

Music Analysis and Synthesis

Joint Mathematics Meetings
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<https://www.halfaya.org>

Chinese Remainder Theorem (JMM 2024)

C

Induction Comaximality. $1 = (-1) \cdot 5 + 1 \cdot 6$

Musical score for Pipa and Zhongruan. The score consists of two staves. The top staff is for Pipa, starting with dynamic *f*. The bottom staff is for Zhongruan, also starting with *f*. Both staves have measures numbered 61 through 65. Measure 61 shows eighth-note patterns. Measures 62-64 show sixteenth-note patterns. Measure 65 shows eighth-note patterns again. Dynamics include *f*, *mf*, and *sf*. A blue annotation "accompaniment to yangqin" is placed under the Zhongruan staff between measures 61 and 65.

Musical score for Yangqin. The score consists of two staves. The top staff is for Yangqin, starting with eighth-note patterns. The bottom staff is for bass, providing harmonic support. Measures are numbered 61 through 65. Dynamics include *sf* and *sfp*. A pink annotation "continuity, then echo of A theme, 5 bars" is placed under the Yangqin staff between measures 61 and 65.

Coinduction (math+fiction+art) with Amy Zhu (UW)



The first book, by mathematician Andrea Emily, is named *Animals*, a curious title given that it is entirely about cats. However it argues that cats are representative of all animals. In fact it focuses only on a single cat, Yu, the queen of the jungle, who is in turn representative of all cats.

Outline

1. Past work:

- Music Tools
- Analysis and Synthesis of Counterpoint

2. Recent work in progress:

- Lean and proofs
- Equivalence

Music Tools

Brief History

- Originally written in Haskell and then Agda / Cubical Agda.

Agda is written in Haskell and has a Haskell FFI (Foreign Function Interface), which gives access to high quality libraries for MIDI, (Music)XML, and interfaces to SMT (Satisfiability Modulo Theories) solvers.

- Currently porting to Lean 4.

Lean 4 is written in Lean 4 (bootstrapped by C++), but the FFI is still unstable and requires a C interface.

Allows use of Mathlib, an extremely powerful mathematics library.

Music Tools

Functionality

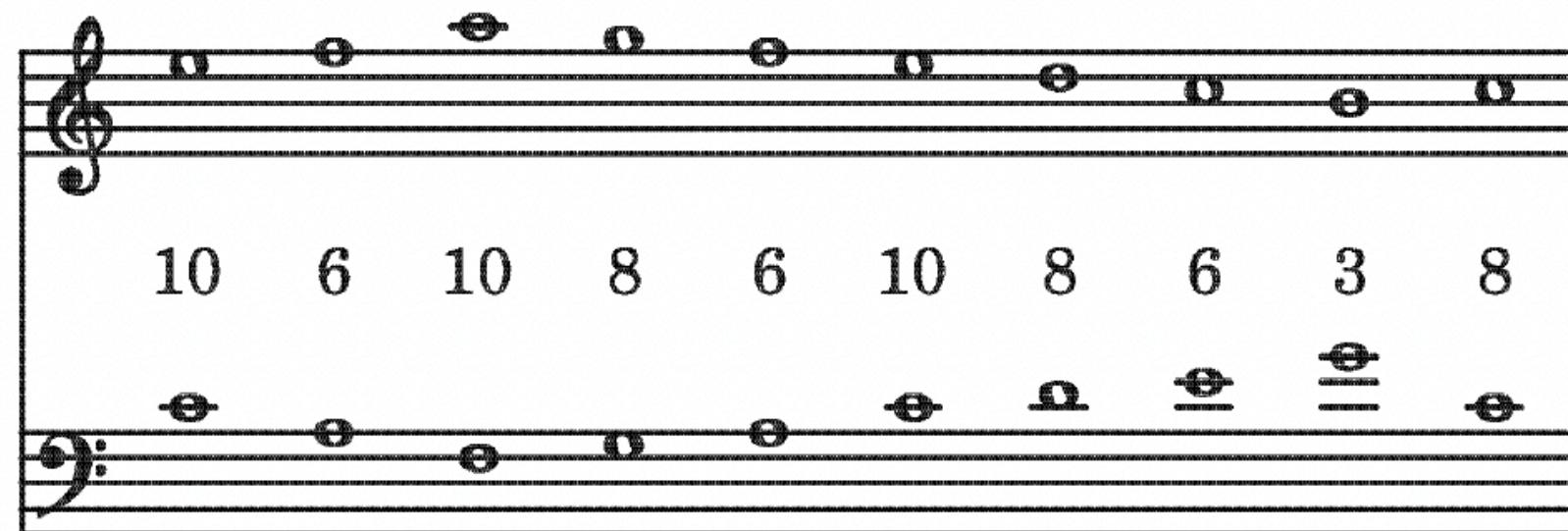
- Small, composable tools for representation, analysis and synthesis of music.
- Reads and writes MusicXML; also outputs MIDI.
- Interfaces with an SMT solver (Z3) for synthesis.
- Previous presentations:
 - *Logical Soundness* [FARM (Functional Art and Music) 2021]
links at <https://www.halfaya.org/music/icfp>
 - *Counterpoint Analysis and Synthesis* [FARM 2022]
<https://www.youtube.com/watch?v=znjisTSoKCE>

