Ordinary

This value

3/26/14

sch p 261

$y'' + xy' + y = 0$ $\quad y(0) = 1$, $\quad y'(0) = 0$

$\phi(0)$ $\quad a_0 = 1$
$\phi'(0)$ $\quad = 0$
$\phi''(0) \to$ $\quad a_1$
$\phi'''(0)$ $\quad a_2 = -1/2$
$\phi''''(0)$ $\quad a_3 = 0$
$\quad a_4$

$y = \phi$

$a_n = \dfrac{\phi^n}{n!}$

$a_0 = \dfrac{\phi^0}{0!} = 1$

$a_1 = \dfrac{\phi^1}{1!} = 0$

$a_2 = \dfrac{\phi^2}{2!} = \dfrac{-1}{2}$

$y'' = -xy' - y$

$\phi'' = 0 - 1$

$a_3 = \dfrac{\phi'''}{3!}$

$y''' + xy'' + x'y + y' = 0$

$y''' = -xy'' - x'y - y' = 0$

$\phi'''(0) = 0 - 0 - 0$

$y'''' + x'y'' + xy''' + x'y' + x''y' + y'' = 0$

$\phi(0) = 1$

$\phi'(0) = 0$

$\phi''(0) + \cancel{x}\overset{0}{} \phi' \overset{0}{\cancel{(0)}} + \overset{1}{\cancel{\phi(0)}} = 0$

$\phi''(0) + 1 = 0$

$\phi''(0) = -1$

$\phi(0)$ $\quad a_0$
$\phi'(0)$ $\quad a_1$
$\phi''(0)$ $\quad a_2$
$\phi'''(0)$ $\quad a_3$
$\phi''''(0)$ $\quad a_4$

$y = \sum_{n=}^{\infty} a_n x^n$

$1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$y''' + y' + xy'' + y' = 0$

$\phi'''(0) + \underset{0}{\phi'(0)} + \underset{(0)}{x}\,\underset{-1}{\phi''(0)} + \underset{0}{\phi'(0)} = 0$

$\phi'''(0) = 0$

$y'''' + y'' + y'' + xy''' + y'' = 0$

$\phi''''(0) + \underset{-1}{\phi''(0)} + \underset{-1}{\phi''(0)} + \underset{0}{x}\,\underset{0}{\phi'''(0)} + \underset{-1}{\phi''(0)} = 0$

$\phi''''(0) = $

3/27/14

diff y twice and plug into LHS of Eqn

end up with $v_1'y_1' + v_2'y_2' = g(t)$ ← RHS

System of Equations for $v_1'y_1 + v_2'y_2 = 0$

$$v_1'y_1' + v_2'y_2' = g(t)$$

gen sol $= y_p + y_h$

$\uparrow v_1y_1 + v_2y_2$

Frobenius 3/31/14

$2x^2y'' + 7x(x+1)y' - 3y = 0$

1. Check $x_0 =$ is regular singular Hidden asymptote at 0
2. $y = x^r = \sum_{n=0}^{\infty} a_n x^{n+r}$

$\ldots\ 2r(r-1)a_0x^r + 7r a_0 x^r + 7r a_0 x^r - 3a_0x^r$

$+ \sum_{n=1}^{\infty} [2(n+r)(n+r-1)a_n + 7(n+r-1)a_{n-1} + 7(n+r)a_n - 3a_n]x^{n+r} = 0$

$(2r(r-1) + 7r - 3)a_0x^r + \quad ''$

$\overset{\shortparallel}{0}$

$2r^2 + 5r - 3 = 0$

$(2r-1)(r+3) = 0 \quad r = 1/2, -3$ indices for method

$r = 1/2$

$$a_n = \frac{-7(n-\frac{1}{2})a_{n-1}}{2(n+1/2)(n-1/2) + 7(n+1/2)}$$ Recursion Relation

a_0 arb

$$a_1 = \frac{-7(\frac{1}{2})a_0}{2(\frac{3}{2})(\frac{1}{2}) + 7(\frac{3}{2}) - 3} = -\frac{7}{18}a_0$$

$$a_2 = \frac{-7(\frac{3}{2})a_1}{2(\frac{5}{2})(\frac{3}{2}) + 7(\frac{5}{2}) - 3} = \boxed{}\,a_0$$

