Chinese Remainder Theorem

Joint Mathematics Meetings
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Background

PhD in number theory at UCLA (2008)

Played 中阮 (zhongruan) with the

UCLA Music of China Ensemble and currently with the Seattle Chinese Orchestra

Presenter at and Organizer of FARM (Functional Art and Music) the last several years (functional-art.org)

Current Research

Mathematics

Proof Sonification

Type Theory /
Formal Verification

Music

Computer Science

Music Analysis and Synthesis
(e.g. FARM 2022
www.youtube.com/watch?v=znjisTSoKCE)

Assertions

1. Mathematics is an art form, but one which requires serious study to fully appreciate.
2. Mathematics and music are intimately related.
3. Previous compositions based on mathematics have used relatively simple mathematics.

Thesis

There is still much to explore to use more advanced mathematics to structure music.

Goals

Increasing Difficulty

1. Use music to help understand mathematical proofs.
2. Use mathematics to help create new musical expression.
3. Create new music I personally enjoy.

Increasing Importance

Expressing Math as Music

One Approach

- Gödel encode a theorem as a positive integer n.
- Play middle C on the piano n times.

Previous Work

Logical Soundness

(FARM 2021)

- Music: www.youtube.com/watch?v=GmWG5ZXMxFE
- Presentation: https://www.youtube.com/watch?v=g0wx4My75zY&t=11041s

Programmatically translated a mechanized soundness proof into music using leitmotifs.

Chinese Remainder Theorem

Let $I_1, \ldots, I_N$ be pairwise comaximal ideals of commutative ring R ($I_i + I_j = R$ for all $i \neq j$). Then there is a ring isomorphism $R/(\bigcap_k I_k) \simeq \prod_k (R/I_k)$.

In particular if $n_k \in \mathbb{Z}$ are pairwise coprime then $\mathbb{Z}/[(\prod_k n_k)\mathbb{Z}] \simeq \prod_k (\mathbb{Z}/n_k\mathbb{Z})$.

Example: $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}3\mathbb{Z}$

$0 \leftrightarrow (0,0) \quad 1 \leftrightarrow (1,1) \quad 2 \leftrightarrow (0,2) \quad 3 \leftrightarrow (1,0) \quad 4 \leftrightarrow (0,1) \quad 5 \leftrightarrow (1,2)$

$2 + 5 = 1 \quad (0,2) + (1,2) = (1,1)$

$2 \cdot 5 = 4 \quad (0,2) \cdot (1,2) = (0,1)$

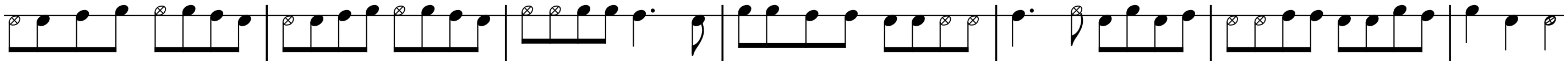
CRT as Music

Plan

- Score for a small Chinese ensemble.
- Key of G, but don't use C (4th note): "major hexatonic scale".
- Divide the proof into 7 parts, one minute per part.
- The music should both reflect the math of the proof, and also demonstrate the theorem in action.
- Different instruments use 2, 3, 5 and 7 bar phrases, and the piece ends in $2 \cdot 3 \cdot 5 \cdot 7 = 210$ bars.
- At 120 beats per minute (4/4 time), this is exactly 7 minutes with 7 one-minute sections.

Instrumentation

- Bowed Strings (“2”): Erhu and Zhonghu
- Plucked Strings (“3”): Pipa and Zhongruan
- Yangqin [hammered dulcimer] (“5”)
- Percussion (“7”)
  - Xiaoluo [small gong]: plays 7 times, each 30 bars, counts section number
  - Paigu [drums]: repeating pattern of 7 bars



Proof Outline (1)

1. Bézout's Lemma: Coprime implies comaximal. If $(m, n) = 1$ then $\exists a, b$ s.t. $1 = am + bn$.

2. Surjectivity. Base case $n = 2$. Given $(c, d) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, let $x = dam + cbn$.

3. Induction: For all k, $n_k\mathbb{Z}$ is comaximal to the product of the other $n_i\mathbb{Z}$: $n_k\mathbb{Z} + \prod_{I \neq k} n_k\mathbb{Z} = \mathbb{Z}$.

4. By the theorem with $n = 2$, for all k we can find y_k satisfying $y_k \equiv 1 \bmod n_k$, $y_k \equiv 0 \bmod \prod_{I \neq k} n_i$.

Proof Outline (2)

5. Then given $(x_1, \ldots, x_N) \in \prod_k (\mathbb{Z}/n_k\mathbb{Z})$, $\sum_k x_k y_k$ maps to $(x_1, \ldots, x_N)$.
6. The map $f : \mathbb{Z} \to \prod_k (\mathbb{Z}/n_k\mathbb{Z})$ defined by $f(x) = \prod_k (x \bmod n_k)$ is a (surjective) ring homomorphism.
7. The kernel of f is $(\prod_k n_k)\mathbb{Z}$.

Assessment

1. Use music to help understand mathematical proofs.
   - ✗ Doesn't help understand the proof.
   - ✓ Does illustrate some aspects of the theorem, but hard to perceive.
2. Use mathematics to help create new musical expression.
   - ✗ Nothing groundbreaking.
   - ✓ Was able to use the math to help structure conventional music.
3. ✓ Create new music I personally enjoy.

Resources

Github: https://github.com/halfaya/CRT

Music Video: https://youtu.be/8aoA-wpgEos