Martin solutions for Exercises

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1 Chapter 5

5.1 From the definition of 2-form $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots + \mathrm{d}x^{2n-1} \wedge x^{2n}$, it follows

$$\begin{aligned}\omega \wedge \omega = {} & \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge (\mathrm{d}x^3 \wedge \mathrm{d}x^4 + \cdots + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}) \\ & + \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}) \\ & + \cdots \\ & + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n} \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots \mathrm{d}x^{2n-3} \wedge \mathrm{d}x^{2n-2}) \\ = {} & 2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \cdots + 2\mathrm{d}x^{2n-3} \wedge \mathrm{d}x^{2n-2} \wedge \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}\end{aligned}$$

Also, in the similar manner, we have

$$\begin{aligned}\omega \wedge \omega \wedge \omega = {} & 2\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge \omega \\ & + \cdots \\ & + 2\mathrm{d}x^{2n-3} \wedge \mathrm{d}x^{2n-2} \wedge \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n} \wedge \omega \\ = {} & 2^2\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^6 + \cdots + 2^2\mathrm{d}x^{2n-5} \wedge \cdots \wedge \mathrm{d}x^{2n}\end{aligned}$$

Eventually, we will obtain

$$\therefore \underbrace{\omega \wedge \cdots \wedge \omega}_{n} = 2^n \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n}$$

5.2 For $f,\, g \in C^\infty(\mathbb{R}^2)$, their exterior derivatives are

$$\mathrm{d}f = \partial_i f \, \mathrm{d}x^i, \quad \mathrm{d}g = \partial_j g \, \mathrm{d}x^j \qquad (i, j = 1, 2)$$

Then, the wedge product of those two gives

$$\begin{aligned}\mathrm{d}f \wedge \mathrm{d}g &= \partial_i f \mathrm{d}x^i \wedge \partial_j g \mathrm{d}x^j \\ &= (\partial_i f)(\partial_j g) \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= [\partial_1 f \partial_2 g - \partial_2 f \partial_1 g] \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= \frac{\partial(f,\, g)}{\partial(x^1,\, x^2)} \mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

5.3 From given 2-form $\omega = \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j (i < j)$,

$$\begin{aligned}\mathrm{d}\omega &= \partial_k \omega_{ij} \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \partial_k \omega_{ij} [\mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j - \mathrm{d}x^i - \mathrm{d}x^k \wedge \mathrm{d}x^j + \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k]/3 \\ &= \frac{1}{3}(\partial_k \omega_{ij} - \partial_j \omega_{ik} + \partial_i \omega_{jk}) \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= 0\end{aligned}$$

since ω is closed. Hence, we obtain

$$\therefore \partial_k \omega_{ij} - \partial_j \omega_{ik} + \partial_i \omega_{jk} = 0$$

5.4 On a vector space V_4, a form, ω is defined as

$$\omega = x^1\mathrm{d}x^1 + x^2\mathrm{d}x^2 + x^3\mathrm{d}x^3 + \left[1 + (x^1)^2 + (x^2)^2 + (x^3)^2\right]\mathrm{d}x^4.$$

Then, its exterior derivative reads

$$\begin{aligned}\mathrm{d}\omega &= 2\left[x^1\mathrm{d}x^1 + x^2\mathrm{d}x^2 + x^3\mathrm{d}x^3\right]\wedge\mathrm{d}x^4\\ &= 2\omega\wedge\mathrm{d}x^4\\ &\neq 0\end{aligned}$$

for non-zero x^1, x^2, x^3. The goal is to find $f = f(x^1, x^2, x^3)$ such that $f\omega = \mathrm{d}\varphi$. Then we have

$$\mathrm{d}f\wedge\omega + f\,\mathrm{d}\omega = 0.$$

Since $\mathrm{d}f = \partial_i f\mathrm{d}x^i$ $(i = 1, 2, 3)$,

$$\partial_i f\,\mathrm{d}x^i\wedge\omega + 2f\omega\wedge\mathrm{d}x^4 = 0.$$

By matching each components, we have

$$\begin{aligned}&(\partial_i f)\left[1 + x_i x^i\right] + 2x^i f = 0\\ &\varepsilon_{ijk}(\partial_i f)x^j = 0,\end{aligned}$$

i.e., f is the function of $x_i x^i$ since it is curl-free. Thus, from the first equation, we have

$$\therefore\ f(x^1, x^2, x^3) = 1 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$$

