Chinese Remainder Theorem

John Leo

December 29, 2023

1 Instrumentation

1. Xiaoluo [small gong] (30 bar phrases)
2. Paigu [drums] (7 bar phrases)
3. Yanqin (5 bar phrase)
4. Plucked Strings (3 bar phrases)
   (a) Zhongruan
   (b) Pipa
5. Bowed Strings (2 bar phrases)
   (a) Erhu
   (b) Zhonghu

2 Musical Structure

- There are $2 \cdot 3 \cdot 5 \cdot 7 = 210$ bars of music.
- Tempo is 120 bpm. 4/4 time, so 840 beats and exactly 7 minutes of music. Divide into 7 one-minute sections of 30 bars of music.
- The key signature is G major. However they key could also be viewed as D major, as C/C# is avoided.
- All instruments play only scale tones.
  - Yangqin was originally to play the pentatonic scale omitting 4 and 7 (G, A, B, D, E).
    (Major Pentatonic: https://ianring.com/musictheory/scales/661)
    However it ended up playing the same hexatonic scale as the others.
  - All non-percussion instruments play a hexatonic scale omitting only 4 (G, A, B, D, E, F#).
    (https://ianring.com/musictheory/scales/2709)
    If the key is viewed as D major then 7 is omitted and the scale is the major diatonic hexachord (https://ianring.com/musictheory/scales/693)

3 Chinese Remainder Theorem

Theorem (Chinese Remainder Theorem)**.** *Let $I_1, \ldots, I_N$ be pairwise comaximal ideals of commutative ring R ($I_i + I_j = R$ for all $i \neq j$). Then there is a ring isomorphism $R/(\cap_k I_k) \simeq \prod_k (R/I_k)$. In particular if $n_k \in \mathbb{Z}$ are pairwise coprime then $\mathbb{Z}/[(\prod_k n_k)\mathbb{Z}] \simeq \prod_k (\mathbb{Z}/n_k\mathbb{Z})$.*

Proof Outline. The primary source for the proof is Serge Lang, *Algebra, Revised Third Edition* (2002), pages 94-95.

1. Bézout's Lemma: Coprime implies comaximal. If $(m, n) = 1$, then $\exists a, b$ s.t. $1 = am + bn$.
2. Surjectivity. Base case $n = 2$. Given $(c, d) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, let $x = dam + cbn$.
3. Induction: For all k, $n_k\mathbb{Z}$ is comaximal to the product of the other $n_i\mathbb{Z}$: $n_k\mathbb{Z} + \prod_{i \neq k} n_k\mathbb{Z} = \mathbb{Z}$.
4. By the theorem with $n = 2$, for all k we can find y_k satisfying $y_k \equiv 1 \bmod n_k$, $y_k \equiv 0 \bmod \prod_{i \neq k} n_i$.
5. Then given $(x_1, \ldots, x_N) \in \prod_k (\mathbb{Z}/n_k\mathbb{Z})$, $\sum_k x_k y_k$ maps to $(x_1, \ldots, x_N)$.
6. The map $f : \mathbb{Z} \to \prod_k (\mathbb{Z}/n_k\mathbb{Z})$ defined by $f(x) = \prod_k (x \bmod n_k)$ is a (surjective) ring homomorphism.
7. The kernel of f is $(\prod_k n_k)\mathbb{Z}$.

□

4 Musical Realization

Xiaoluo introduces each 30 bar section, counting the section number. Paigu plays patterns changing every 7 bars so that the full cycle is completed in 210 bars. Each section is described below.

1. Bézout's Lemma (EuclideanDomain.gcd_eq_gcd_ab) is illustrated by the special case $m = 2, n = 3$ following the extended Euclidean algorithm (EuclideanDomain.xgcd). Erhu/Zhonghu represent 2, Pipa/Zhongruan represent 3, and Yangqin does additional work. The sequence (following EuclideanDomain.xgcdAux) is:

   (a) $2 = 1 \cdot 2 + 0 \cdot 3$

   (b) $3 = 0 \cdot 2 + 1 \cdot 3$

   (c) $2 = 1 \cdot 2 + 0 \cdot 3$

   (d) $1 = (-1) \cdot 2 + 1 \cdot 3$

   Each step takes 6 bars. The final 6 bars is devoted to a special case of the surjectivity base case: $2 = (-2) \cdot 2 + 2 \cdot 3$

2. Surjectivity. Base case $n = 2$. Again for $m = 2, n = 3$, five of the six cases (omitting only $(0, 0)$) are illustrated using different themes in this order:

   (a) $2 = 1 \cdot 2 + 0 \cdot 3$ $[2 \leftrightarrow (0, 2)]$

   (b) $4 = 2 \cdot 2 + 0 \cdot 3$ $[4 \leftrightarrow (0, 1)]$

   (c) $3 = 0 \cdot 2 + 1 \cdot 3$ $[3 \leftrightarrow (1, 0)]$

   (d) $5 = 1 \cdot 2 + 1 \cdot 3$ $[5 \leftrightarrow (1, 2)]$

   (e) $7 = 2 \cdot 2 + 1 \cdot 3$ $[1 \leftrightarrow (1, 1)]$

3. Illustration of the beginning of induction using $2, 3, 5$: Each number is comaximal to the product of the other two.

   (a) $1 = (-1) \cdot 5 + 1 \cdot 6$
   (b) $1 = (-3) \cdot 3 + 1 \cdot 10$
   (c) $1 = (-7) \cdot 2 + 1 \cdot 15$

   The sections are 5, 10, 15 measures long respectively. The basic idea is that the "lone" instrument (or pair) plays in opposition to the other two.

4. Inductive hypothesis with $n = 2$ for each of the three combinations.

   (a) $1 \cdot 6 \equiv 1 \bmod 5, 0 \bmod 6$
   (b) $1 \cdot 10 \equiv 1 \bmod 3, 0 \bmod 10$
   (c) $1 \cdot 15 \equiv 1 \bmod 2, 0 \bmod 15$

   Each section is 10 measures long. Previous themes are used by the instruments. In the first five measures, the solo instrument plays in counterpoint to the other two. In the second 5 measures, the olo instrument changes to play the same melody, except inverted or retrgrade so it "candels out".

5. 5

   (Below saved for later just in case I want to use it.)

   (a) $11 = 1 \cdot 5 + 1 \cdot 6$ $[11 \leftrightarrow (1, 5)]$
   (b) $16 = 2 \cdot 5 + 1 \cdot 6$ $[16 \leftrightarrow (1, 4)]$
   (c) $17 = 1 \cdot 5 + 2 \cdot 6$ $[17 \leftrightarrow (2, 5)]$
   (d) $22 = 2 \cdot 5 + 2 \cdot 6$ $[22 \leftrightarrow (2, 4)]$
   (e) $27 = 3 \cdot 5 + 2 \cdot 6$ $[27 \leftrightarrow (2, 3)]$

6. 6

7. 7