Counterpoint Rules

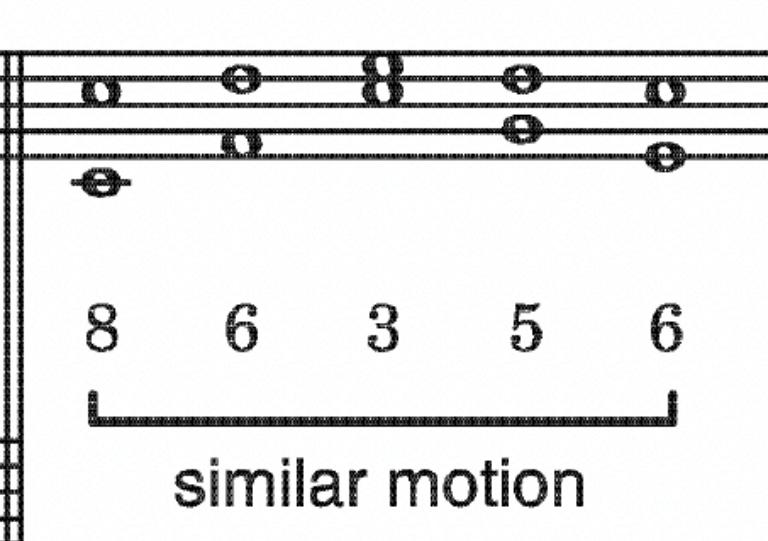
from Laitz: *The Complete Musician* (2012)

EXAMPLE 2.9 Contrapuntal Motions

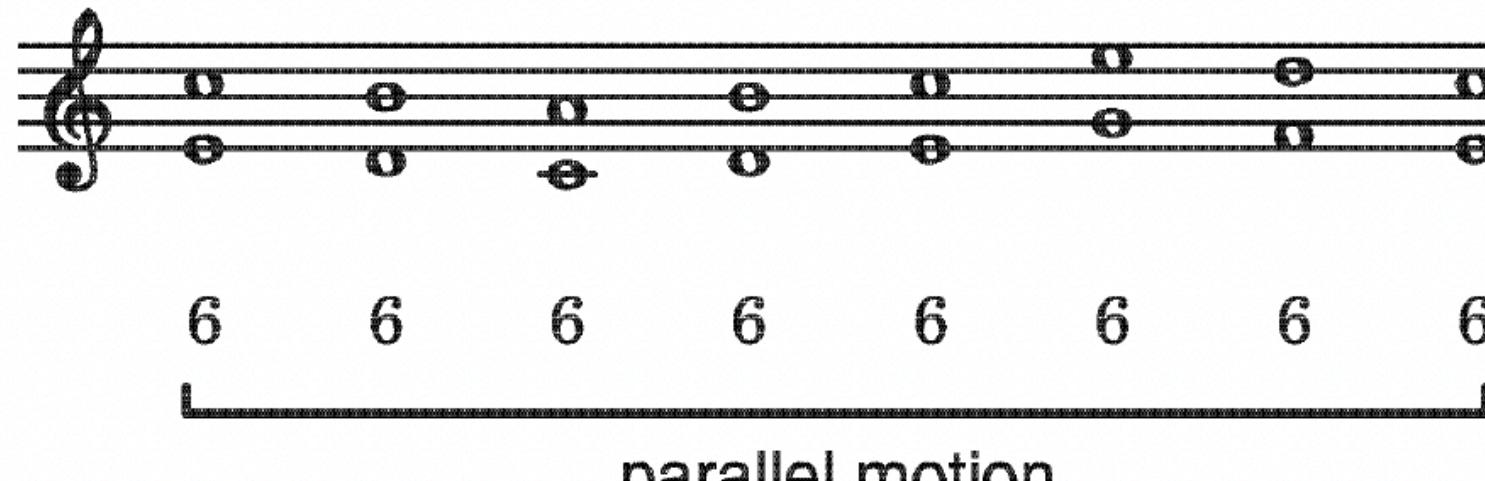
A.