5.5

$$\omega = \frac{1}{r^n}\sum_{i=1}^{n}(-1)^{i-1}x^i\,\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n$$

where $r^2 = \sum_{i=1}^n (x^i)^2$. Then, its exterior derivative is

$$\begin{aligned}\mathrm{d}\omega &= \frac{n}{r^n}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n - \frac{n}{r^{n+2}}x^i\,\mathrm{d}x^i\wedge\sum_{i=1}^{n}(-1)^{i-1}x^i\,\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n\\ &= \left[\frac{n}{r^n} - \frac{n}{r^{n+2}}\sum_{i=1}^{n}(x^i)^2\right]\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\\ &= 0\end{aligned}$$

Hence, ω is a closed form. For the case $n = 3$,

$$\begin{aligned}\omega &= \frac{1}{r^3}\left(x^1\mathrm{d}x^2\wedge\mathrm{d}x^3 - x^2\mathrm{d}x^1\wedge\mathrm{d}x^3 + x^3\mathrm{d}x^1\wedge\mathrm{d}x^2\right)\\ &= \mathrm{d}\theta\end{aligned}$$

where θ is a 1-form on an open set starlike with regard to $(1, 1, 0)$. If we write θ as $\theta = \theta_i\mathrm{d}x^i$,

$$\mathrm{d}\theta = (\partial_1\theta_2 - \partial_2\theta_1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 + (\partial_2\theta_3 - \partial_3\theta_2)\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + (\partial_3\theta_1 - \partial_1\theta_3)\,\mathrm{d}x^3\wedge\mathrm{d}x^1$$

It means that when we write $\vec{\theta} = (\theta_1,\, \theta_2,\, \theta_3)$, then we have

$$\nabla\times\vec{\theta} = \frac{1}{r^2}\hat{r} \quad\text{or}\quad \varepsilon_{ijk}\partial_i\theta_j = \frac{1}{r^3}x_k$$

5.6 Given $\omega = f(y^1, \cdots, y^n)\mathrm{d}y^{i_1}\wedge\cdots\wedge\mathrm{d}y^{i_p}\in\Lambda^p(N)$, according to the theorem 5.1.1, a pull-back operation is given by

$$\begin{aligned}\varphi^*\omega &= f(y^1,\, \cdots,\, y^n)\varphi^*(\mathrm{d}y^{i_1}\wedge\cdots\wedge\mathrm{d}y^{i_p})\\ &= f(x^1, \cdots, x^m)\varphi^*\mathrm{d}y^{i_1}\wedge\cdots\wedge\varphi^*\mathrm{d}y^{i_p}\\ &= f(x^1, \cdots, x^m)\frac{\partial y^{i_1}}{\partial x^{j_1}}\cdots\frac{\partial y^{i_p}}{\partial x^{j_p}}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\\ &= f(x^1, \cdots, x^m)\frac{\partial(y^{i_1}\cdots y^{i_p})}{\partial(x^{i_1}\cdots x^{i_p})}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\end{aligned}$$

5.7 Since

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \qquad (x^1, \cdots, x^n) \mapsto (y^1, \cdots, y^n),$$

we have

$$\begin{aligned}\Phi^*(\mathrm{d}y^1 \wedge \mathrm{d}y^2) &= \sum_{i,j} \frac{\partial y^1}{\partial x^i}\frac{\partial y^2}{\partial x^j}\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i<j} \frac{\partial(y^1, y^2)}{\partial(x^i, x^j)}\mathrm{d}x^i \wedge \mathrm{d}x^j\end{aligned}$$

5.8 Let $\omega = (\omega_1 \wedge \omega_2) + (\omega_3 \wedge \omega_4)$ where $\omega_i \in \mathcal{D}_1(M)$ are linearly independent. Then,

$$\begin{aligned}\omega \wedge \omega &= \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 + \omega_3 \wedge \omega_4 \wedge \omega_1 \wedge \omega_2 \\ &= 2\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4\end{aligned}$$

since ω_i are linearly independent. If ω were decomposable, we may write $\omega = \alpha \wedge \beta$ and

$$\omega \wedge \omega = (\alpha \wedge \beta) \wedge (\alpha \wedge \beta) = 0.$$

Hence, ω is not decomposable.

