Notes on Pairs Trading

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Abstract

This study aims to reproduce and analyze the findings presented in [1], with a focus on validating the methodology and extending the original research implications.

1 Theoretical Approach for Pairs Trading

The pairs trading framework is established upon two financial assets that exhibit strong historical correlation in their price movements. Successful implementation of this strategy requires robust methodologies for both the identification of suitable asset pairs and the development of appropriate pricing models. These methodologies are essential as they form the foundation for effective trading execution and risk management.

1.1 System Definition

For a risk-free asset $M(t)$ with a risk-free rate of r compounded continuously, it satisfies

$$\mathrm{d}M(t) = rM(t)\mathrm{d}t \tag{1}$$

Let the two paired assets be denoted as $A(t)$ and $B(t)$ at time t. For modeling the asset dynamics, we assume that stock $B(t)$ follows a geometric Brownian motion with drift μ and volatility σ, given by:

$$\mathrm{d}B(t) = \mu B(t)\mathrm{d}t + \sigma B(t)\mathrm{d}Z(t). \tag{2}$$

Here, $Z(t)$ is a standard Brownian motion. Let $X(t)$ denote the logarithmic spread between the two stocks at time t, defined as

$$X(t) = \ln\big(A(t)\big) - \ln\big(B(t)\big) \tag{3}$$

This logarithmic transformation ensures that the spread is scale-invariant and more likely to exhibit mean-reverting behavior. We assume that the spread follows an Ornstein-Uhlenbeck process

$$\mathrm{d}X(t) = k(\theta - X(t))\mathrm{d}t + \eta \mathrm{d}W(t), \tag{4}$$

where $k > 0$ is the mean reversion rate determining the speed at which the spread returns to its mean, θ is the long-term equilibrium level to which the spread reverts, and $\eta > 0$ represents the volatility of the spread process. The term $k(\theta - X(t))$ is the drift term that represents the expected instantaneous change in the spread at time t. Here $W(t)$ is a standard Brownian motion where ρ denotes the instantaneous correlation coefficient between $Z(t)$ and $W(t)$, i.e., $\mathbb{E}\mathrm{d}W(t)\mathrm{d}Z(t) = \rho\mathrm{d}t$.

From Eq.(3), we can express $A(t)$ in terms of $B(t)$ and the spread $X(t)$ as

$$A(t) = B(t)e^{X(t)} \tag{5}$$

Applying Itô's Lemma to this composite function of two stochastic processes, we obtain

$$\mathrm{d}A(t) = B(t)\mathrm{d}\big(e^{X(t)}\big) + e^{X(t)}\mathrm{d}B(t) + \mathrm{d}B(t)\mathrm{d}\big(e^{X(t)}\big) \tag{6}$$

where the last term represents the quadratic covariation between $B(t)$ and $e^{X(t)}$. Also, when we apply Ito's formula to $e^{X(t)}$, we have

$$\mathrm{d}\big(e^{X(t)}\big) = e^{X(t)} \left(\mathrm{d}X(t) + \frac{1}{2}(\mathrm{d}X(t))^2\right) \tag{7}$$

Since $(\mathrm{d}X(t))^2 = \eta^2 \mathrm{d}t$, we get:

$$\mathrm{d}\left(e^{X(t)}\right) = e^{X(t)} \left(k(\theta - X(t))\mathrm{d}t + \eta \mathrm{d}W(t) + \frac{1}{2}\eta^2 \mathrm{d}t\right) \tag{8}$$

and the only non-negligible cross term is

$$\begin{aligned}
\mathrm{d}B(t)\mathrm{d}\left(e^{X(t)}\right) &= e^{X(t)}\sigma B(t)\mathrm{d}Z(t) \cdot \eta \mathrm{d}W(t) \\
&= e^{X(t)}\sigma B(t)\eta \mathbb{E}\mathrm{d}Z(t)\mathrm{d}W(t) \\
&= e^{X(t)}\sigma B(t)\eta\rho \mathrm{d}t.
\end{aligned}$$

