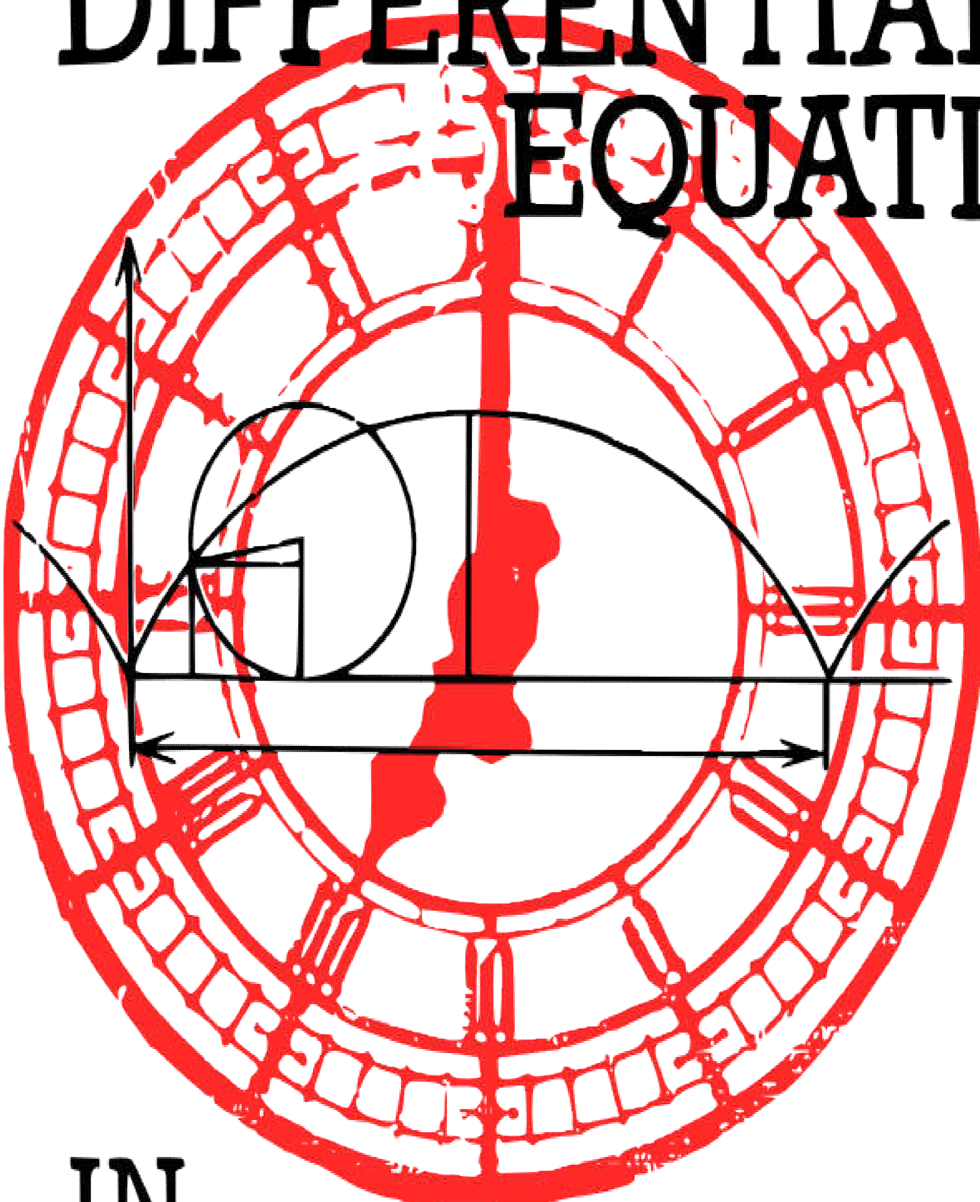
DIFFERENTIAL EQUATIONS



IN APPLICATIONS

FINAL PROJECT

SUBMITTED BY: GROUP NO# 01

SUBMITTED TO: DR. SABA INAAM

BS MATHEMATICAL SCIENCES

SEMESTER NO: 04

COURSE TITLE: INTRODUCTION TO DIFFERENTIAL EQUATIONS

TOPIC OF PROJECT:-

"POPULATION GROWTH MODEL"

