RANDOM WALK ON RANDOM INFINITE LOOPTREES

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Abstract. Looptrees have recently arisen in the study of critical percolation on the uniform infinite planar triangulation. Here we consider random infinite looptrees defined as the local limit of the looptree associated with a critical Galton–Watson tree conditioned to be large. We study simple random walk on these infinite looptrees by means of providing estimates on volume and resistance growth. We prove that if the offspring distribution of the Galton–Watson process is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2]$ then the spectral dimension of the looptree is $2\alpha/(\alpha + 1)$.

1. Introduction

Random graphs have been used as a discretization of continuous manifolds in a 'sum over histories' approach to quantum gravity [3]. In physics these graphs go by the name of *dynamical triangulations* since the basic building blocks of the graphs are triangles (if the manifold is two-dimensional) and their higher dimensional analogs. It is not important for large scale properties how one chooses these building blocks and in two dimensions one may e.g. replace triangles by polygons of higher degree and the general graph is in this case called a *planar map*.

An important observable one would like to have in such theories is some notion of dimension of the random graphs. Although planar maps are locally two-dimensional in the sense that their building blocks are polygons they may on large scales be far from two-dimensional. As an example, consider the planar map with vertex set $\{0, 1, \ldots, n\} \times \{0, 1\}$ and edges between (i, j) and (i', j') iff $|i - i'| + |j - j'| = 1$. The graph consists of n squares connected together in a linear chain and when n is large it is in some ways more one-dimensional than two-dimensional. There are different ways to define a dimension of a random graph and one way is through the behaviour of simple random walk on the graph. The *spectral dimension* $d_{\mathrm{s}}(G)$ may be defined for any connected, locally finite graph G as the limit

$$d_{\mathrm{s}}(G) = -2 \lim_{n\to\infty} \frac{\log p^G_{2n}(x, x)}{\log n} \tag{1.1}$$

provided this exists ($d_{\mathrm{s}}(G)$ is then easily seen to be independent of x). Here $p^G_{2n}(x, x)$ is the probability that simple random walk on G, started at x, returns to x after $2n$ steps. The spectral dimension of the d-dimensional lattice $\mathbb{Z}^d$ equals d which motivates its definition.

Date: August 31, 2018.
2000 *Mathematics Subject Classification.* 05C80, 05C81, 05C05, 60J80, 60K37.
Key words and phrases. Looptrees, random trees, random walk, spectral dimension.
Research supported by the Knut and Alice Wallenberg Foundation.

The last decades have seen substantial progress on the understanding of random walk on random graphs. Much of this work has been motivated by the Alexander–Orbach conjecture [2], which concerns random walk on critical percolation clusters in $\mathbb{Z}^d$, which are supposed to be 'tree-like'. In a seminal 1986 paper [19], Kesten initiated a rigorous study of random walk on critical percolation clusters and large critical random trees. More recently Barlow and Kumagai [5] studied simple random walk on the incipient infinite cluster of percolation on the d-regular tree, which we denote here by $\mathcal{T}$. Among other things, Barlow and Kumagai proved that $d_{\mathrm{s}}(\mathcal{T}) = 4/3$ for almost every realization of $\mathcal{T}$.

The incipient infinite cluster $\mathcal{T}$ of the d-regular tree may be constructed as the large N local limit of a Galton–Watson tree with critical offspring distribution $\mathrm{Bin}(d,1/d)$, conditioned to have size N. Recall that a Galton–Watson tree with offspring distribution $\pi = (\pi_i)_{i\geq 0}$ is a random tree constructed recursively by starting with a single individual who gives birth to a number of offspring according to the distribution π and then each of the offspring has independently offspring of their own according to the same distribution and so on, see e.g. [4]. The offspring distribution and the process are said to be *critical* if the expected number of offspring is 1. We let $\mathcal{T}_N$ denote a Galton–Watson tree conditioned on having N vertices.

Fuji and Kumagai [13] generalized the result of [5] by showing that the same spectral dimension 4/3 is obtained for *any* tree $\mathcal{T}$ arising as the local limit of $\mathcal{T}_N$ when the offspring distribution π is critical and has finite variance. Croydon and Kumagai [7] studied the case when the offspring distribution may have infinite variance: more precisely they considered the case when π is in the domain of attraction of a stable distribution of index $\alpha \in (1, 2]$ which is equivalent to

$$\sum_{i=0}^{n} i^2 \pi_i = n^{2-\alpha} L_1(n) \tag{1.2}$$

where L_1 is slowly varying at infinity (see Section 3). They showed that then the spectral dimension equals

$$d_{\mathrm{s}}(\mathcal{T}) = \frac{2\alpha}{2\alpha - 1}, \quad \text{almost surely.} \tag{1.3}$$

This includes the result by Fuji and Kumagai for $\alpha = 2$.

The methods of the above-mentioned papers relied on connections between the asymptotics of the random walk with volume and resistance growth in the graph. A general formulation of these methods, which applies not only to trees but in principle to any random, strongly recurrent graphs, was given by Kumagai and Misumi in [20]. The general scheme of [20] is to show that for random infinite graphs in which the volume of graph balls of radius r grows roughly like r^a and in which the resistance from the center to the boundary of the balls grows roughly like r^b, the spectral dimension is

$$d_{\mathrm{s}} = 2a/(a + b), \tag{1.4}$$

cf. [20, Eq. (1.1)]. The results of the current paper rely on the general methods of [20]. Other applications of these methods include determining the spectral dimension of the local limit of random bipartite planar maps in a certain 'condensation phase', see [6]. In parallel to the above-mentioned progress, mathematical physicists have developed generating function methods to calculate the spectral dimension of random trees, see e.g. [10, 11, 16, 17, 24]. The benefits of those methods are their simplicity but the disadvantages are that they do not apply as generally and give somewhat weaker results regarding the existence of d_{s}.

Recently a new tree-like random structure called a *looptree* was introduced by Curien and Kortchemski [8]. Given a finite rooted tree τ embedded in the plane, one may informally define the corresponding rooted looptree $\mathrm{Loop}(\tau)$ as follows (more detail is given in Section 2). The vertex set of $\mathrm{Loop}(\tau)$ is the same as that of τ, but instead of connecting each vertex to all its children, edges connect *siblings* in the order they appear in the embedding, and also there are edges from the leftmost and rightmost children to the parent. This is illustrated in Fig. 1. (What we call $\mathrm{Loop}(\tau)$ is called $\mathrm{Loop}(\tau)$ in [9] and $\mathrm{Loop}'(\tau)$ in [8].)

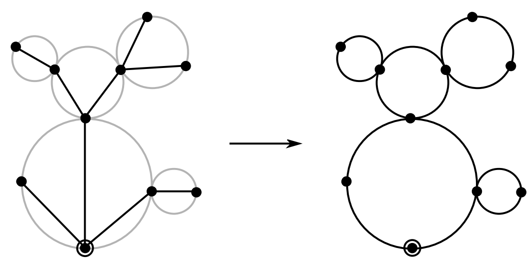

FIGURE 1. On the left is a tree τ in black and the edges in its associated looptree in gray. On the right is $\mathrm{Loop}(\tau)$. The genealogical order in τ is defined with respect to the root which is circled.

The recent paper [8] studied the *scaling limit* of $\mathrm{Loop}(\mathcal{T}_N)$, where $\mathcal{T}_N$ is a critical Galton–Watson tree conditioned to have size N and whose offspring distribution $\pi = (\pi_i)_{i\geq 0}$ satisfies (1.2). By scaling limit we mean convergence in the Gromov–Hausdorff sense of a sequence of graphs with a properly renormalized graph metric. For each α, the scaling limit was described directly in terms of an α–stable Lévy process and was referred to as a *stable looptree with index* α. Furthermore it was proved that the stable looptrees arise as the scaling limits of certain random dissections of a polygon. In [9] it was shown that the scaling limit of the boundary of a critical percolation cluster in the uniform infinite planar triangulation equals the stable looptree with index $\alpha = 3/2$ and it was conjectured that the family of stable looptrees appears generically in the study of interfaces in statistical mechanical models on planar maps. Therefore, it is of interest to understand the properties of stable looptrees in detail.

1.1. **Main results.** In this paper we study the *local* limit $\mathcal{L}$ of $\mathrm{Loop}(\mathcal{T}_N)$ as $N \to \infty$ where $\mathcal{T}_N$ is a random tree which belongs to the family of so-called simply generated random trees. In most cases of interest one may view $\mathcal{T}_N$ as a critical or sub-critical Galton–Watson tree conditioned on having N vertices as we will explain in Section 2.1. Local convergence of a sequence of rooted graphs means that for any finite R, the graph ball of radius R

around the root is eventually constant. The existence and construction of the local limit of $\mathrm{Loop}(\mathcal{T}_N)$ is straightforward, given a recent general result on local limits of simply generated trees by Janson [15], see Theorem 2.2 and Corollary 2.3.

After providing the general construction of $\mathcal{L}$ we will mainly focus on the case when $\mathcal{T}_N$ is a critical size conditioned Galton–Watson tree whose offspring distribution π is in the domain of a attraction of a stable law with index $\alpha \in (1,2]$, i.e. satisfies (1.2). By using the results of [20], and establishing suitable bounds on volume and resistance growth in $\mathcal{L}$, we then prove the following:

Theorem 1.1. *If π is critical and satisfies* (1.2) *then for almost every realization of $\mathcal{L}$, the spectral dimension satisfies*

$$d_{\mathrm{s}}(\mathcal{L}) = \frac{2\alpha}{\alpha+1}.$$

Further results on the escape time from the graph ball of radius R around the root will be stated in Theorem 3.1. Our strategy for proving Theorem 1.1 is to establish the volume growth exponent $a = \alpha$ and the resistance growth exponent $b = 1$, cf. (1.4). We note that the resistance growth exponent is in fact the same as in the critical tree $\mathcal{T}$. An intuitive explanation for this resistance growth is that $\mathcal{T}$ consists of exactly one infinite simple path from the root (a *spine*) which has finite independent critical Galton-Watson outgrowths. The height of the outgrowths is sufficiently small so that only $O(1)$ number of vertices at level $r/2$ in $\mathcal{T}$ are connected to level r in $\mathcal{T}$ via a path which does note intersect the spine. We will show that roughly the same picture applies to the looptrees.

Theorem 1.1 is a statement about the spectral dimension which holds for almost every realization of the random infinite looptrees and in that case d_{s} is often referred to as the *quenched spectral dimension*. Another way to view a quantitative property of a random walk on a random graph is to average it over all the realizations of the graphs. In that way, we define the so-called *annealed spectral dimension* of the random looptrees as

$$\bar{d}_{\mathrm{s}} = -2 \lim_{n\to\infty} \frac{\log \mathbb{E}(p_{2n}^{\mathcal{L}}(x,x))}{\log n}. \tag{1.5}$$

In general, the quenched and annealed spectral dimensions need not be equal and one can exist while the other is not defined. We will show:

Theorem 1.2. *If π is critical and satisfies* (1.2) *then the annealed spectral dimension of the infinite random looptree $\mathcal{L}$ is*

$$\bar{d}_{\mathrm{s}} = \frac{2\alpha}{\alpha+1}.$$

The annealed result relies on the same type of resistance and volume estimates as the quenched result but requires stronger bounds. Furthermore, one needs bounds on the expected volume of the graph ball $B(n;\mathcal{L})$ of radius n centered on the root in $\mathcal{L}$. This bound is provided in Lemma 3.4 which also shows that the so-called annealed volume growth exponent of $\mathcal{L}$ is

$$\lim_{n\to\infty} \frac{\log \mathbb{E}(|B(n;\mathcal{L})|)}{\log n} = \alpha. \tag{1.6}$$

This value coincides with the (quenched) Hausdorff dimension of random stable looptrees, cf. [8, Thm. 1]. We remark that for the infinite tree $\mathcal{T}$ with offspring distribution (1.2) studied in [7] it was not possible to obtain annealed results corresponding to Theorem 1.2 and Eq. (1.6) when the variance of the offspring distribution is infinite. The reason is that the vertices in the trees tend to have very large degrees which causes trouble when averaging over the trees, e.g. $\mathbb{E}(|B(n;\mathcal{T})|) = \infty$. The degrees of vertices in the looptrees are however bounded by 4 which eliminates this problem.