Hence, by combining all terms, we obtain

$$\begin{aligned}
\mathrm{d}A(t) &= A(t) \left(k(\theta - X(t))\mathrm{d}t + \eta \mathrm{d}W(t) + \frac{1}{2}\eta^2 \mathrm{d}t + \mu \mathrm{d}t + \sigma \mathrm{d}Z(t) + \sigma\eta\rho \mathrm{d}t\right) &(9)\\
&= \left[k(\theta - X(t)) + \mu + \frac{1}{2}\eta^2 + \rho\sigma\eta\right] A(t)\mathrm{d}t + \sigma A(t)\mathrm{d}Z(t) + \eta A(t)\mathrm{d}W(t) &(10)
\end{aligned}$$

where the $\rho\sigma\eta$ term comes from the covariance between the Wiener processes $W(t)$ and $Z(t)$.

Let $V(t)$ be the value of a self-financing pairs-trading portfolio and let $h(t)$ and $\tilde{h}(t)$ denote the portfolio weights for stocks A and B at time t, respectively. Additionally, we only allow ourselves to trade stocks A and B as a pair, i.e., we require that

$$h(t) = -\tilde{h}(t). \tag{11}$$

By always putting the portfolio weight on the risk-free asset as 1, the wealth dynamics of the portfolio value is given by

$$\begin{aligned}
\mathrm{d}V(t) &= V(t) \left\{h(t)\frac{\mathrm{d}A(t)}{A(t)} + \tilde{h}(t)\frac{\mathrm{d}B(t)}{B(t)} + \frac{\mathrm{d}M(t)}{M(t)}\right\} &(12)\\
&= V(t) \left\{h(t)\left(\frac{\mathrm{d}A(t)}{A(t)} - \frac{\mathrm{d}B(t)}{B(t)}\right) + r\mathrm{d}t\right\} &(13)\\
&= V(t) \left\{\left[h(t)(k(\theta - X(t)) + \frac{1}{2}\eta^2 + \rho\sigma\eta) + r\right] \mathrm{d}t + h(t)\eta \mathrm{d}W(t)\right\}. &(14)
\end{aligned}$$

1.2 Stochastic Control

For a stochastic control problem, we consider two state variables $X(t)$ and $V(t)$ with their respective initial conditions:

$$X(0) = x_0, \qquad V(0) = v_0, \tag{15}$$

Our objective is to solve the problem where the control variable $h(t)$ represents the portfolio weight, i.e., we want to maximize the expected utility of the terminal wealth $V(T)$ at time T:

$$\sup_{h(t)} \mathbb{E}\left[U\big(V(T)\big)\right], \tag{16}$$

where the utility function $U(x)$ is given by:

$$U(x) = \frac{1}{\gamma}x^\gamma, \quad \gamma \neq 1. \tag{17}$$

1.3 Hamilton-Jacobi-Bellman Equation

We define our optimization objective as a value function $G(t, v, x)$, where v represents the portfolio value and x denotes the spread between two assets:

$$G(t, v, x) = \sup_{h(\cdot)\in\mathcal{A}} \mathbb{E}\left[U\big(V(T)\big)\,\Big|\, V(t) = v, X(t) = x\right], \tag{18}$$

Following the principle of dynamic programming, we derive the Hamilton-Jacobi-Bellman(HJB) equation:

$$G_t + \sup_{h(t)} \left\{\mathcal{L}^h G(t,\, v,\, x)\right\} = 0 \tag{19}$$

where $\mathcal{L}^h$ represents the infinitesimal generator (Ito's generator) associated with the stochastic process. To derive the explicit form of the HJB equation, we apply Ito's formula to obtain $\mathrm{d}G$:

$$\mathrm{d}G = G_t \mathrm{d}t + G_x \mathrm{d}X + G_v \mathrm{d}V + \frac{1}{2} G_{xx} (\mathrm{d}X)^2 + G_{xv} \mathrm{d}X \mathrm{d}V + \frac{1}{2} G_{vv} (\mathrm{d}V)^2. \tag{20}$$

where quadratic variation gives

$$(\mathrm{d}X)^2 = \eta \mathrm{d}t, \quad (\mathrm{d}V)^2 = h^2 V^2 \eta^2 \mathrm{d}t, \quad \mathrm{d}V \mathrm{d}X = hV\eta^2 \mathrm{d}t. \tag{21}$$

