FACULTY OF ENGINEERING

The Systems Control and Optimization Laboratory

Optimal Control Problem:

A Simplified Moon Mission

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1 Abstract

In this project we aim to optimally control the trajectory of a shuttle to the Moon. Starting at a geostationary orbit around Earth the shuttle should reach the moon while minimizing the used energy and therefore minimizing the additional acceleration that is applied by the shuttle. We describe the formulation of a nonlinear programming (NLP) problem and the reformulations required to deal with the nonlinear equality constraints. For generating solutions to the reformulated NLP the solver CasADi IPOPT (Andersson u. a., 2019) is used.

2 Problem description

Let us first describe the movement of the Moon and space shuttle in an ordinary differential equation. We use a simplified 2D model centering Earth at the origin. Indices are used for clarification of which object we are referencing, e.g. M_E will describe the mass of Earth, v_M the speed vector of the Moon, and p_S the position of the space shuttle. Further, we are using the gravitational constant $G = 6.6743 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$. In order to keep variables and parameters at a reasonable value, we express distance and time by means of astronomical units 1 [AU] = 384403000m and 1 [AT] = 2551400s, such that 1 [AU] is the distance between Earth and the Moon and 1 [AT] is the time of one lunar orbit around Earth. All other parameters are scaled using these units.

2.1 Underlying dynamics

Using ordinary differential equations based on (Giordano u. a., 2006), we first describe the motion of the Moon due to the mass of Earth by the system

$$\dot{v}_M = -\frac{GM_E(p_M - p_E)}{\|p_M - p_E\|_2^3} \tag{1a}$$

$$\dot{p}_M = v_M. \tag{1b}$$

We need the position and velocity of the Moon in its orbit in order to determine the impact of the mass of the Earth and the Moon on the shuttle. Therefore, a simulation of the lunar orbit is done prior to the optimal control process. This is only possible with the simplification that the mass of the shuttle is negligible compared to the mass of Earth and the Moon which is a reasonable simplification. Additionally, we assume that the mass of the Moon does not impact the Earth in order to keep the Earth in the origin. We will need to monitor the position and velocity of the Moon when calculating the space shuttle dynamics, which are determined by the impact of the mass of Earth, the mass of the Moon, and most importantly the two-dimensional acceleration control variables. The resulting differential equations are of the following form

$$\dot{v}_S = -\frac{GM_E(p_S - p_E)}{\|p_S - p_E\|_2^3} - \frac{GM_M(p_S - p_M)}{\|p_S - p_M\|_2^3} + u \tag{2a}$$

$$\dot{p}_S = v_S. \tag{2b}$$

3 The optimal control problem

Having described the underlying dynamics we can formulate the optimal control problem.

3.1 States

For a clear formulation, we define the used states and controls at first. The state x, as shown in equation (3), is a four-dimensional vector with the position and velocity of the shuttle. The control u is the acceleration of the shuttle in both dimensions. Further, we make the assumption that our shuttle can accelerate in any direction.

$$x = \begin{pmatrix} p_x \\ p_y \\ v_x \\ v_y \end{pmatrix}, \quad u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{3}$$

3.2 Discretization

The discretization of the dynamical system is done by choosing two parameters in the main script. One can alter the total duration T [AT] of space travel and Δ_t [AT] of discretization step size. From these values we can derive the number $N = \lfloor T/\Delta_t \rfloor$ of discretization steps, where $\lfloor \cdot \rfloor$ is the floor function. We will discretize both states and controls, and use the classic Runge-Kutta method (RK4) for solving the dynamical system.

3.3 Basic approach

Let f denote the integration of the underlying dynamical system. Given an initial state x_0, we shall optimize

$$\begin{aligned}
&\underset{x,u}{\text{minimize}} && \sum_{i=0}^{N-1} \|u_i\|_2^2 && \text{(4a)}\\
&\text{subject to} && x_0 = \bar{x}_0 && \text{(4b)}\\
& && x_{k+1} = f(x_k, u_k, \Delta_t) \qquad \text{for} \quad k = 0, \dots, N-1 && \text{(4c)}\\
& && v_{S,N} = v_{M,N} && \text{(4d)}\\
& && p_{S,N} = \tilde{p}_{S,N}, && \text{(4e)}
\end{aligned}$$

where $\tilde{p}_{S,N}$ denotes the intended final position of the shuttle and r_M describes the radius of the Moon. The chosen objective function is used to minimize the energy that is used during the flight. However, we use the squared Euclidean norm to ensure differentiability. The equality constraints (4d) and (4e) force the shuttle to reach a point close to the Moon with the Moon's velocity. Due to the strong final state equality constraints, this approach does not terminate in a reasonable number of iterations. This problem is solved by weakening the constraint in the following manner.

