# Chapter 1

# $NK_2(\mathbb{F}_2[C_4])$

$$NK_i(R) = \ker(NK_i(R[x]) \xrightarrow{x\mapsto 0} K_i(R)).$$

**目标**

1. 证明 $NK_2(\mathbb{F}_2[C_4])$ 中有四阶元，
2. 确定它的结构和生成元。
3. 推广到 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 和 $NK_p(\mathbb{F}_{p^f}[C_{p^n}])$。
4. 未来推广到 $NK_2(\mathbb{F}_{p^f}[G])$，其中 $G$ 是有限交换群，$G = C_{p^n} \times H$。

## 1.1 思路

已经有 $NK_2(\mathbb{F}_2[C_2])$ 和 $NK_2(\mathbb{F}_p[C_p])$ 的结果 [2]。这种方法详细见下文1.3.1。

考虑 $\mathbb{F}_{p^f}[C_{p^n}]$ 时，可以将它写成截断多项式 $F_{p^f}[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$。

考虑它的 $NK_2$ 时，可以转化成相对 $K_2$ 群：

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})).$$

利用 van der Kallen [7] 中的方法对于这种情形的相对 $K_2$ 群，有这样的结论 (符号说明见后文):

**Theorem.** $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+xk[x]/(x^{[\alpha,i]}))^\times.$$

接下来的任务就是确定两件事情：

- 确定集合$\Lambda^{00}$，确定数值$[\alpha,i]$。
- 确定右边这个乘法群的结构。

实际上对于第一件事情，我们只需要考虑这个集合的一部分，因为 $(1+xk[x]/(x))^\times$ 是平凡的，所以只要考虑 $[\alpha,i] > 1$ 所对应的 $(\alpha,i)$ 全体（后文详细解释）。

对于第二件事情，有一些例子是可以直接计算的，如 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$。对于一般的情况引入 big Witt vectors，

$$BigWitt_n(\mathbb{F}_q) := (1+x\mathbb{F}_q[\![x]\!])^\times/(1+x^{n+1}\mathbb{F}_q[\![x]\!])^\times \cong (1+x\mathbb{F}_q[x]/(x^{n+1}))^\times$$

由 big Witt vectors分解成 typical Witt vectors有

$$\begin{aligned}(1+x\mathbb{F}_{p^f}[x]/(x^{n+1}))^\times &\cong BigWitt_n(\mathbb{F}_{p^f})\\ &\cong \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} W_{1+\lfloor \log_p \frac{n}{m}\rfloor}(\mathbb{F}_{p^f})\\ &= \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor \log_p \frac{n}{m}\rfloor}\mathbb{Z})^f,\end{aligned}$$

其中 $\lfloor x\rfloor$ 表示不超过 $x$ 的最大整数。

结合这个同构2.1就可以得到结论。

**结论** 已知的结果

**Theorem.** *(1)$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$,*
*(2)$NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t,x]/(t^2),(t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i,x\rangle \mid i\ge 0\}$ 与 $\{\langle tx^i,t\rangle \mid i\ge 1$为奇数$\}$ 生成的，这样的符号均为 2 阶元。*

$NK_2(\mathbb{F}_2[C_4])$ 的结果

**Theorem.** *(1)$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$,*
*(2)$NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号*

$$\{\langle tx^{i-1},x\rangle \mid i\ge 1\},\{\langle tx^i,t\rangle \mid i\ge 1\text{为奇数}\},\{\langle t^3x^{i-1},x\rangle \mid i\ge 1\},\{\langle t^3x^i,t\rangle \mid i\ge 1\text{为奇数}\}$$

*生成的。*

$NK_2(\mathbb{F}_q[C_{2^n}])$ 的结果

$$NK_2(\mathbb{F}_q[C_{2^n}]) \cong \bigoplus_\infty \bigoplus_{k=1}^n \mathbb{Z}/2^k\mathbb{Z}.$$

## 1.2 预备知识和引理

这一节主要是介绍前文提到的如何将 $NK_2$ 转化成相对 $K_2$ 群，[7] 中符号说明与相对 $K_2$ 群，以及 Witt 向量的分解。

令 $k$ 是特征为 $p>0$ 的有限域，考虑两个变元的多项式环 $k[t_1,t_2]$，令 $I$ 是 $k[t_1,t_2]$ 的一个真理想，满足以下条件

1. $I$ 是由 $k[t_1]$ 中的单项式生成的，
2. 对于某个 $n$，$t_1^n\in I$.

实际上这样的 $I$ 具有形式 $(t_1^n)$, 令

$$A=k[t_1,t_2]/I,$$

$M$ 是 $A$ 的 nil 根 (小根)，即 $M=(t_1)$，那么有 $A/M=k[t_2]$.

**Proposition 1.1.** $K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \cong K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$.

*Proof.* 首先有下面两个相对 $K$ 群的正合列

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k[t_2]) \longrightarrow 0$$

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k) \longrightarrow 0$$

由于 $k$ 是有限域，它是正则环，于是有 $K_2(k)=0, K_2(k[t_2]) = K_2(k)\oplus NK_2(k)=0$. 从而可以得到

$$K_2(k[t_1,t_2]/(t_1^n),(t_1)) \cong K_2(k[t_1,t_2]/(t_1^n)) \cong K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)).$$

□

当 $k=\mathbb{F}_{p^f}$ 时，$k[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$，其中 $C_{p^n}$ 是 $p^n$ 阶循环群。有以下可裂正合列

$$0 \longrightarrow NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow 0,$$

由于 $K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) = K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}))$，并且 $K_2(\mathbb{F}_{p^f}[C_{p^n}])=0$，从而

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)).$$

### 1.2.1 Dennis-Stein 符号

回到一般情形，$K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$ 可以用 Dennis-Stein 符号表示
生成元 $\langle a,b\rangle, (a,b)\in A\times M\cup M\times A$;
关系 $\langle a,b\rangle = -\langle b,a\rangle$,
$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc,b\rangle$,
$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ 其中 $(a,b,c)\in A\times M\times A\cup M\times A\times M$.

