Felix Baumgartner's Jump from Space

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1 Introduction

Felix Baumgartner's space jump was a breathtaking feat that captured the world's imagination. On a crisp October day in 2012, the Austrian daredevil ascended to the edge of space in a pressurized capsule dangling from a massive helium balloon. As he rose through the atmosphere, the curvature of the Earth became visible, and the sky darkened to an inky black. At an altitude of 39 kilometers above the Earth's surface, Baumgartner took a deep breath and stepped into the void. He plummeted through the thin air of the stratosphere, quickly accelerating to supersonic speeds. His body pierced the sound barrier, reaching a top speed of about 1357 km/h, faster than a commercial airliner. For over four minutes, Baumgartner free-fell through the sky, fighting to maintain a stable position as he plunged back toward Earth. The extreme cold and low pressure at high altitudes posed significant risks, and a special pressurized suit protected him from the harsh conditions. As he neared the ground, Baumgartner deployed his parachute, gracefully gliding to a landing in the New Mexico desert. The entire jump lasted just over 10 minutes, but it set multiple world records and provided valuable scientific data on human performance in extreme environments. The Red Bull Stratos project, as it was officially known, was more than just a publicity stunt. It pushed the boundaries of human endurance and gathered crucial information for the development of emergency escape systems for high-altitude aircraft and spacecraft. Baumgartner's daring leap from the stratosphere remains a testament to human courage and the relentless pursuit of pushing limits.

2 Theory

2.1 Free Fall in the Vacuum

Free fall in a vacuum is a captivating phenomenon that strips away the complexities of real-world physics, revealing the pure essence of gravitational motion. Imagine dropping a feather and a bowling ball in a perfect vacuum - they would fall side by side, accelerating at exactly the same rate. This scenario, while rare on Earth, is the reality of objects moving in the vast emptiness of space.

In this idealized environment, objects succumb solely to the relentless pull of gravity. Their motion is elegantly described by a set of simple equations that predict position, velocity, and acceleration with perfect accuracy. As an object begins its descent, its velocity increases linearly with time, governed by the equation:

$$v(t) = gt + v_0 \tag{1}$$

Here, v represents (time-dependent) velocity, g is the gravitational acceleration (approximately $-9.81\,\mathrm{m/s^2}$ on Earth's surface), v_0 is the initial velocity at time $t = 0$ and t is the time elapsed. The distance traveled, meanwhile, follows a quadratic relationship with time:

$$d(t) = \frac{1}{2}gt^2 + v_0 t + d_0 \tag{2}$$

This equation reveals that an object falls four times as far in twice the time, a non-intuitive result that highlights the accelerating nature of free fall. The acceleration experienced by all objects in this vacuum-sealed world is constant, expressed simply by Newton's law, i.e.

$$F_g = m \cdot a(t), \quad a(t) = g. \tag{3}$$

Constant acceleration leads to an ever-increasing velocity, with no terminal velocity to halt the object's acceleration - a stark contrast to falls in air-filled environments (see section 2.2). While true vacuum free fall is rarely experienced on Earth, it's approximated in specialized drop towers used for scientific experiments. More significantly, it's the principle that governs the motion of celestial bodies and spacecraft, making it fundamental to our understanding of orbital mechanics and space exploration. The concept of free fall in a vacuum serves as a cornerstone of physics, providing a simplified model that allows us to understand more complex scenarios. It strips away the extraneous forces that often complicate real-world problems, leaving us with the pure, mathematical beauty of objects in motion under gravity's influence. The position is related to the velocity and acceleration by the following set of differential equations:

$$v(t) = d'(t) = \frac{\mathrm{d}}{\mathrm{d}t} d(t), \tag{4}$$

$$a(t) = v'(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t). \tag{5}$$

The velocity is therefore defined as the change of distance at a given time, and the acceleration is the change of velocity at a given time. Solving this differential equation is performed by the mathematical concept of integration. Unfortunately, we won't have enough time to learn the fundamentals of integration, but it can also be understood geometrically as the area underneath a curve.

