MPC with Externally-Triggered Dual Switching in Dynamics and Disturbance

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Abstract— **This work explores systems with an additive disturbance where an imperfect preview is provided. Two external switching signals alter the structure and constraints of the system – one switches the system's dynamics, constraints, and/or cost function, the other switches an size of error allowed in the preview. This system is motivated by applications to distributed systems with external switching. The feasibility of such a system cannot be ensured using current research and naive expansions require set operations in high dimensions. This work presents a techniques to reformat the problem as a linear disturbed problem and reduce the dimension of the required set operations. The resulting controller is persistently feasible under all possible combinations of both switching signals.**

UPDATE HORIZON SO THAT IT IS CONSISTENT ACROSS SWITCHES AND NODES

MAKE SURE TO REMOVE THE MATHFRAK W AND REPLACE WITH P

I. Introduction

This work explores externally triggered switched linear systems under model predictive control (MPC) that are provided with a preview of future disturbances. The preview is imperfect though this is nominally by a relatively small amount. Occasionally, however, the preview can have a much larger error. The system, then, has two sources of switching. The first is in the local dynamics while the second is in the allowable preview error set.

A system receiving a preview of future disturbances has been used in a number of fields to improve system performance. It has been used style of preview arises in certain distributed control frameworks where the individual nodes receive a preview from their neighbors to better select their inputs [**?**].

When the nodes are allowed to switch by some external signal, the preview may change suddenly. This generates two

Model predictive control (MPC) has shown itself to be a valuable control scheme in a wide variety of applications. Its value stems from its inherent near-optimal nature and constraint satisfaction. These desirable properties come at the cost, however, of increased complexity. This complexity makes the application of MPC to certain classes of systems difficult. For this reason, much of the recent work in MPC during recent years has focused on overcoming this complexity burden in a wide variety of applications.

One such class of systems are "imperfect" distributed systems. Two possible sources of these imperfections are sudden changes to the systems dynamics and additive disturbances. Thus far, the literature has examined any

In [**?**], the authors apply MPC to unswitched, disturbed, linear systems with a preview of the system's disturbances. The authors were also motivated by this system's application to distributed MPC. This work established the system's feasibility using bounds on the system's cost function's max rate of change.

Distributed MPC with switching was handled in [**?**]. The authors used the principles of decentralized control were the effects of a node's neighbors are treated as an unknown disturbance. The set that this disturbance could be drawn experienced external switching when the communication topology changed. This work relies on weak coupling so the disturbance sets are not prohibitively large.

II. Preliminaries

A. Notation

- Blackboard font is used to denote a vector space
- Calligraphy font is used to denote subsets. In this work, these usually denote polyhedron
- Given an ordered vector space, $\mathbb{S}$ and elements $a, b \in \mathbb{S}$, the subset $\mathbb{S}_{[a,b]}$ is the set from a to b inclusive To denote exclusion of the bounds, parenthesis are used instead of brackets. $\mathbb{S}_{\geq a}$ is the set of all elements of $\mathbb{S}$ that satisfy the inequality.
- The Fraktur font is used to denote a sequence of vectors. For example, $\mathfrak{u} = \{u_0, ..., u_{N-1}\}$ is a sequence of inputs.

B. MPC Overview

Given a constrained system, $x(t+1) = f(x(t), u(t))$, state constraints, $x \in \mathcal{X}$, and input constraints, $u \in \mathcal{U}$, a model predictive controller will look for the best input sequence over the next $N \in \mathbb{Z}_{\geq 1}$ time steps. To do this, it uses the model of the system to generate predictions of future states. The set of allowable input sequences that respect the system's constraints over the trajectory starting at x_0 is denoted $\mathcal{U}^N(x_0)$. Given a feasible input sequence, $\mathfrak{u} \in \mathcal{U}^N(x(t))$, and initial condition, $x(t)$, the predicted state after $k \in \mathbb{Z}_{[0,N]}$ steps is denoted $\phi_\sigma(x(t), \mathfrak{u} : k)$ or $x_{k|t} \triangleq \phi_\sigma(x(t), \mathfrak{u} : k)$ when the input sequence and initial value are clear from context.