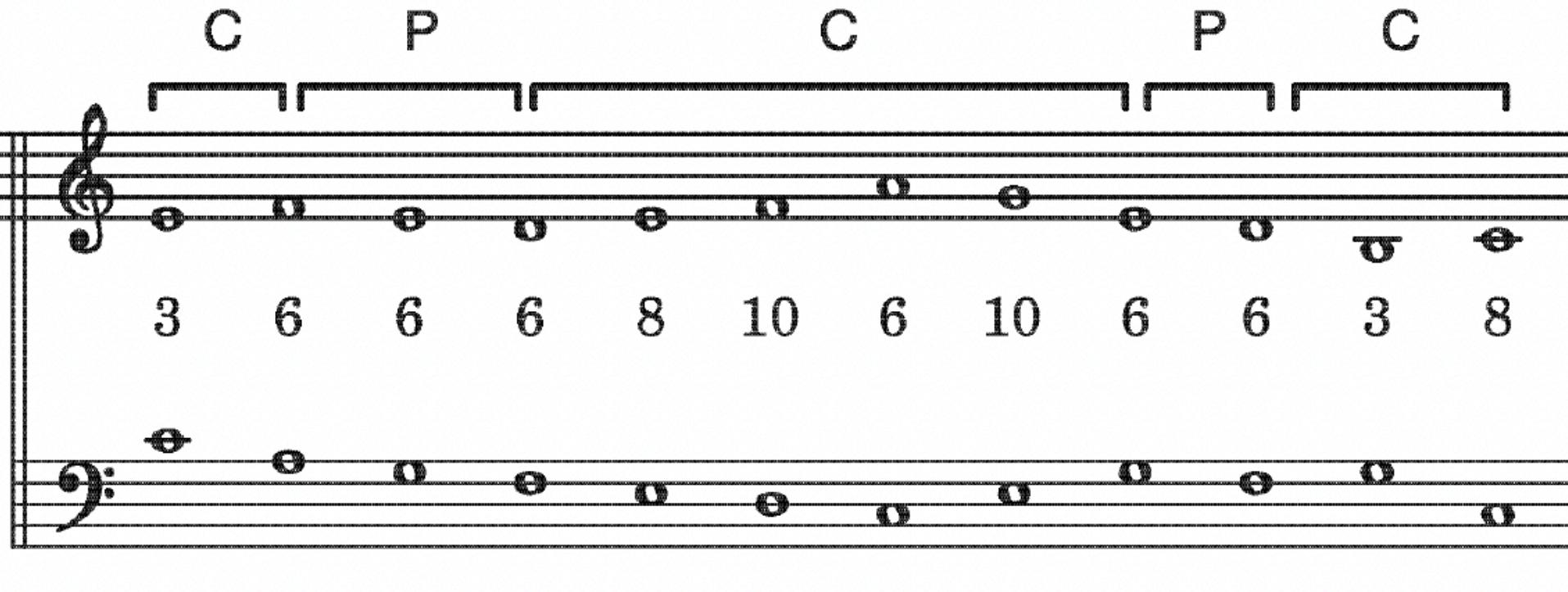
B.



C.



D.



Counterpoint Example

First Species (exercise 146 by Haydn for Beethoven)

A musical staff consisting of five horizontal lines. A treble clef is positioned at the top left. A vertical brace on the far left groups the first two lines. A 3/4 time signature is located in the upper-middle section. A bass clef is placed at the far right end of the staff.