DATE OF SUBMISSION: 22ND JUNE, 2020

GROUP MEMBERS:-

- ✓ SHAFIA MARYAM-044
- ✓ TOOBA KHALID-046
- ✓ NAGINA REHMAN-031
- ✓ MAMOONA BATOOL-025
- ✓ UMM-E-HABIBA-048
- ✓ HIRA-015
- ✓ HALEEMA MUNIR-011

Fatima Jinnah Women University

Opening Portals of Excellence Through Higher Education

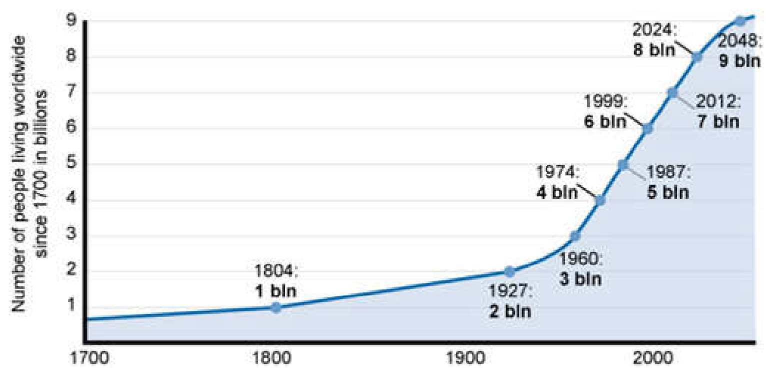
CONTENT PAGE:

Population Growth Model

❖ Population Growth:-

Population Growth is the change in Population over time or change in the number of individuals in a Population per unit time. Population Growth rate is the rate at which the individuals in a Population increases. Typically, both for human and non-human Populations; we want to know the average growth rate to predict future years of growth or decline for Population.

General Human Population growth (estimated by growth models):



❖ General Equation:-

To understand the differential models that is used to represent population dynamic. Let's start by looking at a General Equation for population growth rate(change in the number of individuals over time):

$$\frac{dN}{dt} = rN$$

In this equation, $\frac{dN}{dt}$ is the growth rate of the population in a given instant, N is the Population size is time; r is the per capita rate of increase that is how quickly the population grows per individuals already in the Population.

❖ Types of Population Growth model:-

Environmental Scientists use two models to describe how population grows over time:

- Exponential Growth model
- Logistic Growth Model

➢ Concepts related to these growth models are:

✓ ***Carrying Capacity (K):-***

Carrying capacity is the number of individuals that the available resources can support.

In equations and models, it is represented by K.

- ***Limiting resources:-***

A limiting resource is a resource that organism must have in order to survive and that is available in fewer amounts.

LOGISTIC GROWTH MODEL

"When the growth rate of population decreases as the number of individuals increase, this is called Logistic Population Growth".

As long as there are enough resources available, there will be an increase in the number of individuals in a population over time, or a positive growth rate. However, most populations cannot continue to grow forever because they will eventually run out of water, food, sunlight, space or other resources. As these resources begin to run out, population growth will start to slow down.

- **The logistic model:-**

A fundamental growth model in ecology is Logistic Growth Model. In one respect Logistic Growth Model is more realistic than exponential growth because Logistic growth is not unbounded. We can write logistic model as:

$$P(t) = \frac{K.P_\circ}{(K - P_\circ).e^{-rt} + P_\circ}$$

Where P(t) is the population size at time t, assuming that time is measured in days. $P_\circ$ Is the initial population size, K is the carrying capacity of environment and r is the constant representing the rate of the population growth.

- **Explanation:-**

Verhulst proposed a model, called the logistic model, for population growth in 1838. It does not assume unlimited resources. Instead, it assumes there is a carrying capacity K for the population. This carrying capacity is the stable population level. If the population is above K, then the population will decrease, but if below, then it will increase.

For this model it is assumed that the rate of change $\frac{dy}{dt}$ of the population 'y' is proportional to the product of the current population y and K − y, or what is the same thing, proportion to the product y(1 − y/K). That gives us the logistic differential equation:-

$$\boldsymbol{\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)}$$

Here, r is a positive constant. Note that when y < K, $\frac{dy}{dt}$is positive, so y increases; but when y < K, $\frac{dy}{dt}$is negative, so y decreases. We can solve this differential equation by the method of separation of variables. First, separate the variables to get

$$\frac{1dy}{y\left(1 - \frac{y}{K}\right)} = rdt$$

Now, by integration; we get:-

$$\int \frac{1dy}{y\left(1-\frac{y}{K}\right)} = \int rdt \qquad \ldots\ldots.(*)$$

Now, we know that

$\int rdt = rt + c$ 'Remember c is any constant of integration'

What about the integral on the left side of the equation $(*)$?