Then we have

$$\begin{aligned}
\mathrm{d}G = {} & G_t \mathrm{d}t \\
& + G_x \left[k(\theta - X(t))\mathrm{d}t + \eta \mathrm{d}W(t)\right] \\
& + G_v V(t) \left[a(t)\mathrm{d}t + \eta \mathrm{d}W(t)\right] \\
& + \frac{1}{2} G_{xx} \eta \mathrm{d}t + G_{xv} hV\eta^2 \mathrm{d}t + \frac{1}{2} G_{vv} h^2 V^2 \eta^2 \mathrm{d}t \\
= {} & \left[G_t + G_x k(\theta - X(t)) + G_v V(t) a(t) + \frac{1}{2}\eta G_{xx} + h\eta^2 V G_{xv} + \frac{1}{2} h^2 \eta^2 V^2 G_{vv}\right] \mathrm{d}t \\
& + \eta \left[G_x + V G_v\right] \mathrm{d}W(t)
\end{aligned} \tag{22}$$

For this value function to be optimized, we need the following operation

$$\mathcal{L}^h G = G_t + k\big(\theta - X(t)\big) G_x + a(t)V(t) G_v + \frac{1}{2}\eta G_{xx} + \eta^2 hV G_{xv} + \frac{1}{2} h^2 V^2 \eta^2 G_{vv} \tag{23}$$

where $a(t) = h(t)(k(\theta - X(t)) + \frac{\eta^2}{2} + \rho\sigma\eta) + r$. Now, HJB equation is written as

$$G_t + \sup_{h(t)} \left\{ a(t)V(t)G_v + k(\theta - X(t))G_x + \frac{1}{2}\eta^2 h^2 V^2 G_{vv} + \frac{1}{2}\eta^2 G_{xx} + \eta^2 hV G_{xv} \right\} = 0. \tag{24}$$

subject to the terminal condition $G(T, v, x) = v^\gamma$. This equation ensures that at every point in time, the control $h(t)$ is chosen to maximize the expected infinitesimal increase in the value function. For the notational ease, let us write $b = k(\theta - X(t)) + \frac{\eta^2}{2} + \rho\sigma\eta$. Then, Eq.(24) is rewritten as

$$G_t + \sup_h \left\{ \frac{1}{2}\eta^2 (h^2 v^2 G_{vv} + G_{xx} + 2hv G_{xv}) + (hb + r)v G_v + (b - \frac{\eta^2}{2} - \rho\sigma\eta) G_x \right\} = 0. \tag{25}$$

Or, to exapnd in the order of h,

$$G_t + \sup_h \left\{ \frac{1}{2}\eta^2 G_{vv} v^2 h^2 + \left(\eta^2 G_{xv} + b G_v\right) vh + \frac{1}{2}\eta^2 G_{xx} + \left(b - \frac{\eta^2}{2} - \rho\sigma\eta\right) G_x \right\} = 0. \tag{26}$$