$y_1 = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \ldots$

$= a_0x^{\frac{1}{2}}(1 + -\frac{7}{18}x + \boxed{}x^2 + \ldots$

Laplace 3/31/14

$D[f(t)] = f'(t)$

$D[\sin(t)] = \cos t$

$J[f(t)] = \int_1^t f(\tau)\,d\tau = g(t)$

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{1\}$

$\mathcal{L}\{t\}$

$\boxed{GE}$

3/31/14

ODE

ordinary $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

singular

regular

~~irregular~~

Euler $\alpha \cdot x^2 \cdot y'' + \beta \cdot x \cdot y' + \gamma y = 0$

Frobenius S.S $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$

Index same

$y' = \sum_{n=0}^{\infty} a_n (n+r)(x-x_0)^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-x_0)^{n+r-2}$

r is some unknown

$2x^2 y'' - xy' + (1+x)y = 0$ Ordinary or Singular? Regular or Irregular.

Singular points 0

$y'' - \frac{x}{2x^2} y' + \frac{(1+x)}{2x^2} y = 0 \qquad x_0 = 0$

$\lim_{x \to 0} (x-0) \cdot \frac{-1}{2x} = -\frac{1}{2}$ Both limit exist $\Rightarrow$ regular singular point

$\lim_{x \to 0} (x-0)^2 \left(\frac{1+x}{2x^2}\right) = \frac{1}{2}$

Ch.6 Laplace Transformation 3/31/14

$$\boxed{\text{DEs }(t,x)\ \text{Initial value}} \xrightarrow{\mathcal{L}} \boxed{\text{Algebraic Problem}}$$

$\cancel{\downarrow}$ I.F. u.c. VOP

$\downarrow$ Partial Fraction Decomposition

$$\boxed{\text{Solution}} \xleftarrow{\mathcal{L}^{-1}} \boxed{\text{Solve Algebra}}$$

...ble $\dfrac{1}{s^3+5s} = \dfrac{1}{s(s^2+5)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+5}$ 1 power lower

Factor

...ld ...ions $\left(\dfrac{1}{(s^2+5)s} = \dfrac{A}{s} + \dfrac{B\cdot s+C}{s^2+5}\right) \times s(s^2+5)$

$1 = A(s^2+5) + (Bs+c)s$

$1 = As^2 + 5A + Bs^2 + cs$

$0\cdot s + 1 = (A+B)s^2 + C\cdot s + 5A$

$A+B=0$
$C=0$
$5A=1$
$A=1/5$
$B=-1/5$

$$\frac{1}{s^2(s^2+5)} + \frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s+0}{s^2+5} = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\left(\frac{s}{s^2+5}\right)$$

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① $\dfrac{6}{\cancel{s}^2-1} =$

② $\dfrac{x-1}{x^2+x} =$

③ $\dfrac{2x-3}{x^2-x-6} =$

④ $\dfrac{5-3x}{x^2+9} =$

$$= \frac{Ax+B}{y^2+9}$$

3/31/14

(11) $\frac{5-3x}{x^2+9} = \frac{5}{x^2+9} - \frac{3x}{x^2+9}$

x^2

Discussion

4/1/14

4.19 Find sing pts. Irregular or Regular?

$$\underbrace{x^2(1-x)}_{P(x)}y'' + \underbrace{(x-2)}_{Q(x)}y' \underbrace{- 3}_{R(x)}y = 0$$

$0 = P(x) = x^2(1-x \Rightarrow x=0,\ x=1$

Reg or Irregular

$=0$ $\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{(x-2)}{x^2(1-x)} = \infty$