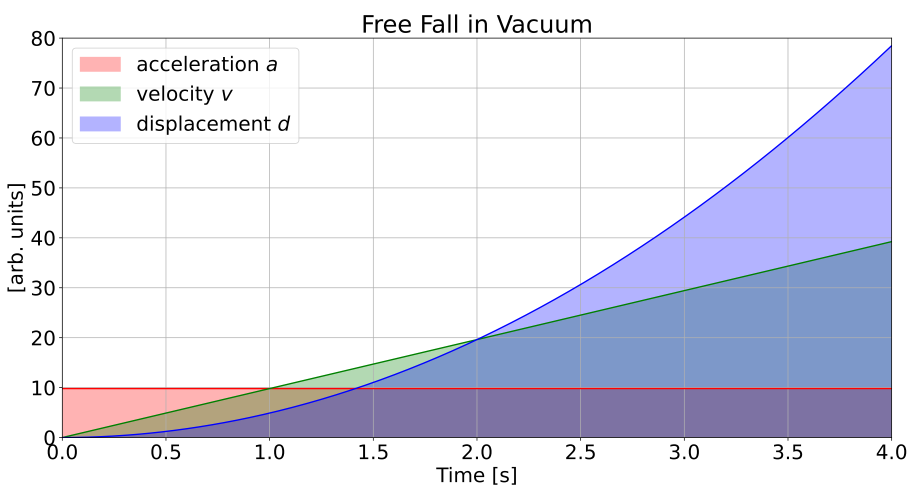

Figure 1: Acceleration $a(t)$, velocity $v(t)$ and displacement $d(t)$ as a function of time t for a free-falling body in vacuum.

If we plot the function $a(t) = g$ versus time, we get a horizontal line. As velocity is defined as the area corresponding to this line, we get $v(t) = a(t) \cdot t = gt$. Compared to the general solution eq. (1), we missed the initial velocity v_0, which, mathematically, is known as an integration constant. It accounts for a different choice of initial conditions, such as a non-vanishing velocity prior the acceleration at $t = 0$. Similarly, we can now find the traveled distance $d(t)$ given the velocity $v(t) = gt + v_0$. The first term defines a triangle in time, which corresponds to an area of $\frac{1}{2}gt^2$. The second term is again a constant in time, and we already saw how a horizontal line defines an area – in this case $v_0 t$ instead of gt. Altogether, we get $d(t) = \frac{1}{2}gt^2 + v_0 t$, that is equal to eq. (2) up to the integration constant d_0 which accounts for a non-vanishing initial position.

2.2 Free Fall in a Fluid

Free fall in a fluid presents a more complex and nuanced scenario than its vacuum counterpart, introducing a dynamic interplay between gravity and the resistant forces of the surrounding medium.

Picture a pebble sinking in water or a skydiver plummeting through the atmosphere - these are everyday examples of objects in free fall through a fluid environment. The motion is governed by a total force that accounts for both gravitational pull and fluid drag

$$F_{\text{tot}} = F_g + F_d \tag{6}$$

Here, m is the mass of the object, g is gravitational acceleration, and F_d represents the drag force exerted by the fluid.