**Proposition 1.2.** *对任意环 $R$，$q\geq 1$，$K_2(R[t]/(t^q),(t))$ 由 Dennis-Stein 符号 $\langle at^i,t\rangle$ 和 $\langle at^i,b\rangle$ 生成，其中 $a,b\in R, 1\leq i<q$。*

*Proof.* 参见文献 [6]。 □

### 1.2.2 符号说明

为了表述方便，遵从 [7] 的符号详述如下

- $\mathbb{Z}_+$: 非负整数全体，
- $\epsilon^1 = (1,0)\in\mathbb{Z}_+^2, \epsilon^2=(0,1)\in\mathbb{Z}_+^2$,
- 对于 $\alpha\in\mathbb{Z}_+^2$, 记 $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2}$, 于是有 $t^{\epsilon^1}=t_1, t^{\epsilon^2}=t_2$,
- $\Delta = \{\alpha\in\mathbb{Z}_+^2 \mid t^\alpha\in I\}$,
- $\Lambda = \{(\alpha,i)\in\mathbb{Z}_+^2\times\{1,2\} \mid \alpha_i\geq 1, t^\alpha\in M\}$, 若 $\delta\in\Delta$, 则有 $\delta+\epsilon^i\in\Delta, I=1,2$,
- 对于 $(\alpha,i)\in\Lambda$, 令 $[\alpha,i] = \min\{m\in\mathbb{Z} \mid m\alpha-\epsilon^i\in\Delta\}$, 若 $(\alpha,i),(\alpha,j)\in\Lambda$, 有 $[\alpha,i]\leq[\alpha,j]+1$,
- 若 $gcd(p,\alpha_1,\alpha_2)=1$, 令 $[\alpha] = \max\{[\alpha,i] \mid \alpha_i\not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \{(\alpha,i)\in\Lambda \mid gcd(\alpha_1,\alpha_2)=1, i\neq\min\{j \mid \alpha_j\not\equiv 0 \bmod p, [\alpha,j]=[\alpha]\}\}$.

若 $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, 令

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i\rangle,$$

若 $g(t_1, t_2) = t_i h(t_1, t_2) \in \sqrt{I} = (t_1)$, 令

$$\Gamma_i(1 - g(t_1, t_2)) = \langle h(t_1, t_2), t_i\rangle,$$

且有

$$\Gamma_{\alpha,i}(1 - xf(x)) = \Gamma_i(1 - t^\alpha f(t^\alpha)).$$

由于 $t_1 \in \sqrt{I}$, $\Gamma_1$ 诱导了同态

$$\begin{aligned}(1 + t_1 k[t_1, t_2]/t_1 I)^\times &\longrightarrow K_2(A, M)\\ 1 - g(t_1, t_2) &\mapsto \langle h(t_1, t_2), t_1\rangle\end{aligned}$$

$\Gamma_2$ 诱导了同态

$$\begin{aligned}(1 + t_2\sqrt{I}/t_2 I)^\times &\longrightarrow K_2(A, M)\\ 1 - g(t_1, t_2) &\mapsto \langle h(t_1, t_2), t_2\rangle\end{aligned}$$

若 $(\alpha, i) \in \Lambda$, $\Gamma_{\alpha,i}$ 诱导了同态

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

**Theorem 1.3.** $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

*Proof.* 参见文献 [7]。 □

### 1.2.3 Witt 向量

令 $R$ 是一个交换环，big Witt 环 (the ring of universal/big Witt vectors over $R$, 泛 Witt 环)$BigWitt(R)$ 作为 Abel 群同构于 $(1 + xR[\![x]\!])^\times$，即常数项为 1 的形式幂级数全体在乘法运算下形成的交换群，

$$\begin{aligned}BigWitt(R) &\longrightarrow (1 + xR[\![x]\!])^\times\\ (r_1, r_2, \cdots) &\mapsto \prod_i (1 - r_i x^i)^{-1}.\end{aligned}$$

考虑子群 $(1 + x^{n+1}R[\![x]\!])^\times$，定义 $BigWitt_n(R) = (1 + xR[\![x]\!])^\times/(1 + x^{n+1}R[\![x]\!])^\times$。显然 $BigWitt_1(R) = R$，并且当 $n \geq 3$ 时，$BigWitt_n(\mathbb{F}_2)$ 不是循环群。

**Lemma 1.4.** $BigWitt_n(\mathbb{F}_q) \cong (1 + x\mathbb{F}_q[x]/(x^{n+1}))^\times$.

*Proof.* 由定义 $BigWitt_n(\mathbb{F}_q) := (1 + x\mathbb{F}_q[\![x]\!])^\times/(1 + x^{n+1}\mathbb{F}_q[\![x]\!])^\times$，且有同态

$$\begin{aligned}(1 + x\mathbb{F}_q[\![x]\!])^\times &\longrightarrow (1 + x\mathbb{F}_q[x]/(x^{n+1}))^\times\\ 1 + \sum_{i\geq 1} a_i x^i &\mapsto 1 + \sum_{i=1}^{n} a_i x^i\end{aligned}$$

容易看出核是 $(1 + x^{n+1}\mathbb{F}_2[\![x]\!])^\times$，从而 $BigWitt_n(\mathbb{F}_q) \cong (1 + x\mathbb{F}_q[x]/(x^{n+1}))^\times$。 □

**Example 1.5.** $BigWitt_3(\mathbb{F}_2) \cong (1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

*Proof.* 考虑 $1 + x \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ 是 4 阶元，由它生成的子群 $\langle 1 + x\rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$，且 $1 + x^3$ 是二阶元，令 $\sigma, \tau$ 分别是 $\mathbb{Z}/4\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned}\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3.\end{aligned}$$

□

**Example 1.6.** $BigWitt_4(\mathbb{F}_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

*Proof.* 考虑 $1 + x \in BigWitt_5(\mathbb{F}_2)$, 它是 8 阶元，由它生成的子群 $\langle 1 + x\rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3, 1 + x^4, 1 + x + x^4, 1 + x^2 + x^4, 1 + x + x^2 + x^3 + x^4\}$，另外 $1 + x^3$ 是二阶元，令 $\sigma, \tau$ 分别是 $\mathbb{Z}/8\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned}\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3 + x^4\end{aligned}$$

于是 $(\sigma^i, \tau^j), 0 \le i < 8, 0 \le j < 2$ 对应于 $(1 + x)^i(1 + x^3)^j$, 详细的对应如下

$$\begin{aligned}
(1, \tau) &\mapsto 1 + x^3, & (\sigma, \tau) &\mapsto 1 + x + x^3 + x^4,\\
(\sigma^2, \tau) &\mapsto 1 + x^2 + x^3, & (\sigma^3, \tau) &\mapsto 1 + x + x^2 + x^4,\\
(\sigma^4, \tau) &\mapsto 1 + x^3 + x^4, & (\sigma^5, \tau) &\mapsto 1 + x + x^3,\\
(\sigma^6, \tau) &\mapsto 1 + x^2 + x^3 + x^4, & (\sigma^7, \tau) &\mapsto 1 + x + x^2,\\
(1, 1) &\mapsto 1, & (\sigma, 1) &\mapsto 1 + x,\\
(\sigma^2, 1) &\mapsto 1 + x^2, & (\sigma^3, 1) &\mapsto 1 + x + x^2 + x^3,\\
(\sigma^4, 1) &\mapsto 1 + x^4, & (\sigma^5, 1) &\mapsto 1 + x + x^4,\\
(\sigma^6, 1) &\mapsto 1 + x^2 + x^4, & (\sigma^7, 1) &\mapsto 1 + x + x^2 + x^3 + x^4.
\end{aligned}$$

