4th Feb; 2010 Spring 2010

NATIONAL UNIVERSITY
OF COMPUTER AND EMERGING SCIENCES
PESHAWAR CAMPUS

27834

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Course: Prob. & Random Process Serial No. 4th Feb; 2010.

Name: Mubashir & Zaryab Date:

Roll No. Student's Signature

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

Rough Work

Q./Part No.

Mathematical Models as tools in analysis and designing

Model:- A model is approximate representation of a physical situation. (See P#18; Prob. & random process for more details)

Deterministic models:- In deterministic models the conditions under which an experiment is carried out determine the exact outcome of the experiment. for e.g $I = \frac{V}{R}$, here the voltage and current in any circuit consisting of an interconnection of batteries and resistors can be found by solving a system of simultaneous linear equations that are found by applying Ohm's law. (See P#20; Garcia for more details)

Probability models:- In probability models the conditions under which an experiment is carried out are not completely specified

determine the random outcomes of the experiment
(See book for more details)

Experiment:- the term experiment means a planned activity or process whose result yield a set of data. A single performance of an experiment is called trial. the results obtained from an experiment or a trial is called an outcome.

Random experiment:- the experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.
for e.g (See Garcia book P#40 for the examples of r. experiment)

Sample space:- the sample space of random experiment is defined as the set of all possible outcomes. we denote sample space with S. e.g random experiment of flip a coin twice has sample space.

$$S = \{HH, HT, TH, TT\}.$$

Elements of set S are called outcomes of random experiment. (See P#41; Garcia for more details)

Types of Sample space:- (See P#41,42,43 for more details)

(i) Discrete S.S:- If S is countable i.e its outcomes can be written in a sequence

be put into one to one correspondence with the +ve integers. e.g Select a ball from an urn containing balls numbered from 1 to 50.

or $S = \{HH, HT, TH, TT\}$. etc. (see book for details)

(ii) Continuous S.S:- If S is not countable e.g

$$S = \{t : t \geq 0\} = [0, \infty).$$

Note:- Sample space can be multidimensional. for eg

Determine the values of a voltage wave form at time t_1 and t_2. i.e → Two dimensional form.

$$S = \{(v_1, v_2) / -\infty < v_1 < \infty ; -\infty < v_2 < \infty\}.$$

Another example is $S_1 = \{H, T\}$; $S_2 = \{1, 2, 3, 4, 5, 6\}$

$$S_0 = S_1 \times S_2 = \{(H,1)(H,2), \ldots, (H,6), (T,1), \ldots (T,6)\}$$

$$S_1 = S_1 \times S_1 = \{(H,H), (H,T), (T,H), (T,T)\}. \text{ etc (See P\# 43 of book.)}$$

Event:- A subset of a sample space is called event. e.g $S = \{(H,H), HT, TH, TT\}$.

Then events may be at least one head i.e

$$A = \{HH, HT, TH\}. \Rightarrow A \subset S$$

At most one head → $B = \{TT, HT, TH\}$. etc.

$$\Rightarrow B \subset S.$$ See P\# 43 for more examples of events in book

by ^nP_r and defined as

Another example is, $S = \{a, b, c\}$. Then all possible events of S are

ϕ, $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

ϕ → Null event or impossible event

$\{a\}, \{b\}, \{c\}$ → Simple events or Single tom events

$\{a, b\}, \{a, c\}, \{b, c\}$ → Composite events.

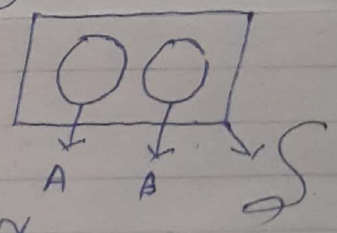
$\{a, b, c\}$ → Sure event or Certain event

(See book p# 44 for further details)

Mutually exclusive events:- (disjoint event)

Let A and B are two events. These events will be disjoint / mutually exclusive if

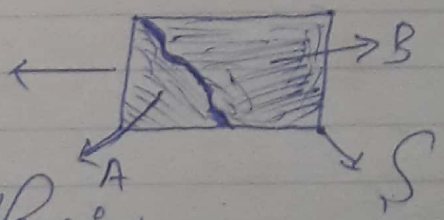
$$A \cap B = \phi$$



Mutually exhaustive events:-

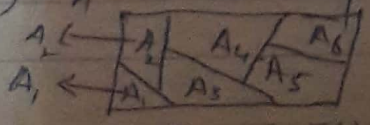
A & B are mutually exhustive events iff

$$A \cup B = S.$$



Partition of a Sample Space:-

Let $A_1, A_2, A_3, \ldots, A_n$ be pairwise mutually exclusive and also mutually exhustive events. Then events $A_1, A_2, \ldots, A_n$ make a partition of Sample space, i.e



Then $A_1, A_2, \ldots, A_6$ are pairwise mutually exclusive also

$A_1 \cup A_2 \cup \cdots \cup A_6 = S$.

Set and Set operations:

Set: Collection of well defined and distinct objects. e.g Collection of students in Prob: of random process course in FAST.

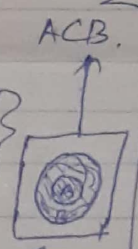
Subset: Let A & B be two sets. then $A \subset B$ if all the elements of A are in B. e.g

Students in Course Prob: of random process Course in FAST

also Students of FAST. or

$A = \{1, 2, 3\}$.

$B = \{1, 2, 3, \ldots, 15\}$

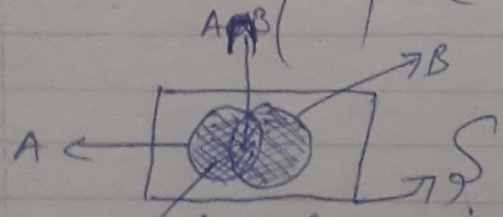


$A \subset B$ $\Rightarrow$ A is Subset of B or B is upper set of A.

Clearly $A \subset B$ → proper subset

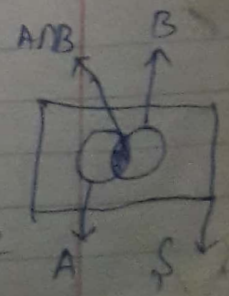
(Proper and improper subsets) → (See Set the

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.



shaded area is $A \cup B$

Intersection: $A \cap B = \{x \mid x \in A \text{ \& } x \in B\}$.



(See book pg 44, 45, and 46 for more details and examples)

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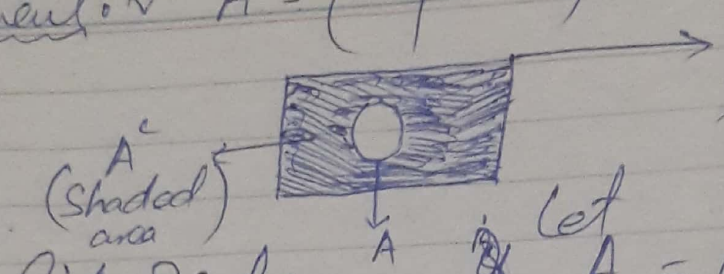
Complement:- $A^c = \{x / x \in S \text{ and } x \notin A\}$.



Equality of sets:- Let $A = B$ if $A \subseteq B$ & $B \subseteq A$.

Properties and laws of Set operations:-

(i) Commutative laws:-
$A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(ii) Associative laws:-
$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) Distributive Property laws:-

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Demorgan's laws:-

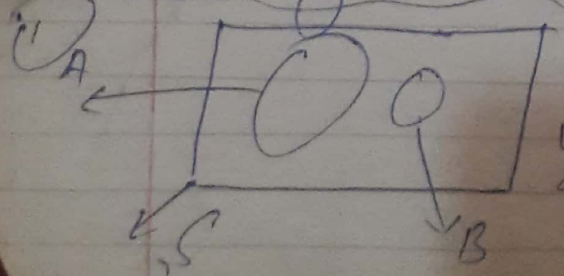
(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$.

(See examples related to set theory on pg 45).

Venn Diagram and Tree diagram:-

(i)



Venn diagram

Tree diagram

$S_1 = \{H, T\}$

$S_2 = \{H, T\}$

$\Rightarrow S = S_1 \times S_2 = \{HH, HT, TH, TT\}$.

H → HH, HT

T → TH, TT

Counting Sample Points:-

(i) If a Compound experiment Consists of two experiments such that the first experiment has exactly m distinct outcomes and if corresponding to each outcome of the first experiment there can be n distinct outcomes of the 2nd experiment. then the Compound experiment has exactly mn outcomes. eg

$S_1 = \{H, T\} \Rightarrow$ Two outcomes

$S_2 = \{1, 2, 3, 4, 5, 6\} \Rightarrow$ 6 outcomes.

then $S = S_1 \times S_2 = \{(H,1), (H,2), \ldots, (H,6), (T,1), (T,2), \ldots, (T,6)\}$.

$\Rightarrow S$ has 12 outcomes.

i.e No. of outcomes in $S = 8$

Rule of Permutation:- the no. of permutations of r objects, selected in a definite order from n distinct objects is denoted by ^nP_r and defined as

Rough Work

Q./Part No.

$$^{n}P_{r} = \frac{n!}{(n-r)!} \quad \text{where } r \leq n.$$

For example no. of ways to open a four digits locks of brief case etc.

Rule of Combination: The Combination is any subset of r objects selected without regard to their order, from a set of n distinct objects. The total no. of Combinations is denoted by $^{n}C_{r}$ and defined as

$$^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

e.g. Out of 40 students in a Class 5 students are selected for some Competition. Then we use the above mentioned formula.

Permutation for arrangement & Time

A Club Consists of four members. How ways are in the sample space when three Officers; President, Secretary and treasure, are to be Choosen?

Sample points

Sample

Exp #2: A three person Comittee is to be formed from a list of four persons. How many sample points are associated with the experiment? → Combination, as selection is

(B) (i)

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Course: Prob: & Random Processes — Serial No. 27035

Name: Mubashir Sayyam — Date: 10/02/10.

Roll No. — Student's Signature — Semester: Spring, 2010.

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

Q./Part No. — Rough Work

Definitions of Probability:-

(i) Classical definition:- If a random experiment can produce n mutually exclusive and equally likely outcomes, and if m out of these outcomes are considered favourable to the occurrence of a certain event A, then Prob: of event A, denoted by P(A) is defined as:

$$P(A) = \frac{m}{n} = \frac{\text{No. of favourable outcomes}}{\text{Total No. of possible outcomes}}$$

For e.g Flipping two coins or testing two integrated circuits. Then (ex) $S = \{aa, ar, ra, rr\}$.

Since we know all outcomes of S are mutually exclusive and equally likely as nothing is common in these outcomes + they have equal chances of occurrence.

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Now let A be the event that at least one circuit is acceptable. then

$$m = \{aa, ar, ra\} = \{\text{No. of favourable outcomes}\} \Rightarrow m = 3$$

$$n = \{aa, ar, ra, rr\} = S = \{\text{Total No. of outcomes}\} \Rightarrow n = 4$$

So $P(A) = 3/4$.

(ii) Relative frequency definition of Prob:-

If a random experiment is repeated a large no. of times, say n, under identical conditions and if an event A is observed to occur m-times, then prob: of the event A is defined as the limit of the relative frequency $\frac{m}{n}$ as n tends to infinity. Symbolically

$$P(A) = \lim_{n \to \infty} \frac{m}{n}.$$

(See p#22, 23, 24 Garcia Fig: 1.3, 1.4)

(iii) Axiomatic definition of Prob: (See p#24, Garcia for more detail)

~~Let S be a sample space with some sample points.~~

The modern theory of prob: begins with the construction of a set of axioms that specify that prob: assignment must satisfy these properties:

It is suppose (a) A random experimental Rasham

ed, and a set S of all possible outcomes has been identified (b) a class of subset of S called events has been specified (c) Each event has been assigned a number, $P(A)$, in such a way that the following axioms are satisfied.

(i) $0 \leq P(A) \leq 1$ (ii) $P(S) = 1$ (iii) If A & B are mutually exclusive events. then $P(A \cup B) = P(A) + P(B)$

See P#47, Garcia for more details

For exp: (*) we observed

(i) $0 \leq \underset{\boxed{3/4}}{P(A)} \leq 1$ (ii) $P(S) = 1$

Let B be the event that no circuit is acceptable

then $B = \{rr\} \Rightarrow P(B) = 1/4$ $\left\{ \because A \text{ \& } B \text{ are mutually exclusive} \right.$

Also $A \cup B = \{aa, ar, ra\} \cup \{rr\}$ $\left\{ \text{as } A \cap B = \{\ \} \right.$

$A \cup B = \{aa, ar, ra, rr\}$.

$P(A \cup B) = 4/4 = 1$.

Also $P(A) + P(B) = 3/4 + 1/4 = 4/4 = 1$.

So $P(A \cup B) = P(A) + P(B)$.

Satisfied all the three conditions/axioms of Prob.

Note:- we can generalized axiom (iii) as well i.e $P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$ when $A_1, A_2, \ldots, A_n$ are (*)

Pairwise disjoint.

Prob. of an event:- Let A be an event of sample space S. Then prob. of A will be equal to the sum of the probabilities of all sample points that are included in A. i.e

$$P(A) = \sum_i P(E_i)$$ where E_i are sample points included in A.

e.g $S = \{aa, ar, ra, rr\}$.

$A = \{aa, ar, ra\}$.

$$P(A) = P(aa) + P(ar) + P(ra)$$
$$= 1/4 + 1/4 + 1/4 = \boxed{3/4}$$

Laws of Probability:- → Complementation law or rule.

(i) Corollary $P(A^c) = 1 - P(A)$

Proof:- Since we know that A & A^c are mutually exclusive i.e $A \cap A^c = \phi$ So from axiom (iii)

$$P(A \cup A^c) = P(A) + P(A^c).$$
$$\Rightarrow P(S) = P(A) + P(A^c)$$
$$\Rightarrow 1 = P(A) + P(A^c) \quad \{\because P(S) = 1$$

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$\Rightarrow P(A^c) = 1 - P(A)$ (Proved)

Corollary 2:- $P(A) \leq 1$ (See P#49, 50, 51 Garcia for more details)

Proof:- Since we know from Cor-1 that

$$P(A) = 1 - P(A^c)$$

$$\Rightarrow P(A) \leq 1 \text{ Proved} \quad \left\{ \because P(A^c) \geq 0 \right.$$

Corollary 3:- $P(\phi) = 0$

Proof:- Since we know $S \cup \phi = S$ $\left\{ \because A \cap \phi = A \right.$

So $P(S \cup \phi) = P(S)$ —— (*)

Now $P(S \cup \phi) = P(S) + P(\phi)$ $\left\{ \begin{array}{l} \because S \cap \phi = \phi \\ \text{Mutually exclusive} \\ \because \text{axiom(iii)} \end{array} \right.$

So $P(S \cup \phi) = P(S)$

$\Rightarrow \cancel{P(S)} + P(\phi) = \cancel{P(S)}$

$\Rightarrow P(\phi) = 0$ (Proved).

Note:- Cor-4 is extension of axiom (iii) which we are going to prove.

Corollary-4:- If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive. then $P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P(A_k)$ for $n \geq 2$.

Proof:- We prove this Cor-4 by mathematical induction

For $k = 2$, result is clearly true by axiom(ii)

$\because P(A_1 \cup A_2) = P(A_1) + P(A_2)$ $\left\{ \because A_1, A_2 \text{ mutually exclusive} \right.$

Let us suppose that result is true for $k = m$

i.e $P\left[\bigcup_{k=1}^{m} A_k\right] = \sum_{k=1}^{m} P(A_k)$ —— (*)

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Now we prove that result is also true for $m+1$

we have

$$P\left[\bigcup_{k=1}^{m+1} A_k\right] = P\left[\left\{\bigcup_{k=1}^{m} A_k\right\} \cup A_{m+1}\right]$$

$$= P\left[\bigcup_{k=1}^{m} A_k\right] + P(A_{m+1}) \quad \left\{ \because A_i\text{'s are pairwise mutually exclusive given} \right.$$

$$= \sum_{k=1}^{m} P(A_k) + P(A_{m+1}) \quad \left\{ \because \text{by eq } (*) \right.$$

$$= \sum_{k=1}^{m+1} P(A_k)$$

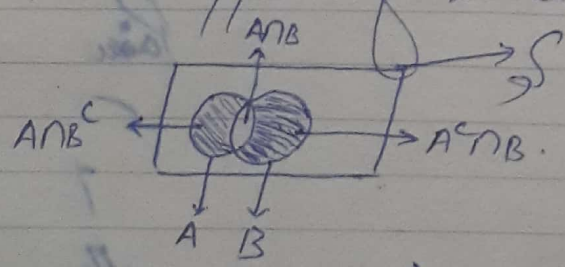
So $P\left[\bigcup_{k=1}^{m+1} A_k\right] = \sum_{k=1}^{m+1} P(A_k)$

Thus by mathematical induction result is true for all the integers greater than or equal to 2.