5.9 The annihilator $\mathrm{Ann}(\omega)$ of $\omega \in \Lambda^p(V_n)$ is

$$\mathrm{Ann}(\omega) = \{\varphi \in V_n^* : \varphi \wedge \omega = 0\}$$

Then, for a coordinate $\{x^i\}$, we may write

$$\omega = \omega_{i_1,\cdots,i_p}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}.$$

To have $\varphi \wedge \omega = 0$, φ should have one of the basis $\{\mathrm{d}x^{i_k}\}$, $k = 1, \cdots, p$.

2 Chapter 6

6.1 When $\omega \in \Lambda^1(M)$, then $\omega(X) \in C^\infty$ where $X \in \mathcal{D}^1(M)$. From the assumption, it follows that

$$\begin{aligned}\mathcal{L}_Y(\omega X) &= (\mathcal{L}_Y\omega)X + \omega(\mathcal{L}_Y X) \\ &= (\mathcal{L}_Y\omega)X + \omega[Y,\, X] \\ &= Y(\omega(X))\end{aligned}$$

,i.e.,

$$(\mathcal{L}_Y\omega)X = Y(\omega(X)) - \omega[Y,\, X]$$

where $Y \in \mathcal{D}^1(M)$. For $X = \partial_i$ in natural basis,

$$\begin{aligned}(\mathcal{L}_Y\omega)_i &= Y(\omega_i) - \omega[Y,\, \partial_i] \\ &= Y^j\partial_j\omega^i + \omega_j\partial_i Y^j\end{aligned}$$

Hence we obtain

$$\mathcal{L}_X Y_a = X^b\partial_b Y_a + Y_b\partial_a X^b.$$

6.2 Let $T \in \mathcal{D}^r_s(M)$ and $X, Y \in \mathcal{D}^1(M)$.
(i) For $a, b \in \mathbb{R}$, write $Z = aX + bY$. In a natural coordinate, $\{x^i\}$, we have

$$\begin{aligned}&T = T^{i_1\cdots i_r}_{j_1\cdots j_s}\,\partial_{i_1} \otimes \cdots \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \\ &(\mathcal{L}_Z T)^{i_1\cdots i_r}_{j_1\cdots j_s} = ZT^{i_1\cdots i_r}_{j_1\cdots j_s} - T^{ii_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial Z^{i_1}}{\partial x^i} - \cdots + T^{i_1\cdots i_r}_{jj_2\cdots j_s}\frac{\partial Z^j}{\partial x^{j_1}} + \cdots\end{aligned}$$

