General Discussion

The Gauss – Seidel Method

Recall Jacobi's Method in which we have used the vector $\vec{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)})$ for finding $\vec{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)})$ by substituting $k=1$ in the scheme. For finding $\vec{x}^{(3)}$ we have used $\vec{x}^{(2)}$ and so on.

- While calculating $x_2^{(1)}$ we have already calculated $x_1^{(1)}$ so it is good to use this updated value rather than using $x_1^{(0)}$ in calculation.
- Similarly while calculating $x_3^{(1)}$ we have already calculated $x_1^{(1)}, x_2^{(1)}$ so it is good to use these updated values rather than using $x_1^{(0)}, x_2^{(0)}$ in calculation of finding $x_3^{(1)}, x_4^{(1)}$.

Problem

Use the Gauss-Seidel Iterative technique to find approximate solutions to

$$\boxed{\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}}$$

Starting with $\vec{x} = (0,0,0,0)^t$ and iterating until

$$\frac{\|\vec{x}^{(k)} - \vec{x}^{(k-1)}\|}{\|\vec{x}^{(k)}\|_\infty} < 10^{-3}.$$

Stopping Criteria

Sol: Eq:1 $\quad 10x_1 - x_2 + 2x_3 = 6$

$$\Rightarrow \boxed{x_1 = \frac{1}{10}(x_2 - 2x_3 + 6)} = \boxed{\frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}}$$

Eq:2 $\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25$

$$\Rightarrow x_2 = \frac{1}{11}(x_1 + x_3 - 3x_4 + 25)$$

$$\Rightarrow \boxed{x_2 = \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}}$$

Eq:3 $\quad 2x_1 - x_2 + 10x_3 - x_4 = -11$

$$\Rightarrow x_3 = \frac{1}{10}(-2x_1 + x_2 + x_4 - 11)$$

$$\Rightarrow \boxed{x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}}$$

Eq: 4 $\quad 3x_2 - x_3 + 8x_4 = 15$

$$\Rightarrow x_4 = \frac{1}{8}(-3x_2 + x_3 + 15)$$

$$\Rightarrow \boxed{x_4 = -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}}$$

So

$$x_1 = \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}$$

$$x_2 = \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}$$

$$x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}$$

$$x_4 = -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}$$

Scheme of Gauss-Seidel Method is:

$\boxed{K \geq 1}$

$$x_1^{(K)} = \frac{1}{10}x_2^{(K-1)} - \frac{1}{5}x_3^{(K-1)} + \frac{3}{5}$$

$$x_2^{(K)} = \frac{1}{11}x_1^{(K)} + \frac{1}{11}x_3^{(K-1)} - \frac{3}{11}x_4^{(K-1)} + \frac{25}{11} \quad (\star\star)$$

$$x_3^{(K)} = -\frac{1}{5}x_1^{(K)} + \frac{1}{10}x_2^{(K)} + \frac{1}{10}x_4^{(K-1)} - \frac{11}{10}$$

$$x_4^{(K)} = -\frac{3}{8}x_2^{(K)} + \frac{1}{8}x_3^{(K)} + \frac{15}{8}$$

Given Initial Guess $\boxed{\vec{x}^{(0)} = (0,0,0,0)^t}$. By Comparing with $\vec{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)})$ we get

$x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$ and $x_4^{(0)} = 0$

Substituting $k=1$ in $(**)$ we get

$$x_1^{(1)} = \frac{1}{10} x_2^{(0)} - \frac{1}{5} x_3^{(0)} + \frac{3}{5}$$

$$x_2^{(1)} = \frac{1}{11} x_1^{(1)} + \frac{1}{11} x_3^{(0)} - \frac{3}{11} x_4^{(0)} + \frac{25}{11}$$

$$x_3^{(1)} = -\frac{1}{5} x_1^{(1)} + \frac{1}{10} x_2^{(1)} + \frac{1}{10} x_4^{(0)} - \frac{11}{10}$$

$$x_4^{(1)} = -\frac{3}{8} x_2^{(1)} + \frac{1}{8} x_3^{(1)} + \frac{15}{8}$$

Now substituting values in formula for $x_1^{(1)}$ we get

$$x_1^{(1)} = \frac{1}{10}(0) - \frac{1}{5}(0) + \frac{3}{5} = \frac{3}{5} = 0.6$$

So $\boxed{x_1^{(1)} = 0.6}$

Next

$$x_2^{(1)} = \frac{1}{11}(0.6) + \frac{1}{11}(0) - \frac{3}{11}(0) + \frac{25}{11} = 2.3273$$

So $\boxed{x_2^{(1)} = 2.3273}$

Next $x_3^{(1)} = -\frac{1}{5}(0.6) + \frac{1}{10}(2.3273) + \frac{1}{10}(0) - \frac{11}{10}$

$\boxed{x_3^{(1)} = -0.98727}$

Finally

$$x_4^{(1)} = \frac{-3}{8}(2.3273) + \frac{1}{8}(-0.98727) + \frac{15}{8}$$

$$\boxed{x_4^{(1)} = 0.87885}$$

So

$$\vec{x}^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\right)$$

$$\Rightarrow \boxed{\vec{x}^{(1)} = (0.6, 2.3273, -0.98727, 0.87885)^t}$$

1st Approximation.

Now given stopping criteria is to iterate until

$$\frac{\| \vec{x}^{(K)} - \vec{x}^{(K-1)} \|_\infty}{\| \vec{x}^{(K)} \|_\infty} < 10^{-3} \text{ is True.}$$

Now for K=1, we have

$$\frac{\| \vec{x}^{(1)} - \vec{x}^{(0)} \|_\infty}{\| \vec{x}^{(1)} \|_\infty} < 10^{-3}$$

To check above, let's calculate $\|\vec{x}^{(1)}\|_\infty$ and then $\|\vec{x}^{(1)} - \vec{x}^{(0)}\|_\infty$.

Now

$$\|\vec{x}^{(1)}\|_\infty = \max\{|0.6|, |2.3273|, |-0.98727|, |0.87885|\}$$

$$= \boxed{2.3273}$$

and

$$\|\vec{x}^{(1)} - \vec{x}^{(0)}\|_\infty = \max\left\{ |x_1^{(1)} - x_1^{(0)}|, |x_2^{(1)} - x_2^{(0)}|, |x_3^{(1)} - x_3^{(0)}|, |x_4^{(1)} - x_4^{(0)}| \right\}$$

$$= \max\left\{ |0.6 - 0|, |2.3273 - 0|, |-0.98727 - 0|, |0.87885 - 0| \right\}$$

$$= \max\left\{ 0.6, 2.3273, 0.98727, 0.87885 \right\}$$

$$= 2.3273$$

Since

$$\frac{\|\vec{x}^{(1)} - \vec{x}^{(0)}\|_\infty}{\|\vec{x}^{(1)}\|_\infty} = \frac{2.3273}{2.3273} = 1$$

Since $1 < 10^{-3}$ (Not True)

So we will do next iteration and will calculate $\vec{x}^{(2)}$ and will then then check

$$\frac{\|\vec{x}^{(2)} - \vec{x}^{(1)}\|_\infty}{\|\vec{x}^{(2)}\|_\infty} < 10^{-3}$$

(IF True then we will stop otherwise we will calculate $\vec{x}^{(3)}$ and continue the process)