1 Homework03

3.1

$$
\begin{aligned}
\rho_x(1) &= \frac{\gamma(x_t, x_{t-1})}{\gamma(0)} \\
&= \frac{\mathbb{E}\left[x_t x_{t-1}\right]}{\mathrm{var}(w_t + \theta w_{t-1})} \\
&= \frac{\theta \sigma_w^2}{(1+\theta^2)\sigma_w^2} \\
&= \frac{\theta}{1+\theta^2} \\
|\rho_x(1)| &= \frac{|\theta|}{1+\theta^2}
\end{aligned}
$$

The equality holds if and only if

$$|\theta| = 1$$

That is, for $\theta = -1$, $\rho_x(1)$ attains its minimum $-\frac{1}{2}$; for $\theta = 1$, $\rho_x(1)$ attains its maximum $\frac{1}{2}$.

3.2

(a)

$$
\begin{aligned}
x_t &= \sum_{i=0}^{t-1} \phi^i w_{t-i} \\
\mu_x(t) &= 0 \\
\mathrm{var}_x(t) &= \sum_{i=0}^{t-1} \phi^{2i} \sigma_w^2 \\
&= \frac{1-\phi^{2t}}{1-\phi^2} \sigma_w^2
\end{aligned}
$$

$\mathrm{var}_x(t)$ is not a constant, so x_t is nonstationary.

(b)

$$
\begin{aligned}
\mathrm{corr}(x_t, x_{t-h}) &= \frac{\mathbb{E}\left[x_t x_{t-h}\right]}{\sqrt{\mathrm{var}_x(t)\mathrm{var}_x(t-h)}} \\
&= \phi^h \frac{\mathrm{var}_x(t-h)}{\sqrt{\mathrm{var}_x(t)\mathrm{var}_x(t-h)}} \\
&= \phi^h \sqrt{\frac{\mathrm{var}_x(t-h)}{\mathrm{var}_x(t)}}
\end{aligned}
$$

(c) From (a),

$$\begin{aligned}
\mathrm{var}_x(t) &= \frac{1-\phi^{2t}}{1-\phi^2}\sigma_w^2 \\
&\approx \frac{\sigma_w^2}{1-\phi^2} \\
\mathrm{corr}(x_t, x_{t-h}) &= \phi^h \sqrt{\frac{\mathrm{var}_x(t-h)}{\mathrm{var}_x(t)}} \\
&\approx \phi^h
\end{aligned}$$

(d) The simulated observation of x_t has mean 0 and varaince $\frac{\sigma_w^2}{1-\phi^2}$, so we can multiply the iid values by $\sqrt{\frac{\sigma_w^2}{1-\phi^2}}$.

(e) If $x_1 = w_1/\sqrt{1-\phi^2}$,

$$\begin{aligned}
x_t &= \sum_{i=0}^{t-2} \phi^i w_{t-i} + \phi^{t-1} w_1/\sqrt{1-\phi^2} \\
\mathrm{var}_x(t) &= \frac{1-\phi^{2t-2}}{1-\phi^2}\sigma_w^2 + \frac{\phi^{2t-2}}{1-\phi^2}\sigma_w^2 \\
&= \frac{\sigma_w^2}{1-\phi^2} \\
\mathrm{corr}(x_t, x_{t-h}) &= \phi^h \sqrt{\frac{\mathrm{var}_x(t-h)}{\mathrm{var}_x(t)}} \\
&= \phi^h
\end{aligned}$$

so this process is stationary.

3.4

(a)

$$\begin{aligned}
(1-0.8B+0.15B^2)x_t &= (1-0.3B)w_t \\
(1-0.5B)(1-0.3B)x_t &= (1-0.3B)w_t \\
(1-0.5B)x_t &= w_t \\
x_t &= 0.5x_{t-1} + w_t
\end{aligned}$$

it is an AR(1) model. It is causal and invertible.

(b)

$$(1 - B + 0.5B^2)x_t = (1 - B)w_t$$

it is an ARMA(2,1) model, it is causal but not invertible.

3.5

In this problem, z_1, z_2 can't be zero.

(a) If $|z_1| > 1, |z_2| > 1$,

If $|z_1| > 1$ and $|z_2| > 1$, if $z_2 > 0$,

$$\begin{aligned}\phi_1 + \phi_2 &< \left|\frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{z_1 z_2}\right| \\ &= \frac{|z_1 + z_2 - 1|}{|z_1 z_2|} \\ &< \frac{|z_1|}{|z_1 z_2|} < 1\end{aligned}$$

if $z_2 < 0, z_1 > 0$, in a similar way, $\phi_1 + \phi_2 < 1$, if $z_2 < 0, z_1 < 0$, $\phi_1 + \phi_2 < 0 < 1$.