$\lim_{x\to 0} x^2\frac{R(x)}{P(x)}$ $x=0$ Irregular Sing pt

$x=1$ $\lim_{x\to 1} (x-1)\frac{Q(x)}{P(x)} = \lim_{x\to 1}\frac{(x-1)(x-2)}{x^2(1-x)} = \overbrace{\frac{-(1-2)}{1^2}}^{\text{finite}}$

$\lim_{x\to 1} (x-1)^2\frac{R(x)}{P(x)} = \lim_{x\to 1}(x-1)^2\frac{(-3x)}{x^2(1-x)} = \overbrace{0\cdot\frac{3}{1^2}}^{\text{finite}}$

Frobenius needs regular singular pts

5.1

$y = a_0 + a_1 x$ $\quad y = a_1 x^1 + a_1 x^{r-1}$ $\quad r a_0 x^{r-1}$

$y = x^r \sum_{n=0}^{\infty} a_n x^n$

$y = x^r(a_0 + a_1 x + \dots)$ 4/1/14

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$0 = xy'' + (1-x)y' - y$

$0 = x \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + (1-x)\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$- \sum_{n=0}^{\infty} a_n x^{n+r}$

$0 = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1}$

$- \sum_{n=0}^{\infty} a_n x^{n+r}$

~~$0 = \sum_{n=0}^{\infty} a_n r$~~ $\quad 0 = a_0 r(r-1) x^{r-1} + \sum_{n=1}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1}$

$+ a_0 r x^{r-1} + \sum_{n=1}^{\infty} a_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} [a_n(n+r+1)] x^{n+r}$

$0 = x^{r-1} a_0 r(r-1+1)$

$\int e^{at} \sin bt = \dfrac{e^{at}(a \sin bt - b \cos bt)}{a^2+b^2}$ 4/2/14

deriv respect to t

$\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty} \dfrac{t^n}{e^{st}} \dots = \lim \dfrac{const}{s^n e^{st}} = 0$

LH

this grows faster

$D(f(t)) = f'(t)$

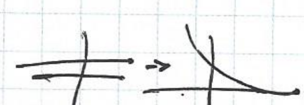
like dot product

$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$= F(s)$

memorize

$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\, 1\, dt = \dfrac{e^{-st}}{-s}\Big|_0^{\infty} = 0 - \dfrac{e^0}{-s} = \dfrac{1}{s}$



$\mathcal{L}\{t\} = \int_0^{\infty} \underset{dv}{e^{-st}}\, \underset{u}{t}\, dt = t\dfrac{e^{-st}}{-s}\Big|_0^{\infty} - \int_0^{\infty} \dfrac{e^{-st}}{-s} dt = 0 - 0 + \dfrac{1}{s}\dfrac{e^{-st}}{-s}\Big|_0^{\infty} =$

$0 - \dfrac{1}{-s^2} = \dfrac{1}{s^2}$

4/2/14

$$\boxed{\mathcal{L}\{f'(t)\}} = \int_0^\infty \underset{u}{e^{-st}}\, \underset{dv}{f'(t)\,dt} = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$$

$$= 0 - f(0)\cdot 1 + s\boxed{\int_0^\infty e^{-st}f(t)\,dt} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(sF(s) - f(0)) - f'(0)$$
$$= s^2F(s) - sf(0) - f'(0)$$

$y'' + 3y' + 2y = 0, \quad y(0) = 1, \; y'(0) = 2$

Take laplace of both sides

$$(s^2\bar{y}(s) - sy(0) - y'(0)) + 3(s\bar{y}(s) - y(0)) + 2\bar{y} = 0$$
$$s^2\bar{y} - s - 2 + 3s\bar{y} - 3 + 2\bar{y} = 0$$
$$(s^2+3s+2)\bar{y} = s + 5$$
$$\bar{y} = \frac{s+5}{s^2+3s+2}$$ Laplace transform of solution

$\boxed{GE}$ Ch 6 Laplace Transform 4/2/14

$\mathcal{L}$ ← Linear function

$$\begin{tikzcd}
\boxed{\text{DEs IV}} \arrow[r, "\mathcal{L}"] \arrow[d] & \boxed{\text{Algebraic Problem}} \arrow[d] \\
\boxed{\text{Ans}} \arrow[r, dash, "\text{UC, VP}\quad \mathcal{L}^{-1}"] & \boxed{\text{Algebra}}
\end{tikzcd}$$

