代数 K 理论讨论班笔记

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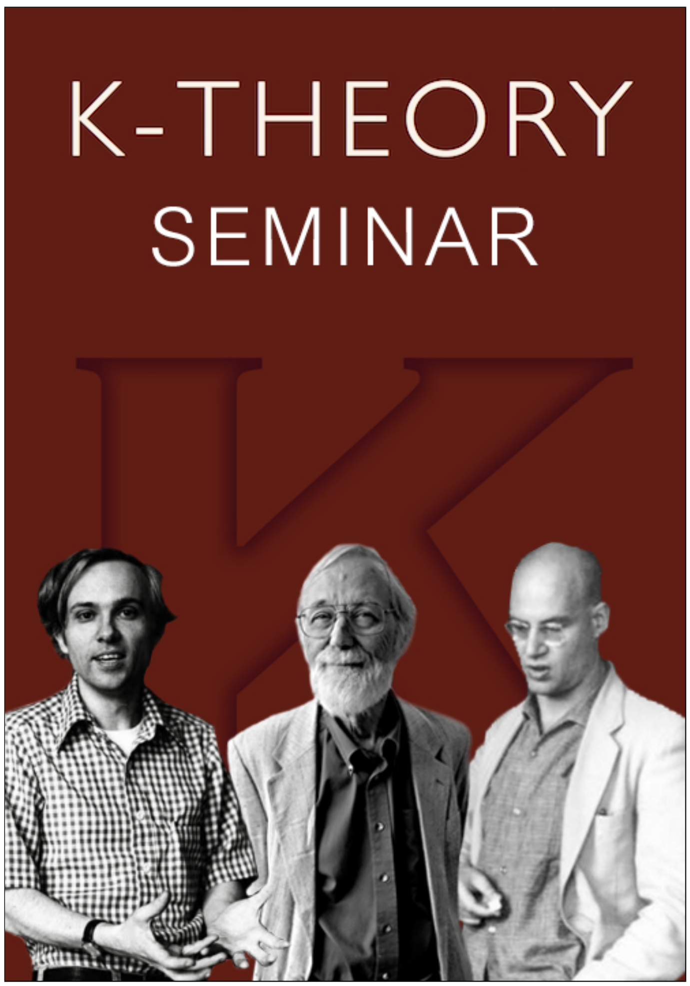

从左至右依次为
Quillen Milnor Grothendieck

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目录

Chapter 1

K 理论简介

1.1 释题

初见题目，大概最先问的问题就是“K 理论”中的“K”是什么含义，因而我们从释题开始：

“K” “K”源于德文“klassen”，中文意为“分类”。从而简略地说，“K 理论”就是分类的理论。1957 年 Grothendieck[1]在 Riemman-Roch 定理的工作中引入了函子 $K(\mathcal{A})$, 这就是 K 理论的开端。他之所以用 K 而不用 C(英语“class”的首字母) 是由于 Grothendieck 在泛函分析中做的许多工作里 $C(X)$ 通常表示连续函数空间，因此用他的母语---德语“分类”的首字母。

对于历史感兴趣的读者请参考 C.Weibel《The development of algebraic K-theory before 1980》。

分类 对于分类的思想，在数学中并不陌生，下表举了一些例子：

	例子	备注
表示论	有限群的不可约表示分类	Brauer 群，Voevodsky
代数几何	代数簇分类	Riemann-Roch-Hirzebruch-Grothendieck
代数拓扑	向量丛、拓扑空间的分类	拓扑 K-理论，Atiyah-Singer 指标定理
泛函分析	C^* 代数分类	算子 K-理论
代数数论	理想类群	Picard 群，Dedekind 环
几何拓扑	CW 复形	Whitehead 挠元
其它联系	非交换几何，上同调，谱序列等等	

[1] 1928.3.28-2014.11.13,1966 Fields Medal

1.2 历史

粗略地讲，K-理论是研究一系列函子：

$$K_n : \text{好的范畴} \longrightarrow \text{交换群范畴}, n \in \mathbb{Z}$$

$$\mathcal{C} \longrightarrow K_n(\mathcal{C})$$

- $n < 0$: 负 K-理论
- $n = 0, 1, 2$: 经典（低阶）K-理论
- $n \geq 3$: 高阶 K-理论

想法 构造环 R 的代数不变量 $K_i(R)$，称之为 K-群，这可以看作是环上的“线性代数”，更一般的看成某个空间的同伦群。构造高阶 K-群时有不同的构造方式，另外从广义上同调理论看，可以构造代数 K-理论谱 (Spectrum)，使得它的同伦群就是 K-群。