To integrate the left hand side, we will use the method of ***partial integration***. Fraction $\frac{1dy}{y\left(1-\frac{y}{K}\right)}$ can be written as the sum of two simpler rational partial fractions:-

$$\frac{1\mathrm{dy}}{\mathrm{y}\left(1-\frac{\mathrm{y}}{\mathrm{K}}\right)} = \frac{\mathrm{A}}{\mathrm{y}} + \frac{\mathrm{B}}{1-\frac{\mathrm{y}}{\mathrm{K}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Here, A and B are the coefficients that are to be determined. Now clearing the denominators, we get:-

$$1 = \mathrm{A}\left(1 - \frac{\mathrm{y}}{\mathrm{K}}\right) + \mathrm{By} = \mathrm{A} - \frac{\mathrm{A}}{K}\mathrm{y} + \mathrm{By}$$

From here, by solving we get: A=1 and B=1/K

Thus substituting in equation 1:

$$\frac{1\mathrm{dy}}{\mathrm{y}\left(1 - \frac{\mathrm{y}}{\mathrm{K}}\right)} = \frac{1}{\mathrm{y}} + \frac{\frac{1}{\mathrm{K}}}{1 - \frac{\mathrm{y}}{\mathrm{K}}} = \frac{1}{y} + \frac{1}{K - y}$$

So, putting this value in Left hand side of equation $(*)$:-

$$\int \frac{1dy}{y\left(1-\frac{y}{K}\right)} = \int \frac{dy}{y} + \int \frac{dy}{K-y}$$

$$= \ln|y| - \ln|K-y|$$

$$= \ln\left|\frac{y}{y-K}\right|$$

Now, we have the left and right hand side of equation $(*)$:-

$$\ln\left|\frac{y}{y-K}\right| = rt + c$$

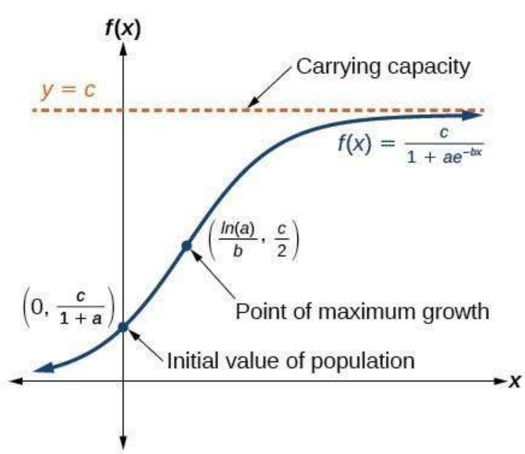
Now using little algebra, we get the ***general solution of logistic equation***:-

$$\boldsymbol{y = \frac{K}{1 + Ae^{-rt}}}$$

And here, $A = \frac{K - P_{\circ}}{P_{\circ}}$

Here, A is a constant.

