d'Inverno solutions for problems

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Chapter 6

6.1 (i) Eq.(6.17) says

$$L_X T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c.$$

Then, for the dirac-delta δ^a_b,

$$\begin{aligned} L_X \delta^a_b &= X^c \partial_c \delta^a_b - \delta^c_b \partial_c X^a + \delta^a_c \partial_b X^c \\ &= -\partial_b X^a + \partial_b X^a \\ &= 0 \end{aligned}$$

Hence, it follows that

$$\begin{aligned} L_X(\delta^a_b T^a{}_b) &= \delta^a_b L_X(T^a{}_b) + T^a{}_b L_X(\delta^a_b) \\ &= \delta^a_b L_X(T^a{}_b) \\ &= L_X(T^a{}_a) \end{aligned}$$

from the Leibniz rule.
(ii) From the definition of the Lie derivative,

$$L_X T^a{}_b = \lim_{\delta u \to 0} \frac{T^a{}_b(\tilde{x}) - \tilde{T}^a{}_b(\tilde{x})}{\delta u}$$

where

$$\tilde{x}^a = x^a + \delta u X^a(x)$$

which is a point transformation. Then,

$$L_X \delta^a_b = \lim_{\delta u \to 0} \frac{\delta^a_b(\tilde{x}) - \tilde{\delta}^a_b(\tilde{x})}{\delta u} = 0$$

since δ^a_b is an invariant numerical tensor, $\tilde{\delta}^a_b = \delta^a_b$. Hence, just like (i), we have $\delta^a_b L_X T^a{}_b = L_X T^a{}_a$.

6.2 From Eq.(6.17), it follows that

$$L_X Z_{bc} = X^d \partial_d Z_{bc} + Z_{dc} \partial_b X^d + Z_{bd} \partial_c X^d$$

and

$$\begin{aligned} L_X(Y^a Z_{bc}) &= X^d \partial_d (Y^a Z_{bc}) - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \\ &= Z_{bc} X^d \partial_d Y^a + Y^a X^d \partial_d Z_{bc} - Y^d Z_{bc} \partial_d X^a + Y^a Z_{dc} \partial_b X^d + Y^a Z_{bd} \partial_c X^d \\ &= Z_{bc}(X^d \partial_d Y^a - Y^d \partial_d X^a) + Y^a (X^d \partial_d Z_{bc} + Z_{bc} \partial_b X^d + Z_{bd} \partial_c X^d) \\ &= Z_{bc} L_X Y^a + Y^a L_X Z_{bc} \end{aligned}$$

since Eq.(6.15) says $L_X Y^a = [X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a$. Hence, the Leibniz property, Eq.(6.12), is verified.

6.3 Covariant derivative of a tensor field X^a,

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b$$

is a tensor of valence $(1,1)$. If we write the Jacobian as

$$J^p{}_a = \frac{\partial x'^p}{\partial x^a} \quad \text{and} \quad \tilde{J}^a{}_p = \frac{\partial x^a}{\partial x'^p}$$

(p, q, r, $\cdots$ for x' and a, b, c, $\cdots$ for x) then,

$$\begin{aligned} J^a{}_p \tilde{J}^q{}_b (\nabla_q X^p) &= J^a{}_p \tilde{J}^q{}_b (\partial_q X^p + \Gamma^p_{rq} X^r) \\ &= \tilde{J}^q{}_b \partial_q (J^a{}_p X^p) + \Gamma'^a_{cb} J^c{}_r X^r \\ &= \tilde{J}^q{}_b (\partial_q J^a{}_p) X^p + \tilde{J}^q{}_b J^a{}_p \partial_q X^p + \Gamma'^a_{cb} J^c{}_r X^r \end{aligned}$$

Or,

$$\Gamma'^a_{cb} J^c{}_r X^r = J^a{}_p \tilde{J}^q{}_b \Gamma^p_{rq} X^r - \tilde{J}^q{}_c (\partial_q J^a{}_r) X^r$$

,i.e.,

$$\Gamma'^a_{bc} = J^a{}_p \tilde{J}^r{}_b \tilde{J}^q{}_c \Gamma^p_{rq} - \tilde{J}^r{}_b \tilde{J}^q{}_c \partial_q J^a \quad (\because J^a{}_p \tilde{J}^p{}_b = \delta^a_b)$$

From $J^a{}_r \tilde{J}^r{}_b = \delta^a_b$, one can easily verify $\tilde{J}^r{}_b \partial_q J^a{}_r = -J^a{}_r \partial_q \tilde{J}^r{}_b$. Therefore, we have

$$\Gamma'^a_{bc} = J^a{}_p \tilde{J}^q{}_b \tilde{J}^r{}_c \Gamma^p_{qr} + J^a{}_r \tilde{J}^q{}_c \partial_q \tilde{J}^r{}_b.$$

6.4

$$\begin{aligned} \nabla_c \delta^a_b &= \nabla_c (J^a{}_p \tilde{J}^p{}_b) \\ &= \tilde{J}^p{}_b \nabla_c J^a{}_p + J^a{}_p \nabla_c \tilde{J}^p{}_b \quad (\because \text{Leibniz rule}) \\ &= \tilde{J}^p{}_b (\partial_c J^a{}_p + \Gamma^a_{dc} J^d{}_p - \Gamma^q_{cp} J^a{}_q) + J^a{}_p (\partial_c \tilde{J}^p{}_b + \Gamma^p_{qc} \tilde{J}^q{}_b - \Gamma^d_{cb} \tilde{J}^p{}_d) \\ &= \tilde{J}^p{}_b \partial_c J^a{}_p + J^a{}_p \partial_c \tilde{J}^p{}_b + \Gamma^a_{dc} (J\tilde{J})^d{}_b - \Gamma^q_{cp} \tilde{J}^p{}_b J^a{}_q + \Gamma^p_{qc} \tilde{J}^q{}_b J^a{}_p - \Gamma^d_{cb} (J\tilde{J})^a{}_d \\ &= \Gamma^a_{dc} - \Gamma^a_{cb} \\ &= 0 \quad (\because \text{torsion free}) \end{aligned}$$

6.5 Since $X_a Y^a$ is a scalar, it follows that

$$\nabla_b (X_a Y^a) = \partial_b (X_a Y^a).$$

From the Leibniz rule, it follows that $\nabla_b (X_a Y^a) = (\nabla_b X_a) Y^a + X_a (\nabla_b Y^a)$. Then, from the covariant derivative of a contravariant vector,

$$\begin{aligned} (\nabla_b X_a) Y^a + X_a (\partial_b Y^a + \Gamma^a_{cb} Y^c) &= (\nabla_b X^a) Y^a + X_a \partial_b Y^a + X_a \Gamma^a_{cb} Y^c \\ &= (\partial_b X_a) Y^a + X_a \partial_b Y^a \end{aligned}$$

,i.e.,

$$(\nabla_b X_a) Y^a = (\partial_b X_a - \Gamma^d_{ab} X_d] Y^a$$

Since Y^a is arbitrary, we have

$$\therefore\ \nabla_b X_a = \partial_b X_a - \Gamma^d_{ab} X_d$$

6.6

$$\begin{aligned} L_X Y^a &= X^b \nabla_b Y^a - Y^b \nabla_b X^a \\ &= X^b(\partial_b Y^a + \Gamma^a_{db} Y^d) - Y^b(\partial_b X^a + \Gamma^a_{db} X^d) \\ &= X^b \partial_b Y^a - Y^b \partial_b X^a + (\Gamma^a_{db} - \Gamma^a_{bd}) X^b Y^d \end{aligned}$$

Hence, if torsion free, we get

$$X^b \nabla_b Y^a - Y^b \nabla_b X^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

6.7 X, Y, $Z \in \mathcal{X}(M)$, f, g are C^∞, and λ, $\mu \in \mathbb{C}$.
(i)

$$\begin{aligned} \nabla_X(\lambda Y + \mu Z) &= X^c \nabla_c(\lambda Y + \mu Z) \\ &= \lambda X^c \nabla_c Y + \mu X^c \nabla_c Z \\ &= \lambda \nabla_X Y + \mu \nabla_X Z \end{aligned}$$

(ii)

$$\begin{aligned} \nabla_{fX+gY} Z &= (f X^c + g Y^c) \nabla_c Z \\ &= f X^c \nabla_c Z + g Y^c \nabla_c Z \\ &= f \nabla_X Z + g \nabla_Y Z \end{aligned}$$

(iii)

$$\begin{aligned} \nabla_X(fY) &= X^c \nabla_c(fY) \\ &= (X^c \partial_c f) Y + f X^c \nabla_c Y \\ &= (Xf) Y + f \nabla_X Y \end{aligned}$$

6.8 From Eq.(6.33),

$$\begin{aligned} \nabla_X X^a = X^c \nabla_c X^a &= X^c(\partial_c X^a + \Gamma^a_{dc} X^d) \\ &= X^c \partial_c X^a + \Gamma^a_{dc} X^d X^c \\ &= \lambda X^a \end{aligned}$$

Since $X^a = \frac{\mathrm{d}x^a}{\mathrm{d}u}(u)$, $\frac{\partial}{\partial x^c} = \frac{\partial u}{\partial x^c} \frac{\mathrm{d}}{\mathrm{d}u}$, we have

$$X^c \partial_c X^a = \frac{\mathrm{d}x^c}{\mathrm{d}u} \frac{\partial u}{\partial x^c} \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{\mathrm{d}}{\mathrm{d}u} x^a \right) = \frac{\mathrm{d}^2}{\mathrm{d}u^2} x^a(u)$$

Hence, it leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} x^a + \Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}u} \frac{\mathrm{d}x^c}{\mathrm{d}u} = \lambda \frac{\mathrm{d}x^a}{\mathrm{d}u}.$$

6.9 If s is an affine parameter, then

$$\frac{\mathrm{d}x^a}{\mathrm{d}s^2} + \Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}s} \frac{\mathrm{d}x^c}{\mathrm{d}s} = 0.$$

If this equation is covariant under the transformation

$$s \to \bar{s} = \bar{s}(s),$$

then the transformation is called an affine parameter. To $\bar{s}$ be an affine parameter, it must satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\bar{s} = 0$$

since $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{s}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\bar{s}}$. Hence, $\bar{s} = \alpha s + \beta$ for some $\alpha, \beta \in \mathbb{R}$.

6.10

$$\begin{aligned}
\nabla_c\nabla_d X^a{}_b &= \partial_c(\nabla_d X^a{}_b) - \Gamma^e_{cd}\nabla_e X^a{}_b + \Gamma^a_{ec}\nabla_d X^e{}_b - \Gamma^e_{cb}\nabla_d X^a{}_e \\
&= \partial_c[\partial_d X^a{}_b + \Gamma^a_{ed}X^e{}_b - \Gamma^e_{db}X^a{}_e] \\
&\quad - \Gamma^e_{cd}[\partial_e X^a{}_b + \Gamma^a_{fe}X^f{}_b - \Gamma^f_{eb}X^a{}_f] \\
&\quad + \Gamma^a_{ec}[\partial_d X^e{}_b + \Gamma^e_{fd}X^f{}_b - \Gamma^f_{db}X^e{}_f] \\
&\quad - \Gamma^e_{cb}[\partial_d X^a{}_e + \Gamma^a_{fd}X^f{}_e - \Gamma^f_{de}X^a{}_f] \\
&= \partial_c\partial_d X^a{}_b + (\partial_c\Gamma^a_{ed})X^e{}_b + \Gamma^a_{ed}(\partial_c X^e{}_b) - (\partial_c\Gamma^e_{db})X^a{}_e - \Gamma^e_{db}\partial_c X^a{}_e \\
&\quad - \Gamma^e_{cd}\nabla_e X^a{}_b \\
&\quad + \Gamma^a_{ec}\partial_d X^e{}_b + \Gamma^a_{ec}\Gamma^e_{fd}X^f{}_b - \Gamma^a_{ec}\Gamma^f_{db}X^e{}_f \\
&\quad - \Gamma^e_{cb}\partial_d X^a{}_e - \Gamma^e_{cb}\Gamma^a_{fd}X^f{}_e + \Gamma^e_{cb}\Gamma^f_{de}X^a{}_f \\
&= (\partial_c\mathbf{\Gamma}_d + \mathbf{\Gamma}_c\mathbf{\Gamma}_d)^a{}_e X^e{}_b - (\partial_c\mathbf{\Gamma}_d - \mathbf{\Gamma}_d\mathbf{\Gamma}_c)^e{}_b X^a{}_e - \Gamma^e_{cd}\nabla_e X^a{}_b
\end{aligned}$$

Hence, since torsion free, it follows that

$$[\nabla_c,\, \nabla_d]X^a{}_b = R^a{}_{ecd}X^e{}_b - R^e{}_{bcd}X^a{}_e$$

6.11

$$\begin{aligned}
\nabla_X(\nabla_Y Z^a) = X^c\nabla_c(Y^b\nabla_b Z^a) &= X^c[(\nabla_c Y^b)\nabla_b Z^a + Y^b(\nabla_c\nabla_b Z^a)] \\
&= X^c[(\partial_c Y^b + \Gamma^b_{dc}Y^d)(\partial_b Z^a + \Gamma^a_{eb}Z^e) \\
&\quad + Y^b\{\partial_c(\partial_b Z^a + \Gamma^a_{eb}Z^e) - \Gamma^e_{cb}\nabla_e Z^a + \Gamma^a_{ec}\nabla_b Z^e\}] \\
&= X^c[\partial_c Y^b\partial_b Z^a + \Gamma^a_{eb}\partial_c Y^b Z^e + \Gamma^b_{dc}\partial_b Z^a Y^d + \Gamma^a_{eb}\Gamma^b_{dc}Y^d Z^e \\
&\quad + Y^b\partial_c\partial_b Z^a + Y^b\partial_c(\Gamma^a_{eb}Z^e) - \Gamma^e_{cb}\partial_e Z^a Y^b - \Gamma^e_{cb}\Gamma^a_{de}Z^d Y^b \\
&\quad + \Gamma^a_{ec}Y^b\partial_b Z^e + \Gamma^a_{ec}\Gamma^e_{db}Y^b Z^d] \\
&= X^c(\partial_c Y^b)\partial_b Z^a + \Gamma^a_{db}X^c(\partial_c Y^b)Z^d + X^c(\Gamma^b_{dc} - \Gamma^b_{cd})Y^d\partial_b Z^a \\
&\quad + X^c(\Gamma^a_{eb}\Gamma^b_{dc} - \Gamma^a_{eb}\Gamma^b_{cd})Y^d Z^e + X^c Y^b\partial_c\partial_b Z^a \\
&\quad + X^c Y^b(\partial_c\Gamma^a_{eb})Z^e + (X^c Y^b\Gamma^a_{eb}\partial_c + X^c Y^b\Gamma^a_{ec}\partial_b)Z^e + X^c\Gamma^a_{ec}\Gamma^e_{db}Y^b Z^d
\end{aligned}$$

Then, we exchange X and Y, thereby subtracting two terms gives

$$\begin{aligned}
&X^c(\partial_c Y^b)\partial_b Z^a - Y^c(\partial_c X^b)\partial_b Z^a + \Gamma^a_{db}X^c(\partial_c Y^b)Z^d - \Gamma^a_{db}Y^c(\partial_c X^b)Z^d \\
&+ X^c Y^b(\partial_c\Gamma^a_{eb})Z^e - Y^c X^b(\partial_c\Gamma^a_{eb})Z^e + X^c\Gamma^a_{ec}\Gamma^e_{db}Y^b Z^d - Y^c\Gamma^a_{ec}\Gamma^e_{db}X^b Z^d \\
&= \{X^c(\partial_c Y^b) - Y^c(\partial_c X^b)\}\nabla_b Z^a + \{\partial_c\Gamma^a_{db} - \partial_d\Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}\}Z^b X^c Y^d \\
&= [X,\, Y]^b\nabla_b Z^a + R^a{}_{bcd}Z^b X^c Y^d
\end{aligned}$$

Therefore,

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]}Z^a = R^a{}_{bcd}Z^b X^c Y^d$$

6.12 To show whether the connection is integrable, we need to show Riemann tensor vanishes when given manifold is affine flat. Suppose a manifold M is affine flat, then for any point P in M, we have a patch such that

$$\Gamma^a_{bc}|_P \stackrel{*}{=} 0$$

Then, from the definition of Riemann tensor

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc}$$

Also, when $\Gamma^a_{bc} = 0$,

$$\partial_b \partial_c X^a = \partial_c \partial_b X^a$$

for any vector field X^a. It implies two relations:

$$\Gamma^a_{bc} = \Gamma^a_{cb}, \quad \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} = 0,$$

i.e., the connection is symmetric. Also, the Riemann tensor vanishes. Hence, the connection is integrable since P was arbitrary.

