代数 K 理论讨论班笔记

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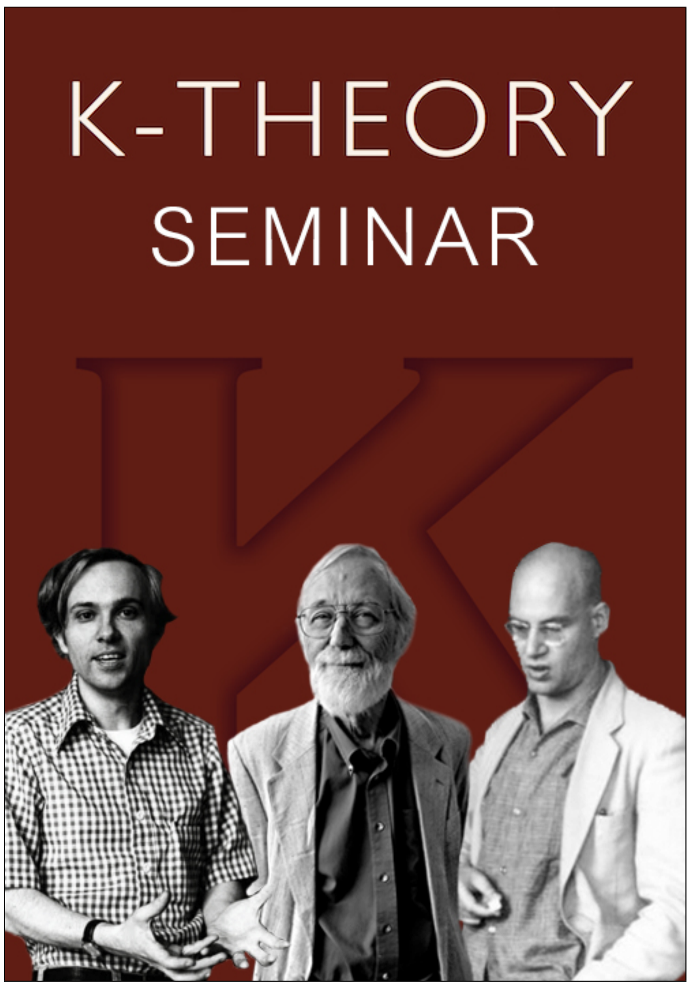

从左至右依次为
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2016 年 3 月 11 日

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Chapter 1

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [5].

1.1 Outline

Definition 1.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x \mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty} \mathbb{F}_2 x^i = \oplus_{i=1}^{\infty} \mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\, dx = \oplus_{i=1}^{\infty} \mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\, dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

1.2 Preliminaries

1.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)

R-mod: the category of R-modules,

$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,

$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 1.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

1.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[[x]])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$
\begin{aligned}
[a] &\colon \alpha(x) \mapsto \alpha(ax) \\
V_m &\colon \alpha(x) \mapsto \alpha(x^m) \\
F_m &\colon \alpha(x) \mapsto \sum_{\zeta^m = 1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m &\colon 1 - rx \mapsto 1 - r^m x
\end{aligned}
$$

Remark 1.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [1].

Proposition 1.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$
\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a] V_m &= V_m [a^m] \\
F_m [a] &= [a^m] F_m \\
[a][b] &= [ab] \\
V_m F_k = F_k V_m&, \text{ if } (k, m) = 1
\end{aligned}
$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[[x]]$ is not.

1.2.3 Dennis-Stein symbols

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_12(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 1.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem.*

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a,b\rangle \in K_2(R,I)$ satisfy following relations

(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$ if any of a,b,c are in I.

Theorem 1.6. *1. If R is a commutative local ring, then $D(R) \stackrel{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R,I) \stackrel{\cong}{\to} K_2(R,I)$.

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows

$r,s \in R$ commute and $1-rs$ is a unit, that is $rs = sr$ and $1-rs \in R^*$,

$$\langle r,s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r,s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r,s\rangle \in K_2(R,I)$

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$,

(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of r,s,t are in I).

Note that $\langle r,1\rangle = 0$ for any $r \in R$ and $\langle r,s\rangle_{version2} = \langle -r,s\rangle_{version1}$.

Theorem 1.7. *1. If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r,s\rangle$ satisfying D1, D2, D3.*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R,I)$ is generated by $\langle r,s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3.

1.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 1.4.