If $z_2 < 0$,

$$\begin{aligned}|\phi_2 - \phi_1| &= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_1 z_2}\right| \\ &= \frac{|z_1 + z_2 + 1|}{|z_1 z_2|} \\ &< \frac{|z_1|}{|z_1 z_2|} < 1\end{aligned}$$

if $z_2 > 0, z_1 < 0$, in a similar way, $\phi_1 + \phi_2 < 1$, if $z_2 > 0, z_1 > 0$, $\phi_1 + \phi_2 < 0 < 1$.

$$|\phi_2| = \frac{1}{z_1 z_2} < 1$$

(b) If $\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1, |\phi_2| < 1$,

$$|z_1 z_2| = \frac{1}{\phi_2} > 1 \tag{1}$$

$$\frac{z_1 + z_2 + 1}{z_1 z_2} = -(\phi_2 - \phi_1) > -1 \tag{2}$$

$$\frac{z_1 + z_2 - 1}{z_1 z_2} = \phi_1 + \phi_2 < 1 \tag{3}$$

if $|z_1| < 1, |z_2| > 1$, if $z_1 < 0$,

$$\frac{z_1 + z_2 + 1}{z_1 z_2} > \frac{z_2}{z_1 z_2} = \frac{1}{z_1} < -1$$

if $z_1 > 0$

$$\frac{z_1 + z_2 - 1}{z_1 z_2} < \frac{z_2}{z_1 z_2} = \frac{1}{z_1} > 1$$

if $|z_1| > 1, |z_2| < 1$, in a similar way, it's contrary with (2) or (3). If $|z_1| < 1, |z_2| < 1$, it's contrary with (1). Thus $|z_1| > 1, |z_2| > 1$.

2 Homework04

3.6

The autoregressive polynomial for this model is

$$\phi(z) = 1 + 0.9z^2$$

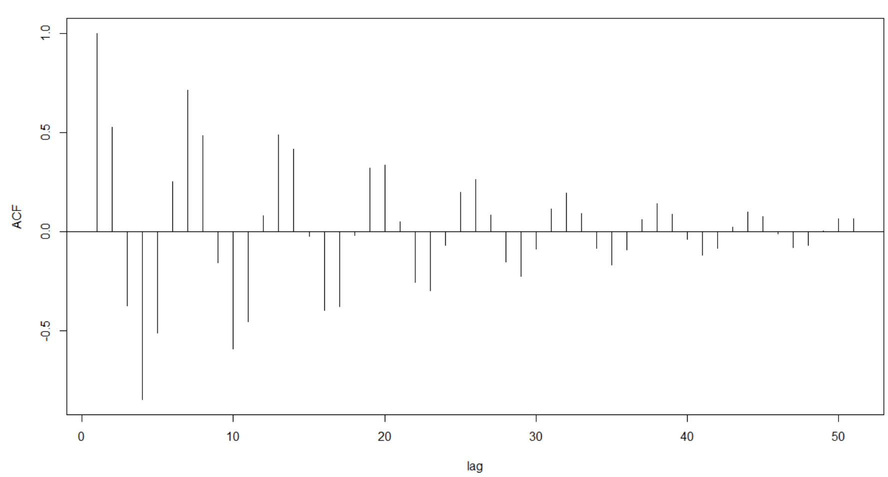
The roots of $\phi(z)$ are

$$z = \pm\sqrt{0.9}\mathrm{i}$$

solving the arg with R, we get

$$1/\arg = 4$$

The ACF plot is as follows.



$$\rho(h) = a0.9^{-h/2}\cos(\frac{2\pi}{4}h + b)$$

2

$$\begin{aligned}
\rho(h) &= c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h} \\
&= c_1 |z_1|^{-h} e^{-\mathrm{i}\theta h} + \bar{c}_1 |z_1|^{-h} e^{\mathrm{i}\theta h} \\
&= a|z_1|^{-h} \left(e^{-\mathrm{i}\theta h} + e^{\mathrm{i}\theta h} + e^{\gamma} + e^{-\gamma}\right) && (\gamma \text{ is related to } c_1, a \text{ is a constant}) \\
&= a|z_1|^{-h} \cos(h\theta + b) && (b \text{ is a constant})
\end{aligned}$$

This is the plot of ACF of Example 3.10, it is exponential decay with sinusoid pattern.