Find the laplace transform of f using the given table

1. $f(t) = \sin 2t$ 2. $f(t) = e^{3t}\sin 3t$ 3. $f(t) = \cosh(t)$

4. $f(t) = t^3 + e^{3t} + \cos(5t)$ 5. $f(t) = te^t$ 6. $f(t) = (1+t)^{\cancel{3}} = t^2 + 2t + 1$

1. $f(t) = \sin 2t$ $\mathcal{L} = \frac{2}{s^2+2^2}$

(4) $\frac{3!}{s^4} + \frac{1}{s-2} + \frac{s}{s^2+5^2}$

2. $\mathcal{L} = \frac{3}{(s-2)^2+3^2}$

(5) $\frac{1!}{(s-1)^2}$

(3) $\mathcal{L} = \frac{s}{s^2-1^2}$

(6) $\frac{2!}{s^3} + \frac{2\cdot 1!}{s^2} + \frac{1}{s}$

(4) $f(t)$ ... 3! [illegible]

6.2 4/2/14

(2) $\frac{2s}{(s-3)(s+2)} - \frac{3}{(s-3)(s+2)}$

$\frac{2s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$

$2s = A(s+2) + B(s-3)$
$2s = As + 2A + Bs - 3B$
$2s = (A+B)s + (2A-3B)$

$A+B=2$
$2A-3B=0$

(1) $\frac{1}{s^3+5s} = \frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$) Num 1 less power than denom

$1 = A(s^2+5) + (Bs+C)s$
$= As^2 + 5A + Bs^2 + Cs$
$= (A+B)s^2 + Cs + 5A$

$\frac{1}{5}\cdot\frac{1}{s} + \frac{1}{5}\ \frac{s}{s^2+5}$

$\mathcal{L}\{f(t)\} = \frac{1}{5}\cdot 1 + \frac{-1}{5}\cos\sqrt{5}t$

$A+B=0$ $B=-\frac{1}{5}$
$C=0$
$A=\frac{1}{5}$

(2) $F(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}$

$2s-3 = As + 2A + Bs - 3B$ $\mathcal{L}^{-1} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$
$= (A+B)s + 2A - 3B$

$A+B=2$ $A=\frac{3}{5}$
$2A-3B=-3$
$B=\frac{7}{5}$

6.1

$\mathcal{L}\{f(t)\}$? $f(t)=1$

$\mathcal{L}\{1\} = \frac{1}{s}$

Laplace Definition $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = \mathcal{L}(1) = \frac{1}{s} = \int_0^\infty e^{-st} f(t)\,dt$

4/3/14

Formula: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t\} = \frac{1}{s}$

$\mathcal{L}\{t\} = \frac{1}{s} \cdot \mathcal{L}\{t^{n-1}\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$

$\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$

Show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$n = 1$ (Base of induction)

$\mathcal{L}\{t^1\} = \frac{1}{s^2} = \frac{1}{s^{1+1}}$

Induction Hypothesis - Assume it happens for n: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Prove it holds for $n+1$:

$\mathcal{L}\{t^{n+1}\} = \frac{(n+1)}{s}\mathcal{L}\{t^n\}$

$\mathcal{L}\{t^{n-1}\} = \frac{(n+1)}{s} + \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{(n+1)+1}}$

$P(x)y'' + Q(x)y' + R(x)y = 0$

Singular pt: x_0 S.T. $0 = P(x_0)$

$\left[y'' + \frac{Q}{P}y' + \frac{R}{P}y = 0\right](x-x_0)^2$

Euler

$x^2y'' + xby' + cy = 0, \quad x_0 = 0$

$(x^2 - x_0)^2 y'' + (x-x_0)^2 \frac{Q}{P}y' + (x-x_0)^2\frac{R}{P}y = 0$

$0 = (x-x_0)^2 y'' + (x-x_0)\left[(x-x_0)\frac{Q(x)}{P(x)}\right]y' + \left[(x-x_0)^2\frac{R(x)}{P(x)}\right]y$