In addition to the system model, the MPC controller is equipped with an objective function that quantifies the merits of an input trajectory and its resulting state trajectory. The

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one considered here is

$$J_{\sigma_x}(x_0, \mathfrak{u}) \triangleq \underbrace{x_{N|t}^T P x_{N|t}}_{\text{Terminal Cost}} + \sum_{k=0}^{N-1} \underbrace{x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t}}_{\text{Running Cost}}. \tag{1}$$

The weight matrices, $Q,\ R$ and P, are assumed to be positive definite. MPC uses this objective function to find an optimal input sequence, $\mathfrak{u}^*(x_0) \in \mathcal{U}^N(x_0)$, that minimizes $J(x_0, \mathfrak{u})$ by solving

$$J^*(x_0) = \min_{\mathfrak{u} \in \mathcal{U}^N(x_0)} J(x_0, \mathfrak{u}) \tag{2}$$

Denote the set of all states at which (**??**) is feasible as $\mathcal{F} \subseteq \mathcal{X}$. After finding $\mathfrak{u}^*(x_0)$, its first element is applied to the system. Then (**??**) is re-solved at the new state and the process is repeated.

C. System Description

This work explores systems taking the form

$$x(t+1) = A_{\sigma_x(t)} x(t) + B_{\sigma_x(t)} u(t) + E p_0(t) \in \mathbb{R}^n \tag{3}$$

where $p_0(t) \in \mathcal{P}_0 \subset \mathbb{R}^m$ is some bounded disturbance and the dynamics and input matrices switch between M modes according to $\sigma_x : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{[1,M]}$. For brevity, the switching signal's explicit time dependence will be dropped in the future. The system has both state constraints, $x(\cdot) \in \mathcal{X}_{\sigma_x}$, and input constraints, $u(\cdot) \in \mathcal{U}_{\sigma_x}$. Note that the constraints also switch according to $\sigma_x(t)$. Additionally, the objective cost weights are also allowed to switch. In total, each mode can be defined as the tuple,

$$\mathcal{M}_i \triangleq \{\underbrace{A_i, B_i}_{\text{Dynamics}}, \underbrace{\mathcal{X}_i, \mathcal{U}_i}_{\text{Constraints}}, \underbrace{Q_i, R_i, P_i}_{\text{Cost Weights}}\}$$

At time $t \in \mathbb{Z}_{\geq 0}$, the system receives a preview of the next N disturbances, $\mathfrak{p}(t) = \{p_0(t), ..., p_N(t)\}$. The next previewed disturbance to be "felt" by the system, $p_0(t)$, is called the system's immediate preview. The preview evolves according to the disturbed dynamics

$$\begin{aligned} p_i(t+1) &= p_{i+1}(t) + \omega_i(t) \quad \forall i \in \mathbb{Z}_{[0,N-1]} \\ p_N(t+1) &= G_f p_N(t) + \omega_N(t) \end{aligned} \tag{4}$$

where $\omega_i \in \mathcal{W}_{\sigma_p}$ is a bounded error on the disturbance preview. The error set, $\mathcal{W}_{\sigma_p}$, switches between a nominal set, $\mathcal{W}_n$ and a larger set, $\mathcal{W}_l$ according to a second switching signal, $\sigma_p : \mathbb{Z}_{\geq 0} \to \{s, l\}$. The preview is constrained within the set $\mathcal{P} = \{\mathcal{P}_0 \times ... \times \mathcal{P}_N\}$ where $\mathcal{P}_{i-i} \subseteq \mathcal{P}_i$ and $\mathcal{P}_N$ is positive invariant under $G_f \in \mathbb{R}^{m \times m}$.

The switching signals, $\sigma_x(t)$ and $\sigma_p(t)$, are external to the system and provide no preview. Their values at the current time, however, are known immediately and without error. Furthermore, they are constrained both in the rate of switching and the possible switches. The constraints on σ_x are given as the mode dependent minimum dwell times (M-MDT), $d_x = \{d_{x,0}, ..., d_{x,M}\} \in \{\mathbb{Z}_{\geq 0}\}^M$. These dwell times imply that if $\sigma_x(t-1) \neq \sigma_x(t) = i$, then $\sigma_x(t+\tau) = i \;\forall\, \tau \in \mathbb{Z}_{[0,d_{x,i}]}$. In words, the system will dwell for $d_{x,i}$ time steps before switching out of mode i.