But since a partial derivative is also a linear operator, we have

$$\mathcal{L}_{aX+bY}T = a\mathcal{L}_X T + b\mathcal{L}_Y T$$

(ii) If we calculate $\mathcal{L}_X(\mathcal{L}_Y T)$,

$$\begin{aligned}
\mathcal{L}_X(\mathcal{L}_Y T)^{i_1\cdots i_r}_{j_1\cdots j_s} &= X(\mathcal{L}_Y T)^{i_1\cdots i_r}_{j_1\cdots j_s} - (\mathcal{L}_Y T)^{i i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial X^{i_1}}{\partial x^i} - \cdots + (\mathcal{L}_Y T)^{i_1\cdots i_r}_{j j_2\cdots j_s}\frac{\partial X^j}{\partial x^{j_1}} + \cdots \\
&= XYT^{i_1\cdots i_r}_{j_1\cdots j_s} + X\left(-T^{i i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial Y^{i_1}}{\partial x^i} + \cdots + T^{i_1\cdots i_r}_{j j_2\cdots j_s}\frac{\partial Y^j}{\partial x^{j_1}} + \cdots\right) \\
&\quad - (\mathcal{L}_Y T)^{i i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial X^{i_1}}{\partial x^i} - \cdots + (\mathcal{L}_Y T)^{i_1\cdots i_r}_{j j_2\cdots j_s}\frac{\partial X^j}{\partial x^{j_1}} + \cdots \\
&= XYT^{i_1\cdots i_r}_{j_1\cdots j_s} + X\left(-T^{i i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial Y^{i_1}}{\partial x^i} + \cdots + T^{i_1\cdots i_r}_{j j_2\cdots j_s}\frac{\partial Y^j}{\partial x^{j_1}} + \cdots\right) \\
&\quad - YT^{i i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial X^{i_1}}{\partial x^i} + T^{k i_2\cdots i_r}_{j_1\cdots j_s}\frac{\partial Y^i}{\partial x^k}\frac{\partial X^{i_1}}{\partial x^i} + \cdots \\
&\quad + YT^{i_1\cdots i_r}_{j j_2\cdots j_s}\frac{\partial X^j}{\partial x^{j_1}} - T^{k i_2\cdots i_r}_{j j_2\cdots j_s}\frac{\partial Y^{i_1}}{\partial x^k}\frac{\partial X^j}{\partial x^{j_1}} + \cdots
\end{aligned}$$

Then, we obtain the following result

$$\begin{aligned}
\mathcal{L}_X(\mathcal{L}_Y T)^{i_1\cdots i_r}_{j_1\cdots j_s} - \mathcal{L}_Y(\mathcal{L}_X T)^{i_1\cdots i_r}_{j_1\cdots j_s} &= [X,Y]T^{i_1\cdots i_r}_{j_1\cdots j_s} - T^{i i_2\cdots i_r}_{j_1\cdots j_s}(X\partial_i Y^{i_1} - Y\partial_i X^{i_1}) + \cdots \\
&= [X,Y]T^{i_1\cdots i_r}_{j_1\cdots j_s} - T^{i_1 i_2\cdots i_r}_{j_1\cdots j_s}\partial_i[X,Y]^{i_1} + \cdots \\
&= \mathcal{L}_{[X,Y]}T^{i_1\cdots i_r}_{j_1\cdots j_s}
\end{aligned}$$

Hence,

$$\therefore \mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) = \mathcal{L}_{[X,Y]}T$$

6.3 For $X, Y, Z \in \mathcal{D}^1(M)$, the Lie derivative is defined by

$$\mathcal{L}_X Y = [X,Y]$$

,i.e., in natural basis, $(\mathcal{L}_X Y)^i = [X,Y]^i = X^j\partial_j Y^i - Y^j\partial_j X^i$. Then,

$$\begin{aligned}
\mathcal{L}_X\mathcal{L}_Y Z = \mathcal{L}_X([Y,Z]) &= [X,[Y,Z]] \\
&= X[Y,Z] - [Y,Z]X \\
&= XYZ - XZY - YZX + ZYX \\
&= -YXZ + XYZ + YXZ - YZX + ZYX - ZXY + ZXY - XZY \\
&= [X,Y]Z + Y[X,Z] + Z[Y,X] + [Z,X]Y \\
&= -[Z,[X,Y]] - [Y,[Z,X]] \\
&= -\mathcal{L}_Z\mathcal{L}_X Y - \mathcal{L}_Y\mathcal{L}_Z X
\end{aligned}$$

Hence,

$$\therefore\ \mathcal{L}_X\mathcal{L}_Y Z + \mathcal{L}_Y\mathcal{L}_Z X + \mathcal{L}_Z\mathcal{L}_X Y = 0$$