1.2. **Outline and notation.**

The rest of this paper is organized as follows. In Section 2 we precisely define our model and present the local limit theorem along with some basic estimates on looptrees. Theorems 1.1 and 1.2 are proved in Section 3. In the Appendix we collect some results about slowly varying functions and their application to some of the random variables which we study in this paper.

The following notation will be used throughout the paper. $\mathbb{E}$ denotes expected value and $\mathbb{P}$ denotes probability, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, the minimum of a and b is written $a\wedge b$ and the maximum $a \vee b$, the indicator of an event A is written $\mathbf{1}_A$, and $f(x) \sim g(x)$ denotes that $f(x)/g(x) \to 1$ for the appropriate limiting value of x (usually $x \to \infty$ or $x \to 1^-$).

2. Plane trees and looptrees

In this paper we will define rooted plane trees in the same, slightly unconventional, way as in [6]. Let T_∞ be an infinite tree with vertex set $V(T_\infty) = \bigcup_{n\geq 0} \mathbb{Z}^n$ i.e. the set of all finite sequences of integers. The unique element in $\mathbb{Z}^0$ is called the root and is denoted by $\varnothing$. The concatenation of two elements $u, v \in V(T_\infty)$ is denoted by uv. The edges in T_∞ are defined by connecting each vertex vi, $i \in \mathbb{Z}$, $v \in V(T_\infty)$ to the vertex v by an edge and we say that v is the parent of vi and conversely that vi is a child of v. When vertices share the same parent v we call them siblings and they are ordered from left to right by declaring that vi is to the left of vj if

- $i = 0$ ($v0$ is the leftmost child) or
- $ij > 0$ and $i < j$ or
- $i > 0$ and $j < 0$.

Two siblings are said to be adjacent if one is immediately to the left of the other, i.e. if they are of the form vi and $v(i-1)$, with $i \neq 0$. We say that v is a descendant of u, written $v \succeq u$, if there is a self-avoiding path from $\varnothing$ to v containing u. If $v \succeq u$ and $v \neq u$ we write $v \succ u$.

A rooted plane tree τ is defined as a subtree of T_∞ which contains the root $\varnothing$ and has the property that for every vertex $v \in V(\tau)$ there is a number $\mathrm{out}(v) \in \{0, 1, \ldots\} \cup \{\infty\}$, called the *outdegree* of v, such that $vi \in V(\tau)$ if and only if $\lfloor -\mathrm{out}(v)/2\rfloor < i \leq \lfloor \mathrm{out}(v)/2\rfloor$. The child vi of v in τ is said to be the *rightmost* child of v if it has no siblings in τ to its right (thus the rightmost child is $v0$ if $\mathrm{out}(v) = 1$, $v1$ if $\mathrm{out}(v) = 2$, or $v(-1)$ if $\mathrm{out}(v) \geq 3$). Siblings vi, vj of v in τ are said to be *adjacent* if they are either adjacent in T_∞, or if $\mathrm{out}(v) < \infty$ and $i = \lfloor \mathrm{out}(v)/2\rfloor$ and $j = \lfloor -\mathrm{out}(v)/2\rfloor$. From now on we assume that all trees are rooted plane trees and we will simply refer to them as trees.

The set of trees is denoted by $\mathfrak{T}$ and the set of trees with n vertices is denoted by $\mathfrak{T}_n$. A metric is defined on $\mathfrak{T}$ as follows. For every $R \geq 0$ define the set

$$V^{[R]} = \bigcup_{n=0}^{R} \{\lfloor -R/2 \rfloor + 1, \lfloor -R/2 \rfloor + 2, \dots, \lfloor R/2 \rfloor - 1, \lfloor R/2 \rfloor\}^n \qquad (2.1)$$

and for $\tau \in \mathfrak{T}$ let $\tau^{[R]}$ be the finite subtree of τ with vertex set $V(\tau) \cap V^{[R]}$. Define the metric

$$d_{\mathfrak{T}}(\tau_1, \tau_2) = \left(1 + \sup\left\{R \ : \ \tau_1^{[R]} = \tau_2^{[R]}\right\}\right)^{-1}, \qquad \tau_1, \tau_2 \in \mathfrak{T}. \qquad (2.2)$$

The set $\mathfrak{T}$ is equipped with the Borel σ-algebra generated by $d_{\mathfrak{T}}$.

To each tree τ we associate a looptree $\mathrm{Loop}(\tau)$ as follows. The vertex set of $\mathrm{Loop}(\tau)$ is the same as that of τ and the edges are constructed by connecting adjacent siblings with an edge and by connecting each parent to both its leftmost and rightmost child with an edge. Thus each vertex of finite outdegree $k \geq 1$ in the tree corresponds to a cycle of length $k+1$ in the looptree and a vertex of infinite degree corresponds to a doubly–infinite path. Note that our definition of $\mathrm{Loop}(\tau)$ coincides with what is called $\mathrm{Loop}(\tau)$ in [9] and $\mathrm{Loop}'(\tau)$ in [8]. We denote the set of looptrees by $\mathfrak{L}$.

For any rooted graph G with root $\varnothing$ and graph metric d_G, the graph ball of radius R centered on the root of G is defined to be

$$B(R; G) := \{x \in G \ : \ d_G(x, \varnothing) < R\}. \qquad (2.3)$$

We identify $B(R; G)$ with the induced subgraph of G spanned by its vertices. A metric is defined on $\mathfrak{L}$ by

$$d_{\mathfrak{L}}(L_1, L_2) = (1 + \sup\{R \ : \ B(R; L_1) = B(R; L_2)\})^{-1}, \qquad L_1, L_2 \in \mathfrak{L} \qquad (2.4)$$

and the set $\mathfrak{L}$ is equipped with the Borel σ-algebra generated by $d_{\mathfrak{L}}$. Limits of sequences in metrics of the type $d_{\mathfrak{T}}$ and $d_{\mathfrak{L}}$ are often referred to as *local limits* since convergent sequences are eventually constant in every finite 'neighbourhood' of the root, where the precise definition of neighbourhood depends on the context. We have the following important result:

Lemma 2.1. *The function* $\mathrm{Loop} : \mathfrak{T} \to \mathfrak{L}$ *is a homeomorphism.*

Proof. By construction, Loop is a bijection. We will outline the proof of the continuity of Loop and the continuity of its inverse may be shown in a similar way. Let $(\tau_i)_{i\geq 0}$ be a sequence of trees in $\mathfrak{T}$ which has a limit τ. This means that for every $R \geq 0$, eventually (for all i large enough) $\tau_i^{[R]} = \tau^{[R]}$. Thus, for each $R \geq 0$, eventually $\mathrm{Loop}(\tau_i^{[R]}) = \mathrm{Loop}(\tau^{[R]})$. For each $R' \geq 0$ one may choose R large enough ($R \geq 2R' + 1$ suffices) such that eventually $B(R'; \mathrm{Loop}(\tau_i^{[R]})) = B(R'; \mathrm{Loop}(\tau_i))$ and such that $B(R'; \mathrm{Loop}(\tau^{[R]})) = B(R'; \mathrm{Loop}(\tau))$. Therefore, eventually

$$B(R'; \mathrm{Loop}(\tau_i)) = B(R'; \mathrm{Loop}(\tau_i^{[R]})) = B(R'; \mathrm{Loop}(\tau^{[R]})) = B(R'; \mathrm{Loop}(\tau)).$$

□

2.1. **Simply generated trees and the local limit theorem.** Let $(w_i)_{i\geq 0}$ be a sequence of non-negative numbers. Simply generated trees with n vertices are random trees $\mathcal{T}_n$ in $\mathfrak{T}$ defined by assigning a weight

$$W(\tau) = \prod_{v\in V(\tau)} w_{\mathrm{out}(v)} \tag{2.5}$$

to each tree $\tau \in \mathfrak{T}$ and letting

$$\mathbb{P}(\mathcal{T}_n = \tau) = \frac{W(\tau)}{\sum_{\tau'\in\mathfrak{T}_n} W(\tau')}, \quad \tau \in \mathfrak{T}_n \quad \text{(0 otherwise)}. \tag{2.6}$$

The random looptrees with n vertices with which we are concerned in this paper are defined by $\mathcal{L}_n = \mathrm{Loop}(\mathcal{T}_n)$.

Janson [15] showed in full generality that the sequence of random trees $\mathcal{T}_n$ converges weakly towards an infinite random tree $\mathcal{T}$ and we state this as Theorem 2.2 below. Before stating the theorem we describe the limit $\mathcal{T}$ which requires a few definitions. Denote the generating function of $(w_i)_{i\geq 0}$ by

$$g(z) = \sum_{i=0}^{\infty} w_i z^i \tag{2.7}$$

and denote its radius of convergence by ρ. Note that if $\rho > 0$ then for each $0 < t < \rho$, the sequence of probabilities

$$p_i(t) = \frac{t^i w_i}{g(t)} \tag{2.8}$$

when inserted in (2.5) and (2.6), defines the same random tree $\mathcal{T}_n$ as w_i. The largest mean of $p_i(t)$ is given by

$$\gamma = \lim_{t\nearrow\rho} \frac{tg'(t)}{g(t)}. \tag{2.9}$$

In the case $\rho = 0$ we define $\gamma = 0$. Let τ be

(1) the solution $\tau \in (0, \rho)$ to $\tau g'(\tau)/g(\tau) = 1$, if $\gamma > 1$ (which is unique since the ratio in (2.9) is increasing, see e.g. [15, Lemma 3.1]); or
(2) $\tau = \rho$, if $\gamma \leq 1$.

Then define $\pi_i = \lim_{t\uparrow\tau} p_i(t)$ if $\rho > 0$ or $\pi_i = \delta_{i,0}$ if $\rho = 0$. (Note that the sequence $(\pi_i)_{i\geq 0}$ is well-defined in all cases since $g(t)$ is monotonically increasing in $t > 0$.) Let ξ be a random variable distributed by $(\pi_i)_{i\geq 0}$. One may view ξ as an offspring distribution of a Galton–Watson process and we denote its expected value by μ. By definition

$$\mu = \min\{\gamma, 1\} \leq 1 \tag{2.10}$$

and so ξ is either critical or sub-critical. For $\rho > 0$, $\mathcal{T}_n$ is a Galton–Watson tree with offspring distribution ξ conditioned on having n vertices. Define also the random variable $\hat{\xi}$ by

$$\mathbb{P}(\hat{\xi} = k) = \begin{cases} k\pi_k & \text{if } k < \infty \\ 1-\mu & \text{if } k = \infty. \end{cases} \tag{2.11}$$

(If $\mu = 1$ then $\hat{\xi}$ is simply the size-biased version of ξ.)

The tree $\mathcal{T}$ will now be introduced, following a construction by Janson [15]. It is a modified Galton–Watson tree which contains two types of vertices, called normal and special, which independently give birth to vertices in the following recursive way. First of all the root is declared to be special. Special vertices give birth to vertices according to the offspring distribution $\hat{\xi}$. If the number of children of a special vertex is finite, one of them is chosen uniformly and declared to be special and the rest are declared to be normal. If the number of children is infinite (which may happen when $\mu < 1$) then all the children are declared to be normal. Normal vertices give birth to normal vertices according to the offspring distribution ξ.

Theorem 2.2 (Janson [15])**.** *For any sequence $(w_i)_{i\geq 0}$ such that $w_0 > 0$ and $w_k > 0$ for some $k \geq 2$ it holds that $\mathcal{T}_n \xrightarrow{\mathrm{d}} \mathcal{T}$ as $n \to \infty$.*

Special cases of Theorem 2.2 have been proven independently by many authors. The original proof in the case $\mu = 1$ was implicitly given by Kennedy [18] and later Aldous and Pitman [1] gave an explicit proof. The case $0 < \mu < 1$ was originally proven, for weights obeying a power law, by Jonsson and Stefánsson [17] and the case $\mu = 0$ was proved by Janson, Jonsson and Stefánsson [14] in almost complete generality.

