Chinese Remainder Theorem

John Leo

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1 Instrumentation

1. Xiaoluo [small gong] (30 bar phrases)
2. Paigu [drums] (7 bar phrases)
3. Yanqin (5 bar phrase)
4. Plucked Strings (3 bar phrases)
   (a) Zhongruan
   (b) Pipa
5. Bowed Strings (2 bar phrases)
   (a) Erhu
   (b) Zhonghu

2 Musical Structure

- There are $2 \cdot 3 \cdot 5 \cdot 7 = 210$ bars of music.
- Tempo is 120 bpm. 4/4 time, so 840 beats and exactly 7 minutes of music. Divide into 7 one-minute sections of 30 bars of music.
- Key is G major.
- All instruments play only scale tones.
  - Yangqin plays the pentatonic scale omitting 4 and 7 (G, A, B, D, E). (Major Pentatonic: https://ianring.com/musictheory/scales/661
  - All other strings play a hexatonic scale omitting only 4 (G, A, B, D, E, F#). (Raga Kumud: https://ianring.com/musictheory/scales/2709

3 Chinese Remainder Theorem

Theorem (Chinese Remainder Theorem)**.** *Let* $I_1, \ldots, I_N$ *be pairwise comaximal ideals of commutative ring* R *(*$I_i + I_j = R$ *for all* $i \neq j$*). Then there is a ring isomorphism* $R/(\cap_k I_k) \simeq \prod_k (R/I_k)$*. In particular if* $n_k \in \mathbb{Z}$ *are pairwise coprime then* $\mathbb{Z}/[(\prod_k n_k)\mathbb{Z}] \simeq \prod_k (\mathbb{Z}/n_k\mathbb{Z})$*.*

Proof Outline. The primary source for the proof is Serge Lang, *Algebra, Revised Third Edition* (2002), pages 94-95.

1. Bézout's Lemma: Coprime implies comaximal. If $(m, n) = 1$. then $\exists a, b$ s.t. $1 = am + bn$.
2. Surjectivity. Base case $n = 2$. Given $(c, d) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, let $x = dam + cbn$.
3. Check that $x \equiv c \bmod m$ and $x \equiv d \bmod n$.
4. Induction: For all k, $n_k\mathbb{Z}$ is comaximal to the product of the other $n_i\mathbb{Z}$: $n_1\mathbb{Z} + \prod_{i \neq k} n_k\mathbb{Z} = \mathbb{Z}$.
5. By the theorem with $n = 2$, for all k we can find y_k satisfying $y_k \equiv 1 \bmod n_k$, $y_k \equiv 0 \bmod \prod_{i \neq k} n_i$.
6. Then given $(x_1, \ldots, x_N) \in \prod_k (\mathbb{Z}/nk\mathbb{Z})$, $\sum_k x_k y_k$ maps to $(x_1, \ldots, x_N)$.
7. Injectivity. The kernel of the homomorphism is $(\prod_k n_k)\mathbb{Z}$.

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