□

固定素数 $p$，考虑局部环 $\mathbb{Z}_{(p)} = \mathbb{Z}[1/\ell \mid \text{所有素数}\ell \neq p]$，即 $\mathbb{Z}$ 在素理想 $(p) = p\mathbb{Z}$ 的局部化，于是一个 $\mathbb{Z}_{(p)}$-代数 $R$ 就是除 $p$ 外的素数均可逆的交换环，如 $\mathbb{F}_{p^n}$ 是 $\mathbb{Z}_{(p)}$-代数。

考虑 $p$-Witt 环 $W(A)$ 与截断 $p$-Witt 环 $W_n(A)$，$p$-Witt 向量为 $(a_0, a_1, \cdots)$，加法用 Witt 多项式定义，以下仅考虑用加法定义的 Abel 群结构，例如 $W(\mathbb{F}_p) = \mathbb{Z}_p$，作为 Abel 群 $W_n(\mathbb{F}_{p^f})$ 同构于 $(\mathbb{Z}/p^n\mathbb{Z})^f$。

Artin-Hasse 级数定义为

$$AH(x) = \exp(-\sum_{n\ge 0} \frac{x^{p^n}}{p^n}) = 1 - x + \cdots \in 1 + x\mathbb{Q}[\![x]\!],$$

实际上 $AH(x) \in 1 + x\mathbb{Z}_{(p)}[\![x]\!]$。对于 $BigWitt(R) = 1 + xR[\![x]\!]$ 中的任一元素 $\alpha$ 存在以下写成无穷乘积的表法

$$\alpha = \prod_{n\ge 1}(1 - r_n x^n),\ r_n \in R,$$

若 $A$ 是 $\mathbb{Z}_{(p)}$-代数，$BigWitt(A) = 1 + xA[\![x]\!]$ 中的任一元素 $\alpha$ 还有如下表法

$$\alpha = \prod_{n\geq 1} AH(a_n x^n),\ a_n \in A.$$

将整数 $n$ 写成 $n = mp^a$，使得 $gcd(m,p) = 1, a \geq 0$，由于 $A$ 是 $\mathbb{Z}_{(p)}$-代数，$m$ 可逆，从而 $[x \mapsto x^{1/m}] \in \mathrm{End}(BigWitt(A))$ 是双射，于是我们可以将 $\alpha \in BigWitt(A)$ 以如下的形式表出

$$\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}.$$

另一方面对于 $\mathbb{Z}_{(p)}$-代数 $A$，下列映射是群同态

$$W(A) \longrightarrow BigWitt(A)$$
$$(a_0, a_1, \cdots) \mapsto \prod_{i\geq 0} AH(a_i x^i).$$

$BigWitt_n(A)$ 可以分解为 $p$-Witt 环的直和，实际上有以下同构

$$BigWitt(A) \cong \prod_{\substack{m\geq 1\\ gcd(m,p)=1}} W(A),$$

元素 $\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}$ 对应于一个 $m$-分量为 $(a_m, a_{mp}, a_{mp^2}, \cdots) \in W(A)$ 的 Witt 向量。对于截断的 Witt 环，有同构

$$BigWitt_n(A) \cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(A),$$

其中 $\ell(m,n)$ 是一个整数，定义为

$$\ell(m,n) = 1 + \text{使得 } mp^k \leq n \text{ 成立的最大整数 } k.$$

考虑特征为 $p$ 的有限域 $\mathbb{F}_q$，有同构 [3]

$$BigWitt_n(\mathbb{F}_q) \cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(\mathbb{F}_q),$$

注意到两边都是 $q^n$ 阶的群，因为 $\sum_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} \ell(m,n) = n$。

**Corollary 1.7.** *若有限域 $\mathbb{F}_{p^f}$ 的特征 $ch(\mathbb{F}_{p^f}) = p$，则作为 Abel 群有*

$$BigWitt_n(\mathbb{F}_{p^f}) \cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{1+\lfloor \log_p \frac{n}{m} \rfloor}(\mathbb{F}_{p^f}) = \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor \log_p \frac{n}{m} \rfloor}\mathbb{Z})^f,$$

*其中 $\lfloor x \rfloor$ 表示不超过 $x$ 的最大整数。*

**Example 1.8.** $BigWitt_3(\mathbb{F}_2) = W_{\ell(1,3)}(\mathbb{F}_2) \oplus W_{\ell(3,3)}(\mathbb{F}_2) = W_2(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$，

$BigWitt_4(\mathbb{F}_2) = W_{\ell(1,4)}(\mathbb{F}_2) \oplus W_{\ell(3,4)}(\mathbb{F}_2) = W_3(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$，

$BigWitt_2(\mathbb{F}_3) = W_{\ell(1,2)}(\mathbb{F}_3) \oplus W_{\ell(2,2)}(\mathbb{F}_3) = W_1(\mathbb{F}_3) \oplus W_1(\mathbb{F}_3) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$。

## 1.3 $NK_2(\mathbb{F}_2[C_2])$ 的计算

方法一是在讲 Weibel 文章 [9] 时讲过的。方法二是基于上面的思路给出来的详细证明。

### 1.3.1 方法一

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{1.8}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all $i$.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0.$

$$\begin{tikzcd} & NK_3(\mathbb{Z},(2)) = 0 \arrow[d] & & \\ & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\ 0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\ & NK_2(\mathbb{Z},(2)) = 0 & & \end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2], (\sigma+1), (\sigma-1))$, from Guin-Loday-Keune [2], $NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

**Theorem 1.9.** $NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.

In fact, when $p$ is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

**Example 1.10** ($\mathbb{Z}[C_p]$). $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p] & \xrightarrow{\sigma\mapsto\zeta} & \mathbb{Z}[\zeta] \\ {\scriptstyle \sigma\mapsto 1}\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order $p$ and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R, I)$$
$$(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1, 1\rangle^p = 1$$

Also see [5].

**Example 1.11** ($\mathbb{Z}[C_p][x]$). There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p][x] & \longrightarrow & \mathbb{Z}[\zeta][x] \\ \downarrow & & \downarrow \\ \mathbb{Z}[x] & \longrightarrow & \mathbb{F}_p[x] \end{array}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

### 1.3.2 方法二

计算 $k = \mathbb{F}_2, p = 2, n = 2$ 的情形，即 $NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^2), (t_1))$.

