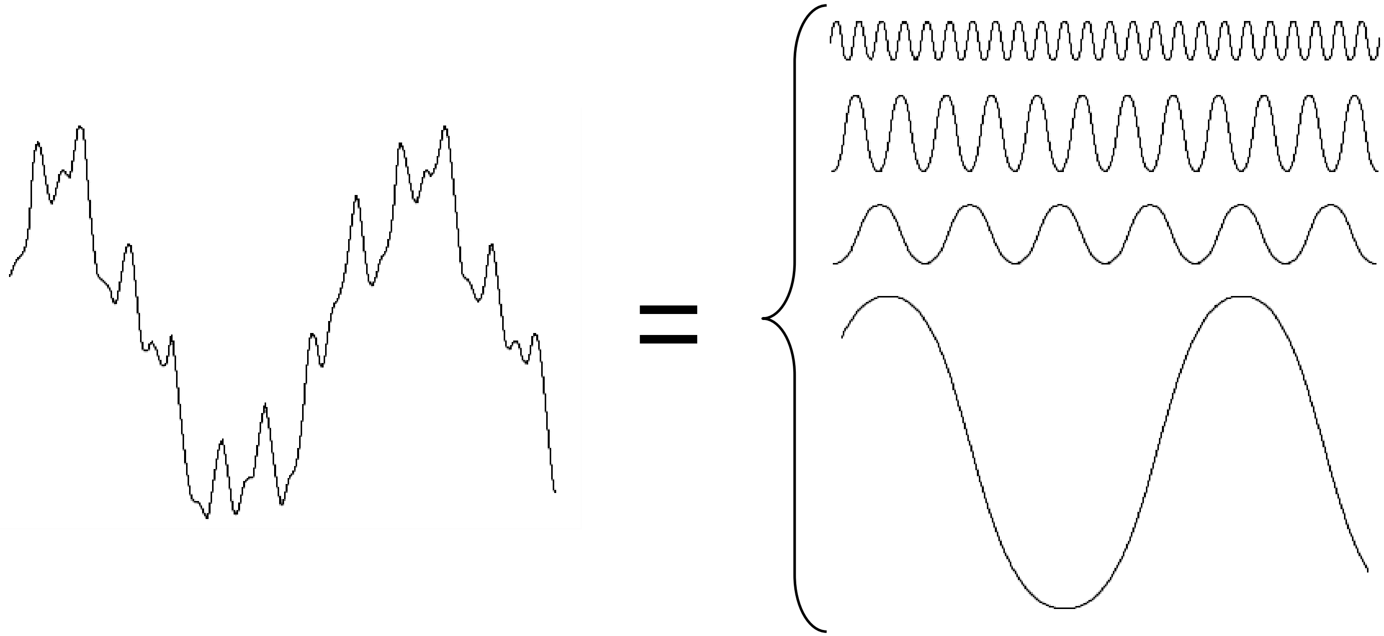


Images have frequencies…really?



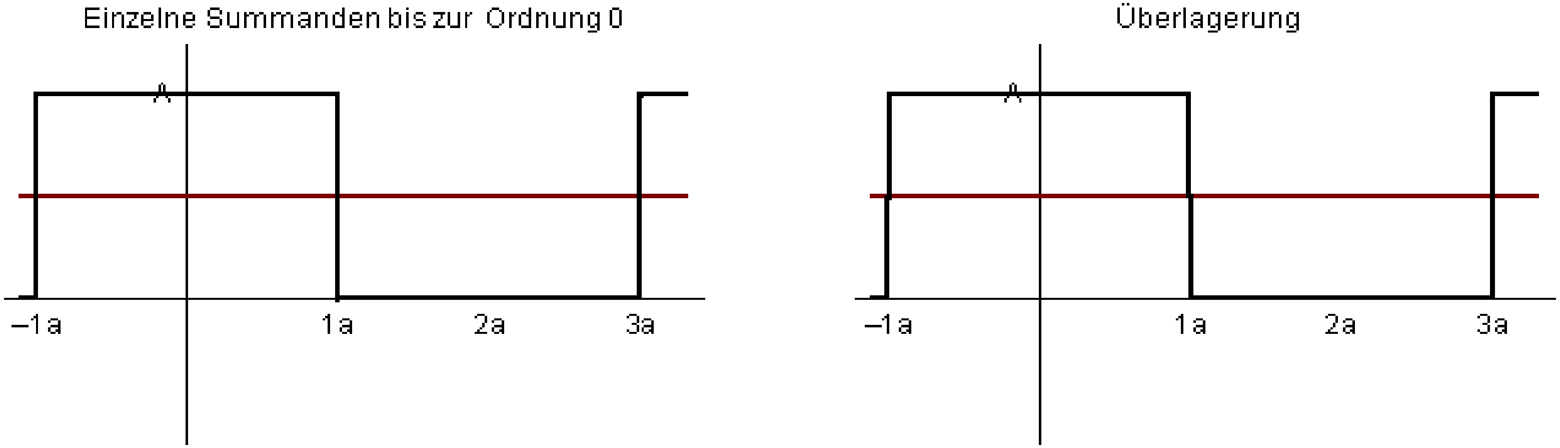
Spatial frequency. (a) frequency = 0, gray level = 51, (b) frequency = 0, gray level = 204, (c) frequency = 1, horizontal sine wave, (d) frequency = 1, horizontal square wave, (e) frequency = 20, horizontal sine wave, (f) frequency = 20, horizontal square wave.

The Big Idea



Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

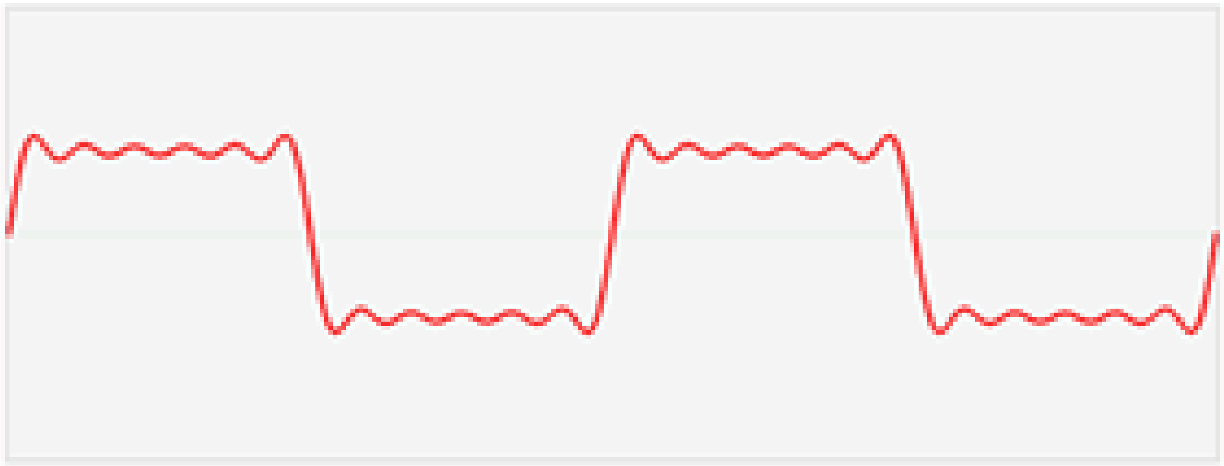
The Big Idea (cont…)



