CSCI 301, Winter 2017
Math Exercises #3

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Exercises for Section 11.1

12. Prove that the relation | (divides) on the set $\mathbb{Z}$ is reflexive and transitive.

Proof. Suppose the relation | (divides) on the set $\mathbb{Z}$ is both reflexive and transitive.

Transitivity The expression x|z impiles that the second argument, z, will be equal to xn where n is an integer.
Thus follows $x = xn$ so $n = 1$.

Reflexivity To prove reflexivity we must show that x|y and y|z implies x|z.
Recall that x|y implies $y = xa$ for some integer a and y|z implies $z = yb$ for some integer b.
Reason as follows.

$$z = yb = xab = x(ba)$$

Therefore z is a multiple of x making x|z is a true statement thus making this relation reflexive.

Exercises for Section 11.2

8. Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 + y^2$ is even. Prove R is an equivalence relation (prove from the definitions).

Proof. Suppose $\{(x, y) \in \mathbb{Z} : x^2 + y^2 = 2a, x, y, a \in \mathbb{Z}\}$ is an equivalence realtion.
Reflexivity The expression $x^2 + x^2$ must be even to prove reflexivity.
Reason follows that $x^2 + x^2 = 2x^2 = 2(x^2)$ thus proving that $x^2 + x^2$if even for $a = x^2$.

Symmetry The expressions $x^2 + y^2 = y^2 + x^2 = 2a$ is a valid expression thus proving the relation to be symmetric.

Transitivity To prove the realtion transitive we must prove that $x^2 + y^2 = 2a$ and $y^2 + z^2 = 2a$ imply $x^2 + z^2 = 2a$.
Reason as follows.

$$x^2 + y^2 = 2a \text{ and } y^2 + z^2 = 2a \; x^2 = 2a - y^2 \text{ and } z^2 = 2a - y^2$$

Exercises for Section 11.4

4. Write the addition and multiplication tables for $\mathbb{Z}_6$

+	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

$*$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

Exercises for Section 12.2

8. A function $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is defined as $f(m,n) = (m+n, 2m+n)$. Verify whether this function is injective and whether it is surjective.
Injectivity Suppose $a, b, c, d \in A$ and $(a, b) \neq (c, d)$. Then $f(a, b) = (a + b, 2a + b)$ and $f(c, d) = (c + d, 2c + d)$. So $a + b \neq c + d$ and $2a + b \neq 2b + c$.
Therefore $f(a, b) \neq f(c, d)$ and $f(m, n)$ is injective.

Surjectivity Since the expression $2m + n$ will only be even if n is even or odd only if n is odd it is possible that every number in the domain of $\mathbb{Z}$ can be represented by this function making the function surjective.

Exercises for Section 12.3

2. Prove that if a is a natural number, then there exist two unequal natural numbers k and ℓ for which $a^k - a^\ell$ is divisible by 10.
Proof. Let A be the set of possible powers of a $\{a, a^2, a^3, a^4...a^{11}\}$ and let B be the set of powers of a with the same last digit. When two powers of a have the same last digit their difference will create a number divisible by 10. Notice that $|A| = 11$ and $|B| = 10$, thus $|A| > |B|$.
Thus the pigeon hole principle asserts that this relation is not injective.
We can then assume that there exist some powers of a, l and k, for which when a is evaluated at those powers the last digits are the same thus making their difference divisible by 10.

Exercises for Section 12.5

6. The function $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ defined by the formula $f(m,n) = (5m+4n, 4m+3n)$ is bijective. Find its inverse.
Proof. First we prove the function is injective. Assume $f(m_1, n_1) = f(m_2, n_2)$. Then $5m_1 + 4n_1 = 5m_2 + 4n_2$ and $4m_1 + 3n_1 = 4m_2 + 3n_2$.
We can rewrite f in terms of a and b to create a systems of equations.

$$f(m_1, n_1) = (m_2, n_2) \text{ thus } (m_1, n_1) = f(m_2, n_2) = (5m_2 + 4n_2, 4m_2 + 3n_2).$$

$$5m_2 + 4n_2 = m_1 \;\; 4m_2 + 3n_2 = n_1.$$

$$20m_2 + 16n_2 = 4m_1 \;\; 20m_2 + 15n_2 = 5n_1.$$

Solve for m_2 and n_2 in terms of m_1 and n_1 for later substitution.