**Theorem 1.12.** *(1)$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$,*
*(2)$NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t, x]/(t^2), (t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1$为奇数$\}$ 生成的，这样的符号均为 2 阶元。*

*Proof.* (1) 令 $A=\mathbb{F}_2[t_1,t_2]/(t_1^2)=\mathbb{F}_2[C_2][x]$, 此时 $I=(t_1^2)$, $M=(t_1)$, $A/M=\mathbb{F}_2[x]$.

$$\begin{aligned}\Delta&=\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2 \mid t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1^2)\}\\&=\{(\alpha_1,\alpha_2)\mid \alpha_1\geq 2,\alpha_2\geq 0\},\end{aligned}$$

$$\begin{aligned}\Lambda&=\{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid \alpha_i\geq 1,\text{且}t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1)\}\\&=\{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid \alpha_i\geq 1,\alpha_1\geq 1,\alpha_2\geq 0\}\\&=\{((\alpha_1,\alpha_2),1)\mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2)\mid \alpha_1\geq 1,\alpha_2\geq 1\},\end{aligned}$$

若 $(\alpha,i)\in\Lambda$, $[\alpha,i]=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^i\in\Delta\}$, 于是有

$$\begin{aligned}[\alpha,1]&=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^1\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid (m\alpha_1-1,m\alpha_2)\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid m\alpha_1\geq 3\}.\end{aligned}$$

$$\begin{aligned}[\alpha,2]&=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^2\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid (m\alpha_1,m\alpha_2-1)\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid m\alpha_1\geq 2\}.\end{aligned}$$

此时

$$\begin{aligned}[(1,\alpha_2),1]&=3,\ \alpha_2\geq 0,\\ [(2,\alpha_2),1]&=2,\ \alpha_2\geq 0,\\ [(\alpha_1,\alpha_2),1]&=1,\ \alpha_1\geq 3,\alpha_2\geq 0,\\ [(1,\alpha_2),2]&=2,\ \alpha_2\geq 1,\\ [(\alpha_1,\alpha_2),2]&=1,\ \alpha_1\geq 2,\alpha_2\geq 1.\end{aligned}$$

若 $gcd(2,\alpha_1,\alpha_2)=1$，即 $\alpha_1,\alpha_2$ 中至少一个是奇数，令 $[\alpha]=\max\{[\alpha,i]\mid \alpha_i\not\equiv 0 \bmod 2\}$, $\alpha=(\alpha_1,\alpha_2)$, 若仅 $\alpha_1$ 是奇数，$[\alpha]=[\alpha,1]$, 若仅 $\alpha_2$ 是奇数，$[\alpha]=[\alpha,2]$，若两者均为奇数，则 $[\alpha]=\max\{[\alpha,1],[\alpha,2]\}$，有

$$\begin{aligned}[(1,\alpha_2)]&=\max\{[(1,\alpha_2),1],[(1,\alpha_2),2]\}=3,\alpha_2\geq 1\text{ 是奇数}\\ [(1,\alpha_2)]&=[(1,\alpha_2),1]=3,\alpha_2\geq 0\text{ 是偶数}\\ [(3,\alpha_2)]&=\max\{[(3,\alpha_2),1],[(3,\alpha_2),2]\}=1,\alpha_2\geq 1\text{ 是奇数}\\ [(3,\alpha_2)]&=[(3,\alpha_2),1]=1,\alpha_2\geq 0\text{ 是偶数}\\ [(2,1)]&=[(2,1),2]=1,\\ [\alpha]&=1,\text{其它符合条件的 }\alpha.\end{aligned}$$

为了方便我们把上面的计算结果列表如下

| $(\alpha_1,\alpha_2)$ | $[(\alpha_1,\alpha_2),1]$ | $[(\alpha_1,\alpha_2),2]$ | $[(\alpha_1,\alpha_2)]$ |
|---|---|---|---|
| $(1,\alpha_2)$ | $3,\alpha_2\geq 0$ | $2,\alpha_2\geq 1$ | 3 |
| $(2,\alpha_2)$ | $2,\alpha_2\geq 0$ | $1,\alpha_2\geq 1$ | 1, 当 $\alpha_2$ 是奇数时 |
| $(3,\alpha_2)$ | $1,\alpha_2\geq 0$ | $1,\alpha_2\geq 1$ | 1 |
| $(\alpha_1,0),\alpha_1\geq 3$ | 1 | 无定义 | 1, 当 $\alpha_1$ 是奇数时 |
| $(\alpha_1,\alpha_2),\alpha_1\geq 3,\alpha_2\geq 1$ | 1 | 1 | 1, 当 $(\alpha_1,\alpha_2)=1$ 时 |

下面我们计算 $\Lambda^{00}=\{(\alpha,i)\in\Lambda\mid gcd(\alpha_1,\alpha_2)=1,i\neq\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[\alpha,j]=[\alpha]\}\}$，分情况来讨论