The preview error's switching signal has both a minimum dwell time, $d_p = \{d_{p,s}, 0\}$ and a maximum dwell time, $D_p = \{\infty, 0\}$. Note that these constraints imply that, when $\sigma_p(t) = l$, the signal will immediately switch back to $\sigma_p(t+1) = s$ and dwell for at least $d_{p,s}$ time steps before the next switch.

In addition to the switching rate constraints above, the allowable switches are also constrained. These constraints are represented by a binary matrix $\mathcal{G} \in \{0,1\}^{M \times M}$. If an element, $\mathcal{G}(i,j) = 1$, then the system can switch from mode i to mode j. Else, it cannot switch between them. For notational convenience, we say that $(i,j) \in \mathcal{G}$ if $\mathcal{G}(i,j) = 1$.

The set of all dynamic switching signals that respect a set of M-MDTs, d_x, and the possible switches in $\mathcal{G}$ is denoted $\Sigma_x(d_x, \mathcal{G})$. Likewise, the set of all preview switching signals respecting the M-MDT d_p is denoted $\Sigma_p(d_p)$.

Assume d_x, d_p, and $\mathcal{G}$ are provided. The objective of this work is to implement MPC that will remain feasible under all the allowable combinations of $\sigma_x \in \Sigma_x(d_x, \mathcal{G})$ and $\sigma_p \in \Sigma(d_p)$. This problem is motivated by distributed systems experiencing external switching. If the subsystems are loosely coupled, then a viable control scheme is to treat the coupling as local bounded disturbances **CITE**. Performance can be improved if the distributed controllers share their local, state-trajectory predictions with the coupled nodes so that the local controllers can better select their inputs **CITE**. At each time-step, these predictions can be expected to change only a small amount (hence the small error set for the disturbance previews). When local switching is introduced, however, its trajectory preview can change suddenly and, as a result, the disturbance preview being shared with any neighbors can change as well. Since each system only switches at a rate slower than their dwell times, this large error will only occur at a bounded rate and for a single step. This motivates the intermittent property of the disturbance prediction error.

Remark. *This idea that the disturbance preview comes from a neighboring, stable node motivates the assumption that the terminal preview evolves according to stable, linear dynamics G_f.*

D. Transformation to Linear Disturbed System

The model, as written in (**??**), is not convenient for analysis under MPC due to its time-varying affine nature from the disturbance preview. It can be transformed, however, into a higher dimension, disturbed linear system. First, notice that, from (**??**), the preview evolves according to the linear disturbed system,

$$\mathfrak{p}(t+1) = \underbrace{\begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & G_f \end{bmatrix}}_{G} \mathfrak{p}(t) + \mathfrak{w}(t)$$

where $\mathfrak{w}$ is a column vector made by the stacking the preview errors. Define a new state variable, $z \triangleq [x; \mathfrak{p}]$. Then the entire system can be written as

$$\begin{aligned} z(t+1) &= \begin{bmatrix} A_{\sigma_x} & [E,\ 0] \\ 0 & G \end{bmatrix} z(t) + \begin{bmatrix} B_{\sigma_x} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \mathfrak{w}(t) \end{bmatrix} \\ &= \hat{A}_{\sigma_x} z(t) + \hat{B}_{\sigma_x} u(t) + \hat{\mathfrak{w}}(t). \end{aligned} \tag{5}$$

Written in this form, the system becomes a linear disturbed system. The new disturbance, $\hat{\mathfrak{w}} \in \widehat{\mathbb{W}}_{\sigma_p} \triangleq \{0 \times \{\mathbb{W}_{\sigma_p}\}^{N+1}\}$ represents the error on the previewed disturbance from the initial system. Note that the new system is in $\mathbb{R}^{n+(N+1)m}$ compared with the original dimension of $\mathbb{R}^n$. This increased dimension will be problematic in future sections where set operations are required. It will be shown, however, how the structure of $\hat{A}_{\sigma_x}$ can be leveraged to circumvent the issues that arise.

