MA1102R Cheatsheet

1. Functions

$$cos^2(\theta) + sin^2(\theta) = 1$$
$$1 + tan^2(\theta) = sec^2(\theta)$$
$$1 + cot^2(\theta) = csc^2(\theta)$$
$$csc(\theta) = \frac{1}{sin(\theta)} \quad sec(\theta) = \frac{1}{cos(\theta)} \quad cot(\theta) = \frac{1}{tan(\theta)}$$
$$cos^2(\theta) = \frac{1+cos(2\theta)}{2} \quad sin^2(\theta) = \frac{1-cos(2\theta)}{2}$$
$$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$$
$$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$$
$$cos(2\theta) = cos^2(\theta) - sin^2(\theta)$$
$$sin(2\theta) = 2sin(\theta)cos(\theta)$$

	$\pi/6$	$\pi/4$	$\pi/3$
sin	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$
cos	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
tan	$1/\sqrt{3}$	1	$\sqrt{3}$

2. Limits and Continuity

Limit

Let $f(x)$ be defined on an open interval around c.

$\lim_{x\to c} f(x) = L$ if $\forall\epsilon > 0\, \exists\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

Finding limits

1. Solve $|f(x) - L| < \epsilon$ to find interval (a, b) containing c, $\forall x \neq c$
2. Find $\delta > 0$ such that $(c - \delta, c + \delta)$ is within (a, b)

(T4) Sandwich Theorem

Let $g(x) \leq f(x) \leq h(x)\, \forall x$ in some open interval around c.

If $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$

Continuity

f is continuous at c if left and right-limits agree with function value at $x = c$, i.e.:

$$\lim_{x\to c} f(x) = f(c)$$

(T8) Properties of Continuous Fns

Let f and g be continuous functions at $x = c$. Then these are also continuous at $x = c$: sums and differences $(f \pm g)$, constant multiples, products and quotients, powers and roots

(T9) Compositions of Continuous Fns

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c

(T10) Limits of Continuous Fns

If $\lim_{x\to c} f(x) = b$ and g is continuous at b, then $\lim_{x\to c} g(f(x)) = g(b)$

(T11) Intermediate Value Theorem for Continuous Functions

If f is continuous on $[a, b]$ and $y_0 \in [f(a), f(b)]$, then $y_0 = f(c)$ for some $c \in [a, b]$

Limits involving Infinity

$\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0\, \exists M$ such that $|f(x) - L| < \epsilon$ whenever $x > M$

$\lim_{x\to c} f(x) = \infty$ if $\forall B > 0\, \exists\delta$ such that $f(x) > B$ whenever $0 < |x - c| < \delta$

3. Derivatives

Derivative

Derivative exists at $x = x_0$ if left and right derivatives exist there, and are equal

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$

(T1) Differentiability implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$

Differentiation Rules

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(cu) = c \cdot \frac{du}{dx}$$
$$\frac{d}{dx}x^n = nx^n - 1 \qquad \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Trigo Derivatives

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$
$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$
$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

(T2) Chain Rule

Let $g(x)$ be differentiable at x, and $f(u)$ be differentiable at $u = g(x)$.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

i.e. Let $y = f(u)$ and $u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Linearization

Linearization of f at a, $L(x) = f(a) + f'(a)(x - a)$

4. Applications of Derivatives

Extreme Values

f has absolute maximum at c if $f(x) \leq f(c)\, \forall x \in D$

f has absolute minimum at c if $f(x) \geq f(c)\, \forall x \in D$

Critical point: an *interior* point of f, where $f'(x) = 0$/undefined.

(T1) Extreme Value Theorem

If f is continuous on $[a, b]$, then it has absolute max M and absolute min m in $[a, b]$.

(T2) 1st Derivative of Local Extreme Values = 0

If f has local min/max at an *interior* point $c \in D$, and f' is defined at c, then $f'(c) = 0$.

Finding Absolute Extrema

- Find all critical points of f
- Evaluate f at all *critical* points and *endpoints*
- Take largest and smallest values

(T3) Rolle's Theorem

Let f be a *continuous* function over $[a, b]$ and *differentiable* at every point of its interior (a, b).

