Problem 1

4.6

(a)

$$
\begin{aligned}
\gamma(0) &= \mathrm{var}(s_t) + A^2\mathrm{var}(s_t) + 2A\gamma_s(D) + \mathrm{var}(n_t) \\
\int_{-1/2}^{1/2} f_x(\omega)\,\mathrm{d}\,\omega &= \int_{-1/2}^{1/2} (1 + A^2) f_s(\omega) + f_n(\omega) + f_s(\omega)\mathrm{e}^{2\pi \mathrm{i}\omega D}\,\mathrm{d}\,\omega \\
&= \int_{-1/2}^{1/2} (1 + A^2 + 2A\cos(2\pi\omega D)) f_s(\omega) + f_n(\omega)\,\mathrm{d}\,\omega
\end{aligned}
$$

Thus,

$$
f_x(\omega) = (1 + A^2 + 2A\cos(2\pi\omega D)) f_s(\omega) + f_n(\omega)
$$

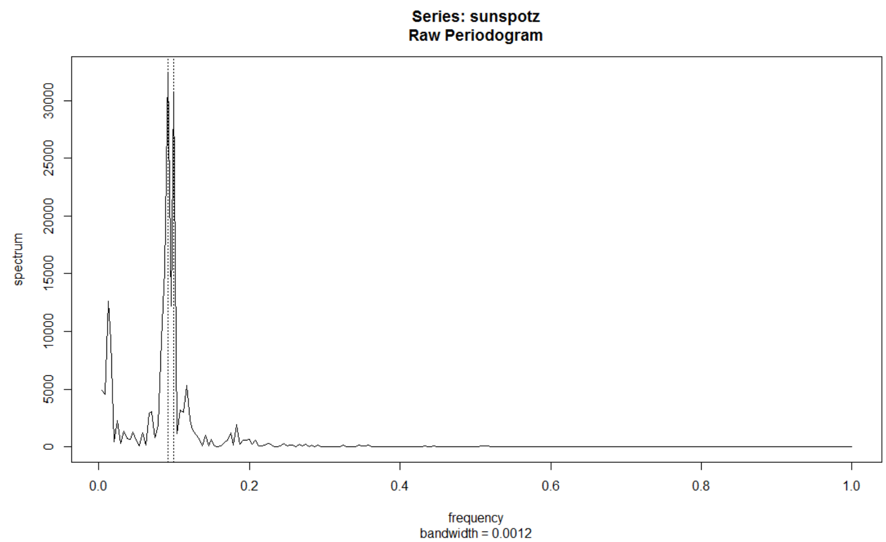
(b)

Observe the periodically change in $f_x(\omega)$. Besides the period of $f_s(\omega)$, there should be another period of $1/D$.

4.8

```
library(astsa)
ss = sunspotz
ss.per = spec.pgram(sunspotz, taper = 0, log = 'no', detrend = T)
abline(v = 1/10, lty = 'dotted')
abline(v = 22/240, lty = 'dotted')

U = qchisq(0.025,2)
L = qchisq(1 - 0.025,2)

# Output
> ss.per$spec[24]
[1] 30583.29
> 2*ss.per$spec[22]/L
[1] 8804.265
> 2*ss.per$spec[22]/U
[1] 1282807
> 2*ss.per$spec[24]/L
[1] 8290.672
> 2*ss.per$spec[24]/U
[1] 1207975
```

The periodogram is shown below.

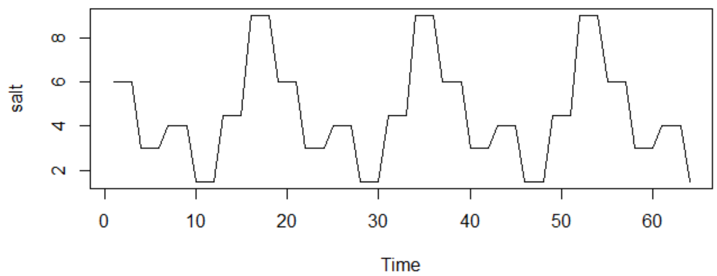


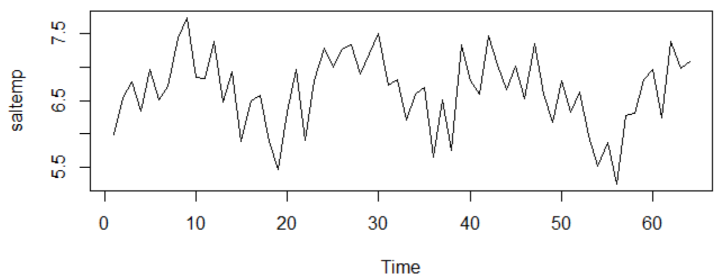
The result shows that the periodogram as an estimator is susceptible to large uncertainties.