Corollary-5: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

↳ Addition law of Prob. for two events (See pg. 50, Exercise)

Proof:-



Since

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$$

$$\Rightarrow P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) \quad \left\{ \because \text{mutually exclusive} \right.$$

$(*)$

Now since

$$A = (A \cap B^c) \cup (A \cap B)$$

$$\Rightarrow P(A) = P(A \cap B^c) + P(A \cap B) \quad \left\{ \because \text{Mutually exclusive} \right.$$

$\Rightarrow P(A \cap B^c) = P(A) - P(A \cap B)$ (i)

Also since $B = (A \cap B) \cup (A^c \cap B)$

$\Rightarrow P(B) = P(A \cap B) + P(A^c \cap B)$ } ∵ Mutually exclusive

$\Rightarrow P(A^c \cap B) = P(B) - P(A \cap B)$ (ii)

Now putting values from (i) & (ii) in eq (*), we get

$P(A \cup B) = P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cap B)$ (Proved).

Addition law of Prob. for three events.

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

Proof:- Try by your own.

Hint:- Let $B \cup C = D$.

then $P(A \cup D) = P(A) + P(D) - P(A \cap D)$.

then put values of D and try to find the required result by using addition law of two events and distributive law.

↳ Don't try to go with Venn diagram like in Previous Corollary.

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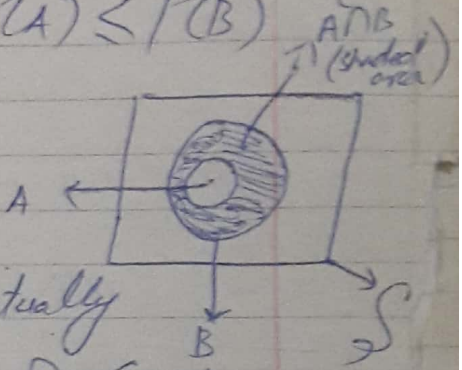
Corollary-5 $P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P(A_j) - \sum_{j<k} P(A_j \cap A_k) + \cdots + (-1)^{n+1} P[A_1 \cap \cdots \cap A_n]$

Proof can be seen by induction by not included in your course.

Corollary-7:- If $A \subset B$ then $P(A) \le P(B)$

Proof:- Since $B = A \cup (A^c \cap B)$



$\Rightarrow P(B) = P(A) + P(A^c \cap B)$

as A & $A^c \cap B$ are mutually exclusive.

$\Rightarrow P(B) \ge P(A)$ $\{\because P(A^c \cap B) \ge 0$

(Proved).

Exp# 2.6 (#53 Garcia)

$S = \{0, 1, 2, \ldots, 9\} \Rightarrow n(S) = 10$

$A = \{\text{odd ball}\} = \{1, 3, 5, 7, 9\}$.

$\Rightarrow n(A) = 5$.

Sol:-

Now $P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \boxed{1/2}$

$P(A)$ can also find by the following method

$P(A) = P\{1\} + P\{3\} + P\{5\} + P\{7\} + P\{9\}$

$= 1/10 + 1/10 + 1/10 + 1/10 + 1/10 = 5/10 = \boxed{1/2}$

Also $B = \{\text{No. multiple of } 3\} = \{3, 6, 9\}$.

$\Rightarrow n(B) = 3$.

Now $P(B) = \frac{n(B)}{n(S)} = \boxed{3/10}$.

Try remaining parts by yourself.

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(C)

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Continuation Sheet

Course _____ Serial No C- 1945

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No #10 An integer is choosen at random from first 300 positive integers. what is the probability that the integer choosen is divisible by 7 or by 9?

Sol:- Try.

Hint:- $\left[\frac{300}{7}\right] = k$, $\left[\frac{300}{9}\right] = k$, $\left[\frac{300}{63}\right] = k$

Q #11. A pair of fair dice are thrown. Find the probability of getting total of 5 or 6. Try.

Q #12:- Three Computer Systems A, B, C are in efficiency test. Computer A is twice faster than as like to B to complete certain operation and B is twice faster as likely Computer C to perform that particular operation. what is the probability that

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System A or B perform that operation earliest?

Sol:- Try.

Continuous Sample Space:-

Exp#2.10. Suppose that the lifetime of computer P#56 memory chips is measured, and we find that "the proportion of chips whose lifetime exceeds t decreases exponentially at a rate α". Find the appropriate prob. law.

Sol:- Let the sample space in experiment be

$$S = (0, \infty).$$

Now $P[(t, \infty)] = e^{-\alpha t}$ for $t > 0$.

↳ by using given statement

where $\alpha > 0$.

Since $e^{-\alpha t}$ will be b/w 0 and 1 for $t > 0$

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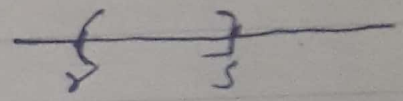
So axiom I is satisfied. Axiom II is also satisfied because

$$P[S] = P[(0,\infty)] = e^{-\alpha(0)} = e^{0} = 1.$$

The Probability that the life time is in the interval $(r,s]$ is found by



So by axiom III $\qquad (r,\infty) = (r,s] \cup (s,\infty)$

$$P[(r,\infty)] = P[(r,s]] + P[(s,\infty)]$$

$$\Rightarrow P[(r,s]] = P[(r,\infty)] - P[(s,\infty)]$$

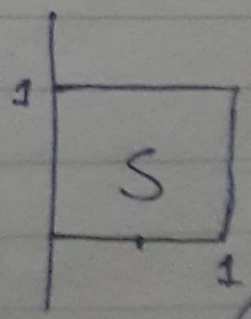
$$= e^{-\alpha r} - e^{-\alpha s} \quad (*)$$

Thus $(*)$ is the Prob: Rule of orbitary intervals in S.

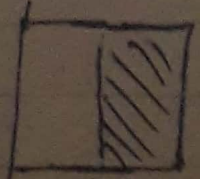
Exp#2.11

P#57.

Sol:- (a) $A = \{x > 0.5\}$

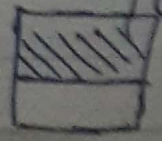


(a)



$x > \frac{1}{2}$

(b) $B = \{y > 0.5\}$



$y > \frac{1}{2}$

(c) $\{x > y\}$

$x > y$.

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$P[A] = 1/2$; $P[B] = 1/2$; $P[C] = 1/2$.

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Peshawar Campus

Continuation Sheet

Serial No C- 1943

Course

Roll No Section Date

Name Main Answer Book Number

Sample with and without replacement:-
and with order:-

Q.No

Ex 2.12:-

P#59 Sol:-

Garcia Sampling with replacement

Total number of ordered pairs $= 5^2 = 25$

(1,1) (1,2) ——— (1,5)
(2,1) (2,2) ——— (2,5)
(5,1) (5,2) ——— (5,5)

$P(A) = P$(Two draws yeids the same number)

$= 5/25 = 1/5$

Ex P# 2.13

P#60 Sol:-

Garcia Sampling without replacement

Total no. of ordered pairs $= 5 \times 4 = 20$

So $P(A) = P$(that the first ball has a number larger then that of the 2nd ball)

$= 10/20 = 1/2$

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Note that

Permutation of Combinations already discussed.

Exp# 2.19:- Quality Control p# 63 Garcia

$$n(S) = \binom{50}{10} = \frac{50!}{40!\,10!} = \star'$$

$$n(A) = \binom{40}{5}\binom{10}{5} = \frac{40!}{35!\,5!} \cdot \frac{10!}{5!\,5!} = \star$$

So $P(A) = \frac{n(A)}{n(S)} = \frac{\star}{\star'} = 0.016$.

Imp:- Permutation rule:- For similar gps in n objects.

The number of permutations of n objects, selected all at a time, when n objects consists of n_1 of one kind, n_2 of 2nd kind, ... n_k of kth kind.

Also $\sum n_i = n$

$$P = \frac{n!}{n_1!\,n_2! \cdots n_k!}$$

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Conditional Probability P#64:- Garcia Book

(Formula) $P(A/B) = \cancel{\text{mutual of}} \dfrac{P(A \cap B)}{P(B)}$.

for $P(B) > 0$.

Some axioms:-

(i) $0 \leq P(A/B) \leq 1$ (ii) $P(S/B) = 1$.

(iii) $P(A_1 \cup A_2 / B) = P(A_1/B) + P(A_2/B)$.

where A_1 & A_2 are mutually exclusive.

Exp# 2.21:- $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.

P# 65 Garcia

Sol:- Here $S = \{(1,b), (2,b), (3,w), (4,w)\}$

$A \cap B = \{(2,b)\}$.

$P(A \cap B) = 1/4$.

$P(B) = 2/4$ $\therefore B = \{(2,b), (4,w)\}$

Now $P(A/B) = \dfrac{1/4}{2/4} = 1/2 = 0.5 = P(A)$

$\therefore P(A) = 1/2$

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Also $P(A/C) = \frac{P(A \cap C)}{P(C)}$.

$P(A \cap C) = 0 \quad (\because A \cap C = \phi)$

$P(C) = 2/4$.

So $P(A/C) = \frac{0}{2/4} = 0 \neq P(A)$.

<u>Note</u>: In 1st Case knowledge of B didnot affects the Prob. of A. But in second case knowledge of C implied that A had not occurred.

Also Note that

$$P(A/B) \neq P(B/A) \quad \text{in general}$$

<u>Q#1</u>:- Two Coins are tossed what is the Prob. that two heads results given that there is at least one head?

Try.

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Continuation Sheet

Course _____ Serial No C- 1942

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No #2. A man tossed two fair dice. what is the Conditional Prob: that the sum of two dice will be 7, given that (i) sum is odd, (ii) the sum is greater than 6. (iii) the two dices had the same outcome?

Try.

Main key:- $P(A/B) = \frac{P(A \cap B)}{P(B)}$ (*)

Q #3. A missile is fired at a target and the Prob: that the target is hit is 0.7. find how many missiles should fired so that the Prob: that the target is hit at least once, is greater than 0.995.

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Page No: 2.

Hint: Let A be the event that missile hit the target. then

$P(A) = 0.7$.

$P(\bar{A}) = 0.3$

$\therefore 0.7 + 0.3 = 1$.

Total Prob.

~~Now let the missile is fired n times~~

~~$P[\text{at least one hit}]$~~

~~$= P(A) = 1 - P(\bar{A})$.~~

~~$\Rightarrow P(A) = 1 - [0.3]^n$~~

Let the missile is fired n times to obtained the Prob: of at least one hit is greater then 0.995.

then $1 - (0.3)^n \geq 0.995$

Solve for n and get result.

Try.

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(C)-I

(1)

NATIONAL UNIVERSITY
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PESHAWAR CAMPUS

ANSWER BOOK

27536

Course: Probability Serial No.

Name: Mubashir Rayyan Date: 11/02/10.

Roll No. Student's Signature

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

(for Combination Rule/Law)

Q./Part No. Exp.

Rough Work

Suppose three integrated circuits are testing for acceptance or reject. What is the Prob: of at least one acceptance?

Sol:- In this situation, $S = \{aaa, aar, ara, raa, rra, rar, arr, rrr\}$.

So $n(S) = 8$ As $2^3 = 8$
(Total no. of sample points)

Let $A = \{$At least 1 accept$\}$.

Now $A^c = \{$No accept$\} = \{rrr\} \Rightarrow n(A) = 1$

Since we know that $P(A) = 1 - P(A^c)$. (*)

Also $P(A^c) = 1/8$ So $P(A) = 1 - \frac{1}{8} = \boxed{7/8}$

Here is another approach to solve such a problem.

Let suppose we are interested in acceptance of circuits. So just count out no. of acceptance

in this experiment. So Sample Space S be

$$S = \{0, 1, 2, 3\}$$
$$\quad \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}\ \tfrac{1}{4}$$

Let A be the event that two acceptance then $P(A) = \boxed{1/4}$ which is different from the method we discussed earlier.

Both the approaches are Ok but in Prob: theory in this Course "Probability of Random Process" we take the first approach which we already discussed. (See example #2.7, Garcia, P#53 for more details)

Few of the Problems on Probability:-

Q#1:- A Card is drawn from a Pack of 52 Playing Cards, find the Prob: of the following events (i) the Card is Spade Card (ii) the Card is King Card (iii) Even number Card etc (Try)

Q#2:- Consider an experiment that consists of testing four integrated circuits that comes from the same Silicon wafer, and observe in each case whether a circuit is accepted (a) or rejected (r). Let A denotes the event that the second circuit is a

find the Prob. of A.

#2:- Find Sample Space for this experiment by using tree diagram. (Try)

#3:- A Pair of ~~two~~ dice is thrown, what is the Prob. of getting (i) a sum of 9 or more (ii) Sum at least 3 (iii) A sum at most 11 (iv) Sum of those Pairs having same no. on both etc

(Quiz#3 in Lecture Mon 1 Sept)

#3:- A Student's test Score T is an integer between 0 and 100 corresponding to the experimental outcomes $s_0, s_1, \ldots, s_{100}$. A Score of 90 to 100 is an A, 80 to 89 is a B, 70 to 79 is a C, 60 to 69 is a D, and below 60 is a failing grade F. Given that all Scores b/w 51 and 100 are equally likely and a Score of 50 or less cannot occur, please find the following Probs.

(a) $P[\{s_{79}\}]$ (b) $P[\{s_{100}\}]$

(c) $P[A]$ (d) $P[F]$ (e) $P[T \geq 80]$

(f) $P[T < 90]$ (g) $P[\text{a C grade or better}]$

(h) $P[\text{Student Pass}]$ (Try)

Hint:- There are exactly 50 equally likely outcomes S_{51} through S_{100}. Each of these outcomes has a probability $1/50 = 0.02$.

So (a) $P[\{S_{79}\}] = 0.02$ (b) $P[\{S_{100}\}] = 0.02$

(c) $P[A] = P[S_{90}, \ldots, S_{100}] = 11 \times 0.02$

$= 0.22$

$\because P[S_{90}, S_{91}, \ldots, S_{100}] = P(\{S_{90}\}) + P(\{S_{91}\}) + \ldots + P(\{S_{100}\})$

Further explination:-

$= 0.02 + 0.02 + \ldots + 0.02$

$= 11(0.02) = 0.22$ (Try remaining parts)

(in Good Man's Book)

Quiz #14 on attached pages, we need to discuss how to solve this problem.

	L	B
V	VL	VB
D	DL	DB

Also given from the prob. like:

$P[L] = 0.6$, $P[V] = 0.7$

$P[VL] = 0.35$ $\{\because VL = (V \cap L)$

Now we want to find the required probs.

$P[DL] = P[D \cap L] = P[LV^c] = P[L] - P[LV]$

$= 0.6 - 0.35 = 0.25$

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Illustrations:- → from table contructed at start

Since $L = (LV) \cup (LD)$

$\Rightarrow P(L) = P(LV) + P(LD)$ {∵ Mutually exclusive}

$\Rightarrow P(LD) = P(L) - P(LV)$

$= 0.6 - 0.35$ {∵ $LV = VL$, $LD = DL$ etc Commutative Property holds in intersection.}

$= 0.25$

In a Similar way we Can find.

(c) $P[VB] = ?$ (Try)

(b) $P[D \cup L] = P[V^c \cup L]$

$= P[V^c] + P[L] - P[V^c \cap L]$

$= P[V^c] + (P[L] - P[DL])$.

$= P[V^c] + P[LV]$ {∵ $P(L) = P(LV) + P(LD)$ $\Rightarrow P(L) - P(LD) = P(LV)$}

$= 0.3 + 0.35$

$= 0.65$

(d) $P(V \cup L) = P(V) + P(L) - P(VL)$ {∵ $P[V] = 0.7$ So $P[V^c] = 1 - 0.7 = 0.3$ ∵ $V^c = D$, $D^c = V$}

$= 0.7 + 0.6 - 0.35$

$= 0.95$

(e) $P[V \cup D] = P[V \cup V^c] = P[S] = 1$

(f) we Can solve (f) in Similar way as (e). (Try)

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Q#4:- 7 green and 3 yellow balls, which are indistinguishable apart from their colour, are placed in a bag. Six balls are taken from a bag. Find the prob. of their being 3 green and 3 yellow.

Sol:- Total No. of possible outcomes in a sample space $= \binom{10}{6} = \frac{10!}{(10-6)!\,6!} = \frac{10!}{4!\,6!}$

$\Rightarrow n(S) = 210$

Let A be the event represents the three green and three yellow balls taken from bag. Then no. of outcomes that corresponds to event A are

$$n(A) = \binom{7}{3} \times \binom{3}{3} \quad \left\{ \because {}^{n}C_{r} = \binom{n}{r} \right\}$$

$$= \frac{7!}{(7-3)!\,3!} \times \frac{3!}{(3-3)!\,3!}$$

$$= \frac{7!}{4!\,3!} \times \frac{3!}{0!\,3!} = 35$$

Therefore $P(A) = \frac{n(A)}{n(S)} = \frac{35}{210} = 0.1667$ Ans.

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FAST-NUCES Peshawar Campus wishes to hire people in Electrical Engineering from a group of 120 applicants, 80 men and 40 women, all of whom are equally qualified to fill the position. If administration selects the two at random, what is the probability that (i) both will be men (ii) at least one will be women?

Soln:- Try (Same prob. like previous one)

Q#6:- Four Components are taken at random from a machine of 12 Components and inspected. The machine is rejected if more than 1 Component is found to be defective. If there are 3 defective Components in a machine. Find the probability that the machine is accepted.

Soln: $n(S) = \binom{12}{4} = 495$ sample points.

Since machine contains 3 defective and 9 good Components. The machine is accepted if there is (i) no defective Component (ii) One defective Component in a sample

Q./Part No. Component Selected.

Let A be the event that machine is accepted

i.e the no. of defective item choosen is 0 or 1.

