[redacted] T1

Наугад выбрали номер из n номеров [redacted]

a) $\xi \sim R(0, \theta)$ $\theta > 0$ – вероятн. модель

$\vec{X_n}$ – выборка

$\tilde{\theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n} \sum_{i=1}^{n} x_i$

$\tilde{\theta}_2 = x_{min}$

$\tilde{\theta}_3 = x_{max}$

$\tilde{\theta}_4 = x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i$

1) $\tilde{\theta}_1$: несмещ. $M[\tilde{\theta}_1] = \theta$

$M[2 \cdot \frac{1}{n} \sum_{i=1}^{n} x_i] = \frac{2}{n} M[\sum_{i=1}^{n} x_i] = 2M[\xi] = 2M\xi$

$x_i \sim R(0, \theta)$

$p(x) = \frac{1}{\theta - 0} \{(0, \theta)\} = \frac{1}{\theta} \{(0, \theta)\}$ – равномерное распред. на $(0, \theta)$

$M\xi = \int_0^\theta \frac{1}{\theta} x dx = \frac{\theta}{2} \Rightarrow$

$\Rightarrow M[\tilde{\theta}_1] = \theta \Rightarrow$ <u>несмещ.</u>

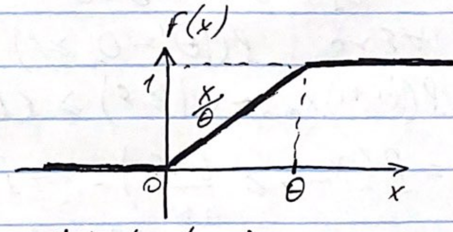
$\mathcal{D}[\tilde{\theta}_1] = \mathcal{D}[\frac{2}{n} \cdot \sum_{i=1}^{n} x_i] = \frac{4}{n^2} \cdot \mathcal{D}[\sum_{i=1}^{n} x_i] \overset{\text{независ. сл. вел.}}{=} \frac{4}{n^2} \sum_{i=1}^{n} \mathcal{D}x_i = \frac{4}{n} \cdot \mathcal{D}\xi$

$\mathcal{D}\xi = M[\xi^2] - M^2[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12} \Rightarrow \mathcal{D}[\tilde{\theta}_1] = \frac{\theta^2}{3n} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow$ По дост. условию состоят. $\tilde{\theta}_1$ – <u>состоятельна.</u>

2) $\tilde{\theta}_2 = x_{min} = x_{(1)}$

$M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} y q(y) dy$



$x_i \sim R(0, \theta)$ $x_i \sim F(x)$

$x_{(1)} \sim \underbrace{1 - (1 - F(y))^n}_{\varphi(y)}$

$q(y) = n(1 - F(y))^{n-1} \cdot F'(y) = n(1 - \frac{y}{\theta})^{n-1} \cdot \frac{1}{\theta} \{(0, \theta)\}$

$M[\tilde{\theta}_2] = \int_0^\theta y n (1 - \frac{y}{\theta})^{n-1} \frac{1}{\theta} dy = \{ t = 1 - \frac{y}{\theta} \} =$

$= -\int_1^0 \theta(1-t) n t^{n-1} \cdot \frac{1}{\theta} \theta dt = n\theta(\int_0^1 t^{n-1} dt - \int_0^1 t^n dt) = n\theta \cdot \frac{1}{n} - n\theta \frac{1}{n+1} =$

$= \theta \frac{1}{n+1}$ – <u>смещенная</u>

Возьмем $\tilde{\Theta}_2' = (n+1)\tilde{\Theta}_2 = (n+1)x_{min} \Rightarrow M[\tilde{\Theta}_2'] = \Theta$ – несмещ.

$$\mathcal{D}[\tilde{\Theta}_2] = M[\hat{\Theta}_2^2] - M^2[\tilde{\Theta}_2] = \frac{2\Theta^2}{(n+1)(n+2)} - \frac{\Theta^2}{(n+1)^2} = \frac{\Theta^2 n}{(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

$$M[\hat{\Theta}_2^2] = \frac{2\Theta^2}{(n+1)(n+2)}$$

! Однако оценка $\tilde{\Theta}_2$ – смещена, и достаточное условие не работает

$$\mathcal{D}[\tilde{\Theta}_2'] = \mathcal{D}[(n+1)\tilde{\Theta}_2] = (n+1)^2 \frac{\Theta^2 n}{(n+1)^2(n+2)} \underset{n\to\infty}{\not\to} 0$$