1. 对于任何的 $\alpha_2\geq 0$，$((1,\alpha_2),1)\notin\Lambda^{00}$，这是因为 $1\not\equiv 0 \bmod 2$ 且 $[(1,\alpha_2),1]=3=[(1,\alpha_2)]$，从而 $\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[(1,\alpha_2),j]=[(1,\alpha_2)]\}=1$；
2. 对于任何的奇数 $\alpha_2\geq 0$，$((2,\alpha_2),1)\in\Lambda^{00}$，偶数 $\alpha_2\geq 0$，$((2,\alpha_2),1)\notin\Lambda^{00}$，因为 $\alpha_2\not\equiv 0 \bmod 2$ 并且 $[(2,\alpha_2),2]=1=[(2,\alpha_2)]$，故 $\{j\mid\alpha_j\not\equiv 0 \bmod 2,[(2,\alpha_2),j]=[(2,\alpha_2)]\}=2\neq 1$，此时 $[(2,\alpha_2),1]=2$；
3. 对于偶数 $\alpha_1\geq 3$ 和奇数 $\alpha_2\geq 1$，$((\alpha_1,\alpha_2),1)\in\Lambda^{00}$，其余情况当 $\alpha_1\geq 3$ 为奇数或 $\alpha_1,\alpha_2$ 均为偶数时 $((\alpha_1,\alpha_2),1)\notin\Lambda^{00}$。由于要求 $1\neq\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[(\alpha_1,\alpha_2),j]=[(\alpha_1,\alpha_2)]\}$，当 $\alpha_1\geq 3$ 为奇数时上式不成立，$2=\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[(\alpha_1,\alpha_2),j]=[(\alpha_1,\alpha_2)]\}$ 当且仅当 $\alpha_1\geq 3$ 为偶数且 $\alpha_2\geq 1$ 为奇数，此时 $[(\alpha_1,\alpha_2),1]=1$；
4. 对于任何的 $\alpha_2\geq 1$，$((1,\alpha_2),2)\in\Lambda^{00}$，由于此时 $[(1,\alpha_2),1]=3=[(1,\alpha_2)]$，$\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[\alpha,j]=[\alpha]\}=1$，此时 $[(1,\alpha_2),2]=2$；
5. 对于任何的奇数 $\alpha_2\geq 1$，$((2,\alpha_2),2)\notin\Lambda^{00}$，由于 $[(2,\alpha_2),2]=1=[(2,\alpha_2)]$，与 $2\neq\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[\alpha,j]=[\alpha]\}$ 矛盾；
6. 对于奇数 $\alpha_1\geq 3$ 和任意 $\alpha_2\geq 1$，$((\alpha_1,\alpha_2),2)\in\Lambda^{00}$，其余情况只要当 $\alpha_1\geq 3$ 为偶数时 $((\alpha_1,\alpha_2),2)\notin\Lambda^{00}$。要求 $2\neq\min\{j\mid\alpha_j\not\equiv 0 \bmod 2,[(\alpha_1,\alpha_2),j]=[(\alpha_1,\alpha_2)]\}$，当 $\alpha_1$ 为偶数时上式不成立，而当 $\alpha_1$ 为奇数时，任意 $\alpha_2\geq 1$，$[(\alpha_1,\alpha_2),1]=1=[(\alpha_1,\alpha_2)]$，此时 $[(\alpha_1,\alpha_2),2]=1$。

从而

$$\begin{aligned}\Lambda^{00}=&\{((2,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}\\&\cup\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}\\&\cup\{((\alpha_1,\alpha_2),1)|\alpha_1\geq 3\text{为偶数},\alpha_2\geq 1\text{为奇数}\}\\&\cup\{((\alpha_1,\alpha_2),2)|\alpha_1\geq 3\text{为奇数},\alpha_2\geq 1\}.\end{aligned}$$

记 $\Lambda^{00}_1=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]=1\}$，$\Lambda^{00}_2=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]=2\}$，我们有

$$\Lambda^{00}_1=\{((\alpha_1,\alpha_2),1)|\alpha_1\geq 3\text{为偶数},\alpha_2\geq 1\text{为奇数}\}\cup\{((\alpha_1,\alpha_2),2)|\alpha_1\geq 3\text{为奇数},\alpha_2\geq 1\}$$

$$\Lambda^{00}_2=\{((2,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}\cup\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}$$

$$\Lambda^{00}=\Lambda^{00}_1\sqcup\Lambda^{00}_2.$$

若 $[\alpha,i]=1$ 时，$(1+x\mathbb{F}_2[x]/(x))^\times$ 是平凡的，$[\alpha,i]=2$ 时，$(1+x\mathbb{F}_2[x]/(x^2))^\times\cong\mathbb{Z}/2\mathbb{Z}$，从而由定理2.1得

$$\begin{aligned}
NK_2(\mathbb{F}_2[C_2]) \cong K_2(A,M) \cong & \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
= & \bigoplus_{(\alpha,i)\in\Lambda^{00}_2} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
= & \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
= & \bigoplus_{\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z},
\end{aligned}$$

作为 Abel 群，

$$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z}.$$

(2) 由2.1，对于任意 $(\alpha,i)\in\Lambda^{00}$，$\Gamma_{\alpha,i}$ 诱导了同态

$$\begin{aligned}
\Gamma_{\alpha,i}\colon (1+xk[x]/(x^{[\alpha,i]}))^\times &\longrightarrow K_2(A,M) \\
1-xf(x) &\mapsto \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i\rangle.
\end{aligned}$$

此时只需考虑 $\Lambda^{00}_2 = \{((2,\alpha_2),1)\mid \alpha_2\geq 1\text{为奇数}\}\cup\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}$，对于任意 $(\alpha,i)\in\Lambda^{00}_2$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2\geq 1$，

$$\begin{aligned}
\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle,
\end{aligned}$$

对任意 $\alpha_2\geq 1$ 为奇数，

$$\begin{aligned}
\Gamma_{(2,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle,
\end{aligned}$$

我们作简单的替换令 $t=t_1, x=t_2$，于是 $\langle t_1t_2^{\alpha_2-1},t_2\rangle = \langle tx^{\alpha_2-1},x\rangle$，$\langle t_1t_2^{\alpha_2},t_1\rangle = \langle tx^{\alpha_2},t\rangle$。由同构2.1可知 $NK_2(\mathbb{F}_2[C_2])$ 是由 Dennis-Stein 符号 $\{\langle tx^i,x\rangle \mid i\geq 0\}$ 与 $\{\langle tx^i,t\rangle \mid i\geq 1\text{为奇数}\}$ 生成的，由于 $t^2=0$ 故 $\langle tx^i,x\rangle+\langle tx^i,x\rangle = \langle tx^i+tx^i-t^2x^{2i+1},x\rangle = 0$，$\langle tx^i,t\rangle+\langle tx^i,t\rangle = \langle tx^i+tx^i-t^3x^{2i},t\rangle = 0$。 □

**Remark 1.13.** 对于 $i\geq 1$为偶数，$\langle tx^i,t\rangle = \langle x^{i/2},t\rangle+\langle x^{i/2},t\rangle = \langle x^{i/2}+x^{i/2}+tx^i,t\rangle = 0$。

Weibel 在文献 [9] 中给出了以下可裂正合列

$$0\longrightarrow V/\Phi(V)\xrightarrow{F} NK_2(\mathbb{F}_2[C_2])\xrightarrow{D}\Omega_{\mathbb{F}_2[x]}\longrightarrow 0,$$

其中 $V=x\mathbb{F}_2[x]$，$\Phi(V)=x^2\mathbb{F}_2[x^2]$ 是 $V$ 的子群，$\Omega_{\mathbb{F}_2[x]}\cong\mathbb{F}_2[x]\,dx$ 是绝对 Kähler 微分模，$F(x^n)=\langle tx^n,t\rangle$，$D(\langle ft,g+g't\rangle)=f\,dg$。显然 $D(\langle tx^i,t\rangle)=0$,$D(\langle tx^i,x\rangle)=x^i\,dx$,可以看出 $NK_2(\mathbb{F}_2[C_2])$ 的直和项 $\bigoplus_{((2,\alpha_2),1),\alpha_2\geq 1\text{为奇数}}\mathbb{Z}/2\mathbb{Z}$ $V/\Phi(V)$，直和项 $\bigoplus_{((1,\alpha_2),2),\alpha_2\geq 1}\mathbb{Z}/2\mathbb{Z}\cong\mathbb{F}_2[x]\,dx$。