By Theorem 2.2 and Lemma 2.1 we immediately get the corresponding convergence for the random looptrees. Define $\mathcal{L} = \mathrm{Loop}(\mathcal{T})$.

Corollary 2.3. *For any sequence $(w_i)_{i\geq 0}$ such that $w_0 > 0$ and $w_k > 0$ for some $k \geq 2$ it holds that $\mathcal{L}_n \xrightarrow{\mathrm{d}} \mathcal{L}$ as $n \to \infty$.*

The tree $\mathcal{T}$ is qualitatively different depending on whether $\mu = 1$ or $\mu < 1$. In the former case, $\mu = 1$, the special vertices will form an infinite path starting at the root, which we refer to as an *infinite spine*. The normal children of the special vertices are the roots of independent critical Galton–Watson trees with offspring distribution ξ, and we will refer to these trees as *outgrowths* from the spine. This construction is initially due to Kesten [19]. In the latter case, $\mu < 1$, the special vertices form an a.s. finite path, which we refer to as a *finite spine* and its length ℓ is distributed by

$$\mathbb{P}(\ell = i) = (1-\mu)\mu^i, \quad i \geq 0 \tag{2.12}$$

(where we understand $0^0 = 1$). The finite spine starts at the root and ends at a special vertex having infinite degree. The outgrowths from the finite spine are defined as in the former case and are now sub-critical Galton–Watson trees. This construction is due to Jonsson and Stefánsson [17]. In the special case $\mu = 0$ the spine has length 0 and the outgrowths are all empty and thus $\mathcal{T}$ is deterministic.

From the description of $\mathcal{T}$ one may arrive at a similar description of $\mathcal{L} = \mathrm{Loop}(\mathcal{T})$. The spine of $\mathcal{T}$ corresponds to what may be called a *loopspine* in $\mathcal{L}$. When $\mu = 1$ the loopspine consists of an infinite sequence of cycles $(C_i)_{i\geq 1}$ whose lengths are independent copies of $\hat{\xi} + 1$. These cycles form a chain $\mathcal{C}$ by recursively attaching C_{i+1} to C_i at a point x_{i+1} chosen uniformly (independently of all other choices) on $C_i \setminus \{x_i\}$. The recursion starts with $x_1 = \varnothing$. At each vertex of $\mathcal{C}\setminus\{x_i : i \geq 1\}$ we then attach an independent copy of $\mathrm{Loop}(\tau)$, where τ is a Galton–Watson tree with offspring distribution ξ.

We call these copies of $\mathrm{Loop}(\tau)$ *outgrowths*. The case $\mu < 1$ is similar however the sequence $(C_i)_{1\le i\le \ell+1}$ is almost surely finite and the last element is not a cycle but a bi-infinite path (the graph with vertex set $\mathbb{Z}$ and an edge between i and j iff $|i-j| = 1$). In the extreme case when $\mu = 0$ the looptree is deterministic and equals a rooted bi-infinite path.

In the following two subsections we provide basic estimates on the volume growth of $\mathcal{L}$. These results do not rely on any assumptions about ξ being in the domain of attraction of a stable distribution, hence we separate them from the main proofs in Section 3.

2.2. **A bound on the height of a random looptree.** Let G be a finite graph, and single out a root vertex in G, which we denote $\varnothing$. Letting d_G denote the graph metric in G, we define the *height* of G as

$$\mathrm{Height}(G) = \max_{v\in G} d_G(\varnothing, v). \tag{2.13}$$

In this section we will prove the following result; to avoid trivialities we assume that ξ is not identially 1.

Lemma 2.4. *Let τ be a Galton–Watson tree with critial offspring distribution ξ (i.e., $\mathbb{E}(\xi) = 1$). Then there is a constant $c > 0$ such that for all $m \ge 2$,*

$$\mathbb{P}(\mathrm{Height}(\mathrm{Loop}(\tau)) \ge m) \le \frac{c}{m}. \tag{2.14}$$

Before proving this we state some general facts about (plane) trees and their associated looptrees. Let τ be a finite tree with n vertices and let $\varnothing = u_0, u_1, u_2, \dots, u_{n-1}$ denote the vertices of τ in lexicographical (depth-first-search) order. We define the *Lukasiewicz path* $W(\tau) = (W_j(\tau) : 0 \le j \le n)$ of τ by

$$W_0(\tau) = 0, \quad W_{j+1}(\tau) = W_j(\tau) + \mathrm{out}(u_j) - 1 \text{ for } 0 \le j \le n-1. \tag{2.15}$$

When τ is clear from the context we simply write W and W_j in place of $W(\tau)$ and $W_j(\tau)$. We have that $W_n = -1$ and that $W_j \ge 0$ for $j \le n-1$; also, $W_{j+1} \ge W_j - 1$ for all $0 \le j \le n-1$.

Note that u_1 is the leftmost child of the root, and that the process $(W_j : 1 \le j < n)$ achieves a record minimum at step j if and only if u_j is a child of the root. Introducing the number $M_j = \min_{1\le i\le j} W_i$, for $1 \le j < n$, and denoting the children of $\varnothing$ by $v_1, \dots, v_m$, numbered from left to right, it follows that

$$u_j \succeq v_k \Leftrightarrow M_j = m - k. \tag{2.16}$$

Fix an arbitrary vertex u_j in τ and for $u_i \prec u_j$ write

$$x(i,j) = \min_{i+1\le \ell\le j} W_\ell(\tau) - W_i(\tau) + 1. \tag{2.17}$$

By applying (2.16) iteratively to the subtrees rooted at the vertices on the unique path from $\varnothing$ to u_j in τ, we see that

$$d_{\mathrm{Loop}(\tau)}(\varnothing, u_j) \le \sum_{u_i \prec u_j} x(i,j). \tag{2.18}$$

(The right-hand-side of (2.18) is the length of a path in $\mathrm{Loop}(\tau)$ from $\varnothing$ to u_j which traverses each 'loop' in the anticlockwise direction; cf. the more general exact expression for $d_{\mathrm{Loop}(\tau)}$ in [8, Eq. (40)]).

As observed in [8, Eq. (46)] we have for any fixed $j \geq 1$ that

$$\sum_{u_i \prec u_j} x(i,j) = d_\tau(\varnothing, u_j) + W_j(\tau). \tag{2.19}$$

Combining (2.19) with (2.18) it follows that

$$\text{Height}(\text{Loop}(\tau)) \leq \text{Height}(\tau) + \max_{0 \leq j \leq n} W_j(\tau). \tag{2.20}$$

Proof of Lemma 2.4. By (2.20)

$$\mathbb{P}(\text{Height}(\text{Loop}(\tau)) \geq m) \leq \mathbb{P}(\text{Height}(\tau) \geq m/2) + \mathbb{P}(\max W(\tau) \geq m/2). \tag{2.21}$$

Writing $\xi_i = \text{out}(u_i)$ for the summands appearing in (2.15), the ξ_i are simply independent copies of ξ and $W_j = \sum_{i=1}^{j}(\xi_i - 1)$ is a random walk. The event $\{\max W(\tau) \geq m/2\}$ equals the event that the random walk W_j reaches level $\geq m/2$ before it reaches level -1. Letting

$$T = \min\{j \geq 0 : W_j \leq -1 \text{ or } W_j \geq m/2\},$$

it follows from Wald's identity (see e.g. [12, page 601]) that $\mathbb{E}(W_T) = \mathbb{E}(T)\mathbb{E}(\xi - 1) = 0$. Since $W_T + 1 \geq 0$ we have by Markov's inequality that

$$\mathbb{P}(\max W(\tau) \geq m/2) = \mathbb{P}(W_T \geq m/2) \leq \frac{\mathbb{E}(W_T) + 1}{m/2 + 1} = \frac{1}{m/2 + 1}. \tag{2.22}$$

Moreover, it follows from [4, Theorem I.9.1] (and the remark below it) that

$$\mathbb{P}(\text{Height}(\tau) \geq m/2) \leq \frac{c'}{m},$$

for a constant $c' > 0$. The result follows. □

2.3. **A bound on the volume of a random loopspine.** As in the description of $\mathcal{L}$ in Section 2.1, let $(C_i)_{i\geq 1}$ be a sequence of cycles, each having a marked vertex x_i. Assume that C_i has a random length $X_i + 1$, where the X_i are independent and distributed as a random variable X taking values in the positive integers $\{1, 2, \ldots\}$. Construct an infinite chain $\mathcal{C}$ by identifying the point x_{i+1} in cycle C_{i+1} to a uniformly chosen point in $C_i \setminus \{x_i\}$ for all $i \geq 1$. We call x_i the root of the cycle C_i, and let $\varnothing = x_1$ be the root of $\mathcal{C}$. The loopspine of $\mathcal{L}$ is constructed like this with $X = \hat{\xi}$, but the following bound holds regardless of the distribution of X (as long as $X \geq 1$ a.s.). Define the set

$$A_n = \{v \in \mathcal{C} \; : \; d_{\mathcal{C}}(\varnothing, v) \leq n\} \tag{2.23}$$

where $d_{\mathcal{C}}$ denotes the graph metric in $\mathcal{C}$.

Lemma 2.5. *For all $n \geq 0$*

$$\mathbb{E}(|A_n|) \leq 16n + 1. \tag{2.24}$$

Proof. Given $(X_i)_{i\geq 1}$, let $(U_i(X_i))_{i\geq 1}$ denote independent random variables, $U_i(X_i)$ being uniformly chosen from the set $\{1, 2, \ldots, X_i\}$. Define

$$Y_i^{(n)} = \min\{U_i(X_i), X_i - U_i(X_i) + 1, n\}. \tag{2.25}$$

Letting

$$T_n = \min\Big\{N \ : \ \sum_{i=1}^{N} Y_i^{(n)} \geq n\Big\}, \tag{2.26}$$

we see that T_n is the smallest number N for which the root of a cycle C_{N+1} is at a distance at least n from the root $\varnothing$. Writing $X_i^{(n)} = \min\{X_i, 2n\}$ we have that

$$|A_n| \leq \sum_{i=1}^{T_n} X_i^{(n)} + 1. \tag{2.27}$$

Using this along with Wald's identity we get

$$\mathbb{E}(|A_n|) \leq \mathbb{E}(T_n)\mathbb{E}(X_1^{(n)}) + 1. \tag{2.28}$$

Again, by Wald's identity and the definitions of T_n and $Y_i^{(n)}$ we find that

$$\mathbb{E}(T_n) = \frac{\mathbb{E}\left(\sum_{i=1}^{T_n} Y_i^{(n)}\right)}{\mathbb{E}(Y_1^{(n)})} = \frac{\mathbb{E}\left(\sum_{i=1}^{T_n-1} Y_i^{(n)} + Y_{T_n}^{(n)}\right)}{\mathbb{E}(Y_1^{(n)})} \leq \frac{2n}{\mathbb{E}(Y_1^{(n)})} \tag{2.29}$$

and thus

$$\mathbb{E}(|A_n|) \leq 2n\frac{\mathbb{E}(X_1^{(n)})}{\mathbb{E}(Y_1^{(n)})} + 1. \tag{2.30}$$

We conclude by showing that the ratio of expected values in (2.30) is bounded from the above by 8. Fix $\epsilon \in (0, 1/2)$ and note that

$$\begin{aligned} Y_1^{(n)} &\geq Y_1^{(n)} \mathbf{1}_{\{\epsilon X_1 \leq U_1(X_1) \leq (1-\epsilon)X_1 + 1\}} \\ &\geq \epsilon \min\{X_1, n/\epsilon\} \mathbf{1}_{\{\epsilon X_1 \leq U_1(X_1) \leq (1-\epsilon)X_1 + 1\}} \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{E}(Y_1^{(n)}) &\geq \epsilon \mathbb{E}(\min\{X_1, n/\epsilon\} \mathbf{1}_{\{\epsilon X_1 \leq U_1(X_1) \leq (1-\epsilon)X_1 + 1\}}) \\ &= \epsilon \mathbb{E}\big(\min\{X_1, n/\epsilon\} \mathbb{P}(\epsilon X_1 \leq U_1(X_1) \leq (1-\epsilon)X_1 + 1 \mid X_1)\big) \\ &= \epsilon \mathbb{E}\Big(\min\{X_1, n/\epsilon\} \frac{\lfloor (1-\epsilon)X_1 \rfloor - \lceil \epsilon X_1 \rceil + 2}{X_1}\Big) \\ &\geq \epsilon(1-2\epsilon)\mathbb{E}(\min\{X_1, n/\epsilon\}). \end{aligned} \tag{2.31}$$

Also, $\mathbb{E}(X_1^{(n)}) \leq \mathbb{E}(\min\{X_1, n/\epsilon\})$ since $\epsilon < 1/2$. Taking the optimal $\epsilon = 1/4$ yields the desired result. □

We conclude this section by establishing a lower bound on the number of outgrowths (possibly empty) from the loopspine $\mathcal{C}$ up to and including distance n from the root. We use the same notation as was introduced in the beginning of this section. We will call the vertices $\varnothing = x_1, x_2, \ldots$, *closed* and the vertices in $\mathcal{C} \setminus \{x_1, x_2, \ldots\}$ *open*. By definition, outgrowths only emanate from the open vertices and thus we want a lower bound on the number of open vertices up to and including distance n from the root. Denote their number by $\mathcal{R}_n$. For each $i \geq 0$, define $Y_i = \min\{U_i(X_i), X_i - U_i(X_i) + 1\}$ with the same $U_i(X_i)$ as in (2.25).

