6.2/6.3 Ch 6 Laplace Transform 4/7/14

$\boxed{\text{LgE}}$

☆ $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$\mathcal{L}\{y'''\} = s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)$

$\mathcal{L}\{y''''\} = s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$

$y' + 3y = t \qquad y(0) = -1$

Int Factor $\mu = e^{\int 3\,dt} = e^{3t}$

$\mu y = \int \mu g = \int e^{3t}$

$e^{3t}y = \int e^{3t}t\,dt$

IBP $e^{3t}y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + c$

$y = \frac{1}{3}t - \frac{1}{9} + ce^{3t}$

$y(0) = -1 = -\frac{1}{9} + c$

$c = -\frac{8}{9}$

$y = \frac{1}{3}t - \frac{1}{9} - \frac{8}{9}e^{-3t}$

~

$y' + 3y = t \quad y(0) = -1$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{t\}$

$s\cdot\mathcal{L}\{y\} - y(0) + 3\mathcal{L}\{y\} = \mathcal{L}\{t\}$ from table

$\quad '' \quad + 1 \quad '' \quad = \frac{1}{s}$ ''

$(s+3)\mathcal{L}\{y\} = \frac{1}{s^2} - 1 = \cancel{\frac{1}{s^2} - \frac{s^2}{s^2}}$

$\mathcal{L}\{y\} = \frac{1}{s^2(s+3)} - \frac{1}{s+3}$ ← Table (#2)

Not on table so partial fraction

4/3/14

cont

$$0 = (x-x_0)^2 y'' + (x-x_0)\,p_0\,y' + q_0 y$$ Euler

$$y = (x-x_0)^r$$

Indicial Eqn $\quad 0 = r(r-1) + r\,p_0 + q_0$ $\quad$ Indicial Eq

4/7/14

18 $\quad y^{(4)} - y = 0, \quad y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0$

$$s^{(4)}\bar{y} - s^3 y(0) - s^2 \cancel{y'(0)} - s\,\cancel{y''(0)} - \cancel{y'''(0)} - \bar{y} = 0$$

$$s^4 \bar{y} - s^3 - \bar{y} = 0$$

$$(s^4 - 1)\,\bar{y} = s^3$$

$$\bar{y} = \frac{s^3}{s^4-1} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1}$$

$(s^2+1)(s-1)(s+1)$

$$s^3 = (As+B)\underbrace{(s-1)(s+1)}_{s^2-1} + C(s^2+1)(s+1) + D(s^2+1)(s-1)$$

$s=1 \quad 1 = C(2)(2) \quad C = \frac{1}{4}$

$s=-1 \quad -1 = D(2)(-2) \quad D = 1/4$

$s=i \quad -i = (Ai+B)(-1-1) = -2B - 2Ai$

$0 = -2B \qquad -1 = -2A$

$B = 0 \qquad A = \frac{1}{2}$

Tabular IBP

$$\bar{y} = \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{4}}{s-1} + \frac{\frac{1}{4}}{s+1}$$

soln to DE $\quad \underline{y(t) = \frac{1}{2}\cos t + \frac{1}{4}e^t + \frac{1}{4}e^{-t}}$

#25 $\quad y'' + y = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases}, \quad y(0) = y'(0) = 0$

B/C formula

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 \underbrace{e^{-st}}_{dv}\,\underbrace{t}_{u}\,dt + \int_1^\infty \cancel{e^{-st}\,0}\,dt$$

$$= t\,\frac{e^{-st}}{-s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\Big|_0^1 = \frac{e^{-s}}{-s} - \frac{1}{s^2}\left(e^{-s} - 1\right) = \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$s^2\bar{y} - \cancel{s\,y(0)} - \cancel{y'(0)} + \bar{y} = 1 - e^{-s} \quad -e^{-s} \qquad \bar{y} = 1 - e^{-s} \quad e^{-s}$

4/2/14

$f(t) = e^{2t}$

$\mathcal{L} = \int_0^\infty e^{-st} e^{2t} \, dt = \int_0^\infty e^{2t-st} dt = \int_0^\infty e^{t(2-s)} dt$

$= \frac{1}{2-s} e^{t(2-s)} \Big|_0^\infty = \cancel{\frac{1}{2-s} e^{(2-s)\infty}} - \frac{1}{2-s} e^{(2-s)0} = \frac{1}{s-2}$