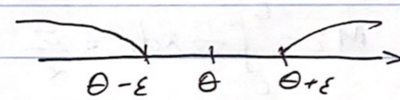
! Достаточное условие не работает, т.к. $\mathcal{D} \not\to 0 \quad n\to\infty$

Попробуем посчитать по определению состоятельности:

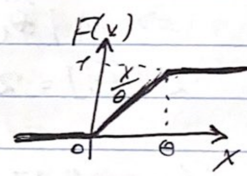
$$\tilde{\Theta}_2 \xrightarrow{P} \Theta \quad \forall \Theta > 0$$

$$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2 - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$\underset{x_{min}}{\tilde{\Theta}_2} \geq \Theta + \varepsilon \qquad \tilde{\Theta}_2 \leq \Theta - \varepsilon$$



$$x_i \sim R(0,\Theta) \Rightarrow P(x_{min} \geq \Theta + \varepsilon) = 0 \quad \varepsilon > 0$$



$$P(x_{min} \leq \Theta - \varepsilon) = P(x_{min} < \Theta - \varepsilon) = \Phi(\Theta - \varepsilon)$$

$$\Phi(y) = 1 - (1 - F(y))^n = 1 - \left(1 - \frac{\Theta - \varepsilon}{\Theta}\right)^n =$$

$$= 1 - \left(\frac{\varepsilon}{\Theta}\right)^n \xrightarrow[n\to\infty]{} 1$$ не является состоятельной

$$0 < \Theta - \varepsilon < \Theta \Rightarrow 0 < \varepsilon < \Theta$$

Рассмотрим несмещенную $\tilde{\Theta}_2$

$$\tilde{\Theta}_2' \xrightarrow{P} \Theta \quad \forall \Theta > 0$$

$$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2' - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|(n+1)x_{min} - \Theta| \geq \varepsilon) \geq P(x_{min}(n+1) \geq \Theta + \varepsilon) =$$

$$= P\left(x_{min} \geq \frac{\Theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\Theta+\varepsilon}{n+1}\right) = 1 - \Phi\left(\frac{\Theta+\varepsilon}{n+1}\right) =$$

$$= \left(1 - F\left(\frac{\Theta+\varepsilon}{n+1}\right)\right)^n \underset{n \geq n_0}{=} \left(1 - \frac{\Theta+\varepsilon}{\Theta(n+1)}\right)^n \longrightarrow e^{-\frac{\Theta+\varepsilon}{\Theta}} > 0$$

не является состоятельной даже несмещ.

3) $\tilde{\theta}_3 = X_{max}$

$$M[\tilde{\theta}_3] = \int\limits_{-\infty}^{\infty} y\,q(y)\,dy$$

$\Psi(y) = (F(y))^n$. Тогда $q(y) = \Psi'(y) = n\,F(y)^{n-1} \cdot F'(y) =$
$= n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\,\{(0,\theta)\}$

$M[\tilde{\theta}_3] = \int\limits_0^{\theta} y^n \cdot \frac{n}{\theta^n}\,dy = \frac{\theta n}{n+1}$ - смещенная оценка, поэтому

$\tilde{\theta}_3' = \frac{n+1}{n}\,\tilde{\theta}_3 = \frac{n+1}{n}\,x_{max} \qquad M[\tilde{\theta}_3'] = \theta$

$$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3] = \theta^2 \frac{n}{n+2} - \theta^2 \frac{n^2}{(n+1)^2} = \theta^2 \frac{n}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_3'] = D\left[\frac{n+1}{n}\,x_{max}\right] = \frac{(n+1)^2}{n^2}\,\theta^2 \frac{n}{(n+2)(n+1)^2} \xrightarrow[n \to \infty]{} 0$$

+ несмещ.

$\tilde{\theta}_3'$ - состоят. и несмещ.

4) $\tilde{\theta}_4 = X_1 + \frac{1}{(n-1)} \sum\limits_{i=1}^{n} x_i$

$$M[\tilde{\theta}_4] = M\left[x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i\right] = M[x_1] + \frac{1}{n-1} \sum_{i=2}^{n} M[x_i] =$$
$= M\xi + M\xi = 2M\xi = \theta \Rightarrow$ несмещ.

$$D[\tilde{\theta}_4] = D[x_1] + \frac{1}{(n-1)^2} \sum_{i=2}^{n} Dx_i = \frac{\theta^2}{12} + \frac{1}{(n-1)^2}\frac{\theta^2}{12} \xrightarrow[n\to\infty]{} \frac{\theta^2}{12}$$