- **<u>Graph of Logistic Growth Model:-</u>**

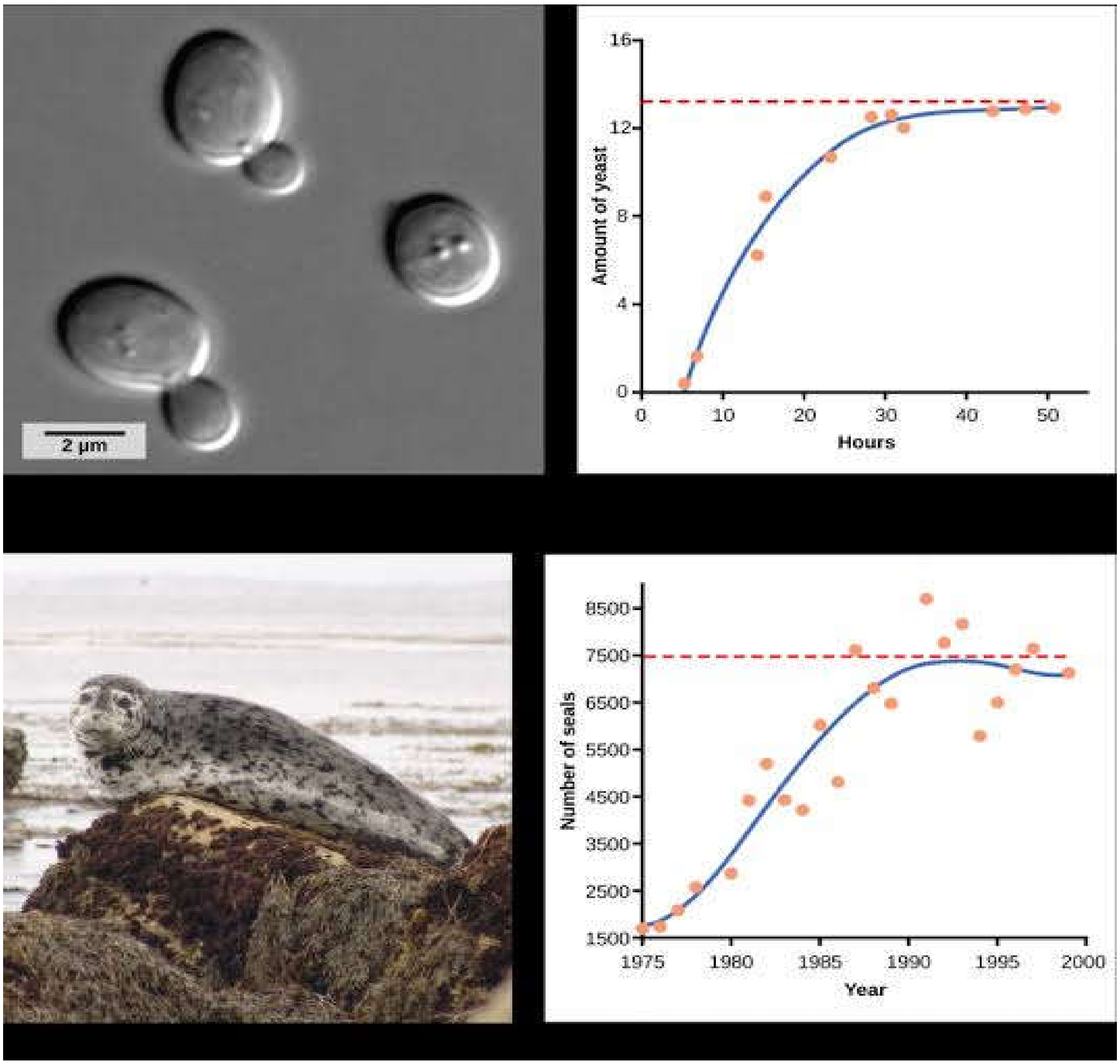


If we look at the graph undergoing the logistic population growth, it will have a characteristic S-shaped curve. The population grows in size slowly when there are only a few individuals. Then, the population grows faster when there are more individuals. Finally lots of individuals in the population cause growth to slow because resources are limited. In Logistic growth, a population will continue to grow until it reaches the carrying capacity which is maximum number of individuals, the environment can support.

- **<u>Example of Logistic Growth:-</u>**

Yeast, a microscopic fungus used to make bread and alcoholic beverages, exhibits the classical S-shaped curve when grown in a test tube. It's growth levels off as the population depletes the nutrients that are necessary for its growth. In the real world, however, there are variations to this idealized curve. Examples in wild populations include sheep and harbor seals.

In both examples, the population size exceeds the carrying capacity for short periods of time, and then falls below the carrying capacity afterwards. This fluctuation in population size continues to occur as the population oscillates around its carrying capacity. Still, even with this oscillation, the logistic model is confirmed.



Numerical Problems:-

- **Numerical No 01:**

If dp/dt=0.08(1-$\frac{P}{1000}$) and P(0)=100; hence find P(40) and P(80). As well as find the time when the population of specie reaches 900? Use Logistic population model.