Then

$$n(A) = \binom{3}{0}\binom{9}{4} + \binom{3}{1}\binom{9}{3}$$

$$= 126 + 252 = 378 \text{ sample points.}$$

Therefore $P(A) = \frac{n(A)}{n(S)} = \frac{378}{495} = 0.76$ Ans.

Hence the probability that the machine is accepted is 0.76 i.e 76%. Ans.

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For example, from the table we can read that the probability of a brief data call is $P[BD] = 0.08$. Note that $\{V, D, F\}$ is an event space corresponding to $\{B_1, B_2, B_3\}$ in Theorem 1.8. Thus we can apply Theorem 1.8 to find

$$P[L] = P[LV] + P[LD] + P[LF] = 0.57$$

Quiz 1.4. Monitor a phone call. Classify the call as a voice call (V), if someone is speaking; or a data call (D) if the call is carrying a modem or fax signal. Classify the call as long (L) if if the call lasts for more than three minutes; otherwise classify the call as brief (B). Based on data collected by the telephone company, we use the following probability model: $P[V] = 0.7$, $P[L] = 0.6$, $P[VL] = 0.35$. Please find the following probabilities:

(a) $P[DL]$
(b) $P[D \cup L]$
(c) $P[VB]$
(d) $P[V \cup L]$
(e) $P[V \cup D]$
(f) $P[LB]$

1.5 CONDITIONAL PROBABILITY

As we suggested earlier, it is sometimes useful to interpret $P[A]$ as our knowledge of the occurrence of event A before an experiment takes place. If $P[A] \approx 1$, we have advance knowledge that A will almost certainly occur. $P[A] \approx 0$ reflects strong knowledge that A is unlikely to occur when the experiment takes place. With $P[A] \approx 1/2$, we have little knowledge about whether or not A will occur. Thus $P[A]$ reflects our knowledge of the occurrence of A prior to performing an experiment. Sometimes, we refer to $P[A]$ as the *a priori* probability of A.

In many practical situations, it is not possible to find out the precise outcome of an experiment. Rather than the outcome s_i itself, we obtain information that the outcome is in the set B. That is, we learn that some event B has occurred, where B consists of several outcomes. Conditional probability describes our knowledge of A when we know that B has occurred but we still don't know the precise outcome.

Example 1.15. Consider an experiment that consists of testing two integrated circuits that come from the same silicon wafer, and observing in each case whether a circuit is accepted (a) or rejected (r). There are four outcomes of the experiment rr, ra, ar, and aa. Let A denote the event that the second circuit is a failure. Mathematically, $A = \{rr, ar\}$.

The circuits come from a high-quality production line. Therefore the prior probability of A is very low. In advance, we are pretty certain that the second circuit will be acceptable. However, some wafers become contaminated by dust, and these wafers have a high proportion of defective chips. Let $B = \{rr, ra\}$ denote the event that the first chip tested is rejected.

Given the knowledge that the first chip was rejected, our knowledge of the quality of the second chip changes. With the first chip a reject, the likelihood that the second chip will also be rejected is higher than it was a priori. Thus we would assign a higher probability to the rejection of the second chip when we know that the first chip is a reject.

The notation for this new probability is $P[A|B]$. We read this as "the probability of A given B." Mathematically, the definition of conditional probability is as follows.

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(e) $P[T \geq 80]$ (f) $P[T < 90]$

(g) $P[\text{a } C \text{ grade or better}]$ (h) $P[\text{student passes}]$

1.4 SOME CONSEQUENCES OF THE AXIOMS

Here we list some properties of probabilities that follow directly from the three axioms. While we do not supply the proofs, we suggest that students prove at least some of these theorems in order to gain experience working with the axioms.

Theorem 1.7. *The probability measure $P[\cdot]$ satisfies*

(a) $P[\phi] = 0$.

(b) $P[A^c] = 1 - P[A]$.

(c) *For any A and B (not necessarily disjoint),*

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

(d) *If $A \subset B$, then $P[A] \leq P[B]$.*

The following theorem is a more complex consequence of the axioms. It is very useful. It refers to an event space $B_1, B_2, \ldots, B_m$ and any event, A. It states that we can find the probability of A by adding the probabilities of the parts of A that are in the separate components of the event space.

Important

Theorem 1.8. *For any event A, and event space $\{B_1, B_2, \ldots, B_m\}$,*

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$

Proof The proof follows directly from Theorem 1.2 and Theorem 1.4. In this case, the disjoint sets are $C_i = \{A \cap B_i\}$.

Example 1.14. *A company has a model of telephone usage. It classifies all calls as either long (L), if they last more than three minutes, or brief (B). It also observes whether calls carry voice (V), data (D) or fax (F). This model implies an experiment in which the procedure is to monitor a call. The observation consists of the nature of the call, V, D, or F, and the length, L or B. The sample space has six outcomes $S = \{LV, BV, LD, BD, LF, BF\}$. The probabilities can be represented in the table in which the rows and columns are labeled by events and a table entry represents the probability of the intersection of the corresponding row and column events. In this case, the table is*

	V	F	D
L	0.3	0.15	0.12
B	0.2	0.15	0.08

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(D)

(1)

NATIONAL UNIVERSITY
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PESHAWAR CAMPUS

ANSWER BOOK

Course: Probability Serial No. 27633

Name: Mubashir Qayyum. Date: 18/02/10

Roll No. Student's Signature

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

Q./Part No. Rough Work

Q#6:- A Pair of fair dice is thrown. Find the Probability of getting total of 9 or 11. (Try)

Q#7:- Three Computer Systems A, B, C are in efficiency test. Computer A is twice faster than as likely B to complete certain operation, and B is twice faster as likely Computer C to perform that particular operation. What is the probability that system A or B perform that operation earliest? (Try)

Continuous Sample space:- (P#54,55) Read it for introduction.

Example #2.10:- Let the sample space in P#56 (Garcia) this experiment be $S = (0, \infty)$.

Sol:- Also given $P[(t, \infty)] = e^{-\alpha t}$ for $t > 0$

where $\alpha > 0$.

Since Note: $e^{-\alpha t}$ will be b/w 0 and 1

So axiom-I satisfied.

Also axiom-II is satisfied b/c $\because P[(t,\infty)] = e^{-\alpha t}$

$$P[S] = P[(0,\infty)] = e^{-\alpha(0)} = 1$$

Also Probability that the life time is in the interval $(r,s]$ is found by,

So $P[(r,\infty)]$

$= P[(r,s]] + P[(s,\infty)]$ $\quad (r,\infty) = (r,s] \cup (s,\infty)$

by axiom III.

$\Rightarrow$

$$P[(r,s]] = P[(r,\infty)] - P[(s,\infty)]$$

$$= e^{-\alpha r} - e^{-\alpha s} \quad (*)$$

Thus eq (*) is the Prob: rule for arbitrary intervals in S.

Exp# 2.11:- (Try)

P#57:- (Garcia)

Exp# 2.19 Quality Control (Try)

P#63 (Garcia) Same as Prob.s solved in random (C).

See

Probability :- (P# 64, 65, Garcia) for more details

$$P(A/B) = \frac{P(A\cap B)}{P(B)} \; ; \; P(B) \geq 0;$$

or $$P(B/A) = \frac{P(A\cap B)}{P(A)} \; ; \; P(A) > 0.$$

Important Axioms of Conditional Prob:-

(i) $0 \leq P(A/B) \leq 1$ (ii) $P(S/B) = 1$

(iii) $P[(A_1 \cup A_2)/B] = P(A_1/B) + P(A_2/B)$

where A_1, A_2 are mutually exclusive events.

↳ These properties/conditions can be easily proved.

Exp# 2.21 (P# 65, Garcia)

Sol:- Here $S = \{(1,b), (2,b), (3,w), (4,w)\}$

Also given $A = \{(1,b), (2,b)\} = \{$"black ball selected"$\}$

$B = \{$Even no. ball selected$\} = \{(2,b), (4,w)\}$.

Now we try to find $P(A/B)$.

Since $P(A/B) = \frac{P(A\cap B)}{P(B)}$ (*)

Since A & B given, So

$A\cap B = \{(2,b)\}$.

So $P(A\cap B) = \frac{n(A\cap B)}{n(S)} = 1/4$

Also $P(B) = \frac{n(B)}{n(S)} = 2/4$, Put values in eq (*)

$P(A/B) = \frac{1/4}{2/4} = \boxed{1/2}$ (Try remaining parts)

Also $P(A) = \frac{n(A)}{n(S)} = 2/4 = 1/2$

Now Since

$$P(A/B) = P(A) = 1/2$$

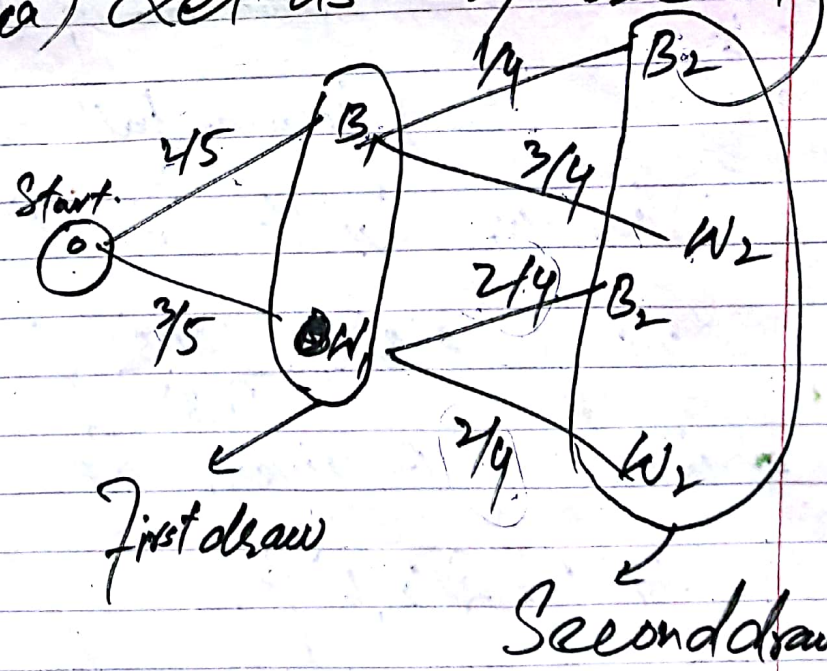
Note:- This shows in this case occurance of event B does not effects the Prob. of A. But this is not true in general. Check that $P(A/C)$, that will be different from $P(A)$.

Note also $P(A/B) \neq P(B/A)$ (In general)

Exp#2.22 (P#66, Garcia) Let us try tree diagram.

Sol:-

See fig. 2.12:- P# 66



Let B_1 and B_2 be the events that the outcome is a black ball in the first and second draw respectivily. So we rep:

$(B_1 \cap B_2) = ?$

Now Since $P(B_2/B_1) = \dfrac{P(B_1 \cap B_2)}{P(B_1)}$ (*)

$\Rightarrow P(B_1 \cap B_2) = P(B_1) \cdot P(B_2/B_1)$

Multiplication Law of Probability:

$= (2/5) \cdot \left(\frac{1}{4}\right)$ (See tree diagram)

$= \frac{2}{20} = \boxed{\frac{1}{10}}$ Ans.

Q1:- Two Coins are tossed, what is the Prob. that two heads results, given that there is at least one head? (try)

Q2:- A man tossed two fair dices. what is the prob. that the Sum of two dices will be 9, given that

(i) Sum is odd (ii) Sum is less then 6

(iii) the two dices had the same outcome?

(Try) Hint: $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.

Multiplication Law of Probability:-

Since we know that

$P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ where $P(B) > 0$.

$\Rightarrow P(A \cap B) = P(B) \cdot P(A/B)$.

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Q:- A box contains 15 items, 4 of which are defective and 11 are good. Two items are selected without replacement. What is the probability that the first is good and second is defective?

Sol:- Let A represents the event that the first item selected is good and B be the event that second item is defective.

Then we need to find $P(A \cap B)$.

Since $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B/A)$ (*)

Since $P(A) = 11/15$

$P(B/A) = 4/14$

Hence $P(A \cap B) = (11/15)(4/14) = 0.16$ Ans.

Q:- (*) Two cards are taken from a pack of ordinary playing cards. Find the probability that the 2nd card taken is heart card?

Sol:- Let us assume that A_1 be the event that first card is heart card and let A_2 be the event that second card is heart card.

Hint:-

Then $A_2 = (A_1 \cap A_2) \cup (\bar{A}_1 \cap A_2)$ (*)

$P(A_2) = P(A_1 \cap A_2) + P(\bar{A}_1 \cap A_2)$ (∵ Mutually exclusive)

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Now Since $P(A_2/A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$

$\Rightarrow P(A_1 \cap A_2) = P(A_1) \cdot P(A_2/A_1)$

~~Ans~~ $\Rightarrow P(A_1 \cap A_2) = (13/52) \cdot (12/51)$

Also $P(A_2/\bar{A}_1) = \frac{P(\bar{A}_1 \cap A_2)}{P(\bar{A}_1)}$ (i)

$\Rightarrow P(\bar{A}_1 \cap A_2) = P(\bar{A}_1) \cdot P(A_2/\bar{A}_1)$

$= (39/52) \cdot (13/51)$

Now Putting values from (i) & (ii) in eq (*), we get the required result.

Q: Box A box contains 5 green and 7 red balls. Box B contains 3 green, 3 red and 6 yellow balls. A ~~ball~~ box is selected at random and a ball is drawn from it. what is the prob. that the ball drawn is green?

Soln: $P(\text{Green ball}) = P[\text{box A \& green}] + P[\text{box B \& green}]$.

i.e $P(\alpha) = P(A \cap \alpha) + P(B \cap \alpha)$

where α is the event that green ball is selected.

(Try).

Problem related to Conditional Prob:-

Q:- Roll two foursided dice. Let denote the no. of dots that appear on and die 2 respectively. Draw the 4by4 space. Let A be the event $X_2 \geq 2$. what is $P(A)$? Let B denote the event $X_2 > X_1$. what is $P(B)$? what is $P(A/B)$?

Soln: Here

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{Bmatrix} \Rightarrow n(S) = 4^2 = 16$$

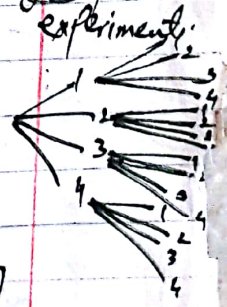
A, B

Now $A = \{X_2 \geq 2\} = \begin{Bmatrix} (1,2), & (1,3) & (1,4) \\ (2,2) & (2,3) & (2,4) \\ (3,2) & (3,3) & (3,4) \\ (4,2) & (4,3) & (4,4) \end{Bmatrix}$

$\Rightarrow n(A) = 12$

& $P(A) = \dfrac{n(A)}{n(S)} = 12/16 = \boxed{3/4}$

Tree for this experiment:



Also $B = \{X_2 > X_1\} \Rightarrow n(B) = 6$

& $P(B) = \dfrac{n(B)}{n(S)} = 6/16 = \boxed{3/8}$

Now $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ (Try).

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(E)

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FAST

[redacted]

Course: Probability

Name: Mubashir Rayyan

Serial No. 27637

Date: 18/02/10

Roll No.

Student's Signature

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

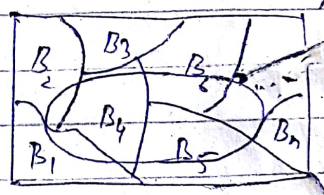
Q./Part No. (P#67, 68, Garcia)

Rough Work

Total Probability Law:

Statement: Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events whose union equal to the whole sample space S. i.e. $B_1, B_2, \ldots, B_n$ makes partition of S. then

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n).$$



Proof:- Since we know that

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n)$$

$$\Rightarrow A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \cdots \cup (A \cap B_n)$$

$$\Rightarrow P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$

$$\Rightarrow P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)$$

{ $\because$ B_i's are mutually exclusive: by Using Multiplication Law of Prob.

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A \cap B_i)$$

i.e. required total Prob: law.