To have the maximum, we need to assume $G_{vv} < 0$, i.e., it should be concave. Since this quadratic equation $\frac{1}{2}ah^2 + bh + c = 0; (a < 0)$ has a negative leading coefficient, it attains its global maximum at $h^* = -\frac{b}{a}$ with a maximum value of $-\frac{b^2}{2a} + c$. The first-order condition for the maximization will be obtained by differentiating with respect to $h(t)$: (Note. functional derivative)

$$vh^* \eta^2 G_{vv} + \eta^2 G_{xv} + bG_v = 0, \tag{27}$$

which gives

$$vh^* = -\frac{\eta^2 G_{xv} + bG_v}{\eta^2 G_{vv}}. \tag{28}$$

Then, the supremum will be

$$\begin{aligned} &-\frac{1}{2\eta^2 G_{vv}} \left(\eta^2 G_{xv} + bG_v\right)^2 + \frac{1}{2}\eta^2 G_{xx} + \left(b - \frac{\eta^2}{2} - \rho\sigma\eta\right) G_x \\ &= -\frac{1}{2\eta^2 G_{vv}} \left[\eta^4 G_{xv}^2 + 2b\eta^2 G_{xv} G_v + b^2 G_v - \eta^4 G_{xx} G_{vv} + 2k(x-\theta)\eta^2 G_x G_{vv}\right]. \end{aligned} \tag{29}$$

By substituting into Eq.(26), we have the final form:

$$\begin{aligned} 0 = {}& \eta^2 G_t G_{vv} - \frac{1}{2}\eta^4 G_{vx}^2 - \frac{1}{2} b^2 G_v^2 - b\eta^2 G_v G_{vx} \\ &+ \frac{1}{2}\eta^4 G_{vv} G_{xx} + r\eta^2 v G_v G_{vv} - k(x-\theta)\eta^2 G_x G_{vv}. \end{aligned} \tag{30}$$

The solution to this PDE will yield the optimal strategy for pairs trading.

1.3.1 Closed form solution

We propose an ansatz based on the separation of variables:

$$G(t, v, x) = f(t, x) v^\gamma \tag{31}$$

with the condition that $f(T, x) = 1$ for all x. Then, by following calculations,

$$\eta^2 G_t G_{vv} - \frac{1}{2}\eta^4 G_{vx} - \frac{1}{2} b^2 G_v^2 = \eta^2 \gamma(\gamma-1) f f_t v^{2\gamma-2} - \frac{1}{2}\eta^4 \gamma^2 f_x^2 v^{2\gamma-2} - \frac{1}{2} b^2 \gamma^2 f^2 v^{2\gamma-2},$$

$$-b\eta^2 G_v G_{vx} - k(x-\theta)\eta^2 G_x G_{vv} = -b\eta^2 \gamma^2 f_x f v^{2\gamma-2} - k(x-\theta)\eta^2 \gamma(\gamma-1) f f_x v^{2\gamma-2}$$

and

$$\frac{1}{2}\eta^4 G_{vv} G_{xx} + r\eta^2 v G_v G_{vv} = \frac{1}{2}\eta^4 \gamma(\gamma-1) f f_{xx} v^{2\gamma-2} + r\eta^2 \gamma^2 (\gamma-1) f^2 v^{2\gamma-2},$$

we obtain

$$\begin{aligned} 0 &= \eta^2 \gamma(\gamma-1) f f_t v^{2\gamma-2} - \frac{1}{2}\eta^4 \gamma^2 f_x^2 v^{2\gamma-2} - \frac{1}{2} b^2 \gamma^2 f^2 v^{2\gamma-2} - b\eta^2 \gamma^2 f_x f v^{2\gamma-2} \\ &\quad - k(x-\theta)\eta^2 \gamma(\gamma-1) f f_x v^{2\gamma-2} + \frac{1}{2}\eta^4 \gamma(\gamma-1) f f_{xx} v^{2\gamma-2} + r\eta^2 \gamma^2 (\gamma-1) f^2 v^{2\gamma-2} \\ &= \gamma f v^{2\gamma-2} \Big[(\gamma-1)\eta^2 f_t - \frac{1}{2}\gamma\eta^4 \frac{f_x^2}{f} - \frac{1}{2}\gamma b^2 f - \frac{1}{2}\gamma\eta^4 f_x - \gamma\rho\sigma\eta^3 f_x + \frac{1}{2}(\gamma-1)\eta^4 f_{xx} \\ &\qquad + r\gamma(\gamma-1)\eta^2 f + k(x-\theta)\eta^2 f_x\Big]. \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} \therefore (\gamma-1)\eta^2 f_t &- \frac{1}{2}\gamma\eta^4 \frac{f_x^2}{f} - \frac{1}{2}\gamma b^2 f - \frac{1}{2}\gamma\eta^4 f_x - \gamma\rho\sigma\eta^3 f_x \\ &+ \frac{1}{2}(\gamma-1)\eta^4 f_{xx} + r\gamma(\gamma-1)\eta^2 f + k(x-\theta)\eta^2 f_x = 0. \end{aligned} \tag{33}$$

