微分方程数值解
第十四周作业

于慧倩
14300180118

2017 年 6 月

1. P222.2

跳蛙格式:

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}+c\frac{u_{i+1}^n-u_{i-1}^n}{2h}=f_i^n$$

(a) 截断误差

$$\begin{aligned}
R_{i+1} &= \frac{u(t_{n+1},x_i)-u(t_{n-1},x_i)}{2\tau}+c\frac{u(t_n,x_{i+1})-u(t_n,x_{i-1})}{2h}-f(t_n,x_i)\\
&= \frac{u(t_{n+1},x_i)-u(t_{n-1},x_i)}{2\tau}+c\frac{u(t_n,x_{i+1})-u(t_n,x_{i-1})}{2h}\\
&\quad -\frac{\partial u(t_n,x_i)}{\partial t}-c\frac{\partial u(t_n,x_i)}{\partial x}\\
&= \frac{u(t_{n+1},x_i)-u(t_{n-1},x_i)}{2\tau}-\frac{\partial u(t_n,x_i)}{\partial t}\\
&\quad +c\frac{u(t_n,x_{i+1})-u(t_n,x_{i-1})}{2h}-c\frac{\partial u(t_n,x_i)}{\partial x}\\
&= R_t+R_x
\end{aligned}$$

$$\begin{aligned}
R_t &= \frac{u(t_{n+1},x_i)-u(t_{n-1},x_i)}{2\tau}-\frac{\partial u(t_n,x_i)}{\partial t}\\
&= \frac{\tau^2}{3!}\frac{\partial u(t-n,x_i)}{\partial t}+O(\tau^2)
\end{aligned}$$

$$\begin{aligned}
R_x &= c\frac{u(t_n,x_{i+1})-u(t_n,x_{i-1})}{2h}-\frac{\partial u(t_n,x_i)}{\partial x}\\
&= c\frac{h^2}{3!}\frac{\partial u(t-n,x_i)}{\partial x}+O(h^2)
\end{aligned}$$

$$\begin{aligned}
R_{i+1} &= R_t+R_x\\
&= \frac{\tau^2}{3!}\frac{\partial u(t-n,x_i)}{\partial t}+c\frac{h^2}{3!}\frac{\partial u(t-n,x_i)}{\partial x}+O(\tau^2)+O(h^2)
\end{aligned}$$

(b) 数值稳定性

利用传播因子法估计

代入 $u_i^n = g(n)e^{i\omega x_i}$ 得

$$c\frac{g(n+1)-g(n-1)}{2\tau}e^{i\omega x_i} + c\frac{g(n)e^{i\omega x_{i+1}} - g(n)e^{i\omega x_{i-1}}}{2h} = 0$$

两边提取 $e^{i\omega x_i}$ 得

$$\frac{g(n+1)-g(n-1)}{2\tau} + cg(n)\frac{e^{i\omega h} - e^{-i\omega h}}{2h} = 0$$