6.13 g^{ab} is defined by

$$g_{ab} g^{bc} = \delta^c_a$$

Since g_{ab} is a tensor of valence $(0,2)$, it transforms as

$$g_{ab} \mapsto g'{}_{pq} = \tilde{J}^a{}_p \tilde{J}^b{}_q g_{ab}$$

and we have

$$g'{}_{pq} g'^{qr} = \tilde{J}^a{}_p \tilde{J}^b{}_q g_{ab} g'^{qr} = \delta^r_p$$

since δ^r_q is the numerical tensor. Or,

$$g_{ab} \left(\tilde{J}^b{}_q g'^{qr} \right) = J^p{}_a \delta^r_p = J^r{}_a$$

By multiplying both sides by g^{ca},

$$\delta^c_b \tilde{J}^b{}_q g'^{qr} = \tilde{J}^c{}_q g'^{qr} = J^r{}_a g^{ac}$$

,i.e.,

$$\therefore\ g'^{qr} = J^q{}_c J^r{}_a g^{ca}$$

Hence, the inverse metric is the tensor of valence $(2,0)$.

6.14 Since g_{ab} is diagonal, we can write $g_{ab} = \lambda^a \delta_{ab}$ where λ^a are some constants. It means that

$$g_{ab} g^{bc} = \lambda^a \delta_{ab} g^{bc} = \lambda^a g^{ac} = \delta^a_c$$

,i.e.,

$$\therefore\ g^{ac} = (\lambda^a)^{-1} \delta^c_a$$

6.15 For $\mathbb{R}^3$, from the definition of the metric $g = g_{ab}(x) \mathrm{d}x^a \otimes \mathrm{d}x^b$,
(i) $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$

$$g_{ab} = g^{ab} = \delta^b_a, \quad g = 1$$

(ii) $\mathrm{d}s^2 = \mathrm{d}R^2 + R^2 \mathrm{d}\phi^2 + \mathrm{d}z^2$

$$g_{ab} = \mathrm{diag}[1,\ R^2,\ 1],\ \ g^{ab} = \mathrm{diag}[1,\ R^{-2},\ 1] \quad g = R^2$$

(iii) $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$

$$g_{ab} = \mathrm{diag}[1,\ r^2,\ r^2 \sin^2\theta],\ \ g^{ab} = \mathrm{diag}[1,\ r^{-2},\ r^{-2} \sin^{-2}\theta], \quad g = r^4 \sin^2\theta$$

6.16 Since indices of tensors are raised or lowered by the metric tensor, we have

$$T_{ab} = g_{ac} g_{bd} T^{cd}$$

6.17 The law of the tensor transformation can be written as

$$g_{ab}(x) \mapsto g'_{ab}(x') = \tilde{J}^p{}_a \tilde{J}^q{}_b g_{pq}(x)$$

where $\tilde{J}^p{}_a = \partial x^a / \partial x'^p$. From

$$\left\{ \begin{array}{c} a \\ bc \end{array} \right\} = \frac{1}{2} g^{ad} \{\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\}$$

we have

$$\begin{aligned}
&\frac{1}{2} J^a{}_p J^d{}_q g^{pq} \left\{ \tilde{J}^r{}_b \partial_r (\tilde{J}^s{}_c \tilde{J}^t{}_d g_{st}) + \tilde{J}^r{}_c \partial_r (\tilde{J}^s{}_b \tilde{J}^t{}_d g_{st}) - \tilde{J}^r{}_d \partial_r (\tilde{J}^s{}_b \tilde{J}^t{}_c g_{st}) \right\} \\
&= \frac{1}{2} J^a{}_p J^d{}_q g^{pq} \{ \tilde{J}^r{}_b (\partial_r \tilde{J}^s{}_c) \tilde{J}^t{}_d g_{st} + \tilde{J}^r{}_b \tilde{J}^s{}_c (\partial_r \tilde{J}^t{}_d) g_{st} + \tilde{J}^r{}_b \tilde{J}^s{}_c \tilde{J}^t{}_d \partial_r g_{st} \\
&\qquad + \tilde{J}^r{}_c (\partial_r \tilde{J}^s{}_b) \tilde{J}^t{}_d g_{st} + \tilde{J}^r{}_c \tilde{J}^s{}_b (\partial_r \tilde{J}^t{}_d) g_{st} + \tilde{J}^r{}_c \tilde{J}^s{}_b \tilde{J}^t{}_d \partial_r g_{st} \\
&\qquad - \tilde{J}^r{}_d (\partial_r \tilde{J}^s{}_b) \tilde{J}^t{}_c g_{st} - \tilde{J}^r{}_d \tilde{J}^s{}_b (\partial_r \tilde{J}^t{}_c) g_{st} - \tilde{J}^r{}_d \tilde{J}^s{}_b \tilde{J}^t{}_c \partial_r g_{st} \} \\
&= \frac{1}{2} J^a{}_p g^{pq} \{ \tilde{J}^r{}_b (\partial_r \tilde{J}^s{}_c) g_{sq} + \tilde{J}^r{}_c (\partial_r \tilde{J}^s{}_b) g_{sq} - (\partial_q \tilde{J}^s{}_b) \tilde{J}^t{}_c g_{st} \\
&\qquad + \tilde{J}^r{}_b \tilde{J}^s{}_c J^d{}_q (\partial_r \tilde{J}^t{}_d) g_{st} + \tilde{J}^r{}_c \tilde{J}^s{}_b J^d{}_q (\partial_r \tilde{J}^t{}_d) g_{st} - \tilde{J}^s{}_b (\partial_q \tilde{J}^t{}_c) g_{st} \\
&\qquad + \tilde{J}^r{}_b \tilde{J}^s{}_c \partial_r g_{sq} + \tilde{J}^r{}_c \tilde{J}^s{}_b \partial_r g_{sq} - \tilde{J}^s{}_b \tilde{J}^t{}_c \partial_q g_{st} \} \\
&= \frac{1}{2} J^a{}_p \tilde{J}^q{}_b \tilde{J}^r{}_c g^{pt} \{ \partial_q g_{rt} + \partial_r g_{qt} - \partial_t g_{qr} \} \\
&\quad + \frac{1}{2} J^a{}_p \{ \tilde{J}^r{}_b (\partial_r \tilde{J}^p{}_c) + \tilde{J}^r{}_c (\partial_r \tilde{J}^p{}_b) - (g^{pq} \partial_q \tilde{J}^s{}_b) \tilde{J}^t{}_c g_{st} \\
&\qquad + \tilde{J}^r{}_b \tilde{J}^s{}_c J^d{}_q (\partial_r \tilde{J}^t{}_d) g^{pq} g_{st} + \tilde{J}^r{}_c \tilde{J}^s{}_b J^d{}_q (\partial_r \tilde{J}^t{}_d) g^{pq} g_{st} - (g^{pq} \partial_q \tilde{J}^t{}_c) \tilde{J}^s{}_b g_{st} \}
\end{aligned}$$

But, $\tilde{J}^r{}_b \tilde{J}^s{}_c J^d{}_q \partial_r \tilde{J}^t{}_d = \tilde{J}^r{}_b \tilde{J}^s{}_c J^d{}_r \partial_q \tilde{J}^t{}_d = \delta^d_b \tilde{J}^s{}_c \partial_q \tilde{J}^t{}_d = \tilde{J}^s{}_c \partial_q \tilde{J}^t{}_b$ and finally we have

$$J^a{}_p \tilde{J}^q{}_b \tilde{J}^r{}_c \left\{ \begin{array}{c} p \\ qr \end{array} \right\} + J^a{}_p \tilde{J}^r{}_b \partial_r \tilde{J}^p{}_c$$

which is the transformation law of the connection since

$$J^a{}_p \tilde{J}^r{}_b \partial_r \tilde{J}^p{}_c = \frac{\partial x'^a}{\partial x^p} \frac{\partial x^r}{\partial x'^b} \frac{\partial}{\partial x^r} \frac{\partial x^p}{\partial x'^c} = \frac{\partial x'^a}{\partial x^p} \frac{\partial^2 x^p}{\partial x'^b \partial x'^c}$$

6.18 In cylindrical polars, we choose coordinates $\{r,\, \theta,\, z\}$. The metric is given by

$$g_{ab} = \mathrm{diag}[1,\, r^2,\, 1]$$

,i.e., the line element reads

$$\mathrm{d}s^2 = g_{ab} \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + \mathrm{d}z^2$$

From the action principle, we have

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(2 g_{ab} \frac{\mathrm{d}x^b}{\mathrm{d}u} \right) - \left(\frac{\partial}{\partial x^a} g_{bc} \right) \frac{\mathrm{d}x^b}{\mathrm{d}u} \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0$$

which is the geodesic equation. Then,

$$\begin{aligned}
&\text{for } a = r, && 2\frac{\mathrm{d}}{\mathrm{d}u}(g_{rr}\dot{r}) - \partial_r g_{bc}\dot{x}^b\dot{x}^c = 2\ddot{r} - 2r\dot{\theta}^2 = 0 \quad \cdot \equiv \mathrm{d}/\mathrm{d}u \\
&\text{for } a = \theta, && 2\frac{\mathrm{d}}{\mathrm{d}u}(g_{\theta\theta}\dot{\theta}) = 2r^2\ddot{\theta} + 2(2r\dot{r}\dot{\theta}) = 0 \\
&\text{for } a = z, && 2\frac{\mathrm{d}}{\mathrm{d}u}(g_{zz}\dot{z}) = 2\ddot{z} = 0.
\end{aligned}$$

Hence,

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = 0 \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0 \\ \ddot{z} = 0 \end{cases}$$

6.19 (maybe typo in problem 6.19) For 3-space $(x^a) = (x^1, x^2, x^3)$ with the metric given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$$

(since the null geodesic is defined when the metric is indefinite). For the null geodesic, we have

$$\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 = 0.$$

If we parametrize coordinates by an affine parameter, u, then

$$\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 = 0$$

with

$$x = lu + l', \quad y = mu + m', \quad z = nu + n'.$$

i.e.,

$$\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 = (l^2 + m^2 - n^2)\mathrm{d}u^2 = 0$$

which says

$$\therefore\ l^2 + m^2 - n^2 = 0$$

6.20

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad} \\
&= \partial_c g_{ab} - \frac{1}{2}\delta^e_b\{\partial_c g_{ae} + \partial_a g_{ce} - \partial_e g_{ca}\} - \frac{1}{2}\delta^e_a\{\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{cb}\} \\
&= \partial_c g_{ab} - \frac{1}{2}\{\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ca}\} - \frac{1}{2}\{\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{cb}\} \\
&= 0
\end{aligned}$$

Hence, $\nabla_c g_{ab} = 0$. By exploiting the Leibniz rule, we have

$$\nabla_b X_a = \nabla_b(g_{ac}X^c) = (\nabla_b g_{ac})X^c + g_{ac}\nabla_b X^c = g_{ac}\nabla_b X^c \quad (\because \nabla_b g_{ac} = 0)$$

6.21 From the definition of the connection,

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ab} = 0$$

we have

$$\partial_c g_{ab} = \Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{ad} \quad \cdots \quad (1)$$

Similarly,

$$\partial_a g_{bc} = \Gamma^d_{ab} g_{dc} + \Gamma^d_{ac} g_{bd} \quad \cdots \quad (2)$$
$$\partial_b g_{ca} = \Gamma^d_{bc} g_{da} + \Gamma^d_{ba} g_{cd} \quad \cdots \quad (3)$$

By $-(1)+(2)+(3)$ and using symmetry condition,

$$\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} = 2\Gamma^d_{ab} g_{dc}$$

Hence,

$$\therefore \Gamma^d_{ab} = \frac{1}{2} g^{dc} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})$$

6.22 The line element in coordinates $(x^a) = (t,\, x,\, y,\, z)$ is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

(i) Since $g_{ab} = \mathrm{diag}[+1, -1, -1, -1]$, the signature is -2.
(ii) Since the metric is diagonal and constant, the metric is non-singular.
(iii) Since given metric is constant, then all the connnection terms are 0 which indicates the Riemann tensor is also 0. Hence, the metric is flat.