For this PDE, we consider the following ansatz for $f(t,x)$:

$$f(t,x) = g(t)\exp\left[x\beta(t) + x^2\alpha(t)\right] \tag{34}$$

From the boundary condition,

$$f(T,x) = g(T)\exp\left[x\beta(T) + x^2\alpha(T)\right] = 1, \tag{35}$$

it follows that

$$g(T) = 1, \quad \beta(T) = \alpha(T) = 0 \tag{36}$$

for all x. From the following calculation,

$$\begin{aligned} f_t &= g_t \exp[x\beta + x^2\alpha] + g\exp[x\beta + x^2\alpha](x\beta_t + x^2\alpha_t) \\ &= \frac{g_t}{g} f + f(x\beta_t + x^2\alpha_t) \end{aligned} \tag{37}$$

$$f_x = g\exp[x\beta + x^2\alpha](\beta + 2x\alpha) = f(\beta + 2x\alpha) \tag{38}$$

$$f_{xx} = f(\beta + 2x\alpha)^2 + 2\alpha f \tag{39}$$

we have

$$(\gamma-1)\eta^2 f_t/f = (\gamma-1)\eta^2\left[\frac{g_t}{g} + x\beta' + x^2\alpha'\right] \tag{40}$$

$$\begin{aligned} -\frac{1}{2}\gamma\eta^4\frac{f_x^2}{f^2} &= -\frac{1}{2}\gamma\eta^4(\beta + 2x\alpha)^2 \\ &= -\frac{1}{2}\gamma\eta^4(\beta^2 + 4x\alpha\beta + 4x^2\alpha^2) \end{aligned} \tag{41}$$

$$-\gamma\left(\frac{1}{2}\eta^4 + \rho\sigma\eta^3\right)\frac{f_x}{f} = -\gamma\left(\frac{1}{2}\eta^4 + \rho\sigma\eta^3\right)(\beta + 2x\alpha) \tag{42}$$

$$\begin{aligned} \frac{1}{2}(\gamma-1)\eta^4\frac{f_{xx}}{f} &= \frac{1}{2}\eta^4(\beta + 2x\alpha)^2 + \alpha(\gamma-1)\eta^4 \\ &= \frac{1}{2}(\gamma-1)\eta^4(\beta^2 + 4x\alpha\beta + 4x^2\alpha^2) + \alpha(\gamma-1)\eta^4 \end{aligned} \tag{43}$$

$$\begin{aligned} k(x-\theta)\eta^2\frac{f_x}{f} &= k(x-\theta)\eta^2(\beta + 2x\alpha) \\ &= -k\theta\beta\eta^2 + (k\beta\eta^2 - 2k\eta^2\alpha\theta)x + 2k\eta^2\alpha x^2. \end{aligned} \tag{44}$$

