FACULTY OF ENGINEERING

The Systems Control and Optimization Laboratory

Optimal control problem:

A simplified moon mission

David Pellhammer and Jonas Hall

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Contents

1 Abstract

In this project we aim to optimally control a flight of a shuttle to the moon. Starting at a geostationary orbit around the earth the shuttle should reach the moon while minimizing the used energy and therefore minimizing the additional acceleration that is applied by the shuttle. We describe the formulation of a Non Linear Programming problem and the reformulations required to deal with the non-linear equality constraints. For the solution CasAdi IPOPT (Andersson u. a., In Press, 2018) solver is used.

2 Problem description

Les us first describe the movement of the moon and space shuttle in a partial differential equation. We use a simplified 2D model centering earth at the origin. For clarification of which object we are referencing we use indices, e.g. M_E will describe the mass of the earth, v_M the speed vector of the moon, and p_S the position of our space shuttle. Further we are using the gravitational constant $G = 6.674310^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$. In order to keep variables and parameters at a reasonable value we express distance and time by means of astronomical units 1AU = 384403000m and 1AT = 2551400s, such that 1AU is the distance between earth and moon and 1AT is the time of one moon orbit around the earth. All other parameters are scaled using these units.

2.1 Underlying dynamics

Using partial differential equations based on (Giordano u. a., 2006) we first describe the motion of the moon due to the mass of earth by the system

$$\frac{dv_M}{dt} = -\frac{G_E M_E (p_M - p_E)}{d(p_E, p_M)^3} \tag{1}$$

$$\frac{dp_M}{dt} = v_M. \tag{2}$$

We need the position and velocity of the moon on its orbit in order to determine the impact of the mass of the earth and moon on the shuttle. Therefore a simulation of the moon orbit is done prior to the optimal control formulation. This is only possible with the simplification that the mass of the shuttle is negligible compared to the mass of earth and moon which is a reasonable simplification. Additionally we assume that the mass of moon does not impact the earth in order to keep the earth in the origin. We will need to keep the position and velocity when calculating our space shuttle dynamics, which are determined by the impact of mass of earth, mass of moon, and most importantly our two-dimensional acceleration control variables. The resulting differential equations are of the following form

$$\frac{dv_S}{dt} = -\frac{G_E M_E (p_S - p_E)}{d(p_E, p_S)^3} - \frac{G_M M_M (p_S - p_M)}{d(p_M, p_S)^3} + u \tag{3}$$

$$\frac{dp_S}{dt} = v_S. \tag{4}$$

The integration of these systems is done by means of an RK4-integrator.

3 The optimal control problem

Having described the underlying dynamics we can formulate the optimal control problem.

3.1 States

For a clear formulation we define the used states and controls at first. The state x is a four dimensional vector with the position and velocity of the shuttle. The control u is the acceleration of the shuttle in both dimensions. Further we make the assumption, that our shuttle can accelerate in any direction.

$$x = \begin{pmatrix} p_x \\ p_y \\ v_x \\ v_y \end{pmatrix}, u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{5}$$

3.2 Basic approach

Let f denote the RK4-integration of our dynamic system. Given an initial state x_0 we shall optimize

$$\begin{aligned} \underset{x,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \|u_i\|_2^2 && (6)\\ \text{subject to} \quad & x_0 = \bar{x}_0 && (7)\\ & x_{k+1} = f(x_k, u_k) \qquad \text{for} \quad k = 0, \dots, N-1 && (8)\\ & v_{S,N} = v_{M,N} && (9)\\ & p_{S,N} = \tilde{p}_{S,N}, && (10) \end{aligned}$$

where $\tilde{p}_{S,N}$ denotes the intended final position of the shuttle and r_M describes the radius of the moon. The chosen objective function is used to minimize the energy that is used during the flight. The equality constraints (9) and (10) ensure that the shuttle reaches a point close to the moon and has the same velocity as the moon. Due to the strong final state equality constraints this approach does not terminate in a reasonable number of iterations. This Problem is solved by weakening the constraint in the following subsection.