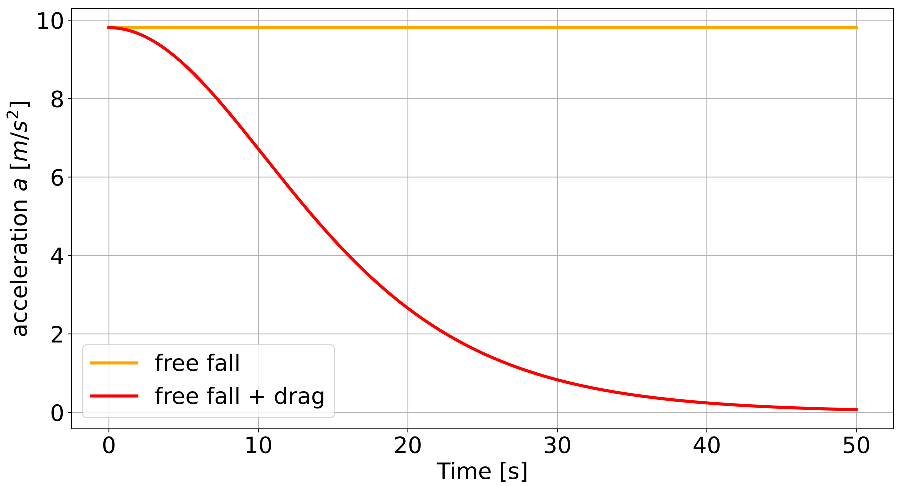

Figure 2: Acceleration a as function of time t for a free-fall motion with and without the drag force given by eq. (8). For the numerical solution with drag force, we used $C_d = 0.47$, $A = 0.1\,\text{m}^2$ and $\rho = 1.225\,\text{kg/m}^3$.

The total acceleration is given by

$$a_{\text{tot}} = F_{\text{tot}}/m \tag{7}$$

and the drag force itself is typically modeled as

$$F_d = -\frac{\text{sign}(v)}{2}\rho v^2 C_d A \tag{8}$$

Where ρ is the fluid density, v is the object's velocity, C_d is the drag coefficient (dependent on the object's shape), and A is the object's cross-sectional area. Note that the direction of the drag force is always opposite to the velocity and it thus hinders acceleration – eventually balancing the gravitational force. This leads to a key characteristic of fluid free fall: terminal velocity. The terminal velocity (v_t) can be expressed as

$$v_t = \sqrt{\frac{2m|g|}{\rho C_d A}} \tag{9}$$

Unlike in a vacuum, objects of different masses and shapes fall at different rates in a fluid. A feather and a bowling ball dropped in air will have drastically different trajectories due to their varying mass-to-surface-area ratios. The journey to terminal velocity is an exponential approach, described by the equation

$$v(t) = v_t(1 - e^{-\frac{t}{\tau}}) \tag{10}$$

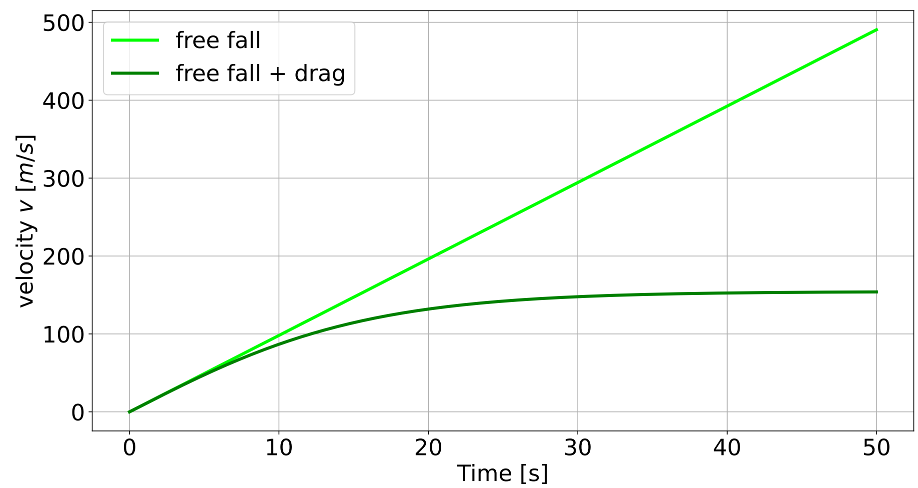

Figure 3: Velocity v as function of time t for a free-fall motion with and without the drag force given by eq. (8). For the numerical solution with drag force, we used $C_d = 0.47$, $A = 0.1\,\mathrm{m}^2$ and $\rho = 1.225\,\mathrm{kg/m}^3$.