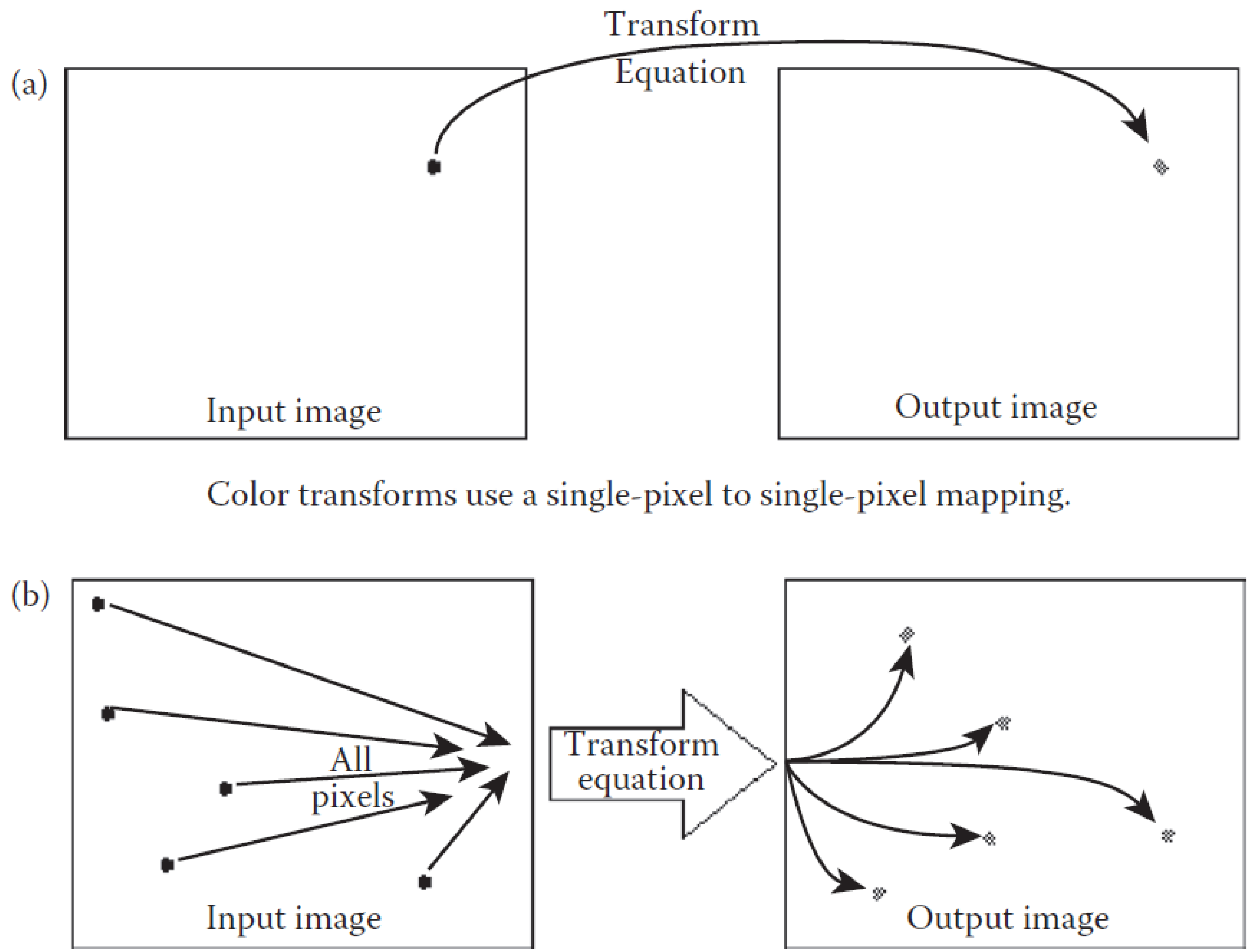
Notice how we get closer and closer to the original function as we add more and more frequencies

Taken from www.tfh-berlin.de/~schwenk/hobby/fourier/Welcome.html

The Big Idea (cont…)



Frequency Transformations



All pixels in the input image contribute to each value in the output image for frequency transforms.

Frequency Transformations

The general form of the transformation equation, assuming an $M \times N$ image, is given by

$$T(u,v) = k \sum_{r=0}^{M-1} \sum_{c=0}^{N-1} I(r,c) B(r,c;u,v)$$

Here, u and v are the frequency domain variables, k is a constant that is transform dependent, $T(u,v)$ are the transform coefficients, and $B(r,c;u,v)$ correspond to the basis images. The notation $B(r,c;u,v)$ defines a set of basis images, corresponding to each different value for u and v, and the size of each is r by c (Figure 5.1-4). The transform coefficients, $T(u,v)$, are the projections of $I(r,c)$ onto each $B(u,v)$.

Example Basis Images

$$\text{Let } I(r,c) = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\text{And let } B(u,v,r,c) = \left\{ \begin{matrix} \begin{bmatrix} +1 & +1 \\ +1 & +1 \end{bmatrix} \begin{bmatrix} +1 & -1 \\ +1 & -1 \end{bmatrix} \\ \begin{bmatrix} +1 & +1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \end{matrix} \right.$$

1D Fourier Example

Example 5.2.1

Given the simple rectangle function shown in Figure 5.2-2a, we can find the Fourier transform by applying the equation defined above:

$$F(v) = \int_{-\infty}^{\infty} I(c) e^{-j2\pi vc} dc$$

$$= \int_{0}^{C} A e^{-j2\pi vc} dc$$

$$= -\frac{-A}{j2\pi v}\left[e^{-j2\pi vc}\right]_0^C = \frac{-A}{j2\pi v}\left[e^{-j2\pi vC} - 1\right]$$

$$= \frac{A}{j2\pi v}\left[e^{j\pi vC} - e^{-j\pi vC}\right] e^{-j\pi vC}$$

then we use the trigonometric identity, $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$

$$= \frac{A}{\pi v}\sin(\pi vC) e^{-j\pi vC}$$

This result is complex function, and here we are interested in the magnitude (defined in the next section), which is

$$|F(v)| = \left|\frac{A}{\pi v}\right| |\sin(\pi vC)| \left|e^{-j\pi vC}\right|$$

Now we multiply through by C/C, and the magnitude of $e^{-j\pi vC} = 1$,
we can get it in the form of a sinc function:

$$= AC\left|\frac{\sin(\pi vC)}{(\pi vC)}\right| = AC|\mathrm{sinc}(vC)|$$

1D Fourier Example

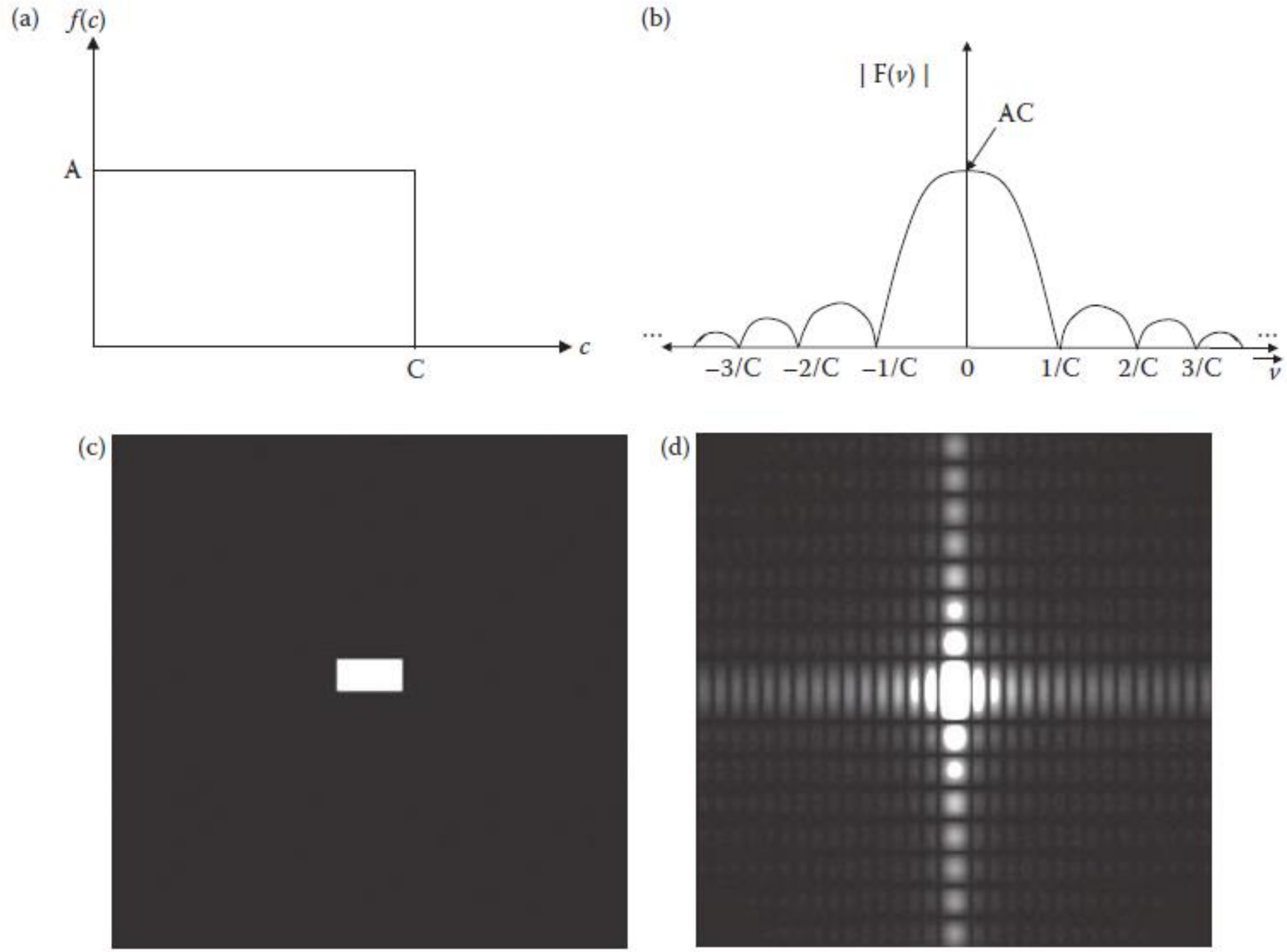


FIGURE 5.2-2
Fourier transform example. (a) The one-dimensional rectangle function, (b) the magnitude of Fourier transform of the 1-D rectangle function: $|F(v)| = AC|\sin(\pi vC/\pi vC)| = AC|\text{sinc}(vC)|$ (c) Two-dimensional rectangle function as an image, (d) the magnitude of the Fourier transform, called the Fourier spectrum, of the 2-D rectangle.