6.23 Given line element in $\mathbb{R}^3$

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \left(\mathrm{d}x^2\right)^2 + \left(x^1 \sin x^2\right)^2 \left(\mathrm{d}x^3\right)^2,$$

the metric is written as

$$g_{ab} = \mathrm{diag}\left(1,\, \left(x^1\right)^2,\, \left(x^1 \sin x^2\right)^2\right)$$

(i) This is identical to spherical coordinates.
(ii) To show whether the metric is flat, we first need to obtain the connection terms

$$\begin{aligned}
&\Gamma^r_{rr} = \Gamma^r_{r\theta} = \Gamma^r_{r\varphi} = 0 \\
&\Gamma^\theta_{rr} = \Gamma^\theta_{r\varphi} = 0, \quad \Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = \frac{1}{2} r^{-2} \partial_r r^2 = \frac{1}{r} \\
&\Gamma^\varphi_{rr} = \Gamma^\varphi_{r\theta} = 0, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{2} g^{\varphi\varphi} \partial_r g_{\varphi\varphi} = \frac{1}{2} r^{-2} \sin^{-2}\theta \partial_r (r^2 \sin^2\theta) = \frac{1}{r} \\
&\Gamma^r_{\theta r} = \Gamma^r_{\theta\varphi} = 0, \quad \Gamma^r_{\theta\theta} = -\frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -\frac{1}{2} \partial_r r^2 = -r \\
&\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = 0, \quad \Gamma^\theta_{\theta r} = \frac{1}{r} \\
&\Gamma^\varphi_{\theta r} = \Gamma^\varphi_{\theta\theta} = 0, \quad \Gamma^\varphi_{\theta\varphi} = \frac{1}{2} g^{\varphi\varphi} \partial_\theta g_{\varphi\varphi} = \frac{1}{2} r^{-2} \sin^{-2}\theta \partial_\theta (r^2 \sin^2\theta) = \cot\theta \\
&\Gamma^r_{\varphi r} = \Gamma^r_{\varphi\theta} = 0, \quad \Gamma^r_{\varphi\varphi} = -\frac{1}{2} g^{rr} \partial_r g_{\varphi\varphi} = -\frac{1}{2} \partial_r (r^2 \sin^2\theta) = r \sin^2\theta \\
&\Gamma^\theta_{\varphi r} = \Gamma^\theta_{\varphi\theta} = 0, \quad \Gamma^\theta_{\varphi\varphi} = -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\varphi\varphi} = -\frac{1}{2} r^{-2} \partial_\theta (r^2 \sin^2\theta) = \sin\theta\cos\theta \\
&\Gamma^\varphi_{\varphi\varphi} = 0, \quad \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta
\end{aligned}$$

In matrix notation,

$$\mathbf{\Gamma}_r = \begin{pmatrix} 0 & & \\ & 1/r & \\ & & -1/r \end{pmatrix}, \quad \mathbf{\Gamma}_\theta = \begin{pmatrix} 0 & -r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & \cot\theta \end{pmatrix}, \quad \mathbf{\Gamma}_\varphi = \begin{pmatrix} 0 & 0 & r\sin^2\theta \\ 0 & 0 & \sin\theta\cos\theta \\ 1/r & \cot\theta & 0 \end{pmatrix}$$

Now we can calculate the Riemann tensor

$$
\begin{aligned}
\mathbf{R}_{r\theta} &= \partial_r \Gamma_\theta + \mathbf{\Gamma}_r \mathbf{\Gamma}_\theta - \mathbf{\Gamma}_\theta \mathbf{\Gamma}_r \\
&= \begin{pmatrix} 0 & -1 & \\ -1/r^2 & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1/r^2 & 0 & 0 \\ 0 & 0 & -\cot\theta/r \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\cot\theta/r \end{pmatrix} \\
&= 0 \\
\mathbf{R}_{r\varphi} &= \partial_r \mathbf{\Gamma}_\varphi + \mathbf{\Gamma}_r \mathbf{\Gamma}_\varphi - \mathbf{\Gamma}_\varphi \mathbf{\Gamma}_r \\
&= \begin{pmatrix} 0 & 0 & \sin^2\theta \\ 0 & 0 & 0 \\ -1/r^2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r}\sin\theta\cos\theta \\ 1/r^2 & \cot\theta/r & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \sin^2\theta \\ 0 & 0 & \sin\theta\cos\theta/r \\ 0 & \cot\theta/r & 0 \end{pmatrix} \\
&= 0 \\
\mathbf{R}_{\theta\varphi} &= \partial_\theta \mathbf{\Gamma}_\varphi + \mathbf{\Gamma}_\theta \mathbf{\Gamma}_\varphi - \mathbf{\Gamma}_\varphi \mathbf{\Gamma}_\theta \\
&= \begin{pmatrix} 0 & 0 & 2r\sin\theta\cos\theta \\ 0 & 0 & \cos^2\theta - \sin^2\theta \\ 0 & -\csc^2\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -r\sin\theta\cos\theta \\ 0 & 0 & \sin^2\theta \\ \frac{1}{r}\cot\theta & \cot^2\theta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & r\sin\theta\cos\theta \\ 0 & 0 & \cos^2\theta \\ \frac{1}{r}\cot\theta & -1 & 0 \end{pmatrix} \\
&= 0
\end{aligned}
$$

Hence, given metric is flat.

6.24 Riemann tensor is written as

$$R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb}$$

Since connections are coefficient artifact, we choose geodesic coordinates at some point P to simplify the above equation

$$R^a{}_{bcd} \stackrel{*}{=} \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{cd}$$

Then, from the direct calculation,

$$
\begin{aligned}
R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} &= \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \partial_d \Gamma^a_{bc} - \partial_b \Gamma^a_{dc} + \partial_b \Gamma^a_{cd} - \partial_c \Gamma^a_{bd} \\
&= 0
\end{aligned}
$$

we can verify Eq.(6.79) which can also be written as $R^a{}_{[bcd]} \equiv 0$ since $R^a{}_{bcd} = -R^a{}_{dbc}$. By using this result,

$$
\begin{aligned}
R_{abcd} &= -R_{acdb} - R_{adbc} \\
&= -R_{acdb} + R_{adcb} \\
&= -2R_{a[cd]b} \\
&= -R_{a[cd]b} + R_{b[cd]a}
\end{aligned}
$$

Also,

$$
\begin{aligned}
R_{cdab} &= -R_{cabd} - R_{cbda} \\
&= -R_{acdb} + R_{bcda} \\
&= -R_{a[cd]b} + R_{b[cd]a}
\end{aligned}
$$

Hence, $R_{abcd} = R_{cdab}$.

6.25

$$
\begin{aligned}
\nabla_a R_{debc} = g_{df} \nabla_a R^f{}_{ebc} \to g_{df}(\partial_a R^f{}_{ebc}) &= g_{df}\partial_a \left[\partial_b \Gamma^f_{ce} - \partial_c \Gamma^f_{de}\right] \quad \text{(at } P \text{ in geodesic coordinates)} \\
&= g_{df}\left[\partial_a \partial_b \Gamma^f_{ce} - \partial_a \partial_c \Gamma^f_{be}\right]
\end{aligned}
$$

But,

$$\begin{aligned}\partial_a\partial_b\Gamma^f_{ce} - \partial_a\partial_c\Gamma^f_{be} + \partial_b\partial_c(\Gamma^f_{ae} - \Gamma^f_{ae}) &= -\partial_c(\partial_a\Gamma^f_{be} - \partial_b\Gamma^f_{ae}) - \partial_b(\partial_c\Gamma^f_{ae} - \partial_a\Gamma^f_{ce}) \\ &= -\partial_c R^f{}_{eab} - \partial_b R^f{}_{eca} \\ &= -\nabla_c R^f{}_{eab} - \partial_b R^f{}_{eca}\end{aligned}$$

Hence, $\nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab} = 0$, i.e., $R_{de[ab;c]} \equiv 0$.

6.26 $p : G_{ab} = 0$, $q : R_{ab} = 0$.
(i) $p \to q$
From the definition of the Einstein tensor,

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 0, \quad R_{ab} = \frac{1}{2}g_{ab}R$$

Then, by considering the trace,

$$R = \mathrm{tr}(R_{ab}) = \frac{1}{2}R\,\mathrm{tr}(g_{ab}) = \frac{n}{2}R$$

where n is the dimension of the manifold. Hence, $R = 0$ and naturally $R_{ab} = \frac{1}{2}g_{ab}R = 0$. Notice that when $n = 1$, R_{ab} is trivially zero since $R^a{}_{bcd} = 0$.
(ii) $q \to p$. trivial.

6.27 From the definition, the Weyl tensor or conformal tensor is written as

$$\begin{aligned}C_{abcd} =& R_{abcd} - \frac{1}{n-2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{db}R_{ca}) \\ &+ \frac{1}{(n-1)(n-2)}(g_{ac}g_{db} - g_{ad}g_{cb})R\end{aligned}$$

where $n \geq 4$. By taking trace, i.e., multiplying g^{ca}, the second and the third terms are

$$\begin{aligned}&g^{ca}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) = (R_{db} + R_{db} - nR_{db} - g_{bd}R) = -(n-2)R_{db} - g_{bd}R \\ &g^{ca}(g_{ac}g_{db} - g_{ad}g_{cb})R = (ng_{db} - g_{db})R = (n-1)g_{db}R\end{aligned}$$

Then,

$$C^a{}_{bad} = R^a{}_{bad} - R_{db} = 0.$$

Since the Weyl tensor possesses the same symmetries as the Riemann tensor, it is trace-free on all pairs of indices.

6.28 Two metrics g_{ab} and $\bar{g}_{ab}$ are said to be conformally realated if

$$\bar{g}_{ab} = \Omega^2 g_{ab}.$$

The length of vectors is defined as $X^2 = g_{ab}X^aX^b$. Then, angles between two vectors can be written as

$$\cos(X,\,Y) = \frac{g_{ab}X^aY^b}{|g_{cd}X^cX^d|^{1/2}|g_{ef}Y^eY^f|^{1/2}} \longrightarrow \frac{\Omega^2 g_{ab}X^aY^b}{\Omega|g_{cd}X^cX^d|^{1/2}\Omega|g_{ef}Y^eY^f|^{1/2}} = \cos(X,\,Y)$$

,i.e., conformally invariant. Also, for the ratios of lengths of vectors,

$$(X/Y)^2 = \frac{g_{ab}X^aX^b}{g_{cd}Y^cY^d} \longrightarrow \frac{\Omega^2 g_{ab}X^aX^b}{\Omega^2 g_{cd}Y^cY^d} = (X/Y)^2.$$

6.29 Null geodesics is defined as

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = 0$$

for an indefinite metric g_{ab}. Then, for a conformally related metrics, where $\bar{g}_{ab} = \Omega g_{ab}$,

$$\mathrm{d}\sigma^2 = \bar{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \Omega^2 g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \Omega^2\mathrm{d}s^2 = 0$$

and they coincide each other.

6.30 Given $\bar{g}_{ab} = \Omega^2 g_{ab}$ and $W = \ln\Omega$, $W_c = \nabla_c(\ln\Omega)$, $W_{cd} = \nabla_c(\nabla_d(\ln\Omega))$,
(i)

$$W_{cd} = \nabla_c(\nabla_d(\ln\Omega)) = \partial_c W_d - \Gamma^e_{cd}W_e$$

But,

$$W_d = \nabla_d(\ln\Omega) = \partial_d(\ln\Omega)$$

and $\partial_c W_d = \partial_c\partial_d(\ln\Omega) = \partial_d W_c$. Hence, we can write

$$W_{cd} = \partial_c W_d - \Gamma^e_{cd}W_e = \partial_d W_c - \Gamma^e_{dc}W_e = W_{dc}$$

,i.e., $W_{cd} = W_{dc}$.
(ii)

$$\begin{aligned}
\bar{\Gamma}^a_{bc} &= \frac{1}{2}\bar{g}^{ad}(\partial_b\bar{g}_{cd} + \partial_c\bar{g}_{bd} - \partial_d\bar{g}_{bc}) \\
&= \frac{1}{2}\Omega^{-2}g^{ad}\left[\partial_b(\Omega^2 g_{cd}) + \partial_c(\Omega^2 g_{bd}) - \partial_d(\Omega g_{bc})\right] \\
&= \Gamma^a_{bc} + \frac{1}{2}\Omega^{-2}g^{ad}[g_{cd}\partial_b\Omega^2 + g_{bd}\partial_c\Omega^2 - g_{bc}\partial_d\Omega^2] \\
&= \Gamma^a_{bc} + \delta^a_c W_b + \delta^a_b W_c - g_{bc}g^{ad}W_d
\end{aligned}$$

Therefore,

$$\therefore \bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_c W_b + \delta^a_b W_c - g_{bc}W^a$$

(iii)

$$\begin{aligned}
\bar{R}^a{}_{bcd} &= \partial_c\bar{\Gamma}^a_{db} - \partial_d\bar{\Gamma}^a_{cb} + \bar{\Gamma}^a_{ce}\bar{\Gamma}^e_{db} - \bar{\Gamma}^a_{de}\bar{\Gamma}^e_{cb} \\
&= \partial_c(\Gamma^a_{db} + \delta^a_d W_b + \delta^a_b W_d - g_{db}W^a) - \partial_d(\Gamma^a_{cb} + \delta^a_c W_b + \delta^a_b W_c - g_{cb}W^a) \cdots (1) \\
&\quad + (\Gamma^a_{ce} + \delta^a_c W_e + \delta^a_e W_c - g_{ce}W^a)(\Gamma^e_{db} + \delta^e_d W_b + \delta^e_b W_d - g_{db}W^e) \cdots (2) \\
&\quad - (\Gamma^a_{de} + \delta^a_d W_e + \delta^a_e W_d - g_{de}W^a)(\Gamma^e_{cb} + \delta^e_c W_b + \delta^e_b W_c - g_{cb}W^e) \cdots (3)
\end{aligned}$$

For Eq.(1),

$$\begin{aligned}
&R^a{}_{bcd} + \delta^a_d\partial_c W_d + \delta^a_b\partial_c W_d - (\partial_c g_{db})W^a - g_{db}\partial_c W^a - \delta^a_c\partial_d W_b - \delta^a_b\partial_d W_c + (\partial_d g_{cb})W^a + g_{cb}\partial_d W^a \\
&= R^a{}_{bcd} + \delta^a_d\partial_c W_b + \delta^a_b\partial_c W_d - (\Gamma^e_{cd}g_{eb} + \Gamma^e_{cb}g_{de})W^a - g_{db}\partial_c W^a - \delta^a_c\partial_d W_b - \delta^a_b\partial_d W_c \\
&\quad + (\Gamma^e_{dc}g_{eb} + \Gamma^e_{db}g_{ce})W^a + g_{cb}\partial_d W^a
\end{aligned}$$

For Eq.(2),

$$\begin{aligned}
&\Gamma^a_{cd}W_b + \Gamma^a_{cb}W_d - \Gamma^a_{ce}g_{db}W^e + \delta^a_c\Gamma^e_{db}W_e + \delta^a_c W_b W_d + \delta^a_c W_b W_d - \delta^a_c g_{db}W_e W^e \\
&+ \Gamma^a_{db}W_c + \delta^a_d W_b W_c + \delta^a_b W_c W_d - g_{bd}W_c W^a - \Gamma^e_{db}g_{ce}W^a - g_{cd}W^a W_b - g_{cb}W^a W_d + g_{db}W^a W_c
\end{aligned}$$

and Eq.(3) is just $(c \leftrightarrow d)$ from Eq.(2). From Eq.(1) + Eq.(2) + Eq.(3), the remaining terms are

$$\begin{aligned}
&\Gamma^a_{de}g_{cb}W^e - \delta^a_d\Gamma^e_{cb}W_e - \delta^a_d W_b W_c - \delta^a_d W_b W_c + \delta^a_d g_{cb}W_e W^e \\
&- \delta^a_c W_b W_d + g_{bc}W_d W^a + \Gamma^e_{cb}g_{de}W^a + g_{db}W^a W_c - g_{cb}W^a W_d
\end{aligned}$$

Finally,

$$\begin{aligned}
\therefore \bar{R}^a{}_{bcd} &= R^a{}_{bcd} + \delta^a_d W_{cb} - \delta^a_c W_{db} - g_{db}W^a_c + g_{cb}W^a_d + \delta^a_c W_b W_d - \delta^a_c g_{db}W_e W^e + g_{db}W^a W_c \\
&\quad - \delta^a_d W_b W_c + \delta^a_d g_{cb}W_e W^e - g_{cb}W^a W_d
\end{aligned}$$

