代数 K 理论讨论班笔记

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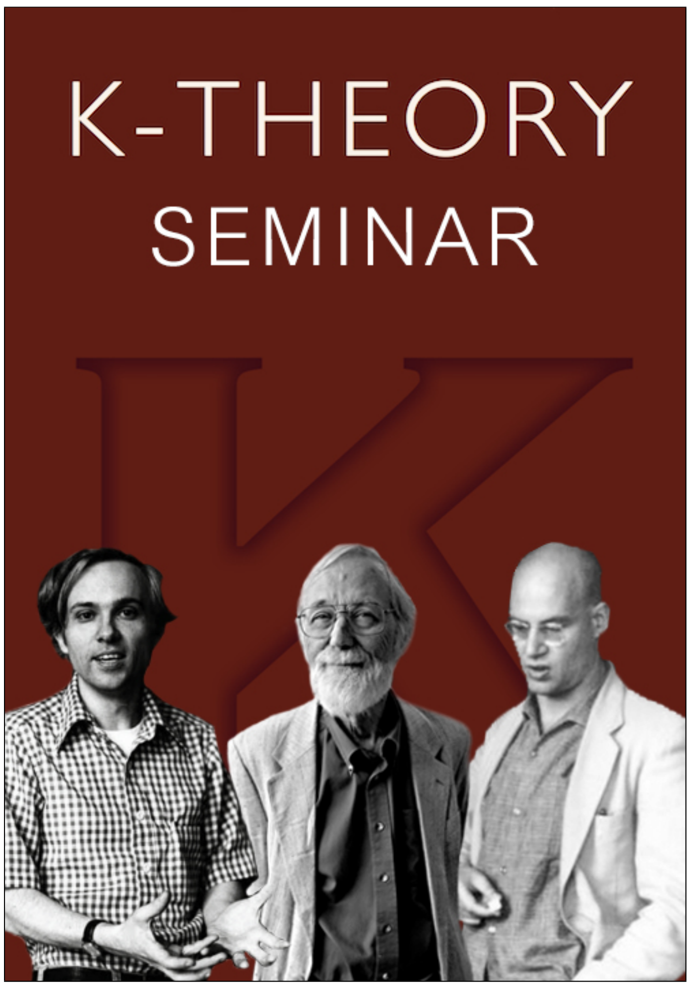

从左至右依次为
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Chapter 1

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [3].

1.1 Outline

Definition 1.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty}\mathbb{F}_2 x^i = \oplus_{i=1}^{\infty}\mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\,dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

1.2 Preliminaries

1.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 1.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

1.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[[x]])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$
\begin{aligned}
[a]\colon \alpha(x) &\mapsto \alpha(ax) \\
V_m\colon \alpha(x) &\mapsto \alpha(x^m) \\
F_m\colon \alpha(x) &\mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m\colon 1 - rx &\mapsto 1 - r^m x
\end{aligned}
$$

Remark 1.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [1].

Proposition 1.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$
\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a]V_m &= V_m[a^m] \\
F_m[a] &= [a^m]F_m \\
[a][b] &= [ab] \\
V_m F_k &= F_k V_m, \ \textit{if } (k, m) = 1
\end{aligned}
$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[[x]]$ is not.

1.2.3 Double relative group

You can skip this subsection for first reading. We will use the results in 1.4.

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R,I) \longrightarrow K_3(R/I, I{+}J/J) \longrightarrow I/I^2 {\otimes_{R^e}} J/J^2 \xrightarrow{\psi} K_2(R,I) \longrightarrow K_2(R/I, I{+}J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b\rangle$, see [4] 3.5.10 or [2] p. 195.

In the case $I \cap J = 0$, $K_2(R, I, J) \cong I/I^2 \otimes_{R^e} J/J^2$.

我的疑问：if R is commutative, whether $K_2(R, I, J) = I \otimes_R J$ or not?

Lemma 1.5. *Let (R, I, J) satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I + J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R,I) \longrightarrow K_2(R/I, I + J/J) \longrightarrow 0$$

is split exact.

Relative group For relative K-group, there is an exact sequence

$$\begin{aligned} &K_3(R,I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R,I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow \\ \longrightarrow &K_1(R,I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \end{aligned}$$

1.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [1] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$
$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$-module structure on $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [1]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 1.6. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 1.7. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 1.8. 一个不知道有没有用的结论, see [3]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$D\colon x\mathbb{F}_2[x] \longrightarrow \Omega_{\mathbb{F}_2[x]}$$
$$x^i \mapsto ix^{i-1}dx$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty} \mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

1.4 NK_i of the group C_2

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By Mayer–Vietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

1.5 NK_i of the group C_4

1.6 NK_i of the group D_4

References

[1] B. H. Dayton and C. A. Weibel. Module structures on the Hochschild and cyclic homology of graded rings. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, pages 63–90. Kluwer Acad. Publ., Dordrecht, 1993.

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