CSCI 301, Winter 2017
Math Exercises #2

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Exercises for Chapter 4 Use the method of direct proof to prove the following statements.

16. If two integers have the same parity, then their sum is even.

Proposition If two integers have the same parity, then their sum is even.
Proof. Without loss of generality, suppose m and n are even integers.
Thus $m = 2a$ and $n = 2b$ for some integers a and b.
So $m + n = 2a + 2b = 2(a + b)$.
Therefore the sum of two integers of the same parity is even, by definition of an even number.

Exercises for Chapter 5 Use the method of contrapositive proof to prove the following statements.

12. Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

Proposition Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.
Proof. Suppose a is not odd.
Thus a is even, so $a = 2b$ for some integer b.
Then $2b^2 = 2b * 2b = 4b^2 = 4(b^2)$.
Therefore $a^2 = 4z$, where z is the integer b^2.
Consequently, a^2 is divisible by 4 by definition of divides.
Therefore a^2 is not *not* divisible by 4.

Exercises for Chapter 6 Use the method of proof by contradiction to prove the following statements.

18. Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

Proposition Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd. *Proof.* For the sake of contradiction, suppose $4 \mid (a^2 + b^2)$ and a and b are both not *not* odd.
Then $4 \mid (a^2 + b^2)$ and a and b are both odd.
Since a and b are odd there is an integer c for which $a = 2c + 1$ and an integer g for which $b = 2g + 1$.
By definition of divides $(a^2 + b^2) = 4k$ for some integer k.
Now reason as follows.
$4k = (2c + 1)^2 + (2g + 1)^2$
$4k = 4c^2 + 4c + 1 + 4g^2 + 4g + 1$
$4k = 4(c^2 + g^2 + 4c + 4g + \frac{1}{2})$
$k = c^2 + g^2 + 4c + 4g + \frac{1}{2}$
So it follows that k will always be a rational number due to the addition of a rational number to an integer.
Thus $4 \mid (a^2 + b^2)$ is divisible by an integer k and a rational number k, a contradiction.

Exercises for Chapter 7 State clearly which method of proof you are using.

24. If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$.

Proposition If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$.
Proof. (contrapositive) Suppose $4 \nmid (a^2 - 3)$.
Thus $a^2 - 3 = 4k$ for some integer k.
Now reason as follows.

$4(\frac{a^2}{4} - \frac{3}{4}) = 4k$
$\frac{a^2-3}{4} = k$
Since k is an integer a^2, and thus a, must be a rational number in order to make this a true statement.
Thus proving a to be a non-integer.
Therefore $a \notin \mathbb{Z}$.

Exercises for Chapter 8

20. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Proof. Suppose $a \in \{9^n : n \in \mathbb{Q}\}$. This means $a = 9^n$ for some $n \in \mathbb{Q}$.
Therefore $a = 9^n = 9^{n/2} = (9^{\frac{1}{2}})^n = 3^n$.
Since $a = 3^n$ for $n = \frac{n}{2}$ it follows that $a \in \{3^n : n \in \mathbb{Q}\}$.
This establishes that $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.

Suppose $a \in \{3^n : n \in \mathbb{Q}\}$. This means $a = 3^n$ for some $n \in \mathbb{Q}$.
Therefore $a = 3^n = 3^{2/n} = (3^2)^n = 9^n$.
Since $a = 9^n$ for $n = 2$ it follows that $a \in \{9^n : n \in \mathbb{Q}\}$.
This establishes that $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$.

Since $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$ and $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$, it follows that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Exercises for Chapter 9 Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it.

18. If $a, b, c \in \mathbb{N}$, then at least one of $a - b$, $a + c$, and $b - c$ is even.

This is a true statement. Observe that the expression $b - c$ evaluated when $b = 9$ and $c = 1$, $1, 9 \in \mathbb{N}$, the result, 8, is even.

