MA1102R Cheatsheet

1. Functions

$$csc(\theta) = \frac{1}{sin(\theta)}$$
$$sec(\theta) = \frac{1}{cos(\theta)}$$
$$cot(\theta) = \frac{1}{tan(\theta)}$$
$$cos^2(\theta) + sin^2(\theta) = 1$$
$$1 + tan^2(\theta) = sec^2(\theta)$$
$$1 + cot^2(\theta) = csc^2(\theta)$$
$$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$$
$$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$$
$$cos(2\theta) = cos^2(\theta) - sin^2(\theta)$$
$$sin(2\theta) = 2sin(\theta)cos(\theta)$$
$$cos^2(\theta) = \frac{1+cos(2\theta)}{2}$$
$$sin^2(\theta) = \frac{1-cos(2\theta)}{2}$$

2. Limits and Continuity

Limit

Let $f(x)$ be defined on an open interval around c.

$\lim_{x\to c} f(x) = L$ if $\forall \epsilon > 0\, \exists \delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

Finding limits

1. Solve $|f(x) - L| < \epsilon$ to find interval (a, b) containing c, true for all x≠c
2. Find $\delta > 0$ such that $(c - \delta, c + \delta)$ is within (a, b)

(T4) Sandwich Theorem

Let $g(x) \leq f(x) \leq h(x)\, \forall x$ in some open interval around c.

If $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$, then $\lim\limits_{x\to c} f(x) = L$

Continuity

f is continuous at c if left and right-limits agree with function value at $x = c$, i.e.:

$$\lim_{x\to c} f(x) = f(c)$$

(T8) Properties of Continuous Functions

Let f and g be continuous functions at $x = c$. Then these are also continuous at $x = c$: sums and differences $(f \pm g)$, constant multiples, products and quotients, powers and roots

(T9) Compositions of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c

(T10) Limits of Continuous Functions

If $lim_{x\to c} f(x) = b$ and g is continuous at b, then $lim_{x\to c} g(f(x)) = g(b)$

(T11) Intermediate Value Theorem for Continuous Functions

If f is a continuous function on $[a, b]$ and y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c \in [a, b]$

Limits involving Infinity

$\lim_{x\to\infty} f(x) = L$ if $\forall \epsilon > 0\, \exists M$ such that $|f(x) - L| < \epsilon$ whenever $x > M$

$\lim_{x\to c} f(x) = \infty$ if $\forall B > 0\, \exists \delta$ such that $f(x) > B$ whenever $0 < |x - c| < \delta$

3. Derivatives

Derivative

Derivative exists at $x = x_0$ if left and right derivatives exist there, and are equal

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \end{aligned}$$

(T1) Differentiability implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$

Differentiation Rules

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(cu) = c \cdot \frac{du}{dx}$$
$$\frac{d}{dx}x^n = nx^n - 1 \qquad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Trigo Derivatives

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$
$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$
$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

(T2) Chain Rule

Let $g(x)$ be differentiable at x, and $f(u)$ be differentiable at $u = g(x)$.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

i.e. Let $y = f(u)$ and $u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Implicit Differentiation

1. Differentiate both sides of the equation wrt x, treating y as a differentiable function of x.
2. Collect terms with $\frac{dy}{dx}$ on one side of the equation to solve for it.

Linearization

Linearization of f at a,
$L(x) = f(a) + f'(a)(x - a)$

4. Applications of Derivatives

Extreme Values

f has absolute maximum at c if $f(x) \leq f(c)\, \forall x \in D$

f has absolute minimum at c if $f(x) \geq f(c)\, \forall x \in D$

Critical point: an *interior* point of f, where $f'(x) = 0$/undefined.

(T1) Extreme Value Theorem

If f is continuous on $[a, b]$, then it has absolute max M and absolute min m in $[a, b]$.

(T2) 1st Derivative of Local Extreme Values = 0

If f has local min/max at an *interior* point $c \in D$, and f' is defined at c, then $f'(c) = 0$.

Finding Absolute Extrema

- Find all critical points of f
- Evaluate f at all *critical* points and *endpoints*
- Take largest and smallest values

(T3) Rolle's Theorem

Let f be a *continuous* function over $[a, b]$ and *differentiable* at every point of its interior (a, b).

If $f(a) = f(b)$, then $\exists c \in (a, b)$ at which $f'(c) = 0$

(T4) Mean Value Theorem

(Same conditions as above)

$\exists c \in (a, b)$ at which $\frac{f(b)-f(a)}{b-a} = f'(c)$

Corollary 1: If $f'(x) = 0\, \forall x \in [a, b]$, then $f(x) = C$ for all such x

Corollary 2: If $f'(x) = g'(x)\, \forall x \in [a, b]$, then $f(x) = g(x) + C$ for all such x

Corollary 3a: If $f'(x) > 0\, \forall x \in [a, b]$, then f is increasing on $[a, b]$. If $f'(x) < 0$, then f is decreasing.

1st Derivative Test

- Local minimum: f' moves from $-ve$ to $+ve$
- Local maximum: f' moves from $+ve$ to $-ve$
- Local extremum: f' does not change sign

Concavity

- Concave up: f' *increasing* on I — f'' $+ve$
- Concave down: f' *decreasing* on I — f'' $-ve$

Point of inflection: point where graph has tangent line, and *concavity* changes. Here, $f''(c) = 0$/undefined.

But $f''(c) = 0$ alone does not guarantee point of inflection.

(T5) 2nd Derivative Test

- Local maximum: $f'(c) = 0$ and $f''(c) < 0$
- Local minimum: $f'(c) = 0$ and $f''(c) > 0$
- TEST FAILS IF: $f'(c) = 0$ and $f''(c) = 0$

Newton's Method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, with initial guess x_0

5. Integrals

Riemann sum

Partition a closed interval $[a, b]$, $P = \{x_0 ... x_n\}$.

For each k from 1 to n, choose $c_k \in [x_{k-1}, x_k]$.

Riemann sum $= \sigma_{k=1}^{n} \delta x_k \cdot f(c_k)$

Norm of a partition, $||p|| = \max_{k=1..n} \delta x_k$, i.e. the largest sub-interval

Definite Integral

Definite integral J is the limit of Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$, whereby $\forall \epsilon > 0, \exists \delta > 0$ such that for any partition $P = \{x_0 ... x_n\}$ with $||p|| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$:

$$|\sum_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$$

$$\int_a^b f(x)\,dx = \lim_{||p||\to 0}\sum_{k=1}^{n}(c_k)\Delta x_k$$