Combinatorics Seminar

Combinatorial Geometry

Based on "Extremal problems for pairs of triangles in a convex polygon"

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Convex Geometric Hypergraph

- A convex geometric hypergraph or cgh is a family of subsets of a set of points in strictly convex position in the plane
- We assume these points, denoted by $\Omega_n = v_0, v_1, ..., v_{n-1}$, are the vertices of some regular n-gon with the clockwise cylic ordering $v_0 < v_1 < \cdots < v_{n-1} < v_0$.
- Let the extremal function $ex_{\circlearrowright}(n, F)$ denote the maximum number of edges in an r-uniform cgh on n points that does not contain F – *an F-free cgh*
- In this paper, we concentrated on intersection patterns of pairs of triangles (r = 3)

Theorem 1 (partial)

For all $n \geq 3$, if H is an extremal n-vertex M_1-free cgh, then $H \in \mathcal{H}'(n)$ and:

$$ex_{\circlearrowright}(n, M_1) = \triangle(n) + \frac{n(n-3)}{2}$$

where:

Construction 1 - $\mathcal{H}^*(n)$

For $n \geq 3$ odd, let $\mathcal{H}^*(n)$ comprise the single cgh consisting of triangles which contain in their interior the centroid of Ω_n.

[FIGURA 1]

- There are i + 1 choices of $v\prime, v\prime\prime$ so that the shorter arc has exactly i vertices of Ω_n in the interior, and $i \leq n{-}3$.
- Summing over i (and dividing by 3 because we count each triangle 3 times), we get that the number of triangles containing x equals:

[FIGURA 2]

$$\frac{n}{3}(1+2+...+\frac{n-1}{2}) = \frac{(n+1)n(n-1)}{24}$$

For $n \geq 4$ even, each $H \in \mathcal{H}^*(n)$ consists of all triangles which contain the centroid of Ω_n and, for each diameter $\{v_i, v_{i+n/2}\}$ of Ω_n, we either add all triangles $\{v_i, v_j, v_{i+n/2}\}$ where $v_i < v_j < v_{i+n/2}$, or all triangles $\{v_i, v_j, v_{i+n/2}\}$ where $v_{i+n/2} < v_j < v_i$.

FIGURA 3

- **If n is even** then we do the same count, but first excluding the triangles with one side being the diameter of the circle. We get:

$$\frac{n(n-2)(n-4)}{24}$$

- Next, out of triangles with one side being the diameter exactly half of them contain x, which equals $\frac{n(n-2)}{4}$ Summing up, we get that the number of triangles containing x is

$$\frac{n(n-2)(n-2)}{24}$$

- As no two triangles in any $H \in \mathcal{H}^*(n)$ have disjoint interiors, we have:

$$ex_{\circlearrowright}(n, \{M_1, S_1, D_1\}) \geq \triangle(n)$$

- Frankl, Holmsen and Kupavskii posed the following problem: *What happens if one relaxes the intersecting condition and allows triangles to intersect on the boundary?*
- One way to see it is we allow S_1 and D_1 and look for $ex_{\circlearrowright}(n, M_1)$

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Construction 2

For $n \geq 3$ odd, the unique cgh in H?(n) is obtained by adding all triangles containing a pair {vi, vi+(n−1)/2} to any cgh in H?(n) (left diagram in Figure 2). For $n \geq 4$ even, H?(n) is obtained by adding all triangles containing a diameter of Ωn, plus all triangles containing a pair from a set of n/2 pairwise intersecting pairs of the form {vi, vi+n/2−1} (right diagram in Figure 2).

[FIGURA PDF]

- By inspection, every cgh in H?(n) is M1-free, and has size ·?(n) + n(n − 3)/2.

FIGURAS 4 e 5

Let n ≥ 3 be odd.

- If H ∈ H?(n) then we are done, so we may assume H contains a triangle T(i, j, k) = {vi, vj, vk} with vi < vj < vk < vi+(n−1)/2.
- Moreover, we may assume that among all such triangles, T(i, j, k) is the triangle where the longest edge {vi, vk} is as short as possible.
- Replace all triangles T(i, j?, k) ∈ H with i < j? < k with all triangles T(i − 1, k + 1, l) where j and l are on opposite sides of the edge {vi, vk} as shown in Figure 4.

FIGURA PDF

- Since T(i, j, k) and T(i − 1, k + 1, l) form a copy of M1, T(i − 1, k + 1, l) ?∈ H for all such l. Moreover, since vi < vk < vi+(n−1)/2, the number of triangles T(i − 1, k + 1, l) that we added is greater than the number of triangles T(i, j, k) that we deleted.
  - Note that none of $T(i-1,k+1,l)$ is in H, otherwise they would form a copy of M_1 with $T(i,j,k)$
- This produces a cgh H? with |H?| > |H|. Since H is extremal M1-free, there exists a copy of M1 in H?, which must contain a triangle T(i − 1, k + 1, l) ∈ H?