The contributions of this work are twofold. First, we use the transformed form above to extend previous works, such as [**?**], to ensure feasibility in the face of the dual switching signals. Second, it is shown that the transformation to higher dimensions does not increase the computational load at the exponential rate that would be expected given a naive implementation.

III. Establishing Persistent Feasibility

Without care, switches in either the local system or the disturbance error could cause the system to become infeasible. This will be addressed by first examining the case when $\sigma_p(\cdot) = s$ for all time (the disturbance set never switches from mode s). This scenario was addressed in [**?**]. Next, the possibility of large preview errors dictated by the signal σ_p will be added.

A. Set Operations

Before moving on, basic set definitions and operations should be reviewed. Consider a system

$$x^+ = f_i(x,u) + \omega$$

where $\omega \in \mathbb{W}$ and i denotes the active mode. A set, $\mathcal{S}$, is robust control invariant (RCI) if $x \in \mathcal{S}$ implies that there exists a $u \in \mathcal{U}$ such that $x^+ \in \mathcal{S}$ for all possible disturbances. If the system is instead autonomous under a control law $u = \kappa(x)$ but still satisfies the above condition, then it is robust positive invariant (RPI). Finally, a set is admissible RPI if it is RPI, $\mathcal{S} \subseteq \mathcal{X}$ and $\kappa(\mathcal{S}) \subseteq \mathcal{U}$ where $\mathcal{X}$ and $\mathcal{U}$ are some state and input constraints. Robust pre-sets of a set $\mathcal{S}$ under the dynamics of mode $i \in \mathbb{Z}_{[1,M]}$ are those that satisfy

$$\begin{aligned} &\mathrm{Pre}_i^0(\mathcal{S}, \mathbb{W}) \triangleq \mathcal{S} \\ &\mathrm{Pre}_i^k(\mathcal{S}, \mathbb{W}) \triangleq \{x \mid \exists\ u \in \mathcal{U} \text{ s.t.} \\ &\quad x^+ = (f_i(x,u) + \omega) \in \mathrm{Pre}_i^{k-1}(\mathcal{S}),\ \forall \omega \in \mathbb{W}\}\ \forall k \in \mathbb{Z}_{\geq 1} \end{aligned}$$

Autonomous versions follow immediately. If the disturbance set is empty, then it can be omitted entirely from the pre-set notation. The pre-set operator is used to compute the RCI and RPI sets mentioned above [**?**].

B. Switched MPC with Additive Disturbance

Omitting the occasional large preview errors, this section focuses on switched linear systems with additive bounded disturbances. This was the focus of [**?**] which built upon [**?**]. The key ideas are reproduced here. For notational convenience, consider the simplified model

$$x(t+1) = A_\sigma x(t) + B_\sigma u(t) + \omega(t), \quad \omega \in \mathcal{W}. \tag{6}$$

This has had much of the superfluous notation stripped compared with (**??**).

Suppose (**??**) is controlled using MPC. Without any disturbance, feasibility can be ensured by finding the switch-robust control invariant (switch-RCI) sets $\{\mathcal{C}_i\}_{i=1}^M$ that correspond the the system's modes [**?**].

Definition 1 (Switch-RCI Sets)**.** *Given a collection of modes, $\mathcal{M} = \{\mathcal{M}_i\}_{i=0}^M$, a M-MDT set, d_x, and the allowable switches $\mathcal{G}$, a collection of M sets, $\{\mathcal{C}_i\}_{i=1}^M$ are Switch-RCI if they satisfy the following*

1) *$\mathcal{C}_i$ is RCI under the dynamics of mode i.*
2) *$\mathcal{C}_i \subseteq Pre_j^{d_{x,j}}(\mathcal{C}_j)$ for all $(i,j) \in \mathcal{G}$.*

