Prob 1 $\mu = E[x]$, $f(x)$ DF

1) Show $\text{cov}(x,y) = E[g(x)(x-\mu)]$

$$\text{cov}(x,y) = E[xy] - E[x]E[y$$
$$= E[x\,g(x)] - \mu E[g(x)]$$
$$= E[g(x)(x-\mu)]$$

Used $E[y] = E[E[y|x]] = E[g(x)]$

$x > \mu \Rightarrow g(x) > g(\mu)$ by definition

multiplying by $(x-\mu) > 0$

$$g(x)(x-\mu) > g(\mu)(x-\mu)$$

$x \leq \mu \Rightarrow g(x) \leq g(\mu)$ multiply by $(x-\mu) < 0$

$$g(x)(x-\mu) \geq g(\mu)(x-\mu)$$

Therefore, $g(x)(x-\mu) \geq g(\mu)(x-\mu)$ $\forall x$

$$E[g(x)(x-\mu)] \geq E[g(\mu)(x-\mu)]$$
$$\geq E[g(\mu)x] - E[g(\mu)(\mu)]$$
$$\geq E[x]g(\mu) - g(\mu)E[\mu]$$
$$\geq 0$$

Therefore $\text{cov}(x,y) = E[g(x)(x-\mu)] \geq 0$

<u>problem 1</u>

a) $y_i = \beta_0 + \beta_1 x_i + \mu_i$

$$\beta^* = \underset{\beta}{\text{argmin}} \frac{1}{N}\sum_{1} [(y_i - \beta_0 - \beta_1 x_i)^2]$$

FC: $\frac{1}{N}\sum x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$ (1)

$\frac{1}{N}\sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$ (2)

(2) $\frac{1}{N}\sum_i y_i = \hat{\beta}_0 + \hat{\beta}_1 \frac{1}{N}\sum_i x_i$

$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$ □

b) $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + u_i$

$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

$y_i - \bar{y} = \hat{\beta}_0 - \hat{\beta}_0 - \hat{\beta}_1 (x_i - \bar{x}) + u$

$= \beta_1 (x_i - \bar{x})$ □

c) $\frac{1}{N}\sum (y_i - \bar{y}) u_i = \left[\frac{1}{N}\sum \hat{\beta}_1 (x_i - \bar{x}) + u_i\right] u_i$

By FC $u \perp\!\!\!\perp x_i \quad \forall i$

$= \frac{1}{N}\sum_i u_i^2$

$\frac{1}{N}\sum_i (y_i - \bar{y}) u_i = \frac{1}{N}\sum_i u_i^2$ (3)

d) solve for $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{(y - \bar{y}) - u_i}{(x_i - \bar{x})}$$

plug back into

$$\beta_1 \frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x}) = \frac{1}{N}\sum_i (y_i - \bar{y})\left[\frac{(y - \bar{y}) - u_i}{(x_i - \bar{x})}\right](x_i - \bar{x})$$

$$= \frac{1}{N}\sum_i (y_i - \bar{y})^2 - \underbrace{\frac{1}{N}\sum_i (y_i - \bar{y}) u_i}_{(7)}$$

$= \frac{1}{N}\sum_i (y_i - \bar{y})^2 - \frac{1}{N}\sum_i u_i^2$ □

e)

$$R^2 = 1 - \frac{\frac{1}{N}\sum u_i^2}{\frac{1}{N}\sum_i (y_i - \bar{y})^2}$$

$$\hat{\beta}_1 = \frac{\frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{N}\sum_i (x_i - \bar{x})^2}$$

$$\underbrace{\left(\frac{\frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{N}\sum_i (x_i - \bar{x})^2}\right)}_{\hat{\beta}_1} \frac{1}{N}\sum_i (y_i - \bar{y})(x_i - \bar{x}) = \frac{1}{N}\sum_i (y_i - \bar{y})^2 - \frac{1}{N}\sum_i u_i^2$$

$$\left(\frac{\left[\frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x})\right]^2}{\frac{1}{N}\sum_i (x_i - \bar{x})^2}\right) = \frac{1}{N}\sum (y_i - \bar{y})^2 - \frac{1}{N}\sum u_i^2$$

Lets multiply LHS & RHS by $\frac{1}{\frac{1}{N}\sum_i (y_i - \bar{y})^2}$

$$\frac{1}{\frac{1}{N}\sum_i (y_i - \bar{y})^2}\left(\frac{\left[\frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x})\right]^2}{\frac{1}{N}\sum_i (x_i - \bar{x})^2}\right) = \left(\frac{1}{N}\sum (y_i - \bar{y})^2 - \frac{1}{N}\sum u_i^2\right)\frac{1}{\frac{1}{N}\sum_i (y_i - \bar{y})^2}$$

$$= \left(\frac{\frac{1}{N}\sum (y_i - \bar{y})^2}{\frac{1}{N}\sum (y_i - \bar{y})^2} - \frac{\frac{1}{N}\sum u_i^2}{\frac{1}{N}\sum (y_i - \bar{y})^2}\right)$$

$$\underbrace{\left(\frac{\left[\frac{1}{N}\sum (y_i - \bar{y})(x_i - \bar{x})\right]}{\left[\frac{1}{N}\sum_i (x_i - \bar{x})^2 (y_i - \bar{y})^2\right]^{1/2}}\right)}_{\rho_{x,y}}{}^2 = \underbrace{1 - \frac{\frac{1}{N}\sum u_i^2}{\frac{1}{N}\sum (y_i - \bar{y})^2}}_{R^2}$$

Thus $\rho^2_{x,y} = R^2$

Modify answer in 1c)

$$\frac{\frac{1}{N}\sum(y_i-\bar{y})(x_i-\bar{x})}{\frac{1}{N}\sum_i(y_i-\bar{y})^2} \quad \frac{1}{N}(y_i-\bar{y})(x_i-\bar{x}) = \frac{1}{N}\sum(x_i-\bar{x})^2 - \frac{1}{N}\sum u_i^2$$

If we instead multiply by $\dfrac{1}{\frac{1}{N}\sum(x_i-\bar{x})}$

$$\frac{\left(\frac{1}{N}\sum_i(y_i-\bar{y})(x_i-\bar{x})\right)^2}{\frac{1}{N}\sum_i(x_i-\bar{x})(y_i-\bar{y})} = 1 - \frac{\frac{1}{N}\sum_i u_i^2}{\frac{1}{N}\sum_i(x_i-\bar{x})^2}$$

By taking the square root of the denom on the left hand side and then square the LHS

$$\Rightarrow \left[\frac{\frac{1}{N}\sum(y_i-\bar{y})(x_i-\bar{x})}{\left(\frac{1}{N}\sum_i(x_i-\bar{x})^2(y_i-\bar{y})^2\right)^{1/2}}\right] = R^2$$

$$\Rightarrow \rho_{xy}^2 = R^2 = \rho_{yx}^2 \quad \square$$