Stability from ADT

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Theorem 1 (Stability by MM-ADT). *Suppose the system switches according to $\boldsymbol{\sigma}(\cdot) \in \Sigma_T$ where $\tau_{i,i} = 1$ and $\tau_{i,j}$ satisfies*

$$\tau_{i,j} = -\log \tilde{S}_{[i,j]} / \log \gamma_i \quad i \neq j. \tag{1}$$

Then the system is asymptotically stable.

Proof. Denote the amount the cost-to-go is scaled during the k^{th} dwelling period and the subsequent switch as

$$\Gamma_k \triangleq \gamma_{\boldsymbol{\sigma}(t_k)}^{\delta_k} S_{\boldsymbol{\sigma}(t_k),\boldsymbol{\sigma}(t_{k+1})}.$$

Recall that δ_k is the dwell time of the k^{th} dwelling instance and t_k is the time at which the k^{th} dwelling instance began. Using the above definition, the cost-to-go at the end of the k^{th} dwelling instance can be upper bound by

$$J(t_k) \leq J(0) \left(\prod_{i=0}^{k-1} \Gamma_k \right).$$

Use $\overline{\mathcal{M}} \in \{\mathbb{Z} \times \mathbb{Z}\}^{\overline{M}}$ to denote all the valid pairs of source and destination modes. For an element $\overline{m} \in \overline{\mathcal{M}}$, use $\overline{m}_s$ to represent the source mode and $\overline{m}_d$ to represent the destination mode. Then the previous upper bound can be rewritten as

$$J(t_k) \leq J(0) \left(\prod_{\overline{m} \in \overline{\mathcal{M}}} \prod_{\substack{i=0 \\ \boldsymbol{\sigma}(t_i)=\overline{m}_s \\ \boldsymbol{\sigma}(t_{i+1})=\overline{m}_d}}^{k-1} \Gamma_k \right).$$

From this, it is clear that if each of the second product operators is upper bound by 1 as $k \to \infty$, then the cost-to-go approaches 0. Note that this product operator multiplies each of the scaling terms of the dwelling instances when the system was in mode $\overline{m}_s$ and the switched to $\overline{m}_d$. The number of switching times that satisfy these requirements is upper bound by $N_{\boldsymbol{\sigma}}(0, t_k, \overline{m}_s, \overline{m}_d)$. The time spent in these instances is given by $t_{\overline{m}_s,\overline{m}_d} : \mathbb{Z} \to \mathbb{Z}$. Using these two facts, the second product can be rewritten and upper bound by 1

$$\prod_{\substack{i=0 \\ \boldsymbol{\sigma}(t_i)=\overline{m}_s \\ \boldsymbol{\sigma}(t_{i+1})=\overline{m}_d}}^{k-1} \Gamma_k \leq \gamma_{\overline{m}_s}^{t_{\overline{m}_s,\overline{m}_d}(t_k)} S_{\overline{m}_s,\overline{m}_d}^{N_{\boldsymbol{\sigma}}(0,t_k,\overline{m}_s,\overline{m}_d)} < 1.$$

For ease of notation, we will continue with $\overline{m}_s = i$ and $\overline{m}_d = j$. Rearranging the above inequality and using the definition of $N_{\boldsymbol{\sigma}}(\cdot)$ concludes the proof.

$$\begin{aligned}
\gamma_i^{t_{i,j}(t_k)} S_{i,j}^{N_{\boldsymbol{\sigma}}(0,t_k,i,j)} &< 1 \\
S_{i,j}^{N_{\boldsymbol{\sigma}}(0,t_k,i,j)} &< \gamma_i^{-t_{i,j}(t_k)} \\
N_{\boldsymbol{\sigma}}(0, t_k, i, j) \cdot \ln(S_{i,j}) &< -t_{i,j}(t_k) \cdot \ln(\gamma_i) \\
\left(N_0 + \frac{t_{i,j}(t_k)}{\tau_{i,j}} \right) \ln(S_{i,j}) &< -t_{i,j}(t_k) \cdot \ln(\gamma_i) \\
-\frac{\ln(S_{i,j})}{\ln(\gamma_i)} &> \tau_{i,j} + \frac{\tau_{i,j} N_0 \ln(S_{i,j})}{\ln(\gamma_s) t_{i,j}(t_k)} \underset{k \to \infty}{=} \tau_{i,j}
\end{aligned}$$

$\square$