(iv) From the previous result,

$$
\begin{aligned}
\bar{R}_{bd} &= R_{bd} + W_{db} - nW_{db} - g_{db}W^a_a + W_{bd} + nW_bW_d - ng_{db}W_eW^e \\
&\quad + g_{db}W_eW^e - W_bW_d + g_{db}W_eW^e - W_bW_d \\
&= R_{bd} - (n-2)W_{bd} - g_{db}W^a_a + (n-2)W_bW_d - (n-2)g_{db}W_eW^e \\
&= R_{bd} - g_{db}W^a_a - (n-2)[W_{db} - W_bW_d + g_{db}W_eW^e]
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\bar{R} = g^{bd}\bar{R}_{bd} &= R - nW^a_a - (n-2)[W^b_b - W_eW^e + nW_eW^e] \\
&= R - 2(n-1)W^a_a - (n-2)(n-1)W_eW^e
\end{aligned}
$$

Now from the definition,

$$
\begin{aligned}
C_{abcd} = R_{abcd} &+ \frac{1}{n-2}(g_{ad}R_{bc} + g_{bc}R_{da} - g_{ac}R_{bd} - g_{bd}R_{ca}) \\
&+ \frac{1}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{cb})R
\end{aligned}
$$

and a conformal transformation gives

$$
\begin{aligned}
\bar{C}_{abcd} &= R_{abcd} + g_{ad}W_{cb} - g_{ac}W_{db} - g_{db}W_{ac} + g_{cb}W_{ad} + g_{ac}W_bW_d - g_{ac}g_{db}W_eW^e + g_{db}W_aW_c - g_{ad}W_bW_c \\
&\quad + g_{ad}g_{cb}W_eW^e - g_{cb}W_aW_d + \frac{1}{n-2}\big[g_{ad}(R_{bc} - g_{bc}W^a_a - (n-2)(W_{bc} - W_bW_c + g_{bc}W_eW^e) \\
&\quad + g_{bc}(R_{da} - g_{da}W^a_a - (n-2)[W_{da} - W_dW_a + g_{da}W_eW^e]) \\
&\quad - g_{ac}(R_{bd} - g_{bd}W^a_a - (n-2)[W_{bd} - W_bW_d + g_{bd}W_eW^e]) \\
&\quad - g_{bd}(R_{ca} - g_{ca}W^a_a - (n-2)[W_{ca} - W_cW_a + g_{ca}W_eW^e])\big] \\
&\quad + \frac{1}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc})(R - 2(n-1)W^a_a - (n-2)(n-1)W_eW^e) \\
&\quad = C_{abcd}
\end{aligned}
$$

and hence,

$$
\therefore\ \bar{C}_{abcd} = C_{abcd}
$$

6.31 In two dimensional Lorentzian manifold, the line elment is written as

$$
\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2
$$

In null curves, changing to new coordinates $\lambda = \lambda(x^0,\, x^1)$ and $\nu = \nu(x^0,\, x^1)$ will satisfy

$$
g^{ab}\lambda_{,a}\lambda_{,b} = g^{ab}\nu_{,a}\nu_{,b} = 0
$$

and the line element becomes

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left[\left(\frac{\partial x^0}{\partial \lambda}\right)\mathrm{d}\lambda + \left(\frac{\partial x^0}{\partial \nu}\right)\mathrm{d}\nu\right]^2 + \left[\left(\frac{\partial x^1}{\partial \lambda}\right)\mathrm{d}\lambda + \left(\frac{\partial x^1}{\partial \nu}\right)\mathrm{d}\nu\right]^2 \\
&= -\left[\left(\frac{\partial x^0}{\partial \lambda}\right)^2 - \left(\frac{\partial x^1}{\partial \lambda}\right)^2\right](\mathrm{d}\lambda)^2 - \left[\left(\frac{\partial x^0}{\partial \nu}\right)^2 - \left(\frac{\partial x^1}{\partial \nu}\right)^2\right](\mathrm{d}\nu)^2 + 2\left[-\left(\frac{\partial x^0}{\partial \lambda}\right)\left(\frac{\partial x^0}{\partial \nu}\right) + \left(\frac{\partial x^1}{\partial \lambda}\right)\left(\frac{\partial x^1}{\partial \nu}\right)\right]\mathrm{d}\lambda\mathrm{d}\nu \\
&= 2\left[-\left(\frac{\partial x^0}{\partial \lambda}\right)\left(\frac{\partial x^0}{\partial \nu}\right) + \left(\frac{\partial x^1}{\partial \lambda}\right)\left(\frac{\partial x^1}{\partial \nu}\right)\right]\mathrm{d}\lambda\mathrm{d}\nu \\
&= -2\det(\tilde{J})\mathrm{d}\lambda\mathrm{d}\nu
\end{aligned}
$$

where $\tilde{J}$ is a coordinate transformation from $(\lambda\,\nu)$ to $(x^0,\, x^1)$. Given that the orientation is preserving, we may write

$$
\mathrm{d}s^2 = -e^{2\mu}\mathrm{d}\lambda\mathrm{d}\nu.
$$

By introducing $Q = \frac{1}{2}(\lambda + \nu)$ and $P = \frac{1}{2}(\lambda - \nu)$, $\mathrm{d}\lambda = \mathrm{d}Q + \mathrm{d}P$ and $\mathrm{d}\nu = \mathrm{d}Q - \mathrm{d}P$. Hence,

$$\mathrm{d}s^2 = -2e^2 2\mu \left[(\mathrm{d}Q)^2 - (\mathrm{d}P)^2\right]$$

which implies that any two-dimensional Lorentzian manifold is conformally flat.

6.32
(i)

$$\begin{aligned} g_{ab} &= \operatorname{diag}\left[e^{\nu},\ -e^{\lambda},\ -r^2,\ -r^2\sin^2\theta\right] \\ g^{ab} &= \operatorname{diag}\left[e^{-\nu},\ -e^{-\lambda},\ -r^{-2},\ -r^{-2}\sin^{-2}\theta\right] \\ g &\equiv \det(g) = -e^{\nu+\lambda}r^4\sin^2\theta \end{aligned}$$

Chapter 7

7.1 Let Φ be a scalar density of weight $+1$. Its expression for the covariant derivative will be written as

$$\nabla_a \Phi(x) = \partial_a \Phi(x) - \Gamma^d_{da}\Phi(x)$$

7.2 Levi-Civia alternating symbol ε^{abcd} is a completely anti-symmetric tensor density of weight $+1$. Then, its covariant deriavative can be written as

$$\begin{aligned} \nabla_e \varepsilon^{abcd} &= \partial_e \varepsilon^{abcd} + \Gamma^a_{fe}\varepsilon^{fbcd} + \Gamma^b_{fe}\varepsilon^{afcd} + \Gamma^c_{fe}\varepsilon^{abfd} + \Gamma^d_{fe}\varepsilon^{abcf} - \Gamma^f_{fe}\varepsilon^{abcd} \\ &= \Gamma^a_{fe}\varepsilon^{fbcd} + \Gamma^b_{fe}\varepsilon^{afcd} + \Gamma^c_{fe}\varepsilon^{abfd} + \Gamma^d_{fe}\varepsilon^{abcf} - \Gamma^f_{fe}\varepsilon^{abcd} \end{aligned}$$

By multiplying both sides by ε_{abcd},

$$\begin{aligned} \varepsilon_{abcd}\nabla_e \varepsilon^{abcd} &= \varepsilon_{abcd}\left(\Gamma^a_{fe}\varepsilon^{fbcd} + \Gamma^b_{fe}\varepsilon^{afcd} + \Gamma^c_{fe}\varepsilon^{abfd} + \Gamma^d_{fe}\varepsilon^{abcf} - \Gamma^f_{fe}\varepsilon^{abcd}\right) \\ &= 3!\Gamma^f_{fe} + 3!\Gamma^f_{fe} + 3!\Gamma^f_{fe} + 3!\Gamma^f_{fe} - 4!\Gamma^f_{fe} (\because\ \varepsilon_{abcd}\varepsilon^{fbcd} = 3!\delta^f_a) \\ &= 0 \end{aligned}$$

Since ε_{abcd} is a constant, $\nabla_e \varepsilon^{abcd} = 0$. Also, whenever $\varepsilon_{abcd} = 0$, its derivative is obviously 0.

$$\nabla_e(\varepsilon_{abcd}\varepsilon^{abcd}) = (\nabla_e \varepsilon_{abcd})\varepsilon^{abcd} + \varepsilon_{abcd}\nabla_e \varepsilon^{abcd} = 0$$

and naturally $\nabla_e \varepsilon_{abcd} = 0$ since $\nabla_a g_{bc} = 0$.

7.3 Write Jacobian matrix as

$$J^a{}_b = \partial x'^a/\partial x^b, \quad \text{and} \quad \tilde{J}^c{}_d = \partial x^c/\partial x'^d.$$

From the definition, we know that $J^a{}_b\tilde{J}^b{}_d = \delta^a_d$. If we write $J = \det(J^a{}_b)$, by fixing i,

$$J = \sum_j J^i{}_j(\mathcal{J})_i{}^j$$

where $(\mathcal{J})_i{}^j$ is the cofactor of $J^i{}_j$. It means

$$(\mathcal{J})_i{}^j = \partial J/\partial(J^i{}_j).$$

The components of given Jacobian matrix are all functions of coordinates x^k. Then the determinant of a functional of the $J^i{}_j$ is

$$J = J\left(J^i{}_j(x^k)\right)$$

and it gives

$$\partial_c J = \frac{\partial J}{\partial (J^i{}_j)} \frac{\partial (J^i{}_j)}{\partial x^c} = (\mathcal{J})_i{}^j \partial_c (J^i{}_j) = J \tilde{J}^i{}_j \partial_c J^i{}_j$$

since $\tilde{J}^i{}_j = (\mathcal{J})_j{}^i / J$. Hence we obtain

$$\partial_c J = J \tilde{J}^i{}_j \partial_c J^i{}_j$$

In the textbook notation, $\partial_c J = J J^{ab} \partial_c J_{ba}$.

7.4 For an arbitrary vector field T^a,

$$\nabla_a \left[\sqrt{-g}\, T^a \right] = \partial_a \left(\sqrt{-g}\, T^a \right).$$

By exploiting Leibniz rule, the right hand side can be written as

$$\begin{aligned}
\nabla_a [\sqrt{-g}\, T^a] &= T^a \nabla_a [\sqrt{-g}] + \sqrt{-g} \nabla_a T^a \\
&= T^a \nabla_a [\sqrt{-g}] + \sqrt{-g} \partial_a T^a + \sqrt{-g} \Gamma^a_{ba} T^b
\end{aligned}$$

and the left hand side will be $\partial_a (\sqrt{-g}) T^a + \sqrt{-g}\, \partial_a T^a$. Then we have

$$T^a \nabla_a [\sqrt{-g}] = T^a \partial_a (\sqrt{-g}) - \sqrt{-g} \Gamma^a_{ba} T^b,$$

or since T^a was arbitrary,

$$\therefore\ \nabla_a [\sqrt{-g}] = \partial_a (\sqrt{-g}) - \Gamma^b_{ba} \sqrt{-g}.$$

Hence, $\sqrt{-g}$ is a scalar density of weight $+1$.

7.5 For any vector field T^a,

$$\begin{aligned}
\int_\Omega \nabla_a T^a \sqrt{-g} \mathrm{d}^4 x &= \int_\Omega \nabla_a (\sqrt{-g} T^a)\, \mathrm{d}^4 x \quad (\because\ \nabla_c (\sqrt{-g}) = 0) \\
&= \int_\Omega \partial_a (\sqrt{-g} T^a) \mathrm{d}^4 x \\
&= \int_{\partial\Omega} T^a \sqrt{-g} \mathrm{d} S_a
\end{aligned}$$

Hence, one verifies

$$\int_\Omega \nabla_a T^a \sqrt{-g} \mathrm{d}^4 x = \int_{\partial\Omega} T^a \sqrt{-g} \mathrm{d} S_a$$

7.6 (i) $L(y, y', x) = y^2 + y'^2$.

$$\begin{aligned}
\frac{\partial L}{\partial y} &= 2y \\
\frac{\partial L}{\partial y'} &= 2y'.
\end{aligned}$$

The equations of motion says,

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 2y'' - 2y = 0.$$

Hence,

$$\therefore\ y'' - y = 0.$$

(ii) $L(y_1,\, y_2,\, {y_1}',\, {y_2}',\, x) = xy_1^3 + y_1y_2 + y_1({y_1'}^2 + {y_2'}^2)$. Notice, in this case, given Lagrangian explicitly depends on x.

$$\begin{aligned}\frac{\partial L}{\partial y_1} &= 3xy_1^3 + y_2 + ({y_1'}^2 + {y_2'}^2)\\ \frac{\partial L}{\partial y_2} &= y_1\\ \frac{\partial L}{\partial {y_1}'} &= 2y_1{y_1}'\\ \frac{\partial L}{\partial {y_2}'} &= 2y_1{y_2}'\end{aligned}$$

The equations of motion says

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial {y_1}'}\right) - \frac{\partial L}{\partial y_1} &= 2y_1{y_1}'' + 2{y_1'}^2 - 3xy_1^3 - y_2 - ({y_1'}^2 + {y_2'}^2) = 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial {y_2}'}\right) - \frac{\partial L}{\partial y_2} &= 2y_1{y_2}'' + 2{y_1}'{y_2}' - y_1 = 0.\end{aligned}$$

Hence,

$$\therefore \begin{cases}2y_1{y_1}'' + {y_1'}^2 - 3xy_1^3 - y_2 - {y_2'}^2 = 0\\ 2y_1{y_2}'' + 2{y_1}'{y_2}' - y_1 = 0\end{cases}$$

7.7 Consider the Lagrangian functional $L = L(x^a,\, \dot{x}^a,\, u)$ where u is a spacelike affine parameter, i.e, $\mathrm{d}L/\mathrm{d}u = 0$, which is

$$L(x^a,\, \dot{x}^a) = \sqrt{g_{ab}(x)\dot{x}^a\dot{x}^b}, \quad (\cdot \equiv \mathrm{d}/\mathrm{d}u).$$

Then, in the textbook notation, we may write

$$K := \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}L^2 = -\frac{1}{2}$$

since we are using a spacelike affine parameter, $g_{ab}\dot{x}^a\dot{x}^b = -1$. When the Lagrangian functional, L, does not explicitly depend on its parameter, L and L^2 give the same geodesic equation and geodesic is obtained from the action principle $\delta S = 0$. However, there is an ambiguity in the sign of the action and hence, the form of the geodesic equation itself does not change whether a parameter is timelike or spacelike.