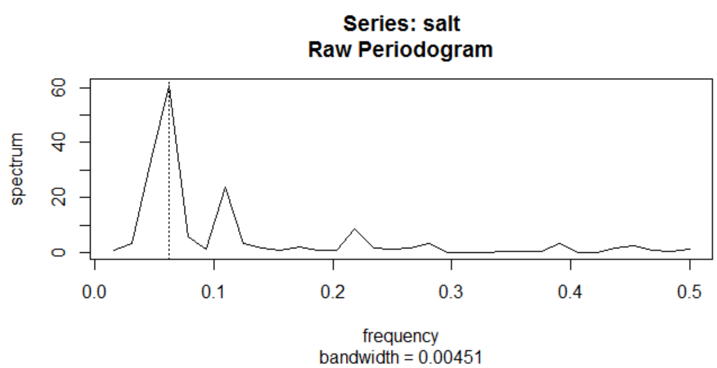
4.9

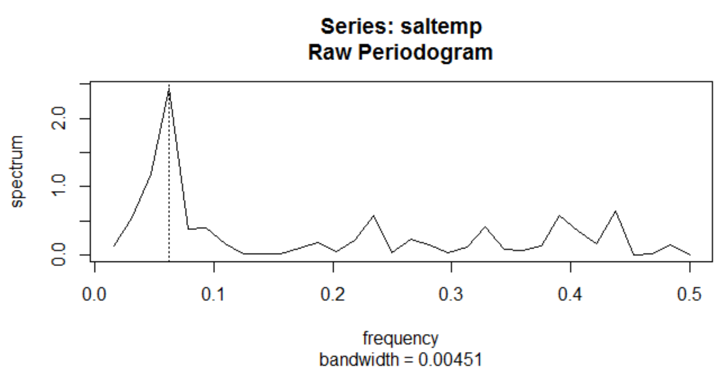
```
library(astsa)

par(mfrow=c(2,1))
plot(salt)
plot(saltemp)

par(mfrow=c(2,1))
salt.per = spec.pgram(salt, taper = 0, log = 'no', detrend = T)
abline(v = 1/16, lty = 'dotted')
saltemp.per = spec.pgram(saltemp, taper = 0, log = 'no', detrend = T)
abline(v = 1/16, lty = 'dotted')

> salt.per$spec[4]
[1] 60.66648
> saltemp.per$spec[4]
[1] 2.43787
U = qchisq(0.025,2)
L = qchisq(1 - 0.025,2)
> 2*salt.per$spec[4]/L
[1] 16.44577
> 2*salt.per$spec[4]/U
[1] 2396.198
>
> 2*saltemp.per$spec[4]/L
[1] 0.6608701
> 2*saltemp.per$spec[4]/U
[1] 96.29072
```

It’s shown that the level of salt is corresponding to the average tempearture for the soil.









Problem 2

(a)

Since $\gamma(h)$ is non-negative definite, define

$$
\begin{aligned}
f_n(\omega) &= \frac{1}{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\exp(-2\pi i\omega s)\gamma(r-s)\exp(2\pi i\omega r) \\
&= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}(n-|h|)\exp(-2\pi i\omega h)\gamma(h) \geq 0
\end{aligned}
$$

Since $\gamma(h)$ is absolutely summable,

$$
\begin{aligned}
f(\omega) &= \lim_{n\to\infty} f_n(\omega) \\
&= \sum_{h=-\infty}^{\infty}\gamma(h)\exp(-2\pi i\omega h) \geq 0
\end{aligned}
$$

(b)&(c)

Since $\gamma(h) = \gamma(-h)$,

$$
\begin{aligned}
f(\omega) &= \sum_{h=-\infty}^{\infty}\gamma(h)\exp(-2\pi i\omega h) \\
&= \gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos(2\pi\omega h)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f(-\omega) &= \gamma(0) + 2\sum_{h=1}^{\infty}\gamma(-h)\cos(-2\pi\omega h) = f(\omega) \\
f(1-\omega) &= \gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)\cos(2\pi h - 2\pi\omega h) = f(\omega)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\int_{-1/2}^{1/2}\exp(2\pi i\omega h)f(\omega)\,\mathrm{d}\,\omega &= \int_{-1/2}^{1/2}\sum_{k=-\infty}^{\infty}\gamma(h)\exp(-2\pi i\omega(h-k))\,\mathrm{d}\,\omega \\
&= \sum_{k=-\infty}^{\infty}\int_{-1/2}^{1/2}\gamma(h)\exp(-2\pi i\omega(h-k))\,\mathrm{d}\,\omega
\end{aligned}
$$

if $k \neq h$, $\int_{-1/2}^{1/2}\gamma(h)\exp(-2\pi i\omega(h-k))\,\mathrm{d}\,\omega = 0$, so

$$
\begin{aligned}
\int_{-1/2}^{1/2}\exp(2\pi i\omega h)f(\omega)\,\mathrm{d}\,\omega &= \int_{-1/2}^{1/2}\gamma(h)\,\mathrm{d}\,\omega \\
&= \gamma(h)
\end{aligned}
$$