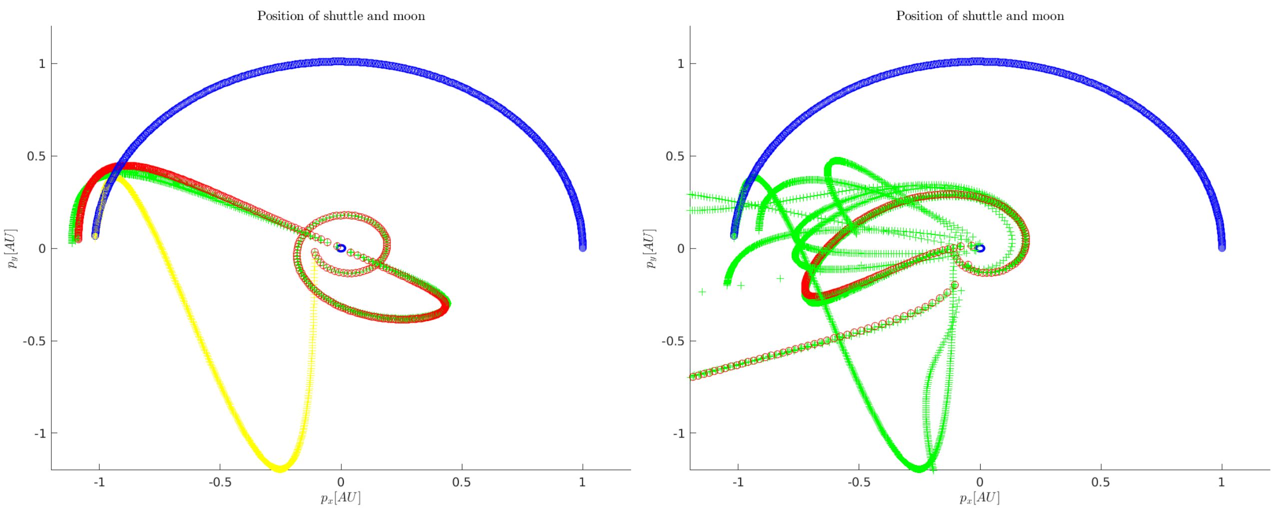

Figure 1: Left: A first solution using a homotopy parameter. Right: Constraining shuttle to not 'hit' earth may not be strict enough.

3.3 Using a homotopy parameter ε

Since the final state equality constraints are hard to fulfill we replace those by only requesting the final states to be close to the desired state. This is done by introducing a two-dimensional parameter ε that weakens the final state constraints by giving it a margin. We get the optimization problem

$$\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \|u_i\|_2^2 && && (11)\\
\text{subject to} \quad & x_0 = \bar{x}_0 && && (12)\\
& x_{k+1} = f(x_k, u_k) && \text{for} \quad k = 0, \dots, N-1 && (13)\\
& \|p_{S,N} - \tilde{p}_{S,N}\|_2^2 \leq \varepsilon_{pos} && && (14)\\
& \|v_{S,N} - v_{M,N}\|_2^2 \leq \varepsilon_{vel} && && (15)
\end{aligned}$$

The convex constraints (14) and (15) are added to reach the final state. The idea of the homotopy is to start with an initial ε and solve the NLP. We then update $\varepsilon \leftarrow \varepsilon\beta$ for some $\beta \in (0, 1)$ and solve the NLP again using the prior solution as initial guess. This procedure can be repeated as often as desired. An example to a solution using this method is shown in 1.
The solution obtained by this approach can be seen in figure (1).

3.4 Avoiding a potential crash

In order to improve the model some additional constraints are added. The shuttle is not supposed to fly to close or even through the moon and earth so some non-convex inequality constraints are added to the problem

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \|u_i\|_2^2 && && (16)\\
\text{subject to} \quad & x_0 = \bar{x}_0 && && (17)\\
& x_{k+1} = f(x_k, u_k) && \text{for} \quad k = 0, \dots, N-1 && (18)\\
& \|p_{S,N} - \tilde{p}_{S,N}\|_2^2 \le \varepsilon_{pos} && && (19)\\
& \|v_{S,N} - v_{M,N}\|_2^2 \le \varepsilon_{vel} && && (20)\\
& \|p_{S,N} - p_{M,N}\|_2^2 \ge r_M^2 && && (21)\\
& \|p_{S,k} - p_E\|_2^2 \ge r_E^2 && \text{for} \quad k = 0, \dots, N-1, && (22)
\end{aligned}
$$

where r_M and r_E are the radius of moon and the initial distance between earth and shuttle, respectively. Specifically the latter is a reasonable value since we are initializing with orbit speed and we want to diverge away from earth. If we simply chose r_E to be the radius of earth we experienced that given our step size the trajectory computation may fail. The shuttle will pick up a velocity that will be enough to 'jump across' earth with a single step and the solutions might become unstable (see figure 1).