$V$ 和 $\Omega_{\mathbb{F}_2[x]}$ 作为 Abel 群是同构的，但作为 $W(\mathbb{F}_2)$-模是不同的。$V=x\mathbb{F}_2[x]$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$V_m(x^n)=x^{mn},$$

$$F_d(x^n)=\begin{cases} dx^{n/d}, & \text{若 } d|n \\ 0, & \text{其它}\end{cases},$$

$$[a]x^n=a^nx^n.$$

$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx,$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx.$$

结合两者我们可以得到 $NK_2(\mathbb{F}_2[C_2])$ 的 $W(\mathbb{F}_2)$-模结构为

$$V_m(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{mn}, t\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1 \text{ 为奇数}$$
$$V_m(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{mn-1}, x\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1$$
$$F_d(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{n/d}, t\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1 \text{ 为奇数}$$
$$F_d(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{n/d-1}, x\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1$$
$$[1]\langle tx^n, t\rangle = \langle tx^n, t\rangle, \quad n \geq 1 \text{ 为奇数}$$
$$[1]\langle tx^{n-1}, x\rangle = \langle tx^{n-1}, x\rangle, \quad n \geq 1.$$

## 1.4 $NK_2(\mathbb{F}_2[C_4])$ 的结构

用同样的方法计算 $NK_2(\mathbb{F}_2[C_{2^2}])$，继而对于任意 $n$ 可以得到类似的结果。

**Theorem 1.14.** $NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$。

*Proof.* $\mathbb{F}_2[t_1, t_2]/(t_1^4) = \mathbb{F}_2[C_4][t_2]$，此时 $I = (t_1^4)$，$M = (t_1)$ 不变，我们直接写出以下集合

$$\Delta = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 4, \alpha_2 \geq 0\},$$
$$\Lambda = \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\},$$

用 $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 $x$ 的最小整数，

$$[\alpha, 1] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 5\} = \lceil 5/\alpha_1 \rceil,$$
$$[\alpha, 2] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 4\} = \lceil 4/\alpha_1 \rceil.$$

例如

$$
\begin{aligned}
[(1,\alpha_2),1] &= 5,\ \alpha_2 \geq 0,\\
[(2,\alpha_2),1] &= 3,\ \alpha_2 \geq 0,\\
[(3,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(4,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(\alpha_1,\alpha_2),1] &= 1,\ \alpha_1 \geq 5, \alpha_2 \geq 0,\\
[(1,\alpha_2),2] &= 4,\ \alpha_2 \geq 1,\\
[(2,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(3,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(\alpha_1,\alpha_2),2] &= 1,\ \alpha_1 \geq 4, \alpha_2 \geq 1.
\end{aligned}
$$

| $(\alpha_1,\alpha_2)$ | $[(\alpha_1,\alpha_2),1]$ | $[(\alpha_1,\alpha_2),2]$ | $[(\alpha_1,\alpha_2)]$ |
|---|---|---|---|
| $(1,\alpha_2)$ | $5,\alpha_2 \geq 0$ | $4,\alpha_2 \geq 1$ | 5 |
| $(2,\alpha_2)$ | $3,\alpha_2 \geq 0$ | $2,\alpha_2 \geq 1$ | 2, 当 $\alpha_2$ 是奇数时 |
| $(3,\alpha_2)$ | $2,\alpha_2 \geq 0$ | $2,\alpha_2 \geq 1$ | 2 |
| $(4,\alpha_2)$ | $2,\alpha_2 \geq 0$ | $1,\alpha_2 \geq 1$ | 1当 $\alpha_2$ 是奇数时 |
| $(\alpha_1,0),\alpha_1 \geq 5$ | 1 | 无定义 | 1, 当 $\alpha_1$ 是奇数时 |
| $(\alpha_1,\alpha_2),\alpha_1 \geq 5,\alpha_2 \geq 1$ | 1 | 1 | 1, 当 $(\alpha_1,\alpha_2)=1$ 时 |

记 $\Lambda_d^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = d\}$，$\Lambda_{>1}^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] > 1\}$

由于 $(\alpha,i) \in \Lambda_1^{00}$ 均有 $[(\alpha,i)] = 1$，实际上要计算 $(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ 只需确定 $\Lambda_{>1}^{00}$。由同样的方法可得 $\Lambda_4^{00} = \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$，$\Lambda_3^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}$，$\Lambda_2^{00} = \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}$，

$$
\begin{aligned}
\Lambda_{>1}^{00} = &\{((1,\alpha_2),2) \mid \alpha_2 \geq 1\} \cup \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\}\\
&\cup \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}.
\end{aligned}
$$

由定理2.1，

$$
\begin{aligned}
NK_2(\mathbb{F}_2[C_4]) \cong K_2(A,M) \cong & \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\
&= \bigoplus_{(\alpha,i)\in\Lambda_{>1}^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\
&= \bigoplus_{\substack{((3,\alpha_2),2)\\ gcd(3,\alpha_2)=1\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((4,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^2))^\times\\
&\oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^3))^\times \oplus \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^4))^\times.
\end{aligned}
$$

由1.5有 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$，$(1+x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$，于是 $NK_2(\mathbb{F}_2[C_4])$ 作为 Abel 群有

$$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}.$$

对于任意 $(\alpha,i)\in\Lambda^{00}_{>1}$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2\geq 1$，$gcd(3,\alpha_2)=1$

$$\begin{aligned}\Gamma_{(3,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1^3t_2^{\alpha_2-1},t_2\rangle,\end{aligned}$$

对任意 $\alpha_2\geq 1$，

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^4))^\times &\rightarrowtail K_2(A,M)\\ 1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2-1},t_2\rangle,\\ 1+x^3(\text{二阶元}) &\mapsto \langle t_1^3t_2^{3\alpha_2-1},t_2\rangle,\end{aligned}$$

对任意 $\alpha_2\geq 1$ 为奇数，

$$\begin{aligned}\Gamma_{(4,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1^3t_2^{\alpha_2},t_1\rangle,\\ \Gamma_{(2,\alpha_1),1}\colon (1+x\mathbb{F}_2[x]/(x^3))^\times &\rightarrowtail K_2(A,M)\\ 1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2},t_1\rangle.\end{aligned}$$

