LOW GENUS CURVES WITH MANY ISOLATED POINTS

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Abstract. Bourdon et al. [BEL+19] in 2019 introduced the concept of *isolated points* on varieties, a generalization of rational points, and proved that any "nice" curve over a number field has finitely many isolated points. We examine the state of the art on finding low genus curves with a large number of isolated points.

1. Introduction

Expand the abstract a bit here.

2. Background

We provide here some fundamental definitions and facts. For details see [VV24].

Let k be a number field and $\overline{k}$ be its algebraic closure. Let C/k denote a *nice* (smooth, projective, and geometrically integral over k) curve (variety of dimension 1 over k). A *closed point* on C is the Galois orbit of a geometric point in $X(\overline{k})$, and its *degree* is the cardinality of this orbit. A rational point is the special case in which the degree is 1.

The number of rational points on a curve is governed by its genus:

(1) **Genus 0.** There are either zero or infinitely many rational points.
(2) **Genus 1.** The set of ratonal points is a finitely generated abelian group (the Mordell-Weil Theorem). Furthermore the possible torsion groups are completely understood for a curve over $\mathbb{Q}$ ([Maz77, Maz78]) and are known to be bounded in general ([Mer96]).
(3) **Genus 2.** There are finitely many rational points (Faltings's Theorem (add citation)).

When $d > 1$, the number of degree d points can be infinite even when the genus is ≥ 2. However all but finitely many of these points can be paramerized in one of two ways.

(1) $\mathbb{P}^1$**-parameterized points**. A closed point $x \in C$ is $\mathbb{P}^1$-parameterized if there exists a morphism $\pi : \mathbb{C} \to \mathbb{P}^1$ with $\deg(\pi) = \deg(x)$ and $\pi(x) \in \mathbb{P}^1(k)$.
(2) **AV-parameterized points**. A degree d closed point $x \in C$ is AV-parameterized if there exists a positive rank abelian subvariety $A \subset \mathrm{Pic}^0_C$ such that $[x] + A \subset W^d = \mathrm{im}(\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C)$.

A point that is neither $\mathbb{P}^1$- nor AV-parameterized is called *isolated*, and [BEL+19, Theorem 4.2] prove that there are only finitely many isolated points (considering all possible degrees) on a given curve, and furthermore there are infinitely many points of degree d iff there exists a degree d $\mathbb{P}^1$-parameterized point. These are referenced as Corollary 4.3 and Theorem 4.4 of [VV24].

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Given that any curve has only finitely many isolated points, a natural question to ask is how to find specific curves with large numbers of such points. Since the concept of isolated points is a relatively new one, introduced in [BEL+19], we first consider the rational (degree 1) special case, which has been studied for a longer period of time.

3. Curves with Many Rational Points

As noted above, finiteness of the number of rational points is controlled by the genus.

3.1. **Genus** < 2**.** A curve of genus 0 either has zero or infinitely many rational points, so this case is uninteresting.

For the genus 1 case, by the Mordell-Weil Theorem the set of rational points is a finitely-generated abelian group and thus finite iff the rank of the group is zero. The number of rational points is then equal to the size of the torsion group. In the case $k = \mathbb{Q}$, Mazur's theorem states that the largest such group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ of order 16. There are currently three such curves in LMFDB ([LMF24]), which can be found by searching for "Elliptic Curves over $\mathbb{Q}$" with the given rank and torsion.

For arbitrary nunber fields, the situation is more complicated.

3.2. **Genus** ≥ 2**.** xxx

References

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