Note that Problem Solved in #6 of 7 with (*) Signs, actually using

Exp #2.24 P#68 Garcia

(Try)

Hint

$$W_2 = (W_1 \cap W_2) \cup (\overline{W_1} \cap W_2)$$

Sol:

$$W_2 = (W_1 \cap W_2) \cup (B_1 \cap W_2)$$

$$\Rightarrow P(W_2) = P(W_1 \cap W_2) + P(B_1 \cap W_2)$$

$$\Rightarrow P(W_2) = P(W_1)P(W_2/W_1) + P(B_1)P(W_2/B_1)$$

Exp #2.25 P#68 Garcia

Let C be the event that "chip is functional after t seconds" and let G be the event that "chip is good" and B be the event that "chip is bad". So by total probability law

Sol:

$$P(C) = P[(G \cap C) \cup (B \cap C)]$$

$$= P[G \cap C] + P[B \cap C].$$

$$= P(G)P(C/G) + P(B)P(C/B)$$

$$= (1-p)P(C/G) + p \cdot P(C/B) \quad \left\{ \because P(G) = P(B) = p \right.$$

$$= (1-p)e^{-\alpha t} + p \cdot e^{-1000\alpha t}$$

$$\because P(C/G) = e^{-\alpha t}$$

$$\& \; P(C/B) = e^{-1000\alpha t}$$

(See P#56, Garcia Text)

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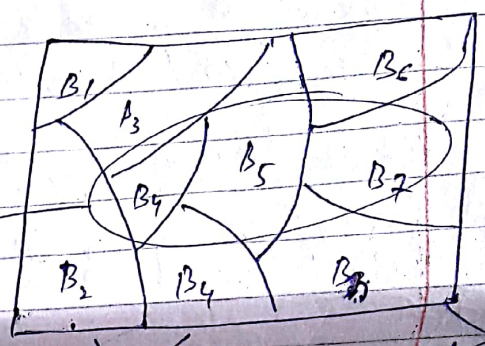
Bayes' Rule (P# 69; Garcia):-

Let $B_1, B_2, \ldots, B_n$ be a partition of a sample space, S. Suppose that event A occurs (which may be different from B_i's). If one of the B_j occurs, then for any j

$$P(B_j/A) = \frac{P(B_j)\,P(A/B_j)}{\sum_{j=1}^{n} P(B_j)\,P(A/B_j)} \quad \text{for } j=1,2,\ldots,n.$$

Proof:-



Let us take the L.H.S

$$= P(B_j/A) = \frac{P(A\cap B_j)}{P(A)} \quad \text{(By def. of Conditional Prob.)}$$

$$P(B_j/A) = \frac{P(B_j)\,P(A/B_j)}{P(A)} \quad (*)$$

Now since we know $B_1, B_2, \ldots, B_n$ are making the partition of S. So

$$A = A\cap S = A\cap(B_1\cup B_2\cup \ldots \cup B_n)$$

$$A = (A\cap B_1)\cup(A\cap B_2)\cup \ldots \cup(A\cap B_n)$$

$$\Rightarrow P(A) = P(A\cap B_1)+P(A\cap B_2)+\ldots+P(A\cap B_n)$$

(∵ mutually exclusive.)

$$\Rightarrow P(A) = P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+\ldots+P(B_n)P(A/B_n)$$

(∵ by multiplication Law of Prob.)

$$P(A) = \sum_{j=1}^{n} P(B_j)P(A/B_j)$$

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Putting value in eqn (*), we get

$$P(B_j/A) = \frac{P(B_j)P(A/B_j)}{\sum_{j=1}^{n} P(B_j)P(A/B_j)} \quad \text{for } j = 1, \ldots$$

Hence the required proof.

Q In a bolt factory machine A, B, C manufactures 25, 35 and 40 percent of total output respectively. Of their outputs 5, 4 and 2 percent respectively are defective bolts. A bolt is selected at random and found to be defective. What is the prob. that the bolt came from machine A? B? C?

Sol:- Given that

$P(A) = 25\% = 0.25$; $P(B) = 35\% = 0.35$;

$P(C) = 40\% = 0.40$.

Let D be the event that bolt is defective.

Then the conditional probs. are

$P(D/A) = 0.05$; $P(D/B) = 0.04$; $P(D/C) = 0.02$.

On the basis of the above mentioned information we need to find (i) $P(A/D) = ?$ (ii) $P(B/D) = ?$ (iii) $P(C/D) = ?$

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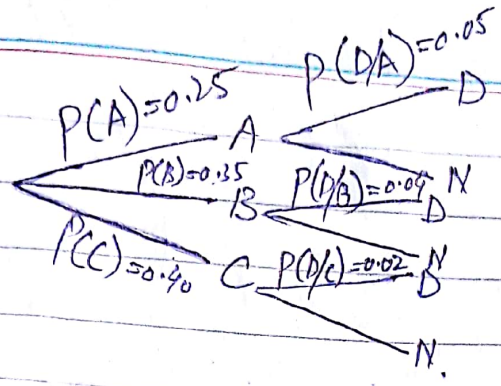

Since by Bayes's rule

$$P(A/D) = \frac{P(A)P(D/A)}{P(A)P(D/A)+P(B)P(D/B)+P(C)P(D/C)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}$$

$$= \frac{0.0125}{0.0345} = 0.362$$

Similarly try remaining parts. (Try) Try EXPH 2.27 Garcia Book.

Q An Urn contains 4 balls which are known to be either (i) all white or (ii) two white and two black. A ball is drawn at random and is found to be white. (a) What is the prob. that all the balls are white? (b) Also what is the prob. that two black and two white

Sol:- Let A_1 be the event hypothesis that all the balls are white and A_2 be the hypothesis that two are white and two black. Then

$$P(A_1) = P(A_2) = 1/2$$

(50%, 50% chance for both hypothesis)

Let B be the event that the ball

ite. Then $P(B/A_1) = \binom{4}{1} \Big/ \binom{4}{1} = \boxed{1}$.

$$P(B/A_2) = \binom{2}{1} \Big/ \binom{4}{1} = \boxed{1/2}$$

Therefore by Baye's Rule

(a) $$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{P(A_1)P(B/A_1) + P(A_2)P(B/A_2)}$$

$$P(A_1/B) = \frac{(1/2)(1)}{(1/2)(1) + (1/2)(1/2)} = \boxed{2/3}$$

Also

(b) $$P(A_2/B) = \frac{P(A_2)P(B/A_2)}{P(A_2)P(B/A_2) + P(A_1)P(B/A_1)}$$

$$= \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)(1)} = \boxed{1/3}$$

Hence the first hypothesis i.e all white balls is preferred. Ans.

(P#70, Garcia)

Independence of events:- Two events A & B are said to be independent if the knowledge of accurance of an event B does not effect/alter the prob. of A.

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$P(A/B) = P(A)$ & $P(B/A) = P(B)$.

Therefore two events A & B are independent iff $P(A \cap B) = P(A) \cdot P(B)$.

$\because P(A) = P(A/B)$ & independent

$\Rightarrow P(A) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A) \cdot P(B) = P(A \cap B)$.

Important Note:- Two mutually exclusive events are independent iff $P(A) \cdot P(B) = 0$.

which is true when either $P(A) = 0$ or $P(B) = 0$. Thus two events are independent can't be mutually exclusive.

~~Ex# 2.28~~ (Try)

~~P#71, 72~~

~~(Garcia Book)~~

~~Ex# 2.29~~

~~(P# 72, Garcia)~~

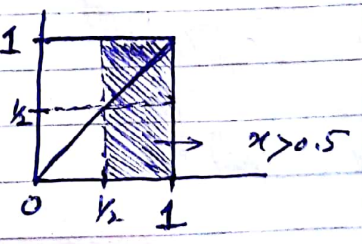
Here $A = \{x > 0.5\}$.

$B = \{y > 0.5\}$, $C = \{x > y\}$.

Sol:-

we need to check (i) A & B are independent (ii) A & C are def: or indep?

Now if $P(A/B) = P(A)$ then A & B independent. (according to def).



Q./Part No. Rough Work

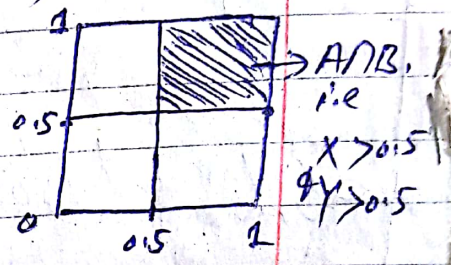
Now $P(A) = (\frac{1}{2})(1) = \boxed{\frac{1}{2}}$

Also $P(B) = (1)(\frac{1}{2}) = \boxed{\frac{1}{2}}$

& $P(C) = \frac{1}{2}(1)(1) = \boxed{\frac{1}{2}}$

Now $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ (*)

$P(A \cap B) = (\frac{1}{2})(\frac{1}{2}) = \boxed{\frac{1}{4}}$



So putting in eq (*), we get

$P(A/B) = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \boxed{\frac{1}{2}} = P(A)$.

Thus A & B are independent.

Check part (ii), (Try)

Q. Two events A & B are such that $P(A) = \frac{1}{4}$, $P(A/B) = \frac{1}{2}$ and $P(B/A) = \frac{1}{2}$.

(i) Are A & B independent?

(ii) Are A & B mutually exclusive

(iii) Find $P(A \cap B)$ and $P(B)$.

Sol:- Hint (iii) Hint No. 1:- Since $P(A \cap B) = P(A)P(B/A)$

$= (\frac{1}{4})(\frac{1}{2})$

Hint No. 2:- Also Since $P(A)P(B/A) = P(B)P(A/B)$

By def: $P(A/B)$ ...

(E) (F) NEW (1)

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Continuation Sheet

Course _____ Serial No C- 513

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

(Question on total Prob: law)

Q.No

Q#1:- A Company has three machines B_1, B_2 and B_3 for making 1KΩ resistors. It has been observed that 80% of resistors produced by B_1 are within 50Ω of nominal value. Machine B_2 produces 90% of resistors within 50Ω of the nominal value. The percentage of machine B_3 is 60%. Each hour machine B_1 produces 3000 resistors, B_2 produces 4000 resistors and B_3 produces 3000 resistors. All the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships a resistor that is within 50Ω of the nominal value?

Sol:- Let $A = \{$resistor is with in 50Ω of nominal value$\}$

So $P(A/B_1) = 80\% = 0.8$; $P(A/B_2) = 90\% = 0.90$

$P(A/B_3) = 60\% = 0.60$.

Since the production figure states that $3000+4000+3000 = 10000$ resistors per hours are produced. Now

PTO.

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$P(B_1) = 3000/10{,}000 = 0.3$; $P(B_2) = \frac{4000}{10{,}000} =$

$P(B_3) = 3000/10{,}000 = 0.3$.

Now for finding $P(A)$ we apply Total law Probability.

$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + P(A/B_3)P(B_3)$$

$$= (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = \boxed{0.78}$$

For the whole factory, 78% of resistors are within 5% of the nominal value Ans.

(Question on Bayes law)

Q#2:- In Q#1 if all the condition are same & find the Probability that acceptable resistors comes from machine B_3? (Use Baye's law)

Sol:-

$$P(B_3/A) = \frac{P(A/B_3)P(B_3)}{P(A/B_3)P(B_3) + P(A/B_2)P(B_2) + P(A/B_1)P(B_1)}$$

$$= \frac{P(A/B_3)P(B_3)}{P(A)}$$

$$= (0.6)(0.3)/0.78 = \boxed{0.23} \text{ Ans}$$

$\because$ In Q#1

$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + P(A/B_3)P(B_3)$

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Exp # 2.22, 2.23, 2.24
P# 66, 67 and 68. (Garcia Book).

Q#3. Consider three traffic lights encountered driving down a road. We say a light was red if the driver was required to come to a complete stop at that light; otherwise we call the light green. For the sake of simplicity, these definitions were carefully choosen to exclude the case of the yellow light. An out come of the experiment is a discription of whether each light was red or green. A sequence of three lights is equally likely. Are the event R_2 that 2nd light was red and G_2 that the 2nd light was green independent? Are the event R_1 and R_2 are independent?

PTO.

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Course

Roll No

Sol: - Here $S = \{rrr, rrg, rgr, grr, grg, ggr, ggg\}$

Then $R_2 = \{rrr, rrg, grr, grg\} \Rightarrow P(R_2) = 4/8 = 1/2$

$G_2 = \{rgr, ggr, rgg, ggg\} \Rightarrow P(G_2) = 4/8 = 1/2$

Also $G_2 \cap R_2 = \phi \Rightarrow P(G_2 \cap R_2) = 0$.

Now since $P(G_2 \cap R_2) \neq P(G_2) \cdot P(R_2)$

So G_2 and R_2 are not independent.

Check $P(R_1 \cap R_2) = P(R_1) \cdot P(R_2)$

Q#4 Suppose traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability 0.8 a driver will find the second light to have the same colour as first. Assume that 1st light is equally likely to be red or green, what is the prob: $P[G_2]$ that

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Continuation Sheet

Course _____ Serial No C- 0364

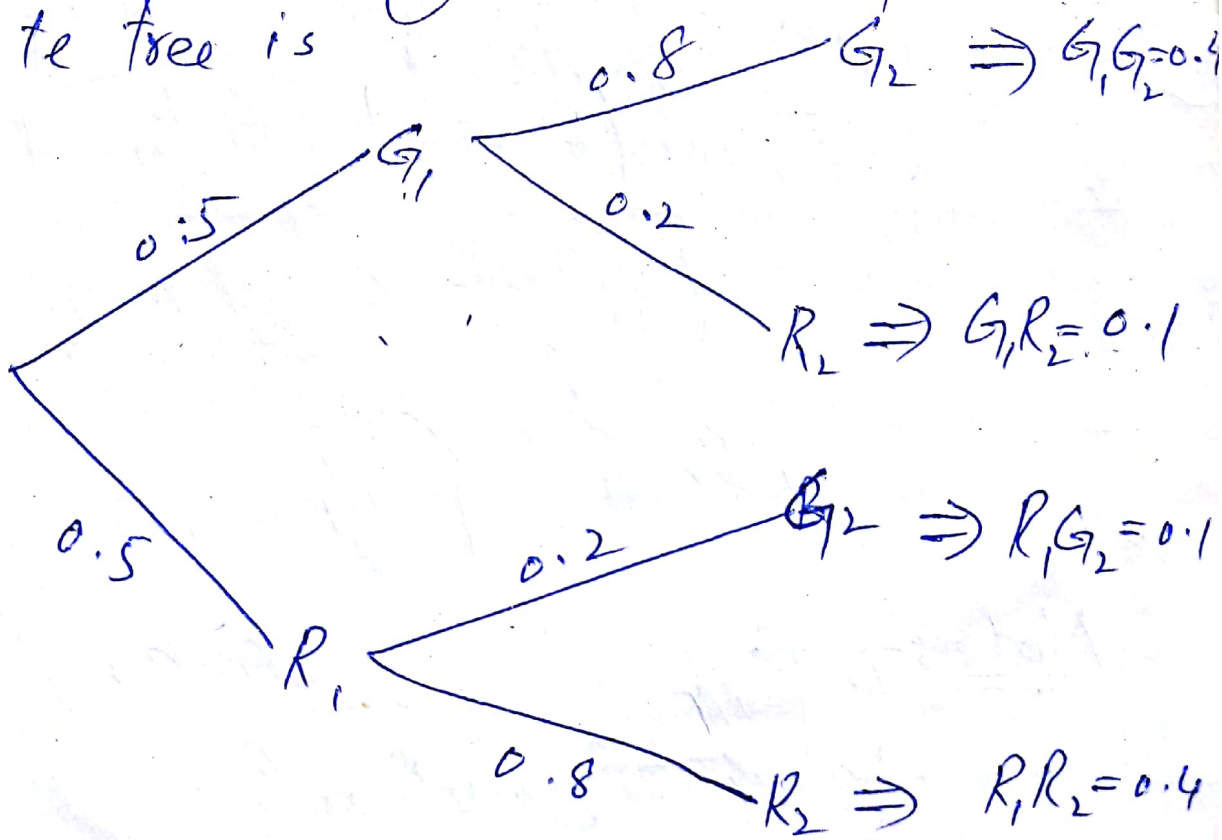
Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No

the 2nd light is green? Also what is the Probability P[W] that you wait for at least one light? lastly what is $P[G_1/R_2]$ the Conditional Prob. of a green first light given a red second light?

Sol:- In the case of two light experiment/ Complete tree is



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Now the Prob: that 2nd light

$$P[G_2] = P[G_1 G_2] + P[R_1 G_2]$$

$$P[G_2] = 0.4 + 0.1 = \boxed{0.5}$$

let the event W that you wait for at least one light is

$$W = \{R_1 G_2 \cup G_1 R_2 \cup R_1 R_2\}.$$

$$\Rightarrow P[W] = P[R_1 G_2] + P[G_1 R_2] + P[R_1 R_2]$$

$$= 0.1 + 0.1 + 0.4 = \boxed{0.6}$$

Now to find $P[G_1/R_2]$ we need $P[R_2]$

$$R_2 = \{G_1 R_2 \cup R_1 R_2\}$$

$$\Rightarrow P[R_2] = P[G_1 R_2] + P[R_1 R_2].$$

$$P[R_2] = 0.1 + 0.4 = \boxed{0.5}$$

Now $P[G_1/R_2] = \dfrac{P[G_1 R_2]}{P[R_2]} = \dfrac{0.1}{0.5}$

$$= \boxed{0.2}$$

Ans

Note: $P[G_1/R_2]$ Can be calculated by using Baye's rule.

Other explination

$$G_2 = (G_1 \cap G_2) \cup (R_1 \cap G_2)$$

$$\Rightarrow P(G_2) = P(G_1 \cap G_2) + P(R_1 \cap G_2)$$

$$= P(G_1)P(G_2/G_1) + P(R_1)P(G_2/R_1)$$

$$= (0.5)(0.8) + (0.5)(0.2)$$

$$= 0.4 + 0.1 = \boxed{0.5}$$ Same as previously obtained

Also

$$N = (R_1 \cap G_2) \cup (G_1 \cap R_2) \cup (R_1 \cap R_2)$$

$$\Rightarrow P(N) = P(R_1 \cap G_2) + P(G_1 \cap R_2) + P(R_1 \cap R_2)$$

$$= P(R_1)P(G_2/R_1) + P(G_1)P(R_2/G_1) + P(R_1)P(R_2/R_1)$$

$$= (0.5)(0.2) + (0.5)(0.2) + (0.5)(0.8)$$

$$= 0.1 + 0.1 + 0.4 = \boxed{0.6}$$ (As previously)

Note:- $P(G_1 R_1) = P(G_1 \cap R_1)$

In a cellular phone system, a mobile phone must be paged to recieve a phone call. However paging attempts don't always succeed because the mobile phone may not receive the paging signal clearly. Consequently, the system will page

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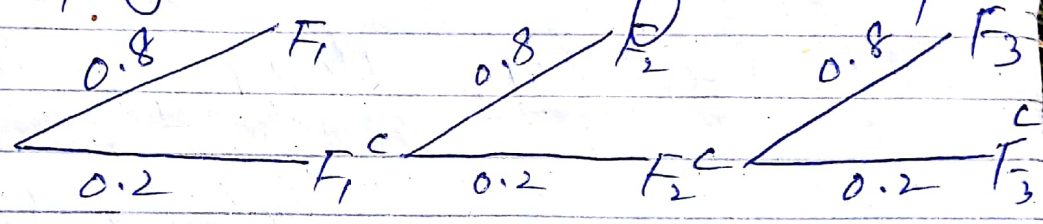
a Phone up to three times before giving a Signal Paging attempt Succeeds with probability 0.8, Sketch a probability tree for this experiment and find the Probability $P[F]$ that the Phone ~~call~~ is found?

