THE NATURE AND USE OF THE HARMONIC MEAN

BY WIRTH F. FERGER, *University of North Carolina*

It appears from an examination of the treatment of the harmonic mean (the reciprocal of the arithmetic mean of the reciprocals of the items being averaged)[1] in a number of the popular textbooks on statistical methods, that there is a good deal of doubt and misapprehension in the minds of even our experts as to its true nature and usefulness. The current confusion was vividly brought home to the writer a few years ago the first time he tried to explain its use to a class of college juniors. Having been taught that the harmonic mean should be employed in averaging time rates, he used the example of the average rate at which three men load dirt into a wagon, one loading two, another three, and another four wagons per hour. Seeking to show that the arithmetic mean of three loads per hour was incorrect, he found, in front of the class, that it was perfectly correct—if it be agreed that the men all work the same length of time.

Several texts consulted at that time failed to elucidate the problem; some have no mention of the harmonic mean, while others merely give its definition with no consideration of its nature or use. Some texts give illustrations which are valid, but incomplete for a full understanding of the problem; a few give misleading and even impossible illustrations. Space limitations prevent detailed criticism: suffice it to say that all but four[2] of the 32 texts so far consulted leave much to be desired in adequacy, clarity or accuracy, in their treatment of the harmonic mean.

In presenting to a class the nature and usefulness of the harmonic mean, the writer has found the following illustration his favorite:

Let us consider three automobiles, A, B, and C.

Method I $\left\{\begin{array}{l} A \text{ travels at the rate of 10 miles per hour.} \\ B \text{ travels at the rate of 20 miles per hour.} \\ C \text{ travels at the rate of 30 miles per hour.} \end{array}\right.$

What is the average rate of speed for these cars? The arithmetic mean is 20 m.p.h., while the harmonic mean is 16.4 m.p.h. Which is

[1] The formula is $H = \dfrac{1}{\dfrac{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n}}{N}} = \dfrac{N}{\sum\frac{1}{x}}$, where $x_1, x_2 \ldots x_n$ represent the N values taken by the variable.

[2] Robert Wilber Burgess, *Introduction to the Mathematics of Statistics;* Truman L. Kelley, *Statistical Method;* C. W. Odell, *Educational Statistics;* Harold O. Rugg, *Statistical Methods Applied to Education.*

correct? The problem is indeterminate until it be agreed whether the cars shall be equally weighted in the sense of (T) all cars travelling the same length of *time*, or (D) all cars travelling the same *distance*. To put the matter differently, are the cars taxicabs *operating continuously*, say eight hours a day (T); or are the cars private vehicles carrying a group of people to a point, say 30 miles away, *each car making one trip?* The correct average will differ according to the condition agreed upon. Why?

How do we usually interpret the concept "average speed"? We mean that speed at which all three cars would travel if they maintained equal speeds and traversed the same total distance in the same total time. In case T, the total distance per day is $80+160+240=480$ miles; the total time is 24 hours, so the average speed is 20 miles per hour. In case D the total distance is 90 miles, the total time $3+1\frac{1}{2}+1=5\frac{1}{2}$ hours, so the average speed is 16.4 miles per hour. Then with speeds stated as they are in method I above,

$$\text{Method I}\ \begin{cases}\text{Case } T \text{ requires the use of the arithmetic mean.}\\ \text{Case } D \text{ requires the use of the harmonic mean.}\end{cases}$$

But the speeds of the cars could just as logically be stated as follows:

$$\text{Method II}\ \begin{cases}A \text{ travels at the rate of 6 minutes to the mile.}\\ B \text{ travels at the rate of 3 minutes to the mile.}\\ C \text{ travels at the rate of 2 minutes to the mile.}\end{cases}$$

Now the arithmetic mean is 11/3 minutes to the mile (or 16.4 miles per hour), while the harmonic mean is 3 minutes to the mile (or 20 miles per hour). Taking the same cases T and D above, with speeds stated as in method II,

$$\text{Method II}\ \begin{cases}\text{Case } T \text{ requires the use of the harmonic mean.}\\ \text{Case } D \text{ requires the use of the arithmetic mean.}\end{cases}$$

It is evident, then, that the blanket statement, "the harmonic average must be used in the averaging of time rates" is false. It must be used in *certain* cases of averaging ratios (not only time rates) where certain conditions are agreed upon. It differs from the arithmetic mean in giving more weight to the smaller items being averaged. (The reciprocal of a large number is small, and the reciprocal of a small number is large, relatively.) This is evidently appropriate in case I D, where the slow car (A) exercises more influence on the total time, and thus on the average speed. It is likewise appropriate in case II T, where the fast car (C) exercises so much more influence on the total distance travelled.

To summarize the rule, the principal use of the harmonic mean is in certain cases of averaging rates. Rates, being ratios, may always be stated in two forms, keeping one or the other of the factors constant —in the above illustration, (T) time, or (D) distance. The conditions of the problem for which an average is sought will determine which of the factors *should* be kept constant (whether the cars shall all travel equal distances or equal times). If it is desired to keep constant that factor which is constant in the rates as stated, then the arithmetic mean should be used: if, on the contrary, the recorded rates make variable the factor desired to be constant, then the harmonic mean is the correct average to be employed.