A musical score for three voices: Treble, Alto, and Bass. The score consists of three staves. The top staff (Treble) has a treble clef, a key signature of one sharp (F#), and a common time signature. It contains six measures of music. The middle staff (Alto) has an alto clef and a common time signature. The bottom staff (Bass) has a bass clef and a common time signature. The vocal parts are separated by vertical bar lines. The vocal parts are as follows:

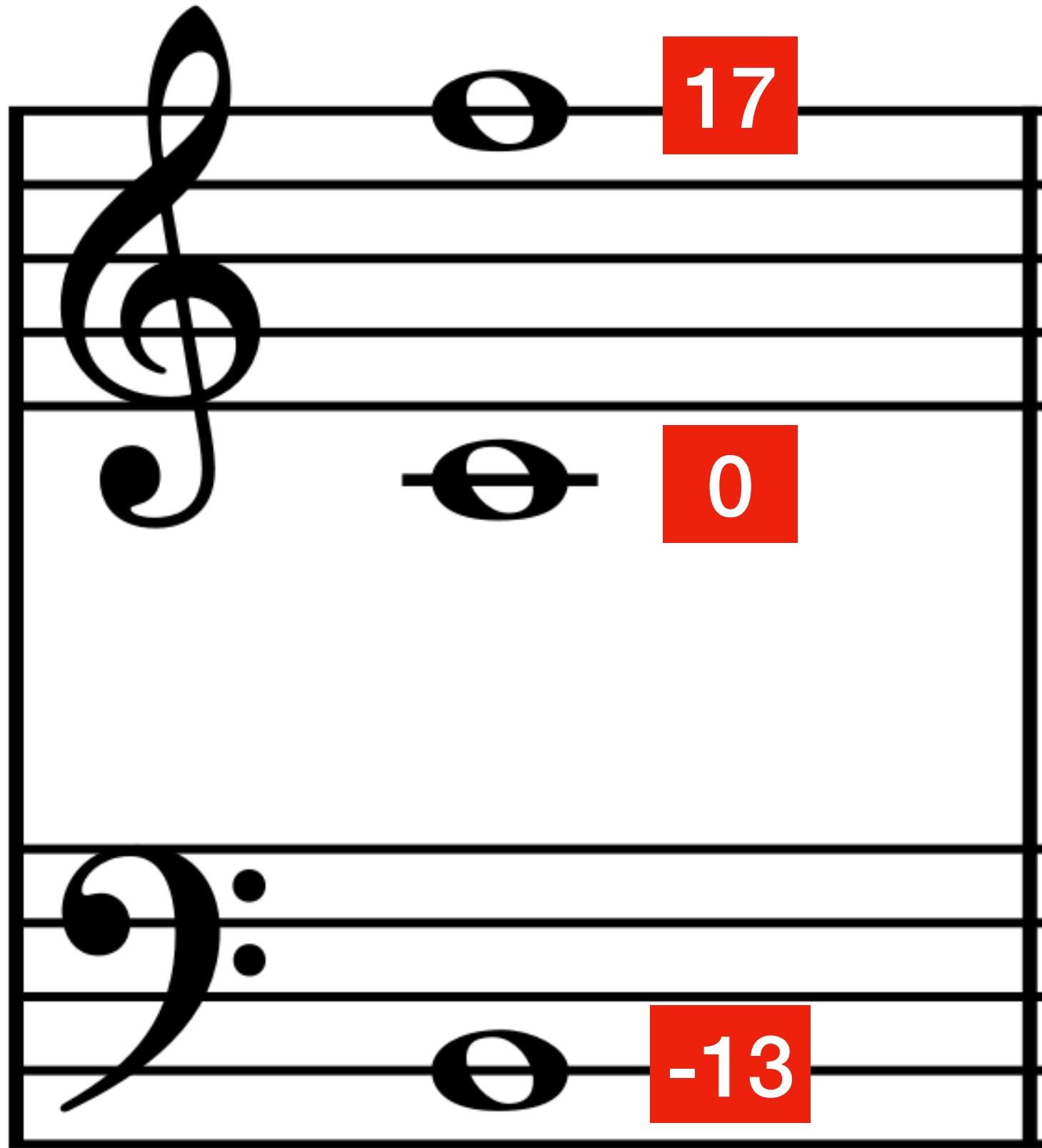
- Treble:** Measures 1-2: G4 (quarter note). Measures 3-4: F#4 (quarter note). Measures 5-6: E4 (quarter note).
- Alto:** Measures 1-2: B3 (quarter note). Measures 3-4: A3 (quarter note). Measures 5-6: G3 (quarter note).
- Bass:** Measures 1-2: D3 (quarter note). Measures 3-4: C3 (quarter note). Measures 5-6: B2 (quarter note).