7.8

7.9 The metric is given by

$$g_{ab} = \begin{pmatrix} x^2 & 0\\ 0 & x\end{pmatrix},$$

where $(x^a) = (x^0,\, x^1) = (x, y)$. Then, it can be easily seen that given metric is independent of y. Hence, the Killing vector solution will be

$$\therefore\ Y = \frac{\partial}{\partial y}$$

7.10 We have

$$\begin{aligned}L_X g_{ab} &= X^e\partial_e g_{ab} + g_{ad}\partial_b X^d + g_{bd}\partial_a X^d\\ &= X^e\partial_e g_{ab} + \partial_b X_a - X^d\partial_b g_{ad} + \partial_a X_b - X^d\partial_a g_{bd}\\ &= \partial_b X_a + \partial_a X_b - X^e(\partial_b g_{ae} + \partial_a g_{be} - \partial_e g_{ab})\\ &= \partial_b X_a + \partial_a X_b - 2X^e g_{ec}\Gamma^c_{ab}\\ &= (\partial_b X_a - X_c\Gamma^c_{ba}) + (\partial_a X_b - X_c\Gamma^c_{ab}) \quad (\because\ \Gamma^c_{ab} = \Gamma^c_{ba})\\ &= \nabla_b X_a + \nabla_a X_b.\end{aligned}$$

Hence,

$$\therefore\ L_X g_{ab} = \nabla_a X_b + \nabla_b X_a.$$

7.11 From the line element,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

the metric can be written as $g_{ab} = \mathrm{diag}[+1, +1, +1]$. By using the Killing's equations

$$\nabla_a X_b + \nabla_b X_a = \partial_a X_b + \partial_b X_a = 0$$

we can find Killing vector fields, X^a. But, by differentiate on both sides, one finds that

$$\begin{aligned}\partial_c\partial_a X_b + \partial_c\partial_b X_a &= \partial_a(-\partial_b X_c) + \partial_c\partial_b X_a \\ &= -\partial_b(-\partial_c X_a) + \partial_c\partial_b X_a \\ &= 2\partial_b\partial_c X_a \\ &= 0\end{aligned}$$

,i.e., $\partial_b\partial_c X_a = 0$. By integrating, eventually,

$$X^a = \omega^a{}_b x^b + t^a$$

where $\omega^a{}_b$ and t^a are constants of integration and $\omega_{ab} + \omega_{ba} = 0$ from the Killing's equations. There are six independent parameters three from $\omega^a{}_b$ and three from t^a. Then the general solution for X^a will

$$\begin{aligned}\overset{1}{X^a} &= (1, 0, 0) \\ \overset{2}{X^a} &= (0, 1, 0) \\ \overset{3}{X^a} &= (0, 0, 1) \\ \overset{4}{X^a} &= (0, z, -y) \\ \overset{5}{X^a} &= (-z, 0, x) \\ \overset{6}{X^a} &= (y, -x, 0)\end{aligned}$$

where the first three represent translation and the last three represent rotation.

7.12 Let X^a and Y^a be Killing vector fields. Then, from the Killing's equation, it follows that

$$\begin{aligned}\nabla_a X_b + \nabla_b X_a = 0 \cdots (1) \\ \nabla_a Y_b + \nabla_b Y_a = 0 \cdots (2)\end{aligned}$$

From the linearity of ∇, λ * Eq.(1) + μ * Eq.(2) gives

$$\begin{aligned}\lambda(\nabla_a X_b + \nabla_b X_a) + \mu(\nabla_a Y_b + \nabla_b Y_a) &= \nabla_a(\lambda X_b + \mu Y_b) + \nabla_b(\lambda X_a + \mu Y_a) \\ &= 0\end{aligned}$$

Hence, $\lambda X_a + \mu Y_a$ is also a Killing vector field.

7.13

$$L_X L_Y - L_Y L_X = L_{[X,Y]}$$

where L represents the Lie derivative.
(i) Since Lie derivative is a type preserving operator,

$$L_X L_Y f - L_Y L_X f = (XY - YX)f = [X, Y]f$$

(ii) Let m^a be arbitrary vector field. From the definition of the Lie derivative

$$\begin{aligned}
L_Y m^a &= [Y,m]^a \\
L_X L_Y m^a &= [X,[Y,m]]^a
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
(L_X L_Y - L_Y L_X) m^a &= [X,[Y,m]]^a - [Y,[X,m]]^a \\
&= -[Y,[m,X]]^a - [m,[X,Y]]^a - [Y,[X,m]]^a \\
&= [[X,Y],m]^a \\
&= L_{[X,Y]} m^a
\end{aligned}$$

(iii) If we write $f = V_a m^a$, from the Leibniz rule, we have

$$\begin{aligned}
L_Y f &= (L_Y V_a) m^a + V_a L_Y m^a \\
L_X L_Y f &= (L_X L_Y V_a) m^a + (L_Y V_a)(L_X m^a) + (L_X V_a)(L_Y m^a) + V_a L_X L_Y m^a
\end{aligned}$$

From the above relation, we may write

$$\begin{aligned}
(L_X L_Y - L_Y L_X) f &= \{(L_X L_Y - L_Y L_X) V_a\} m^a + V_a \{(L_X L_Y - L_Y L_X) m^a\} \\
&= L_{[X,Y]} f \\
&= (L_{[X,Y]} V_a) m^a + V_a (L_{[X,Y]} m^a)
\end{aligned}$$

Hence, from the result of (i) and (ii), we have $(L_X L_Y - L_Y L_X) V_a = L_{[X,Y]} V_a$.
The Killing equation is given by $L_X g_{ab} = 0$. Then, if X and Y are Killing vector fields, we have

$$L_X L_Y g_{ab} - L_Y L_X g_{ab} = 0 = L_{[X,Y]} g_{ab}$$

Hence, $[X,Y]$ is also a Killing vector field. From this result, if $X = \partial/\partial x$ and $Y = -y\partial/\partial x + x\partial/\partial y$, then $\partial/\partial y$ is also an another Killing vector since

$$[\partial/\partial x, -y\partial/\partial x + x\partial/\partial y] = \partial/\partial y$$

7.14 From

$$\begin{aligned}
\nabla_c(\nabla_b X_a) &= \partial_c(\nabla_b X_a) - \Gamma^d_{cb}\nabla_d X_a - \Gamma^d_{ca}\nabla_b X_d \\
&= \partial_c(\partial_b X_a - \Gamma^d_{ba} X_d) - \Gamma^d_{cb}(\partial_d X_a - \Gamma^e_{da} X_e) - \Gamma^d_{ca}(\partial_b X_d - \Gamma^e_{bc} X_e) \\
&= \partial_c\partial_b X_a + (\partial_c \Gamma^d_{ba}) X_d - \Gamma^d_{ba}\partial_c X_d - \Gamma^d_{cb}\partial_d X_a + \Gamma^d_{cb}\Gamma^e_{da} X_e - \Gamma^d_{ca}\partial_b X_d + \Gamma^d_{ca}\Gamma^e_{bd} X_e,
\end{aligned}$$

we can calculate

$$\begin{aligned}
(\nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a) &= (\partial_c \Gamma^d_{ba} - \partial_b \Gamma^d_{ca}) X_d + (\Gamma^d_{ca}\Gamma^e_{bd} - \Gamma^d_{ba}\Gamma^e_{cd}) X_e \\
&= R^d{}_{acb} X_d
\end{aligned}$$

Then, if X^a is a Killing vector field, then from the definition,

$$\nabla_a X_b + \nabla_b X_a = 0$$

we can differentiate both side covariantly

$$\nabla_c \nabla_a X_b + \nabla_c \nabla_b X_a = 0$$

By using the previous result,

$$\nabla_c \nabla_a X_b + (\nabla_b \nabla_c X_a + R^d{}_{acb} X_d) = 0$$

and contracting a and b,

$$g^{ab}\nabla_c \nabla_a X_b + g^{ab}\nabla_b \nabla_c X_a + (g^{ab} R^d{}_{acb}) X_d = g^{ab}\nabla_b \nabla_c X_a + R_{dc} X^d$$

since $g^{ab}\nabla_a X_b = 0$. Hence,

$$g^{ab}\nabla_b\nabla_c X_a + R_{dc}X^d = 0.$$

7.15 From the above result, it follows that

$$\begin{aligned}
(\nabla_c\nabla_b - \nabla_b\nabla_c)X_a &= R^d{}_{acb}X_d \\
(\nabla_b\nabla_a - \nabla_b\nabla_a)X_c &= R^d{}_{cba}X_d \\
(\nabla_a\nabla_c - \nabla_c\nabla_a)X_b &= R^d{}_{bac}X_d
\end{aligned}$$

Chapter 8

8.1 In spherical coordinate, we have

$$g_{ab} = \mathrm{diag}[1,\, r^2,\, r^2\sin^2\theta]$$

where $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$. Then the action principle says

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0, \quad \cdot \equiv \frac{\mathrm{d}}{\mathrm{d}t}$$

where $L = g_{ab}\dot{x}^a\dot{x}^b$ and t is an affine parameter. It gives

$$2g_{ab}\ddot{x}^b + 2\partial_b g_{ac}\dot{x}^b\dot{x}^c - \partial_a g_{bc}\dot{x}^b\dot{x}^c = 0.$$

For $a = r$,

$$2\ddot{r} - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2 = 0,$$

or,

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0$$

and we have

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta$$

For $a = \theta$,

$$2r^2\ddot{\theta} + 2(2r)\dot{r}\dot{\theta} - 2r^2\sin\theta\cos\theta\dot{\phi}^2 = 0,$$

or,

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0.$$

and we have

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.$$

For $a = \phi$,

$$2r^2\sin^2\theta\ddot{\phi} + 2(2r\sin^2\theta)\dot{r}\dot{\phi} + 2(2r^2\sin\theta\cos\theta)\dot{\theta}\dot{\phi} = 0,$$

or,

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} = 0.$$

and we have

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r, \quad \Gamma^\phi_{\theta\phi} = \cot\theta.$$

If we write $\Gamma^a_{bc} = (\mathbf{\Gamma}_b)^a{}_c$,

$$\mathbf{\Gamma}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/r \end{pmatrix}, \quad \mathbf{\Gamma}_\theta = \begin{pmatrix} 0 & -r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & \cot\theta \end{pmatrix}, \quad \mathbf{\Gamma}_\phi = \begin{pmatrix} 0 & 0 & -r\sin^2\theta \\ 0 & 0 & -\sin\theta\cos\theta \\ 1/r & \cot\theta & 0 \end{pmatrix}$$

then the Riemann tensor will be written as

$$\mathbf{R}_{cd} = \partial_c \mathbf{\Gamma}_d - \partial_d \mathbf{\Gamma}_c - [\mathbf{\Gamma}_c,\, \mathbf{\Gamma}_d]$$

where $R^a{}_{bcd} = (\mathbf{R}_{cd})^a{}_b$. From the direct calculation, for $r\theta$-components,

$$\begin{aligned}
\mathbf{R}_{r\theta} &= \partial_r \mathbf{\Gamma}_\theta - \partial_\theta \mathbf{\Gamma}_r + [\mathbf{\Gamma}_r,\, \mathbf{\Gamma}_\theta] \\
&= \begin{pmatrix} 0 & -1 & \\ -1/r^2 & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ 1/r^2 & 0 & 0 \\ 0 & 0 & \cot\theta/r \end{pmatrix} - \begin{pmatrix} 0 & -1 & \\ & 0 & \\ 0 & 0 & \cot\theta/r \end{pmatrix} \\
&= O
\end{aligned}$$

where O is the zero matrix. For $r\phi$-components

$$\begin{aligned}
\mathbf{R}_{r\phi} &= \partial_r \mathbf{\Gamma}_\phi - \partial_\phi \mathbf{\Gamma}_r + [\mathbf{\Gamma}_r,\, \mathbf{\Gamma}_\phi] \\
&= \begin{pmatrix} 0 & 0 & -\sin^2\theta \\ & 0 & \\ -1/r^2 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & -\sin\theta\cos\theta/r \\ 1/r^2 & \cot\theta/r & 0 \end{pmatrix} - \begin{pmatrix} 0 & & -\sin^2\theta \\ & 0 & -\sin\theta\cos\theta/r \\ 0 & \cot\theta/r & 0 \end{pmatrix} \\
&= O.
\end{aligned}$$

For $\theta\phi$-components,

$$\begin{aligned}
\mathbf{R}_{\theta\phi} &= \partial_\theta \mathbf{\Gamma}_\phi - \partial_\phi \mathbf{\Gamma}_\theta + [\mathbf{\Gamma}_\theta,\, \mathbf{\Gamma}_\phi] \\
&= \begin{pmatrix} 0 & & -r\sin 2\theta \\ & 0 & -\cos 2\theta \\ 0 & -\csc^2\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & & r\sin\theta\cos\theta \\ & 0 & -\sin^2\theta \\ \cot\theta/r & \cot^2\theta & 0 \end{pmatrix} - \begin{pmatrix} 0 & & -r\sin\theta\cos\theta \\ & 0 & -\cos^2\theta \\ \cot\theta/r & -1 & 0 \end{pmatrix} \\
&= O.
\end{aligned}$$

Hence, verified that the Riemann tensor vanishes.

8.2 Let X^a be a timelike vector and orthogonal to some vector Y^a, i.e.,

$$g_{ab} X^a Y^b = 0.$$

Then, we have

$$Y^0 = \frac{\vec{X} \cdot \vec{Y}}{X^0}$$

since $X^0 \neq 0$. By squaring both sides,

$$\begin{aligned}
\left(Y^0\right)^2 &= \frac{\left(\vec{X} \cdot \vec{Y}\right)^2}{(X^0)^2} \\
&< \frac{\left(\vec{X} \cdot \vec{Y}\right)^2}{\vec{X} \cdot \vec{X}} \quad \left(\because (X^0)^2 > \vec{X} \cdot \vec{X}\right) \\
&\leq \vec{Y} \cdot \vec{Y} \quad \left(\because \vec{X} \cdot \vec{Y} \leq \left(\vec{X} \cdot \vec{Y}\right)^2\right).
\end{aligned}$$

Hence, given vector Y^a must be spacelike. In the similar way, if we write two null vectors as u^a and v^b, we obtain the following relation

$$(u_i v^i)^2 = (u_i)^2 (v_i)^2,$$

which says that u^i and v^i are parallel. also, since two vectors are null-like, u^a and v^a are parallel.