A few further illustrations will show that this problem is frequently encountered, and should be thoroughly understood by all statisticians and students in research. This is being written on the day following the Indianapolis Speedway races. What was the average speed of all cars finishing the 500-mile race? Clearly the correct average (if the speed of each car is stated in miles per hour) is the harmonic mean. On the other hand, the average time of several runners in the 100-yard race would be the arithmetic mean (since the speed is stated in seconds per 100 yards). If a time-and-motion study expert records the time taken by different workers to perform a given operation, what is the average time of all workers? Clearly the harmonic mean, if all the workers put in the same number of hours per day. If data of ship speeds be given, as is commonly done, in the form of so many days per turn-around (round trip), the average speed of several ships would obviously be the harmonic mean, since the ships operate continuously. Suppose we record the number of arithmetic problems correctly completed by different pupils in a five-minute test: what is the average rate of speed at which they work? If the pupils be thought of as all working the same length of time, then the arithmetic mean is correct: but if the average rate at which they all prepare the same assignment of home-work be sought, then only the harmonic mean is accurate. Burgess [1] differs with Kelley [2] as to which of these interpretations gives the more meaningful answer.

A still more harassing situation is met in the computation of index numbers of prices. Prices are, of course, ratios, and may be stated either as so much per unit, or so many units to the dollar. Clearly the arithmetic mean of prices in one form is equivalent to the harmonic mean of the same data stated in the other form. (The reason for this is evident when it is noted that the prices stated in one form are merely the reciprocals of the same prices stated in the other form.) But, given prices stated in the customary manner of so many dollars

[1] Robert Wilber Burgess, *Introduction to the Mathematics of Statistics*, p. 95.

[2] Truman L. Kelley. *Statistical Method*, p. 64.

per unit, which is the correct average? What would we be assuming if we used the harmonic mean? We would be assuming that, with the prices of given commodities changing, people spend the same amount of money, getting more or fewer units for their money. The use of the arithmetic mean assumes that people, during rising prices, buy the same *quantities* of goods, paying more for them. Which is the case? The answer is not clear. For a measure of prices over a long period (as, for instance, to show the trends of prices since 1875) it would seem that the latter would more nearly fit the facts. As the prices of all or most commodities change (a change in the price level, or the purchasing power of money), there is a tendency for wages and salaries also to change in the same direction—the price of labor tends to follow the course of other prices—so that people's purchasing power *tends* to remain constant. There are time lags and "economic friction" involved here, but the usual assumption that the quantity of goods remains constant is probably nearer the truth than the opposite assumption. In this case the problem is not in aggravated form, however, because changing weights can be approximately determined and used (as in the "Ideal" index formula).

But for most practical purposes in current month-to-month indexes, constant weights must be employed. Now the problem is insistent: should the weight of a commodity be kept constant from month to month in the sense of equal *quantities* being bought, or in the sense of the *expenditure* remaining the same? The latter would seem to command serious consideration in view of the elasticity of the demand for many, if not most, commodities, and since the incomes of most people do not fluctuate closely or immediately with the cost of living. This latter assumption would obviously require the use of the harmonic mean of relatives of prices stated in the usual form of so much per unit. It matters not whether the elasticity of demand is due to the commodity being a luxury, or to the availability of a satisfactory substitute. Even "necessities" may conform to this tendency: consider the weight which should be given, from month to month, to "eggs, strictly fresh," or potatoes, or oranges, or cabbage. Even necessities such as shoes and clothes may have elastic demands, especially when quality, looks and style are considered.

It should be noted that the problem is not a decision between unweighted and weighted index numbers, but between the senses in which given weights are to be held constant between the base period and the given period. A weighted harmonic mean $\left(H = \frac{\Sigma w}{\sum \frac{w}{x}}\right)$ of relatives is implied as against a weighted arithmetic mean. The formula, using the common notations of zero subscripts to denote base year, and 1 subscripts to denote given year prices, quantities, and values, follows:

Harmonic Average-of-Relatives Price Index

$$= \frac{\Sigma v_0}{\sum \frac{v_0}{p_1/p_0}} = \frac{\Sigma v_0}{\sum \frac{v_0 p_0}{p_1}} = \frac{\Sigma p_0 q_0}{\sum \frac{p_0^2 q_0}{p_1}}.$$

The question may be put in another form: does not the "downward bias" of the harmonic type of formula really represent a closer approach to actuality in measuring prices (with fixed weights) than the formulas in common use—the aggregative and arithmetic or geometric average-of-relatives? Our economic theory explains the ordering of our economic life through the price system—the function of prices is to stimulate or discourage consumption as well as production, on the basis of the elasticity of demand. Should we not recognize in our index numbers that people do *not* buy equal *quantities* of goods as prices vary, but change their purchasing habits with shifting prices. Does not the "downward bias" represent actual economies of the housewife in exercising common sense in the selection of commodities, and so more nearly approximate actual weights?

Two peculiarities of the harmonic mean should be mentioned. Several writers have pointed out the fact that the harmonic mean of a given distribution is always smaller than both the arithmetic and geometric means. In this connection it would be well to correct an error which (doubtless through an oversight) crept into Žižek's study [1] and which has been propagated in at least two subsequent texts. The more exact relationship, that "of the three means of the same set of values the geometric is the geometric mean of the other two" is true only in the special case of averaging just two items.

The second peculiarity is that the harmonic mean cannot be employed for the average of a distribution where the variable even once takes the value zero. Reflection will show the reason: the taking of reciprocals in this case would involve division by zero, an operation excluded from arithmetic and algebra because meaningless. This mathematical limitation exposes a logical inconsistency in our problem: we should not seek the average speed at which several autos travel (all to travel equal distances) when one is standing still!

[1] Franz Žižek, *Statistical Averages, A Methodological Study*, p. 132.