Where τ is the characteristic time constant of the system, dependent on the object's properties and the fluid's characteristics. This fluid free fall model finds applications in diverse fields, from meteorology (studying raindrop formation) to engineering (designing parachutes) and even in understanding the movement of microorganisms in liquids. It serves as a bridge between idealized physics and the complex, resistance-filled world we inhabit, providing crucial insights into the behavior of objects moving through Earth's atmosphere and the depths of its oceans. The study of free fall in fluids thus offers a fascinating glimpse into the intricate balance of forces that shape motion in our everyday world, revealing the subtle interplay between gravity and the resistive embrace of the fluids that surround us.

The comparison between acceleration, velocity and displacement for a free-fall in vacuum and a free-fall in presence of drag due to a fluid (like air) is given in fig. 2 and fig. 3. In fig. 2, you can clearly see that the acceleration is not constant in the presence of drag, but is suppressed until $a = 0$ around $t = 50s$. When the acceleration is suppressed, there is no variation in the velocity of the body, meaning that the terminal velocity has been reached. This is clearly shown in fig. 3, where the dark green line becomes horizontal (i.e. the velocity is constant) around $t = 50s$. The solution for $a(t)$ and $v(t)$ in the presence of drag force were obtained using the Euler method (see Sec.section 2.3.1 for more insights).

2.3 Differential Equations

A differential equation is a type of math problem that involves functions and their rates of change. In math, the rate of change of a function is known as the *derivative* of the function itself, and therefore a differential equation is an equation involving a function and its derivative. For instance, for a body moving in space the velocity quantifies how the position of the body changes in time, and accordingly is defined as the *derivative* of position with respect to time. If the position is indicated as x, its velocity is given by

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t} d(t)\,. \tag{11}$$

Let's make a concrete example. If you take the function $f(t) = \sin(t)$, the derivative with respect to the t variable is given by

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \cos(t)\,, \tag{12}$$

meaning that in each point t the rate of variation of the function $f(t) = \sin(t)$ is given by the cosine function $f'(t) = \cos(t)$. We can visualize this concept with a simple plot. In each point, the change of rate can be visualized through the line which is tangent to the function in the chosen point. The slope

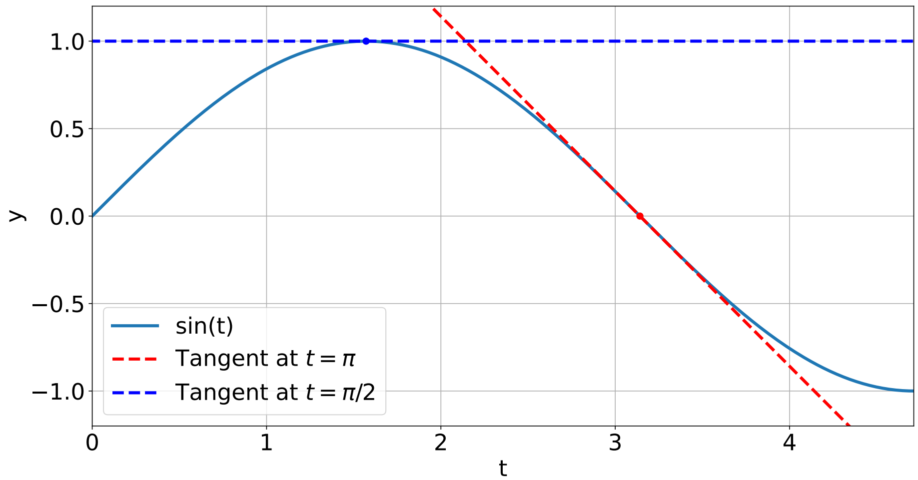

Figure 4: The function $f(t) = \sin(t)$ is plotted in the range $0 \leq t \leq 3\pi/2$. The blue and the red lines are the tangent of the function at the points $t = \pi/2$ and $t = \pi$, respectively. The slope of these straight lines is the derivative of the function in the given point.