8.3 By calculating the norm,

$$
\begin{aligned}
g_{ab}L^aL^b &= \frac{1}{2}g_{ab}(T^aT^b + T^aZ^b + T^bZ^a + Z^aZ^b) = 0 \\
g_{ab}N^aN^b &= \frac{1}{2}g_{ab}(T^aT^b - T^aZ^b - T^bZ^a + Z^aZ^b) = 0 \\
g_{ab}M^aM^b &= \frac{1}{2}g_{ab}(X^aX^b - Y^aY^b + iX^aY^b + iX^bY^a) = 0 \\
g_{ab}\bar{M}^a\bar{M}^b &= \frac{1}{2}g_{ab}(X^aX^b - Y^aY^b - iX^aY^b - iX^bY^a) = 0
\end{aligned}
$$

we can verify that L^a, N^a, M^a, $\bar{M}^a$ are null-like. Since L^a, N^a and $M^a, \bar{M}^a$ live in a different orthogonal subspaces, we need only to calculate the inner product of L^a, N^a and M^a, $\bar{M}^a$ pairs.

$$
\begin{aligned}
g_{ab}L^aN^b &= \frac{1}{2}g_{ab}(T^aT^b - Z^aZ^b) = 1 \\
g_{ab}M^a\bar{M}^b &= \frac{1}{2}g_{ab}(X^aX^b + Y^aY^b) = -1
\end{aligned}
$$

8.4 Lorentz transformation from an observer O to another observer O' can be written as

$$\Lambda : O \to O', \quad x^a \mapsto L^a{}_b x^b.$$

Under this transformation, the induced metric can be written as

$$\eta_{cd} L^c{}_a x^a L^d{}_b x^b = \eta_{cd} L^c{}_a L^d{}_b x^a x^b.$$

Hence, one obtains from the Lorentz invariance,

$$\therefore\ \eta_{ab} = \eta_{cd} L^c{}_a L^d{}_b$$

To show these transformations form a group, consider an additional observer O'' and two more transformations

$$
\begin{aligned}
\Lambda' : O' \to O'', &\quad x'^a \mapsto x''^a = (L')^a{}_b x'^b \\
\Lambda'' : O \to O'', &\quad x^a \mapsto x''^a = (L'')^a{}_b x^b.
\end{aligned}
$$

By a successive transformation, we have

$$(L'')^a{}_b = (L')^a{}_c L^c{}_b$$

and the induced metric will be

$$\eta_{cd}(L')^c{}_e L^c{}_a (L')^d{}_f L^c{}_b = \eta_{ef} L^c{}_a L^d{}_f = \eta_{ab}.$$

Hence, these transformations form a group since the identity map is trivial.
For the Poincare group, write $P(L,\, a)$ and $P'(L',\, a)$. Then, for $P''(L'',\, a'')$, it follows that

$$
\begin{aligned}
x''^a &= L'^a{}_b x'^b + t'^a \\
&= L'^a{}_b (L^b{}_c x^c + t^b) + t'^a \\
&= L'^a{}_b L^b{}_c x^c + L'^a{}_b t^b + t'^a \\
&\equiv L''^a{}_b x^b + t''^a
\end{aligned}
$$

and we have

$$\therefore L''^a{}_b = L'^a{}_c L^c{}_b, \quad t''^a = L'^a{}_b t^b + t'^a.$$

When we write ${L'}^a{}_b = (L^{-1})^a{}_b$ and ${t'}^a = -(L^{-1})^a{}_b t^b$, we can obtain the inverse of $P(L,\, a)$. To show associativity, we only need to check the inhomogeneous part. Since

$$\begin{aligned}
{t'''}^a &= {L''}^a{}_b {L'}^b{}_c x^c + {L''}^a{}_b {t'}^b + {t''}^a \\
&= {L''}^a{}_b ({L'}^b{}_c x^c + {t'}^b) + {t''}^a \\
&= {L''}^a{}_b {t''}^b + {t''}^a,
\end{aligned}$$

associativity follows.

8.5 If X^a is a Killing vector field, then it follow

$$\nabla_a X_b + \nabla_b X_a = 0.$$

But if we are in the Minkowski coordinates, then we can write

$$\partial_a X_b + \partial_b X_a = 0.$$

By partially differentiate on both sides, we have

$$\begin{aligned}
\partial_c \partial_a X_b + \partial_c \partial_b X_a &= \partial_a \partial_c X_b + \partial_c \partial_b X_a \\
&= \partial_a(-\partial_b X_c) + \partial_c \partial_b X_a \\
&= -\partial_b(-\partial_c X_a) + \partial_c \partial_b X_a \\
&= 2\partial_b \partial_c X_a \\
&= 0
\end{aligned}$$

Hence,

$$\therefore\ \partial_b \partial_c X_a = 0.$$

Now if we integrate the result,

$$\partial_c X_a = \omega_{ca}$$

where ω_{ca} is some constant satisfying $\omega_{ca} + \omega_{ac} = 0$ since $\partial_c X_a + \partial_a X_c = 0$. Eventually,

$$\therefore\ X_a = \omega_{ca} x^c + t_a$$

Then, for an n dimensional manifold, the degree of freedom will be

$$n \times (n-1)/2 + n = n(n+1)/2$$

and in the case of Minkowski spacetime, $4 \times 5/2 = 10$. Here, it corresponds to 3 rotations, 3 boosts, and 4 translations.

8.6 We start from the action principle

$$S = \int_M L$$

where M is given manifold. Geodesic means the shortest path between two fixed end points. Naturally thinking, it is reasonable to consider L to be $(g_{ab} x^a x^b)^{1/2}$ which is the length. Then, the action can be written as

$$S = \int_M \mathrm{d}t \sqrt{g_{ab} \dot{x}^a \dot{x}^b}$$

where t is the parameter for the curve on the manifold M. But if the metric, g, is independent of parameter, we can consider the following action

$$S = \int_M \mathrm{d}t g_{ab} \dot{x}^a \dot{x}^b$$

which gives

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = (\ddot{t}/\dot{t})\dot{x}^a$$

When $t = \tau$, i.e., a proper time, it is timelike. Hence, $L = +1$ and it means $(\partial L/\partial\dot{x}^a)(\mathrm{d}L/\mathrm{d}u) = 0$ and it must follow $\ddot{\tau} = 0$, i.e., τ is an affine parameter.

Chapter 10

10.1 According to the Taylor's theorem in 3-dimension, a locally analytic function f can be written as

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \sum_{n=1}^{\infty}\frac{1}{n!}(\mathbf{h}\cdot\nabla)^n f(\mathbf{x})$$

Now consider the three leading orders of the Taylor expansion. For $n = 1$,

$$\mathbf{h}\cdot\nabla f = h_1\partial_1 f + h_2\partial_2 f + h_3\partial_3 f$$

and for $n = 2$, since

$$(\mathbf{h}\cdot\nabla)^2 = (h_i\partial_i)(h_j\partial_j) = h_ih_j\partial_i\partial_j$$

we have

$$\frac{1}{2!}\sum_{i,j=1}^{3} h_ih_j\partial_i\partial_j f = \frac{1}{2}(h_1^2\partial_1^2 f + h_2^2\partial_2^2 f + h_3^2\partial_3^2 f) + h_1h_2\partial_1\partial_2 f + h_2h_3\partial_2\partial_3 f + h_3h_1\partial_1\partial_3 f.$$

Similarly, for $n = 3$,

$$\begin{aligned}
\frac{1}{3!}h_ih_jh_k\partial_i\partial_j\partial_k &= \frac{1}{3!}(h_1^3\partial_1^3 f + h_2^3\partial_2^3 f + h_3^3\partial_3^3 f + 3h_1^2h_2\partial_1^2\partial_2 f + 3h_2^2h_3\partial_2^2\partial_3 f \\
&\qquad + 3h_3^2h_1\partial_3^2\partial_1 f + 3!h_1h_2h_3\partial_1\partial_2\partial_3 f) \\
&= \frac{1}{3!}(h_1^3\partial_1^3 f + h_2^3\partial_2^3 f + h_3^3\partial_3^3 f) \\
&\quad + \frac{1}{2}(h_1^2h_2\partial_1^2\partial_2 f + h_2^2h_3\partial_2^2\partial_3 f + h_3^2h_1\partial_3^2\partial_1 f) \\
&\quad + h_1h_2h_3\partial_1\partial_2\partial_3 f
\end{aligned}$$

10.2 (i) Write $P : x^\alpha$ and $QQ : x^\alpha + \eta^\alpha$. Then, at Q, a scalar field can be written as

$$\phi(x^\alpha + \eta^\alpha) = \phi(x^\alpha) + \eta^\beta\partial_\beta\phi(x^\alpha) + O(\eta^2).$$

Now operating partial derivative on both sides will give

$$(-\partial^\alpha\phi)_Q = -(\partial^\alpha\phi)_P - (\eta^\beta\partial_\beta\partial^\alpha\phi)_P + O(\eta^2)$$

since partial derivatives commute.
(ii) The Newtonian equation of deviation says

$$\ddot{\eta}^\alpha + K^\alpha{}_\beta\eta^\beta = 0 \quad \text{where} \quad K_{\alpha\beta} = \partial_\alpha\partial_\beta\phi$$

To connect with the Newtonian field equations in empty space, Laplace equation can be written as

$$\nabla^2\phi = \partial^\alpha\partial_\alpha\phi = K^\alpha{}_\alpha = 0$$

,i.e., K is traceless. It implies the nature of divergence-free meaning that there is no monopole and, hence, we consider the transverse waves.

10.3 When $V^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$ and ξ^a is an arbitrary vector field on a Riemannian manifold,
(i)

$$\nabla_V V^a = \frac{D}{D\tau}\left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x^a + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0$$

since τ is an affine parameter.
(ii) Since the metric is covariantly constant, $\nabla_c g_{ab} = 0$, it follows that

$$\nabla_V V_a = g_{ab}\nabla_V V^b = 0.$$

(iii)

$$V_a \nabla_\xi V^a = \frac{1}{2}\nabla_\xi (V_a V^a) = 0$$

since $V_a V^a$ is a scalar.
(iv) Due to $\nabla_c g_{ab} = 0$, it follows from (iii).

10.4 Let $h^a{}_b = \delta^a_b - V^a V_b$ which is the projection operator. If we define the orthogonal connecting vector η^a by $\eta^a := h^a{}_b \xi^b$ where ξ^a is a connecting vector which satisfies

$$\frac{D^2}{D\tau^2}\xi^a - R^a{}_{bcd}V^b V^c \xi^d = 0$$

,by substituting $\eta^a = \eta^a + (V_b \xi^b)V^a$, we obtain

$$\frac{D^2}{D\tau^2}\eta^a + \frac{D^2}{D\tau^2}(V_b\xi^b)V^a - R^a{}_{bcd}V^b V^c \eta^d = 0 \qquad (\because R_{abcd} = -R_{abdc})$$

But, $\frac{D^2}{D\tau^2}V^a = 0$ since V^a is the tangent vector for geodesic. Hence,

$$\therefore \frac{D^2}{D\tau^2}\eta^a - R^a{}_{bcd}V^b V^c \eta^d = 0$$

10.5 When X^a is the Killing vector, then it follows

$$\nabla_a X_b + \nabla_b X_a = 0.$$

Then, it follows that

$$\nabla_a \nabla_a X_b + \nabla_a \nabla_b X_a = 0.$$

But, we know that in the case of Killing vector fields,

$$\nabla_a \nabla_b X_a = R_{abad}X^d$$

and hence,

$$\nabla_a \nabla_a X_b + R_{abad}X^d = 0$$

or

$$\frac{D^2}{Du^2}X^a - R^a{}_{bcd}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}x^c}{\mathrm{d}u}X^d = 0$$

along any geodesic.

10.6 Let R_{ab} a tensor of valence (0,2). Consider

$$R_{ab}v^a v^b = 0$$

for any timelike vector v^a. Let us decompose this timelike vector into two parts, $v^a = t^a + \lambda s^a$ ($\lambda \in \mathbb{R}$) where $t^a t_a = 1$, $s^a s_a = -1$ and $t^a s_a = 0$. Then,

$$v^a v_a = t^a t_a + \lambda^2 s^a s_a = 1 - \lambda^2.$$

For $0 \le \lambda < 1$, v^a is a timelike vector. Now consider the following

$$R_{ab} v^a v^b = R_{ab} t^a t^b + \lambda R_{ab} t^a v^b + \lambda R_{ab} v^a t^b + \lambda^2 R_{ab} v^a v^b.$$

For a special coordinate $t^a \stackrel{*}{=} \delta^a_0$, $s^a \stackrel{*}{=} \delta^a_i$ $(i = 1, 2, 3)$,

$$R_{00} + \lambda R_{0i} + \lambda R_{i0} + \lambda^2 R_{ii} = R_{00} + 2\lambda R_{0i} + \lambda^2 R_{ii} = 0.$$

It must satisfy for any λ, so when $\lambda = 0$, we have $R_{00} = 0$. Also for $\lambda \neq 0$,

$$2\lambda R_{0i} + \lambda^2 R_{ii} = 0$$

we have $R_{0i} = R_{ii} = 0$ since λ and λ^2 are independent and given equation satisfies for $0 < \lambda < 1$.

10.7 If a frame ${e_i}^a$ is parallely propagated along C, then it means

$$\frac{D}{D\lambda} {e_i}^a = 0$$

where λ is a parameter for a curve C. The dual frame is defined by

$${e_i}^a {e^j}_a = \delta^j_i$$

and it follows that

$$\begin{aligned}\frac{D}{D\lambda}({e_i}^a {e^j}_a) &= {e_i}^a \frac{D}{D\lambda} {e^j}_a + \frac{D}{D\lambda} {e_i}^a {e^j}_a \\ &= {e_i}^a \frac{D}{D\lambda} {e^j}_a \\ &= 0\end{aligned}$$

Hence, we obtain for any frame ${e_i}^a$

$$\frac{D}{D\lambda} {e^j}_a = 0$$

since $g_{ab} {e_i}^a {e_j}^b = \eta_{ij}$.

10.8 From the definition of a frame, ${e_i}^a$ where i is an index for a vector and a is an index for components, we have

$$g_{ab} {e_i}^a {e_j}^b = \eta_{ij}$$

Then, since ${e_i}^a {e^j}_a = \delta^j_i$ or ${e_i}^a {e^i}_b = \delta^b_a$,

$$g_{ab} {e_i}^a {e_j}^b {e^i}_c {e^j}_d = \eta_{ij} {e^i}_c {e^j}_d$$

,i.e.,

$$g_{cd} = \eta_{ij} {e^i}_c {e^j}_d$$

By using $\eta^{ik} \eta_{kj} = \delta^j_i$, we have

$$g_{ab} e^{ia} e^{jb} = \eta^{ij} \quad \text{or} \quad g^{ab} {e^i}_a {e^j}_b = \eta^{ij}$$

Similarly,

$$g^{ab} = \eta^{ij} {e_i}^a {e_j}^b$$

If $(x^a) = (t, r, \theta, \phi)$ and

$$e_0{}^a = (A^{-1/2}, 0, 0, 0) \quad e_1{}^a = (0, A^{1/2}, 0, 0)$$
$$e_2{}^a = (0, 0, 1/r, 0) \quad e_3{}^a = (0, 0, 0, (r\sin\theta)^{-1})$$

where $A = A(r)$ is an arbitrary function. Then,

$$\begin{aligned} g^{ab} &= \eta^{ij} e_i{}^a e_j{}^b \\ &= -e_0{}^a e_0{}^b + e_1{}^a e_1{}^b + e_2{}^a e_2{}^b + e_3{}^a e_3{}^b \end{aligned}$$

,i.e.,

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -A^{-1}\mathrm{d}t^2 + A\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

10.9 From the classical equation of motion

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x^\alpha = -g^{\alpha\beta}\partial_\beta\phi$$

and Eq.(10.8)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x^\alpha = -\frac{1}{2}c^2\frac{\partial g_{00}}{\partial x^\alpha}[1 + O(\varepsilon)]$$

we have

$$-g^{\alpha\beta}\partial_\beta\phi = -\frac{1}{2}c^2\frac{\partial}{\partial x^\alpha}g_{00}[1 + O(\varepsilon)]$$

For the lowest order, we have

$$-\partial_\beta\phi = -\frac{1}{2}c^2\partial_\beta g_{00}$$

Or,

$$\phi + C = \frac{1}{2}c^2 g_{00}, \quad g_{00} = C + \frac{2\phi}{c^2}$$

where C is some constant. But, from the asymptotic flatness, when $\phi \to 0$, we must have $g_{00} \to 1$. Hence, we obtain the Eq.(10.9) which is

$$\therefore\ g_{00} = 1 + \frac{2\phi}{c^2} + O(v/c)$$

10.10 When we consider a coordinate transformation

$$x^a \to x'^a = x^a + \varepsilon X^a(x)$$

where X^a is some vector field, then we have a Jacobian matrix

$$J^a{}_b = \frac{\partial x'^a}{\partial x^b} = \delta^a_b + \varepsilon\partial_b X^a, \qquad \tilde{J}^a{}_b = \frac{\partial x^a}{\partial x'^b} = \delta^a_b - \varepsilon\partial_b X^a.$$

Under this coordinate transformation, the metric transforms as

$$\begin{aligned} g'_{ab} = \tilde{J}^c{}_a\tilde{J}^d{}_b\, g_{cd} &= (\delta^c_a - \varepsilon\partial_a X^c)(\delta^d_b - \varepsilon\partial_b X^d)\, g_{cd} \\ &= g_{ab} - \varepsilon\partial_a X_b - \varepsilon\partial_b X_d + O(\varepsilon^2) \\ &= g_{ab} - \varepsilon(\partial_a X_b + \partial_b X_a) + O(\varepsilon^2) \end{aligned}$$

For 00-th component,

$$\begin{aligned} g'_{00} &= g_{00} - 2\varepsilon\partial_0 X_0 + O(\varepsilon^2) \\ &\sim g_{00} - 2\varepsilon^2\partial_\alpha X_0 + O(\varepsilon^2) \\ &= g_{00} + O(\varepsilon^2) \end{aligned}$$

due to the slow-motion approximation, $\partial_0 X_a \sim \varepsilon\partial_\alpha X_0$.

