Chapter-6: Approximating Functions

6.1 Polynomial Interpolation

x	x_0	x_1	x_2	x_3	…….	x_n
y	y_0	y_1	y_2	y_3	…….	y_n

-We seek a polynomial p of lowest possible degree for which $p(x_i)=y_i \ (0\leq i\leq n)$. Such polynomial said to interpolate the data.

Theorem: If $x_0, x_1, x_2, \ldots, x_n$ are distinct real numbers, then for arbitraray values $y_0, y_1, y_2, \ldots, y_n$ there is a unique polynomial p_n of degree at most n such that $p_n(x_i)=y_i \quad (0\leq i\leq n)$.

Proof: i)Unicity: Assume that there are polynomials p_n and q_n such that $p_n(x_i)=y_i, \quad q_n(x_i)=y_i$ and $p_n\neq q_n \qquad \Rightarrow [p_n-q_n](x_i)=0 \qquad 0\leq i\leq n$

The degree of p_n-q_n can be at most n. Therefore p_n-q_n can have at most n zeros if p_n-q_n is not the zero polynomial. But (p_n-q_n) has n+1 zeros. Therefore, $p_n\equiv q_n$.

ii) Existence (Proof by induction)

For $n=0$ $p_0(x)=y_0, \quad p_0(x_0)=y_0$ (degree 0 polynomial)

For $n=k-1$ assume that $p_{k-1}(x_i)=y_i$ for $0\le i\le k-1$

Let $p_k(x)=p_{k-1}(x)+c_k(x-x_0)(x-x_1)\ldots(x-x_{k-1})$ (polynomial with degree at most k)

$\Rightarrow p_k(x_i)=p_{k-1}(x_i)=y_i \qquad (0\le i\le k-1)$

Lets find the unknown coefficient c_k that $p_k(x_k)=y_k$

$\Rightarrow p_k(x_k)=y_k=p_{k-1}(x_k)+c_k(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})=y_k$

$$\Rightarrow c_k=\frac{y_k-p_{k-1}(x_k)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})}.$$

c_k exists because $(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})$ is nonzero since $x_k\neq x_i$ for $0\le i\le k-1$

Newton Form of the Interpolation Polynomial

$p_0(x)=c_0, \quad p_1(x)=c_0+c_1(x-x_0)$

$\Rightarrow p_k(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)+\ldots+c_k(x-x_0)(x-x_1)\ldots(x-x_{k-1})$

$\Rightarrow p_k(x)=\sum_{i=0}^{k} c_i \prod_{j=0}^{i-1}(x-x_j)$ Assume that $\prod_{j=0}^{m}(x-x_j)=1$ whenever $m<0$

-These polynomials are called the interpolation polynomials in Newton form.

-Example: $p_3(x)=4x^3+35x^2-84x-954$

x	5	-7	-6	0
y	1	-23	-54	-954

$$p_0(x)=c_0=1,\quad p_1(x)=c_0+c_1(x-x_0)$$

$$p_1(x_1)=c_0+c_1(x_1-x_0)=y_1 \qquad \Rightarrow c_1=\frac{y_1-c_0}{x_1-x_0}=\frac{-23-1}{-7-5}=2$$

$$p_2(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)$$

$$p_2(x_2)=y_2=c_0+c_1(x_2-x_0)+c_2(x_2-x_0)(x_2-x_1) \qquad \Rightarrow c_2=\frac{-54-1+22}{-11}=3$$

$$p_3(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)+c_3(x-x_0)(x-x_1)(x-x_2)$$

$$p_3(x_3)=y_3=c_0+c_1(x_3-x_0)+c_2(x_3-x_0)(x_3-x_1)+c_3(x_3-x_0)(x_3-x_1)(x_3-x_2)$$

$$\Rightarrow c_3=\frac{-954-1+10-3(-5)(7)}{(-5)(7)(6)}=4$$

$$\Rightarrow p_3(x)=1+2(x-5)+3(x-5)(x+7)+4(x-5)(x+7)(x+6)$$

Lagrange Form of the Interpolation Polynomial

$p(x)$ can be expressed in the form

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + \ldots + y_n l_n(x) = \sum_{k=0}^{n} y_k l_k(x) = p_n(x)$$

for the data points (x_i, y_y) for $0 \le i \le n$.