4 Solving the NLP

To solve the NLP an interior point method is used. The optimization is done with the CasADi IPOPT NLP solver. Due to sparsity of the hessian of our Lagrangian a multiple-shooting approach is implemented. We noticed that solving the NLP is highly dependent on the initial and final states of the problem. For some setups the problem might even not converge in reasonable time iterations. To solve this non linear non convex problem we use several iterations while gradually changing the constraints from weak to strong. We noticed that for solving the problem for some setups the equality constraint (22) can be removed which simplifies the problem by removing a non convex constraint. This constraint might be added if the obtained solution is unstable because of the singularity in the dynamics.

4.1 Initial guess

The initial guess of the NLP is created by solving the same NLP in a simplified version. Therefore the basic approach is solved by neglecting the mass of earth and moon in the dynamics of the system. The objective function stays the same as in the final problem. Thus the initial guess is a trajectory from earth to moon without the impact of earth and moon acceleration. This might lead to a solution not considering the orbit of the planet but gives us state and control values in the right dimensions close to the solution.

4.2 Problem setup

To solve the above system we need to set the parameters of the problem. We choose that the shuttle should reach the moon after half the moon orbit around the earth. The time step is $\delta t = 0.001$, which

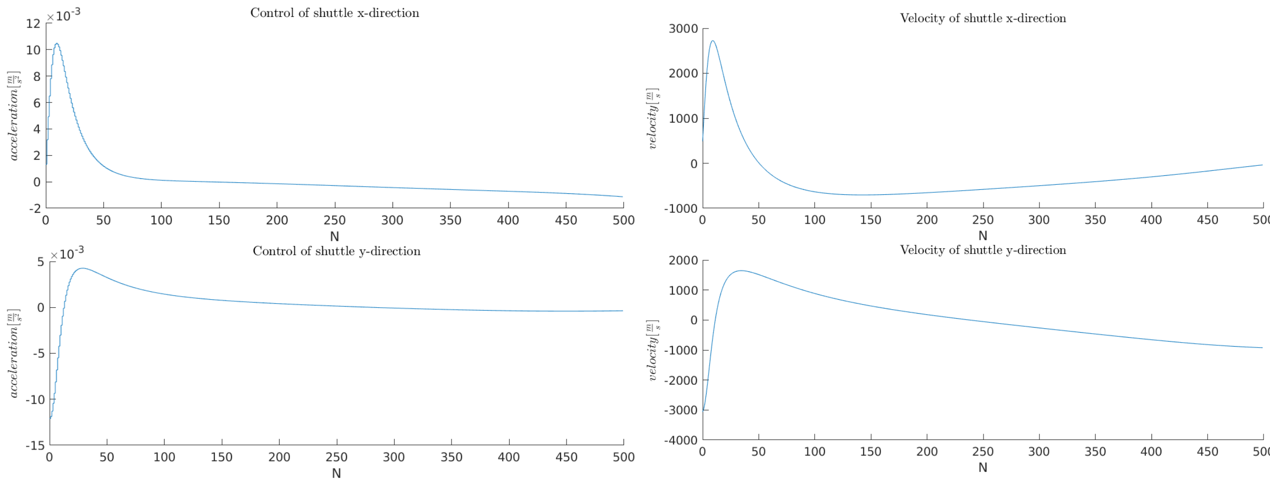

Figure 2: Left: Controls in x and y direction. Right: Velocity in x and y direction.

yields to $N = 500$. The initial homotopy parameters are set to $\varepsilon_{pos} = 0.1$ and $\varepsilon_{vel} = 1$ which are reduced by a factor of $\beta = \frac{1}{2}$ after each homotopy iteration. The initial state meaning the initial position and velocity of the shuttle is set to a geostationary orbit. The final position of the shuttle is set to reach a point close to the moon.

4.3 Final solution

One solution of the NLP can be seen in figure 3. The final solution shows the trajectory of the shuttle reaching the moon orbit. The green trajectories show the intermediate results of the homotopy. It can be seen that the solutions get closer to the final state when the ε- parameter is reduced. The red trajectory shows the final solution. For this solution the inequality constraint to get not to close to the earth is added. The controls and velocity of this final solution are given in figure 2.
We produced another solution by removing both the inequality constraint to the earth and the homotopy method by choosing $\varepsilon = 0s$. The solution is shown in figure 3. At first glance it seems better than the solution above, and indeed it also has a smaller cost value. However, this solution is extremely dependent on the chosen initial guess and the start and end positions and velocities. For small changes in these parameters the optimization does often not terminate at all or gives solutions with much higher cost. The other solution is much more stable in this sense.