$p_0 = \lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \qquad \lim_{x\to x_0}(x-x_0)\frac{R(x)}{P(x)} = q_0$ Need both finite

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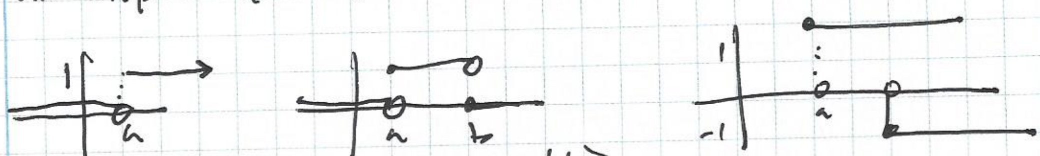
$y = x^r[a_0 + a_1x + \dots], \; x_0 = 0$

$y = (x-x_0)^r[a_0 + a_1(x-x_0) + \dots]$

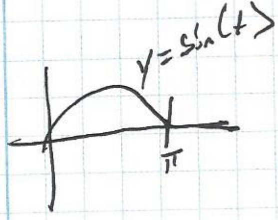
6.3 / 6.4

4/7/14

Unit step: $u_a(t) = u(t-a) = \text{Heaviside}(t-a)$

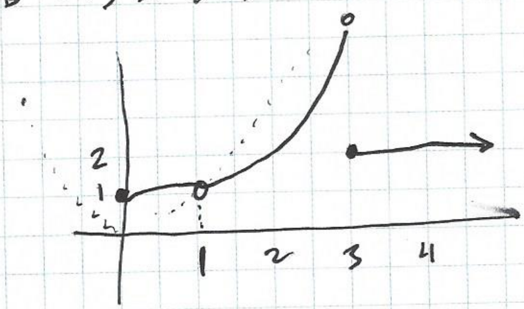


$u_a(t) - u_b(t)$

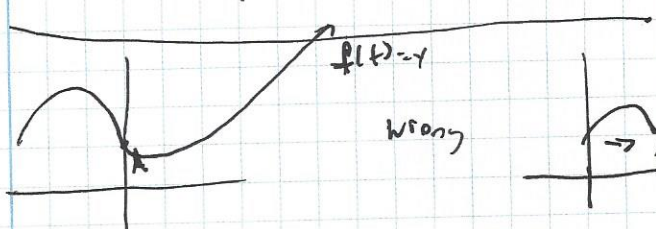


$(u_a(t) - u_b(t))\sin(t)$

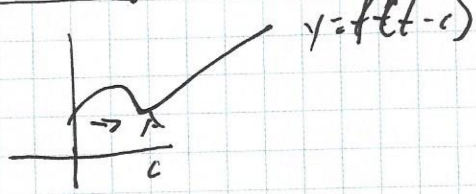
$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ t^2 & 1 \le t < 3 \\ 2 & 3 \le t \end{cases}$$



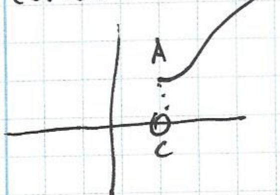
$$f(t) = (u_0 - u_1)1 + (u_1 - u_3)t^2 + (u_3)2$$



wrong



Correct:



$u_c(t) f(t-c)$

same

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c f(t-c)\,dt$$

$$= \int_0^c e^{-st}\, \cancel{0\, f(t-c)}\,dt + \int_c^\infty e^{-st}\, 1\, f(t-c)\,dt$$

$$\tau = t-c \Rightarrow d\tau = dt$$

$$= \int e^{-s(\tau+c)} f(\tau)\,d\tau = \int_0^\infty e^{-s\tau} e^{-sc} f(\tau)\,d\tau$$

$$= e^{-sc}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-sc}F(s)$$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-sc}F(s)$ 2nd shift THM

$\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$ 1st