The Discrete Fourier Transform (DFT)

The *Discrete Fourier Transform* of $f(x, y)$, for x = 0, 1, 2…M-1 and y = 0,1,2…N-1, denoted by $F(u, v)$, is given by the equation:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

for u = 0, 1, 2…M-1 and v = 0, 1, 2…N-1.

In this case the basis functions ($e^{j\theta}$) are sinusoidal (as by Euler's Formula): $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

Sinusoidal help LINK: http://www.regentsprep.org/regents/math/algtrig/att7/sinusoidal.htm

2D DFT

Each DFT coefficient is a complex value

- There is a single DFT coefficient for each spatial sample
- A complex value is expressed by two real values in either Cartesian or polar coordinate space.
    - Cartesian: R(u,v) is the *real* and I(u, v) the *imaginary* component

$$\mathcal{F}(u,v) = R(u,v) + jI(u,v)$$

- Polar: |F(u,v)| is the ***magnitude*** and phi(u,v) the ***phase***

$$\mathcal{F}(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

2D DFT

Representing the DFT coefficients as magnitude and phase is a more useful for processing and reasoning.

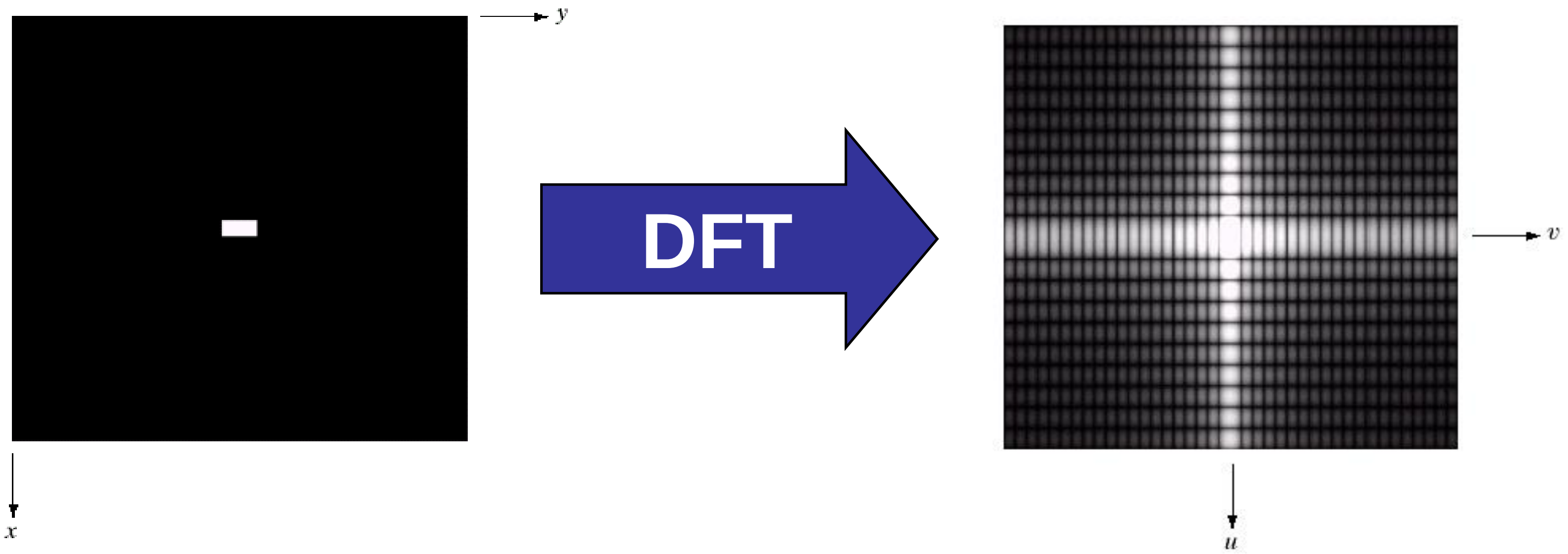
- The magnitude is a measure of strength or length
- The phase is a direction and lies in [-pi, +pi]

The **magnitude** and **phase** are easily obtained from the real and imaginary values

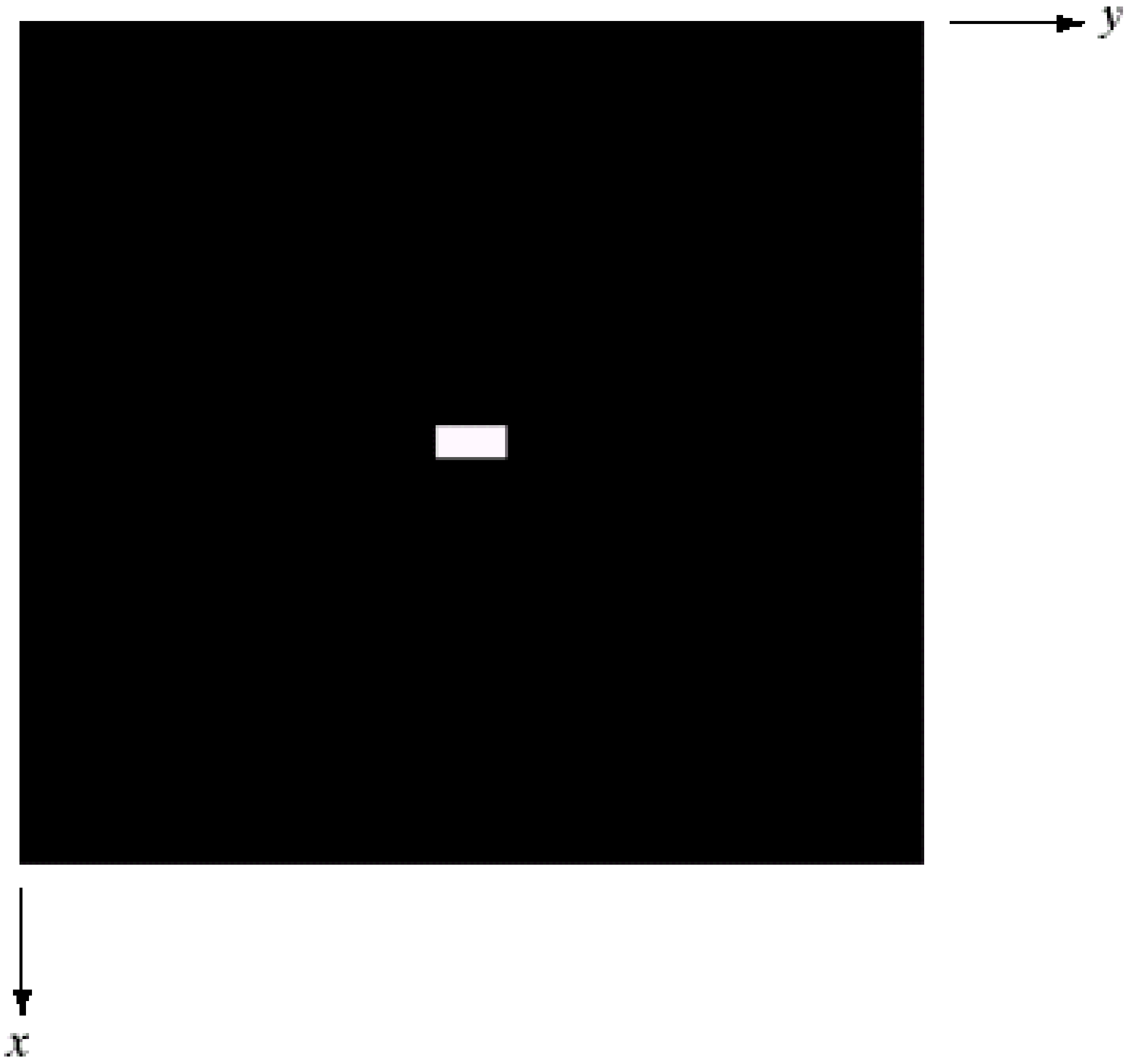
$$
\begin{aligned}
|\mathcal{F}(u,v)| &= \sqrt{R^2(u,v) + I^2(u,v)} \\
\phi(u,v) &= \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right].
\end{aligned}
$$