The vocal parts are separated by vertical bar lines. The vocal parts are as follows:

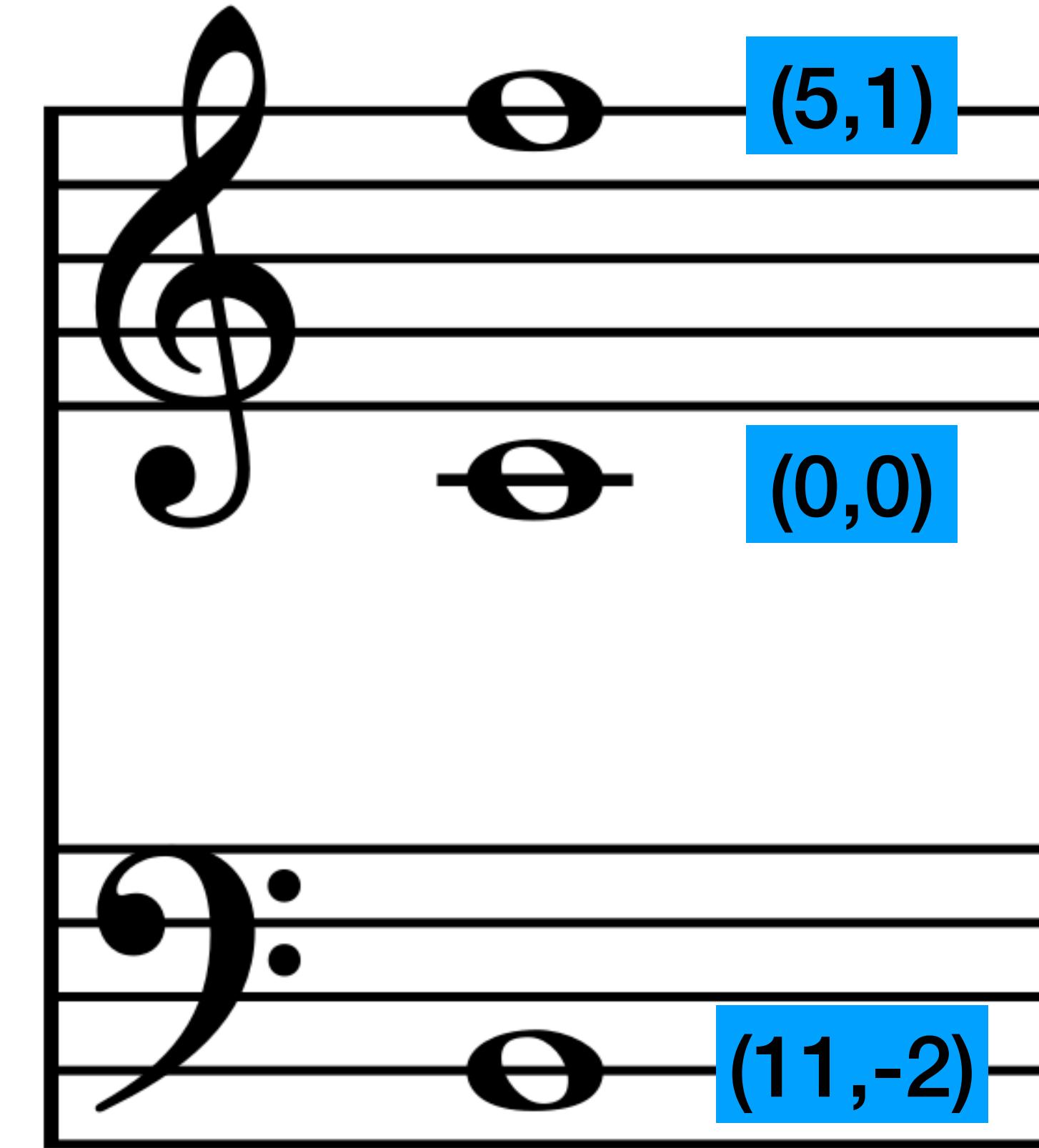
- Treble:** Measures 1-2: G4 (quarter note). Measures 3-4: F#4 (quarter note). Measures 5-6: E4 (quarter note).
- Alto:** Measures 1-2: B3 (quarter note). Measures 3-4: A3 (quarter note). Measures 5-6: G3 (quarter note).
- Bass:** Measures 1-2: D3 (quarter note). Measures 3-4: C3 (quarter note). Measures 5-6: B2 (quarter note).