Lemma 2.6. *Let* $p = \mathbb{P}(Y_i > 1)$. *Then* $\mathcal{R}_n$ *stochastically dominates a binomial random variable* $\mathrm{Bin}(\lfloor n/2 \rfloor, p)$.

Proof. Let K be the index of the first of the cycles $C_1, C_2, \dots$ to reach distance n from the root. Then we have that $\mathcal{R}_n \geq n - K$. For all $k \geq 1$ we have that $\{K \leq k\} \supseteq \{Y_1 + \cdots + Y_k \geq n\}$, and thus

$$\begin{aligned}\mathbb{P}(K \leq k) &\geq \mathbb{P}(Y_1 + \cdots + Y_k \geq n) \\ &= \mathbb{P}((Y_1 - 1) + \cdots + (Y_k - 1) \geq n - k) \\ &\geq \mathbb{P}(\varepsilon_1 + \cdots + \varepsilon_k \geq n - k),\end{aligned} \tag{2.32}$$

where $\varepsilon_i = \mathbf{1}_{\{Y_i > 1\}}$. Thus for each $0 \leq r \leq \lfloor n/2 \rfloor$ we have that

$$\begin{aligned}\mathbb{P}(\mathcal{R}_n \geq r) &\geq \mathbb{P}(K \leq n - r) \\ &\geq \mathbb{P}(\varepsilon_1 + \cdots + \varepsilon_{n-r} \geq r) \\ &\geq \mathbb{P}(\varepsilon_1 + \cdots + \varepsilon_{\lfloor n/2 \rfloor} \geq r).\end{aligned} \tag{2.33}$$

This gives the result since $\varepsilon_1 + \cdots + \varepsilon_{\lfloor n/2 \rfloor}$ has the law $\mathrm{Bin}(\lfloor n/2 \rfloor, p)$. $\square$

3. Random walk and spectral dimension

In this section we prove our main results on random walk on the infinite random looptree $\mathcal{L}$. Given a realization of $\mathcal{L}$, let $(X_n : n \geq 0)$ denote simple random walk on $\mathcal{L}$ started at $\varnothing$. Apart from the return probability $p_{2n}^{\mathcal{L}}(\varnothing, \varnothing) = \mathbb{P}(X_{2n} = \varnothing \mid \mathcal{L})$ we will also consider the escape time from a ball of radius R defined by $\tau_R(\mathcal{L}) := \min\{n \geq 0 : X_n \notin B(R; \mathcal{L})\}$ and the (quenched) expected escape time $T_R(\mathcal{L}) = \mathbb{E}(\tau_R \mid \mathcal{L})$.

We will focus on the case when the offspring distribution ξ (defined above (2.11)) is critical (i.e. $\mu = \mathbb{E}(\xi) = 1$) and is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$ (i.e. satisfies (1.2)). We note briefly that in the sub-critical case $\mu < 1$ the looptree is a bi-infinite path with small outgrowths and a random walker perceives it as one-dimensional: one may easily show that when $\mu = 0$ or $0 < \mu < 1$ and $\mathbb{E}(\xi^{1+\epsilon}) < \infty$ for some $\epsilon > 0$ then almost surely $d_{\mathrm{s}}(\mathcal{L}) = 1$. From now on we thus consider the critical case only.

Denote by $|\tau|$ the total number of individuals in the Galton–Watson tree τ. We begin by summarizing some facts about the tail probabilities and generating functions of ξ and $|\tau|$ which follow from the assumption that ξ is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$. Recall that a function L is said to vary slowly at infinity if it is measurable and for any $\lambda \in \mathbb{R}$ it holds that

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1. \tag{3.1}$$

Common examples of slowly varying functions are powers of logarithms and iterates of logarithms. By [12, Theorem XVII.5.2] one may write

$$\mathbb{E}(\xi^2 \mathbf{1}_{\{\xi \leq x\}}) = x^{2-\alpha} L_1(x) \tag{3.2}$$

where L_1 is slowly varying at infinity, cf. (1.2). When $\alpha < 2$ we then have

$$\mathbb{P}(\xi > x) \sim \frac{2 - \alpha}{\alpha} x^{-\alpha} L_1(x). \tag{3.3}$$

(See [12, Corollary XVII.5.2 and (5.16)].) We will denote the generating function of the offspring probabilities by

$$f(s) = \mathbb{E}(s^{\xi}) = \sum_{n=0}^{\infty} \pi_n s^n. \tag{3.4}$$

It satisfies

$$f(s) = s + (1-s)^{\alpha} L((1-s)^{-1}) \qquad s \in [0,1) \tag{3.5}$$

where

$$L(x) \sim \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} L_1(x) - \frac{1}{2} x^{\alpha-2} \text{ as } x \to \infty. \tag{3.6}$$

This is standard but we sketch a proof in Lemma 4.7 in the Appendix. In the case $\alpha = 2$ one sees immediately from (3.2) that either ξ has finite variance and

$$L_1(x) \to 1 + f''(1) \qquad \text{as } x \to \infty. \tag{3.7}$$

or $L_1(x)$ diverges as $x \to \infty$.

Next, we have by Lemma 4.10 of the Appendix that

$$\mathbb{E}(s^{|\tau|}) = 1 - (1-s)^{1/\alpha} L^*((1-s)^{-1}) \qquad s \in [0,1) \tag{3.8}$$

where L^* is slowly varying at infinity and satisfies

$$\lim_{n\to\infty} L^*(n)^{\alpha} L(n^{\frac{1}{\alpha}} L^*(n)^{-1}) = 1. \tag{3.9}$$

Let a_n be a sequence such that

$$a_n^{-1/\alpha} n L^*(a_n) \to 1 \quad \text{as } n \to \infty. \tag{3.10}$$

Note that (3.9) and [22, Theorem 1.5] imply that

$$a_n \sim n^{\alpha} L(n)^{-1} \tag{3.11}$$

and by [22, 5°, Section 1.5] we may and will choose a_n to be strictly increasing.

3.1. **Proofs of the main results.** To prove our main results Theorems 1.1 and 1.2 we will estimate the volume- and resistance growth in $\mathcal{L}$, and apply the recent results of Kumagai and Misumi [20]. Recall that if $G = (V, E)$ is a locally finite graph with vertex set V and edge set E, and $A, B \subseteq V$ are disjoint subsets of V, then the *effective resistance* $\mathrm{R}_{\text{eff}}(A, B)$ between A and B is defined by letting $\mathrm{R}_{\text{eff}}(A, B)^{-1}$ be the infimum of

$$\sum_{xy\in E} (h(x) - h(y))^2 \tag{3.12}$$

over all functions $h : V \to \mathbb{R}$ satisfying $h(a) = 1$ for all $a \in A$ and $h(b) = 0$ for all $b \in B$. Here we will mainly be using the following two very simple facts about effective resistances:

(1) for all $x, y \in V$ we have $\mathrm{R}_{\text{eff}}(x, y) \le d_G(x, y)$, and
(2) if $C \subseteq V$ *separates* A from B, in the sense that every path in G between some $a \in A$ and $b \in B$ contains some $c \in C$, then $\mathrm{R}_{\text{eff}}(A, B) \ge \mathrm{R}_{\text{eff}}(A, C)$.

We will also use the standard series and parallel laws, for which we refer to [21, Chapter 2]. Recall the graph ball of radius n defined in (2.3). We define its *volume* $V_n = |B(n;\mathcal{L})|$ as the number of vertices in it.

Let $v(n) = a_n$ and $r(n) = n$ and let $\mathcal{I}$ be the inverse function of $v \cdot r$. The functions v and r are both strictly increasing and in light of (3.11) and Lemma 4.2 there exist $1 \leq d_1 < d_2$, $0 < \alpha_1, \alpha_2 \leq 1$ and $C_1, C_2 \geq 1$ such that

$$C_1^{-1}\left(\frac{n}{n'}\right)^{d_1} \leq \frac{v(n)}{v(n')} \leq C_1\left(\frac{n}{n'}\right)^{d_2}, \; C_2^{-1}\left(\frac{n}{n'}\right)^{\alpha_1} \leq \frac{r(n)}{r(n')} \leq C_2\left(\frac{n}{n'}\right)^{\alpha_2} \tag{3.13}$$

for all $0 < n' \leq n < \infty$. Thus, v and r satisfy the basic conditions required in [20]. (In [20] the 'volume' is defined slightly differently than our V_n, but since all vertices in the graph $\mathcal{L}$ have uniformly bounded degrees our definition may be used equivalently.) We will prove that there exist $c_1, c_2, c_3 > 0$, $\lambda_0 > 0$ and $q_1 > 0$, $q_2 > 2$ such that for all $\lambda \geq \lambda_0$ and all $n \geq 1$ we have

$$\mathbb{P}(V_n \leq \lambda v(n), \mathrm{R_{eff}}(\varnothing, B(n;\mathcal{L})^c) \geq \lambda^{-1} r(n)) \geq 1 - \frac{c_1}{\lambda^{q_1}} \tag{3.14}$$

$$\mathbb{P}(V_n \geq \lambda^{-1} v(n), \forall y \in B(n;\mathcal{L}), \mathrm{R_{eff}}(\varnothing, y) \leq \lambda r(d_{\mathcal{L}}(\varnothing, y))) \geq 1 - \frac{c_2}{\lambda^{q_2}} \tag{3.15}$$

$$\mathbb{E}(\mathrm{R_{eff}}(\varnothing, B(n;\mathcal{L})^c) V_n) \leq c_3 v(n) r(n). \tag{3.16}$$

Applying Theorem 1.5 and Proposition 1.4 of [20] we get the following from (3.14)–(3.16):

Theorem 3.1. *There exist $\beta_1, \beta_2 > 0$ such that the following hold.*

(1) *For each realization of $\mathcal{L}$ there exists a number $N(\mathcal{L}) < \infty$ such that*

$$\frac{(\log n)^{-\beta_1}}{v(\mathcal{I}(n))} \leq p_{2n}^{\mathcal{L}}(\varnothing, \varnothing) \leq \frac{(\log n)^{\beta_1}}{v(\mathcal{I}(n))}, \quad \text{for } n \geq N(\mathcal{L}).$$

(2) *For each realization of $\mathcal{L}$ there exists a number $R(\mathcal{L})$ such that*

$$(\log R)^{-\beta_2} v(R) r(R) \leq T_R(\mathcal{L}) \leq (\log R)^{\beta_2} v(R) r(R), \quad \text{for } R \geq R(\mathcal{L}).$$

(3) *There exist $C_1, C_1' > 0$ such that for all $n \geq 1$*

$$\frac{C_1}{v(\mathcal{I}(n))} \leq \mathbb{E}(p_{2n}^{\mathcal{L}}(\varnothing, \varnothing)) \leq \frac{C_1'}{v(\mathcal{I}(n))}.$$

(4) *There exist $C_2, C_2' > 0$ such that for all $R \geq 1$*

$$C_2 v(R) r(R) \leq \mathbb{E}(T_R(\mathcal{L})) \leq C_2' v(R) r(R).$$

Our results Theorem 1.1 and Theorem 1.2 on the spectral dimension follow immediately from the first and third parts of Theorem 3.1, respectively, using (3.11) and standard results on slowly varying functions given in the Appendix. In fact, [20] shows that some more results follow from the inequalities (3.14)–(3.16) which we do not state explicitly here. The rest of this section will be devoted to proving (3.14)–(3.16).