The definition of switch-RCI sets ensure that the system is persistently feasible while dwelling and that, before a switch can occur, the state can enter a set in which a switch won't be problematic. An algorithm to find the switch-RCI sets is given in [**?**]. As suggested in [**?**], this can be expanded to the robust case. The authors merge tube based MPC [**?**] with the concepts described above. Suppose that the set $\mathcal{Z}_i$ is an RPI set for (**??**) given some control law $u = k_{\mathcal{Z}_i} x$. In traditional tube-based MPC, this is used to create a tube around the nominal trajectory that the disturbed trajectory will always be within. In [**?**], these RPI sets where also used to change condition 2 in definition **??** into

$$\mathcal{C}_i \oplus \mathcal{Z}_i \subset \mathrm{Pre}_j^{d_{x,j}}(\mathcal{C}_j) \oplus \mathcal{Z}_j \quad \forall (i,j) \in \mathcal{G}.$$

Sets that satisfy this updated condition are called disturbance-switched-RCI (DS-RCI) sets. It was shown that, by constraining the system with these sets and tightening the input and state constraints (as is typical in tube-MPC), the system could be persistently feasible under switching respecting the dwell times d_x and transitions $\mathcal{G}$.

C. Dual-Switch/Disturbance Robust MPC

The previous section assumed that $\sigma_p(t) = s\ \forall\ t \in \mathbb{Z}_{\geq 0}$. If, as originally stated, this error is allowed to be larger for a single time-step, then the system could possibly become infeasible even with the DS-RCI sets. Here, additional conditions are described that ensure persistent feasibility even under this large occasional disturbance.

At time t, Denote the remaining dwell time until a dynamics switch is allowed as $\zeta_x(t)$ and likewise for the preview errors as $\zeta_p(t)$. For each dynamics mode $i \in \mathbb{Z}_{[1,M]}$, denote its DS-RCI set as $\mathcal{C}_i^0$ and the k-step pre set of this as $\mathcal{C}_m^k$. Finally, use the Minkowski difference to define sets $\tilde{\mathcal{C}}_m^{k,j} \triangleq \mathrm{Pre}_j^m(\mathcal{C}_m^k \ominus \widehat{\mathbb{W}}_l)$. Using these definitions, we have the following observations.

1) To maintain feasibility under dynamics switching, we must always be within $\mathcal{C}_i^{\zeta_x}$.
2) To maintain feasibility under the disturbance impulse, we must always be within $\tilde{\mathcal{C}}_i^{\max\{0,\zeta_x-\zeta_p\},\zeta_p}$. Suppose that $\zeta_p = 0$. Then this is the set $\tilde{\mathcal{C}}_m^{\zeta_x,0}$ (i.e. the set $\mathcal{C}_i^{\zeta_x}$ with an additional "buffer" for the possible large error).
3) Combining these two, we have the constraint that $x(t) \in \mathcal{C}_i^{\zeta_x(t)} \cap \tilde{\mathcal{C}}_i^{\max\{0,\zeta_x(t)-\zeta_p(t)\},\zeta_p(t)}$.
4) This creates the need for many pre-set operations but they can all be done offline.

By supplementing the works such as [?] with the state constraints described above, the system could be made to reject sudden large affine disturbances.

IV. RPI Set Computation

Finding the exact, minimal RPI set, $\mathcal{Z}$, for the system

$$x^+ = Ax + \omega, \quad \omega \in \mathbb{W}$$

requires an incomputable, infinite Minkowski sum [?]. Instead, an arbitrarily close δ-approximation of the system's RPI set, $\mathcal{Z}^\delta$, can be found by terminating the infinite algorithm early and scaling up the result [?]. This process, however, still requires the computation of many Minkowski sums, an expensive prospect for higher dimensional systems. As mentioned in ??, the transformed model proposed in (??) will be in a relatively high dimension even for physically simple systems. In this section, a new process is proposed to find a δ-RPI set for (??) that is much less computationally taxing.