Equivalence in Music

Absolute vs Relative Pitch



(Pitch within octave, Octave)



Equivalence in Music

Horizontal vs Vertical

Allegro

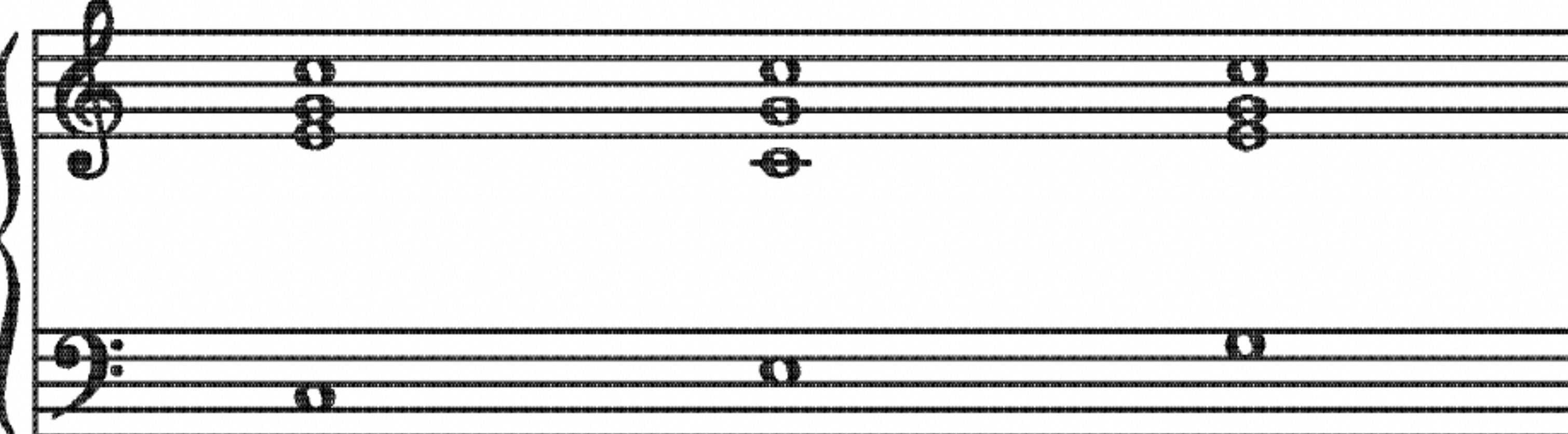
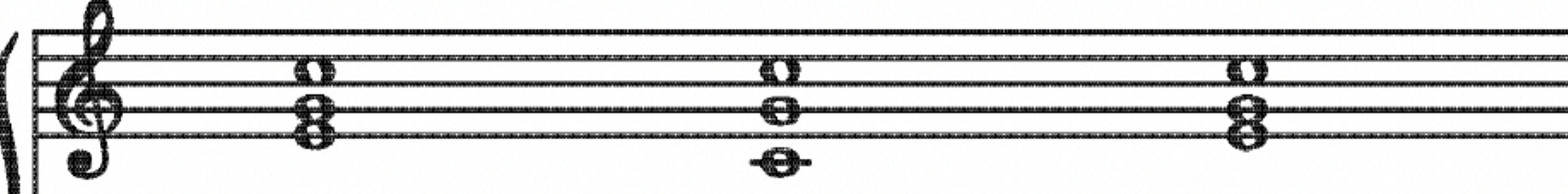
(dol.)