Sol:- Let F_i denotes the event that the user is found on page i. The tree for the experiment is



The User is found unless all three paging attempts fail. Thus the Prob. the user is found is

$$P[F] = 1 - P[F_1^c F_2^c F_3^c] = 1-(0.2)^3$$

$$= 0.992 \text{ Ans.}$$

$F = F_1 \cup (F_1^c \cap F_2) \cup (F_1^c \cap F_2^c \cap F_3)$

Q #6. Suppose you have two Coins, one baised, one fair, but you dont know which Coin is which. Coin 1 is baised. It Comes up heads

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Course _____ Serial No C- 0365

Roll No _____ Section _____ Date _____

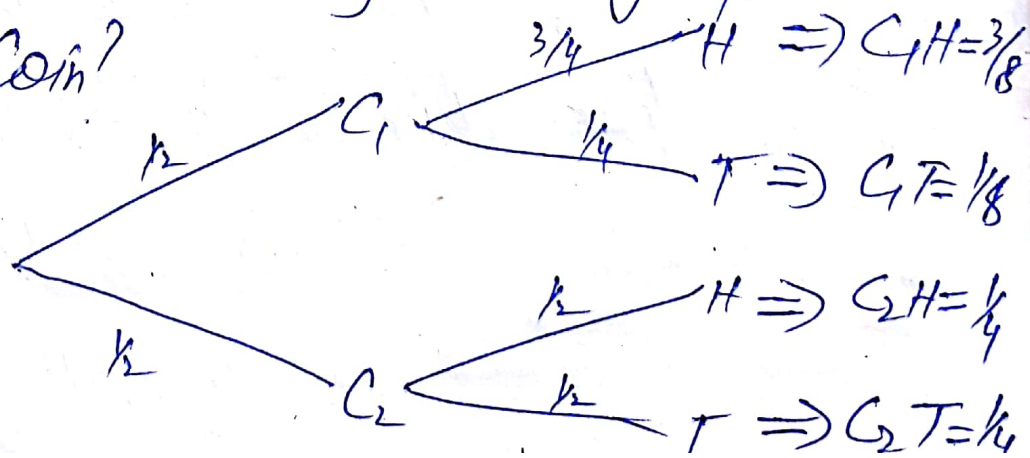
Name _____ Main Answer Book Number _____

Q.No

with Prob: 3/4, while Coin 2 will flip heads with Prob: 1/2. Suppose you pick a Coin at random and flip it. Let C_i denote the event that Coin i is picked. Let H and T denotes the possible out come of the flip. Given that the out come of the flip is head, what is $P[C_1/H]$, the Prob: that you picked a baised Coin? Given that the out come is tail, what is the Prob: $P[C_1/T]$ that you picked the baised Coin?

Hint:

Sol:



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$$P[C_1/H] = \frac{P[C_1 H]}{P[H]} = \frac{3/8}{5/8} = 1$$

Further Explination:

$$P[C_1 H] = P(C_1 \cap H) = P(C_1) \cdot P(H/C_1)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) = \boxed{3/8}$$

Also

$$H = [C_1 H] \cup [C_2 H].$$

or $H = (C_1 \cap H) \cup (C_2 \cap H)$.

$$\Rightarrow P(H) = P(C_1 \cap H) + P(C_2 \cap H).$$

$$= P(C_1)\, P(H/C_1) + P(C_2)\, P(H/C_2).$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= 3/8 + 1/4 = \frac{3+2}{8} = \boxed{5/8}$$

(Try 2nd part).

Q#7:- In a Circuit given below, the Switch open and Closed independently and randomly, the Prob. that a Particular Switch is Closed is "p" which is Same for all Switches. Find the

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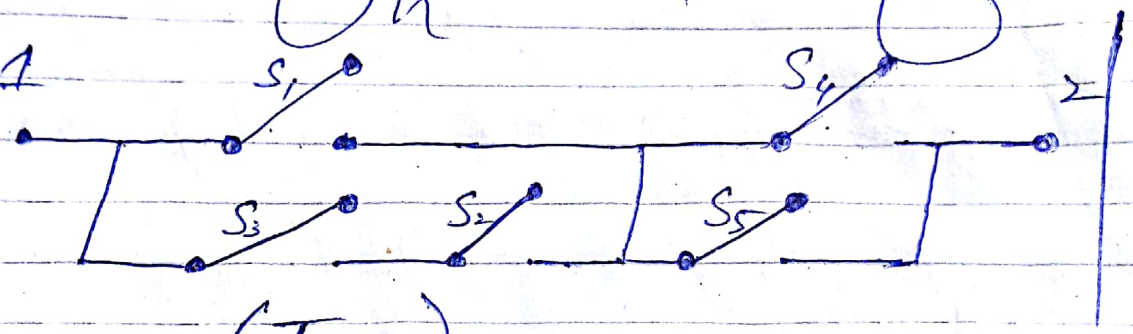
Probability of Closed Path from 1 to 2

(E) Remain

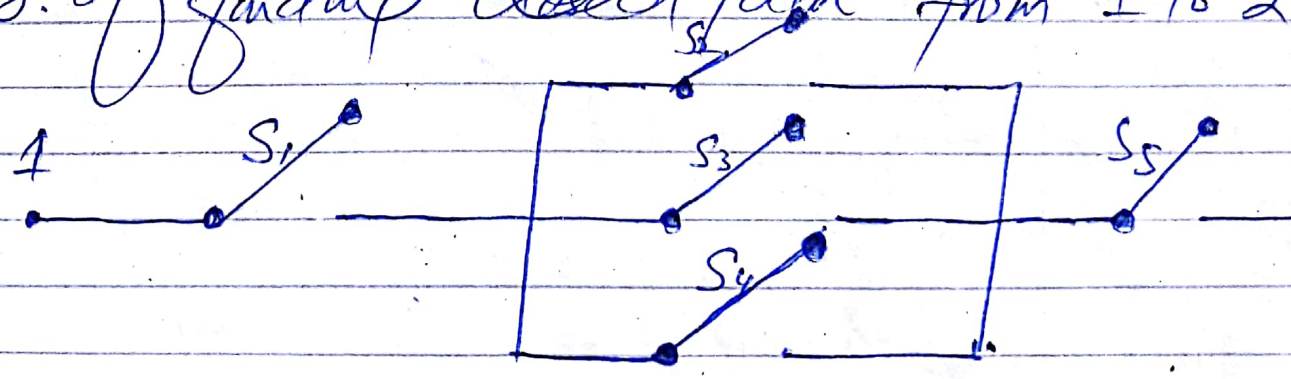


Hint:

(Try)

Use addition law of Probability.

Q#8. In the circuit, given below, the switches open and close independently and randomly. The probability that a particular switch is closed is "p" which is same for all switches. Find the prob. of finding closed path from 1 to 2



Try.

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Continuation Sheet

Course _____ Serial No C- 0426

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

P#77 (Garcia) Q.No

Sequential Experiments:-

Suppose that a random experiment consists of performing experiments $E_1, E_2, \ldots, E_n$ the outcome of this experiment will then be an n-tuple $s = (s_1, s_2, \ldots, s_n)$, where s_k is the outcome of the kth subexperiment. (For more details see Book Page #77)

Random Variable:- It's a real valued fn defined on a sample space. for exp: throwing a coin three time is a random experiment then suppose no. of heads in this experiment is random variable.

X	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

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Again in the experiment of throwing of dice, if we are interested only in the sum of the dots on the upper faces of the two dice and not in the particular dots, we assign to each possible out come of the experiment, one of the number i ($i = 2, 3, \ldots$) corresponding to the sum of the dots appearing on their faces i.e

R.V ← X = Sum of dots

	2	3	4	5	6	7	8	9	10	11	1
Prob: $f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1

Types of r.v:- (i) Discreate r.v (ii) Continuous r.v.

(i) Discrete r.v:- A random variable X is defined to be discrete if it can assume values which are finite or countabaly infinite. when X takes on a finite no. of values, they maybe listed as

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$x_1, x_2, \ldots, x_n$, but in the countably infinite case, values may be listed as $x_1, x_2, \ldots, x_n, \ldots$ The no. of heads obtained in coin tossing experiments, the no. of defective items observed in a consignment etc are the examples of discrete r.v.

Continuous r.v:- A random variable X is defined to be continuous if it can assum every possible value in an interval $[a,b]$, $a<b$, where a and b may be $-\infty$ and $+\infty$ respectively. The hight of the person, the temprature at a place, time to failure for an electronic system or device etc are the exp. of continuous r.v.

The Binomial Probability law/distribution :-

Introduction:- Many experimants consists of repeated independent trial, each trial having two possible out comes. For exp: the two possible out comes of

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the trial may be head and tail, success and good and defective etc.

~~If the outcome of each trial~~

If the probability of each outcome remains the same throughout the trials, then such a trials are called Bernoulli trials and the experiment having n bernoulli trials is called a binomial experiment.

The experiment is called a binomial probability experiment if it possesses the following four properties

(i) The outcomes of each trial maybe classified into one of the two categories called success and failure

(ii) The probability of success remains constant for all trials

(iii) The successive trials are independent.

(iv) The experiment is repeated a fixed number

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(5)

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Continuation Sheet

Course _____ Serial No C- 0427

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No

of times say "n"

Try.

See exp #2.34

(P#78 Garcia)

(Theorem P#78 Garcia)

Let k be the number of successes in n burnu-
lli trials, then the Prob: of k are given by
binomial Probability law as:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n.$$

where $P_n(k)$ is the Probability of k successes
in n trials and $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ is binomial
coefficient

(See Book for further details)

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Exp# 2.35:- Suppose that a Coin is tossed 3 times.
P# 79
(Garcia) assume that the tosses are independent and the Probability of head is p, let "toss results in head" Corresponds to a "Success" then

$$P_3(0) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3.$$

$$P_3(1) = \binom{3}{1} p^1 (1-p)^{3-1} = 3p(1-p)^2.$$

$$P_3(2) = \binom{3}{2} p^2 (1-p)^{3-2} = 3p^2(1-p).$$

$$P_3(3) = \binom{3}{3} p^3 (1-p)^{3-3} = p^3.$$

Exp# 2.36
(P# 80 Garcia)

Try.

Exp# 2.37
(P# 80. Garcia)

Sol:- Here $n = 3$

$k = 2, 3.$

Now $P[k \geq 2] = \binom{3}{2} p^2 (1-p)^{3-2} + \binom{3}{3} p^3 (1-p)^{3-3}$

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$$\Rightarrow P[X \geq 2] = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3$$

$$\cong 3(10^{-6}). \text{ Ans.}$$

Q #1: A and B play a game in which A's probability of winning is 2/3. In a series of 8 games, what is the probability that A will win (i) exactly 4 games (ii) at least 4 games (iii) 6 or mor games (iv) from 1 to 6 games (v) at most 3 games.

Hint.

Sol:- Before going for solution we need to understand the problem.

(a) There are two possible out comes i.e A will win or will not win (b) The probability of A's winning is in each game is $p = 2/3$ (c) The successive games are independently win or lost and (d) There are fixed no. of games i.e 8.

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Therefore binomial Probability dist. with $n=8$ and $p=2/3$ is appropriate.

Let X denotes the number of games won by A. then

(i) $P(X=4) = \binom{8}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{8-4} = \star = 0.1707$

(ii) $P(X \geq 4) = 1 - P(X<4)$ ($\because$ at least means 4 or more)

$= 1 - \sum_{x=0}^{3} \binom{8}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{8-x}$

$= 1 - \left[\binom{8}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^8 + \binom{8}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^7 + \binom{8}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^6 + \binom{8}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5\right]$

$= 0.9121$

(Try remaining Parts.)

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Peshawar Campus

Continuation Sheet

Course _____ Serial No C- 0428

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No

Q#2. The experience of house agent indicates that he can provide suitable accomodation for 75% of the clients who come to him. If on a particular occasion, 6 clients approach him independently, calculate the probability that (i) less than 4 clients (ii) exactly 4 clients (iii) at least five clients (iv) from 3 to 5 clients will get satisfactory accomodation?

Try

Q#1. Try the following problems from Exercise Garcia 71, 72, Section (2.6) (Exercise 2)

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Continuation Sheet

Course _____ Serial No C- 0440

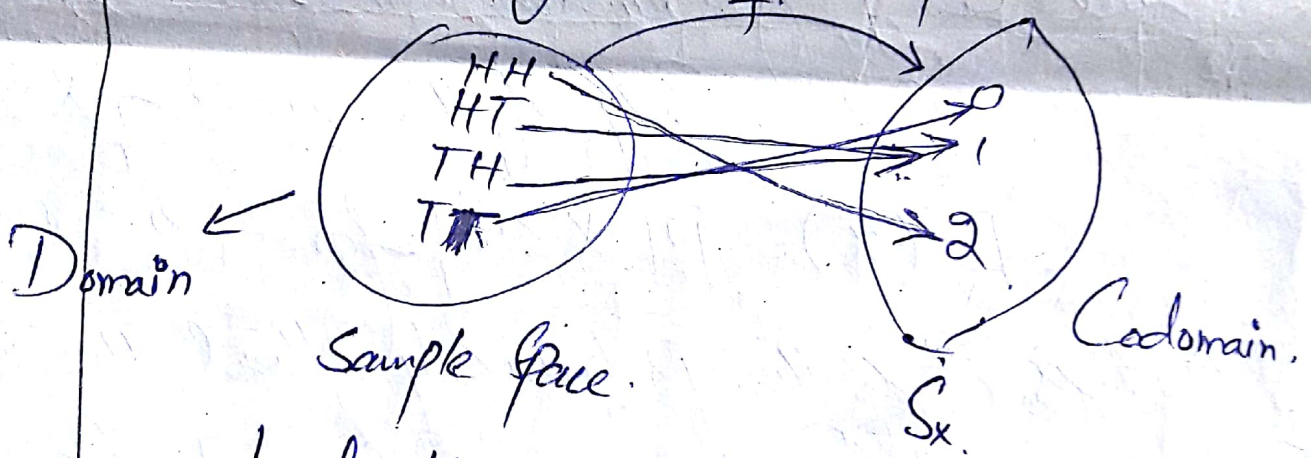
Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

P#100-101

(Garcia) Q.No Random Variable :-

A random variable X is a fn that assigns a real number, $X(\xi)$, to each outcome ξ in the sample space of a random experiment.



Let X is a r.v that shows no. of heads. then $X(HH) = 2$; $X(HT) = 1$, $X(TH) = 1$; $X(TT) = 0$.

For more details see Book P# 101. Also see exp#3.1 for understanding

Example of random variables are

- A, the number of students sleep in the next Probability lecture. (Discrete r.v)
- C, the number of phone calls you answer in the next hour. (Discrete r.v)
- M, the number of minutes you wait until you next answer the phone. (Continuous r.v)

Read

Def#103 Garcia

The Cumulative distribution function:

The cdf of a random variable X is defined as the Probability of the event $\{X \leq x\}$:

$$F_X(x) = P[X \leq x] \text{ for } -\infty < x < +\infty$$

that is, it is the probability that the r.v X takes on a value in the set $(-\infty, x]$. (See Book for more detail)

Imp: Properties of cdf:-

(i) $0 \leq F_X(x) \leq 1$ (ii) $\lim_{x \to \infty} F_X(x) = 1$.

(iii) $\lim_{x \to -\infty} F_X(x) = 0$, (iv) $F_X(x)$ is non decreasing function of x, that is, if $a \leq b$, then $F_X(a) \leq F_X(b)$.

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Exp 3.4

Garcia PH 115

Read last paragraf of PH 102; (Important).