$\tilde{\theta}_4 \xrightarrow{P} \theta$

① $\xi_n \xrightarrow{P} \xi \qquad \eta_n \xrightarrow{P} \eta$

$\xi_n + \eta_n \xrightarrow{P} \xi + \eta$

② ЗБЧ Хинчина: ξ_i - независимы и одинаково распред. и $M\xi_i$ конечно, тогда $\frac{1}{n} \sum\limits_{i=1}^{n} \xi_i \xrightarrow{P} M\xi_i$

$$\tilde{\theta}_4 = \underbrace{x_1}_{\xi_n} + \underbrace{\frac{1}{n-1} \sum_{i=2}^{n} x_i}_{\eta_n} \xrightarrow{P} \xi + \frac{\theta}{2}$$

Не является состоятельной

$\xi_n \xrightarrow{P} x_1$

$\eta_n \xrightarrow{P} M x_i = \frac{\theta}{2}$

$M x_i = M\xi = \frac{\theta}{2}$

б) Какая из исправленных оценок более эффективна?

$\tilde{\Theta}_2' = (n+1)\tilde{\Theta}_2 = (n+1)X_{min}$ $\qquad$ $\tilde{\Theta}_3' = \frac{n+1}{n}\tilde{\Theta}_3 = \frac{n+1}{n}X_{max}$

$D[\tilde{\Theta}_2'] = \frac{\Theta^2}{n+2}\cdot n$ $\qquad$ $D[\tilde{\Theta}_3'] = \frac{\Theta^2}{n+2}\cdot\frac{1}{n}$

$$\frac{\Theta^2}{n+2}\cdot n \overset{>}{\vee} \frac{\Theta^2}{n+2}\cdot\frac{1}{n}$$

$$n \overset{>}{\vee} \frac{1}{n}$$

$$n^2 > 1$$

Значит, $\tilde{\Theta}_3'$ более эффективная оценка, чем $\tilde{\Theta}_2'$

*) Исследовать $\tilde{\Theta}_3 = X_{max}$ на состоятельность по опред.

$\tilde{\Theta}_3 \xrightarrow{P} \Theta \quad \forall\Theta > 0$

$\forall\varepsilon > 0 \quad P(|\tilde{\Theta}_3 - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$\tilde{\Theta}_3 \geq \Theta + \varepsilon \qquad \tilde{\Theta}_3 \leq \Theta - \varepsilon \qquad \tilde{\Theta}_3 = X_{max}$

$\cancel{P(|\tilde{\Theta}_3 - \Theta| \geq \varepsilon) = P(X_{max} \geq \Theta + \varepsilon) = 1 - P(X_{max} < \Theta + \varepsilon) =}$

$\cancel{= 1 - \Phi(\Theta+\varepsilon) = 1 - \left(1 - \left(1 - \frac{\Theta+\varepsilon}{\Theta}\right)^n\right) =}$

$\cancel{= 1 - 1 + \left(1 - \frac{\Theta+\varepsilon}{\Theta}\right)^n = \left(1 - 1 - \frac{\varepsilon}{\Theta}\right)^n = \left(-\frac{\varepsilon}{\Theta}\right)^n}$

$\cancel{0 < \Theta - \varepsilon < \Theta \Rightarrow 0 < \varepsilon < \Theta \quad \left(-\frac{\varepsilon}{\Theta}\right)^n \Bigg| \Rightarrow \left(-\frac{\varepsilon}{\Theta}\right)^n \xrightarrow[n\to\infty]{} 0}$

$\cancel{\Rightarrow \tilde{\Theta}_3 - \text{состоятельная}}$

$P(|\tilde{\Theta}_3 - \Theta| \geq \varepsilon) = P(|X_{max} - \Theta| \geq \varepsilon) =$

$= \underbrace{P(X_{max} \geq \varepsilon + \Theta)}_{0} + P(X_{max} \leq -\varepsilon + \Theta) = P(X_{max} \leq \Theta - \varepsilon) =$

$= \Phi(\Theta - \varepsilon) = 1 - \left(1 - \left(\frac{\Theta-\varepsilon}{\Theta}\right)^n\right) = \begin{cases} \left(1 - \frac{\varepsilon}{\Theta}\right)^n, & 0 \leq \varepsilon < \Theta \\ 0, & \varepsilon \geq \Theta \end{cases} \xrightarrow[n\to\infty]{} 0$

из графика $\Phi(x)$

$\Rightarrow \tilde{\Theta}_3$ – состоятельная

(Т3) Случайная величина имеет экспоненц. з-н распределения:

$$p(x) = \begin{cases} \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}} &, x \geq 0 \\ 0 &, x < 0 \end{cases} \quad \theta > 0 \quad ; \ n = 3 \text{ (выборка)}$$