DFT & Images

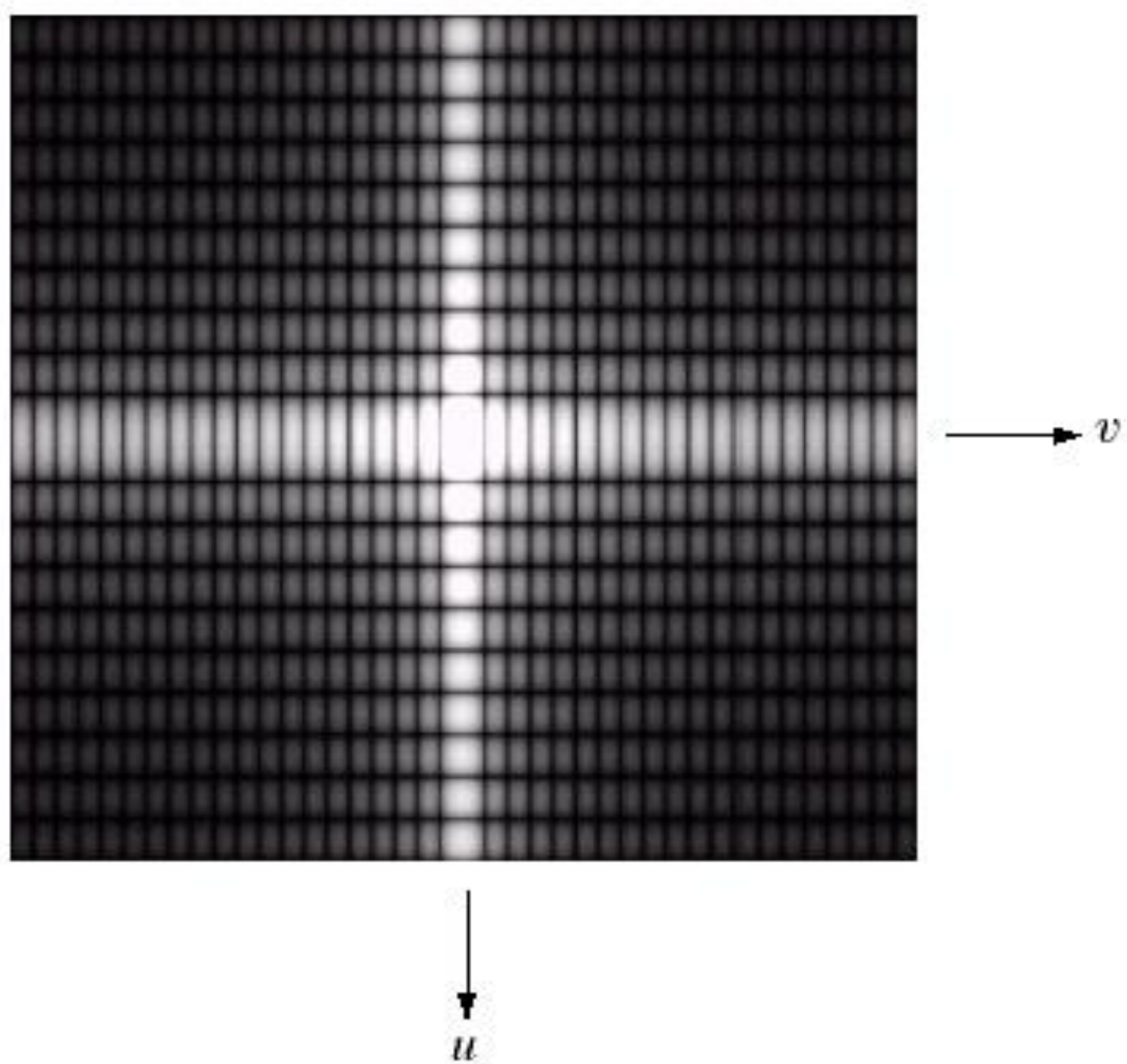
The DFT of a two dimensional image can be visualised by showing the spectrum of the images component frequencies



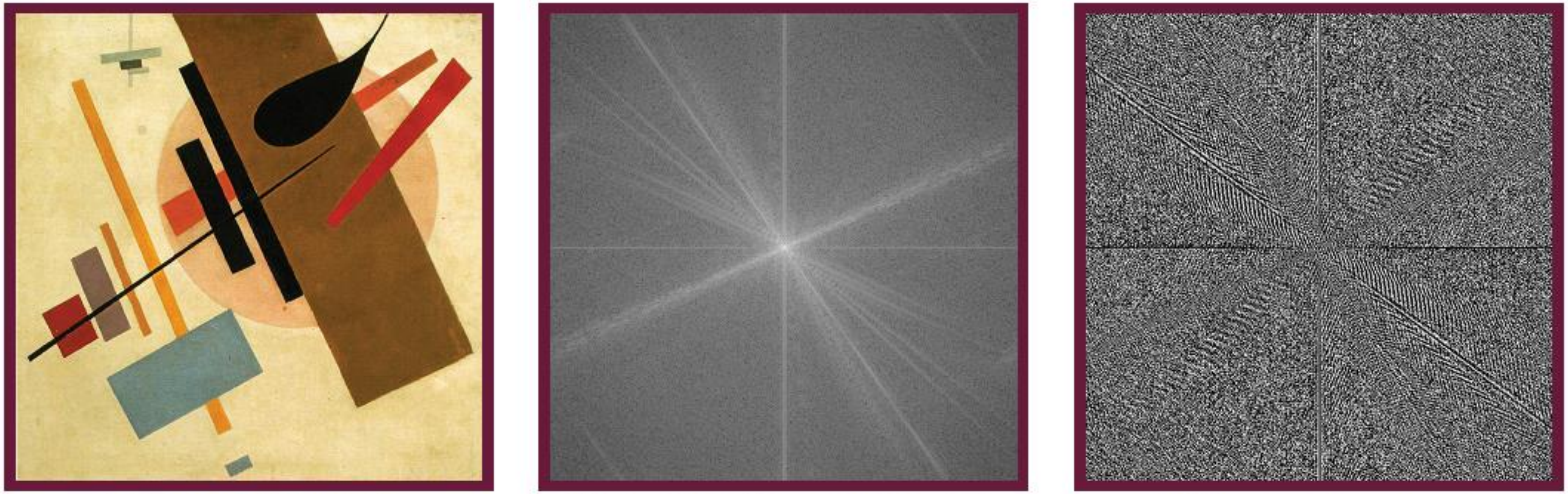
DFT & Images



DFT & Images



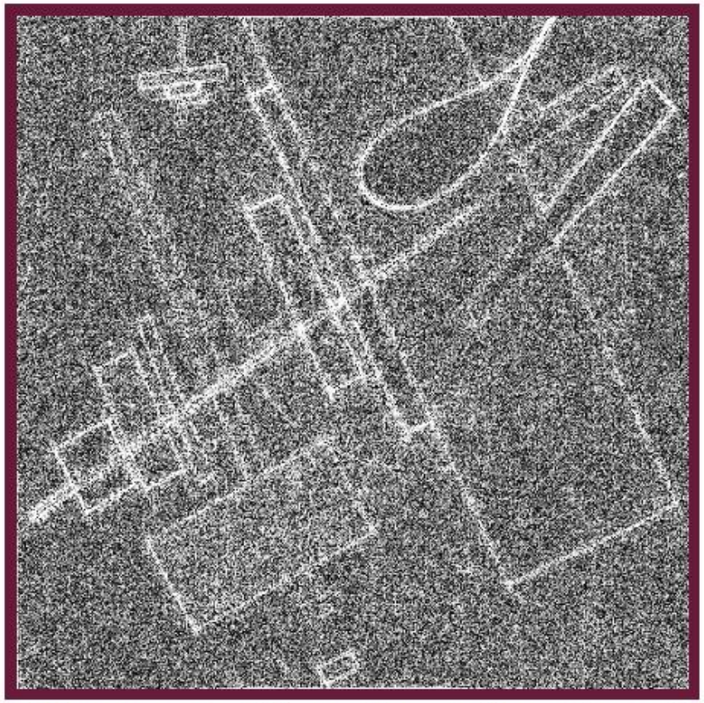
Magnitude Spectrum and Phase Spectrum



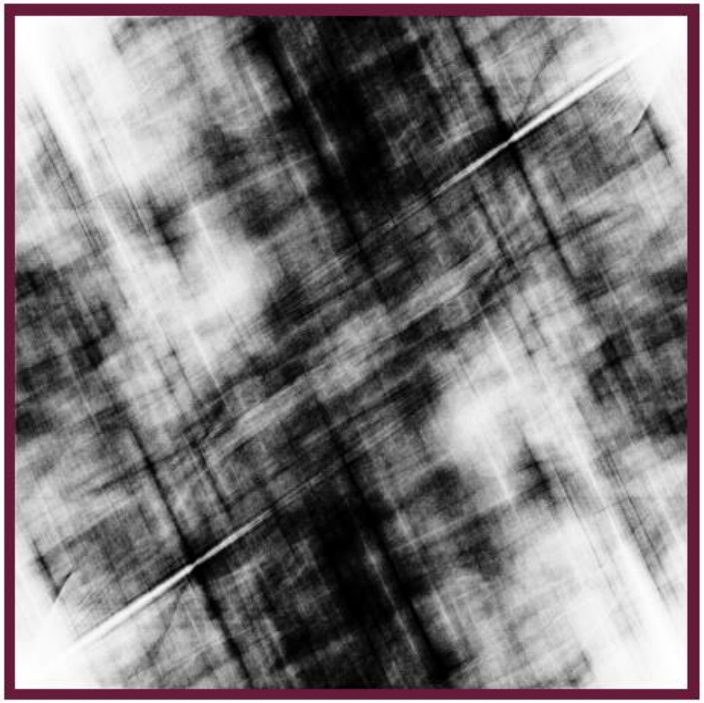
(a) Source image f. (b) Amplitude spectrum, $|\mathcal{F}|$. (c) Phase spectrum, ϕ.