5.4 Euler Method $\quad \alpha x^2 y'' + \beta x y' + \gamma y = 0$

$y = x^r$
$y' = r x^{r-1}$
$y'' = r(r-1) x^{r-2}$

i) 2 root : $C_1 x^{r_1} + C_2 x^{r_2}$
ii) repeated :
iii) complex : —

5.5 Frobenius Method
(regular singular)

$y = \sum_{n=0}^\infty a_n (x - x_0)^{n+r}$

$y' = \sum_{n=0}^\infty a_n (x - x_0)^{n+r-1} (n+r)$

$y'' = \sum_{n=0}^\infty a_n (n+r)(n+r-1)(x-x_0)^{n+r-2}$ $\quad y = y_h + y_p$

plug int eqn $\quad$ solve for 2 r values

4/3/14

Discussion

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = F(s)$

$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} dt$

X $\quad = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty = \frac{-1}{(s-2)} \left(e^{-(s-a)t} \Big|_{t \to \infty} - 1 \right)$

X $\quad$ If $a > s \Rightarrow 0 < a-s = -(s-a)$

Take $s > a$

6.1.5 $\quad \mathcal{L}\{t^n\} \quad n \in \mathbb{N} \ (n = 1, 2, \ldots)$

$F(s) = \mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$ $\quad u = t^n$, $du = n t^{n-1} dt$, $dv = e^{-st} dt$, $v = \frac{e^{-st}}{-s}$

$= \frac{e^{-st} t^n}{-s} \Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{+s} n t^{n-1} dt$

$= \cancel{\frac{e^{-st} t^n}{-s} \Big|_{t \to \infty}} - \cancel{\frac{n e^0 0^n}{(-s)}} + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$

6.2

4/2/14

Find the inverse laplace

(7) $f(s) = \frac{3!}{s^4}$ $\mathcal{L}^{-1}\{F(s)\} = t^3$

(8) $F(s) = \frac{1}{s} - \frac{1}{s^2}$ $= 1 - t$

(9) $F(s) = \frac{5-3s}{s^2+9} = \frac{5-3s}{s^2+3^2} = \frac{5}{s^2+3^2} - \frac{3s}{s^2+3^2} = \frac{\frac{5}{3} \cdot 3}{s^2+3^2} = \frac{5}{3} \cdot \frac{3}{s^2+3^2}$

(10) $F(s) = \frac{1}{s^2+s-20} = \frac{1}{(s+5)(s-4)}$

(9) $\frac{5-3s}{s^2+9} = \frac{5}{3} \cdot \frac{3}{s^2+3^2} - 3\frac{s}{s^2+3^2} = \frac{5}{3}\sin 3t - 3\cos 3t$

Partial Fraction

(10) $\frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$

$1 = As - 4A + Bs + 5B$

$1 = (A+B)s + (-4A+5B)$

$A+B=0$ $A = -\frac{1}{9}, B = \frac{1}{9}$

$-4A+5B=1$

$F(s) = -\frac{1}{9}\frac{1}{s+5} + \frac{1}{9}\frac{1}{s-4}$

$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{9}e^{-5t} + \frac{1}{9}e^{4t}$

(11) $F(s) = \frac{1}{s^3+5s} = \frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{B}{s^2+5}$

(13) $F(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{2s}{(s-3)(s+2)} - \frac{3}{(s-3)(s+2)}$

(11) $1 = As^2 + 5A + Bs$

$= (A)s^2 + (B)s + (5A)$

$A=0$
$B=0$
$5A=1$

4/2/14

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt = t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} n t^{n-1} \, dt$

dv u

$= 0 - 0 + \frac{n}{s} \left| \int_0^\infty e^{-st} t^{n-1} \, dt \right| = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$

$\mathcal{L}\{1\} = \frac{1}{s}$ $\mathcal{L}\{t\} = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$ $\mathcal{L}\{t^2\} = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$

$\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st} \sin bt \, dt$

$= \frac{e^{-st}(-s \sin bt - b \cos bt)}{s^2+b^2}\Big|_0^\infty = \frac{0 - 1(-b \cdot 1)}{s^2+b^2}$

$= \frac{b}{s^2+b^2}$

$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+k^2}$

$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\}$

$= \int_0^\infty e^{-st}(a f(t) + b g(t)) \, dt = \int_0^\infty a e^{-st} f(t) + b e^{-st} g(t) \, dt$

$= a \int_0^\infty e^{-st} f(t) \, dt + b \int_0^\infty e^{-st} g(t) \, dt$

$= a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$

$\mathcal{L}\{3t^7 - 5t^2 + 2t + 8\} = 3 \frac{7!}{s^8} + 5 \frac{2}{s^3} + \frac{2}{s^2} + \frac{8}{s}$

$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt$

Special case

$= \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$

1st Shifting Thm $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) \, dt = \int_0^\infty e^{-(s-a)t} f(t) \, dt$

$= \int_0^\infty e^{-\sigma t} f(t) \, dt = F(\sigma) = F(s-a)$ sigma $\to \sigma = s-a$