Chapter 11

11.1 For a spacetime (M, g), consider an infinitesimal transformation at some point P

$$x^a \to x'^a = x^a + \varepsilon X^a(x)$$

where corresponding Jacobian matrix can be written as

$$J^{a'}{}_b = \frac{\partial x'^a}{\partial x^b} = \delta^a_b + \varepsilon \partial_b X^a(x)$$

Then, under this coordinate transformation, the metric transforms as

$$\begin{aligned}
g'_{ab}(x) &= J^c{}_a J^d{}_b g_{cd}(x') \\
&= (\delta^c_a - \varepsilon \partial_a X^c)(\delta^d_b - \varepsilon \partial_b X^d) g_{cd}(x + \varepsilon X) \\
&= (\delta^c_a - \varepsilon \partial_a X^c)(\delta^d_b - \varepsilon \partial_b X^d)\left[g_{cd}(x) + \varepsilon X^e \partial_e g_{cd}\right] \\
&= g_{ab}(x) - \varepsilon \left[\partial_a X_b + \partial_b X_a - X^e \partial_e g_{ab}\right] + O(\varepsilon^2) \\
&= g_{ab}(x) - \varepsilon L_X g_{ab} + O(\varepsilon^2)
\end{aligned}$$

Hence, we obtain

$$\delta g_{ab}(x) = g'_{ab}(x) - g_{ab} = -\varepsilon(\nabla_b X_a + \nabla_a X_b)$$

since $L_X g_{ab} = \nabla_a X_b + \nabla_b X_a$.

11.2 From $\mathcal{L}_G = \mathfrak{g}^{ab} R_{ab}$ and

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_a \Gamma^c_{ba} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ca}$$

we want the terms related to the 2nd order derivative of the metric. It means we only need to calculate the first two terms. From

$$\begin{aligned}
\partial_c \Gamma^c_{ba} &= \frac{1}{2}\partial_c\left[g^{cd}[\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ba}]\right] \\
&= \frac{1}{2} g^{cd} \partial_c \partial_b g_{da} + \frac{1}{2} g^{cd} \partial_c \partial_a g_{db} - \frac{1}{2} g^{cd} \partial_c \partial_d g_{ba} + \cdots
\end{aligned}$$

and

$$\begin{aligned}
\partial_a \Gamma^c_{bc} &= \frac{1}{2}\partial_a\left[g^{cd}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})\right] \\
&= \frac{1}{2} g^{cd} \partial_a \partial_b g_{dc} + \frac{1}{2} g^{cd} \partial_a \partial_c g_{db} - \frac{1}{2} g^{cd} \partial_a \partial_d g_{bc} + \cdots
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial \mathcal{L}_G}{\partial g_{ab,cd}} &= \mathfrak{g}^{ad}\frac{1}{2}g^{cb} + \mathfrak{g}^{bd}\frac{1}{2}g^{ca} - \mathfrak{g}^{ab}\frac{1}{2}g^{cd} - \mathfrak{g}^{cd}\frac{1}{2}g^{ab} - \mathfrak{g}^{db}\frac{1}{2}g^{ac} + \mathfrak{g}^{bc}\frac{1}{2}g^{ad} \\
&= \sqrt{-g}\left[\frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}) - g^{ab}g^{cd}\right]
\end{aligned}$$

11.3 Notice that

$$g_{ab}g^{bc} = \delta^c_a, \qquad g^{bc} = \delta^c_a (g_{ab})^{-1}.$$

It follows that

$$\begin{aligned}
\frac{\partial g^{cd}}{\partial g_{ab}} = \frac{\partial}{\partial g_{ab}}(g_{cd})^{-1} &= -(g_{cd})^{-2}\frac{1}{2}\frac{\partial}{\partial g_{ab}}[g_{cd} + g_{dc}] \\
&= -\frac{1}{2}(g_{cd})^{-2}[\delta^a_c \delta^b_d + \delta^a_d \delta^b_c] \\
&= -\frac{1}{2}(g^{ad}g^{bc} + g^{ac}g^{bd})
\end{aligned}$$

Hence,

$$\therefore \ \frac{\partial g^{cd}}{\partial g_{ab}} = -\frac{1}{2}(g^{ad}g^{bc} + g^{ac}g^{bd})$$

11.4 From $\mathcal{L}_G = \mathfrak{g}^{ab}R_{ab}$ and

$$R_{ab} = R^c{}_{acb} = \partial_c\Gamma^c_{ba} - \partial_b\Gamma^c_{ca} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ca},$$

we consider terms related to $g_{ab,c}$. From the first two terms on the right hand side,

$$\begin{aligned}
&\partial_c\left[\frac{1}{2}g^{cd}(\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ba})\right] - \partial_b\left[\frac{1}{2}g^{cd}(\partial_c g_{da} + \partial_a g_{dc} - \partial_d g_{ca})\right] \\
&= \frac{1}{2}\partial_c g^{cd}(\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ba}) - \frac{1}{2}\partial_b g^{cd}(\partial_c g_{da} + \partial_a g_{dc} - \partial_d g_{ca}) + \cdots
\end{aligned}$$

We see that when we do partial derivative with regard to $g_{ab,c}$, we will only have terms related only to the g_{ab} and its first derivative since $\Gamma = \Gamma(g, \partial g)$.

11.5 When y_A are dynamical variables and $L_1 = L_1(y_A)$ and $L_2 = L_2(y_A)$,
(i) Write $L_3 = \lambda L_1 + \mu L_2$ where $\lambda, \mu \in \mathbb{C}$. Then, $L_3 = L_3(y_A)$. Notice that

$$\begin{aligned}
L_1' &= L_1 + \delta L_1 + O(\delta^2) \\
L_2' &= L_2 + \delta L_2 + O(\delta^2)
\end{aligned}$$

and it follows that

$$\begin{aligned}
L_3' &= \lambda(L_1 + \delta L_1) + \mu(L_2 + \delta L_2) + O(\delta^2) \\
&= L_3 + \lambda\delta L_1 + \mu\delta L_2 + O(\delta^2)
\end{aligned}$$

Hence,

$$\therefore \ \delta L_3 = \lambda\delta L_1 + \mu\delta L_2$$

(ii)

$$\begin{aligned}
L_1'L_2' &= (L_1 + \delta L_1 + O(\delta^2))(L_2 + \delta L_2 + O(\delta^2)) \\
&= L_1L_2 + L_2\delta L_1 + L_1\delta L_2 + O(\delta^2)
\end{aligned}$$

,i.e.,

$$\therefore \ \delta(L_1L_2) = L_2\delta L_1 + L_1\delta L_2$$

11.6 (i) From $\delta^c_a = g^{cb}g_{ba}$, by using **11.5**(ii),

$$\delta g^{cb}g_{ba} + g^{cb}\delta g_{ba} = 0$$

since δ^c_a is constant. By contracting a and b, we obtain

$$\therefore \ g_{ab}\delta g^{ab} = -g^{ab}\delta g_{ab}$$

(ii) $\det(g_{ab}) \to \det(g_{ab} + \delta g_{ab})$,

$$\begin{aligned}
\det(g_{ab} + \delta g_{ab}) &= \frac{1}{4!}\varepsilon^{abcd}\varepsilon^{efgh}(g_{ae} + \delta g_{ae})\cdots(g_{dh} + \delta g_{dh}) \\
&= \det(g_{ab}) + \frac{1}{4!}\varepsilon^{abcd}\varepsilon^{efgh}(\delta g_{ae}\cdots g_{dh} + \cdots + g_{ae}\cdots\delta g_{dh}) + O(\delta^2) \\
&= g + \frac{1}{3!}\varepsilon^{abcd}\varepsilon^{efgh}\delta g_{ae}\cdots\delta g_{dh} + O(\delta^2) \\
&= g + \frac{1}{3!}\varepsilon^{abcd}\varepsilon^{e}{}_{bcd}\,\delta g_{ae} + O(\delta^2) \\
&= g + g^{ae}\delta g_{ae} + O(\delta^2)
\end{aligned}$$

(iii) It is easy to verify that

$$\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}(-\delta g) = -\frac{1}{2}\frac{g}{\sqrt{g}}g^{ab}\delta g_{ab} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}$$

from the previous result.

11.7 From $\mathcal{L}_G = \mathfrak{g}^{ab}R_{ab}$,
(i)

$$\frac{\delta\mathcal{L}_G}{\delta\mathfrak{g}^{ab}} = R^{ab}$$

(ii)

$$\begin{aligned}\frac{\delta\mathcal{L}_G}{\delta\mathfrak{g}_{ab}} = \frac{\delta\mathfrak{g}^{cd}}{\delta\mathfrak{g}_{ab}}R_{cd} &= -g^{ca}g^{bd}R_{cd} \quad (\because \delta g^{ab}g_{bc} + g^{ab}\delta g_{bc} = 0)\\ &= -R^{ab}\end{aligned}$$

(iii)

$$\begin{aligned}\frac{\delta\mathcal{L}_G}{\delta g^{ab}} &= \frac{\delta}{\delta g^{ab}}(-\sqrt{-g})g^{cd}R_{cd} + \sqrt{-g}\frac{\delta g^{cd}}{\delta g^{ab}}R_{cd} + \sqrt{-g}g^{cd}\frac{\delta R_{cd}}{\delta g^{ab}}\\ &= -\frac{1}{2}\sqrt{-g}g_{ab}R + \sqrt{-g}R_{ab}\\ &= \sqrt{-g}G_{ab}\end{aligned}$$

since $\delta R_{cd} = \nabla_a(\delta\Gamma^a_{cd}) - \nabla_d(\delta\Gamma^a_{ca})$. And their differential constraints are

$$\nabla_b R^{ab} = 0 \quad \text{or} \quad \nabla_b G^{ab} = 0$$

which are a contracted Bianchi identity.

11.8 (i) When $\int_\Omega \mathrm{d}\Omega\,\Phi = 0$, suppose there is at least one point in P in Ω such that $\Phi(P) \neq 0$. Then, there is a neighborhood around P such that $\Phi(Q) \neq 0$ where $Q \in N$. From the continuity of Φ, there exists a subdomain Ω' containing N such that $\int_{\Omega'} \mathrm{d}\Omega'\,\Phi \neq 0$. This contradicts our assumption that $\int_\Omega \mathrm{d}\Omega\,\Phi = 0$ for all arbitrary Ω.
(ii) If $\omega^a X_a = 0$ where X_a is arbitrary, then let us choose X_a some coordinate basis, e_b. It leads to $\omega^b = 0$. Since X_a was arbitrary, we conclude that $\omega^a = 0$.

11.9 When two Lagrangians $L(y,\, y',\, x)$ and $\bar{L}(y,\, y',\, x)$ differ by a divergence, i.e.,

$$L = \bar{L} + \frac{\mathrm{d}}{\mathrm{d}x}Q(y,\, y',\, x)$$

then the action can be written as

$$S = \int \mathrm{d}x\, L = \int \mathrm{d}x\, \bar{L} + \int \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}Q = \bar{S} + Q|_{\text{bdry}}$$

But when we consider a classical equation of motion, it follows the principle of least action, which states

$$\delta S = 0.$$

Since $Q|_{\text{bdry}}$ is a constant, we have $\delta Q = 0$. Hence,

$$\delta S = \delta\bar{S}$$

and both actions will give rise to the same classical equation of motion.

11.10 (i)

$$\begin{aligned}
\mathfrak{g}^{ab}{}_{,c} = \partial_c(\sqrt{-g}g^{ab}) &= \partial_c(\sqrt{-g})g^{ab} + \sqrt{-g}\partial_c g^{ab} \\
&= \frac{1}{2\sqrt{-g}}(-\partial_c g)g^{ab} + \sqrt{-g}\partial_c g^{ab} \\
&= \frac{1}{2}\sqrt{-g}g^{ab}g^{de}\partial_c g_{de} - \sqrt{-g}g^{ad}g^{bd}\partial_c g_{ed} \\
&= \sqrt{-g}\left[\Gamma^d_{cd}g^{ab} - \frac{1}{2}(g^{ad}g^{be} + g^{ae}g^{bd})\partial_c g_{ed}\right] \\
&= \Gamma^d_{cd}\mathfrak{g}^{ab} - \Gamma^a_{cd}\mathfrak{g}^{bd} - \Gamma^b_{cd}\mathfrak{g}^{ad}
\end{aligned}$$

Therefore,

$$\therefore\ \mathfrak{g}^{ab}{}_{,c} = \Gamma^d_{cd}\mathfrak{g}^{ab} - \Gamma^a_{cd}\mathfrak{g}^{bd} - \Gamma^b_{cd}\mathfrak{g}^{ad}$$

Since

$$\begin{aligned}
g^{be}\partial_c g_{ed} &= g^{be}\partial_d g_{ec} - g^{be}\partial_e g_{cd} + \Gamma^b_{cd} \\
g^{ae}\partial_c g_{ed} &= g^{ae}\partial_d g_{ec} - g^{ae}\partial_e g_{cd} + \Gamma^a_{cd},
\end{aligned}$$

we have

$$g^{ad}(g^{be}\partial_d g_{ec} - g^{be}\partial_e g_{cd}) + g^{bd}(g^{ae}\partial_d g_{ec} - g^{ae}\partial_e g_{cd}) = 0.$$