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

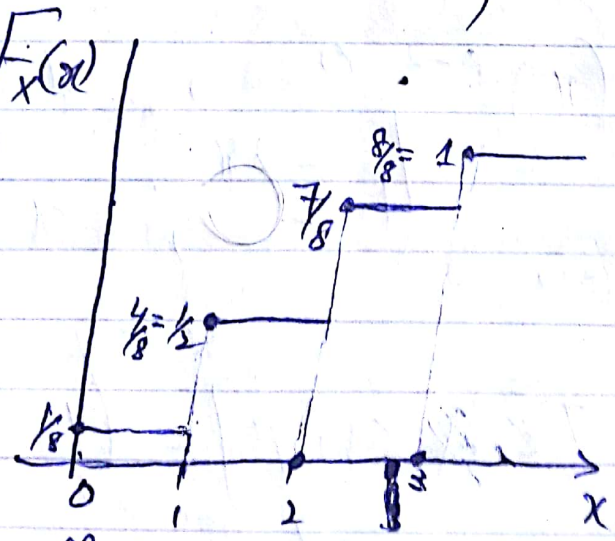
Let X is r.v which defines the no. of heads in three tosses of a fair coin.

then

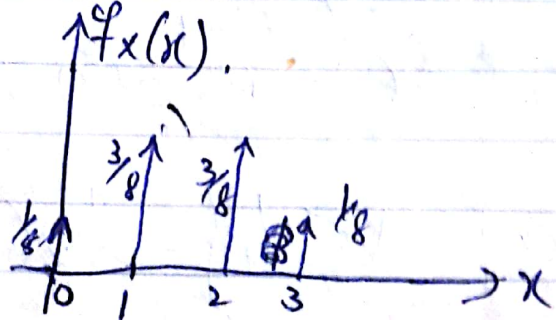
x	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

Now $F_X(x)$ is simply the sum of the prob.s of the outcomes from $\{0,1,2,3\}$ that are less then or equal to x.

fig(a) CDF



fig(b) Pdf.



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Exp H3.5
Garcia
pH 106
Sol:

Given that

$$P[X > x] = e^{-\lambda x}; \quad x > 0.$$

Also $T = 1/\lambda$

we need to find cdf of X.

and $P[T < X \leq 2T]$.

Now $F_X(x) = 0$ if $x < 0$.

$F_X(x) = 1 - P[X > x]$.

$F_X(x) = 1 - e^{-\lambda x}$ when $x \geq 0$.

Because $P[X > x] = 1 - F_X(x) = 1 - P[X \leq x]$.

So $F_X(x) = \begin{cases} 0 & ; \quad x < 0 \\ 1 - e^{-\lambda x} & ; \quad x \geq 0 \end{cases}$ Ans.

Now $P[T < X \leq 2T] = \underbrace{\left(1 - e^{-2}\right) - \left(1 - e^{-1}\right)}_{F_X(b) - F_X(a)}$

$= e^{-1} - e^{-2} \simeq 0.233$ Ans.

So $P[a < X \leq b] = F_X(b) - F_X(a)$.

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Course _____ Serial No C- 0436

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No

Exp #3.6

P#107

Sparcia

$$F_X(x) = P[X \le x].$$

$$[X \le x] = [X \le x;\ \text{idle}] \cup [X \le x;\ \text{busy}]$$

$$= P[X \le x;\ \text{idle}] + P[X \le x;\ \text{busy}]$$

So Mutually exclusive

$$= P(\text{idle})\, P[X \le x / \text{idle}]$$

$$+ P(\text{busy}) \cdot P[X \le x / \text{busy}]$$

By Total Prob. law

$$= p \cdot P[X \le x / \text{busy}] + (1-p) \cdot P[X \le x / \text{busy}]$$

Let $P[X \le x / \text{idle}] = 1$ when $x \ge 0$ and 0 otherwise. we have

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$$F_X(x) = \begin{cases} 0 \; ; & ; x < 0 \\ p(1) + (1-p)(1-e^{-\lambda x}) ; & x \geq 0. \end{cases}$$

$$\therefore \quad F_X(x) = p(1) + (1-p)(1-e^{-\lambda x})$$

where $P[X > x] = e^{-\lambda x}$.

P#109 Garcia

Probability density function:- (pdf) [In case of cts r.v]

$$f_X(x) = \frac{d}{dx} F_X(x)$$

(See Book for more details)

Also $f_X(x) \geq 0$

Or $F_X(x) = \int_{-\infty}^{x} f(x)\,dx \quad \forall x$

Pdf has the following usefull properties:

(i) $f(x) \geq 0 \; \forall x$.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (See Book P#109 Garcia)

(iii) $P[c \leq x \leq d] = F(d) - F(c)$.

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$$= \int_{-\infty}^{d} f(x)\,dx - \int_{-\infty}^{c} f(x)\,dx$$

$$= \int_{c}^{d} f(x)\,dx$$

Also $P(X=k) = \int_{k}^{k} f(x)\,dx = 0$.

So the prob. of a cts r.v X taking any particular value k is always zero.

Also Note that $P[c \le X \le d]$, $P[c < X \le d]$, $P[c \le X < d]$, $P[c < X < d]$ gives the same prob.

See P# 111–112 Exercise. Exp# 3.7, 3.8 shows the pdf of Uniform r.v and Laplace r.v.

Q# (a) Find the value of k so that the function $f(x)$ defined as follows, may be a density function.

PTO

$$f(x) = \begin{cases} kx; & 0 \le x \le 2 \\ 0; & \text{otherwise} \end{cases}$$

(b) Find the probability ~~that both the sam~~ ~~variables will exceed 1 i.e~~ $P(X > 1)$.

(c) Compute the distribution function $F(x)$.

Sol:- By def: $f(x)$ will be density fn if

(i) $f(x) \ge 0$; (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Since (i) satisfied when $k \ge 0$. the second condition will be satisfied if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

i.e if $1 = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$

i.e if $1 = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx\,dx + \int_{2}^{0} 0\,dx$

i.e if $1 = 0 + \left[\frac{kx^2}{2}\right]_0^2 + 0 = 2k^2$

$\Rightarrow 1 = 2k \Rightarrow \boxed{k = 1/2}$

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Continuation Sheet

Course _____ Serial No C- 0487

Roll No _____ Section _____ Date _____

Name _____ Main Answer Book Number _____

Q.No

Hence $f(x) = \begin{cases} x/2 ; & 0 \le x \le 2 \\ 0 ; & \text{otherwise} \end{cases}$

b) $P[X > 1] = \int_1^2 f(x)\,dx = \int_1^2 \frac{x}{2}\,dx$

$= \frac{x^2}{4}\Big|_1^2 = \boxed{3/4}$

c) $F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)\,dx$

Since $\int_{-\infty}^{x} 0\,dx = 0$ when $-\infty < x \le 0$.

If $0 < x \le 2$, we have $F(x) = \int_{-\infty}^{0} 0\,dx + \int_0^x \frac{x}{2}\,dx$

$= 0 + \frac{x^2}{4}\Big|_0^x = \boxed{\frac{x^2}{4}}$

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finally for $x > 2$, we have

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \frac{x}{2}\,dx + \int_{2}^{x} 0\,dx = 1$$

Hence $F(x) = \begin{cases} 0; & \text{for } x < 0 \\ \frac{x^2}{4}; & 0 \le x \le 2 \\ 1; & x > 2 \end{cases}$

Q# A r.v X is cts type with pdf

$$f(x) = 2x; \quad 0 < x < 1$$
$$= 0; \quad \text{otherwise}$$

find (i) $P(X = \frac{1}{2})$ (ii) $P(X \le \frac{1}{2})$

(iii) $P(X > \frac{1}{4})$; (iv) $P(\frac{1}{4} \le X < \frac{1}{2})$

(v) $P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3})$

PTO.

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Sol: Since clearly $f(x)$ is pdf as $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

(i) Since $f(x)$ is cts pdf So

$$P(X = \tfrac{1}{2}) = 0.$$

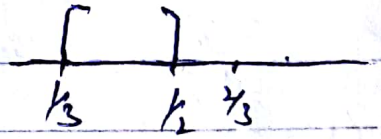
(ii) $$P(X \leq \tfrac{1}{2}) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1/2} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/2} 2x\,dx$$

$$= 0 + \left[x^2\right]_0^{1/2} = \boxed{1/4}$$

(v) $$P\left(X \leq \tfrac{1}{2} \,/\, \tfrac{1}{3} \leq X \leq \tfrac{2}{3}\right) = \frac{P\left[\left(\tfrac{1}{3} \leq X \leq \tfrac{2}{3}\right) \cap \left(X \leq \tfrac{1}{2}\right)\right]}{P\left(\tfrac{1}{3} \leq X \leq \tfrac{2}{3}\right)}$$

$$= \frac{P\left(\tfrac{1}{3} \leq X \leq \tfrac{1}{2}\right)}{P\left(\tfrac{1}{3} \leq X \leq \tfrac{2}{3}\right)}$$



Try remaining parts

$$= \frac{\int_{1/3}^{1/2} 2x\,dx}{\int_{1/3}^{2/3} 2x\,dx} = \frac{\left[x^2\right]_{1/3}^{1/2}}{\left[x^2\right]_{1/3}^{2/3}} = \boxed{5/12}$$

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Q# A Continuous r.v X has the cdf as follows:

$$F(x) = \begin{cases} 0; & \text{for } x<0 \\ \frac{2x^2}{5}; & \text{for } 0<x\leq 1 \\ -\frac{3}{5}+\frac{2}{5}\left(3x-\frac{x^2}{2}\right) & \text{for } 1\leq x\leq 2 \\ 1; & \text{for } x>2. \end{cases}$$

Find pdf and $P(|X|<1.5)$?

Sol: Try Hint $f(x)=\frac{d}{dx}F(x)$

Also $|x|<\alpha$

$\Rightarrow -\alpha<x<\alpha.$

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(J) (1)

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ANSWER BOOK

Course _____ Serial No. 14349

Name _____ Date _____

Roll No. _____ Student's Signature _____

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

(See book for more details) or Mathematica

Garcia. 24/42

Q./Part No. expectation

Rough Work

Expected value or average value of random variables:- Let X be a discrete r.v have possible values $x_1, x_2, \ldots, x_n, \ldots$ with corresponding prob. $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ such that $\sum f(x) = 1$ then expected value of X, denoted by $E(X)$ is defined as:

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n) + \cdots$$

$$= \sum_{i=1}^{\infty} x_i f(x_i)$$, Provided it converges absolutely i.e $E[|x|] = \sum_i |x_i| f(x_i) < \infty$

~~which X takes a finite no of values, we have~~

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If X is Continuous r.v with pdf $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx,$$ Provided the integral converges absolutely. i.e

$$E(|x|) = \int_{-\infty}^{\infty} |x| f(x)\,dx < \infty.$$

Counter exp:-

Q #71 → Sol:- Given that

Q #72

P# 197 exercise

$$F_X(x) = 1 - 1/x \; ; \text{ for } x > 1$$

then

$$f(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx}\left(1 - 1/x\right) = +1/x^2$$

So $f(x) = 1/x^2$ for $x > 1$.

Now since

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{1}^{\infty} x\left(+1/x^2\right)dx$$

$$= +\int_{1}^{\infty} 1/x \, dx = +\left[\ln|x|\right]_{1}^{\infty}$$

So $E(X) \to \infty$ does not exists.

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(a) What is the ~~expected~~ math. expectation of the number of heads when 3 fair coins are tossed?

(b) What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant prob. of success?

Sol: (a) Let X (r.v) represents the number of heads when three fair coins are tossed. Then X has the following p.d

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

Hence $E(X) = 0 \times 1/8 + 1 \times 3/8 + 2 \times 3/8 + 3 \times 1/8$

$= 12/8 = \boxed{1.5}$

Note:- It should be noted that $E(X)$ is 1.5 which is not an integer and not any value X could actually have. It merely means that, if the experiment of tossing 3 coins is repeated a very large no. of times, we would expect on the average to get 1.5 heads.

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(b) Let X be a r.v. denoting the no. of failures preceding the first success. Then X takes the values $0, 1, 2, 3, \ldots$ the respective prob.'s are $p, qp, q^2p, q^3p, \ldots$ where $q = 1-p$.

Hence $E(X) = x_1 f(x_1) + x_2 f(x_2) + \cdots$

$$= 0 \cdot p + 1 \cdot qp + 2 \cdot q^2 p + \cdots$$

$$= qp(1 + 2q + 3q^2 + \cdots)$$

$$= qp(1-q)^{-2} = qp(p)^{-2}$$

$$= q/p \quad \text{Ans.}$$

---

(1) #72 Book (Garcia) p# 197. Hint:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$$

Try

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Variance of r.v's:-

Let X be a r.v. then variance of X is denoted by $V(X)$ and defined as

$$V(X) = E\left[(X-\mu_x)^2\right]. \quad (\because \mu_x = E(X).$$

or $Var(X) = E(X^2) - [E(X)]^2$.

Also standard deviation is the +ve square root of variance.

i.e $\sigma_x = \sqrt{Var[X]}$

Q# Let X be a r.v with pdf

$$f(x) = \begin{cases} 2(x-1) ; & 1<x<2 \\ 0 ; & \text{otherwise} \end{cases}$$

Find the expected value of $H(X) = 2X-1$ and $H(X) = X^2$. for r.v X.

Sol:- Hint:-

$$E(2X-1) = \int_{-\infty}^{\infty} (2x-1) f(x)\,dx$$

& $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

Try. Answers are $E(2X-1) = 7/3$, $E(X^2) = 17/6$

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Q if the Continuous r.v has pdf

$$f(x) = \begin{cases} \frac{3}{4}(3-x)(x-5); & 3 \le x \le 5 \\ 0; & \text{otherwise} \end{cases}$$

Calculate Arithmetic mean, $E(X)$

Variance and Standard deviation of X.

Sol:- Hint. $E(X) = \int_{-\infty}^{\infty} x f(x) dx = 4$ Ans

$\delta(X) = V(X) = E(X^2) - [E(X)]^2 = \boxed{0.2}$ Answer

& $\delta(X) = \sqrt{V(X)} = \boxed{0.447}$ Ans

Also $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

Try. $= \boxed{\frac{81}{5}}$ Ans.

Q A Continuous r.v X has pdf $f(x)$, where $f(x) = \frac{3}{4}(1+x^2); \; 0 \le x \le 1$

If $E(X) = \mu$ & $Var(X) = \delta^2$, find $P(|X-\mu| < \delta)$.

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Hint:-

$$E(X) = \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \boxed{4/6}$$

Ans.

$$Var(X) = E(X^2) - [E(X)]^2 = \boxed{0.0833}$$

Now $P(|X-\mu| < \delta) = P(-\delta < X-\mu < \delta)$

$$= P(\mu - \delta < X < \mu + \delta)$$

$$= \int_{\mu-\delta}^{\mu+\delta} f(x)\,dx = \boxed{0.8521}$$

Ans

Also $\delta = \sqrt{Var(X)}$ Try.

Properties of expected values:-

Let X be a discrete r.v with prob. distribution $(x_i, f(x_i))$; $i = 1, 2, \ldots, n$.

Then (i) $E(X)$ for $X = a$ is

$$E(a) = a.$$

(ii) Let a, b be any two constants & let X is discrete r.v then

$$E(aX+b) = a\,E(X) + b$$ (∵ Expected value of linear transformation)

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Q /Part No (ii) if $b=0$ then

$$E(aX) = a\,E(X).$$

If $a=1$; $b=-\mu$.

then $E(X-\mu) = E(X)-\mu$

$$= \mu-\mu=0.$$

(iii) Let X & Y be two discrete r.vs. then

$$E(X+Y) = E(X)+E(Y).$$

& $E(X-Y) = E(X)-E(Y)$.

(iv) The expected value of product of two independent r.vs is equal to the product of their expected values. i.e

$$E(XY) = E(X)E(Y)$$

Note:- (Some properties of Expected values are for cts r.v.s) Proofs are not included.

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(7) (K)

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See P# 115, 116, 117, 118 for explained values and variances of diff expd r.vs

Course: Stat.

Name

Serial No.

Date

Student's Signature

Roll No.

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

Note: See P# 23, 34 from Exercise P# 193, 194.

Rough Work

Q./Part No.

expected values of Variances of

Binomial & geometric r.vs are.

$E(X) = np$, $Var(X) = npq$ (in case of Binomial r.v)

& $E(X) = \frac{1-p}{p} = \frac{1}{p}$, $V(X) = \frac{1-p}{p^2} = \frac{q}{p^2}$ (In case of Geometric r.vs)

Hint:- See P# 311, 312 & P# 347 & P# 145 (Garcia) → (See, Muhammad Choudry Introduction to Statistical theory part-I)

Mean & Variance of Uniform r.v is $E(X) = \frac{b+a}{2}$, $V(X) = \frac{(a-b)^2}{12}$

See P# 3.29 (Garcia)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right)$$

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(10)

$$\frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{1}{\cancel{(b-a)}}\frac{\cancel{(b-a)}(b+a)}{2}$$

$E(x) = \frac{a+b}{2}$ Proved

If $Var(X) = E(X^2) - [E(X)]^2$

Now $E(X^2) = \int_a^b x^2 f(x)dx$

$$= \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left.\frac{x^3}{3}\right|_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \cancel{\frac{1}{b-a}\left[\frac{1}{3}\right]}$$

$$= \frac{1}{3\cancel{(b-a)}}\left[\cancel{(b-a)}(b^2+ba+a^2)\right]$$

$$E(X^2) = \frac{1}{3}(a^2+ab+b^2)$$

Now $V(X) = E(X^2) - [E(X)]^2$

$$= \frac{1}{3}(a^2+ab+b^2) - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{3}(a^2+ab+b^2) - \frac{(a+b)^2}{4}$$

$$= \frac{1}{12}\left[4(a^2+ab+b^2) - 3(a+b)^2\right]$$

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$$= \frac{1}{12}\left[4a^2+4ab+4b^2-3(a^2+2ab+b^2)\right]$$

$$= \frac{1}{12}\left[4a^2+4ab+4b^2-3a^2-6ab-3b^2\right]$$

$$= \frac{1}{12}\left[a^2-2ab+b^2\right]$$

$$= \frac{(a-b)^2}{12}$$ Proved.