If $f(a) = f(b)$, then $\exists c \in (a, b)$ at which $f'(c) = 0$

(T4) Mean Value Theorem

(Same conditions as above)

$\exists c \in (a, b)$ at which $\frac{f(b)-f(a)}{b-a} = f'(c)$

Corollary 1: If $f'(x) = 0\, \forall x \in [a, b]$, then $f(x) = C$ for all such x

Corollary 2: If $f'(x) = g'(x)\, \forall x \in [a, b]$, then $f(x) = g(x) + C$ for all such x

Corollary 3a: If $f'(x) > 0\, \forall x \in [a, b]$, then f is increasing on $[a, b]$. If $f'(x) < 0$, then f is decreasing.

1st Derivative Test

- Local minimum: f' moves from $-ve$ to $+ve$
- Local maximum: f' moves from $+ve$ to $-ve$
- Local extremum: f' does not change sign

Concavity

- Concave up: f' *increasing* on I — f'' $+ve$
- Concave down: f' *decreasing* on I — f'' $-ve$

Point of inflection: point where graph has tangent line, and *concavity* changes. Here, $f''(c) = 0$/undefined.

But $f''(c) = 0$ alone does not guarantee point of inflection.

(T5) 2nd Derivative Test

- Local maximum: $f'(c) = 0$ and $f''(c) < 0$
- Local minimum: $f'(c) = 0$ and $f''(c) > 0$
- TEST FAILS IF: $f'(c) = 0$ and $f''(c) = 0$

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ with initial guess } x_0$$

5. Integrals

Riemann sum

Partition a closed interval $[a, b]$, $P = \{x_0 ... x_n\}$.

For each k from 1 to n, choose $c_k \in [x_{k-1}, x_k]$.

Riemann sum $= \sigma_{k=1}^{n} \delta x_k \cdot f(c_k)$

Norm of a partition, $||p|| = \max_{k=1..n} \delta x_k$, i.e. the largest sub-interval

Definite Integral

Definite integral J is the limit of Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$, whereby $\forall\epsilon > 0, \exists\delta > 0$ such that for any partition $P = \{x_0 ... x_n\}$ with $||p|| < \delta$ and any choice of $c_k \in [x_{k-1}, x_k]$:

$$|\textstyle\sum_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$$
$$\int_a^b f(x)\, dx = \lim_{||p||\to 0} \sum_{k=1}^{n} (c_k)\Delta x_k = Area$$

Using Riemann sums, with equal sub-intervals:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\tfrac{b-a}{n})(\tfrac{b-a}{n})$$

(T1) Integrability of Continuous Fns

$\int_a^b f(x)\, dx$ exists if f is continuous over $[a, b]$ OR f has *finitely many* jump discontinuities over $[a, b]$

(T2) Definite Integral Rules

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$
$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$
$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$
$$(min f) \cdot (b - a) \leq \int_a^b f(x)\, dx \leq (max f) \cdot (b - a)$$
$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx \text{ if } f(x) \geq g(x) \text{ on } [a, b]$$

(T3) Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then for some $c \in [a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

(T4) Fundamental Theorem of Calculus

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$
$$\int_a^b f(x)\, dx = F(b) - F(a)$$

(T5) Net Change Theorem

Net change in $[a, b] = \int_a^b F'(x)\, dx = F(b) - F(a)$

*NOTE: When asked for area, it should always be positive => divide into sub-intervals where it crosses $x = 0$ => integrate each, add absolute areas

(T6) Substitution Rule

If $u = g(x)$ is differentiable, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

(T7) Substitution in Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

(T8) Even and Odd Functions

Even function: $\int_{-a}^{a} f(x)\, dx = 2 \cdot \int_0^a f(x)\, dx$

Odd function: $\int_{-a}^{a} f(x)\, dx = 0$

Area Between Curves

$$A = \int_a^b [f(x) - g(x)]\, dx$$

*NOTE: Cannot always split into 2 integrals!

6. Applications of Definite Integrals

Volume: Disk Method

$$V_{x-axis} = \int_a^b A(x)\, dx = \int_a^b \pi[R(x)]^2\, dx$$

$$V_{y-axis} = \int_c^d A(y)\, dy = \int_c^d \pi[R(x)]^2\, dy$$

Volume: Washer Method

$$V_{x-axis} = \int_a^b A(x)\, dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\, dx$$

$$V_{y-axis} = \int_c^d A(y)\, dy = \int_c^d \pi([R(x)]^2 - [r(x)]^2)\, dy$$

Volume: Shell Method

$$V_{x=L} = \int_a^b 2\pi \cdot (x - L) \cdot f(x)\, dx$$

Arc Length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \frac{dy}{dx}^2}\, dx$$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_a^b \sqrt{1 + \frac{dx}{dy}^2}\, dy$$