6.4 For X, $Y \in \mathcal{D}^1(M)$ and $f \in C^\infty(M)$, since f and X are smooth, we have

$$fX = f(X) \Rightarrow (f(X))_p = f(p)\cdot X_p,$$

fX is also a smooth vector field. From this definition,

$$\begin{aligned}
\mathcal{L}_{(fX)}Y = [fX,\, Y] &= (fX)Y - Y(fX) \\
&= f(XY - YX) - (Yf)X \\
&= f\mathcal{L}_X Y - (Yf)X
\end{aligned}$$

Hence,

$$\mathcal{L}_{(fX)}Y = f\mathcal{L}_X Y - (Yf)X$$

6.5 By considering the Lie derivative on a tensor field, we have

$$\begin{aligned}\mathcal{L}_X \delta^{i_1\cdots i_r}_{j_1\cdots j_r} = X^k\partial_k\delta^{i_1\cdots i_r}_{j_1\cdots j_r} - \delta^{ki_2\cdots i_r}_{j_1 j_2\cdots j_r}\partial_k X^{i_1} - \cdots - \delta^{i_1\cdots k}_{j_1\cdots j_r}\partial_k X^{i_r} \\ + \delta^{i_1 i_2\cdots i_r}_{k j_2\cdots j_r}\partial_{j_1}X^k + \cdots + \delta^{i_1\cdots i_r}_{j_1\cdots k}\partial_{j_r}X^k\end{aligned}$$

However, $-\delta^{ki_2\cdots i_r}_{j_1 j_2\cdots j_r}\partial_k X^{i_1} + \delta^{i_1 i_2\cdots i_r}_{k j_2\cdots j_r}\partial_{j_1}X^k = 0$. Hence,

$$\mathcal{L}_X \delta^{i_1\cdots i_r}_{j_1\cdots j_r} = 0.$$

6.6 When $\omega \in \Lambda^p(M)$, theorem says

$$\mathcal{L}_X\omega = i_X(\mathrm{d}\omega) + \mathrm{d}(i_X\omega)$$

By substituting $\mathrm{d}\omega \in \Lambda^{p+1}(M)$,

$$\begin{aligned}\mathcal{L}_X(\mathrm{d}\omega) &= i_X(\mathrm{d}^2\omega) + \mathrm{d}(i_X(\mathrm{d}\omega)) \\ &= \mathrm{d}(i_X(\mathrm{d}\omega)) \\ &= \mathrm{d}(i_X(\mathrm{d}\omega) + \mathrm{d}(i_X\omega)) \\ &= \mathrm{d}(\mathcal{L}_X\omega)\end{aligned}$$

Hence,

$$\therefore\ \mathcal{L}_X(\mathrm{d}\omega) = \mathrm{d}(\mathcal{L}_X\omega)$$

6.7 Consider $(M,\, g,\, \nabla)$ where g is a metric and ∇ is a connection. When $\omega \in \Lambda^1(M)$ and $X,\, Y \in \mathcal{D}^1(M)$, define a new connection

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X.$$

Then, for $f,\, g \in C^\infty(M)$ and $a,\, b \in \mathbb{R}$,
(i)

$$\begin{aligned}\tilde{\nabla}_{(fX+gY)}Z &= \nabla_{(fX+gY)}Z + \omega(fX+gY)Z + \omega(Z)(fX+gY) \\ &= f\nabla_X Z + g\nabla_Y Z + f\omega(X)Z + g\omega(Y)Z + f\omega(Z)X + g\omega(Z)Y \\ &= f(\nabla_X Z + \omega(X)Z + \omega(Z)X) + g(\nabla_Y Z + \omega(Y)Z + \omega(Z)Y) \\ &= f\tilde{\nabla}_X Z + g\tilde{\nabla}_Y Z\end{aligned}$$

(ii)

$$\begin{aligned}\text{(ii)}\tilde{\nabla}_X(aY+bZ) &= \nabla_X(aY+bZ) + \omega(X)(aY+bZ) + \omega(aY+bZ)X \\ &= a\nabla_X Y + b\nabla_X Z + a\omega(X)Y + b\omega(X)Z + a\omega(Y)X + b\omega(Z)X \\ &= a(\nabla_X Y + \omega(X)Y + \omega(Y)X) + b(\nabla_X Z + \omega(X)Z + \omega(Z)X) \\ &= a\tilde{\nabla}_X Y + b\tilde{\nabla}_X Z\end{aligned}$$