Let $\quad l_i(x_j) = \begin{Bmatrix} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{Bmatrix} \qquad \Rightarrow p_n(x_i) = y_i$

Clearly l_0 must be of the form :

$$l_0(x) = c(x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n) = c\prod_{j=1}^{n}(x - x_j) \qquad \Rightarrow l_0(x_i) = 0 \quad \text{for } 1 \le i \le n$$

$$l_0(x_0) = 1 = c\prod_{j=1}^{n}(x_0 - x_j) \quad \Rightarrow c = \frac{1}{\prod_{j=1}^{n}(x_0 - x_j)} = \prod_{j=1}^{n}\frac{1}{(x_0 - x_j)} \quad \Rightarrow l_0(x) = \prod_{j=1}^{n}\frac{x - x_j}{(x_0 - x_j)}$$

The general formula for $l_i(x)$ is : $\Rightarrow l_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{(x_i-x_j)}$ $\qquad (0\leq i\leq n)$

$l_i(x)$'s are called cordinal functions. $p(x)=\sum_{k=0}^{n} y_k l_k(x)$ is called Lagrange form of the interpolating polynomial.

Example: What are the cordinal functions and Lagrange for of the interpolating polynomial for the data in the table given below.

x	5	-7	-6	0
y	1	-23	-54	-954

Solution:

$$l_0(x)=\frac{(x+7)(x+6)x}{(5+7)(5+6)5},\quad l_1(x)=\frac{(x-5)(x+6)x}{(-7-5)(-7+6)(-7)},\quad l_2(x)=\frac{(x-5)(x+7)x}{(-6-5)(-6+7)(-6)}$$

$$l_3(x)=\frac{(x-5)(x+7)(x+6)}{(0-5)(0+7)(0+6)} \qquad \Rightarrow p_3(x)=l_0(x)-23\,l_1(x)-54\,l_2(x)-954\,l_3(x)$$

Example: Find an interpolation formula for the table given below.

x	x_0	x_1
y	y_0	y_1

Solution: $\Rightarrow p(x) = y_0 \dfrac{x - x_1}{x_0 - x_1} + y_1 \dfrac{x - x_0}{x_1 - x_0}$

-Another Approach: $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n, \quad p(x_i) = y_i$ for $0 \le i \le n$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^n \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

-But Vandermonde matrix is often ill conditioned.

The Error in Polynomial Interpolation

Theorem: Let f be a function in $C^{n+1}[a,b]$, and let p be the polynomial of degree $\le n$ that interpolates the function f at $n+1$ points $x_0, x_{1,} x_2, \dots, x_n$ in the internal $[a,b]$. To each x in $[a,b]$ there corresponds a point ξ_x in $[a,b]$ such that

$$f(x) - p(x) = \frac{1}{n+1!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Proof: If $x = x_i \quad \Rightarrow f(x_i) - p(x_i) = 0 = \frac{1}{n+1!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i) = 0.$

If $x \neq x_i$ define $w(t) = \prod_{i=0}^{n} (t - x_i) \qquad \phi = f - p - \lambda w$

λ is a real number that makes $\phi(x) = 0 \qquad \Rightarrow \lambda = \frac{f(x) - p(x)}{w(x)}.$

Now $\phi(x) \in C^{n+1}[a,b]$, and $\phi(x)$ vanishes at the $n+2$ points $x, x_0, x_1, x_2, \dots, x_n$. By Rolle's Theorem, $\phi'(x)$ has at least $n+1$ distinct zeros in (a,b). $\phi^2(x)$ has at least n distinct zeros in (a,b). $\phi^{(n+1)}(x)$ has at least one zero say ξ_x in (a,b).

$\Rightarrow \phi^{(n+1)}(x) = f^{(n+1)}(x) - p^{(n+1)}(x) - \lambda w^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\lambda$

$$\Rightarrow \phi^{(n+1)}(\xi_x)=0=f^{(n+1)}(\xi_x)-(n+1)!\,\lambda=f^{(n+1)}(\xi_x)-(n+1)!\,\frac{f(x)-p(x)}{w(x)}=0$$

$$\Rightarrow f(x)-p(x)=\frac{1}{n+1!}f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i)$$

$$\Rightarrow |f(x)-p(x)|=\frac{1}{n+1!}\left|f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i)\right|$$

Example: If the function *f(x)=sin(x)* is approximated by a polynomial of degree 9 that interpolates *f(x)* at ten points in the interval *[0,1]*, how large is the error on this interval?