Relative groups Let R be a ring (not necessarily commutative), $I \subset R$ a two-sided ideal, by definition $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$, there exists a map

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

Definition 1.8. $K(R, I)$ is the homotopy fibre of the map $BGL(R)^+ \longrightarrow BGL(R/I)^+$. $K_i(R, I) := \pi_i(K(R, I)), i \geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots \longrightarrow K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R, I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots .$$

In particular,

$$K_3(R, I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R, I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow$$
$$\longrightarrow K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I)$$

Double relative groups Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals, there is a map

$$K(R, I) \longrightarrow K(R/J, I + J/J).$$

$$\begin{tikzcd}
K(R; I, J) \arrow[d] & & \\
K(R, I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\
K(R/J, I + J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I + J)^+
\end{tikzcd}$$

Definition 1.9. $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I + J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

Remark 1.10. $K_i(R; I, J) \cong K_i(R; J, I)$, $K_i(R; I, I) = K_i(R, I)$.

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R, I) \longrightarrow K_3(R/I, I+J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [6] 3.5.10 or [2] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$.

我的疑问：if R is commutative, whether $K_2(R; I, J) = I \otimes_R J$ or not?

Lemma 1.11. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I+J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

is split exact.

1.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [1] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$
$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$-module structure on $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [1]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 1.12. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 1.13. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 1.14. 一个不知道有没有用的结论, see [5]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$\begin{aligned} D\colon x\mathbb{F}_2[x] &\longrightarrow \Omega_{\mathbb{F}_2[x]} \\ x^i &\mapsto ix^{i-1}dx \end{aligned}$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC^{\mathbb{F}_2}_1(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty} \mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

1.4 NK_i of the group C_2

First, consider the simplest example $G = C_2 = \langle \sigma \rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By Mayer–Vietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma + 1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2],(\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2],(\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1),(\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z},(2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z},(2)) \longrightarrow NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2],(\sigma+1)) \longrightarrow NK_2(\mathbb{Z},(2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z},(2)) \longrightarrow NK_i(\mathbb{Z}) = 0.$

$$\begin{tikzcd}
 & NK_3(\mathbb{Z},(2)) = 0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \arrow[d, "\cong"] & & \\
0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2],(\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\
 & NK_2(\mathbb{Z},(2)) = 0 & &
\end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [3], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1),\sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 1.15. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

1.5 NK_i of the group C_4

1.6 NK_i of the group D_4

Chapter 2

Mackey Functors

参考文献有：[4]

2.1 Introduction

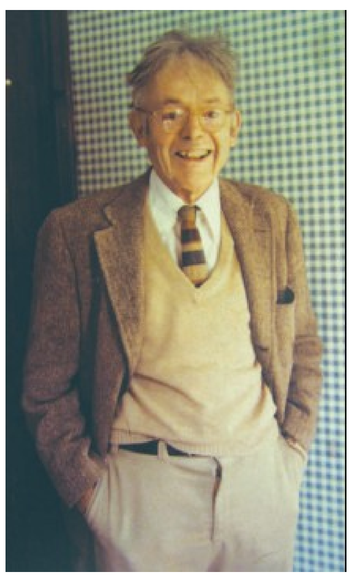

图 2.1: George Mackey

George W. Mackey (1916–2006) was an American mathematician. For more interesting story about him, see https://en.wikipedia.org/wiki/George_Mackey, http://www.ams.org/notices/200707/tx070700824p.pdf or http://www-history.mcs.st-andrews.ac.uk/Biographies/Mackey.html. And I found an interesting fact that Leslie Lamport's advisor Richard Palais was a PhD student of him.[1] And Lamport is best known for the system LaTeX.[2]

Mackey functor is an algebraic structure, related to many constructions from finite groups, such as group cohomology and the algebraic K-theory of group rings.

History: began in 1980s

People: Dress and Green first gave the axiomatic formulation of Mackey functors.

[1] http://www.genealogy.ams.org/id.php?id=35871

[2] https://en.wikipedia.org/wiki/Leslie_Lamport

References

[1] B. H. Dayton and C. A. Weibel. Module structures on the Hochschild and cyclic homology of graded rings. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, pages 63–90. Kluwer Acad. Publ., Dordrecht, 1993.

[2] Eric Friedlander and MR Stein. *Algebraic K-theory. Proc. conf. Evanston, 1980.* Springer, 1981.

[3] Dominique Guin-Waléry and Jean-Louis Loday. *Algebraic K-Theory Evanston 1980: Proceedings of the Conference Held at Northwestern University Evanston, March 24–27, 1980*, chapter Obstruction a l'Excision En K-Theorie Algebrique, pages 179–216. Springer Berlin Heidelberg, Berlin, Heidelberg, 1981.

[4] Zhulin Li. Box product of mackey functors in terms of modules. 2015.

[5] Charles Weibel. NK_0 and NK_1 of the groups C_4 and D_4. *Comment. Math. Helv*, 84:339–349, 2009.

[6] Charles A Weibel. *The K-book: An introduction to algebraic K-theory.* American Mathematical Society Providence (RI), 2013.

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