(iii)

$$\begin{aligned}\tilde{\nabla}_X(fY) &= \nabla_X(fY) + \omega(fY)X + \omega(X)fY \\ &= f\nabla_X Y + (Xf)Y + f\omega(Y)X + \omega(X)fY \\ &= f\tilde{\nabla}_X Y + (Xf)Y\end{aligned}$$

From the definition of torsion,T,

$$(X,\, Y) \mapsto T(X,\, Y) = \nabla_X Y - \nabla_Y X - [X,\, Y],$$

for $\tilde{\nabla}$, it follows that

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = \nabla_X Y - \nabla_Y X$$

,i.e., it has the same torsion as ∇.

6.8 Let Γ define a symmetric affine connection on a manifold M. Then, for $\psi_i \in \mathcal{D}_1(M)$, write

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j$$

Since both Γ and $\bar{\Gamma}$ are defined on M, if $\bar{\Gamma}$ also defines an affine connection, its difference must give a tensor of valence $(1, 2)$. But, manifestly, $\delta^i_j \psi_k + \delta^i_k \psi_j$ is a tensor of valence $(1, 2)$ and hence $\bar{\Gamma}$ is also an affine connection.
Now, a geodesic is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} x^i + \Gamma^i_{jk} \frac{\mathrm{d}}{\mathrm{d}\lambda} x^j \frac{\mathrm{d}}{\mathrm{d}\lambda} x^k = 0$$

Changing of a connection will give

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} x^i + (\Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j) \frac{\mathrm{d}}{\mathrm{d}\lambda} x^j \frac{\mathrm{d}}{\mathrm{d}\lambda} x^k = 0.$$

By changing of an affine parameter, $\lambda \to \lambda'$,

$$\left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)^2 \left[\frac{\mathrm{d}^2}{\mathrm{d}\lambda'^2} x^i + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda'} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda'}\right] + \frac{\mathrm{d}^2 \lambda'}{\mathrm{d}\lambda^2} \frac{\mathrm{d}}{\mathrm{d}\lambda'} x^i + 2 \left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)^2 \frac{\mathrm{d}x^i}{\mathrm{d}\lambda'} \psi_k \frac{\mathrm{d}x^k}{\mathrm{d}\lambda'} = 0.$$

Choosing a parameter such that $\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \lambda' + 2 \left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)^2 \psi_k \frac{\mathrm{d}x^k}{\mathrm{d}\lambda'} = 0$, we can verify that both connections give the same geodesic. Also, in the case of a Riemann tensor,

$$\begin{aligned}
\bar{R}^i{}_{jkl} &= \partial_k \bar{\Gamma}^i_{lj} - \partial_l \bar{\Gamma}^i_{kj} + \bar{\Gamma}^i_{km} \bar{\Gamma}^m_{lj} - \bar{\Gamma}^i_{lm} \bar{\Gamma}^m_{kj} \\
&= \partial_k (\Gamma^i_{lj} + \delta^i_l \psi_j + \delta^i_j \psi_l) - \partial_l (\Gamma^i_{kj} + \delta^i_k \psi_j + \delta^i_j \psi_k) \\
&\quad + (\Gamma^i_{km} + \delta^i_k \psi_m + \delta^i_m \psi_k)(\Gamma^m_{lj} + \delta^m_l \psi_j + \delta^m_j \psi_l) \\
&\quad - (\Gamma^i_{lm} + \delta^i_l \psi_m + \delta^i_m \psi_l)(\Gamma^m_{kj} + \delta^m_k \psi_j + \delta^m_j \psi_k) \\
&= R^i{}_{jkl} + \delta^i_l \partial_k \psi_j - \delta^i_k \partial_l \psi_j + \delta^i_j (\partial_k \psi_l - \partial_l \psi_k) \\
&\quad + \Gamma^i_{kl} \psi_j + \Gamma^i_{kj} \psi_l + \delta^i_k \Gamma^m_{lj} \psi_m + 2\delta^i_k \psi_l \psi_j + \psi_k \Gamma^i_{lj} + \delta^i_l \psi_k \psi_j + \delta^i_j \psi_k \psi_l \\
&\quad - \Gamma^i_{lk} \psi_j - \Gamma^i_{lj} \psi_k - \delta^i_l \Gamma^m_{kj} \psi_m - 2\delta^i_l \psi_k \psi_j - \Gamma^i_{kj} \psi_l - \delta^i_k \psi_l \psi_j - \delta^i_J \psi_l \psi_k \\
&= R^i{}_{jkl} + \delta^i_j (\psi_{kl} - \psi_{lk}) - \delta^i_k \psi_{kl} + \delta^i_l \psi_{jk}
\end{aligned}$$