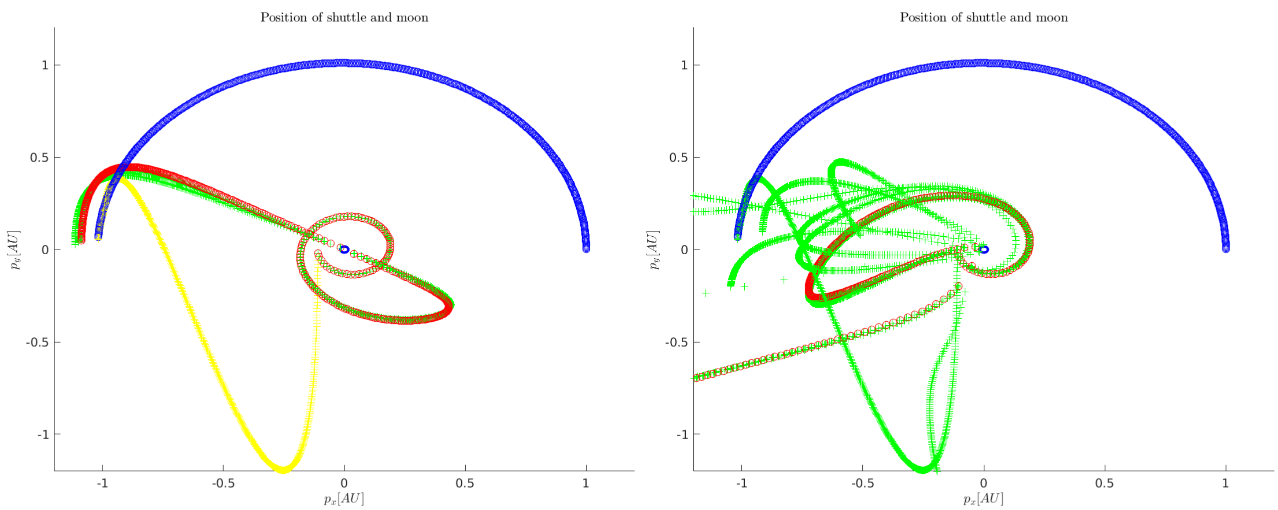

Figure 1: Left: A first solution using a homotopy parameter. Right: Constraining shuttle to not 'hit' Earth may not be strict enough.

3.4 Using a homotopy parameter ε

Since the final state equality constraints are hard to fulfill, we replace those by the inequality constraints (5d) and (5e), and only request the final states to be close to the desired state. This is done by introducing a two-dimensional parameter ε that weakens the final state constraints by providing a margin. We get the following optimization problem:

$$\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \|u_i\|_2^2 && && \text{(5a)}\\
\text{subject to} \quad & x_0 = \bar{x}_0 && && \text{(5b)}\\
& x_{k+1} = f(x_k, u_k, \Delta_t) && \text{for} \quad k = 0, \ldots, N-1 && \text{(5c)}\\
& \|p_{S,N} - \tilde{p}_{S,N}\|_2^2 \leq \varepsilon_{pos} && && \text{(5d)}\\
& \|v_{S,N} - v_{M,N}\|_2^2 \leq \varepsilon_{vel} && && \text{(5e)}
\end{aligned}$$

The idea of the homotopy is to start with an initial ε and solve the NLP. We then update $\varepsilon \leftarrow \varepsilon\beta$ for some $\beta \in (0, 1)$ and solve the NLP again using the prior solution as the initial guess. This procedure can be repeated as often as desired. An example to a solution using this method is shown in Figure 1.

3.5 Avoiding a potential crash

In order to improve the model, some additional constraints are added. The shuttle is not supposed to fly too close, or even through, the Moon and Earth. Thus some non-convex inequality constraints are added to the problem. We introduce the reformulation

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \|u_i\|_2^2 && && \text{(6a)}\\
\text{subject to} \quad & x_0 = \bar{x}_0 && && \text{(6b)}\\
& x_{k+1} = f(x_k, u_k, \Delta_t) && \text{for} \quad k = 0, \dots, N-1 && \text{(6c)}\\
& \|p_{S,N} - \tilde{p}_{S,N}\|_2^2 \leq \varepsilon_{pos} && && \text{(6d)}\\
& \|v_{S,N} - v_{M,N}\|_2^2 \leq \varepsilon_{vel} && && \text{(6e)}\\
& \|p_{S,N} - p_{M,N}\|_2^2 \geq r_M^2 && && \text{(6f)}\\
& \|p_{S,k} - p_E\|_2^2 \geq r_E^2 && \text{for} \quad k = 0, \dots, N-1, && \text{(6g)}
\end{aligned}
$$

where r_M and r_E are the radius of the Moon, and the initial distance between Earth and the shuttle, respectively. Specifically, the latter is a reasonable value since we are initializing with orbit speed and want to diverge away from Earth. When we simply choose r_E to be the radius of Earth, we find that the trajectory computation may fail given our step size. The shuttle will pick up a velocity that is great enough to 'jump across' Earth with a single step and the solutions may become unstable (see Figure 1).