- **<u>Solution:</u>**

The carrying capacity K= 1000

The solution of logistic equation given is :

$$P(t) = \frac{K}{1 + Ae^{-rt}}$$

Where K=1000; r=0.08; $A = \frac{K - P_\circ}{P_\circ}$

Here, A becomes 900/100. So,

A=9

$$P(t) = \frac{1000}{1 + 9e^{-(0.08)t}}$$

So,

$$P(40) = \frac{1000}{1 + 9e^{-(0.08)(40)}}$$

P(40)=732

Now,

$$P(80) = \frac{1000}{1 + 9e^{-(0.08)(80)}}$$

P(80)=985

Now, we find P(t)=900

$$900 = \frac{1000}{1 + 9e^{-(0.08)(t)}}$$

$$1 + 9e^{-0.08t} = 9$$

$$9e^{-0.08t} = 8$$

$$e^{-0.08t} = \frac{8}{9}$$

Now; taking log on both sides:

$$-0.08t = \ln\left|\frac{8}{9}\right|$$

$$\boldsymbol{t = \frac{\ln\left|\frac{8}{9}\right|}{-0.08} \approx -53.46}$$

- **Numerical No 02:**

Suppose a population of butterflies is growing according to the logistic equation. If the carrying capacity is 500 butterflies and r=0.1 individuals (individuals*month), what is the maximum growth rate for the population?

- **Solution:**

To solve this, we must first determine N, population size. From the plot of $\frac{dN}{dt}$ versus N, we know the possible growth rate for population according to the logistic model, occurs when N=$\frac{K}{2}$. Here N = 250 butterflies. Plugging this into the logistic equation:-

$$\frac{dN}{dt} = rN\left[1 - (\frac{N}{K})\right]$$

$$= 0.1(250)[1 - \frac{250}{500}]$$

=12.5 individuals per month

EXPONENTIAL GROWTH MODEL

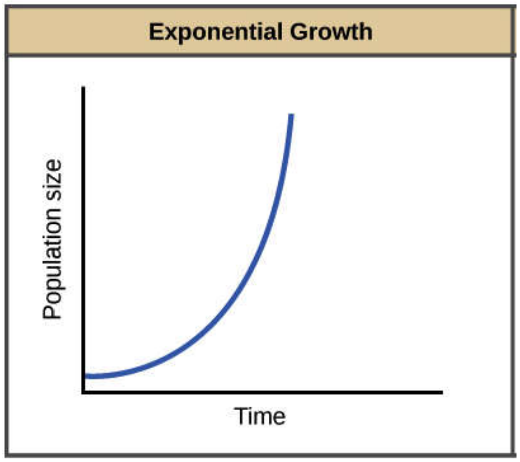
- **Exponential growth:-**

When the per capita rate increase r takes the same positive value regardless of population size, then we get exponential growth

OR

In exponential growth, the population growth rate increases over time; in proportion to the size of the population.

Simple graph of Exponential Growth Model :



Explanation:-

We get exponential growth when r (the per capita rate of increase) for our population is positive and constant. While any positive, constant r can lead to exponential growth, exponential growth represented with an r of r_{max} .

r_{max} is the maximum per capita rate of increase for particular species under ideal conditions. And it varies from specie to specie. The maximum population growth rate for specie also called its biotic potential is expressed in the following equation:-

$$\boldsymbol{\frac{dN}{dt} = r_{max}N}$$

Derivation:-

The differential equation describing the exponential growth is:

$$\frac{dN}{dt} = r_{max}N$$

This can be integrated directly:

$$\int_{N_\circ}^{N} \frac{dN}{N} = \int_{0}^{r} rdt$$

By solving this:

$$\ln |N/N_\circ| = rt$$

Where: $N_\circ = N(t = 0)$

Exponentiation gives:-

$$N(t) = N_\circ e^{rt}$$

This equation is called the law of Growth and in a much more antiquated fashion, the Malthusian Equation; the quantity in this equation is sometimes called Malthusian parameters.

- **Examples:**

Bacteria:-

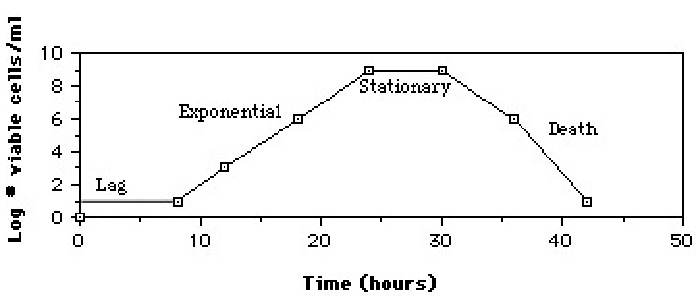
In a laboratory under favorable conditions, a growing bacterial population doubles at regular intervals. Growth is by geometric progression:-

$$1 \cdot 2 \cdot 4 \cdot 8 \text{ or} 2^0 . 2^1 . 2^2 \ldots . 2^n$$

Here n is the number of generations. This is the exponential growth of bacteria. In reality, the exponential growth is only part of bacterial life cycle, and not representative of normal growth of bacteria in nature.