where $\psi_{kl} := \psi_{k;l} - \psi_k \psi_l$. When $\psi_i = \partial \varphi / \partial x^i$ where φ is a scalar field, then,

$$\bar{R}_{jl} - \bar{R}_{lj} = R_{jl} - R_{lj} + \delta^i_j (\psi_{il} - \psi_{li}) - d\psi_{jl} + \psi_{jl} - \delta^i_l (\psi_{ij} - \psi_{ji}) + d\psi_{lj} - \psi_{lj}$$

But,

$$\psi_{i;j} = \partial_i \psi_j - \psi_m \Gamma^m_{ij} = \partial_j \psi_i - \psi_m \Gamma^m_{ji} = \psi_{j;i}$$

since $\psi_i = \partial_i \varphi$. Hence,

$$\therefore \bar{R}^i{}_{jil} - \bar{R}^i{}_{lij} = R^i{}_{jil} - R^i{}_{lij}.$$

6.9

3 Chapter 8

8.1 Let M be a differentiable manifold of dimension n and T^*M be a cotangent bundle. For $p = (x^1, \cdots, x^n) \in M$, then an element of T^*M can be assigned in a canonical way:

$$(x^1, \cdots, x^n, y_1, \cdots, y_n) \in T^*M$$

where $(y_1, \cdots, y_n)$ are the components of the covector with respect to the basis $(\mathrm{d}x^1, \cdots, \mathrm{d}x^n)$ at the point. Then, T^*M can be considered as a differentiable manifold of dimension $2n$ with symplectic structure such that

$$\omega = \sum_{k=1}^{n} \mathrm{d}x^k \wedge \mathrm{d}y_k.$$

where ω is non-degenerate and closed by definition. Due to the nature of non-degeneracy of the symplectic form, ω^n is well-defined, and hence orientable. (Every symplectic manifold is orientable.)

8.2 For two differentiable functions, F and G, on (M, ω) where ω is a symplectic structure, the Poisson bracket is defined by

$$(F, G) := \omega(v_{dF}, v_{dG}) = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q^k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q^k} \right).$$

Here v is a Hamiltonian vector field which is defined by $\mathcal{L}_v \omega = 0$. Then, from the direct calculation,

$$\begin{aligned} -\mathcal{L}_{v_{dF}} G = -v_{dF} G &= -\left(\frac{\partial F}{\partial q^{n+1}}, \cdots, \frac{\partial F}{\partial q^{2n}}, -\frac{\partial F}{\partial q^1}, \cdots, -\frac{\partial F}{\partial q^n} \right) G \\ &= \frac{\partial F}{\partial q^1}\frac{\partial G}{\partial q^{n+1}} + \cdots + \frac{\partial F}{\partial q^n}\frac{\partial G}{\partial q^{2n}} - \frac{\partial F}{\partial q^{n+1}}\frac{\partial G}{\partial q^1} - \cdots - \frac{\partial F}{\partial q^{2n}}\frac{\partial G}{\partial q^n} \end{aligned}$$

which is identical to (F, G). In a similar way, we can also show $\mathcal{L}_{v_{dG}} F$ gives the same result. Hence,

$$(F, G) = -\mathcal{L}_{v_{dF}} G = \mathcal{L}_{v_{dG}} F$$

8.3