LOW GENUS CURVES WITH MANY ISOLATED POINTS

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Abstract. Bourdon et al. [BEL+19] in 2019 introduced the concept of *isolated points* on varieties, a generalization of rational points, and proved that any "nice" curve over a number field has finitely many isolated points. We examine the state of the art related to low genus curves with a large number of isolated points.

1. Introduction

Bourdon et al. [BEL+19] in 2019 introduced the concept of *isolated points* on varieties, a generalization of rational points, and proved that any "nice" curve over a number field has finitely many isolated points. Natural question to ask are how to find specific curves (starting with lower genera) with large numbers of such points, and how to count the points and/or determine them explicity. This report surveys work that has been done for the special case of rational points, as well as more recent work on the general case. In its current state the survey mostly references the papers and does not attempt to summarize the techniques used.

2. Background

We provide here some fundamental definitions and facts. For details see [VV24].

Let k be a number field and $\overline{k}$ be its algebraic closure. Let C/k denote a *nice* (smooth, projective, and geometrically integral over k) curve (variety of dimension 1 over k). A *closed point* on C is the Galois orbit of a geometric point in $X(\overline{k})$, and its *degree* is the cardinality of this orbit. A rational point is the special case in which the degree is 1.

The number of rational points on a curve is governed by its genus:

(1) **Genus 0.** There are either zero or infinitely many rational points.
(2) **Genus 1.** The set of ratonal points is a finitely generated abelian group (the Mordell-Weil Theorem). Furthermore the possible torsion groups are completely understood for a curve over $\mathbb{Q}$ ([Maz77, Maz78]) and are known to be bounded in general ([Mer96]).
(3) **Genus ≥ 2.** There are finitely many rational points (Faltings's Theorem ([Fal83])).

When $d > 1$, the number of degree d points can be infinite even when the genus is ≥ 2. However all but finitely many of these points can be paramerized in one of two ways.

(1) **$\mathbb{P}^1$-parameterized points**. A closed point $x \in C$ is $\mathbb{P}^1$-parameterized if there exists a morphism $\pi : \mathbb{C} \to \mathbb{P}^1$ with $\deg(\pi) = \deg(x)$ and $\pi(x) \in \mathbb{P}^1(k)$.
(2) **AV-parameterized points**. A degree d closed point $x \in C$ is AV-parameterized if there exists a positive rank abelian subvariety $A \subset \mathrm{Pic}^0_C$ such that $[x] + A \subset W^d = \mathrm{im}(\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C)$.

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A point that is neither $\mathbb{P}^1$- nor AV-parameterized is called *isolated*, and [BEL$^+$19, Theorem 4.2] prove that there are only finitely many isolated points (of any degree) on a given curve, and furthermore there are infinitely many points of degree d iff there exists a degree d $\mathbb{P}^1$-parameterized point. These are referenced as Corollary 4.3 and Theorem 4.4 of [VV24].

Furthermore a closed point $x \in C$ is called *sporadic* if there are only finitely many closed points of C with degree at most $\deg(x)$. By [BEL$^+$19, Theorem 4.2], every sporadic point must also be isolated; however, the converse need not hold.

Given that any curve has only finitely many isolated points, a natural question to ask is how to find specific curves with large numbers of such points. Since the concept of isolated points is a relatively new one, introduced in [BEL$^+$19], we first consider the rational (degree 1) special case, which has been studied for a longer period of time.

3. Curves with Many Rational Points

As noted above, finiteness of the number of rational points is controlled by the genus.

3.1. Genus < 2.

A curve of genus 0 either has zero or infinitely many rational points, so this case is uninteresting.

For the genus 1 case, by the Mordell-Weil Theorem the set of rational points is a finitely-generated abelian group and thus finite iff the rank of the group is zero. The number of rational points is then equal to the size of the torsion group. In the case $k = \mathbb{Q}$, Mazur's theorem states that the largest such group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ of order 16. There are currently three such curves in LMFDB ([LMF24a]), which can be found by searching for "Elliptic Curves over $\mathbb{Q}$" with the given rank and torsion.

For arbitrary nunber fields, the situation is more complicated. [Par99] proves that the size of the torsion group is bounded by an exponential in the degree d of the field extension. The bound is conjectured to be polynomial in d; see [CP18].

For quadratic extensions, the combined work of [Kam92a, Kam92b, KM88] shows that the torsion group has size at most 24 and in this case must be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$. For cubic extensions, [DEvH$^+$21] proves that the largest possible torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$ of size 28. Less is known about higher degress. See the introduction to [Gen22] for an overview.

As far as specific examples go, searching LMFDB for "Elliptic curves over $\mathbb{Q}(\alpha)$" of rank 0 and sorting by torsion, one finds the largest group to have order 37, followed by 36, 32, and three curves with 28. The degree of the minimal polynomial for these examples ranges from 3 to 6. There are two curves in the database over quadradic fields with torsion order 24, one over $\mathbb{Q}(\sqrt{6})$ and the other over $\mathbb{Q}(\sqrt{-15})$. There is one cubic curve in the datebase over a cubic field with torsion order 28.

3.2. Genus ≥ 2.

By Faltings's Theorem there are only finitely many rational points. For genus 2 the curve with the most such points in LMFDB has only 39. However Stoll ([MS16]) exhibits a curve with at least 642 rational points, which is apparently still the record. The paper describes how to find such points for a given curve, but does not say how to find curves with likely many points. The referenced announcement [Sto08] merely states "in the course of a systematic search for curves with many rational points in several families constructed by Noam Elkies, I discovered the following curve of genus 2". More details can be found in [Sto15].

For higher genera, [Cap95] is a good, if dated, reference.

4. Isolated and Sporadic Points

As noted above, isolated and sporadic points are relatively new concepts. Work so far has concentrated on the modular curves $X_0(N)$ and $X_1(N)$. These have the advantage that their points correspond to isomorphism classes of elliptic curves, and thus concretely to j-invariants. CM and non-CM cases are distinguished as they require different techniques.

Bourdon and collaborators focus on $X_1(N)$, starting with [BEL+19]. Section 8 of this paper focuses on classifying non-cuspidal non-CM isolated points with rational j-invariant. The conditions on the j-invariant of curves corresponding to such isolated points can then be used to search LMFDB for appropriate elliptic curves, which then correspond to isolated points.

[BN21] focus on conditions for sporadic points and considers both CM and non-CM cases. [BGRW24] focus on isolated points of odd degree, and proves (Theorem 2) that such points can have only two specific (for the non-CM case) or three specific (for the CM case) j-invariants. Only for the two non-CM cases have isolated points been verified to exist.

Finally [BHK+25] develop an algorithm to determine whether a non-CM rational j-invariant is isolated. Running this on LMFDB they determine, given restrictions on the conductor, that at most four such rational j-invariants exist (Theorem 2); they are also known to exist. The authors conjecture that these are the only possible j-invariants (removing the conductor restrictions); the case for odd degree was already proven in [BGRW24], and [Ejd22] handles the case $X_1(\ell^n)$ with ℓ prime.

[BGG23] focus on $X_0(N)$ with several specific values of N, and explicit calculation of isolated points. No more than 4 isolated points are demonstrated for any example curve. [KS24] provide criteria for determining isolated points of more general curves, but do not attempt to find curves with many isolated points.

As of this writing (November 2024), the LMFDB references isolated points only in the beta version [LMF24b] in a new section "Modular Curves"; there is a text box "Isolated" in the section "Modular curve low-degree point search results". However isolated points are not yet computed, so no information can be retrieved.

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