Area of Surface of Revolution

$$S = \int_a^b 2\pi y \sqrt{1 + \frac{dy}{dx}^2}\, dx$$

$$S = \int_c^d 2\pi x \sqrt{1 + \frac{dx}{dy}^2}\, dy$$

Centres of Mass

Vertical strip (wrt dx)

- Find $dm = \delta(x) \cdot (f(x) - g(x)) \cdot dx$
- Find $(\tilde{x}, \tilde{y}) = (x, \frac{f(x)+g(x)}{2})$
- $M = \int_a^b dm$
- $(\bar{x}, \bar{y}) = (\frac{\int_a^b \tilde{x} \cdot dm}{M}, \frac{\int_a^b \tilde{y} \cdot dm}{M})$

7. Transcendental Functions

Inverse Functions and their Derivatives

One-to-one/injective

f is *one-one* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

Inverse function: $f^{-1}(b) = a$ if $f(a) = b$

- If f has domain D and range R, f^{-1} has domain R and range D
- For f^{-1} to exist, f must be one-one
- $f \circ f^{-1} = f^-1 \circ f = id$

(T) Derivative Rule for Inverses

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

(a, b) on f and (b, a) on f^{-1}: reciprocal gradients

Exponentials and Logarithms

e^x and $ln\, x$ are inverse — $e^(ln\, x) = ln(e^x) = x$

a^x and $log_a x$ are inverse — $log_a a^x = a^{log_a x} = x$

$$ln\, x = \int_1^x \frac{1}{t} dt \quad ln\, e = \int_1^e \frac{1}{t} dt = 1$$

$$a^x = e^{x\, ln\, a} \text{ and } x^n = e^{n\, ln\, x}$$

Properties

$$log_a xy = log_a x + log_a y \qquad log_a \frac{x}{y} = log_a x - log_a y$$

$$log_a \frac{1}{y} = -log_a y \qquad log_a x^y = y\, log_a x$$

$$e^x e^y = e^{x+y} \qquad e^{-x} = \frac{1}{e^x} \qquad \frac{e^x}{e^y} = e^{x-y}$$

$$(e^x)^r = e^{rx} \text{ if } r \text{ is rational}$$

Derivatives and integrals

$$(ln\, x)' = \frac{1}{x} \qquad \int \frac{1}{x}\, dx = ln\, |x| + C$$

$$(e^x)' = e^x \qquad \int e^x\, dx = e^x + C$$

$$(a^x)' = a^x\, ln\, a \qquad \int a^x\, dx = \frac{a}{ln\, a} + C$$

$$(log_a x)' = \frac{1}{ln\, a} \cdot \frac{1}{x}$$

L'Hospital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{ln\, f(x)} = e^{\lim_{x \to a} ln\, f(x)}$$

Inverse Trigo Functions

Function	Domain	Range
sin^{-1}	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
cos^{-1}	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
tan^{-1}	$-\infty \leq x \leq \infty$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Some relations

$$cos^{-1}(x) + cos^{-1}(-x) = \pi$$

$$sin^{-1}(x) + cos^{-1}(x) = \frac{\pi}{2}$$

Derivatives of inverse trigo

$$(sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}},\ |x| < 1$$

$$(cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}},\ |x| < 1$$

$$(tan^{-1} x)' = \frac{1}{1+x^2}$$

$$(cot^{-1} x)' = -\frac{1}{1+x^2}$$

$$(sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$$

$$(csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$$

Antiderivatives

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = sin^{-1}(\frac{x}{a}) + C,\ x^2 < a^2$$

$$\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\, tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\, sec^{-1}(\frac{|x|}{a}) + C,\ |x| > a > 0$$

Hyperbolic Functions

$$sinh(x) = \frac{e^x - e^{-x}}{2} \qquad cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$csch(x) = \frac{1}{sinh(x)} \qquad sech(x) = \frac{1}{cosh(x)}$$