Solution:

$$|f^{(10)}(\xi_x)|\le 1, \qquad \prod_{i=0}^{9}|x-x_i|\le 1 \quad \text{for all x in } [0,1]$$

$$|\sin x-p(x)|\le\frac{1}{10!}<2.8*10^{-7}$$

Chebyshev Polynomials

-We want to minimize the error $(\|f(x)-p(x)\|)$ given in the previous theorem by choosing appropriate nodes.

-The Chebyshev polynomials (of the first kind) are defined recursively as follows:

$$\left\{\begin{array}{ll} T_0(x)=1, \quad T_1(x)=x & \\ T_{n+1}(x)=2x\,T_n(x)-T_{n-1}(x) & n\geq 1 \end{array}\right\}$$

$$\Rightarrow T_2(x)=2x^2-1$$

$$T_3(x)=4x^3-3x$$

$$T_4(x)=8x^4-8x^2+1$$

$$T_5(x)=16x^5-20x^3+5x$$

$$T_6(x)=32x^6-48x^4+18x^2-1$$

Theorem: For *x* in the interval *[-1,1]*, the Chebyshev polynomials have this closed-form expression: $T_n(x) = \cos(n.\cos^{-1} x) \qquad n \geq 0$

Proof:

$$\cos(A+B) = cosA\, cosB - sinA \sin B$$
$$\cos(n+1)\theta = \cos\theta \cos n\theta - \sin\theta \sin n\theta$$
$$\cos(n-1)\theta = \cos\theta \cos n\theta + \sin\theta \sin n\theta$$
$$\Rightarrow \cos(n+1)\theta = 2\cos\theta \cos n\theta - \cos(n-1)\theta$$
$$\text{Let } \theta = \cos^{-1} x \qquad \Rightarrow x = \cos\theta$$
$$\text{Define } f_n(x) = \cos(n\theta) = \cos(n\cos^{-1} x) \qquad \Rightarrow f_0(x) = 1, \quad f_1(x) = \cos(\cos^{-1} x) = x$$
$$\Rightarrow f_{n+1}(x) = \cos(n+1)\theta = 2x f_n(x) - f_{n-1}(x) \qquad n \geq 1$$
$$\Rightarrow f_n(x) = T_n(x) \qquad \text{for all n.}$$

Properties of the Chebyshev Polynomials

$$|T_n(x)| \leq 1 \qquad (-1 \leq x \leq 1)$$
$$T_n(\cos\frac{j\pi}{n}) = (-1)^j \qquad (0 \leq j \leq n)$$
$$T_n(\cos\frac{2j-1}{2n}\pi) = 0 \qquad (0 \leq j \leq n)$$

-A monic polynomial is one which the term of highest degree has a coefficient unity.

-The term of highest degree of $T_n(x)$ is 2^{n-1} for $n>0$. Therefore, $2^{1-n}T_n(x)$ is a monic polynomial for $n>0$.

Theorem:

If p is a monic polynomial of degree n, then $\|p(x)\|_\infty = \max\limits_{-1\le x\le 1} |p(x)| \ge 2^{1-n}$

$$\left\{\begin{array}{l} \|f(x)\|_p = \left(\int\limits_a^b |f(x)|^p dx \right)^{1/p} \qquad a\le x\le b \\ \|f(x)\|_\infty = \max\limits_{a\le x\le b} |f(x)| \end{array}\right\}$$

Proof: (By contradiction)

Suppose $|p(x)|<2^{1-n} \qquad (|x|)\le 1$

Let $q=2^{1-n}T_n$ and $x_i=\cos(\frac{i\pi}{n})$ $\quad$ (q is a monic polynomial of degree n.)