of such tangent line is exactly given by the derivative. Mathematically, given $t = t_0$, the equation of the straight line which is tangent to $f(t)$ in $t = t_0$ is

$$y = f'(t_0)(t - t_0) + f(t_0)\,, \tag{13}$$

where $f'(t_0)$ is the derivative of f in $t = t_0$. For example, we plotted the line obtained with the previous equation for $t = \pi$ and $t = \pi/2$. You can see in the picture that the slope of the tangent characterizes the rate of change of the function at the given points. Such slope is given by the derivative of the function at the given points

$$\begin{aligned} f'(t = \pi) &= \cos(\pi) = -1\,, \\ f'(t = -\pi/2) &= \cos(-\pi/2) = 0\,. \end{aligned} \tag{14}$$

So, in general, a differential equation involves a function (like your position) and its derivative (like your speed). The derivative tells us how the function changes. The goal is often to find the original function if we know how it changes, or to understand how it changes if we know the original function. In a nutshell:

i. the *function* represents something that can change, like the altitude of a body in free-fall;

ii. the *derivative* represents the rate of change of that function; for instance, the speed of the body is the derivative of position with respect to time and represents the rate of change of position in time;

iii. the *differential equation* is an equation that involves a function and its derivative.

A differential equation helps us understand the relationship between these things and can be used to predict future behavior.

2.3.1 Solving Differential Equations Numerically

The Euler method is a simple way to find an approximate solution to a differential equation. Here's how it works in very simple terms. Let's assume that we have a function of time $y = y(t)$ whose derivative is given by

$$y'(t) = f(t, y(t))\,, \tag{15}$$

i.e. is a function of time t and of the original function $y(t)$. Knowing the derivative function f, we want to find the original function y. The solution to the previous equation can be found iteratively, through the following steps known as the Euler's method:

1. **Start with a Point**: you begin with an initial point $y_0 = y(t_0)$, which depends on the initial conditions of the given problem. For instance, if the function y is the altitude of a body in free fall, the initial value y_0 is the height from which the body falls.

2. **Take a Small Step**: you move a small step forward along the t-axis, choosing a value h such that each point can be obtained from the first one through the equation $t_n = t_0 + nh$ or equivalently $t_{n+1} = t_n + h$. The quantity h, i.e. how far you move each step along the t-axis is called the "step size".

3. **Find the New Point**: knowing the slope of the function (i.e. its derivative $y' = f$) and the step size, you can calculate a new point. Starting from a point y_n at time n, the following point y_{n+1} at time t_{n+1} can be approximated as

$$y_{n+1} = y_n + hf(t_n, y_n)\,. \tag{16}$$

   We are using the knowledge of the slope of the function (i.e. the knowledge of the derivative function y') to approximate the unknown function $y(t)$.

4. **Repeat**: You repeat this process, using the new point y_{n+1} as the starting point for the next step. You keep doing this until you reach the end of the time interval you're interested in.

In essence, the Euler method consists of approximating iteratively an unknown function using the known derivative. Since the derivative represents the slope of the function, we can use it to find the approximate value of the function close to a given point we already know.