3.2. **Bounds on the volume.** In this section we provide a number of estimates on the volume of the random looptrees under consideration. Recall that $a_n = v(n)$.

Lemma 3.2. *For any $\gamma > 0$ there is a constant K_γ such that*

$$\mathbb{E}[V_n^{-\gamma}] \le K_\gamma a_n^{-\gamma}. \tag{3.17}$$

Proof. We will start by establishing the following: for every $\delta \in (0, 1/\alpha)$ there are constants $c_1, c_2, c_3, c_4 > 0$, possibly depending on δ, such that

$$\mathbb{P}(V_n < \lambda^{-1} a_n) \le c_1 \exp(-c_2 \lambda^{1/\alpha - \delta}) \quad \text{whenever } c_3 \le \lambda \le c_4 a_n. \tag{3.18}$$

We let $X^{(n)}$ denote the number of vertices of $\mathrm{Loop}(\tau)$ at distance at most n from the root of $\mathrm{Loop}(\tau)$, and we let $X_1^{(n)}, X_2^{(n)}, \ldots$ denote independent copies of $X^{(n)}$. Note that V_n dominates a sum $\sum_{i=1}^{\mathcal{R}_{\lfloor n/2 \rfloor}} X_i^{(\lfloor n/2 \rfloor)}$ with $\mathcal{R}_n$ from Lemma 2.6. Thus it suffices to show that there are constants $c_1', c_2', c_3', c_4' > 0$ such that

$$\mathbb{P}\Big(\sum_{i=1}^{\mathcal{R}_n} X_i^{(n)} < \lambda^{-1} a_n \Big) \le c_1' \exp(-c_2' \lambda^{1/\alpha - \delta}) \tag{3.19}$$

whenever $c_3' \le \lambda \le c_4' a_n$. Note that by Lemma 2.6 it holds that for $x \in [0,1]$

$$\mathbb{E}(x^{\mathcal{R}_n}) \le (1 - p(1-x))^{\lfloor n/2 \rfloor} \tag{3.20}$$

and $p > 0$. Let $t = \lambda a_n^{-1}$. Using Markov's inequality, the independence of the $X_i^{(n)}$'s and $\mathcal{R}_n$, Eq. (3.20), Lemma 2.4 and Eq. (3.8) we have for any $t > 0$ that

$$\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^{\mathcal{R}_n} X_i^{(n)} < \lambda^{-1} a_n \Big) &= \mathbb{P}\Big(\exp\Big(-t \sum_{i=1}^{\mathcal{R}_n} X_i^{(n)} \Big) > e^{-1} \Big) \\
&\le e \mathbb{E}\Big(\mathbb{E}\big[\exp\big(-t X^{(n)} \big) \big]^{\mathcal{R}_n} \Big) \\
&\le e \Big(1 - p\big(1 - \mathbb{E}\big[\exp\big(-t X^{(n)} \big) \big] \big) \Big)^{\lfloor n/2 \rfloor} \\
&\le e \Big(1 - p\big(1 - \big[\mathbb{E}(e^{-t|\tau|}) + \mathbb{P}(\mathrm{Height}(\mathrm{Loop}(\tau)) > n) \big] \big) \Big)^{\lfloor n/2 \rfloor} \\
&\le e \Big(1 - p\big((1-e^{-t})^{1/\alpha} L^*((1-e^{-t})^{-1}) - c/n \big) \Big)^{\lfloor n/2 \rfloor}.
\end{aligned} \tag{3.21}$$

Recall that $\lambda \le c_4' a_n$ for some constant $c_4' > 0$. By Lemma 4.1 there is a slowly varying function $\tilde{L}$ asymptotically equal to L^* such that $x^{-1/\alpha} \tilde{L}(x)$ is non-increasing. Hence, for any δ sufficiently small one may choose c_4' and hence t small enough such that

$$\begin{aligned}
(1-e^{-t})^{1/\alpha} L^*((1-e^{-t})^{-1}) &\ge \tfrac{1}{2} (1-e^{-t})^{1/\alpha} \tilde{L}((1-e^{-t})^{-1}) \\
&\ge k_1 t^{1/\alpha} \tilde{L}(2/t) = k_1 \frac{\lambda^{1/\alpha}}{n} \frac{n \tilde{L}(a_n)}{a_n^{1/\alpha}} \frac{\tilde{L}(\frac{2}{t})}{\tilde{L}(\frac{\lambda}{t})} \\
&\ge k_2 \frac{\lambda^{1/\alpha - \delta}}{n}
\end{aligned} \tag{3.22}$$

for some constants $k_1, k_2 > 0$. In the second step we used that $1-e^{-t} > t/2$ for t small enough, and in the last step we used (3.10) and Lemma 4.2.

Finally, for $\lambda \geq c_3'$ with c_3' large enough that $k_2\lambda^{1/\alpha-\delta} \geq c$ (the constant from Lemma 2.4) we have

$$\mathbb{P}\Big(\sum_{i=1}^{\mathcal{R}_n} X_i^{(n)} < \lambda^{-1}a_n\Big) \leq e(1-p(k_2\lambda^{1/\alpha-\delta}-c)/n)^{\lfloor n/2\rfloor} \leq e\exp\Big(-\frac{p}{3}(k_2\lambda^{1/\alpha-\delta}-c)\Big), \tag{3.23}$$

proving (3.19) and hence (3.18).

We now show that (3.18) implies (3.17), using an argument similar to one in [13]. Let m be a fixed integer large enough such that $c_3 \leq c_4 a_m$ and let $\gamma > 0$. Then by (3.18)

$$\begin{aligned}\mathbb{E}[V_n^{-\gamma}] &= \mathbb{E}\Big[V_n^{-\gamma}\mathbf{1}_{\{V_n^{-1}\leq c_4a_m/a_n\}}\Big] + \sum_{k=m}^{n-1}\mathbb{E}\Big[V_n^{-\gamma}\mathbf{1}_{\{c_4a_k/a_n<V_n^{-1}\leq c_4a_{k+1}/a_n\}}\Big] \\ &\quad + \mathbb{E}\Big[V_n^{-\gamma}\mathbf{1}_{\{V_n^{-1}>c_4a_n/a_n\}}\Big] \\ &\leq (c_4a_m)^\gamma a_n^{-\gamma} + c_1c_4^\gamma a_n^{-\gamma}\sum_{k=m}^{n-1} a_{k+1}^\gamma \exp(-c_2(c_4a_k)^{1/\alpha-\delta}) \\ &\quad + c_1\exp(-c_2(c_4a_n)^{1/\alpha-\delta}) \leq K_\gamma a_n^{-\gamma},\end{aligned} \tag{3.24}$$

as required. Here we used that a_n is increasing, and that from (3.11) and Lemma 4.1 it follows that for any $\epsilon > 0$ there are constants $C_1, C_2 > 0$ (possibly depending on ϵ) such that

$$C_1 n^{\alpha-\epsilon} < a_n < C_2 n^{\alpha+\epsilon} \tag{3.25}$$

so $\exp(-c_2(c_4a_n)^{1/\alpha-\delta})$ decays faster than any power of n. The sum in the second last line thus converges as $n\to\infty$ and the term following the sum is negligible. □

Applying Markov's inequality and using the preceding Lemma one finds that for every $\gamma > 0$ there is a constant $c_\gamma > 0$ such that for all $n \geq 0$ and all $\lambda > 1$ we have

$$\mathbb{P}(V_n < \lambda^{-1}a_n) \leq c_\gamma\lambda^{-\gamma}. \tag{3.26}$$

This establishes the part of (3.15) regarding the volume.

As in the preceeding proof, let $X^{(n)}$ denote the number of vertices of $\mathrm{Loop}(\tau)$ at distance at most n from the root. Recall that $f(s) = \mathbb{E}(s^\xi)$. Then the following holds.

Lemma 3.3. *As $n \to \infty$,*

$$\mathbb{E}(X^{(n)}) \sim Mn^{-1}a_n \tag{3.27}$$

where

$$M = \begin{cases} \frac{2f''(1)}{3+h(1)+f''(1)} & \text{if } f''(1) < \infty \\ \frac{2^{\alpha-1}}{\Gamma(\alpha)} & \text{otherwise,}\end{cases} \tag{3.28}$$

and

$$h(x) = \sum_{i=0}^{\infty} \pi_{2i+1} x^i. \tag{3.29}$$

Proof. One has $\mathbb{E}(X^{(0)}) = 1$ and using the independence structure of $\mathrm{Loop}(\tau)$ one immediately arrives at the recursion

$$\begin{aligned}
\mathbb{E}(X^{(n)}) &= \sum_{i=0}^{\infty} \pi_i \Big(1 + 2 \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor \wedge n} \mathbb{E}(X^{(n-j)}) + \mathbf{1}_{\{i \text{ is odd}\}} \mathbf{1}_{\{\lfloor \frac{i}{2} \rfloor + 1 \le n\}} \mathbb{E}(X^{(n - \lfloor \frac{i}{2} \rfloor - 1)})\Big) \\
&= 1 + 2 \sum_{i=0}^{2n-1} \pi_i \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \mathbb{E}(X^{(n-j)}) + 2 \sum_{i=2n}^{\infty} \pi_i \sum_{j=1}^{n} \mathbb{E}(X^{(n-j)}) \\
&+ \sum_{m=0}^{n-1} \pi_{2m+1} \mathbb{E}(X^{(n-m-1)}) = 1 + \sum_{j=1}^{n} \Big(2 \sum_{i=2j}^{\infty} \pi_i + \pi_{2j-1}\Big) \mathbb{E}(X^{(n-j)})
\end{aligned} \tag{3.30}$$

for $n \ge 1$, where in the last step we swapped the j and i sums and renamed $m = j - 1$ in the last sum.

Next, define the generating function

$$F(x) = \sum_{n=0}^{\infty} \mathbb{E}(X^{(n)}) x^n. \tag{3.31}$$

Multiplying both sides of (3.30) with x^n, summing over $n \ge 0$ and swapping the sum over n and j yields a simple equation for $F(x)$ which has solution

$$F(x) = \frac{1}{(1-x)(1-G(x))}. \tag{3.32}$$

with

$$G(x) = \sum_{j=1}^{\infty} \Big(2 \sum_{i=2j}^{\infty} \pi_i + \pi_{2j-1}\Big) x^j. \tag{3.33}$$

Swapping the j and i sum and some rewriting gives

$$G(x) = \frac{2x}{1-x} \Big(1 - f(x^{1/2}) - \frac{(1 - x^{1/2})^2}{2} h(x)\Big) \tag{3.34}$$

with f from (3.4) and h from (3.29). Inserting this expression into (3.32) gives

$$F(x)^{-1} = 1 - x - 2x\Big((1 - x^{1/2}) + (x^{1/2} - f(x^{1/2})) - \tfrac{1}{2}(1 - x^{1/2})^2 h(x)\Big). \tag{3.35}$$

Expanding the first term $1 - x^{1/2}$ to second order and applying (3.5) gives

$$F(x)^{-1} \sim (1-x)^2 (1 - \tfrac{x}{4}) + 2(1 - x^{1/2})^{\alpha} L\big((1 - x^{1/2})^{-1}\big) + (1 - x^{1/2})^2 h(x), \tag{3.36}$$

as $x \to 1^-$. By expanding the terms $1 - x^{1/2}$ to first order this yields

$$F(x) \sim (1-x)^{-\alpha} \Big((3/4 + h(1)/4)(1-x)^{2-\alpha} + 2^{1-\alpha} L((1-x)^{-1})\Big)^{-1} \tag{3.37}$$

as $x \to 1^-$. The expression in the large parentheses is clearly slowly varying as $x \to 1^-$, and $L((1-x)^{-1}) \to \frac{1}{2} f''(1)$ when $f''(1) < \infty$, by (3.6)–(3.7).