$$tanh(x) = \frac{sinh(x)}{cosh(x)} \qquad coth(x) = \frac{cosh(x)}{sinh(x)}$$

$$cosh^2 x - sinh^2 x = 1$$

$$sinh(2x) = 2sinh(x)cosh(x)$$

$$cosh(2x) = cosh^2 x + sinh^2 x$$

$$cosh^2 x = \frac{1}{2}(cosh(2x) + 1)$$

$$sinh^2 x = \frac{1}{2}(cosh(2x) - 1)$$

$$tanh^2 x = 1 - sech^2 x$$

$$coth^2 x = 1 + csch^2 x$$

Derivatives and integrals

$$(sinh\, x)' = cosh\, x \qquad (cosh\, x)' = sinh\, x$$

$$(tanh\, x)' = sech^2\, x \qquad (coth\, x)' = -csch^2\, x$$

$$(sech\, x)' = -sech\, x\, tanh\, x$$

$$(csch\, x)' = -csch\, x\, coth\, x$$

$$\int sinh\, x\, dx = cosh\, x + C$$

$$\int cosh\, x\, dx = sinh\, x + C$$

$$\int sech^2 x\, dx = tanh\, x + C$$

$$\int csch^2 x\, dx = -coth\, x + C$$

$$\int sech\, x\, tanh\, x\, dx = -sech\, x + C$$

$$\int csch\, x\, coth\, x\, dx = -csch\, x + C$$

Inverse Hyperbolic Functions

Identities

$$sech^{-1} x = cosh^{-1} \frac{1}{x}$$

$$csch^{-1} x = sinh^{-1} \frac{1}{x}$$

$$coth^{-1} x = tanh^{-1} \frac{1}{x}$$

Derivatives and integrals

$$(sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}}$$

$$(cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}},\ x > 1$$

$$\int sinh\, x = cosh\, x + C$$

$$\int cosh\, x = sinh\, x + C$$

Relative Rates of Growth

f grows faster than g $\qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

f grows at same rate as g $\qquad \lim_{x \to \infty} \frac{f(x)}{g(x)} = L > 0$

8. Techniques of Integration

Basic Integration Formulas

$$\int \frac{1}{x}\, dx = ln|x| + C \qquad \int a^x\, dx = \frac{1}{ln\, a} a^x + C$$

$$\int sin\, x\, dx = -cos\, x + C \qquad \int cos\, x\, dx = sin\, x + C$$

$$\int sec^2 x\, dx = tan\, x + C \qquad \int csc^2 x\, dx = -cot\, x + C$$

$$\int sec\, x\, tan\, x\, dx = sec\, x + C$$

$$\int csc\, x\, cot\, x\, dx = -csc\, x + C$$

$$\int tan\, x\, dx = ln|sec\, x| + C$$

$$\int cot\, x\, dx = ln\, |sin\, x| + C$$

$$\int sec\, x\, dx = ln\, |sec\, x + tan\, x| + C$$

$$\int csc\, x\, dx = -ln\, |csc\, x + cot\, x| + C$$

Integration by Parts

$$\int u(x) \cdot v'(x)\, dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x)\, dx$$

Trigo Integrals

$$\int sin^m x\, cos^n x\, dx = ?$$

- If m is odd, then $sin^m x = (1 - cos^2 x)^k \cdot sin\, x$ where $m = 2k + 1$, then sub $u = cos\, x$
- If n is odd, then $cos^n x = (1 - sin^2 x)^k \cdot cos\, x$ where $n = 2k + 1$, then sub $u = sin\, x$
- If both are even, then sub $sin^2 x = \frac{1-cos(2x)}{2}$, $cos^2 x = \frac{1+cos(2x)}{2}$

Integration by Partial Fractions

$$\text{E.g. } \frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Reduction formulas

$$\int tan^n x\, dx = \frac{1}{n-1} tan^{n-1} x - \int tan^{n-2}\, dx$$

$$\int (ln\, x)^n\, dx = x(ln\, x)^n - n \int (ln\, x)^{n-1}\, dx$$

$$\int sin^n x\, cos^m x\, dx =$$

$$-\frac{sin^{n-1} x\, cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int sin^{n-2} x\, cos^m x\, dx,$$

$$(n \neq -m)$$

Improper Integrals (∞)

Type I

$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx$, cont. on $[a, \infty)$

$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx$, $(-\infty, b]$

$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$, $(-\infty, \infty)$

Type II

$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx$, discont. at a

$\int_a^b f(x)\, dx = \lim_{a \to b^-} \int_a^c f(x)\, dx$, discont. at b

$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, discont. at c, $a < c < b$

(T2) Direct Comparison Test

Let $0 \leq f(x) \leq g(x)$ where f and g cont. on $[a, \infty)$

- If $\int_a^\infty g(x)\, dx$ converges, $\int_a^\infty f(x)\, dx$ converges
- If $\int_a^\infty f(x)\, dx$ diverges, $\int_a^\infty g(x)\, dx$ diverges

(T3) Limit Comparison Test

Let $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$

- $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ both either converge or diverge