- ***Graphical Explanation:-***

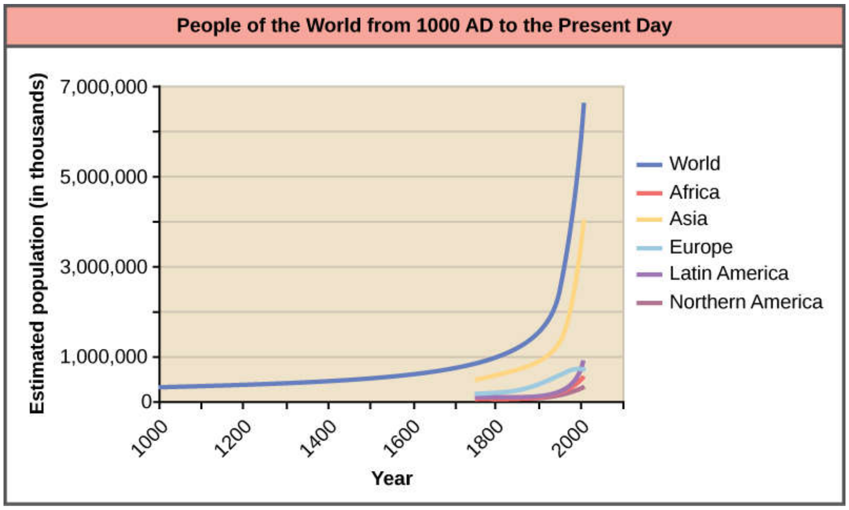
When a fresh medium in inoculated with a given number of cells and the population growth, monitored over a period of time, plotting the data will yield a typical bacterial growth curve. As shown:



It takes bacteria roughly an hour to reproduce through prokaryotic fission. If we placed 100 bacteria in an environment and recorded the population size each hour, we would observe the exponential growth. We would record 200 at start of 2nd hour, 400 at start of 3rd hour and like it so on. Eventually, we would observe a leveling off in population size due to various resources and eco- system constraints.

Human Population:

Human Population is growing exponentially. Human Population is exponential since 1000AD (Dark blue line).



Asia (Yellow line) which has many economically underdeveloped countries is increasing exponentially. Europe (sky blue line) where most of countries are economically developed is growing much more slowly.

Humans have increased the worlds' carrying capacity through migration, agriculture, medical advances and communication. The age structure of population allows us to predict the population growth. Unchecked Human Population could have long-term effects on our environment.

- **Numerical Problems:-**
  - **Numerical no 1:-**

In bacterial culture, there were 2000 bacteria on Tuesday. On Thursday, the number has increased to 4500. Predict the number of bacteria that will be in the culture next Tuesday.

  - **Solution:-**

Our starting quantity is 2000 and the starting time is Tuesday. Thursday is 2 days later, this means:-

a=2000

x=2

y=4500

Putting values in formula:-

$$y = ab^x$$

So we get;

$$4500 = 2000b^2$$

$$b^2 = \frac{4500}{2000}$$

$$b = \sqrt{2.25}$$

So, we get now…

$$b = 1.5$$

Thus our equation is;

$$c = 2000(1.5^t)$$

After interval of 7 days there should be;

$$\boldsymbol{c = 2000(1.5^7)}$$

By calculation of c, we get c as 34,172 colonies.

- **<u>Numerical No 02:-</u>**

Today there are 1000 birds on an island. They breed with a constant continuous growth rate of 10% per year. To 3 significant figures, how many birds on the island after 7 years are present?

- **<u>Solution:-</u>**

Use exponential growth formula;

$$y(t) = a.e^{kt}$$

Here k is a continuous growth rate.

So,

k=10%=0.1

And;

a=1000

$$y(t) = 1000.e^{0.1t}$$

After 7years, the number of birds on the island is:

$$y(7) = 1000.e^{0.1(7)}$$

$= 2010,$ (To 3 significant figures.)