LOW GENUS CURVES WITH MANY ISOLATED POINTS

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Abstract. abstract

1. Introduction

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2. Background

We provide here some fundamental definitions and facts. For details see [VV24].

Let k be a number field and $\overline{k}$ be its algebraic closure. Let C/k denote a *nice* (smooth, projective, and geometrically integral over k) curve (variety of dimension 1 over k). A *closed point* on C is the Galois orbit of a geometric point in $X(\overline{k})$, and its *degree* is the cardinality of this orbit. A rational point is the special case in which the degree is 1.

The number of rational points on a curve is governed by its genus:

(1) **Genus 0.** There are either zero or infinitely many rational points.
(2) **Genus 1.** The set of ratonal points is finitely generated (Mordell's Theorem). Furthermore the possible torsion groups are completely understood ([Maz77, Maz78]).
(3) **Genus 2.** There are finitely many rational points (Faltings's Theorem (add citation)).

When $d > 1$, the number of degree d points can be infinite even when the genus is ≥ 2. However all but finitely many of these points can be paramerized in one of two ways.

(1) $\mathbb{P}^1$**-parameterized points**. A closed point $x \in C$ is $\mathbb{P}^1$-parameterized if there exists a morphism $\pi : \mathbb{C} \to \mathbb{P}^1$ with $\deg(\pi) = \deg(x)$ and $\pi(x) \in \mathbb{P}^1(k)$.
(2) **AV-parameterized points**. A degree d closed point $x \in C$ is AV-parameterized if there exists a positive rank abelian subvariety $A \subset \mathrm{Pic}^0_C$ such that $[x] + A \subset W^d = \mathrm{im}(\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C)$.

A point that is neither PP^1- nor AV-parameterized is called *isolated*, and [BEL+19, Theorem 4.2] prove that there are only finitely many isolated points (considering all possible degrees) on a given curve, and furthermore there are infinitely many points of degree d iff there exists a degree d $\mathbb{P}^1$-parameterized point. These are referenced as Corollary 4.3 and Theorem 4.4 of [VV24].

Given that any curve has only finitely many isolated points, a natural question to ask is how to find specific curves with large numbers of such points. Since the concept of isolated point is a relatively new one, introduced in [BEL+19], we first consider the rational (degree 1) special case, which has been studied for a longer period of time.

3. Curves with Many Rational Points

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3.1. **Genus < 2.** xxx

3.2. **Genus ≥ 2.** xxx

References

[BEL+19] Abbey Bourdon, Özlem Ejder, Yuan Liu, Frances Odumodu, and Bianca Viray, *On the level of modular curves that give rise to isolated j-invariants*, Adv. Math. **357** (2019), 106824, 33. MR4016915 ↑1

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