Exp# 3.31 p# 145 (Garcia)

Sol:-

Expected value and variance of exponential r.v.

Proved

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[x\int_0^{\infty} e^{-\lambda x}\,dx - \int_0^{\infty}\left[\frac{d}{dx}(x)\int_0^{\infty} e^{-\lambda x}\,dx\right]dx\right]$$

$$= \lambda\left[x\frac{e^{-\lambda x}}{-\lambda}\Big|_0^{\infty} - \int_0^{\infty}\frac{1\cdot e^{-\lambda x}}{-\lambda}\,dx\right]$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \left\{\frac{e^{-\lambda x}}{-\lambda}\right\}_0^{\infty}$$

$$= \lim_{x\to\infty} -xe^{-\lambda x} - 0 + \left\{\frac{e^{-\lambda(\infty)}}{-\lambda} - \frac{1}{-\lambda}e^{-\lambda(0)}\right\}$$

$$= \lim_{x\to\infty} -xe^{-\lambda x} + 1/\lambda = \frac{1}{\lambda}$$

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where $\lim_{x\to\infty} e^{-\lambda x} = 0$ & $\lim_{x\to\infty} x e^{-\lambda x} = 0$

Again $Var(X) = E(X^2) - [E(X)]^2$

where $E(X^2) = \int_0^\infty x^2 \lambda . e^{-\lambda x} dx$

$$= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx$$

{∵ Integration by parts}

$$= 0 + \frac{2}{\lambda} E(X)$$

$$= 2/\lambda^2 \quad \because E(X) = 1/\lambda.$$

$$\therefore Var(X) = \sigma^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}.$$

Hence $\sigma(x) = \frac{1}{\lambda}$.

Some properties of Variance of r.v X.

(i) If X always takes the value a, then $Var(X) = 0$. i.e $Var(a) = 0$.

(ii) If $Y = X + b$. then

$$Var[Y] = Var[X].$$

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iii) If $Y = aX$, then

$$Var[Y] = a^2 \, Var[X].$$

~~Proof~~ Proves are not included.

Gamma r.v & its Pdf, $E(X)$, $Var(X)$

Pdf $\rightarrow f(x) = \begin{cases} \frac{1}{\Gamma(m)} x^{m-1} e^{-x} ; & 0 \le x < \infty \\ 0 ; & \text{Otherwise} \end{cases}$

where $\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$.

& $m > 0$.

Also

$E(X) = m$

& $Var(X) = m$

(Proof not required)

Note:- If want to see Proof check Introduction to Statistical theory Part-I by Sher Muhammad Chaudhry

(P#146 - Garcia)

Expected value of $Y = g(X)$

Suppose that if we are interested in finding the expected value of $Y = g(X)$.

then $$E(Y) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx.$$

(Direct Method).

Exp# 3.33 (P# 147 (Garcia)) Given that

$$Y = a\cos(\omega t + \Theta)$$

where a, ω and t are Constants, and Θ is a Uniform random variable in the interval $(0, 2\pi)$.

Now $E(Y) = E[a\cos(\omega t + \Theta)]$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx.$$

$$= \int_{0}^{2\pi} \left[a\cos(\omega t + \theta)\right]\left(\frac{1}{2\pi - 0}\right) d\theta$$

$$= \frac{a}{2\pi} \int_{0}^{2\pi} \cos(\omega t + \theta)\,d\theta$$

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(15)

$$= \frac{a}{2\pi} \sin(\omega t+\theta)\Big|_0^{2\pi}$$

$$= \frac{a}{2\pi}\left[\sin(\omega t+2\pi) - \sin\omega t\right]$$

$$= \frac{a}{2\pi}\left[\sin\omega t - \sin\omega t\right]$$

$$= 0.$$

Also

$$E(y^2) = E\left[a^2\cos^2(\omega t+\theta)\right]$$

$$= \int_0^{2\pi} a^2\cos^2(\omega t+\theta)\left(\frac{1}{2\pi-0}\right)d\theta$$

$$= \frac{a^2}{2\pi}\left(\int_0^{2\pi}\cos^2(\omega t+\theta)\,d\theta\right)$$

$$= \frac{a^2}{2\pi}\int_0^{2\pi}\left[\frac{1+\cos(2\omega t+2\theta)}{2}\right]d\theta$$

$$= \frac{a^2}{4\pi}\int_0^{2\pi}\left[1+\cos(2\omega t+2\theta)\right]d\theta$$

$$= \frac{a^2}{4\pi}\left[\theta + \frac{\sin(2\omega t+2\theta)}{2}\right]$$

$$= \frac{a^2}{4\pi}\left[2\pi + \sin(2\omega t+4\pi) - \sin 2\omega t\right]$$

~~incomplete~~

$\omega t+\theta = x$

$d\theta = dx$

$$\frac{a}{2\pi}\int_0^{2\pi}\cos x\,dx$$

$$\frac{a}{2\pi}\sin x\Big|_0^{2\pi}$$

$$= \frac{a}{2\pi}\left[\sin 2\pi - \sin 0\right]$$

$x = \omega t + 0$

$\Rightarrow x = \omega t$

$x = \omega t + 2\pi$

$\Rightarrow x = 2\pi + \omega t$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\Rightarrow \cos^2\theta - (1-\cos^2\theta) = \cos 2\theta$

$\Rightarrow 2\cos^2\theta - 1 = \cos 2\theta$

$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\Rightarrow \cos^2(\omega t+\theta) = \frac{1+\cos(2\omega t+2\theta)}{2}$

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Rough Work

Q./Part No.

$$= \frac{a^2}{4\pi}\left[2\pi + \cancel{\sin 2\omega t} - \cancel{\sin 2\omega t}\right]$$

$$= \frac{a^2}{\cancel{4\pi}_{2}}(\cancel{2\pi})$$

$$= \boxed{\frac{a^2}{2}}$$ Ans.

Exp#3.32 p#145 (Garcia) (Important)

Let N be the number of times a computer polls a terminal until the terminal has a message ready for transmission. then

$$E(N) = \sum_{k=1}^{\infty} k p q^{k-1} = p \sum_{k=1}^{\infty} k q^{k-1}$$

Cal:-

$$\because E(X) = \sum_{i=1}^{\infty} x_i f(x_i)$$

Now since

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{1}{1-x}$$

$\because$ Infinite geometric series $\because S_n = \frac{1}{1-r}$

$$\text{So } \frac{d}{dx}\sum_{k=0}^{\infty} x^k = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$\Rightarrow \sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

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Rough Work

Letting $x = q$, we get

$$E(N) = p \sum_{k=1}^{\infty} k q^{k-1}$$

$$= p \frac{1}{(1-q)^2} = p \cdot \frac{1}{p^2} = \boxed{\frac{1}{p}}$$

Ans

Note :-

Since if the probability of success in one trial is $p = 1/10$ then we expect that on the average $1/p = 1/(1/10) = 10$ trials are required to obtained success.

~~Alotter direct evaluation is also possibl~~

~~i.e $E(N) = \sum_{k=0}^{\infty} P[N > k]$.~~

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Course _____ Serial No. 14840

Name _____ Date _____

Roll No. _____ Student's Signature _____

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

P#207 (Garcia) Multiple Random Variables

Q./Part No. Rough Work

See P#207 (Introduction) by own.

P#207 (Garcia) Vector Random Variable:- A vector r.v X is a fn that assigns a vector of real no.s to each outcome ξ in S, the sample space of random experiment. for example. (4.1).

Let a random experiment consists of selecting a student's name from an urn. Let ξ denote the outcome of this experiment, and define the following three fns:

$H(\xi)$ = hight of the student ξ in inch

$W(\xi)$ = weight of student ξ in pounds, and

$A(\xi)$ = age of student ξ in years.

The vector $(H(\xi), W(\xi), A(\xi))$ is a vector r.v.

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Joint distribution:-

The distribution of two or more r.vs which observed simultaneously when an experiment performed, is called their join distribution.

Note that the distribution of single r.v is called Univariate. Likewise, a distribution involving two, three or many r.vs simultaneously is referred to as bivariate, trivariate or multivariate.

PH 213-214 See Garcia Book for more details

Bivariate distribution fun:- Let X & Y be two r.v's defined on the same sample space S. Then the fun $F(x,y)$ defined by $F(x,y) = P[X \leq x, Y \leq y]$, where $F(x,y)$ gives the probability that X will take on a values less then or equal to x and at the same time, Y will take on a values less then or equal to y, is called bivariate or joint distribution fun of X & Y.

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Few of well known properties of bivariate distribution fn, $F(x,y)$ are

(i) $F(x,-\infty) = F(-\infty, y) = 0$; $F(+\infty, +\infty) = 1$

(ii) $F(x,y)$ is non decreasing fn of X & Y and is continuous on the right.

PH211 (Garcia) Bivariate Probability fn (Prob. mass fn)

See Book for more details

Let X & Y be two discrete r.v.'s defined on the same sample spaces, X taking the values $x_1, x_2, \ldots, x_m$ and Y taking the values $y_1, y_2, \ldots, y_n$. Then the probability that X takes on the values x_i and at the same time, Y takes on the values y_j, denoted by $f(x_i, y_j)$ is defined to be the joint probability fn or "probability mass fn of X & Y" is given by

$$f(x_i, y_j) = P[X = x_i ; Y = y_j].$$

or $$P(x_i, y_j) = P[\{X = x_i\} \cap \{Y = y_j\}]$$

where $i = 1, 2, \ldots, m$,
$j = 1, 2, \ldots, n$.

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Joint Probability distribution of (X, Y)

X/Y	y_1	y_2	—	y_j	—	y_n	$P(X=x_i)$
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	—	$f(x_1,y_j)$	—	$f(x_1,y_n)$	$g(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	—	$f(x_2,y_j)$	—	$f(x_2,y_n)$	$g(x_2)$
⋮	⋮	⋮	—	⋮	—	⋮	⋮
x_i	$f(x_i,y_1)$	$f(x_i,y_2)$	—	$f(x_i,y_j)$	—	$f(x_i,y_n)$	$g(x_i)$
⋮	⋮	⋮	—	⋮	—	⋮	⋮
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	—	$f(x_m,y_j)$	—	$f(x_m,y_n)$	$g(x_m)$
$P(Y=y_j)$	$h(y_1)$	$h(y_2)$	—	$h(y_j)$	—	$h(y_n)$	1

A joint Probability fn has the following properties:

(i) $f(x_i, y_j) \geq 0 \;\forall\; (x_i, y_j)$ i.e for $i = 1, 2, \ldots, m$ & $j = 1, 2, \ldots, n$.

(ii) $\sum_i \sum_j f(x_i, y_j) = 1$ (See P#211 Garcia for more detail)

P#211 (Garcia) **Marginal Probability fn:**

From joint Probability fn for (X, Y), we can obtained the individual Probability fn of X & Y. Such individual Probability

fns are called "marginal Probability fns".

Let $f(x, y)$ be the joint Probability fn of two discrete r.v's X & Y. Then marginal Prob. fn of X is defined as:

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j)$$

$$= f(x_i, y_1) + f(x_i, y_2) + \cdots + f(x_i, y_n)$$

x_i must occur either with y_1 or y_2 or ... or y_n.

$$= P(X = x_i);$$

Similarly, the marginal Prob. fn for Y is obtained as

$$h(y_j) = \sum_{i=1}^{m} f(x_i, y_j) = P(Y = y_j);$$

Note that

$$\sum_{i=1}^{m} g(x_i) = 1$$

& $$\sum_{j=1}^{n} h(y_j) = 1.$$

(See joint Prob. dist. table of X & Y)

Again

(*) Question:- An Urn Contains 3 black, 2 red and 3 green balls, and 2 balls are selected at random from it. If X is the number of black balls and Y is the number of red balls selected, then find

(i) the joint probability mass fn $f(x,y)$.

(ii) $P(X+Y \leq 1)$;

(iii) the marginal p.d $g(x)$ & $h(y)$.

Sol:- The sample space contains $\binom{8}{2} = 28$ sample points.

The possible values of X are 0, 1, 2 and those for Y are 0, 1, 2. Then (X,Y) can take on are $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$ and $(2,0)$. Our objective is to find $f(x,y)$ for each value (x,y).

(i) Now $f(0,0) = P(X=0 \text{ and } Y=0)$

$$= \frac{\binom{3}{0}\binom{2}{0}\binom{3}{2}}{\binom{8}{2}} = \boxed{3/28}$$

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Again $f(0,1) = \dfrac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{28} = 6/28$

$f(1,1) = \dfrac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{28} = 6/28.$

Similarly calculations give the probs of the values and the joint p.f of X & Y is given as:

(x,y)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,2)$	$(2,0)$
$f(x,y)$	$3/28$	$6/28$	$9/28$	$6/28$	$1/28$	$3/28$

OR these prob. can be written as:

X \ Y	0	1	2	$P(X=x_i)$ $g(x)$
0	$3/28$	$6/28$	$1/28$	$10/28$
1	$9/28$	$6/28$	0	$15/28$
2	$3/28$	0	0	$3/28$
$P(Y=y_j)$ $h(y)$	$15/28$	$12/28$	$1/28$	1

So clearly, this joint p.d of the two r.vs X & Y can be represented by the formula.

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{28}$$

where $x = 0,1,2$
$y = 0,1,2$ & $0 \le x+y \le 2$.

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Q./Part No. (ii) To Compute $P(X+Y \leq 1)$, if we S... $x+y \leq 1$ for the cells $(0,0)$, $(0,1)$, $(1,0)$. the

for $P(X+Y \leq 1) = f(0,0) + f(0,1) + f($

$= 3/28 + 6/28 + 9/28 = 18/$

$= \boxed{9/14}$

(iii) the marginal p.d's are

x	0	1	2
$g(x)$	$10/28$	$15/28$	$3/28$.

y	0	1	2
$h(y)$	$15/28$	$12/28$	$1/28$.

Conditional Probability fn:-

Let X & Y be two discrete r.v's with joint Probability fn $f(x,y)$. then the conditional Prob: fn for X given $Y=y$, denoted as $f(x/y)$, is defined by

$$f(x_i/y_j) = P(X=x_i / Y=y_j)$$

$$= \frac{P(X=x_i \text{ and } Y=y_j)}{P(Y=y_j)}$$

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ANSWER BOOK

Course _____ Serial No. 14841

Name _____ Date _____

Roll No. _____ Student's Signature _____

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Teacher's Signature
Marks obt.												

/Part No. Rough Work

$$= \frac{f(x_i, y_j)}{h(y_j)}, \quad \text{for } i = 1, 2, \ldots \; j = 1, 2, \ldots$$

where $h(y)$ is marginal Prob: of $h(y) > 0$

Similarly $f(y_j / x_i) = \dfrac{f(x_i, y_j)}{g(x_i)}$

where $g(x) > 0$.

<u>Independence</u>:- Two discrete r.v' X & Y are said to be statistically independent if and only if $f(x, y) = P(X = x_i \text{ and } Y = y_j)$

$$= P(X = x_i) \cdot P(Y = y_j)$$

$$= g(x) h(y). \quad \forall i \text{ \& } j.$$

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Question:- For Question (10) (*) on P#6 (Packed.

find (i) Conditional p.d $f(x/1)$

(ii) $P(X=0/Y=1)$ and

(iii) Are X & Y independent?

Sol:- (i) By definition of Conditional P.d

$$\therefore f(x/1) = \frac{f(x,1)}{h(1)} \quad \text{--- (a)}$$

Now $h(1) = \sum_{i=0}^{2} f(x,1) = \frac{6}{28} + \frac{6}{28} + 0$

$$= \frac{12}{28} = \boxed{3/7}$$

Therefore

$$f(x/1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{3/7} = \boxed{\frac{7}{3} f(x,1)}$$

where $x = 0, 1, 2$.

i.e if we want to find

$$P(0/1) = \frac{f(0,1)}{h(1)} = \frac{7}{3} f(0,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right)$$

$$= \boxed{1/2}$$

$$f(1/1) = \frac{7}{3} f(1,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(2/1) = \frac{7}{3} f(2,1) = \left(\frac{7}{3}\right)(0) = 0.$$

Hence Conditional p.d of X given that Y=1 is

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x	0	1	2
$f(x,1)$	$1/2$	$1/2$	0

(ii) $P(X=0/Y=1) = \dfrac{f(0,1)}{h(1)}$

$= \dfrac{(6/28)}{3/7} = \boxed{1/2}$ (~~this prob. find already~~)

(iii) Are X & Y independent?

Since we find that $f(0,1) = 6/28$.

$g(0) = \sum_{y=0}^{2} f(0,y) = 3/28 + 6/28 + 1/28 = \frac{10}{28}$.

& $h(1) = \sum_{x=0}^{2} f(x,1) = 6/28 + 6/28 + 0 = \frac{12}{28}$.

Now $\frac{6}{28} \neq \frac{10}{28} \times \frac{12}{28}$

i.e $f(0,1) \neq g(0)h(1)$.

Therefore X & Y are not statistically independent.

Question:- The joint p.d of two discrete r.v's X & Y is ~~gim~~ given by $f(x,y) = \frac{xy^2}{30}$; for $x = 1,2,3$ & $y = 1,2$.

(Try) Are X & Y independent?

