T4

(a, б) $\xi \sim R(\theta, 2\theta)$

$$p(x,\theta) = \frac{1}{\theta}\{(\theta, 2\theta)\}$$

$\boxed{\text{ОММ}}$

$$\alpha_1 = M\xi = \int_{-\infty}^{+\infty} x\,p(x)\,dx = \frac{1}{\theta}\int_{\theta}^{2\theta} x\,dx = \frac{3\theta}{2}$$

$$\frac{3\theta}{2} = \bar{x} \Rightarrow \tilde{\theta} = \frac{2}{3}\bar{x}$$

Несмещ.

$$M[\tilde{\theta}] = M\left[\frac{2}{3}\bar{x}\right] = \frac{2}{3}M\xi = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$$

оценка несмещена

Состоят.

$$D[\tilde{\theta}] = D\left[\frac{2}{3}\bar{x}\right] = \frac{4}{9}D[\bar{x}] = \frac{4}{9n}D\xi$$

$$M\xi^2 = \int_{\theta}^{2\theta}\frac{1}{\theta}x^2\,dx = \frac{1}{\theta}\frac{x^3}{3}\Big|_{\theta}^{2\theta} = \frac{7\theta^2}{3}$$

$$D[\tilde{\theta}] = \frac{4}{9n}\left(\frac{7\theta^2}{3} - \frac{9\theta^2}{4}\right) = \frac{4\theta^2}{12\cdot 9n} \xrightarrow[n\to\infty]{} 0$$

Оценка $\tilde{\theta}$ состоятельна по дост. условию

$\boxed{\text{ОМП}}$

$$L = \frac{1}{\theta^n}\{\theta \le X_i \le 2\theta \ \forall X_i\}$$

$$X_{max} = 2\theta \Rightarrow \tilde{\theta} = \frac{X_{max}}{2}$$

$$H(x) = (F(x))^n = \left(\int_{\theta}^{x}\frac{1}{\theta}dx\right)^n = \left(\frac{x}{\theta} - 1\right)^n$$

$$M[\tilde{\theta}] = M[X_{max}] = \int_{\theta}^{2\theta}\frac{n}{\theta}\left(\frac{x}{\theta}-1\right)^{n-1}x\,dx = \frac{2\theta n + \theta}{(n+1)} = \theta\cdot\frac{2n+1}{n+1}$$

оценка смещена

$$\tilde{\theta}^* = \cancel{\frac{n+1}{2\theta n+\theta}}\ \frac{n+1}{2n+1}\tilde{\theta} = \frac{n+1}{2n+1}\cdot\frac{X_{max}}{2}$$

$M[\tilde{\theta}^*] = \theta$ — исправленная оценка несмещена

$$D[\hat{\theta}^*] = D\left[\frac{n+1}{2n+1}\cdot\frac{X_{max}}{2}\right] = \left(\frac{n+1}{2n+1}\right)^2 D[X_{max}] =$$

$$= \left(\frac{n+1}{2n+1}\right)^2\left(M[X_{max}^2] - M^2[X_{max}]\cdot\left(\frac{2n+1}{n+1}\right)^2\right)$$

$$M[X_{max}^2] = \int_{\theta}^{2\theta} x^2 \frac{n}{\theta}\left(\frac{x}{\theta}-1\right)^{n-1} dx = 2\theta^2\,\frac{2n^2+4n+1}{(n+2)(n+1)}$$

$$D[\hat{\theta}^*] = \left(\frac{n+1}{2n+1}\right)^2\cdot\frac{4n^2+8n+2}{n^2+3n+2}\,\theta^2 - \theta^2 =$$

$$= \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

(c) $\hat{\theta}^*$ – состоятельна по достаточному условию

$$D[\hat{\theta}_1] = \frac{\theta^2}{27n} \qquad D[\hat{\theta}_1] > D[\hat{\theta}_2^*]$$

$$D[\hat{\theta}_2^*] = \frac{n\theta^2}{(2n+1)^2(n+2)}$$

θ_2^* – эффективнее.

(d) $x_i \in [\theta, 2\theta] \Rightarrow \frac{x_i}{\theta} \in [1, 2]$

$$P(X_{max}) = (F(x))^n = \left(\int_1^x dx\right)^n = (x-1)^n$$

$$X = \theta$$

$$\sqrt[n]{0{,}025}+1 < x < \sqrt[n]{0{,}975}+1$$

$$\sqrt[n]{0{,}025}+1 < \frac{X_{max}}{\theta} < \sqrt[n]{0{,}975}+1$$

$$\frac{X_{max}}{\sqrt[n]{0{,}975}+1} < \theta < \frac{X_{max}}{\sqrt[n]{0{,}025}+1}$$