4 Solving the NLP

The interior point solver CasADi IPOPT NLP (Andersson u. a., 2019) is used to solve the multiple shooting approach. We noticed that solving the NLP is highly dependent on the initial and final states of the problem. For some setups, the solver might not converge in reasonable time. We noticed that the inequality constraint (6g) can be removed for some setups, which simplifies the problem by eliminating a non-convex constraint. This constraint might be added if the obtained solution is unstable due to the singularity in the dynamics.

4.1 Initial guess

The initial guess of the NLP is created by solving the same NLP in a simplified version. Therefore, the basic approach is solved by neglecting the mass of Earth and the Moon in the dynamics of the system. The objective function stays the same as in the final problem. Thus, the initial guess is a trajectory from Earth to the Moon without the impact of Earth and the Moon. This gives us state and control values in the correct dimensions close to the solution.

4.2 Problem setup

To solve the above system, we need to set the parameters of the problem. We choose that the shuttle should reach the Moon after half the Moon orbit around the Earth. The time step is $\Delta_t = 0.001$, which yields to $N = 500$. The initial homotopy parameters are set to $\varepsilon_{pos} = 0.1$ and $\varepsilon_{vel} = 1$ which are reduced by a factor of $\beta = \frac{1}{2}$ after each homotopy iteration. The initial state meaning the initial position and velocity of the shuttle is set to a geostationary orbit. The final position of the shuttle is set to reach a point close to the Moon.

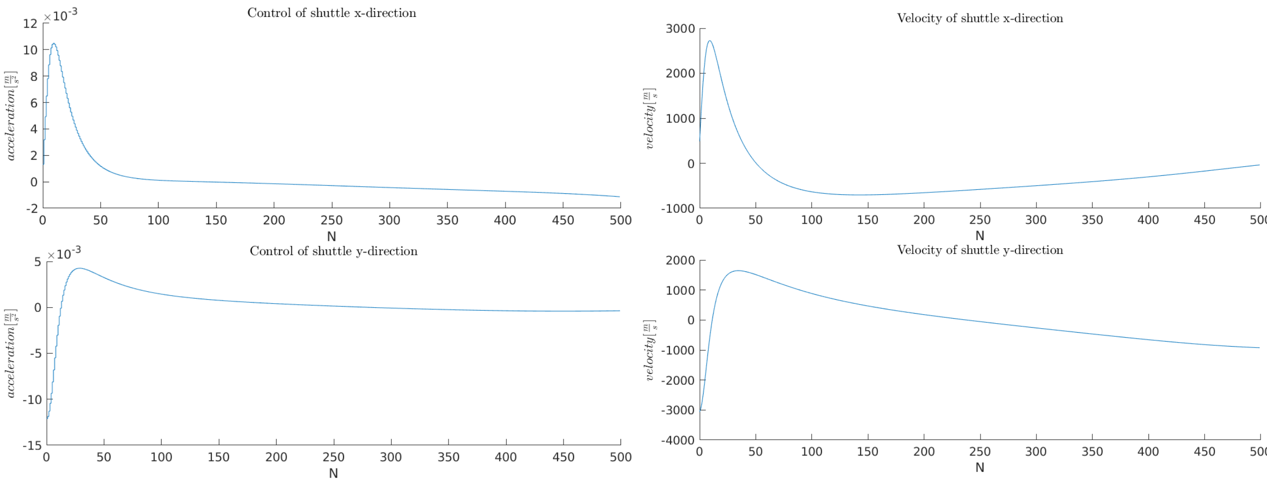

Figure 2: Left: Controls in x and y direction. Right: Velocity in x and y direction.

4.3 Final solution

One solution of the NLP can be seen in Figure 3. The final solution shows the trajectory of the shuttle reaching the lunar orbit. The green trajectories show the intermediate results of the homotopy. It can be seen that the solutions get closer to the final state when the homotopy parameter is reduced. The red trajectory shows the final solution. For this solution the inequality constraint (6g) is added in order to not get to close to Earth. The controls and velocity of this final solution are given in Figure 2.