Оценки θ : $\widehat{\theta}_1 = \overline{X}$, $\widehat{\theta}_3 = X_{(2)}$ - второй член. вариац. ряда

а) Исследуем оценки на несмещённость

$\widehat{\theta}_1 = \overline{X}$:

$$M[\widehat{\theta}_1] = M[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n}\sum_{i=1}^{n} M x_i = M\xi$$

$M\xi = \lambda^{-1}$, где $\lambda = \frac{1}{\theta}$ для данного эксп. расср.

$$M\xi = \left(\frac{1}{\theta}\right)^{-1} = \theta \Rightarrow M[\widehat{\theta}_1] = \theta \Rightarrow \widehat{\theta}_1 - \text{несмещенная}$$

$\widehat{\theta}_2 = X_{(2)}$

$$H(x) = n \cdot p(x) \cdot C_{n-1}^{k-1} \cdot (F(x))^{k-1} \cdot (1 - F(x))^{n-k}$$

$$H(x) = 3 \cdot \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}} \cdot C_2^1 \cdot \left(\int_0^x \frac{1}{\theta} \cdot e^{-\frac{t}{\theta}} dt\right)^1 \left(1 - \int_0^x \frac{1}{\theta} \cdot e^{-\frac{t}{\theta}} dt\right)^{3-2} =$$

$$= 6 \cdot \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}} \cdot \left(-e^{-\frac{t}{\theta}}\Big|_0^x\right) \cdot \left(1 + e^{-\frac{t}{\theta}}\Big|_0^x\right) =$$

$$= 6 \cdot \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}} \cdot \left(-e^{-\frac{x}{\theta}} + 1\right) \cancel{\left(1 + e^{-\frac{x}{\theta}} - 1\right)} \cdot e^{-\frac{x}{\theta}} =$$

$$= \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)$$

$$M[\widehat{\theta}_3] = \int_0^\infty x H(x) dx = \int_0^\infty x \cdot \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right) dx =$$

$$= \frac{6}{\theta}\int_0^\infty x \cdot e^{-\frac{2x}{\theta}} dx - \frac{6}{\theta}\int_0^\infty x \cdot e^{-\frac{3x}{\theta}} dx =$$

$$= \frac{6}{\theta} \cdot \left(-\frac{2}{\theta}\right)^{-1} \int_0^\infty x \, d\left(e^{-\frac{2x}{\theta}}\right) - \frac{6}{\theta} \cdot \left(-\frac{3}{\theta}\right) \int_0^\infty x \, d\left(e^{-\frac{3x}{\theta}}\right) =$$

$$= -3\int_0^\infty x \, d\left(e^{-\frac{2x}{\theta}}\right) + 2\int_0^\infty x \, d\left(e^{-\frac{3x}{\theta}}\right) =$$

$$= -3\left(x \cdot e^{-\frac{2x}{\theta}}\Big|_0^\infty - \int_0^\infty e^{-\frac{2x}{\theta}} dx\right) + 2\left(x \cdot e^{-\frac{3x}{\theta}}\Big|_0^\infty - \int_0^\infty e^{-\frac{3x}{\theta}} dx\right) =$$

$$= -\frac{3}{2}\theta \cdot e^{-\frac{2x}{\theta}}\Big|_0^\infty + \frac{2}{3}\theta \cdot e^{-\frac{3x}{\theta}}\Big|_0^\infty = \left(-\frac{2}{3} + \frac{3}{2}\right)\theta = \frac{5}{6}\theta$$

$$M[\widehat{\theta}_3] = \frac{5}{6}\theta \Rightarrow \widehat{\theta}_3 - \text{смещенная}$$

$\tilde{\theta}_3' = \frac{6}{5}\tilde{\theta}_3$ – несмещенная исправленная

б) Какая оценка более эффективна?

$$M[\xi^2] = \frac{1}{\theta}\int_0^\infty x^2 \cdot e^{-\frac{x}{\theta}}dx = -\int_0^\infty x^2 \cdot (de^{-\frac{x}{\theta}}) =$$

$$= -(\underbrace{x^2 \cdot e^{-\frac{x}{\theta}}\Big|_0^\infty}_{0} - 2\int_0^\infty x \cdot e^{-\frac{x}{\theta}}dx) = 2\int_0^\infty x \cdot e^{-\frac{x}{\theta}}dx =$$

$$= -2\theta\int_0^\infty x \cdot d(e^{-\frac{x}{\theta}}) = -2\theta\left(x \cdot e^{-\frac{x}{\theta}}\Big|_0^\infty - \int_0^\infty e^{-\frac{x}{\theta}}dx\right) =$$