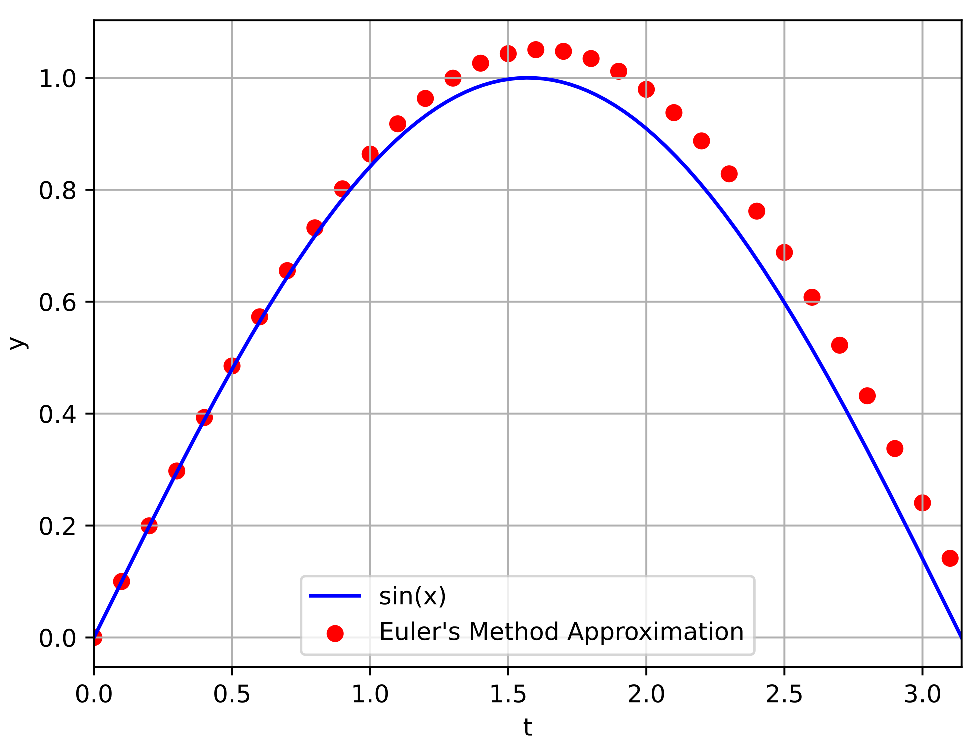

Figure 5: The approximate solution to eq. (17) obtained using the Euler method is plotted with red points. The exact analytical solution $y(t) = \sin(t)$ is plotted with a blue line.

To make a concrete example, I used the Euler method to solve the differential equation

$$y'(t) = \cos(t)\,, \tag{17}$$

meaning that we want to compute the function $y(t)$ whose derivative is $y'(t) = \cos(t)$. You should already know the solution: I told you before that $y'(t) = \cos(t)$ is the derivative of the function $y(t) = \sin(t)$. However, we want to address the problem using the Euler method. We start from an initial point $t_0 = 0$, assume $y_0 = y(t = 0) = 0$, and we evaluate the function in the range $0 \leq t \leq \pi$. Assuming that the step size is $h = 0.1$, the coordinates of the first point can be obtained as

$$\begin{aligned} t_1 &= t_0 + h\,, \\ y_1 &= y_0 + h\cos(t_0)\,, \end{aligned} \tag{18}$$

where we used $y'(t_0) = \cos(t_0)$. The second point can be similarly obtained using the point (t_1, y_1) as starting point

$$\begin{aligned} t_2 &= t_1 + h\,, \\ y_2 &= y_1 + h\cos(t_1)\,, \end{aligned} \tag{19}$$

and so on, until we reach the upper boundary of the range where we want to solve the equation (in this case $t = \pi$). You can see the result in the previous picture. It is clear that the Euler method provides an approximation for the sought function $y(t)$: at the beginning around $t = 0$, you can notice that the approximate solution is very close to the exact one. However, toward the end of the sampling range there is a noticeable difference between the red points and the exact solution plotted with the blue line.