4.4 Comparing parameter setting

For the optimization some parameters are set to reach the given aim. Most importantly the number of optimization steps while decreasing the ε, the β that is multiplied on the ε and the initial ε itself. In table 1 a comparison of different settings are shown. The first row shows a setting that converges quite fast and uses 4 iterations. If the final ε values is used from the beginning doing only one iteration the optimizer does not converge within reasonable time. The third row shows that increasing the number does take more time but is not decreasing the objective. However the distance to the final position is decreased. The last row shows the setting for the solution seen in figure 3. The setting of the first row is chosen to do the optimization.

steps	ε_{pos}	ε_{vel}	β	solution found	constraint on earth	time	objective
4	0.1	1	2	yes	yes	97.7	$1.01 \cdot 10^6$
1	$\frac{0.1}{2^4}$	$\frac{1}{2^4}$	-	no	yes	-	-
8	0.1	1	2	yes	yes	115.1	$1.02 \cdot 10^6$
4	0.1	1	10	yes	yes	113.6	$1.04 \cdot 10^6$
1	0	0	-	yes	no	43.5	$4.39 \cdot 10^5$

Table 1: Comparison of different parameter for the optimization

5 Problems and possible improvements

5.1 Experienced Issues

Since the problem is non linear and non convex the solution is depending on the initial guess. We noticed that the success of the optimization is highly depending on the the set initial and final state. It is also depending on the choice of the parameter ε. Thus it is hard to obtain a good general solution. Another issue was also to deal with the singularities of the given dynamics. Constraining the problem to avoid getting close to them adds non-convexity to the problem.

5.2 Further development

Unfortunately this project was constraint to a short termed period. We have many more ideas for further improvements. For instance one could constrain the maximal acceleration of the shuttle, i.e.

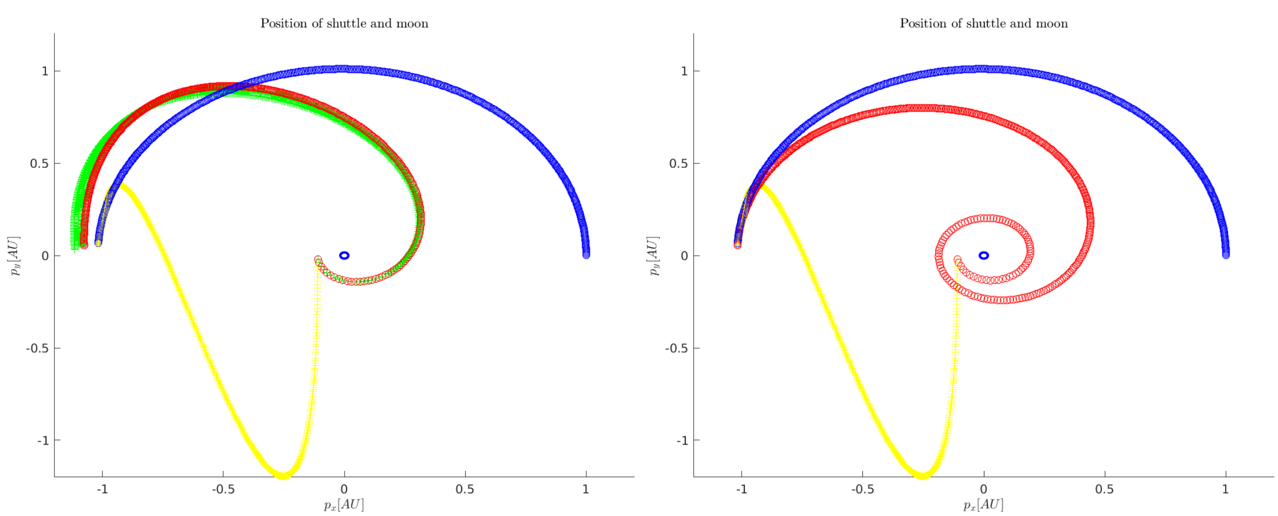

Figure 3: The two different solutions. Left we plot the solution using homotopy (green) and an additional constraint. On the right hand side we eliminated both. The red trajectory are the final trajectories and the yellow lines represent the initial guesses.

a constraint of the form

$$\|u_k\|_2^2 \leq u_{max} \quad \text{for} \quad k = 0, \ldots, N-1.$$

Another possible refinement could be optimizing $\bar{x}_0$. One could add the start position of the shuttle as an optimization variable on a given radius around the earth. Further the time of the moon mission could be optimized or the moon can be controlled on an stable orbit around moon.

References

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