Figure 9.7. DFT Spectrum.

Magnitude Spectrum and Phase Spectrum



(a) Reconstructed from phase information only.



(b) Reconstruction from amplitude information only.

Figure 9.8. Comparison of the contribution of the amplitude and phase spectrum.

Magnitude Spectrum and Phase Spectrum

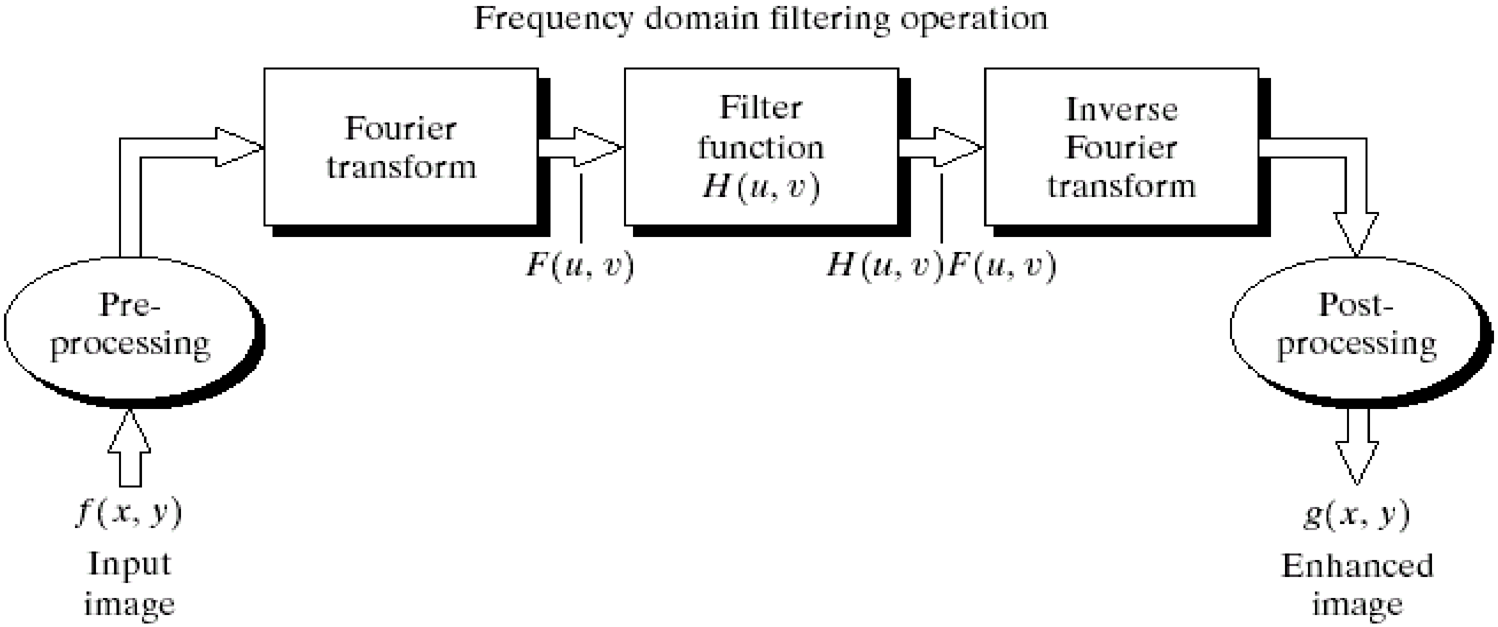
Notes on the magnitude spectrum:

- Magnitudes are generally referred to as the "spectrum" but this should be understood as the magnitude spectrum.
- Typically has an extremely large dynamic range and it is typical to log-compress those values for display (as in the previous slide)
- For presentation, the DC component i.e., F(0,0), is placed at the center. Low frequency components are shown near the center and frequency increases with distance from center.

The DFT and Image Processing

To filter an image in the frequency domain:

1. Compute $F(u,v)$ the DFT of the image
2. Multiply $F(u,v)$ by a filter function $H(u,v)$
3. Compute the inverse DFT of the result



The Inverse DFT

It is really important to note that the Fourier transform is completely **reversible**

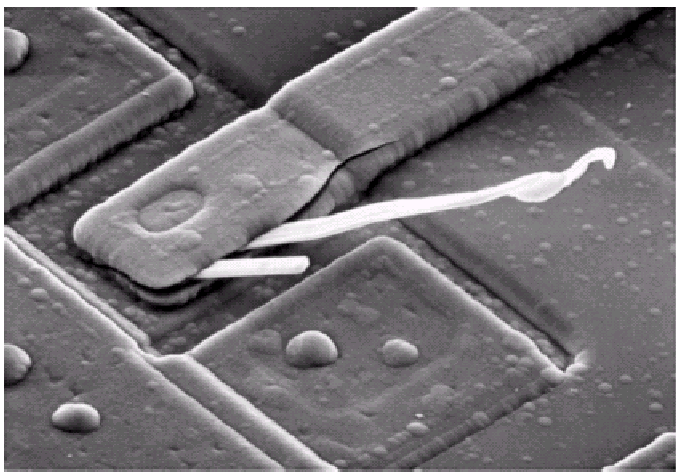
The inverse DFT is given by:

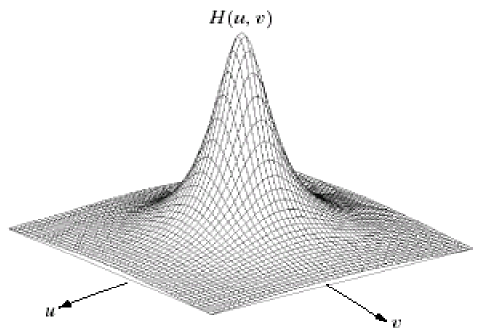
$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

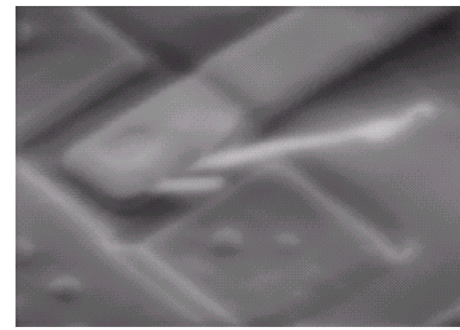
for x = 0, 1, 2…M-1 and y = 0, 1, 2…N-1

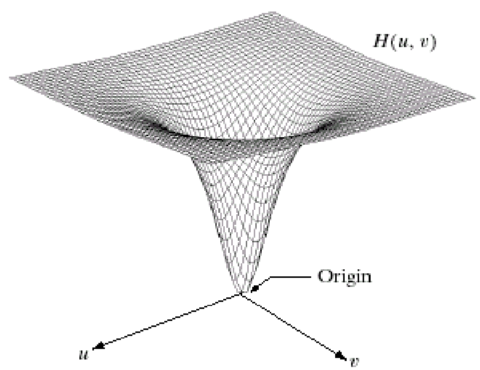
Some Basic Frequency Domain Filters

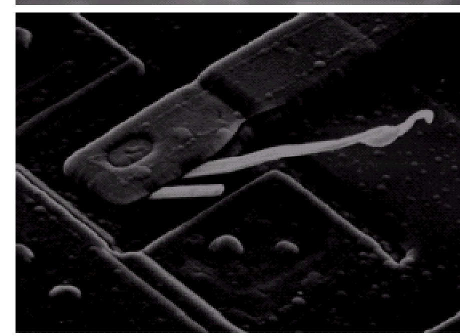
Low Pass Filter











High Pass Filter

Smoothing Frequency Domain Filters

Smoothing is achieved in the frequency domain by dropping out the high frequency components