We produce another solution by removing both the inequality constraint to the Earth and the homotopy method by choosing $\varepsilon = 0$. The solution is shown in Figure 3. At first glance it seems better than the solution above, and indeed it also has a smaller cost value. However, this solution is extremely dependent on the chosen parameters, e.g. the initial and final states, or the initial guess. For small perturbations in these parameters, the optimization may not terminate at all or gives solutions with much higher cost. The other approach is significantly stable in this sense.

4.4 Comparing parameter setting

For the optimization, some parameters are set to reach the given aim. Most importantly, the number of optimization steps while decreasing the homotopy parameter ε, the reduction factor β, and the initial ε itself. A comparison of different settings is given in Table 1. The first row shows a setting that converges quite fast and uses four iterations. Following in the next row, we removed the homotopy and used the margin ε given by the margin used for the optimal solution from row one. Then the solver does not converge within reasonable time, which shows the importance of using the homotopy steps. The third row shows that increasing the number of homotopy loops takes more time but does not decrease the objective. However, the distance to the final position is decreased. The last row shows the setting for the solution seen on the right of Figure 3, where the right side is produced by the setting of the first row.

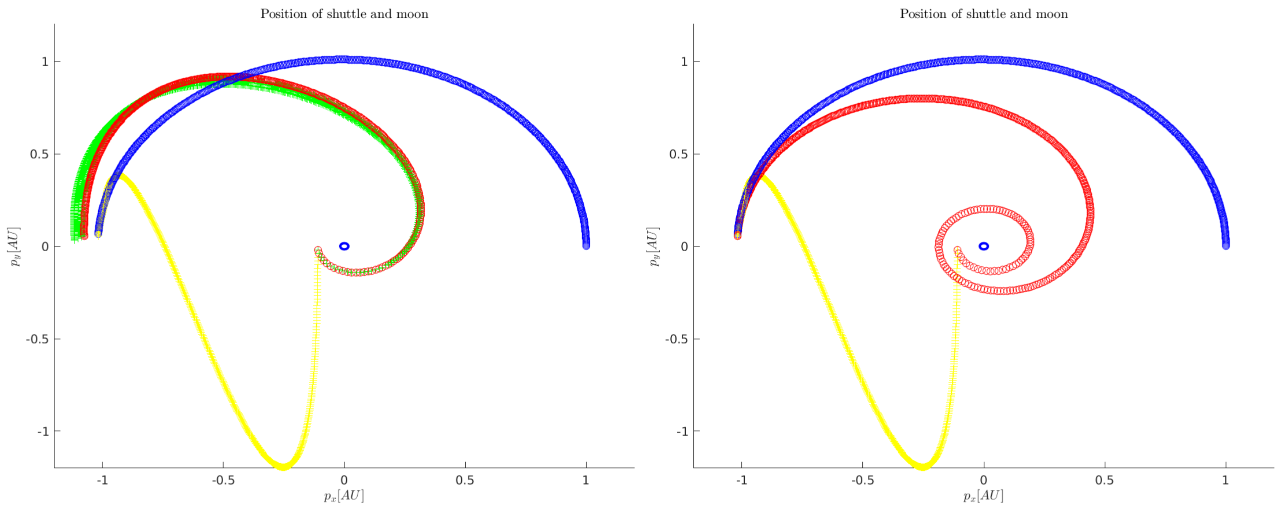

Figure 3: The two different solutions. On the left side, we plot the solution using homotopies (green) and an additional constraint. On the right side, we eliminated both. The red lines are the final trajectories and the yellow lines represent the initial guesses.

steps	ε_{pos}	ε_{vel}	β	solution found	constraint on Earth	time	objective
4	0.1	1	2	yes	yes	97.7	$1.01 \cdot 10^6$
1	$\frac{0.1}{2^4}$	$\frac{1}{2^4}$	-	no	yes	-	-
8	0.1	1	2	yes	yes	115.1	$1.02 \cdot 10^6$
4	0.1	1	10	yes	yes	113.6	$1.04 \cdot 10^6$
1	0	0	-	yes	no	43.5	$4.39 \cdot 10^5$

Table 1: Comparison of different parameter for the optimization

5 Further improvements

Unfortunately, this project was constrained to a short period of time. We have ideas for further improvements. For instance, one could constrain the maximal acceleration of the shuttle, i.e. a constraint of the form

$$\|u_k\|_2^2 \leq u_{max} \quad \text{for} \quad k = 0, \ldots, N-1.$$

Another possible refinement could be optimizing $\bar{x}_0$. One could add the start position of the shuttle as an optimization variable on a given radius around the Earth. Further, the time of the Moon mission could be optimized, or the shuttle could be controlled on a stable orbit around the Moon.

6 Acknowledgement

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