The result therefore follows by applying the Tauberian Theorem 4.5 along with (3.11) and the fact that $\mathbb{E}(X^{(n)})$ is increasing in n. □

From the preceding lemma we get the following bounds on $\mathbb{E}(V_n)$ which prove (1.6).

Lemma 3.4. *There are constants k_1 and k_2 such that*

$$k_1 a_n \leq \mathbb{E}(V_n) \leq k_2 a_n. \tag{3.38}$$

for all $n \geq 0$.

Proof. Let $\mathcal{C}$ be the loopspine of $\mathcal{L}$ and let A_n be defined as in (2.23). Let $(v_i)_i$ be a list of the vertices in $A_n \setminus \{x_j : j \geq 1\}$ and denote the finite outgrowth from v_i by $\mathrm{Loop}(\tau_i)$. Let $X_i^{(n)}$ be the number of vertices in $\mathrm{Loop}(\tau_i)$ at distance at most n from the root of $\mathrm{Loop}(\tau_i)$. Recall that $(\tau_i)_i$ is a sequence of independent Galton–Watson trees with offspring distribution ξ, which are furthermore independent of $|A_n|$. For the upper bound we note that

$$V_n \leq \sum_{i=1}^{|A_n|} X_i^{(n)} + |A_n| \tag{3.39}$$

and thus by Wald's lemma, Lemma 2.5 and Lemma 3.3

$$\mathbb{E}(V_n) \leq \mathbb{E}(|A_n|)\mathbb{E}(X_i^{(n)} + 1) \leq k_2 a_n. \tag{3.40}$$

For the lower bound we note that V_n dominates a sum $\sum_{i=1}^{\mathcal{R}_{\lfloor n/2 \rfloor}} X_i^{(\lfloor n/2 \rfloor)}$ with $\mathcal{R}_n$ from Lemma 2.6 obeying $\mathbb{E}(\mathcal{R}_n) \geq p\lfloor n/2 \rfloor$ and $p > 0$. Since $\mathcal{R}_{\lfloor n/2 \rfloor}$ is independent of the $X_i^{(\lfloor n/2 \rfloor)}$'s the result thus follows from Wald's lemma and Lemma 3.3. □

By applying Markov's inequality and using the previous Lemma one finds that there is a constant $c > 0$ such that for all $n \geq 1$ and all $\lambda > 1$

$$\mathbb{P}(V_n > \lambda a_n) \leq c\lambda^{-1}. \tag{3.41}$$

This establishes the part of (3.14) regarding the volume. Note that, since $\mathrm{R_{eff}}(\varnothing, B(n; \mathcal{L})^c) \leq n = r(n)$ always, the lemma also implies inequality (3.16).

3.3. **Bounds on the resistance.** Having dealt with the volume bounds in (3.14) and (3.15) we now turn to the bounds on the resistance. First we note that the upper bound is trivial: since $\mathrm{R_{eff}}(\varnothing, v) \leq d_{\mathcal{L}}(\varnothing, v)$ for all $v \in V(\mathcal{L})$ we have for all $n \geq 1$ and $\lambda > 1$ that

$$\mathbb{P}(\exists v \in B(n; \mathcal{L}) \ : \ \mathrm{R_{eff}}(\varnothing, v) > \lambda d_{\mathcal{L}}(\varnothing, v)) = 0, \tag{3.42}$$

which together with (3.26) proves (3.15). Therefore (3.14), and hence Theorem 3.1, follows once we prove the following lower bound on the resistance:

Lemma 3.5. *For any $q \in (0, \alpha - 1)$ there is a constant $c > 0$ such that*

$$\mathbb{P}(\mathrm{R_{eff}}(\varnothing, B(n; \mathcal{L})^c) < \lambda^{-1} n) \leq c\lambda^{-q}. \tag{3.43}$$

Proof. Let $(\hat{\xi}_i)_{i \geq 1}$ be a sequence of independent copies of $\hat{\xi}$, and as before let $(C_i)_{i \geq 1}$ be a sequence of cycles with lengths $(\hat{\xi}_i + 1)_{i \geq 1}$ with marked vertices x_i, joined together to form an infinite chain $\mathcal{C}$. Recall that $\mathcal{L}$ is formed by attaching outgrowths at the vertices $\mathcal{C} \setminus \{x_i : i \geq 1\}$, these being independent copies of $\mathrm{Loop}(\tau)$ where τ is a Galton–Watson tree with offspring distribution ξ.

In order to bound the resistance $\mathrm{R_{eff}}(\varnothing, B(n;\mathcal{L})^c)$ we aim to find a set of vertices in $\mathcal{C}$ of size at most 2 which (i) separates $\varnothing$ from all vertices of $\mathcal{L}$ at distance at least n from $\varnothing$, and (ii) is on average sufficiently far away from $\varnothing$ (see (3.46) for a precise statement). Note that if $v \in \mathcal{C}$ then $d_{\mathcal{L}}(\varnothing, v) = d_{\mathcal{C}}(\varnothing, v)$. We will consider only the cycles C_i up to and including the first one which contains a vertex at distance $\geq n/2$ from $\varnothing$. Writing $Y_i = \min\{U_i(\hat{\xi}_i), \hat{\xi}_i - U_i(\hat{\xi}_i) + 1\}$, with the $U_i(\hat{\xi}_i)$ uniform in $\{1, 2, ..., \hat{\xi}_i\}$ as in (2.25), the first C_i intersecting level $\geq n/2$ is C_{I_n}, where

$$I_n = \min\Big\{N : \sum_{i=1}^{N-1} Y_i + \lfloor \hat{\xi}_N/2 \rfloor \geq n/2\Big\}. \tag{3.44}$$

The truncated chain consisting of the cycles $(C_i)_{1\leq i\leq I_n}$ will be denoted by $\mathcal{C}_n$ and we write $\mathcal{C}'_n = \{v \in \mathcal{C}_n : d_{\mathcal{C}}(\varnothing, v) \leq n/2\}$. We will consider outgrowths from $\mathcal{C}'_n$ which have height $\geq n/2$ as candidates for outgrowths which reach level n. It is clear that no other outgrowths from $\mathcal{C}'_n$ can reach level n since their roots are at a distance $< n/2$ from $\varnothing$. The probability that an outgrowth has height $\geq n/2$ will be denoted by p_n, and by Lemma 2.4 we have $p_n \leq c/n$ for some constant $c > 0$. Before proceeding, we define for each vertex $v \in C_i \setminus \{x_i\}$ its 'mirror image' v' as the unique vertex not equal to v in C_i which has the same distance from x_i as v. If there is no such vertex (which may happen when v is at the 'tip' of the cycle) we take $v' = v$.

Now, to each vertex $v \in \mathcal{C} \setminus \{x_1, x_2, \ldots\}$ assign a *mark* if the outgrowth from v has height $\geq n/2$. Thus each vertex is marked independently with probability p_n. We denote the index of the first cycle to contain a mark by K, and consider two main cases. In the first case there is no marked vertex $v \in \mathcal{C}_n$ (i.e. $K > I_n$) and we define the separating set S_n either as $S_n = \{x_{I_n+1}, x'_{I_n+1}\}$ if $d_{\mathcal{C}}(\varnothing, x_{I_n+1}) < n/2$ (Fig. 2, (1a)) or the intersection of $\mathcal{C}_n$ with level $\lfloor n/2 \rfloor$ otherwise (Fig. 2, (1b)). In the second case there is some marked vertex in $\mathcal{C}_n$ (i.e. $K \leq I_n$). We then define the separating set as $S_n = \{x_K\}$ if $K > 1$ (Fig. 2, (2a)) but as the set of neigbours of $x_1 = \varnothing$ if $K = 1$ (Fig. 2, (2b)).

Note that S_n consists of either 1 or 2 vertices, both at the same distance from $\varnothing$. We denote this distance by D_n and it holds that

$$D_n = \begin{cases} \max\{d_{\mathcal{C}}(x_K, \varnothing), 1\} & \text{if } K \leq I_n \\ \min\{d_{\mathcal{C}}(x_{I_n+1}, \varnothing), \lfloor n/2 \rfloor\} & \text{otherwise.} \end{cases} \tag{3.45}$$

We aim to show that for each $\epsilon > 0$ there is a constant $a > 0$ such that

$$\mathbb{P}(D_n \leq i) \leq a((i/n)^{\alpha-1-\epsilon} \wedge 1) + \mathbf{1}_{\{\lfloor n/2 \rfloor \leq i\}}. \tag{3.46}$$

It follows from (3.46) that if $0 < q < \alpha - 1 - \epsilon$ then

$$\begin{aligned} \mathbb{E}(D_n^{-q}) &\leq q \sum_{i=1}^{\infty} i^{-q-1} \mathbb{P}(D_n \leq i) \\ &\leq aq \sum_{i=1}^{n} i^{-q+\alpha-2-\epsilon}/n^{\alpha-1-\epsilon} + q \sum_{i=n+1}^{\infty} i^{-q-1} + q \sum_{i=\lfloor n/2 \rfloor}^{\infty} i^{-q-1} \leq c' n^{-q} \end{aligned} \tag{3.47}$$

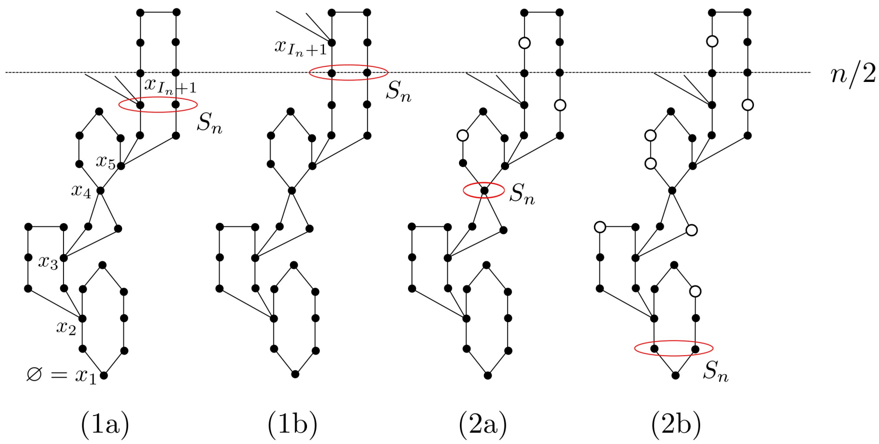

FIGURE 2. Example of a chain $\mathcal{C}_n$ with marks (white circles $\circ$) and the different choices of the separating set S_n.

for some constant $c' > 0$. Since S_n separates $\varnothing$ from $B(n;\mathcal{L})^c$ it follows that

$$\mathrm{R}_{\mathrm{eff}}(\varnothing, B(n;\mathcal{L})^c) \geq \mathrm{R}_{\mathrm{eff}}(\varnothing, S_n) \tag{3.48}$$

and by the series and parallel laws we have

$$\mathrm{R}_{\mathrm{eff}}(\varnothing, S_n) \geq D_n/2. \tag{3.49}$$

It follows that

$$\mathbb{E}(\mathrm{R}_{\mathrm{eff}}(\varnothing, B(n;\mathcal{L})^c)^{-q}) \leq \mathbb{E}((D_n/2)^{-q}) \leq c'' n^{-q} \tag{3.50}$$

and hence by Markov's inequality that

$$\begin{aligned}\mathbb{P}(\mathrm{R}_{\mathrm{eff}}(\varnothing, B(n;\mathcal{L})^c) < \lambda^{-1} n) &= \mathbb{P}(\mathrm{R}_{\mathrm{eff}}(\varnothing, B(n;\mathcal{L})^c)^{-q} > \lambda^q n^{-q}) \\ &\leq \frac{n^q \mathbb{E}(\mathrm{R}_{\mathrm{eff}}(\varnothing, B(n;\mathcal{L})^c)^{-q})}{\lambda^q} \leq c\lambda^{-q},\end{aligned} \tag{3.51}$$

as claimed.