$$= +2\theta^2 = 2\theta^2$$

$$D[\tilde{\theta}_1] = D\left[\frac{1}{n}\sum_{i=1}^n x_i\right] = \frac{1}{n^2}D[\sum x_i] = \frac{D\xi}{n} = \frac{M\xi^2 - M^2\xi}{n}$$

$$D[\tilde{\theta}_1] = \frac{2\theta^2 - \theta^2}{3} = \frac{\theta^2}{3}$$

$$D[\tilde{\theta}_3'] = M[\tilde{\theta}_3'^2] - M^2\tilde{\theta}_3'$$

$$M[\tilde{\theta}_3'^2] = \frac{36}{25} \cdot M[\tilde{\theta}_3^2] = \frac{36}{25} \cdot \int_0^\infty x^2 H(x)dx = \frac{38}{25}\theta^2$$

$$D[\tilde{\theta}_3'] = \frac{38}{25}\theta^2 - \theta^2 = \frac{13}{25}\theta^2$$

$$D[\tilde{\theta}_1] \vee D[\tilde{\theta}_3']$$

$$\frac{\theta^2}{3} \vee \frac{13}{25}\theta^2$$

$$\frac{\theta^2}{3} < \frac{13}{25}\theta^2 \Rightarrow \tilde{\theta}_1 \text{ – эффективнее}$$

в) Исследовать на эффективность с помощью неравенства Крамера – Рао.

$$D[\tilde{\theta}] \geq \frac{(g'(\theta))^2}{n\,I(\theta)}$$

∇ Пусть вероятностная модель регулярна, $\tilde{g}(\vec{x}_n)$ – регулярная оценка дифф. ф-ии $g(\theta)$, тогда:

$$D[\tilde{g}(\vec{x}_n)] \geq \frac{(g'(\theta))^2}{n\,I(\theta)}$$

1) Нужно проверить регулярность модели

$p(x,\theta)$ - непр. дифф. по θ на $\boxminus$

$$\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} p(x,\theta)\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}p(x,\theta)\,dx \quad (A = B)$$

A) $$\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}dx = \frac{\partial}{\partial\theta}\left(-e^{-\frac{x}{\theta}}\Big|_{-\infty}^{+\infty}\right) = 0$$

B) $$\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\left(\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}\right)dx = \int_{-\infty}^{\infty}\left(-\frac{1}{\theta^2}\cdot e^{-\frac{x}{\theta}} + \frac{x\cdot e^{-\frac{x}{\theta}}}{\theta^3}\right)dx =$$

$$= \underbrace{\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}\Big|_{-\infty}^{+\infty}}_{=0} + \int_{-\infty}^{+\infty}\frac{x\cdot e^{-\frac{x}{\theta}}}{\theta^3}dx = -\frac{1}{\theta^2}\int_{-\infty}^{\infty} x\,d\left(e^{-\frac{x}{\theta}}\right) =$$

$$= -\frac{1}{\theta^2}\left(xe^{-\frac{x}{\theta}}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} e^{-\frac{x}{\theta}}dx\right) = -\frac{1}{\theta}e^{-\frac{x}{\theta}}\Big|_{-\infty}^{+\infty} = 0$$

$$I(\theta) = M\left[\left(\frac{\partial \ln\left(\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}\right)}{\partial\theta}\right)^2\right] =$$

$$= M\left[\left(\theta\cdot e^{\frac{x}{\theta}}\left(-\frac{1}{\theta^2}\cdot e^{-\frac{x}{\theta}} + \frac{x}{\theta^3}\cdot e^{-\frac{x}{\theta}}\right)\right)^2\right] =$$

$$= M\left[\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] = \int_0^{\infty}\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\cdot\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}dx = \frac{1}{\theta^2} > 0$$

$I(\theta)$ - непр. на $\boxminus$

Из всего вышенаписанного => модель регулярна.

2) $D[\tilde{\theta}_1]$ и $D[\tilde{\theta}_3']$ - непрерывны на $\forall$ компакте =>
по т. Вейерштрасса
=> огр. на $\forall$ компакте

3) $\tilde{\theta}_1$:

$$D\tilde{\theta}_1 \geq \frac{(\theta')^2}{n\,I(\theta)}$$

$$\frac{\theta^2}{3} = \frac{\theta^2}{3}$$ => оценка эффективна

$\tilde{\theta}_3'$ $$D\tilde{\theta}_3' \geq \frac{(\theta')^2}{n\,I(\theta)}$$

$$\frac{13}{25}\theta^2 > \frac{\theta^2}{3}$$ => оценка не эффективна