The basic model for filtering is:

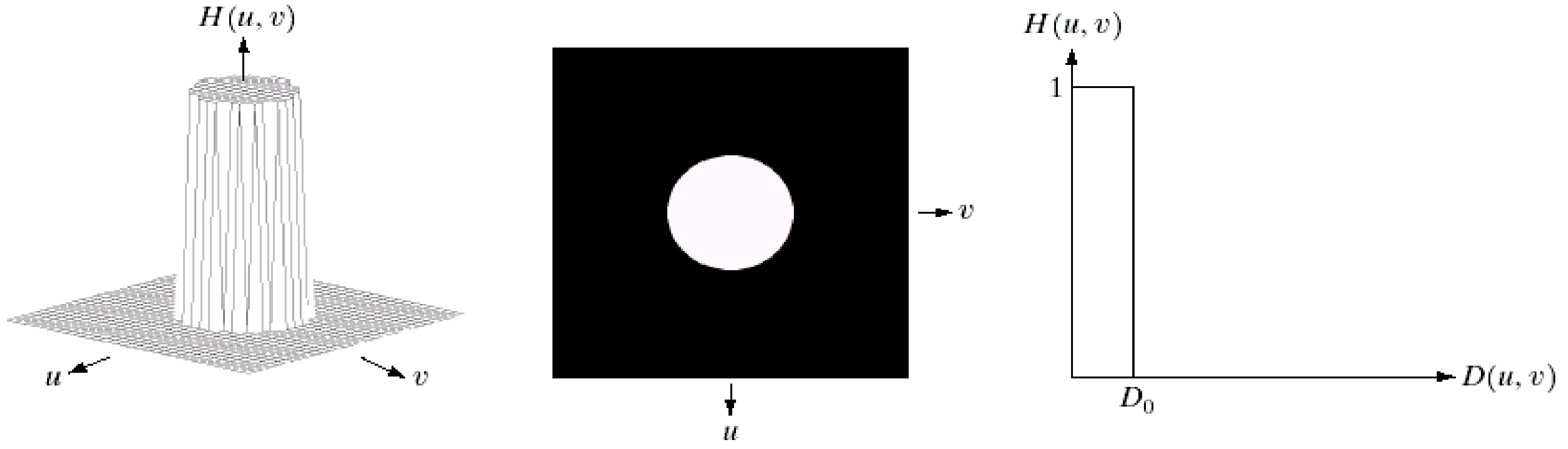
$$G(u,v) = H(u,v) \times F(u,v)$$

where $F(u,v)$ is the Fourier transform of the image being filtered and $H(u,v)$ is the filter transform function

Low pass filters – only pass the low frequencies, drop the high ones

Ideal Low Pass Filter

Simply cut off all high frequency components that are a specified distance D_0 from the origin of the transform



changing the distance changes the behaviour of the filter

Ideal Low Pass Filter (cont…)

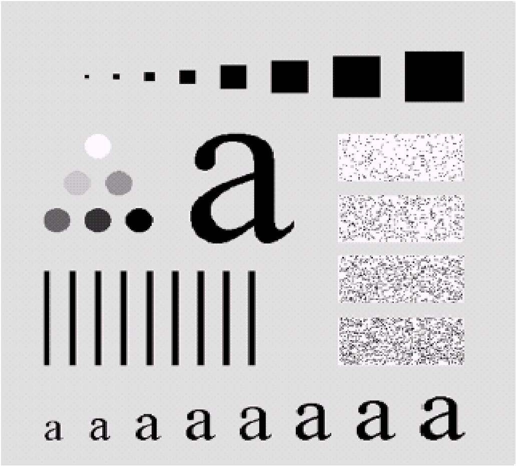
The transfer function for the ideal low pass filter can be given as:

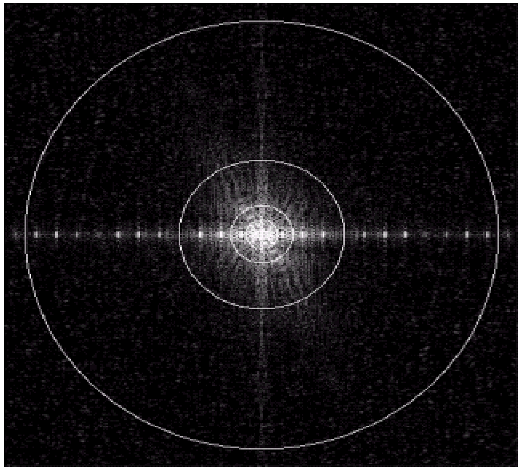
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

where $D(u,v)$ is given as:

$$D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$$

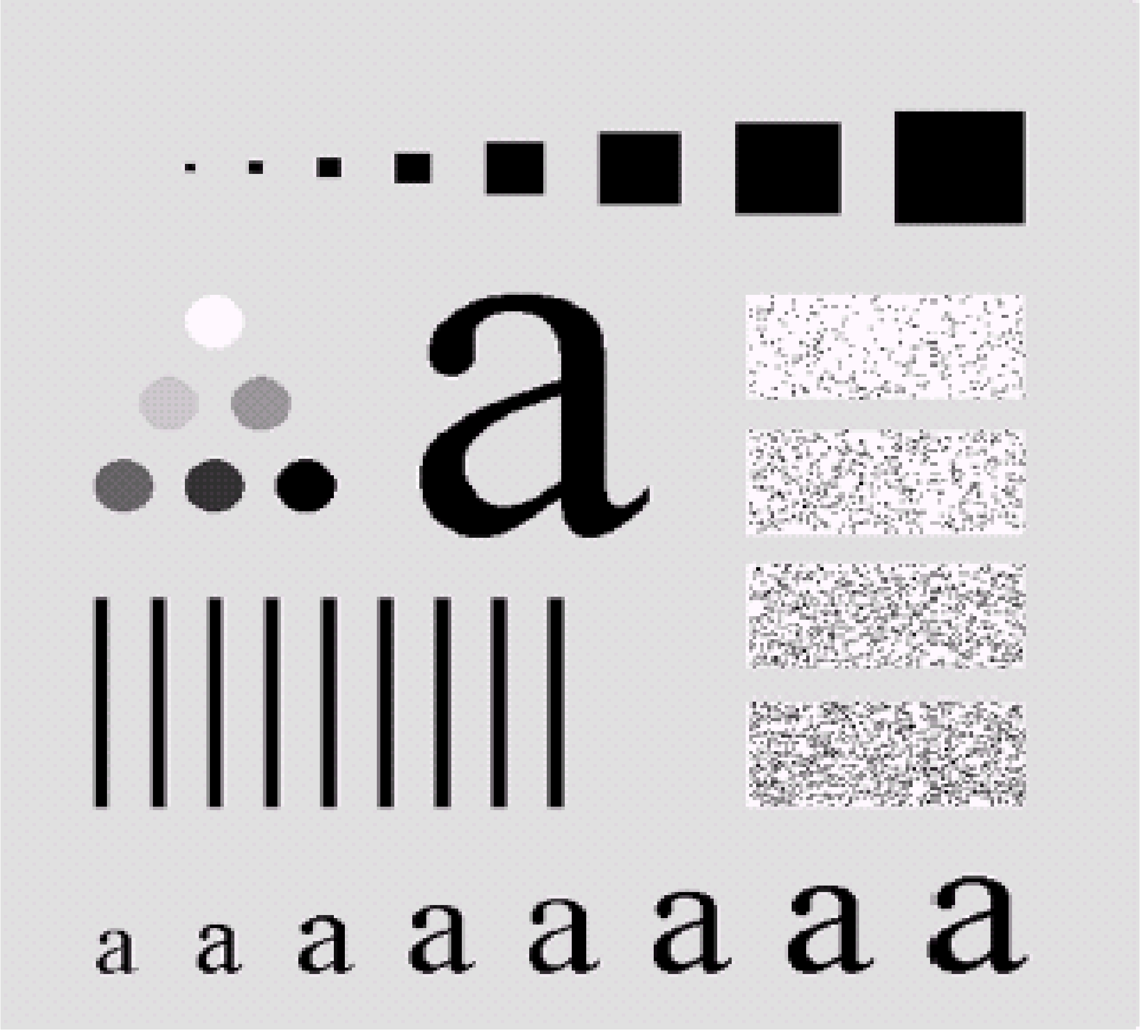
Ideal Low Pass Filter (cont…)