Allegretto

p

Equivalence in Music

Chord Inversions

Root position (Root in bass)	First inversion (3rd in bass)	Second inversion (5th in bass)
		

Equality and Equivalence in Type Theory

Intensional vs Extensional

- Intensional Type Theory (Lean, Agda, Coq/Rocq etc.)
Two kinds of equality:
 - ▶ Definitional Equality – automatic
 - ▶ Propositional Equality – must be proven and manually applied
- Extensional Type Theory

One kind of equality – might have to be proven but automatically applied
 - ▶ Problem: Type checking becomes undecidable

The First Example

```
inductive ℕ : Type where
| zero : ℕ
| succ : ℕ → ℕ

def add : ℕ → ℕ → ℕ
| m, zero => m
| m, succ n => succ (add m n)

theorem add_zero (m : ℕ) : m + 0 = m := rfl

theorem zero_add (m : ℕ) : 0 + m = m := by
  induction m with
  | zero => rfl
  | succ m ih => rewrite [add_succ, ih]; rfl
```

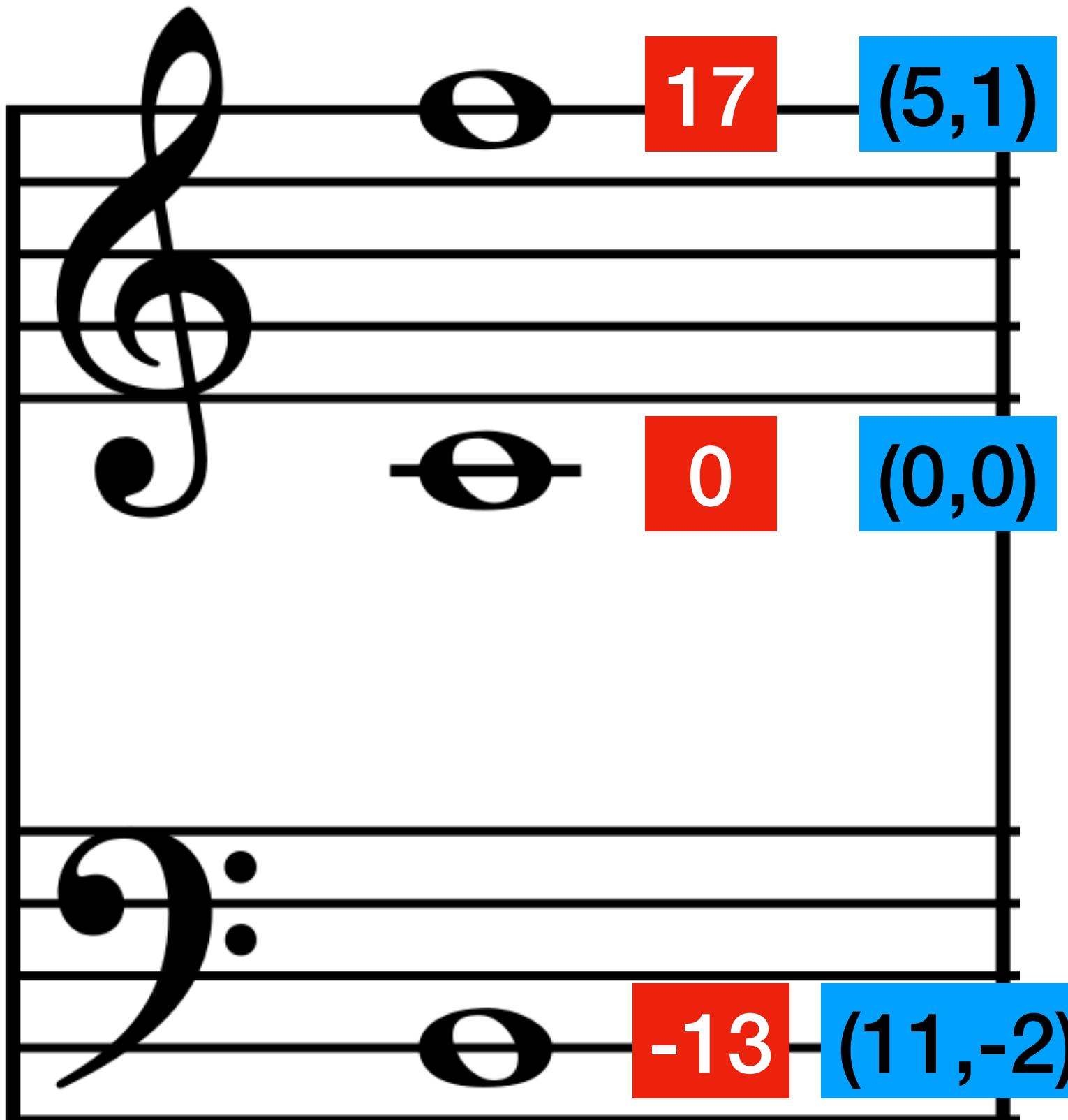
Some Recent Work

with connections to the University of Washington

- Talia Ringer, et. al: *Proof Repair Across Type Equivalences* [PLDI 2021]
<https://dependenttyp.es/pdf/repair.pdf>
- Anjali Pai: *Working with Equivalent Definitions in Rocq*
<https://uwplse.org/2024/08/19/Equivalent-Rocq.html>
- Terry Tao, et. al: *Equational Theories* [Sept-Oct 2024]
https://github.com/teorth/equational_theories
makes use of egg (e-graphs good) [<https://egraphs-good.github.io/>]

Equivalence in Music

Absolute vs Relative Pitch



```
abbrev Pitch := Int
abbrev Octave := Int
def PC := {n : Int // 0 ≤ n ∧ n < 12}
def PCOctave : Type := PC × Octave

def absoluteToRelative (p : Pitch) : PCOctave :=
⟨(p % 12,
  And.intro (Int.emod_nonneg p (by simp))
            (Int.emod_lt_of_pos p (by simp))),  

  p / 12⟩

def relativeToAbsolute : PCOctave → Pitch
| ⟨p, o⟩ => o * 12 + p.1
```

Equivalence of Absolute and Relative Pitch

One direction

```
theorem PitchToPC0ctaveToPitch (p: Pitch) :  
  relativeToAbsolute (absoluteToRelative p) = p :=  
  Int.ediv_add_emod' p 12
```