Let $\mathcal{Z}_N^\delta$ be an δ-RPI set for the system $p_N(t+1) = G_f p_N(t) + \omega_N(t)$. Note that these are the dynamics in the last block-row of (??). Using this, a full δ-RPI set for the preview dynamics can be constructed as follows. If $p_N(t) \in \mathcal{Z}_N^\delta$, then $p_{N-1}(t+1) \in \mathcal{Z}_N^\delta \oplus \widehat{\mathbb{W}}_s \triangleq \mathcal{Z}_{N-1}^\delta$. Continuing this idea, the full preview, $\mathfrak{p}$ call be contained within the RPI set

$$\mathcal{Z}_\mathfrak{p}^\delta \triangleq \{\mathcal{Z}_N^\delta \oplus N\widehat{\mathbb{W}}_s\} \times \{\mathcal{Z}_N^\delta \oplus (N-1)\widehat{\mathbb{W}}_s\} \times \cdots \times \{\mathcal{Z}_N^\delta\}.$$

The set $\mathcal{Z}_\mathfrak{p}^\delta$ creates a tube around the nominal preview trajectory, $\mathfrak{p}(t+1) = A_\omega \mathfrak{p}(t)$. This tube contains all the possible values of the preview as the system evolves under disturbances. The final set, $\mathcal{Z}_0^\delta \triangleq \{\mathcal{Z}_N^\delta \oplus N\widehat{\mathbb{W}}_s\}$, creates a tube around the next preview to be felt by the system, $p_0(t)$. It may not be intuitive that these dimensions would be where the tube has the largest cross-section. After all, it will be "felt" the soonest and so should have the least time to differ from reality. However, this tube does not define the largest error between $\omega_0(t)$ and $\omega_0(t+1)$. Instead, it is the largest error that can occur between the nominal value of ω_0 and the actual disturbed value for all time. After N steps in the nominal system, the disturbed preview will still be within the tube defined by $\mathcal{Z}_0$ around the nominal value.

Next, the immediate preview's δ-RPI set, $\mathcal{Z}_0^\delta$, is used to find the δ-RPI set for the local state space. For some control law, K_x, use $\mathcal{Z}_x^\delta$ to denote a set that is RPI under the dynamics

$$x(t+1) = (A_x - BK_x)x(t) + \omega_x(t) \tag{7}$$

with the constraint $\omega_x(t) \in E\mathcal{Z}_0^\delta$. Denote the nominal values of the local states' trajectory under a certain input trajectory, $\hat{u}(t)$, as $\hat{x}(t)$ and denote the nominal immediate preview trajectory as $\hat{p}(t)$. Suppose that the actual trajectories are given by $x(t) = \hat{x}(t) + e_x(t)$ and $p(t) = \hat{p}(t) + e_p(t)$ and the input is augmented as $u(t) = \hat{u}(t) - K_x e_x(t)$. Then we have

$$\begin{aligned}
&x(t+1)\\
&= A_x x(t) + Ep_0(t) + Bu(t)\\
&= A_x\left(\hat{x}(t) + e_x(t)\right) + E\left(\hat{p}_0(t) + e_p(t)\right)\\
&\quad + B\left(\hat{u}(t) - K_x e_x(t)\right)\\
&= \left(A_x\hat{x}(t) + E\hat{p}_0(t) + B\hat{u}(t)\right)\\
&\quad + \underbrace{(A_x - BK_x)e_x(t) + E\hat{e}_p(t)}_{\text{Matches dynamics in (??)}}\\
&= \hat{x}(t+1) + e_x(t+1)
\end{aligned}$$

By assuming that $e_x(0) \in \mathcal{Z}_x^\delta$, we know that $e_x(t) \in \mathcal{Z}_x^\delta$ for all time. This establishes that the local states are contained within a tube around the nominal trajectory. This set can then be merged with the preview's δ-RPI set to create the full δ-RPI set, $\mathcal{Z} = \{\mathcal{Z}_x^\delta \times \mathcal{Z}_\mathfrak{p}^\delta\}$. This creates a tube around all dimensions that the disturbed dynamics will always be within, assuming the nominal input is augmented with the control law K_x applied to the error.

By breaking the RPI set computation into smaller parts, the computational complexity is greatly reduced without introducing any conservatism. While the original system required finding the RPI set for a system in $\mathbb{R}^{n+(N+1)*m}$, the method proposed above requires finding two RPI sets, one in $\mathbb{R}^n$ and the other in $\mathbb{R}^m$. These, along with N Minkowski sum operations, are all that are required. This allows the transformed systems RPI set to be found without an exponential growth in the nominal dynamics.

V. Conclusion

VI. Conclusion