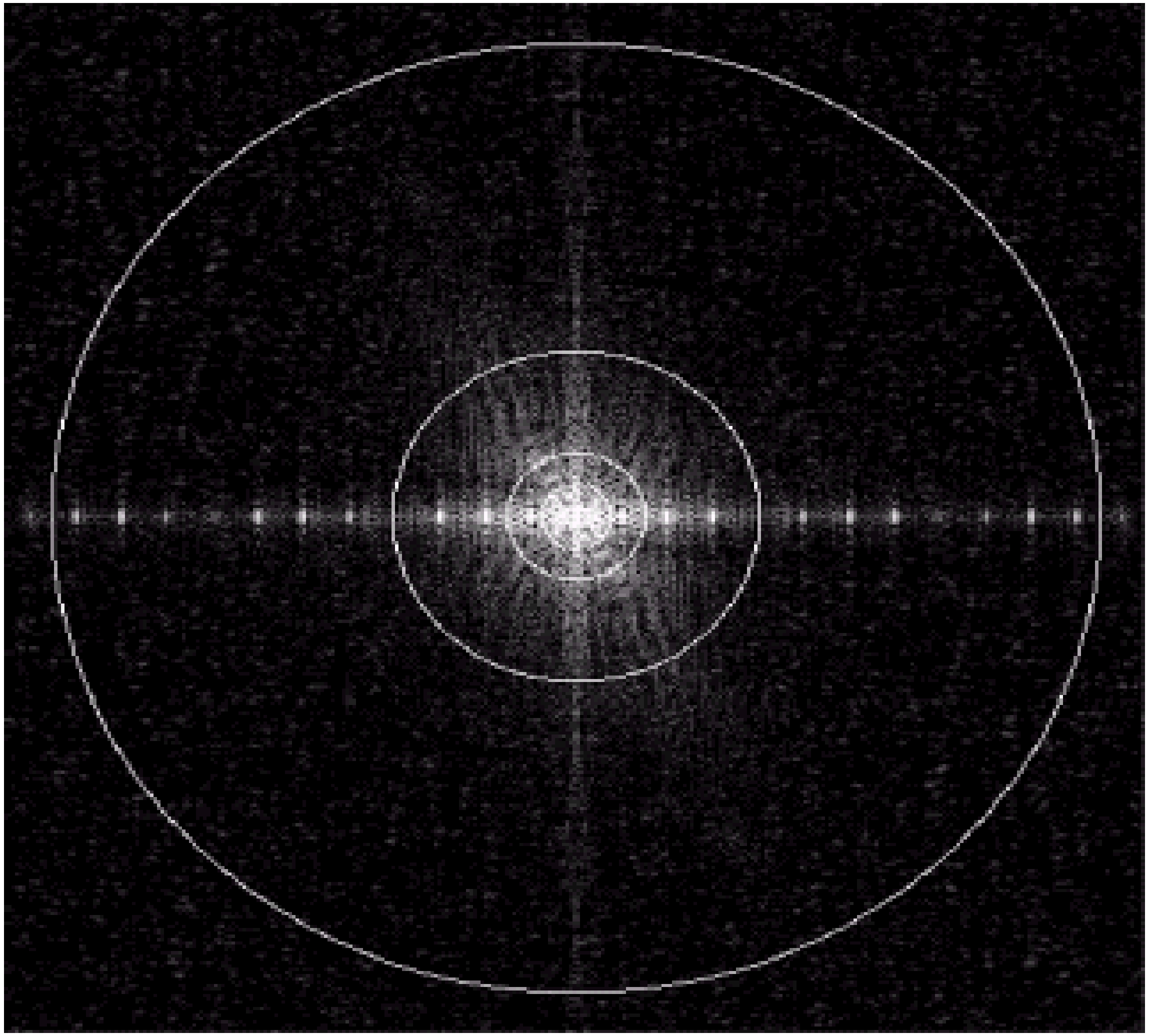


Above we show an image, it's Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it

Ideal Low Pass Filter (cont…)



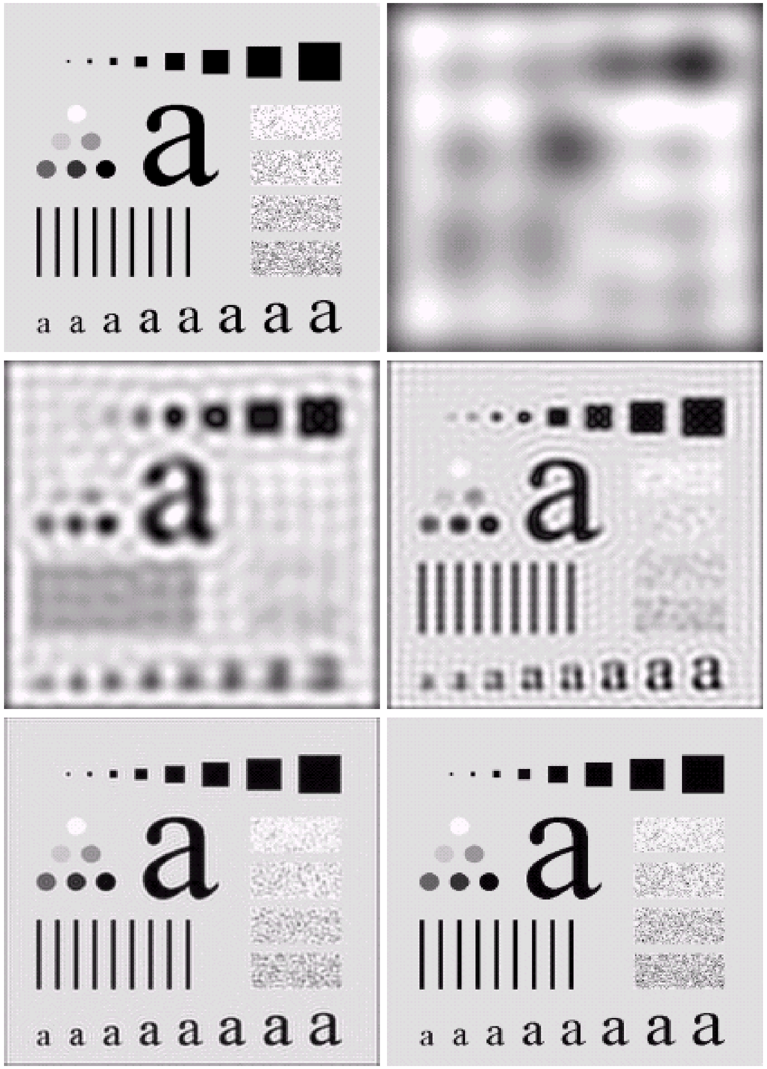
Ideal Low Pass Filter (cont…)



Ideal Low Pass Filter (cont…)

Original image

Result of filtering with ideal low pass filter of radius 5



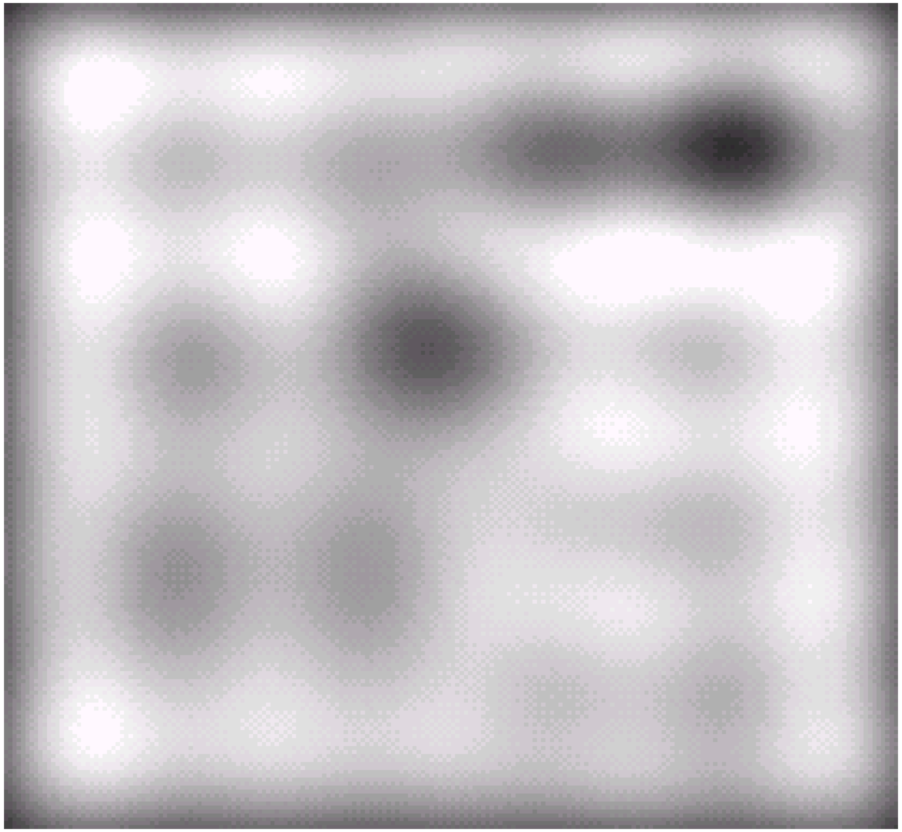
Result of filtering with ideal low pass filter of radius 15