Exercises for Chapter 10

2. For every integer $n \in \mathbb{N}$, it follows that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. We will prove this with mathematical induction.
(1) If $n = 0$, this statement is

$$\sum_{i=1}^{0} i^2 = \frac{0 * (0+1)(2 * 0 + 1)}{6}$$

Since the left-hand side is $0^2 = 0$, and the right-hand side is $\frac{0}{6}$, the equation above holds, as both sides are zero.
(2) Consider any integer $k \geq 0$. We must show that S_k implies S_{k+1}.We use direct proof. So reasons follows:

$$\sum_{i=1}^{k+1} i^2 = (\sum_{i=1}^{k} i^2) + (k+1)^2$$

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k)((k)+1)(2(k)+1)}{6} + (K+1)(k+1)$$
$$6 + 4k + 3k + 2k^2 + 2k^3 + 3k^2 + 4k^2 + 6k = 2k^3 + k^2 + 2k^2 + k + 6k^2 + 12k + 6$$
$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

Therefore $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$.
It follows by induction that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for every integer $n \in \mathbb{N}$.

6. For every natural number n, it follows that

$$\sum_{i=1}^{n}(8i-5) = 4n^2 - n$$

Proof. We will prove this with mathematical induction.
(1) If $n = 0$, this statement is

$$\sum_{i=1}^{1}(8(1)-5) = 4(1)^2 - 1$$

Since the left-handed side is $8(1) - 5 = 3$, and the right-handed side is $4(1)^2 - 1 = 3$, the equation above holds, as both sides are equal to three.
(2) Consider any natural number k. We must show that S_k implies S_{k+1}.We use direct proof. So reasons follows:

$$\sum_{i=1}^{k+1}(8i-5) = \sum_{i=1}^{k}(8i-5) + 4(k+1)^2 - k$$
$$4(k+1)^2 - (k+1) = 4k^2 - k + (8(k+1) - 5)$$
$$4k^2 + 7k + 3 = 4k^2 + 7k + 3$$

Therefore $\sum_{i=1}^{k+1}(8i-5) = 4(k+1)^2 - (k+1)$.
It follows by induction that $\sum_{i=1}^{n}(8i-5) = 4n^2 - n$ for every integer $n \in \mathbb{N}$.

10. For every integer $n \geq 0$, it follows that $3 \mid (5^{2n} - 1)$.
Proposition For every integer $n \geq 0$, it follows that $3 \mid (5^{2n} - 1)$.
Proof. We will prove this with mathematical induction. Observe that the first non-negative integer is 0, so the basis step involves n=0.
(1) If $n = 0$, this statement is $3 \mid (5^{2(0)} - 1)$ or $3 \mid 0$, which is true.
(2) Now assume the statement is true for some integer $n = k \geq 1$, that is assume $3 \mid (5^{2k} - 1)$. This means $5^{2k} - 1 = 3a$ for some integer a, and from this we get $5^{2k} - 1 = 3a + 1$. Now observe that

$$5^{2(k+1)} - 1 =$$
$$5^{2k+2)} - 1 =$$
$$5^2 5^{2k} - 1 =$$
$$5^2(3a+1) =$$
$$25(3a+1) =$$
$$25 * 3a + 25 - 1 =$$
$$75a + 24 = 3(25a + 8)$$

Thus $5^{2(k+1)} - 1 = 3(25a + 8)$, which means $3 \mid (5^{2(k+1)} - 1)$.
So completes the proof by mathematical induction.

14. Suppose $a \in \mathbb{Z}$. Prove that $5 \mid 2^n a$ implies $5 \mid a$ for any $n \in \mathbb{N}$.

Proof. (direct proof) Suppose $5 \mid 2^n a$.
As 2^n can be rewritten as $2_1^1 * 2_2^2 ... 2_n^n$ we know that 2^n is a multiple of 2 and therefore divisible by 2 meaning that 2^n is even.
Therefore 2^n is equivalent to $2c$, $c \in \mathbb{Z}$.
Since we are assuming $5 \mid 2^n a$ we can conclude that $2^n a$ is equivalent to $5b$, $b \in \mathbb{Z}$.
Now we observe that

$$5b = 2^n a$$
$$5b = 2ca$$
$$5\frac{b}{2c} = a$$

Thus proving a is equivalent to some multiple of 5 making a divisible by 5. Therefore $5 \mid a$.