We now prove (3.46). Clearly

$$\mathbb{P}(D_n \leq i) = \sum_{k\geq 1} \mathbb{P}(D_n \leq i; K = k; k \leq I_n) + \mathbb{P}(D_n \leq i; K > I_n). \tag{3.52}$$

Start by considering the case $K > I_n$ and first assume that $d_{\mathcal{C}}(x_{I_n+1}, \varnothing) > \lfloor n/2 \rfloor$ in which case $D_n = \lfloor n/2 \rfloor$. Then

$$\mathbb{P}(D_n \leq i; d_{\mathcal{C}}(x_{I_n+1}, \varnothing) > \lfloor n/2 \rfloor; K > I_n) \leq \mathbf{1}_{\{\lfloor n/2 \rfloor \leq i\}}. \tag{3.53}$$

Secondly (still with $K > I_n$), assume $d_{\mathcal{C}}(x_{I_n+1}, \varnothing) \leq \lfloor n/2 \rfloor$ in which case $D_n = d_{\mathcal{C}}(x_{I_n+1}, \varnothing)$. Write

$$a = d_{\mathcal{C}}(\varnothing, x_{I_n}) \qquad b = \max\{d_{\mathcal{C}}(\varnothing, u) : u \in C_{I_n}\}. \tag{3.54}$$

and note that $b \geq n/2$ by definition of I_n. Conditional on $\mathcal{C}_n$, the point x_{I_n+1} is chosen uniformly on $C_{I_n} \setminus \{x_{I_n}\}$. Since $\mathbb{P}(d_{\mathcal{C}}(\varnothing, x_{I_n+1}) \leq i \mid \mathcal{C}_n) = 0$ if $i \leq a$ we may assume that $a + 1 \leq i \leq b$. The number of vertices on C_{I_n} at distance between $a+1$ and i from $\varnothing$ is at most $2(i-a)$, and the number of vertices on C_{I_n} at distance between $a+1$ and b from $\varnothing$ is at least $b-a$. Hence

$$\mathbb{P}(d_{\mathcal{C}}(\varnothing, x_{I_n+1}) \leq i \mid \mathcal{C}_n) \leq \frac{2(i-a)}{b-a} \leq \frac{2i}{b} \leq 4i/n. \tag{3.55}$$

It follows that

$$\mathbb{P}(D_n \leq i; d_{\mathcal{C}}(x_{I_n+1}, \varnothing) \leq \lfloor n/2 \rfloor; K > I_n) \leq (4i/n) \wedge 1.$$

Finally, consider the case $K \leq I_n$. On the event $\{K = k\}$ we then have that $D_n \geq d_{\mathcal{C}}(\varnothing, x_k) = \sum_{j=1}^{k-1} Y_j$. Therefore

$$\begin{aligned}\sum_{k\geq 1} \mathbb{P}(D_n \leq i; K = k; k \leq I_n) &\leq \sum_{k\geq 1} \mathbb{P}\Big(\sum_{j=1}^{k-1} Y_j \leq i;\ \exists \text{ mark in } C_k\Big) \\ &= \mathbb{E}(1-(1-p_n)^{\hat{\xi}-1}) \sum_{k\geq 1} \mathbb{P}\Big(\sum_{j=1}^{k-1} Y_j \leq i\Big).\end{aligned} \tag{3.56}$$

For the last equality we used the fact that C_k and the process of marks in it is independent of the C_j for $j < k$. To estimate the last expression we will use some results from the Appendix. Using Lemma 4.8 we find that there is a constant C such that

$$\mathbb{E}(1-(1-p_n)^{\hat{\xi}-1}) \leq C n^{1-\alpha} L_1(n), \tag{3.57}$$

where L_1 is the function in (3.2). By Lemma 4.9

$$\mathbb{E}(1-e^{-Y_j/i}) \geq \frac{1}{8} i^{1-\alpha} L_1(i). \tag{3.58}$$

Thus the sum in (3.56) may be estimated as follows

$$\begin{aligned}\sum_{k\geq 1} \mathbb{P}\Big(\sum_{j=1}^{k-1} Y_j \leq i\Big) &= \sum_{k\geq 1} \mathbb{P}\Big(\exp\big(-\sum_{j=1}^{k-1} Y_j/i\big) \geq e^{-1}\Big) \\ &\leq e \sum_{k=1}^{\infty} \mathbb{E}(e^{-Y_j/i})^{k-1} = \frac{e}{1-\mathbb{E}(e^{-Y_j/i})} \\ &\leq 8e i^{\alpha-1} L_1(i)^{-1}.\end{aligned}$$

We have thus shown that

$$\sum_{k\geq 1} \mathbb{P}(D_n \leq i; K = k; k \leq I_n) \leq \left(8eC\left(\frac{i}{n}\right)^{\alpha-1} \frac{L_1(n)}{L_1(i)}\right) \wedge 1. \tag{3.59}$$

Finally, for any $\epsilon > 0$ we have from Lemma 4.2 that there is a constant $C' > 0$ such that for $i \leq n$

$$\frac{L_1(n)}{L_1(i)} \leq C' \left(\frac{i}{n}\right)^{-\epsilon}, \tag{3.60}$$

which gives (3.46). □

Acknowledgement. The authors would like to thank Svante Janson for several interesting discussions about slowly varying functions and stable distributions. SÖS is grateful for hospitality at NORDITA. The authors also thank the anonymous referees whose comments and corrections helped improve the paper.

4. Appendix

In this appendix we collect some results about slowly varying functions and random variables with regularly varying tails. No doubt many of the results stated here are well-known or follow straightforwardly from well-known results, but we include them for completeness.

4.1. **Results on slowly varying functions.** The following lemma is quoted from [22, Section 1.5] (see 1° and the comment on the proof of 5° on p. 23).

Lemma 4.1. *Let L be slowly varying at infinity.*

(1) *For any $\epsilon > 0$ there are constants $x_0, C_1, C_2 > 0$ (possibly depending on ϵ) such that*
$$C_1 x^{-\epsilon} < L(x) < C_2 x^{\epsilon} \tag{4.1}$$
for all $x > x_0$.

(2) *For any $\delta > 0$ there are slowly varying functions $\overline{L}$ and $\underline{L}$ such that (i) $\overline{L}$ and $\underline{L}$ are asymptotically equal to L, (ii) $x^{\delta}\overline{L}(x)$ is strictly increasing in x, and (iii) $x^{-\delta}\underline{L}(x)$ is strictly decreasing in x.*

The following result is a consequence of the second part of Lemma 4.1.

Lemma 4.2. *If L is slowly varying at infinity then for any $\delta > 0$ there exist constants $x_0, C > 0$ (possibly depending on δ) such that*
$$\frac{L(yx)}{L(x)} < C y^{-\delta} \tag{4.2}$$
for all $x > x_0$ and all $y \in (x_0/x, 1]$. Similarly, for any $\delta > 0$ there are constants $x_0', C' > 0$ such that
$$\frac{L(yx)}{L(x)} < C' y^{\delta} \tag{4.3}$$
for all $x > x_0$ and all $y \geq 1$.

The next lemma gives asymptotic expressions for integrals of regularly varying functions.

Lemma 4.3. *Assume that $R(x) \sim x^{-\alpha} L(x)$ where $\alpha \in \mathbb{R}$ and L is slowly varying at infinity. If $\alpha \leq 1$ then*
$$\int_1^y R(x)dx \sim y^{1-\alpha}\hat{L}(y) \quad \text{as } y \to \infty \tag{4.4}$$
where $\hat{L}$ is slowly varying at infinity. Furthermore, if $\alpha < 1$ one may take
$$\hat{L}(n) = \frac{1}{1-\alpha} L(n). \tag{4.5}$$
If $\alpha > 1$ then
$$\int_y^{\infty} R(x)dx \sim \frac{1}{\alpha - 1} y^{1-\alpha} L(y) \qquad \text{as } y \to \infty. \tag{4.6}$$

Proof. We prove the first part, the second part may be proved in a similar way. For any $\epsilon > 0$ there is an $n_0 > 0$ such that

$$(1-\epsilon)\int_{n_0}^{n} x^{-\alpha}L(x)dx < \int_{n_0}^{n} R(x)dx < (1+\epsilon)\int_{n_0}^{n} x^{-\alpha}L(x)dx \tag{4.7}$$

First assume that $\alpha < 1$. In that case, we are done if we show that

$$\int_{n_0}^{n} x^{-\alpha}L(x)dx \sim \frac{1}{1-\alpha}n^{1-\alpha}L(n). \tag{4.8}$$

By changing variables in the integral to $y = x/n$ we find that

$$(n^{1-\alpha}L(n))^{-1}\int_{n_0}^{n} x^{-\alpha}L(x)dx = \int_{n_0/n}^{1} y^{-\alpha}\frac{L(yn)}{L(n)}dy. \tag{4.9}$$

Since $y \leq 1$ then by the first part of Lemma 4.2 it holds that for any $\delta > 0$ there is an n_1 such that

$$\frac{L(yn)}{L(n)} < Cy^{-\delta}. \tag{4.10}$$

for all $n > n_1$ and all $y \in (n_1/n, 1]$. Choosing $n > n_0 > n_1$ and δ small enough such that $\alpha + \delta < 1$ allows us to dominate the integrand in the last expression in (4.9) by an integrable function on $[0, 1]$ and thus we may use the dominated convergence theorem and

$$\begin{aligned}\lim_{n\to\infty}(n^{1-\alpha}L(n))^{-1}\int_{n_0}^{n} x^{-\alpha}L(x)dx &= \int_{0}^{1} y^{-\alpha}\lim_{n\to\infty}\frac{L(yn)}{L(n)}dy \\ &= \int_{0}^{1} y^{-\alpha}dy = \frac{1}{1-\alpha}.\end{aligned} \tag{4.11}$$

When $\alpha = 1$ we need to show that

$$F(n) := \int_{n_0}^{n} \frac{1}{x}L(x)dx \tag{4.12}$$

is slowly varying as $n \to \infty$. This is trivially true if $F(n)$ converges as $n \to \infty$. Thus, we will in the following assume that $F(n)$ diverges as $n \to \infty$. Now, fix a $\lambda > 0$ and choose $n_0 = m_0\lambda$ for some $m_0 > 1$. Then

$$F(\lambda n) = \int_{n_0}^{\lambda n} \frac{1}{x}L(x)dx = \int_{m_0}^{n} \frac{1}{y}L(\lambda y)dy = \int_{m_0}^{n} \frac{1}{x}L(x)\frac{L(\lambda x)}{L(x)}dx. \tag{4.13}$$

For any $\epsilon > 0$ one may choose m_0 large enough such that

$$1-\epsilon < \frac{L(\lambda x)}{L(x)} < 1+\epsilon \tag{4.14}$$

for all $x > m_0$ and thus

$$1-\epsilon \leq \liminf_{n\to\infty}\frac{F(\lambda n)}{F(n)} \leq \limsup_{n\to\infty}\frac{F(\lambda n)}{F(n)} \leq 1+\epsilon, \tag{4.15}$$

Finally, send $\epsilon \to 0$ to get the desired result. □

The following lemma follows from Lemma 4.3 by comparing the sums with integrals and using the second part of Lemma 4.1.