$$\frac{g(n+1)-g(n-1)}{2\tau} + cg(n)\frac{i\mathrm{sin}\omega h}{h} = 0$$

令 $\lambda = i\frac{c\mathrm{sin}\omega h}{h}, \lambda^2 < 0$，则增长因子满足

$$G^2 - 1 + 2\tau\lambda G = 0$$

i. $\tau^2\lambda^2 + 1 \geq 0$ 时

$$G = -\tau\lambda \pm \sqrt{\tau^2\lambda^2 + 1}$$

$$|G|^2 = \tau^2|\lambda|^2 + \tau^2\lambda^2 + 1 = 1$$

ii. 选取 ω 使得 $\tau^2\lambda^2 + 1 < 0$ 时

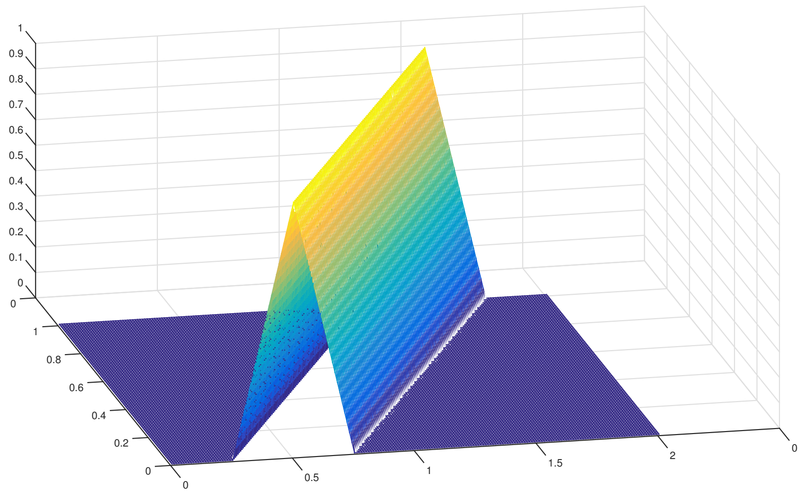
$$G = -\tau\lambda \pm i\sqrt{-\tau^2\lambda^2 - 1}$$

$$|G|^2 = (-\tau|\lambda| \pm \sqrt{-\tau^2\lambda^2 - 1})^2$$

可以选取 ω 对应的某个根的模 $|G| > 1 + \frac{c\tau}{h}$，这时不稳定。
总之，可以选出 ω 使得格式不稳定，所以格式不稳定。

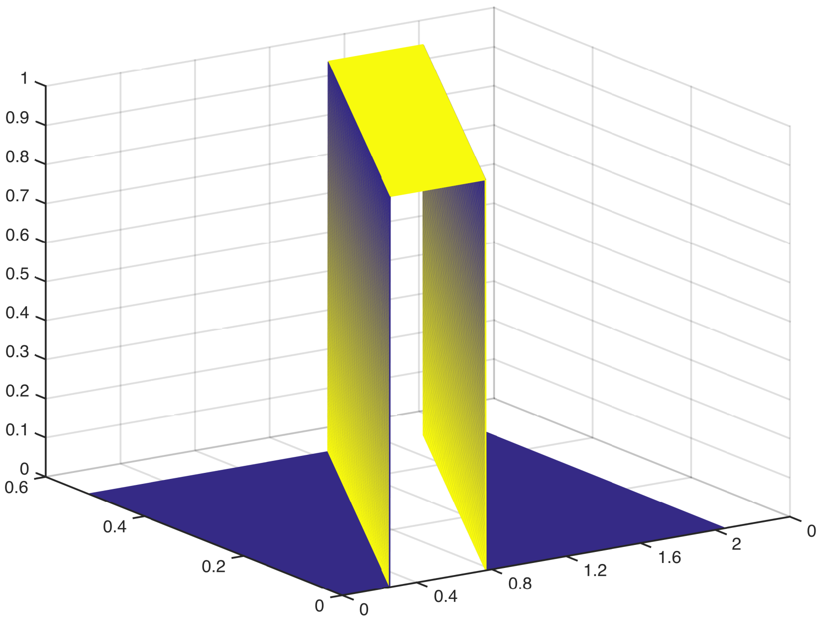
2. P222.5 取初值为方波时，考虑 $X = 2, T = 1$ 的总体情况,$c = 1$ 所以迎风格式取左偏格式。

   (a) 初值为第一种时，无论 $r = 1, r = 1/2$ 都是稳定的，作出下图:



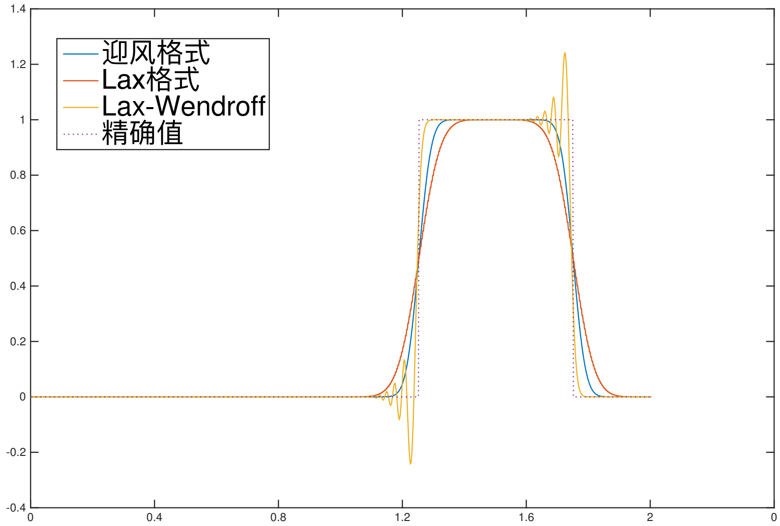
(b) $r = 1$ 时

三种格式都退化成 $u_i^{n+1} = u_{i-1}^n$，取 $X = 2, T = 1$ 时，作出下图：

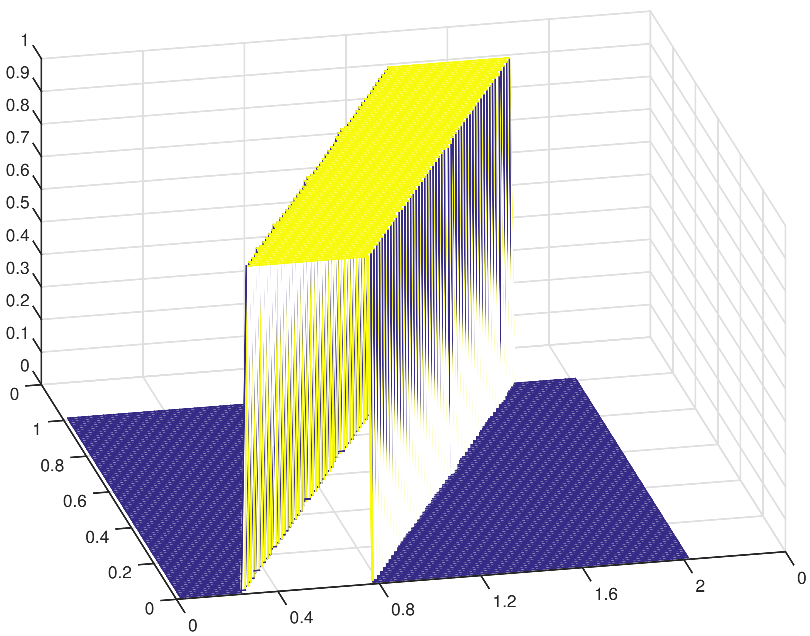


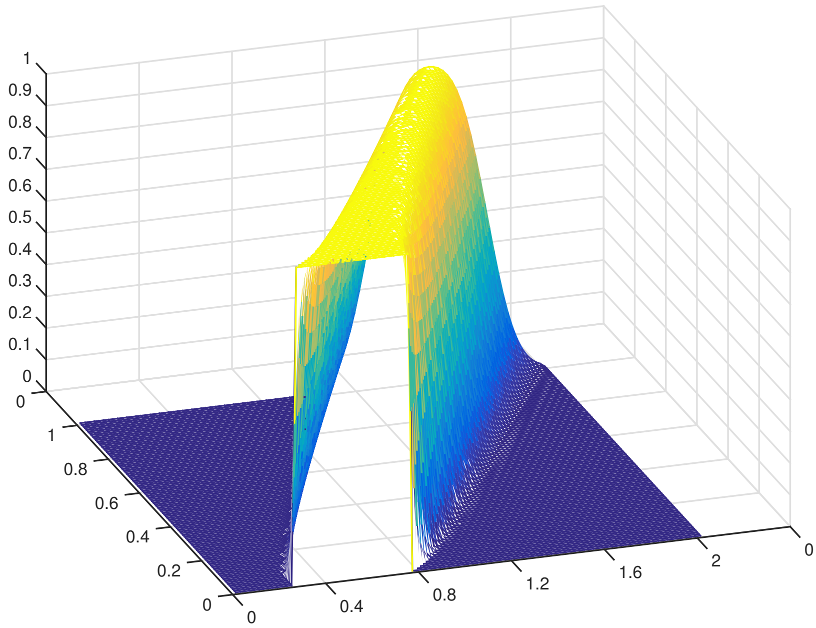
(c) $r = \frac{1}{2}$ 时候

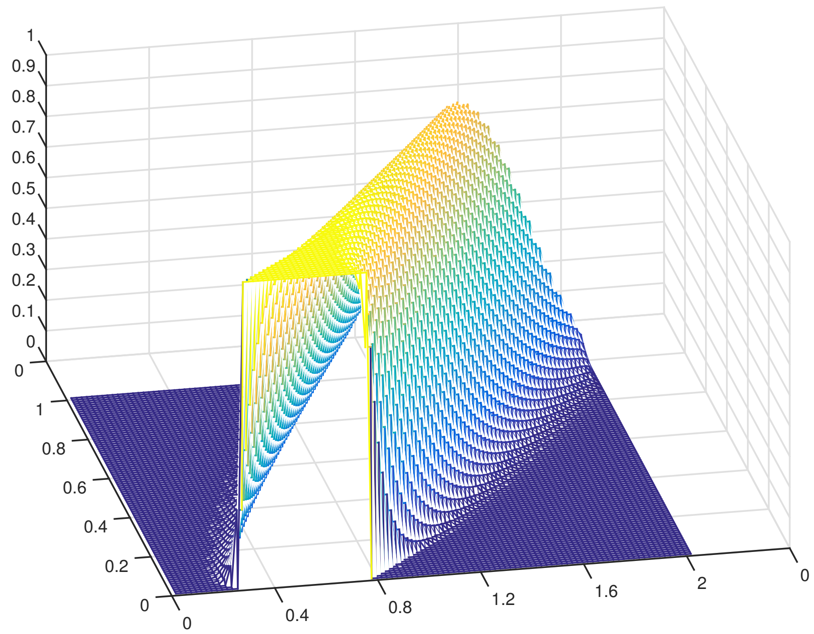
$T=1$ 时三种方法得到的解:

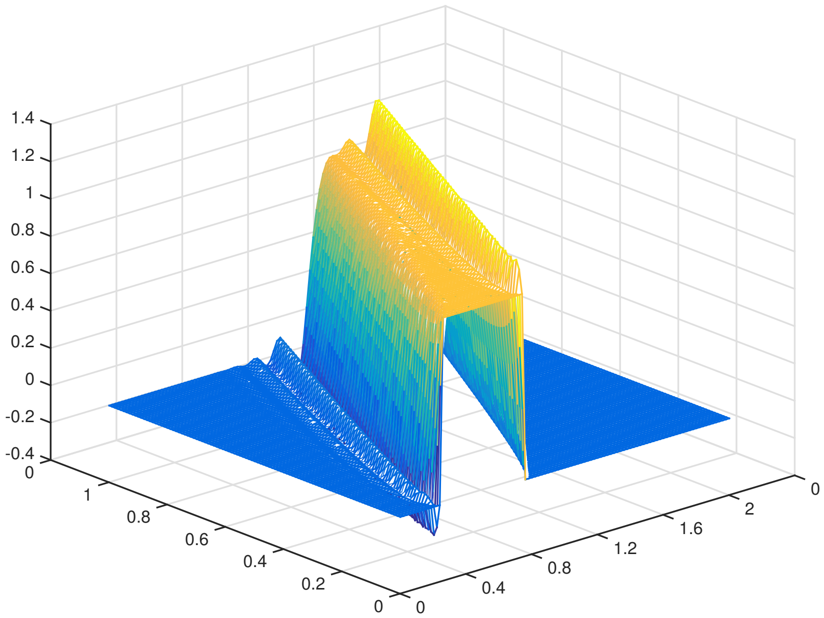


分别作出精确值、迎风格式、Lax 格式、Lax-Wendroff 格式如下:




1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
0.8
0.6
0.4
0.2
0
0
0.4
0.8
1.2
1.6
2
0


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
0.8
0.6
0.4
0.2
0
0
0.4
0.8
1.2
1.6
2
0


1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
0.8
0.6
0.4
0.2
0
0
0.4
0.8
1.2
1.6
2
0