4/1/24

9.5.7 $\overset{P}{x}y'' + \overbrace{(1-x)}^{Q}y' \overset{R}{-}y = 0$

a) Show $x=0$ res sing pt

Regular? $0 = P(x) = x$

something is finite

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} \frac{x(1-x)}{\cancel{x}} = \frac{1}{1-0} < \infty$$

$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{(-1)}{x} = 0(-1) = 0 \text{ finite}$$

Both $\lim$ finite $\Rightarrow$ $x=0$ Regular Sing Pt

b) Indicial Eq & Rec Rel

like Euler characteristic

$xy'' + (1-x)y' - y = 0$

Mult by x

$(0+x)$

$x^2y'' + x(1-x)y' - xy = 0$ take $\lim$

$\lim_{x\to 0}$

Euler Eq $x^2Ay'' + xBy' + Cy = 0$

Euler Eq: $x^2ay'' + xby' + cy = 0$

In the $\lim$ as $x\to 0$ the ode looks like

$x^2y'' + x(1)y' - 0\cdot y = 0$

Magic $y = x^r$ $\quad x^r[r(r-1) + r\cdot 1 + 0] = 0$

$x>0$ $\quad 0 = r(r-1+1) = r^2 \Rightarrow \boxed{r=0}$

$x^2 \underline{f_2(x)}y'' + xf_1(x)y' + f_0(x)y = 0$

indicial Eq

$\lim_{x\to x_0} f_i(x)$ $\quad i = 0,1,2$

3/31/14

① $\dfrac{6}{x^2-1} = \dfrac{6}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$6 = A(x+1) + B(x-1)$
$= Ax + A + Bx - B$ or $6 = Ax - A + Bx + B$
$\cancel{Ax+A}$
$6 = (A+B)x - A + B$

$A + B = 0$
$-A + B = 6$ $B = 3$ $A = -3$

$\dfrac{6}{x^2-1} = \dfrac{-3}{x+1} + \dfrac{3}{x-1}$

~~②~~ $\cancel{\dfrac{x-1}{x^2+x} = \dfrac{x-1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B+C}{x+1}}$

$\cancel{x-1 = A(x+1) + Bx}$ $\cancel{x-1 = A(x+1) + (B+C)x}$

$\cancel{= Ax + A + Bx}$ $\cancel{= Ax + A + Bx + Cx}$

$\cancel{x-1 = (A+B)x + A}$ $\cancel{= (A+B+C)x + A}$

$\cancel{A+B = 1}$ $\cancel{A + B + C = 1}$
$\cancel{A = -1}$ $\cancel{A = -1}$
$\cancel{-1+B = 1}$
$\cancel{B = 2}$

② $\dfrac{x-1}{x^2+x} = \dfrac{x-1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

$x-1 = A(x+1) + B \cdot x$
$= Ax + A + Bx$
$= (A+B)x + A$

$1 = A + B$ $-1 = A$
$2 = B$

$\dfrac{x-1}{x^2+x} = \dfrac{-1}{x} + \dfrac{2}{x+1}$

③ $\dfrac{2x-3}{x^2-x-6} = \left(\dfrac{2x-3}{(x+2)(x-3)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}\right)$

$2x-3 = A(x+2) + B(x-3)$
$= Ax + 2A + Bx - 3B$

$2x-3 = (A+B)x + 2A - 3B$

$2 = A + B$ $A = \cancel{\equiv}$ $\left(\dfrac{3}{5}\right)\left(\dfrac{1}{x-3}\right) + \left(\dfrac{7}{5}\right)\left(\dfrac{1}{x+2}\right)$

$-3 = 2A - 3B$ $B = 7$

3/31/14

Frobenius

$$2x^2y'' - xy' + (1+x)y = 0$$

plug y, y', y'' in

$$2x^2y'' - xy' + y + xy = 0$$

$$2x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x)^{n+r-2} - x\sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} + x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)\underline{x^{n+r}} - \sum_{n=0}^{\infty} a_n(n+r)\underline{x^{n+r}} + \sum_{n=0}^{\infty} a_n \underline{x^{n+r}} + \sum_{n=0}^{\infty} a_n \underbrace{x^{n+r+1}}_{\text{change}} = 0$$