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(See P#217 Gascia for 12 mor details)

Continuous Bivariate distributions:-

The bivariate probability density ftn of Continuous r.v's X & Y is an integrable ftn $f(x,y)$ satisfying the following properties:

(i) $f(x,y) \geq 0 \ \forall (x,y)$.

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ &

(iii) $P[a \leq X \leq b;\ c \leq Y \leq d] = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Also the distribution ftn (d.f) of the bivariate r.v (X,Y) is defined by

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du.$$

Note:- Since $\frac{d}{dx}F(x) = f(x)$

But in case of bivariate r.v (X,Y)

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y),$$ where

F is differentiable.

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The marginal p.d.f of continuous r.v X is $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

If that of the r.v Y is

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Conditional p.d.f of the continuous r.v X given that Y takes the value y, is defined to be

$$f(x/y) = \frac{f(x,y)}{h(y)}$$

where $f(x,y)$ and $h(y)$ are respectively the joint pdf of X & Y, and the marginal p.d.f of Y respectively. Also $h(y) > 0$.

Similarly Conditional p.d.f of the continuous r.v Y given that $X = x$ is

$$f(y/x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0$$

Note:- Conditional pdf's of bivariate r.v, satisfies all the requirements for a univariate density fn

X

(14)

P#214 (Garcia) Marginal Cummulative distribution functions

$$F_X(x) = F_{X,Y}(x,\infty) = P[X \le x; Y < \infty]$$

$$= P[X \le x].$$

$$\& \; F_Y(y) = F_{X,Y}(\infty, y) = P[Y \le y].$$

Exp#4.8 P#215 (Garcia) The joint cdf of X & Y is given by

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}); & x \ge 0, y \ge 0 \\ 0; & \text{elsewhere} \end{cases}$$

Find marginal cdf's.

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = 1 - e^{-\alpha x}; \; x \ge 0$$

$$\& \; F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = 1 - e^{-\beta y}; \; y \ge 0$$

Thus X & Y individually have exponential distributions with parameters α & β respectively.

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(15)

Question:- Given that joint pdf

$$f(x,y) = \frac{1}{8}(6-x-y) \; ; \; 0 \le x \le 2, \; 2 \le y \le 4.$$
$$= 0 \; ; \text{ elsewhere.}$$

(a) Verify that $f(x,y)$ is joint density fn

(b) Calculate (i) $P\left[X \le \frac{3}{2}, Y \le \frac{5}{2}\right]$;

(c) Find the marginal pdf $g(x)$ & $h(y)$.

(d) Find the conditional pdf $f(x/y)$ and $f(y/x)$ & verify that it is pdf.

Sol:- Hints

(a) (i) $f(x,y) \geq 0 \;\; \forall (x,y)$.

Check these two conditions

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

i.e. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{8}\int_{0}^{2}\int_{2}^{4}(6-x-y)\,dy\,dx$

$= 1$ (Check).

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Q./Part No. (16) (i) $P[X \le 3/2,\ Y \le 5/2] = \int_{x=0}^{3/2}\int_{y=2}^{5/2} \frac{1}{8}(6-x-y)\,dy\,dx$ Rough Work

$= 9/32$ (Check ?)

~~(ii) $P[X+Y \le 3] = 5/8$~~

(ii) $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad -\infty < x < \infty$

$= \frac{1}{8}\int_{2}^{4} (6-x-y)\,dy \quad ; \quad 0 \le x \le 2$

$= \frac{1}{4}(3-x) \quad ; \quad 0 \le x \le 2$

(Check ?)

$= 0 \quad ; \quad x < 0$ or $x > 2$

Also $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad ; \quad -\infty < y < \infty$

$= \frac{1}{8}\int_{0}^{2} (6-x-y)\,dx \quad ; \quad 2 \le y \le 4$

(Check this)

$= \frac{1}{4}(5-y) \quad ; \quad 2 \le y \le 4$

$= 0$ elsewhere

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(17) (P)

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ANSWER BOOK

18293

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Q./Part No. Rough Work

(d) $f(x/y) = \dfrac{f(x,y)}{h(y)}$, where $h(y) > 0$

$$= \frac{1/8(6-x-y)}{1/4(5-y)} = \frac{6-x-y}{2(5-y)}$$

Also $f(y/x) = \dfrac{f(x,y)}{g(x)}$; $g(x) > 0$

$$= \frac{1/8(6-x-y)}{1/4(3-x)} = \frac{6-x-y}{2(3-x)}.$$

Question:- Let the bivariate continuous r.v (X, Y) have the joint pdf given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} ; & 0 \le x \le 1 ; 0 \le y \le 2 \\ 0 ; & \text{elsewhere} \end{cases}$$

(a) check that $f(x,y)$ is a pdf.

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(b) Find the marginal pdf's.

(c) Find the conditional pdf and verify that $f(x/y)$ is a pdf

(Try)

Exp# 4.10 P#219 (Garcia)

Given that

$$f_{X,Y}(x,y) = \begin{cases} 1; & 0 \le x \le 1 \ \& \ 0 \le y \le 1 \\ 0; & \text{elsewhere} \end{cases}$$

Find cdf. Try.

Exp# 4.9:- (Garcia) P#216 (Try)

Note: Find Prob of A, B only.

Exp# 4.11 Garcia P#220 Try.

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~~Marks invol.~~

Expected values of two or more r.vs:-

Let X & Y be two discrete r.vs then $E(X+Y) = E(X) + E(Y)$.

Proof:- $E(X+Y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$

$$= \sum_i \sum_j x_i f(x_i, y_j) + \sum_i \sum_j y_j f(x_i, y_j)$$

But $\sum_i \sum_j x_i f(x_i, y_j)$

$$= \sum_i x_i \sum_j f(x_i, y_j)$$

$$= \sum_i x_i \left[f(x_i, y_1) + f(x_i, y_2) + \cdots + f(x_i, y_n) \right]$$

$$= \sum_i x_i g(x_i) \quad \{\because \text{All possible values of } y \text{ are included in } \sum_j\}$$

$$= E(X).$$

Similarly

$$\sum_i \sum_j y_j f(x_i, y_j) = E(Y).$$

Hence $E(X+Y) = E(X) + E(Y)$.

Proved.

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Q./Part No. Rough Work

In a similar way we can prove that $E(X-Y) = E(X) - E(Y)$

Also extend this to n indifferent r.vs. i.e

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

or $E\left[\sum_i X_i\right] = \sum_i \left[E(X_i)\right]$.

Also $E(XY) = E(X) \cdot E(Y)$

if X & Y are independent r.vs.

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Q 1

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FAST

ANSWER BOOK

Serial No. 18298

Course

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Rough Work

Q./Part No.

Question:- If X & Y are two independent r.v's such that $g(x) = \frac{1}{3}$ for $x = 1, 2, 3$ and $h(y) = \frac{1}{2}$ for $y = 0, 1$.

If $Z = 2X - Y$, then verify that $E(Z) = 2E(X) - E(Y)$.

Sol:- The joint dist. of two r.v's is

Y \ X	1	2	3	h(y)
0	1/6	1/6	1/6	1/2
1	1/6	1/6	1/6	1/2
g(x)	1/3	1/3	1/3	1

$\therefore g(x)h(y) = f(x,y)$

Coz X & Y are independent.

Now $E(X) = \sum x g(x) = (1 \times \frac{1}{3}) + (2 \times \frac{1}{3}) + (3 \times \frac{1}{3})$

$= \boxed{2}$

$E(Y) = \sum y h(y) = (0 \times \frac{1}{2}) + (1 \times \frac{1}{2}) = \boxed{\frac{1}{2}}$

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$$E(Z) = E[2X - Y] = 2E(X) - E(Y)$$

$$= 2(2) - \frac{1}{2} = \boxed{7/2}$$

Now for verification, we find $E(Z) = E[2X-Y]$ directly.

$$E(Z) = E[2X-Y] = \sum_i \sum_j (2x_i - y_j) f(x_i, y_j).$$

$$= [2(1)-0]\tfrac{1}{6} + [2(1)-1]\tfrac{1}{6} + [2(2)-0]\tfrac{1}{6}$$

$$+ [2(2)-1]\tfrac{1}{6} + [2(3)-0]\tfrac{1}{6} + [2(3)-1]\tfrac{1}{6}$$

$$= \frac{2}{6} + \frac{1}{6} + \frac{4}{6} + \frac{3}{6} + \frac{6}{6} + \frac{5}{6}$$

$$= \frac{21}{6} = \boxed{7/2}.$$

Hence the result follows.

Question:- Let X & Y be independent r.v's with joint pdf $f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

Find $E(X)$, $E(Y)$, $E(X+Y)$ & $E(XY)$.

Sol:- Hint: Since $E(X) = \int_{-\infty}^{\infty} x g(x)\,dx$

$$E(Y) = \int_{-\infty}^{\infty} y h(y)\,dy$$

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$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx\, dy.$$

Use.

or $E(X+Y) = E(X) + E(Y)$.

Also $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$.

OR Use $E(XY) = E(X)E(Y)$ As X & Y are independent.

Also $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 \frac{x(1+3y^2)}{4}\, dy$

$$= \frac{1}{4}\left[\int_0^1 (x+3xy^2)\, dy\right].$$

$$= \frac{1}{4}\left[xy + \frac{3xy^3}{3}\right]_0^1$$

$$= \frac{1}{4}\left[xy + xy^3\right]_0^1 = \frac{1}{4}\left[x + x - 0\right].$$

$$g(x) = \frac{2x}{4} = \boxed{\frac{x}{2}}; \quad \text{for } 0 < x < 2$$

So $g(x) = x/2$; for $0 < x < 2$.

Similarly $h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^2 \frac{x(1+3y^2)}{4}$

$= \frac{1}{2}(1+3y^2)$; for $0<y<1$

Try

(See P# 249 - Garcia for more detail)

Covariance of two r.vs:- the Covariance of two r.v's X & Y is a numerical measure of the extent to which their values tends to increase or decrease together. It is denoted by σ_{XY} or $Cov(X,Y)$ and is defined as the expected value of the product $[X-E(X)][Y-E(Y)]$. i.e

$$Cov(X,Y) = E\{[X-E(X)][Y-E(Y)]\}$$

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)$$

$$= E(XY) - E(X)\cdot E(Y).$$

If X & Y are independent then

$$E(XY) = E(X)\cdot E(Y) \text{ and}$$

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y) = 0.$$

Note:- It is important to note that Covariance of X & Y is zero when the r.v's X & Y are

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independent but its converse is not generally true. The covariance of a r.v with itself is obviously its variance.

(See Gorakh P#249-250) (for more details)

Correlation Coefficient of random variables:

Let X & Y be two r.v's with non zero variances σ_x^2 and σ_y^2. Then the correlation coefficient which is a measure of linear relationship between X & Y, denoted by ρ_{xy} or $Cor(X,Y)$, is defined as

$$\rho_{xy} = \frac{E\{[X-E(X)][Y-E(Y)]\}}{\sigma_x \cdot \sigma_y} = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

Note: If X & Y are independent, then ρ_{xy} will be zero but zero correlation does not necessarily imply independence.

Correlation coefficient has the following properties:

(i) Correlation coefficient is unitless and symmetric in X & Y i.e $\rho_{xy} = \rho_{yx}$.

(ii) Correlation coefficient remains unchanged if

Constants are added to the r.v's or of the r.v's are multiplied by constants having the same sign.

(iii) Correlation Coefficient is a number between -1 and $+1$ i.e $-1 \leq \rho_{xy} \leq +1$.

for more details see P# 250 (Garcia)

Question:- From the following joint p.d of X & Y, find $Var(X)$, $Var(Y)$, $Cov(X,Y)$ and ρ_{xy}.

X \ Y	0	1	2	3	g(x)
0	0.05	0.05	0.10	0	0.20
1	0.05	0.10	0.25	0.10	0.50
2	0	0.15	0.10	0.05	0.30
h(y)	0.10	0.30	0.45	0.15	1.00

Sol:- Hint:- $Var(X) = E(X^2) - [E(X)]^2$.

$Var(Y) = E(Y^2) - [E(Y)]^2$.

Also $E(X) = \sum_i x_i g(x_i)$; $E(X^2) = \sum_i x_i^2 g(x_i)$

& $E(Y) = \sum_j y_j h(y_j)$; $E(Y^2) = \sum_j y_j^2 h(y_j)$

Also $E(XY) = \sum_i \sum_j (x_i y_j) f(x_i, y_j) = \boxed{1.90}$

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$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y) = \boxed{0.085}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \boxed{0.14}$$

Try

Q $f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 \le x \le 1,\ 0 < y \le 2 \\ 0, & \text{otherwise} \end{cases}$

Find $Var(X)$, $Var(Y)$ and $\rho_{X,Y}$.

Sol:- Hint:- $E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx$

$$E(Y) = \int_{-\infty}^{\infty} y\, h(y)\, dy.$$

$Var(X) = 73/1620$

$Var(Y) = 26/81$

$Cov(X,Y) = -1/162.$

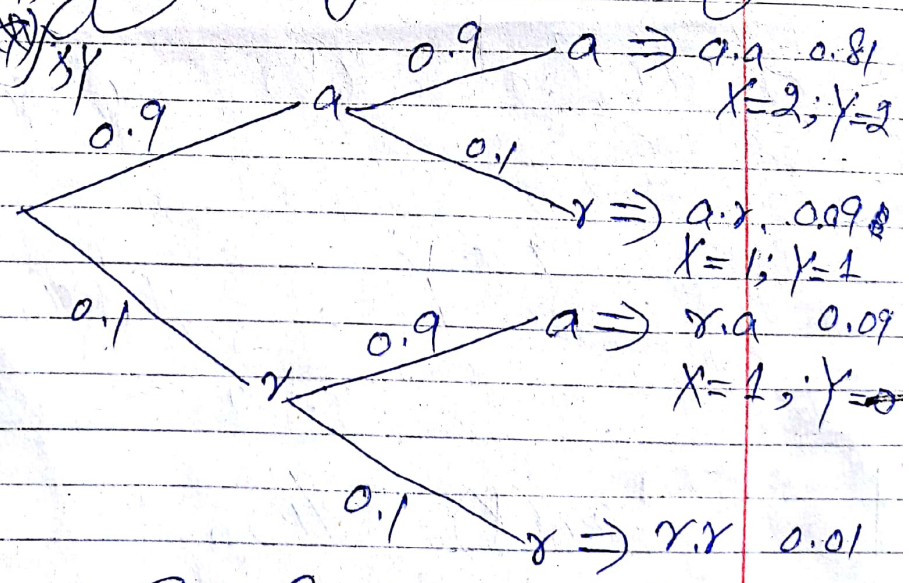
$\rho_{X,Y} = \boxed{-0.05}$

Try

Q./Part No. Rough Work

Question. Test two integrated circuits one after another. On each test, the possible outcomes are "a" (acceptable) and "r" (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of tests, Y before observing the first reject. Draw the diagram for the experiment and find the joint PMF of X & Y. Also find Cov(X,Y).

Sol:-



The sample space of the underlying experiment is $S = \{aa, ar, ra, rr\}$.

Observing the diagram, we compute

$P[aa] = 0.81$; $P[ar] = P[ra] = 0.09$;

$P[rr] = 0.01$

So joint PMF can be written as:

$$f_{X,Y}(x,y) = \begin{cases} 0.81; & x=2;\ y=2 \\ 0.09; & x=1;\ y=1 \\ 0.09; & x=1;\ y=0 \\ 0.01; & x=0;\ y=0 \\ 0; & \text{otherwise} \end{cases}$$

Also table for prob. dist. is

X \ Y	0	1	2	g(x) → Marginal Probs. of X.
0	0.01	0	0	0.01
1	0.09	0.09	0	0.18
2	0	0	0.81	0.81
h(y)	0.1	0.09	0.81	1

Marginal Probs. of Y

Now find $Cov(X,Y)$ & $Corr(X,Y)$.

Hint:- $Cov(X,Y) = E(XY) - E(X)E(Y)$

& $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$

Try

Few of the examples of Correlation Coefficient

(i) X is the hight of the student. Y is the weight of the same student. i.e $0 < \rho_{x,y} < 1$

(ii) X is the distance of a cellular phone from nearest base station. Y is the power of the signal at the cellular phone. i.e $-1 < \rho_{x,y} < 0$.

(iii) X is the temperature of the resistor measured in degrees celsius. Y is the temperature of the same resistor measured in degree kelvin. i.e $\rho_{x,y} = 1$.

(iv) X is the telephone number of a cellular phone. Y is the social security number of the phone's owner. $\rho_{x,y} = 0$.

Question:- A firm sends out two kind of promotional facsimiles. One kind contains only text and requires 40 seconds to transmit each page. The other kind contains

grayscale pictures that takes 60 seconds per page. faxes can be 1, 2 or 3 page long. Let the random variable L represents the length of a fax in pages, $S_L = \{1, 2, 3\}$. Let the random variable T represents the time to send each page, $S_T = \{40, 60\}$. After observing many fax transmissions, the firm defines the following probability model for L and T

L \ T	$t = 40$ sec	$t = 60$ sec
$l = 1$ page	0.15	0.1
$l = 2$ page	0.3	0.2
$l = 3$ page	0.15	0.1

Calculate (i) the mean $E[L]$. (ii) $E[T]$. $E(L)$

(iii) $Var[L]$ (iv) $Var[T]$. (v) the correlation $r_{L,T}$.

(vi) $Cov(L, T)$.

(vii) $\rho_{L,T}$ (Correlation Cofficient).

Try