Result of filtering with ideal low pass filter of radius 30

Result of filtering with ideal low pass filter of radius 80

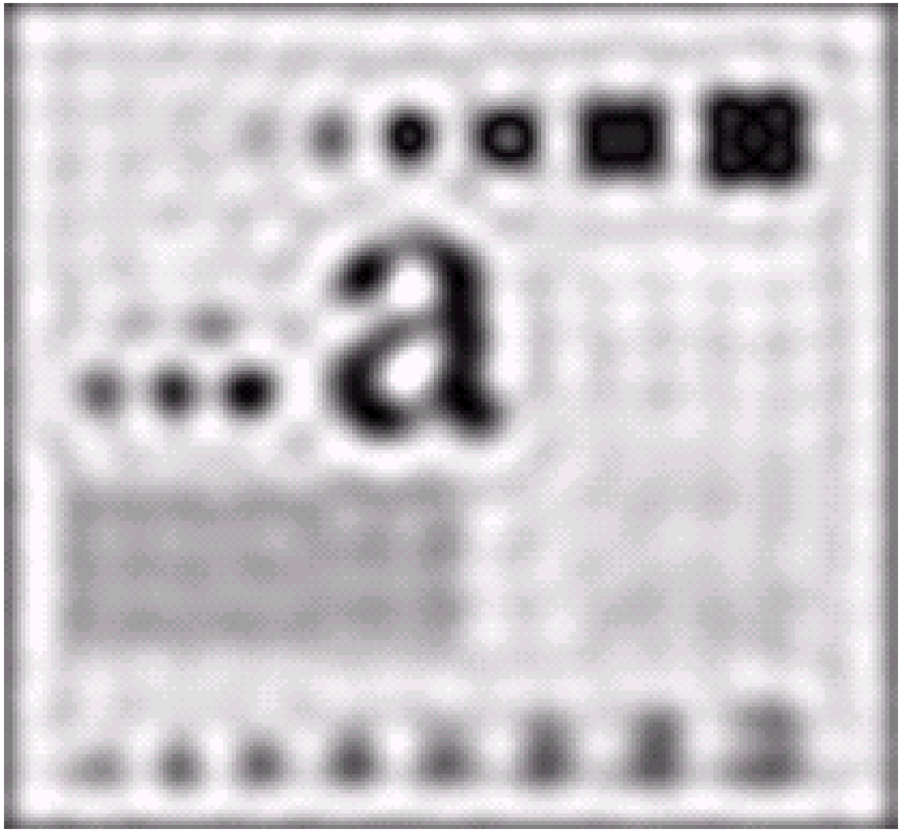
Result of filtering with ideal low pass filter of radius 230

Ideal Low Pass Filter (cont…)



Result of filtering with ideal low pass filter of radius 5

Ideal Low Pass Filter (cont…)



Result of filtering with ideal low pass filter of radius 15

Sharpening in the Frequency Domain

Edges and fine detail in images are associated with high frequency components

High pass filters – only pass the high frequencies, drop the low ones

High pass frequencies are precisely the reverse of low pass filters, so:

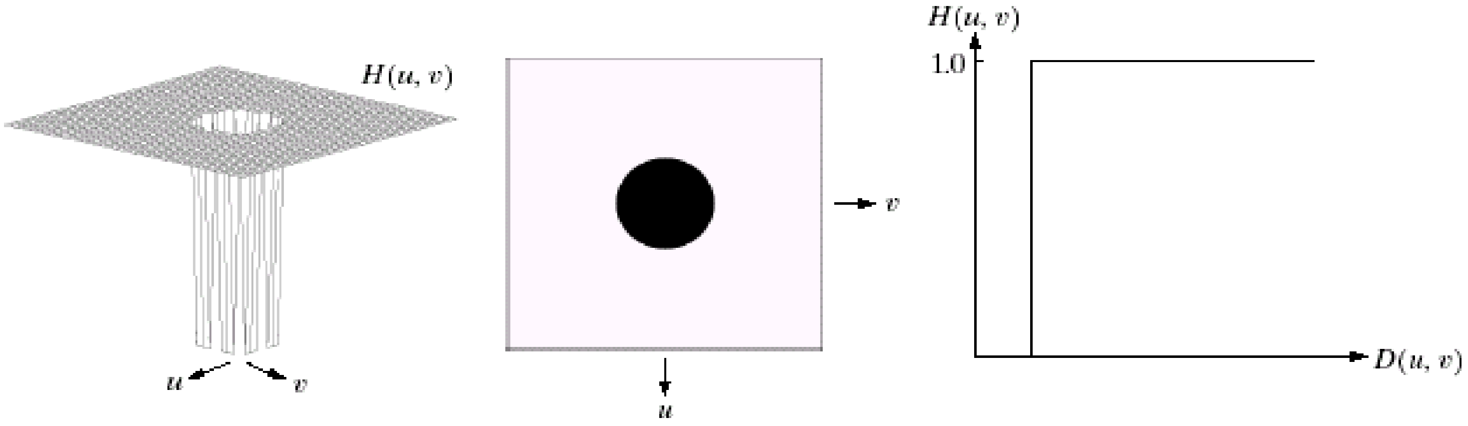
$$H_{hp}(u, v) = 1 - H_{lp}(u, v)$$

Ideal High Pass Filters

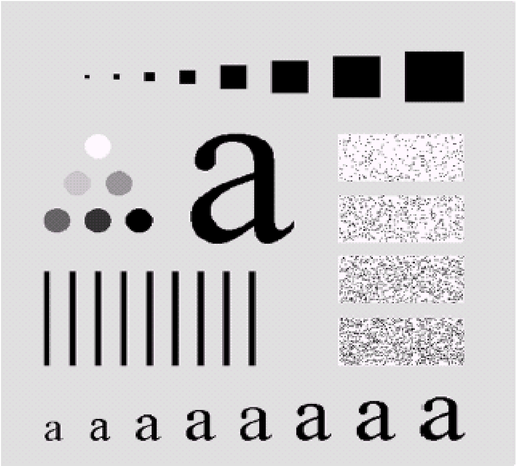
The ideal high pass filter is given as:

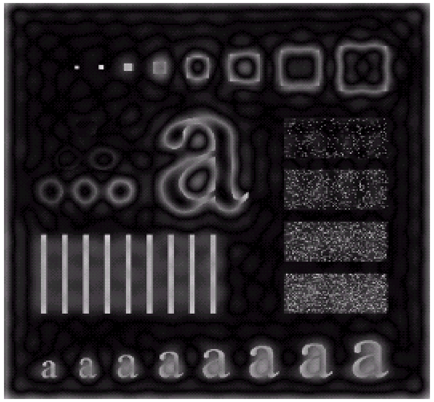
$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

where D_0 is the cut off distance as before

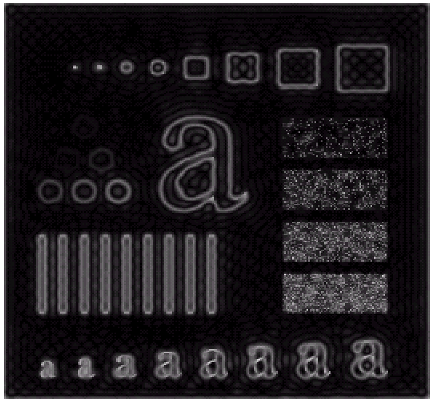


Ideal High Pass Filters (cont…)

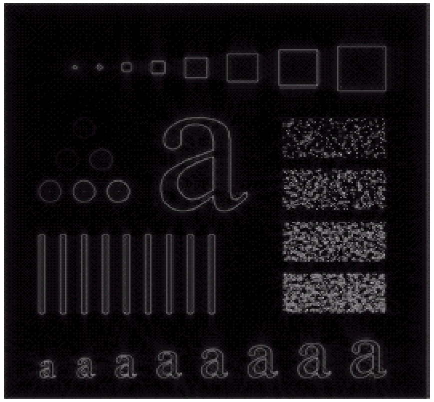




Results of ideal high pass filtering with D_0 = 15



Results of ideal high pass filtering with D_0 = 30



Results of ideal high pass filtering with D_0 = 80

Summary

In this lecture we examined image enhancement in the frequency domain

- The Fourier series & the Fourier transform
- Image Processing in the frequency domain
  - Image smoothing
  - Image sharpening

Next time we will begin to examine image restoration using the spatial and frequency based techniques we have been looking at