$\Rightarrow (-1)^i p(x_i) \le |p(x_i)| < 2^{1-n} = (-1)^i q(x_i).$

$\Rightarrow (-1)^i [q(x_i)-p(x_i)]>0 \qquad 0\le i\le n.$

q(x)-p(x) oscillates in sign *(n+1)* times on the interval *[-1,1]*. Therefore, *q(x)-p(x)* must have at least *n* roots in *(-1,1)*. This is not possible because *q(x)-p(x)* has degree at most *n-1.*

$$\Rightarrow \max_{-1\le x\le 1} |p(x)| \ge 2^{1-n}$$

Choosing the Nodes

If $x \in [-1,1]$ and $\xi_x \in [-1,1]$

$$\max|f(x)-p(x)| \le \frac{1}{(n+1)!} \max_{|x|\le 1} |f^{n+1}(x)| \max_{|x|\le 1} \left|\prod_{i=0}^{n} (x-x_i)\right|$$

By the previous theorem $\max_{|x|\le 1} \left|\prod_{i=0}^{n}(x-x_i)\right| \ge 2^{-n}$

-The minimum value of $\left\{ \max_{|x|\le 1} \left|\prod_{i=0}^{n} (x-x_i)\right| \right\}$ will be attained if $\prod_{i=0}^{n}(x-x_i)$ is the monic multiple of T_{n+1}. That is $2^{-n}T_{n+1}$. The nodes then will be the roots of T_{n+1}.

These are $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right) \quad 0 \le i \le n$.

Theorem: If the nodes x_i are the roots of the Chebyshev polynomials T_{n+1}, then the error formula yields

$$\max|f(x)-p(x)|\leq\frac{1}{2^n(n+1)!}\max_{|t|\leq 1}|f^{n+1}(t)|$$

The Convergence of Interpolating Polynomials

$\|f(x)-p_n(x)\|=\max_{a\leq x\leq b}|f(x)-p_n(x)|$ may not converge to 0 as $n\rightarrow\infty$ for all functions $f(x)$.

6.2 Divided Differences

-We want to develop an easy way to find the interpolation polynomial in Newton's form. Let $q_n(x)=(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$

We can write the interpolation polynomial in Newton's form as:

$$p(x)=\sum_{j=0}^{n}c_j q_j(x) \qquad \Rightarrow p(x_i)=\sum_{j=0}^{n}c_j q_j(x_i)=f(x_i) \qquad 0\leq i\leq n$$

Let $a_{ij}=q_j(x_i) \qquad (0\leq i,\quad j\leq n)$

$$q_j(x)=\prod_{k=0}^{j-1}(x-x_k) \qquad \Rightarrow q_j(x_i)=\prod_{k=0}^{j-1}(x_i-x_k)=0 \quad \text{if } i\leq j-1.$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$ We can easly compute c_i.

c_0 depends on $f(x_0)$, c_1 depends on $f(x_0)$ and $f(x_1)$, and so on. Thus, c_n depends on $f(x)$ at $x_0, x_1, x_2, \ldots, x_n$. Define $c_n = f[x_0, x_1, x_2, \ldots, x_n]$. It means c_n depends on $f(x)$ at $x_0, x_1, x_2, \ldots x_n$.

$$p(x) = \sum_{k=0}^{n} c_k q_k(x) = \sum_{k=0}^{n} f[x_0, x_1, x_2, \ldots, x_k] q_k(x)$$

$q_n(x) = (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}) = x^n +$ lower-order terms.

$\Rightarrow f[x_0, x_1, x_2, \ldots, x_n]$ is the coefficient of x^n in $p(x)$.

$f[x_0, x_1, x_2, \ldots, x_n]$'s are called divided difference of $f(x)$.

$$\Rightarrow f[x_0] = f(x_0), \quad f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

Higher Order Divided Differences

Theorem: Divided differences satisfy the equation

$$f[x_0, x_1, x_2, \ldots, x_n] = \frac{f[x_1, x_2, x_3, \ldots, x_n] - f[x_0, x_1, x_2, \ldots, x_{n-1}]}{x_n - x_0}$$