First, the coefficient of x^2 reads

$$\begin{aligned} x^2 &: (\gamma-1)\eta^2\alpha' - 2\gamma\eta^4\alpha^2 + 2(\gamma-1)\eta^4\alpha^2 + 2k\alpha\eta^2 \\ &= (\gamma-1)\eta^2\alpha' - 2\eta^4\alpha^2 + 2k\eta^2\alpha \end{aligned} \tag{45}$$

and the coefficient for x reads

$$\begin{aligned} x &: (\gamma-1)\eta^2\beta' - 2\alpha\beta\gamma\eta^4 - \alpha\gamma\eta^4 - 2\alpha\gamma\rho\sigma\eta^3 + 2\alpha\beta(\gamma-1)\eta^4 + \beta k\eta^4 - 2\alpha k\eta^2\theta \\ &= (\gamma-1)\eta^2\beta' - 2\alpha\beta\gamma\eta^4 - \alpha\gamma\eta^4 - 2\alpha\gamma\rho\sigma\eta^3 + \beta k\eta^2 - 2\alpha k\theta\eta^2 \end{aligned} \tag{46}$$

Finally, the coefficient for x^0 reads

$$\begin{aligned} x^0 &: (\gamma-1)\eta^2\frac{g'}{g} - \frac{1}{2}\gamma\eta^4\beta^2 - \frac{1}{2}\gamma b^2 - \frac{1}{2}\gamma\eta^4\beta - \gamma\rho\sigma\eta^3\beta + \frac{1}{2}(\gamma-1)\eta^4(\beta^2 + 2\beta) + r\gamma(\gamma-1)\eta^2 - k\theta\eta^2\beta \\ &= (\gamma-1)\eta^2\frac{g'}{g} - \frac{1}{2}\gamma b^2 - \frac{1}{2}\gamma\eta^4\beta - \gamma\rho\sigma\eta^3\beta - \frac{1}{2}\eta^4\beta^2 + (\gamma-1)\eta^4\beta + r\gamma(\gamma-1)\eta^2 - k\theta\eta^2\beta. \end{aligned} \tag{47}$$

Given that the drift term b contains x as $b = -k(x-\theta) + \frac{1}{2}\eta^2 + \rho\sigma\eta$, we must account for this x-dependence in our expansion. Then, the final form will be

$$
\begin{aligned}
0 = &\left\{(\gamma-1)\eta^2\alpha' - 2\eta^4\alpha^2 + 2k\eta^2\alpha - \frac{1}{2}\gamma k^2\right\} x^2 \\
&+ \left\{(\gamma-1)\eta^2\beta' - 2\alpha\beta\gamma\eta^4 - \alpha\gamma\eta^4 - 2\alpha\gamma\rho\sigma\eta^3 + \beta k\eta^2 - 2\alpha k\theta\eta^2 + \gamma k^2\theta + \frac{1}{2}\gamma k\eta^2 + \gamma k\rho\sigma\eta\right\} x \\
&+ \left\{(\gamma-1)\eta^2\frac{g'}{g} - \frac{1}{2}\gamma\eta^4\beta - \gamma\rho\sigma\eta^3\beta - \frac{1}{2}\eta^4\beta^2 + (\gamma-1)\eta^4\beta + r\gamma(\gamma-1)\eta^2 - k\theta\eta^2\beta - \frac{1}{2}\gamma(k\theta + \frac{1}{2}\eta^2 + \rho\sigma\eta)^2\right\}
\end{aligned}
\tag{48}
$$

To obtain x-independent solutions, we need to solve the following two differential equations for α and β:

$$
(\gamma-1)\eta^2\alpha' - 2\eta^4\alpha^2 + 2k\eta^2\alpha - \frac{1}{2}\gamma k^2 = 0 \tag{49}
$$

$$
(\gamma-1)\eta^2\beta' - 2\alpha\beta\gamma\eta^4 - \alpha\gamma\eta^4 - 2\alpha\gamma\rho\sigma\eta^3 + \beta k\eta^2 - 2\alpha k\theta\eta^2 + \gamma k^2\theta + \frac{1}{2}\gamma k\eta^2 + \gamma k\rho\sigma\eta = 0 \tag{50}
$$

2 Numerical Simulation

References

[1] Supakorn Mudchanatonguk, James A. Primbs, and Wilfred Wong. Optimal pairs trading: A stochastic control approach. In *2008 American Control Conference*, pages 1035–1039. IEEE, June 2008.