我们作简单的替换令 $t=t_1, x=t_2$，由同构2.1可知 $NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号 $\{\langle tx^{i-1},x\rangle\mid i\geq 1\}$, $\{\langle tx^i,t\rangle\mid i\geq 1\text{为奇数}\}$, $\{\langle t^3x^{3i-1},x\rangle\mid i\geq 1\}$, $\{\langle t^3x^{i-1},x\rangle\mid i\geq 1, gcd(i,3)=1\}$, $\{\langle t^3x^i,t\rangle\mid i\geq 1\text{为奇数}\}$ 生成的。 □

**Remark 1.15.** $\langle t^3x^{2i},t\rangle = \langle tx^i,t\rangle+\langle tx^i,t\rangle$ 是二阶元。根据 [4]，存在同态

$$\begin{aligned}\rho_1\colon \mathbb{F}_2[x]dx &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^idx &\mapsto \langle t^3x^i,x\rangle\\ \rho_2\colon x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2] &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^i &\mapsto \langle t^3x^i,t\rangle\end{aligned}$$

$\{\langle t^3x^{i-1},x\rangle\mid i\geq 1\}=\{\langle t^3x^{3i-1},x\rangle\mid i\geq 1\}\cup\{\langle t^3x^{i-1},x\rangle\mid i\geq 1, gcd(i,3)=1\}$,从而 $\Omega_{\mathbb{F}_2[x]}\oplus x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2]$ 是 $NK_2(\mathbb{F}_2[C_4])$ 的直和项。

## 1.5 $NK_2(\mathbb{F}_q[C_{2^n}])$

设 $\mathbb{F}_q$ 是特征为 2 的有限域, $q=2^f$, $C_{2^n}$ 是 $2^n$ 阶循环群，这一节计算 $NK_2(\mathbb{F}_q[C_{2^n}])$。假设 $A=\mathbb{F}_q[t_1,t_2]/(t_1^{2^n})=\mathbb{F}_q[C_{2^n}][x]$, 此时 $I=(t_1^{2^n})$, $M=(t_1)$, $A/M=\mathbb{F}_q[x]$。

**Lemma 1.16.** $\Delta=\{(\alpha_1,\alpha_2)\mid \alpha_1\geq 2^n,\alpha_2\geq 0\}$，$\Lambda=\{((\alpha_1,\alpha_2),1)\mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2)\mid \alpha_1\geq 1,\alpha_2\geq 1\}$，对任意 $(\alpha,i)\in\Lambda$，$[\alpha,1]=\lceil(2^n+1)/\alpha_1\rceil$，$[\alpha,2]=\lceil 2^n/\alpha_1\rceil$，其中 $\lceil x\rceil=\min\{m\in\mathbb{Z}|m\geq x\}$ 表示不小于 $x$ 的最小整数。

**Lemma 1.17.** 令 $I_1=\{((\alpha_1,\alpha_2),1)\mid gcd(\alpha_1,\alpha_2)=1, 1<\alpha_1\le 2^n\text{为偶数},\alpha_2\ge 1\text{为奇数}\}$，$I_2=\{((\alpha_1,\alpha_2),2)\mid gcd(\alpha_1,\alpha_2)=1, 1\le\alpha_1<2^n\text{为奇数},\alpha_2\ge 1\}$，则 $\Lambda^{00}_{>1}=I_1\sqcup I_2$。

由定理2.1，

$$\begin{aligned}NK_2(\mathbb{F}_q[C_{2^n}])\cong K_2(A,M)&\cong\bigoplus_{(\alpha,i)\in\Lambda^{00}}(1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times\\&=\bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}}(1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times\\&=\bigoplus_{(\alpha,1)\in I_1}(1+x\mathbb{F}_q[x]/(x^{\lceil(2^n+1)/\alpha_1\rceil}))^\times\\&\oplus\bigoplus_{(\alpha,2)\in I_2}(1+x\mathbb{F}_q[x]/(x^{\lceil 2^n/\alpha_1\rceil}))^\times.\end{aligned}$$

注意到 $BigWitt_k(R)=(1+xR[\![x]\!])^\times/(1+x^{k+1}R[\![x]\!])^\times\cong(1+xR[x]/(x^{k+1}))^\times$，根据公式1.7，

$$\begin{aligned}NK_2(\mathbb{F}_q[C_{2^n}])\cong&\bigoplus_{(\alpha,1)\in I_1}\bigoplus_{\substack{1\le m\le\lceil(2^n+1)/\alpha_1\rceil-1\\ gcd(m,2)=1}}(\mathbb{Z}/2^{1+\left\lfloor\log_2\frac{\lceil(2^n+1)/\alpha_1\rceil-1}{m}\right\rfloor}\mathbb{Z})^f\\&\oplus\bigoplus_{(\alpha,2)\in I_2}\bigoplus_{\substack{1\le m\le\lceil 2^n/\alpha_1\rceil-1\\ gcd(m,2)=1}}(\mathbb{Z}/2^{1+\left\lfloor\log_2\frac{\lceil 2^n/\alpha_1\rceil-1}{m}\right\rfloor}\mathbb{Z})^f.\end{aligned}$$

接下来我们证明对于任意 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 都在 $NK_2(\mathbb{F}_q[C_{p^n}])$ 出现无限多次

**Lemma 1.18.** 对于任意的 $1\le k<n$，$1+\left\lfloor\log_2(\frac{2^n-1}{2^k+1})\right\rfloor=n-k$。

*Proof.* 当 $1\le k<n$ 时，$2^k-1\ge 1\ge\frac{1}{2^{n-k-1}}$，即

$$2^{n-1}-2^{n-k-1}\ge 1,$$

上式等价于 $2^n-1\ge 2^{n-k-1}(2^k+1)$，且 $2^n-1<2^{n-k}(2^k+1)$，于是

$$2^{n-k}>\frac{2^n-1}{2^k+1}\ge 2^{n-k-1},$$

取对数得 $\left\lfloor\log_2(\frac{2^n-1}{2^k+1})\right\rfloor=n-k-1$。 □

考虑 $((1,\alpha_2),2)\in I_2$，

$$\bigoplus_{(\alpha,2)\in I_2}\bigoplus_{\substack{1\le m\le 2^n-1\\ gcd(m,2)=1}}(\mathbb{Z}/2^{1+\left\lfloor\log_2\frac{2^n-1}{m}\right\rfloor}\mathbb{Z})^f$$