''   ''   '' $+\sum_{n=1}^{\infty} a_{n-1} x^{n+r}$

$$\overbrace{2a_0 r(r-1)x^r}^{n=0} + \sum_{n=1}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r}$$

$$-\left[\overbrace{a_0 r x^r}^{n=0} + \sum_{n=1}^{\infty} a_n(n+r)x^{n+r}\right] + \underbrace{a_0x^r}_{n=0} + \sum_{n=1}^{\infty} a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0$$

$$2a_0r(r-1)x^r - a_0 r x^r + a_0x^r + \sum_{n=1}^{\infty} \left[2a_n(n+r)(n+r-1) - a_n(n+r) + a_n + a_{n-1}\right]x^{n+r} = 0$$

$$(\underbrace{2a_0r(r-1) - a_0r + a_0}_{0})x^r + \sum_{n=1}^{\infty} \underbrace{\sim\sim}_{\text{recurrence relation}} = 0$$

$$(2a_0r(r-1) - a_0r + a_0)\overset{\neq 0}{\cancel{x^r}} = 0$$

$a_0(2r(r-1) - r + 1) = 0$ $a_0 \neq 0 \Rightarrow \div a_0$

$2r(r-1) - r + 1 = 0$

$2r^2 - 2r - r + 1 = 0$

$2r^2 - 3r + 1 = 0$

$(r-1)(2r-1) = 0$

$r_1 = 1,\ r_2 = 1/2$

$$2a_n(n+r)(n+r-1) - a_n(n+r) + a_n + a_{n-1} = 0$$

$$a_n[2(n+r)(n+r-1) - (n+r) + 1] + a_{n-1} = 0$$

$$a_n = \frac{-a_{n-1}}{2(n+r)(n+r-1) - (n+r) + 1}$$

$r=1$ $a_n = \dfrac{-a_{n-1}}{2(n+1)(n+1-1) - (n+1) + 1}$

3/31/14

$r = -3$

$$a_n = \frac{-7(n-4)\, a_{n-1}}{2(n-3)(n-4) + 7(n-3) - 3}$$

$$a_1 = \frac{-7(-3)\, a_0}{2(-2)(-3) + 7(-2) - 3} = \frac{21}{-5} a_0 \cancel{= 3a}$$

$a_2 = \triangle a_0$

$$y_2 = a_0 x^{-3}\left(1 + \overset{-\frac{21}{5}x}{} \triangle x^2 + \dots\right)$$

r — $r_1 < r_2$ — $r_1 - r_2 \neq$ int number; $r_1 - r_2$ int

r — $r = -\gamma + \beta i$

r — $r_1 = r_2$ — $r_1 - r_2 = 0$ integer

$$3x^2 y'' + 2xy' + x^2 y = 0$$

$\vdots$

$$\left(3(r-1) r\, a_0 x^r + 2r a_0 x^r\right) + \left(3r(r+1) a_1 x^{1+r} + 2(r+1) a_1 x^{1+r}\right)$$

$$+ \sum_{n=2}^{\infty} \left[3(n+r-1)(n+r) a_n + 2(n+r) a_n + a_{n-2}\right] x^{n+r} = 0$$

$3(r-1)r + 2r = 0$

$3r^2 - r = 0$

$r = 0,\ r = \frac{1}{3}$

$r = 0$

$a_0 =$ arb

$a_1 = 0$

$a_2 = \frac{-a_0}{10}$ $\quad a_4 = \frac{a_0}{440}$

$$\cancel{3 \cdot 0 (1) a_1 x^{1+r}} + \boxed{2(1) a_1} x^{1+r} + \boxed{0} x^{n+1}$$

0

$2a_1 = 0$

$a_1 = 0$

$$a_n = \frac{-a_{n-2}}{3(n-1)n + 2n}$$

$$y_1 = a_0 x^0 \left(1 + -\frac{1}{10}x^2 + 0 + \frac{1}{440}x^4 + \dots\right)$$

$\boxed{r = \frac{1}{3}}$ $\quad y_2 = a_0 x^{\frac{1}{3}}\left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 + \dots\right)$

Discussion Test Review

3/27/14

Constant coeff

Homogeneous eqn's w/ constant coeffs
Find the roots of characteristic polynomial
- distinct real
- double root → write equal sol'ns
- complex roots

Undetermined coeffs
nonhomogeneous
where RHS is a special form $P(t)e^{rt}\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$

Variation of Parameters
RHS can be anything

Reduction of Order
given eqn and solution y_1, find another sol

$y = v y_1$ ← unknown function

Ch 5

$Py'' + Qy' + Ry = 0$ Power Series - radius of convergence (Ratio Test)
5.3 - distance to nearest root of denominator
Ø Method

Ordinary Point $Q(P), R(P)$ $\lim_{x \to 0}$

5.4 Euler Eqns (like constant coeffs)

$W(y_1, y_2)$ to show 2 sol'ns are ≠ 0 (linear independent)

3.6.4
V.P

$y'' + 4y' + 4y = t^{-2}e^{-2t},\ t > 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \quad r = -2$

$y_h = c_1 \underbrace{e^{-2t}}_{y_1},\ c_2 \underbrace{te^{-2t}}_{y_2}$

$y = u_1 y_1 + u_2 y_2$

$y = u_1 e^{-2t} + u_2 te^{-2t}$

assume $u_1' y_1 + u_2' y_2 = 0$

$u_1' e^{-2t} + u_2' te^{-2t} = 0$