3 Application to Baumgartner's Jump from Space

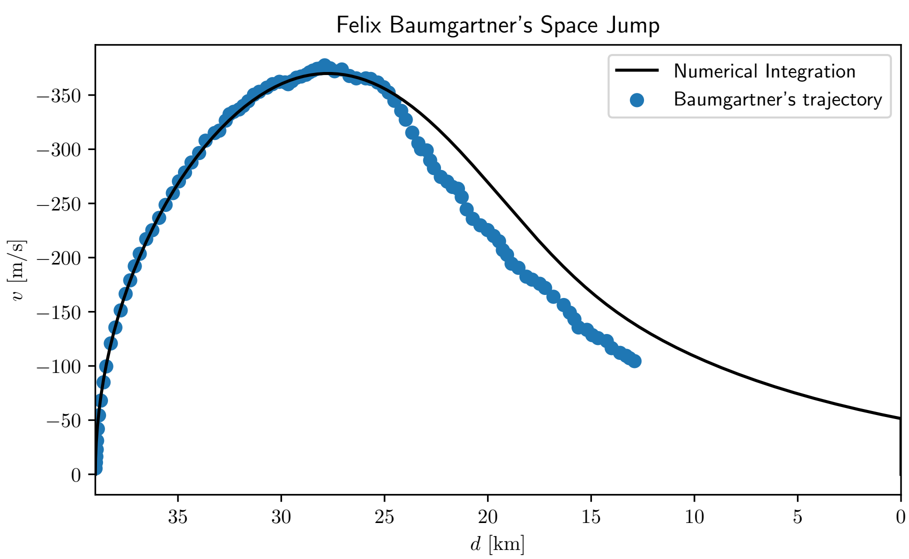

Figure 6: Actual trajectory (blue dots) and approximate numerical trajectory (in black). Constants are chosen as $A = 0.8\,\mathrm{m}^2$ and $H_0 = 7\,\mathrm{km}$ (other values as defined below).

For the jump from space, it is crucial to account for the change of the fluid density in eq. (8)

$$F_d(t, v(t)) = -\frac{\mathrm{sign}(v(t))}{2}\rho_d(t)v^2(t)C_dA \tag{20}$$

where we assume the density changes according to a simplified model

$$\rho_d(t) = \rho \mathrm{e}^{-d(t)/H_0} \tag{21}$$

in which $\rho \approx 1.2\mathrm{kg/m^3}$ is the surface density of air and $H_0 \approx 7000\,\mathrm{m}$ is a height scale. The numerical integration of the differential equation is now performed in two consecutive Euler steps.

1. **Set up all constants and initial conditions**: for $t_0 = 0$, choose Baumgartner's initial altitude of $d_0 = d(t_0) = 39\,\mathrm{km}$ and $v_0 = v(t_0) = 0\,\mathrm{m/s}$. The gravitational constant is $g = -9.81\,\mathrm{m/s^2}$ and Baumgartner's mass is (with equipment) $m = 127\,\mathrm{kg}$. Take $C_d = 1$, $\rho = 1.2\,\mathrm{kg/m^3}$ and pick H_0 between 6000 m and 8000 m. Pick an A between $0\,\mathrm{m^2}$ and $2\,\mathrm{m^2}$.

2. **At step** n: Perform the integration in two consecutive Euler steps.

   2a. **Integrate the acceleration**: Calculate the velocity $v_{n+1} = v_n + h\ a_{\mathrm{tot}}(t_n, v_n)$ by using $a_{\mathrm{tot}}(t, v(t)) = (F_d(t, v(t)) + F_g)/m$. Note that in F_d you need the altitude $d_n = d(t_n)$ in the air density $\rho_d(t_n)$.

   2b. **Integrate the velocity**: Calculate the altitude $d_{n+1} = d_n + h\ v_n$.

3. **Store results and proceed to the next point**: $n \to n + 1$. Repeat step 2..

4. **Visualize output**: Plot the trajectory and compare the line with the data from the actual free fall from space. Change A and H_0 and see if you get realistic values which capture the data to a large extent.

Interestingly, the trajectory in fig. 6 crosses the barrier of sound twice, and the drag due to increasing atmospheric air density decelerates Baumgartner's jump. However, as you can see, clearly not as much that he could spare the parachute at $d = 0$. Despite the apparent differences between the actual trajectory and the numerical simulation, the results based on our model agree with the data pretty well.