Lemma 4.4. *If $R(n) \sim n^{-\alpha}L(n)$ where $\alpha \in (-\infty, 1]$ and L is slowly varying at infinity then*

$$\sum_{i=1}^{n} R(i) \sim n^{1-\alpha}\hat{L}(n) \tag{4.16}$$

where $\hat{L}$ is slowly varying at infinity. Furthermore, if $\alpha < 1$ then

$$\hat{L}(n) = \frac{1}{1-\alpha}L(n). \tag{4.17}$$

Similarly, if $R(n) \sim n^{\alpha-2}L(n)$ where $\alpha \in (-\infty, 1)$ and L is slowly varying at infinity then

$$\sum_{i=n}^{\infty} R(i) \sim \frac{1}{1-\alpha}n^{\alpha-1}L(n). \tag{4.18}$$

4.2. **Random variables with regularly varying tails.** The following Tauberian theorem is essential in the study of random variables in the domain of attraction to a stable distribution. A proof may be found in [12, Thm. XIII.5.5]

Theorem 4.5. *Let $q_n \geq 0$ and suppose that*

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n \tag{4.19}$$

converges for $0 \leq s < 1$. If L is slowly varying at infinity and $\rho \geq 0$ then the following two relations are equivalent:

$$Q(s) \sim \frac{1}{(1-s)^\rho} L\left(\frac{1}{1-s}\right) \qquad \text{as } s \to 1^- \tag{4.20}$$

and

$$\sum_{i=0}^{n} q_i \sim \frac{1}{\Gamma(\rho+1)}n^\rho L(n) \qquad \text{as } n \to \infty. \tag{4.21}$$

Furthermore, if q_n is monotone and $\rho > 0$ then (4.20) is equivalent to

$$q_n \sim \frac{1}{\Gamma(\rho)}n^{\rho-1}L(n) \qquad \text{as } n \to \infty. \tag{4.22}$$

In what follows we let ξ be a random variable taking values in the non-negative integers and belonging to the domain of attraction of a stable distribution with index $\alpha \in (0, 2]$. We let L_1 be a slowly varying function so that

$$\mathbb{E}(\xi^2 \mathbf{1}_{\{\xi \leq n\}}) = n^{2-\alpha}L_1(n), \tag{4.23}$$

cf (3.2).

Using Theorem 4.5 we may arrive at the following asymptotic expressions for the probability generating functions of random variables in the domain of attraction of a stable law. We leave out the details of the proofs of the following two results.

Lemma 4.6. *If $\alpha \in (0, 1]$ then*

$$1 - \mathbb{E}(s^\xi) \sim \frac{\Gamma(3-\alpha)}{\alpha}(1-s)^\alpha \hat{L}\left(\frac{1}{1-s}\right) \qquad \text{as } s \to 1^- \tag{4.24}$$

where $\hat{L}$ varies slowly at infinity. If $\alpha < 1$ then

$$\hat{L}(x) = \frac{1}{1-\alpha} L_1(x). \tag{4.25}$$

Proof. This follows from letting $q_n = \mathbb{P}(\xi > n)$ and applying (3.3), Lemma 4.4 and Theorem 4.5. ☐

Lemma 4.7. *If $\mathbb{E}(\xi) = 1$ and $\alpha \in (1, 2]$ then*

$$\mathbb{E}(s^\xi) - s \sim (1-s)^\alpha L\left(\frac{1}{1-s}\right) \qquad \text{as } s \to 1^-. \tag{4.26}$$

where

$$L(x) = \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} L_1(x) - \frac{1}{2} x^{\alpha-2} \tag{4.27}$$

varies slowly at infinity. (The second term on the right is always negligible when $\alpha < 2$.)

Proof. Let $q_n = n^2 \mathbb{P}(\xi = n)$. Then by (4.23),

$$\sum_{i=0}^{n} q_i = \mathbb{E}(\xi^2 \mathbf{1}_{\{\xi \le n\}}) = n^{2-\alpha} L(n). \tag{4.28}$$

Define

$$Q(s) := \sum_{n=0}^{\infty} q_n s^n = \mathbb{E}(\xi^2 s^\xi) = s \frac{d}{ds}\left(s \frac{d}{ds} \mathbb{E}(s^\xi)\right). \tag{4.29}$$

Then

$$\begin{aligned} \mathbb{E}(s^\xi) &= 1 - \int_s^1 y^{-1} \left(1 - \int_y^1 x^{-1} Q(x) dx\right) dy \\ &= 1 + \log(s) + \int_s^1 y^{-1} \left(\int_y^1 x^{-1} Q(x) dx\right) dy, \end{aligned} \tag{4.30}$$

where we used that $\mathbb{E}(\xi) = 1$. It follows from (4.28) and Thm. 4.5 that

$$Q(s) \sim \Gamma(3-\alpha)(1-s)^{\alpha-2} L\left(\frac{1}{1-s}\right) \qquad \text{as } s \to 1^-. \tag{4.31}$$

Thus, applying Lemma 4.3 twice to (4.30) and expanding $\log(s)$ to second order around $s = 1$ yields the desired result. ☐

The following results are more specific to the situation considered here so we include the details. The first result concerns the size-biased distribution of ξ.

Lemma 4.8. *Suppose $\mathbb{E}(\xi) = 1$ and $\alpha \in (1, 2]$. Then $\hat{\xi}$ defined by $\mathbb{P}(\hat{\xi} = i) = i\mathbb{P}(\xi = i)$ satisfies for any $c > 0$*

$$\mathbb{E}(1 - (1 - cn^{-1})^{\hat{\xi}}) \le C n^{1-\alpha} L_1(n) \tag{4.32}$$

with $C > 0$ a constant.

Proof. We may write

$$\begin{aligned}\mathbb{E}(1-(1-cn^{-1})^{\hat{\xi}}) &= \mathbb{E}((1-(1-cn^{-1})^{\hat{\xi}})\mathbf{1}_{\{\hat{\xi}>n\}}) \\ &\quad + \mathbb{E}((1-(1-cn^{-1})^{\hat{\xi}})\mathbf{1}_{\{\hat{\xi}\leq n\}}) \\ &\leq \mathbb{P}(\hat{\xi}>n) + cn^{-1}\mathbb{E}(\hat{\xi}\mathbf{1}_{\{\hat{\xi}\leq n\}}) \\ &= \mathbb{P}(\hat{\xi}>n) + cn^{-1}\mathbb{E}(\xi^2\mathbf{1}_{\{\xi\leq n\}}) \qquad (4.33)\end{aligned}$$

Let us now consider the first term in the last expression. We find that

$$\begin{aligned}\mathbb{P}(\hat{\xi}>n) = \sum_{i=n+1}^{\infty} i\mathbb{P}(\xi=i) &= n\mathbb{P}(\xi>n) + \sum_{i=n}^{\infty}(i-n)\mathbb{P}(\xi=i) \\ &= n\mathbb{P}(\xi>n) + \sum_{i=n}^{\infty}\sum_{j=n}^{i-1}\mathbb{P}(\xi=i) \\ &= n\mathbb{P}(\xi>n) + \sum_{j=n}^{\infty}\sum_{i=j+1}^{\infty}\mathbb{P}(\xi=i) \\ &= n\mathbb{P}(\xi>n) + \sum_{j=n}^{\infty}\mathbb{P}(\xi>j). \qquad (4.34)\end{aligned}$$

By [12, XVII.5, Eq. (5.16)] we have that for any $\epsilon>0$ there is an $n_0>0$ such that

$$\mathbb{P}(\xi>n) \leq \frac{2-\alpha+\epsilon}{\alpha} n^{-2}\mathbb{E}(\xi^2\mathbf{1}_{\{\xi\leq n\}}) \qquad (4.35)$$

for all $n\geq n_0$. By Lemma 4.4 it holds that

$$\sum_{j=n}^{\infty} j^{-2}\mathbb{E}(\xi^2\mathbf{1}_{\{\xi\leq j\}}) = \sum_{j=n}^{\infty} j^{-\alpha}L_1(j) \sim \frac{1}{\alpha-1}n^{1-\alpha}L_1(n). \qquad (4.36)$$

It now follows that there is a constant $c_1>0$ such that

$$\mathbb{P}(\hat{\xi}>n) \leq c_1 n^{1-\alpha}L_1(n) \qquad (4.37)$$

for all $n>0$, which completes the proof. □

Lemma 4.9. *Let $\hat{\xi}$ be defined as in Lemma 4.8 and define*

$$Y = \min\{U(\hat{\xi}), \hat{\xi}-U(\hat{\xi})+1\} \qquad (4.38)$$

where given $\hat{\xi}$, $U(\hat{\xi})$ is chosen uniformly from $\{1,\ldots,\hat{\xi}\}$. Then

$$\mathbb{E}(1-e^{-Y/n}) \geq \frac{1}{8n}\mathbb{E}(\xi^2\mathbf{1}_{\{\xi\leq n\}}) = \frac{1}{8}n^{1-\alpha}L_1(n). \qquad (4.39)$$

Proof. For $x\in[0,1]$ it holds that

$$1-e^{-x} \geq \frac{x}{2} \qquad (4.40)$$

and thus

$$\mathbb{E}(1-e^{-Y/n}) \geq \mathbb{E}((1-e^{-Y/n})\mathbf{1}_{\{Y\leq n\}}) \geq \frac{1}{2n}\mathbb{E}(Y\mathbf{1}_{\{Y\leq n\}}). \qquad (4.41)$$

We may then write

$$\begin{aligned}\mathbb{E}(Y\mathbf{1}_{\{Y\le n\}}) &= \mathbb{E}(\mathbb{E}(Y\mathbf{1}_{\{Y\le n\}} \mid \hat{\xi}))\\ &= \mathbb{E}\left(\frac{1}{\hat{\xi}}\sum_{i=1}^{\hat{\xi}}(i\wedge(\hat{\xi}-i+1))\mathbf{1}_{\{i\wedge(\hat{\xi}-i+1)\le n\}}\right)\\ &\ge \mathbb{E}\left(\frac{1}{\hat{\xi}}\mathbf{1}_{\{\hat{\xi}\le n\}}\sum_{i=1}^{\hat{\xi}}(i\wedge(\hat{\xi}-i+1))\right)\\ &\ge \frac{1}{4}\mathbb{E}(\hat{\xi}\mathbf{1}_{\{\hat{\xi}\le n\}}) = \frac{1}{4}\mathbb{E}(\xi^2\mathbf{1}_{\{\xi\le n\}}). \end{aligned} \tag{4.42}$$

□

Finally we prove the following:

Lemma 4.10. *Let τ be a Galton–Watson tree with critical offspring distribution ξ satisfying* (4.26) *and let $|\tau|$ denote the total number of individuals in τ. Then*

$$\mathbb{E}(s^{|\tau|}) = 1-(1-s)^{1/\alpha}L^*((1-s)^{-1}) \qquad s\in[0,1). \tag{4.43}$$

where L^* satisfies

$$\lim_{y\to\infty} L^*(y)^\alpha L(y^{\frac{1}{\alpha}}L^*(y)^{-1}) = 1 \tag{4.44}$$

which further implies that it is slowly varying at infinity.

Proof. We define L^* by (4.43) and show that it is slowly varying. It is straightforward to show that

$$\mathbb{E}(s^{|\tau|}) = sf(\mathbb{E}(s^{|\tau|})), \tag{4.45}$$

where $f(s)=\mathbb{E}(s^\xi)$. Using (4.26) and (4.45) gives

$$L^*(y)^\alpha L(y^{\frac{1}{\alpha}}L^*(y)^{-1}) = (1-y^{-1})^{-1}(1-y^{-\frac{1}{\alpha}}L^*(y)). \tag{4.46}$$

Note that when $s=1$ the right hand side of (4.43) should be understood as the limit as $s\to 1^-$ which equals $\mathbb{E}(1)=1$. Thus, L^* satisfies (4.44). Let

$$L_1(y) = L(y^{\frac{1}{\alpha}})^{-1} \tag{4.47}$$

which is clearly slowly varying at infinity and let

$$L_2(y) = L^*(y)^{-\alpha}. \tag{4.48}$$

Then (4.44) is equivalent to

$$\lim_{y\to\infty} L_2(y)L_1(yL_2(y)) = 1. \tag{4.49}$$

Therefore, L_2 is slowly varying by [22, Theorem 1.5] and thus $L^* = L_2^{-1/\alpha}$ is also slowly varying. □

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