(e) по ОММ

$$\sqrt{n}\,\frac{g(\hat{\alpha})-g(\alpha)}{\sigma(\alpha)} \rightsquigarrow N(0,1)$$

$$\sigma(\alpha)=\sqrt{\nabla^T g\, K\, \nabla g}=\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}$$

$$\sqrt{n}\,\frac{\frac{2}{3}\hat{\alpha}-\frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}=\sqrt{n}\,\frac{\hat{\theta}-\theta}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}} \rightsquigarrow N(0,1)$$

$$\sqrt{n}\,\frac{\hat{\theta}-\theta}{\frac{2}{3}\sqrt{\hat{\alpha}_2-\hat{\alpha}_1^2}}=\underbrace{\sqrt{n}\,\frac{\hat{\theta}-\theta}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}}_{\to N(0,1)}\cdot\underbrace{\frac{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}{\frac{2}{3}\sqrt{\hat{\alpha}_2-\hat{\alpha}_1^2}}}_{\to 1} \rightsquigarrow N(0,1)$$

лемма Слуцкого

$$-1{,}96<\sqrt{n}\,\frac{\hat{\theta}-\theta}{\frac{2}{3}\sqrt{\hat{\alpha}_2-\hat{\alpha}_1^2}}<1{,}96$$

$$\frac{-1{,}96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\hat{\alpha}_2-\hat{\alpha}_1^2}+\hat{\theta}<\theta<\frac{1{,}96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\hat{\alpha}_2-\hat{\alpha}_1^2}+\hat{\theta}$$

(Т5)

$$p(x)=\begin{cases}\dfrac{\theta-1}{x^\theta}, & x\geq 1\\ 0, & x<1\end{cases}\qquad \theta>1$$

(a) $$L(x,\theta)=\frac{\theta-1}{x_1^\theta\cdot x_2^\theta\cdots x_n^\theta}$$

$$\ln L=n\ln(\theta-1)-\theta\sum_i \ln x_i$$

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta-1}-\sum_i \ln x_i\ ;\quad \frac{\partial L}{\partial\theta}=0$$

$$\frac{n}{\theta-1}=\sum_i \ln x_i$$

$$\hat{\theta}=1+\frac{n}{\sum_i \ln x_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{(\theta-1)^2} < 0 \Rightarrow L(\theta) \to \sup$$

(6) Дов. интервал для медианы

$$\frac{\partial}{\partial \theta} \int \frac{\theta - 1}{x^{\theta}}\, dx = x^{\cancel{1}-\theta} \ln x$$

$$\int \frac{\partial}{\partial \theta} \left(\frac{\theta - 1}{x^{\theta}} \right) dx = x^{1-\theta} \ln x$$

$\Rightarrow$ сильная регулярность

$$\int_1^{\hat{x}} \frac{\theta - 1}{x^{\theta}}\, dx = -\frac{1}{\hat{x}^{\theta-1}} + 1 = \frac{1}{2}$$

$$\hat{x} = \sqrt[\theta-1]{2} \qquad g(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$$

$$\sqrt{n}\, \frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta})$$

$$\sigma(\tilde{\theta}) = \sqrt{\nabla^T g(\tilde{\theta})\, I^{-1}(\tilde{\theta})\, \nabla g(\tilde{\theta})}$$

$$I(\tilde{\theta}) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = M\left[\left(\frac{\partial(\ln(\theta-1) - \theta \ln x)}{\partial \theta}\right)^2\right] =$$

$$= M\left[\left(\frac{1}{\theta-1} - \ln x\right)^2\right] = \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 p(x,\theta)\, dx =$$

$$= \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 \cdot \frac{\theta-1}{x^{\theta}}\, dx = \frac{1}{(\theta-1)^2}$$ - непр. на $\theta > 1$

$$\nabla g(\tilde{\theta}) = \cancel{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}} = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{(\tilde{\theta}-1)^2}$$

$$\sigma(\tilde{\theta}) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{(\tilde{\theta}-1)}$$

$$\sqrt{n}\,\frac{g(\hat{\theta})-g(\theta)}{\sigma(\hat{\theta})} \rightsquigarrow N(0,1)$$

$$\frac{+1{,}96\cdot\sigma(\hat{\theta})}{\sqrt{n}}+g(\hat{\theta}) < g(\theta) < -\frac{1{,}96\cdot\sigma(\hat{\theta})}{\sqrt{n}}+g(\hat{\theta})$$

(c)

$$\sqrt{n}\,\frac{\tilde{\theta}-\theta}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta})$$

$$\sigma(\tilde{\theta}) = \theta - 1 \quad\Rightarrow\quad \sqrt{n}\,\frac{\tilde{\theta}-\theta}{\theta-1} \rightsquigarrow N(0,1)$$

$$\frac{-1{,}96(\theta-1)}{\sqrt{n}}+1+\frac{n}{\sum_i \ln x_i} < \theta < \frac{1{,}96(\theta-1)}{\sqrt{n}}+1+\frac{n}{\sum_i \ln x_i}$$