Or, it can be obtained from the fact that $\nabla_c\mathfrak{g}^{ab} = 0$, which says

$$\nabla_c\mathfrak{g}^{ab} = \partial_c\mathfrak{g}^{ab} + \Gamma^a_{dc}\mathfrak{g}^{db} + \Gamma^b_{dc}\mathfrak{g}^{ad} - \Gamma^d_{cd}\mathfrak{g}^{ab} = 0$$

since $\mathfrak{g}^{ab}$ is a tensor of density $+1$.
(ii) For the Einstein Lagrangian $\mathcal{L}_G = \sqrt{-g}R = \mathfrak{g}^{ab}R_{ab}$,

$$\begin{aligned}
\mathfrak{g}^{ab}R_{ab} &= \mathfrak{g}^{ab}R^c{}_{acb} \\
&= \mathfrak{g}^{ab}(\partial_c\Gamma^c_{ba} - \partial_b\Gamma^c_{ca} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ca})
\end{aligned}$$

Then to demand that $\bar{\mathcal{L}}_G = \mathfrak{g}^{ab}(\Gamma^c_{bd}\Gamma^d_{ca} - \Gamma^c_{cd}\Gamma^d_{ba})$, see that

$$\mathfrak{g}^{ab}(\partial_c\Gamma^c_{ba} - \partial_b\Gamma^c_{ca}) = \partial_c(\mathfrak{g}^{ab}\Gamma^c_{ba} - \mathfrak{g}^{ac}\Gamma^b_{ba}) - \mathfrak{g}^{ab}{}_{,c}\Gamma^c_{ba} + \mathfrak{g}^{ac}{}_{,c}\Gamma^b_{ba}$$

But, from the previous result,

$$\begin{aligned}
&-(\Gamma^d_{cd}\mathfrak{g}^{ab} - \Gamma^a_{cd}\mathfrak{g}^{db} - \Gamma^b_{cd}\mathfrak{g}^{ad})\Gamma^c_{ba} + (\Gamma^d_{cd}\mathfrak{g}^{ac} - \Gamma^a_{cd}\mathfrak{g}^{dc} - \Gamma^c_{cd}\mathfrak{g}^{ac})\Gamma^b_{ba} \\
&\quad = -\mathfrak{g}^{ab}\Gamma^c_{ba}\Gamma^d_{cd} + \mathfrak{g}^{bd}\Gamma^c_{ba}\Gamma^a_{cd} + \mathfrak{g}^{ad}\Gamma^c_{ba}\Gamma^b_{cd} - \mathfrak{g}^{cd}\Gamma^b_{ba}\Gamma^a_{cd} \\
&= 2(\mathfrak{g}^{bd}\Gamma^c_{ba}\Gamma^a_{dc} - \mathfrak{g}^{ab}\Gamma^c_{ab}\Gamma^d_{cd}) \\
&= 2\bar{\mathcal{L}}_G
\end{aligned}$$

Hence,

$$\mathcal{L}_G = \bar{\mathcal{L}}_G + Q^a{}_{,a} \quad \text{where} \quad Q^a = \mathfrak{g}^{bc}\Gamma^a_{bc} - \mathfrak{g}^{ac}\Gamma^b_{cb}.$$

(iii) From Eq.(11.41) in the textbook, we have

$$\mathfrak{g}^{ab}{}_{,c} = \Gamma^d_{dc}\mathfrak{g}^{ab} - \Gamma^a_{dc}\mathfrak{g}^{db} - \Gamma^d_{bc}\mathfrak{g}^{ad}$$

and it follows that

$$\mathfrak{g}^{ab}{}_{,b} = \Gamma^d_{bd}\mathfrak{g}^{ab} - \Gamma^a_{db}\mathfrak{g}^{db} - \Gamma^d_{db}\mathfrak{g}^{ad} = -\Gamma^a_{bd}\mathfrak{g}^{bd}$$

(iv) Check (ii)

(v) If we define

$$L_{ab} = \Gamma^d_{ac}\Gamma^c_{bd} - \Gamma^c_{ab}\Gamma^d_{cd}$$

then we may write $\bar{\mathcal{L}}_G = \mathfrak{g}^{ab}L_{ab}$, then

$$\delta\bar{\mathcal{L}}_G = \delta\mathfrak{g}^{ab}L_{ab} + \mathfrak{g}^{ab}\delta L_{ab}$$

and from

$$\delta L_{ab} = \delta\Gamma^d_{ac}\Gamma^c_{bd} + \Gamma^d_{ac}\delta\Gamma^c_{bd} - \delta\Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ab}\delta\Gamma^d_{cd}$$

we get

$$\begin{aligned}
\mathfrak{g}^{ab}\delta L_{ab} &= \mathfrak{g}^{ab}\delta\Gamma^d_{ac}(\Gamma^c_{bd} + \Gamma^c_{bd}) - \mathfrak{g}^{ab}\Gamma^d_{cd}\delta\Gamma^c_{ab} - \mathfrak{g}^{ab}\Gamma^c_{ab}\delta\Gamma^d_{cd} \\
&= \mathfrak{g}^{ab}\delta\Gamma^d_{ac}(\Gamma^c_{bd} + \Gamma^c_{bd}) \\
&\quad - \mathfrak{g}^{ab}{}_{,c}\delta\Gamma^c_{ab} - \mathfrak{g}^{bd}\Gamma^a_{cd}\delta\Gamma^c_{ab} - \Gamma^b_{cd}\mathfrak{g}^{ad}\delta\Gamma^c_{ab} \\
&\quad + \mathfrak{g}^{ac}{}_{,a}\delta\Gamma^d_{cd} \\
&= \mathfrak{g}^{ab}\delta\Gamma^d_{ac}(\Gamma^c_{bd} + \Gamma^c_{bd} - \Gamma^c_{db} - \Gamma^c_{db}) - \mathfrak{g}^{ab}{}_{,c}\delta\Gamma^c_{ab} + \mathfrak{g}^{ac}{}_{,a}\delta\Gamma^d_{cd} \\
&= -\mathfrak{g}^{ab}{}_{,c}\delta\Gamma^c_{ab} + \mathfrak{g}^{ac}{}_{,a}\delta\Gamma^d_{cd}
\end{aligned}$$

Hence,

$$\therefore\ \mathfrak{g}^{ab}{}_{,c}\delta\Gamma^c_{ab} - \mathfrak{g}^{ab}{}_{,b}\delta\Gamma^d_{ad} = L_{ab}\delta\mathfrak{g}^{ab} - \delta\bar{\mathcal{L}}_G$$

(iv) From the

$$\mathfrak{g}^{ab}{}_{,c}\Gamma^c_{ab} - \mathfrak{g}^{ab}{}_{,b}\Gamma^c_{ac} = -2\bar{\mathcal{L}}_G$$

we have

$$\begin{aligned}
-2\delta\bar{\mathcal{L}}_G &= \Gamma^c_{ab}\delta\mathfrak{g}^{ab}{}_{,c} + \mathfrak{g}^{ab}{}_{,c}\delta\Gamma^c_{ab} - \Gamma^c_{ac}\delta\mathfrak{g}^{ab}{}_{,b} + \mathfrak{g}^{ab}{}_{,b}\delta\Gamma^c_{ac} \\
&= \Gamma^c_{ab}\delta\mathfrak{g}^{ab}{}_{,c} - \Gamma^c_{ac}\cdot\frac{1}{2}(\delta\mathfrak{g}^{ab}{}_{,a} + \delta\mathfrak{g}^{ab}{}_{,b}) + L_{ab}\delta\mathfrak{g}^{ab} - \delta\bar{\mathcal{L}}_G \\
&= \left[\Gamma^c_{ab} - \frac{1}{2}(\delta^c_a\Gamma^d_{bd} + \delta^c_b\Gamma^d_{ad})\right]\delta\mathfrak{g}^{ab}{}_{,c} + L_{ab}\delta\mathfrak{g}^{ab} - \delta\bar{\mathcal{L}}_G
\end{aligned}$$

,i.e.,

$$\therefore\ \delta\bar{\mathcal{L}}_G = \left[\frac{1}{2}(\delta^c_a\Gamma^d_{bd} + \delta^c_b\Gamma^d_{ad}) - \Gamma^c_{ab}\right]\delta\mathfrak{g}^{ab}{}_{,c} - L_{ab}\delta\mathfrak{g}^{ab}$$

From our build-up, it is natural that $\bar{\mathcal{L}}_G = \bar{\mathcal{L}}_G(\mathfrak{g}^{ab}, \mathfrak{g}^{ab}{}_{,c})$. This result is consistent with two given equations, Eq.(11.42) and Eq.(11.43) in the textbook.

11.11 If

$$\frac{1}{2}\delta^b_c\nabla_d\mathfrak{g}^{ad} + \frac{1}{2}\delta^a_c\nabla_d\mathfrak{g}^{bd} - \nabla_c\mathfrak{g}^{ab} = 0$$

(i)

$$\begin{aligned}
\nabla_d\mathfrak{g}^{ad} &= \partial_d\mathfrak{g}^{ad} + \Gamma^a_{cd}\mathfrak{g}^{cd} + \Gamma^d_{cd}\mathfrak{g}^{ac} - \Gamma^b_{db}\mathfrak{g}^{ad} \\
&= \partial_d\mathfrak{g}^{ad} + \Gamma^a_{cd}\mathfrak{g}^{cd} \\
&= -\Gamma^a_{bc}\mathfrak{g}^{bc} + \Gamma^a_{cd}\mathfrak{g}^{cd} \\
&= 0
\end{aligned}$$

since $\mathfrak{g}^{ab}{}_{,c} = \Gamma^d_{cd}\mathfrak{g}^{ab} - \Gamma^a_{cd}\mathfrak{g}^{bd} - \Gamma^b_{cd}\mathfrak{g}^{ad}$. Hence,

$$\nabla_c \mathfrak{g}^{ab} = 0.$$

(ii) From (i),

$$\begin{aligned}\nabla_c \mathfrak{g}^{ab} = \nabla_c(\sqrt{-g}g^{ab}) &= g^{ab}\nabla_c\sqrt{-g} + \sqrt{-g}\nabla_c g^{ab} \\ &= g^{ab}\nabla_c\sqrt{-g} \\ &= 0\end{aligned}$$

since $\nabla_c g^{ab} = 0$ due to the metric compatibility. Hence for any g^{ab}, $\nabla_c\sqrt{-g} = 0$.
(iii), (iv) From the result (ii),

$$\nabla_c\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\nabla_c g_{ab} = 0.$$

Hence, $\nabla_c g_{ab} = 0$. Since $\delta^a_b = g^{ac}g_{cb}$ and from the result (iii), it follows that $\nabla_c g^{ab} = 0$.

11.12 The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = \sqrt{-g}R^{abcd}R_{abcd} &= \sqrt{-g}g_{ap}g_{bq}g_{cr}g_{ds}R^{abcd}R^{pqrs} \\ &= \sqrt{-g}g^{ap}g^{bq}g^{cr}g^{ds}R_{abcd}R_{pqrs}\end{aligned}$$

with treating g^{ab} and R^{abcd} as independent variables. From the variation, we obtain

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}R^{pqrs}R_{pqrs} + \sqrt{-g}R_a{}^{pqr}R_{bpqr}\delta g^{ab} + \sqrt{-g}R^p{}_a{}^{qr}R_{pbqr}\delta g^{ab} \\ &+ \sqrt{-g}R^{pq}{}_a{}^r R_{pqbr}\delta g^{ab} + \sqrt{-g}R^{pqr}{}_a R_{pqrb}\delta g^{ab} + 2\sqrt{-g}R_{abcd}\delta R^{abcd}\end{aligned}$$

But from the symmetry of the Riemannian tensor, we finally have

$$\therefore\ \delta\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}g_{ab}R^2 + 4R_a{}^{pqr}R_{bpqr}\right]\delta g^{ab} + 2\sqrt{-g}R_{abcd}\delta R^{abcd}$$

11.13 The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}g^{cd}\phi_{,c}\phi_{,d} - V(\phi)$$

(i) The canonical energy-momentum tensor $\Theta^a{}_b$ in Minkowski space is defined by

$$\Theta^a{}_b := -\frac{\partial\mathcal{L}}{\partial(\phi_{,a})}\phi_{,b} + \delta^a_b\mathcal{L}$$

In the case of our Lagrangian,

$$\frac{\partial\mathcal{L}}{\partial\phi_{,a}} = -g^{ac}\phi_{,c}$$

which gives

$$\Theta^a{}_b = g^{ac}\phi_{,c}\phi_{,b} + \delta^a_b\left[-\frac{1}{2}g^{cd}\phi_{,c}\phi_{,d} - V(\phi)\right]$$

Or,

$$\Theta_{ab} = g_{ac}\Theta^c{}_b = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}g^{cd}\phi_{,c}\phi_{,d} - g_{ab}V(\phi)$$

For $a = b = 0$,

$$\Theta_{00} = \frac{1}{2}\left[\dot{\phi}^2 + (\nabla\phi)^2\right] + V(\phi)$$

which is the energy density of given scalar field.
(ii) The principle of minimal coupling says that from SR to GR, certain mathematical terms explicitly involving the curvature tensor should not be added. Thus, in a curved spacetime, the Lagrangian density should be written as

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right]$$

11.14 Consider a variation of the connection Γ^a_{bc} to a new connection $\bar{\Gamma}^a_{bc}$

$$\Gamma^a_{bc} \to \bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta\Gamma^a_{bc}.$$

When the coordinates vary as

$$g_{ab} \to \bar{g}_{ab} = g_{ab} + \delta g_{ab}$$

a connection varies as

$$\begin{aligned}
\bar{\Gamma}^a_{bc} &= \frac{1}{2}\bar{g}^{ad}[\bar{g}_{db,c} + \bar{g}_{cd,b} - \bar{g}_{bc,d}] \\
&= \frac{1}{2}(g^{ad} - g^{ap}g^{dq}\delta g_{pq})[g_{db,c} + \delta g_{db,c} + g_{cd,b} + \delta g_{cd,b} - g_{bc,d} - \delta g_{bc,d}] \\
&= \Gamma^a_{bc} + \frac{1}{2}g^{ad}[\delta g_{db,c} + \delta g_{cd,b} - \delta g_{bc,d}] \\
&\qquad - \frac{1}{2}g^{ap}g^{dq}[g_{db,c} + g_{cd,b} - g_{bc,d}]\delta g_{pq} + O(\delta^2) \\
&= \Gamma^a_{bc} + \frac{1}{2}g^{ad}[\delta g_{db,c} + \delta g_{cd,b} - \delta g_{bc,d}] \\
&\qquad - g^{ap}g^{dq}g_{de}\Gamma^e_{bc}\delta g_{pq} + O(\delta^2) \\
&= \Gamma^a_{bc} + \frac{1}{2}g^{ad}[\delta g_{db,c} + \delta g_{cd,b} - \delta g_{bc,d}] \\
&\qquad - g^{ap}\Gamma^q_{bc}\delta g_{pq} + O(\delta^2)
\end{aligned}$$

But,

$$(\delta g_{ab})_{;c} = (\delta g_{ab})_{,c} - \Gamma^d_{ac}\delta g_{db} - \Gamma^d_{bc}\delta g_{ad} + \Gamma^d_{cd}\delta g_{ab}$$

and in a geodesic coordinates, where all connection coefficients are zero, there is no term related to δg_{ab}. Hence, we obtain

$$\therefore\ \delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}[\nabla_c(\delta g_{db}) + \nabla_b(\delta g_{dc}) - \nabla_d(\delta g_{bc})]$$