$$n_2 = 4m_1 - 5n_1.$$

Now we can substitute n_2 into the original system of equation as follows

$$20m_2 + 16(4m_1 - 5n_1) = 4m_1 \;\; 20m_2 + 64m_1 - 80n_1 = 4m_1 \;\; 20m_2 = -60m_1 + 80n_1$$
$$m_2 = -3m_1 + 4n_1$$

Therefore,

$$f^{-1} = (m_1, n_1) = (m_2, n_2) = (-3m_1 + 4n_1, 4m_1 - 5n_1).$$

Exercises for Section 12.6

6. Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.

Proof. In order for this statement to be true we must prove that all elements of set Y, $Y \subseteq B$, have inputs in set A.
By definition of *preimage* we can assume that the statement $f^{-1}(Y)$ is true since $Y \subseteq B$ and will return an element that is defined in the set of all inputs on the function f in the set A. Since the expression $f^{-1}(Y)$ will return a value in the defined legal inputs of f, set A, or $f^{-1}(Y) \subseteq A$. Thus we can assume that $f(f^{-1}(Y)) = Y$ will indeed always return Y.
Therefore $f(f^{-1}(Y)) = Y$.

Exercises for Section 13.1

A. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).
If a relation on two sets is injective and surjective then the cardinalities of the two sets are equal.
A relation is injective if every input is related to exactly one unique output and a relation is surjective if every output is paired with an input.
So if a relation is both injective and surjective then every input is matched to exactly one unique output and every output has one unique input.
The nature of these definitions implies that any relation on two sets that follow these principles will be of equal cardinality, for every x there is a unique y and for every y there is a unique x.
In terms of a function this bijective realtion would look like $f(x, y) = \{x \in X, y \in Y\}$ or $f : A \to B$ and $|A| = |B|$.

10. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$.
Injectivity Consider the function $f : \{0, 1\} \times \mathbb{N} \to \mathbb{N}$ defined as $f(a, n) = 2n - a$. To prove injectivity we must show that $f(a, n) = f(b, m)$ and so follows $2n - a = 2m - b$. Since a or b can only take on the values 0 or 1 it follows that if one of a or b were 1 and the other 0 that the resulting fucntion would produce one odd and one even result, a contradiction. So $a = b$. Using this we can substitute a for b so $2n - a = 2m - b$ becomes $2n - a = 2m - a$ after adding a to both sides and dividing by 2 we get that $m = n$. Thus $a = b$ and $m = n$, so $(a, n) = (b, m)$. Since $\mathbb{N} \subseteq \mathbb{Z}$ this logic holds. Therefore f is injective.

Surjectivity To prove f's surjectivity let's examine any $b \in \mathbb{N}$. This element can be either even or odd. If b is even then $b = 2n$ for some integer n so $f(0, n) = 2n - 0 = b$. If b is odd then $b = 2n + 1$ for some integer n so $f(1, n + 1) = 2(n + 1) - 1 = 2n + 1 = b$. Therefore f is surjective.

Since f is both surjective and injective it is bijective and thus $|\{0, 1\} \times \mathbb{N}| = |\mathbb{N}|$.

Exercises for Section 13.2

14. Suppose $A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$? Prove or disprove it.
Proof. In order to prove $|\mathbb{N}| = |A|$ we must prove that the realtion on set A for $A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$ is bijective.
Injectivity Suppose $x, y \in A$ and $f(x, y) = f(w, z)$ so $x = \pi y$ and $w = \pi z$ are equivalent.

Surjectivity For any $b \in B$ there is an $a \in A$ such that $f(a) = b$. Because the product of π and natural number can be a natural number or an quotient we can represent all possible outputs of this relation, b, with an input a making this relation surjective.

Also, the inputs of this function $\mathbb{N}$ and the outputs $\mathbb{R}$ are both countably infinite sets making their cardinalities equal.

Since this realtion is both injective and surjective we know that $|\mathbb{N}| = |A|$.

Exercises for Section 13.3

8. Prove or disprove: The set $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is countably infinite.

If $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ then simply observe that $\{(a_1, a_2, a_3, \ldots) \subseteq \mathbb{Z}$. Any subset of a countably infinite set is also countably infinite.