Let $p_n(x)$ be the polynomial of degree at most n that interpolates $f(x)$ at the nodes $x_0, x_1, x_2, \ldots, x_n$, $p_{n-1}(x)$ interpolates $f(x)$ at $x_0, x_1, x_2, \ldots, x_{n-1}$. Let $q(x)$ be the polynomial of degree at most n-1 that interpolates $f(x)$ at $x_1, x_2, x_3, \ldots, x_n$

$$\Rightarrow p_n(x) = q(x) + \frac{x - x_n}{x_n - x_0}[q(x) - p_{n-1}(x)] \quad \text{(degree of both sides are n)}$$

This is true because $p_n(x_i) = f(x_i)$ for $0 \le i \le n$

$q(x_i) = f(x_i)$ for $1 \le i \le n$.

and $\frac{x_i - x_n}{x_n - x_0}[q(x_i) - p_{n-1}(x_i)] = 0$ for $1 \le i \le n$.

and $q(x_n) + \frac{x_n - x_n}{x_n - x_0}[q(x_n) - p_{n-1}(x_n)] = f(x_n)$

$$\Rightarrow p_n(x) = q(x) + \frac{x - x_n}{x_n - x_0}[q(x) - p_{n-1}(x)] \quad \Rightarrow c_n \text{ is same for both sides}$$

$$\Rightarrow c_n = f[x_0, x_1, x_2, \ldots, x_n] = \frac{1}{x_n - x_0}\left(f[x_1, x_2, x_3, \ldots, x_n] - f[x_0, x_1, x_2, \ldots, x_{n-1}]\right)$$

$$\Rightarrow f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \qquad f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

We can generalize the formula as:

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i+j}] - f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}$$

The Table of divided differences:

	c_0	c_1	c_2	c_3	c_4
x_0	$f[x_0]$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3,x_4]$
x_1	$f[x_1]$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	$f[x_1,x_2,x_3,x_4]$	
x_2	$f[x_2]$	$f[x_2,x_3]$	$f[x_2,x_3,x_4]$		
x_3	$f[x_3]$	$f[x_3,x_4]$			
x_4	$f[x_4]$				

Example: Compute a divided difference table and find the Newton interpolating polynomial for function values in the following table.

x	3	1	5	6
f(x)	1	-3	2	4

x_i	$f(x_i)$	c_1	c_2	c_3
3	1	2	-3/8	7/40
1	-3	5/4	3/20	
5	2	2		
6	4			

$$\Rightarrow p(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5) = p_3(x)$$

Theorem: The divided difference is a symmetric function of its arguments. Thus, if $(z_0, z_1, z_2, \ldots, z_n)$ is a permutation of $(x_0, x_1, x_2, \ldots, x_n)$, then $f[z_0, z_1, z_2, \ldots, z_n] = f[x_0, x_1, x_2, \ldots, x_n]$.

Theorem: Let *p* be the polynomial of degree at most *n* that interpolates a function *f(x)* at a set of n+1 distinct nodes $x_0, x_1, x_2, \ldots, x_n$. If t is a point different from the nodes, then

$$f(t) - p(t) = f[x_0, x_1, x_2, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

Proof: Let *q* be the polynomial of degree at most *n+1* interpolates *f(x)* at the nodes $x_0, x_1, x_2, \ldots, x_n, t$

$$\Rightarrow q(x) = p(x) + f[x_0, x_1, x_2, \ldots, x_n, t] \prod_{j=0}^{n} (x - x_j) \quad \text{since } q(t) = f(t) \text{ (by letting x=t)}$$

$$\Rightarrow f(t) - p(t) = f[x_0, x_1, x_2, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

Theorem: If *f(x)* is *n* times continuously differentiable on *[a,b]* and if $x_0, x_1, x_2, \ldots, x_n$ are distinct points in *[a,b]*, then there exists a point ξ in *[a,b]* such that

$$f[x_0, x_1, x_2, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

Proof:

We know that $f(x_n) - p(x_n) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j=0}^{n-1} (x_n - x_j)$

From the previous theorem $f(x_n) - p(x_n) = f[x_0, x_1, x_2, \ldots, x_n] \prod_{j=0}^{n-1} (x_n - x_j)$

$$\Rightarrow f[x_0, x_1, x_2, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi) \qquad a \le \xi \le b$$

6.3 Hermite Interpolation

Example: Find an interpolation formula for the table given below.

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