Equivalence of Absolute and Relative Pitch

The other direction

```
theorem PC0ctaveToPitchToPC0ctave : (pco: PC0ctave) →  
    absoluteToRelative (relativeToAbsolute pco) = pco := by  
    intro pco  
    unfold relativeToAbsolute absoluteToRelative;  
    rw [← Prod.eta pco]  
    apply Prod.ext_iff.2; simp  
    constructor
```

```
def PC := {n : Int // 0 ≤ n ∧ n < 12}
```

```
apply Subtype.ext; simp  
rw [Int.add_emod, Int.mul_emod_left]; simp  
rw [Int.emod_eq_of_lt pco.fst.property.1 pco.fst.property.2]  
  
rw [Int.add_ediv_of_dvd_left (Int.dvd_mul_left pco.snd 12)]  
rw [Int.mul_ediv_cancel pco.snd (by simp)]  
rw [Int.ediv_eq_zero_of_lt pco.fst.property.1 pco.fst.property.2]  
simp
```

Using Equivalence

Given

$$\begin{aligned}\varphi : A &\cong C \\ \psi : B &\cong D\end{aligned}$$

and a function

$f : A \rightarrow B$, derive

$g : C \rightarrow D$ as

$$g = \psi \circ f \circ \varphi^{-1}$$

Using Equivalence in Lean

Automatic coercion

```
instance : Coe Pitch PC0ctave where  
 coe := absoluteToRelative
```

```
instance : Coe PC0ctave Pitch where  
 coe := relativeToAbsolute
```

```
def pitchClass : PC0ctave → PC  
| (p, _) => p
```

```
def f5 : Pitch := 17
```

```
#eval pitchClass f5  
— 5
```

Future Work

- Finish Lean translation
- Port foreign function interfaces to SMT and MusicXML
- Continue work with equivalences and proofs
- Expand functionality to further analysis and synthesis

<https://github.com/halfaya/MusicTools>

Tlön, Uqbar, Orbis Tertius

Jorge Luis Borges, *Ficciones* (1940)

Numismatics, pharmacology and archaeology have been revised. I gather that biology and mathematics are awaiting their avatar. [...]

Then, English, French, and mere Spanish will disappear from this planet. The world will be Tlön. I take no notice. I go on revising, in the quiet of the days in the hotel at Androgué, a tentative translation into Spanish, in the style of Quevedo, which I do not intend to see published, of Sir Thomas Browne's *Urn Burial*.