是 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 的直和项，当 $m=1$ 时 $1+\lfloor\log_2(2^n-1)\rfloor=n$，当 $m=2^k+1(1\le k<n)$ 为奇数时，由1.18，$1+\left\lfloor\log_2\frac{2^n-1}{m}\right\rfloor=n-k$，于是对于任何的 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 均出现在直和项中，且对于任意 $\alpha_2\ge 1$，这样的项总会出现，于是

$$NK_2(\mathbb{F}_q[C_{2^n}])\cong\bigoplus_{\infty}\bigoplus_{k=1}^{n}\mathbb{Z}/2^k\mathbb{Z}.$$

接下来给出一些 $NK_2(\mathbb{F}_q[C_{2^n}])$ 中的 $2^k(1\le k\le n)$ 阶元素。

对任意 $\alpha_2 \geq 1, a \in \mathbb{F}_q$，

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_q[x]/(x^{2^n}))^\times &\rightarrowtail K_2(A,M)\\ 1+ax(2^n \text{ 阶元}) &\mapsto \langle atx^{\alpha_2-1}, x\rangle,\\ 1+ax^3(2^{n-1} \text{ 阶元}) &\mapsto \langle at^3x^{3\alpha_2-1}, x\rangle,\\ 1+ax^{2^k+1}(2^{n-k} \text{ 阶元}) &\mapsto \langle at^{2k+1}x^{(2k+1)\alpha_2-1}, x\rangle.\end{aligned}$$

## 1.6 其他问题和说明

$NK_2(\mathbb{F}_{p^m}[C_{p^n}]) =?$

$\mathbb{F}_2[C_2\times C_2] \cong \mathbb{F}_2[C_2]\otimes\mathbb{F}_2[C_2] \cong \mathbb{F}_2[x,y]/(x^2,y^2)$，可以用同样的方法得到一些结果。

$$0 \longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n)) \longrightarrow K_2(k[t_3]) \longrightarrow 0$$

对于有限域 $k$ 来讲 $K_2(k[t_3]) = 0$,

$$0 \longrightarrow NK_2(\mathbb{F}_2[C_2\times C_2]) \longrightarrow K_2(\mathbb{F}_2[C_2\times C_2][x]) \longrightarrow K_2(\mathbb{F}_2[C_2\times C_2]) \longrightarrow 0,$$

中间那项可以用这篇文章里的方法确定，又 $K_2(\mathbb{F}_2[C_2\times C_2]) = C_2^3$, 于是可以得到 $NK_2(\mathbb{F}_2[C_2\times C_2])$，是 $\oplus_\infty\mathbb{Z}/2\mathbb{Z}$.

另外可以直接用这种方式重新计算 $K_2(\mathbb{F}_2[C_4\times C_4])$，见下一篇笔记。

一个关于模结构的问题，在 Weibel 的文章 [8] 中 5.5 和 5.7 给出的模结构和本文上面的模结构并不一致，用 $V_m$ 作用差一个 $t^m$。

# Chapter 2

# On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

## 2.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative $K_2$-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

## 2.2 Introduction

Let $C_n$ denote the cyclic group of order $n$. Chen et al. [10] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative $K_2$-group $K_2(\mathbb{F}_2 C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2 C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

## 2.3 Preliminaries

Let $k$ be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of $t_i$ in $A$ also as $t_i$. Let $M = (t_1, t_2)$ be the nilradical of $A$. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:

generators: $\langle a, b\rangle$, $(a, b) \in A \times M \cup M \times A$;

relations: $\langle a, b\rangle = -\langle b, a\rangle$,

$\langle a, b\rangle + \langle c, b\rangle = \langle a + c - abc, b\rangle$,

$\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [7]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1, 0) \in \mathbb{N}^2$, $\epsilon^2 = (0, 1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,
- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1, 2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i \rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

**Lemma 2.1.** *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

*Proof.* See Corollary 2.6 in [7]。 □

**Lemma 2.2.** $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

*Proof.* It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3 \rangle = \{1, 1 + x^3\}$. Let $\sigma, \tau$ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

## 2.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

**Lemma 2.3.** $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

*Proof.* The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism $f$ is an isomorphism since $K_2$-group of any finite field is trivial. □

**Theorem 2.4.** *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

*Proof.* Set $A = \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$, then $I = (t_1^4, t_2^4)$, $M = (t_1, t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set $\Lambda^{00}$. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] > 1\}$, and then

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha, 1) \in \Lambda^{00}_{>1}$ and $[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ and $[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1, \alpha_2 \leq 4$ are both odd and $gcd(\alpha_1, \alpha_2) = 1$, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha, 1]$.

By the computation 2.2, we can get the following table

| $(\alpha, i) \in \Lambda^{00}_{>1}$ | $[\alpha, i]$ | $(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ |
|---|---|---|
| $((2,1),1)$ | 3 | $\mathbb{Z}/4\mathbb{Z}$ |
| $((2,3),1)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((4,1),1)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((4,3),1)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((1,2),2)$ | 3 | $\mathbb{Z}/4\mathbb{Z}$ |
| $((1,4),2)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((1,1),2)$ | 4 | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $((1,3),2)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((3,2),2)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((3,4),2)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |
| $((3,1),2)$ | 2 | $\mathbb{Z}/2\mathbb{Z}$ |

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below, the generators of order 4:

$$\langle t_1t_2, t_1\rangle, \langle t_1t_2, t_2\rangle, \langle t_1, t_2\rangle,$$

the generators of order 2:

$$\langle t_1t_2^3, t_1\rangle, \langle t_1^3t_2, t_1\rangle, \langle t_1^3t_2^3, t_1\rangle, \langle t_1t_2^3, t_2\rangle, \langle t_1^3t_2^2, t_2\rangle, \langle t_1t_2^2, t_2\rangle, \langle t_1^3t_2, t_2\rangle, \langle t_1^3t_2^3, t_2\rangle, \langle t_1^3, t_2\rangle.$$

$\square$

**Remark 2.5.** Compared with [10], note that $\langle t_1^3, t_2 \rangle = \langle t_1^2 t_2, t_1 \rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2 \rangle &= \langle t_1^2, t_1 t_2 \rangle - \langle t_1^2 t_2, t_1 \rangle \\
&= \langle t_1, t_1^2 t_2 \rangle - \langle t_1^2 t_2, t_1 \rangle - \langle t_1^2 t_2, t_1 \rangle \\
&= -3 \langle t_1^2 t_2, t_1 \rangle \\
&= -\langle t_1^2 t_2, t_1 \rangle \\
&= \langle t_1^2 t_2, t_1 \rangle,
\end{aligned}$$

since $\langle t_1^2 t_2, t_1 \rangle + \langle t_1^2 t_2, t_1 \rangle = \langle 0, t_1 \rangle = 0$ and $\langle t_1^3, t_2 \rangle = -\langle t_1^3, t_2 \rangle$.

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