代数学分支中很多学科都可以看作线性代数的推广，如同调代数，表示论，李群李代数，矩阵分析，泛函分析等等，这里代数 K-理论某种意义上也是一门线性代数。

1.3 讲了几章 Srinivas 书后的想法

想把 K 理论推广到高阶 K 理论，并且还有类似于经典 K 理论的性质，比如正合列，MV 序列，还有基本定理。首先想得到一个长正合列，从代数上考虑是同调函子可以将复形的短正合列变成一个长正合列。换个角度思考，拓扑上得到一个长正合列除了同调函子还有一个重要的函子是同伦函子，一个 Serre 纤维化序列可以得到一个同伦群的长正合列。这是得到长正合列的方法。Quillen 了不起的想法是对于环 R，构造一个空间，使得这个空间的同伦群就是 K 群。于是他得到了两种定义高阶 K 理论的方法，俗称为“$+$”构造和“Q”构造。首先加法构造是对环 R 的一般线性群 $GL(R)$ 做分类空间 $BGL(R)$，对于任意群都可以找到这样一个相应的拓扑空间叫做分类空间，使得群的同调就是这个拓扑空间的同调。现在有了分类空间还不够，Quillen 发明了加法构造在分类空间的基础上增加相同数目的 2-胞腔和 3-胞腔得到了 $BGL(R)^+$，从这个空间出发求其同伦群就得到了 K 群。为什么说就是 K 群呢？通过计算可以得到，K_1, K_2 的结果正是经典 K 理论里的两个函子，从而这样一次性定义的 K 群就是经典 K 理论的推广。接着 Quillen 在 1972 年的著名论文中给出了 Q 构造，并且这时普遍适用与一大类范畴---正合范畴。对于正合范畴 $\mathcal{C}$，通过做 Q 构造得到 $Q\mathcal{C}$，然后做分类空间得到 $BQ\mathcal{C}$，再然后算 n 阶同伦群也得到了新的函子。可以证明这个函子和经典 K 群是一致的！唯一有些区别在于足标，$n+1$ 阶同伦群得到的是 n 阶 K 群，于是我们对 $BQ\mathcal{C}\mathcal{C}$ 取其 loop space $\Omega BQ\mathcal{C}$ 后，n 阶同伦群就是 n 阶 K 群了。

那这样两个定义是否一致呢？著名的“$+ = Q$”定理说对于环 R 和正合范畴 $\mathcal{P}(R)$ 分别用加法构造和 Q 构造得到的两个拓扑空间是同伦等价的，于是它们取同伦群是一样的！

有了 Q 构造后，高阶 K 群自然而然想推广经典 K 理论中的结论，而恰就是这么巧，很多定理都可以推广，但都不见得是平凡的。高阶 K 群的计算首先就是非常难的一部分，Quillen 在论文里得到了四大定理：加法定理，分解定理，反旋定理和局部化序列，英文分别叫做 Additivity，Resolution，Devissage，Localization。这四大定理再加上推论可以得到一些有趣的结果。首先看加法定理是说正合函子也有类似于 Euler characteristic 的性质，即一个正合函子的短正合列，中间函子诱导的 K 群的映射等于两边函子诱导的映射之和，很容易可以把短正合列推广成长正合列，并且还可以推广到有一个 filtration。对于分解定理和 Devissage，都是通过更简单的满子范畴来替换要研究的正合范畴，并且 K 群不变。局部化序列当然是利用长正合序列从已知来得到未知的信息。

有了这些准备，对于诺特环的 K 理论就会有一个比较深刻的定理，也叫做诺特环的 G 理论，G 理论是说只研究环 R 上的有限生成模的范畴，将 K 理论中投射的要求去掉。对于诺特环的 G 理论，有著名的 homotopy invariance

$$G_n(A[t]) = G_n(A), G_n(A[t,t^{-1}]) = G_n(A) \oplus G_{n-1}(A)$$

对其进行更细致的研究和推广可以得到对于任意环的 K 理论基本定理

$$K_n(A[t,t^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus NK_n(A) \oplus NK_n(A).$$

对于诺特环 G 理论基本定理的证明就是利用局部化序列，并且反复应用四大定理来得到，并且还详细研究了分次环和分次模的一些性质。Srinivas 的书无疑是很好的教材，Quillen 的原文也是值得一看的。

参考

1. David Eisenbud，commutative algebra with a view toward algebraic geometry.

Chapter 2

Notes on Higher K-theory of group-rings of virtually infinite cyclic groups

2.1 Introduction

作者 Aderemi O. Kuku and Guoping Tang

2.1.1 Preliminaries

Definition 2.1 (Virtually cyclic groups)**.** A discrete group V is called virtually cyclic if it contains a cyclic subgroup of finite index, i.e., if V is finite or virtually infinite cyclic.

Virtually infinite cyclic groups are of two types:

1 $V = G \rtimes_\alpha T$ is a semi-direct product where G is a finite group, $T = \langle t \rangle$ an infinite cyclic group generated by t, $\alpha \in Aut(G)$, and the action of T is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

2 V is a non-trivial amalgam of finite groups and has the form $V = G_0 *_H G_1$ where $[G_0 : H] = 2 = [G_1 : H]$.

We denote by $\mathcal{VCYC}$ the family of virtually cyclic subgroups of G.

$$\text{virtually cyclic groups} \begin{cases} \text{finite groups} \\ \text{virtually infinite cyclic groups} \begin{cases} \text{I.} V = G \rtimes_\alpha T, G \text{ is finite}, T = \langle t \rangle \cong \mathbb{Z} \\ \text{II.} V = G_0 *_H G_1, H \text{is finite}, [G_i : H] = 2 \end{cases} \end{cases}$$

若 G 是有限群，V 满足 $1 \to G \to V \to T \to 1$，则 V 是类型 I, $V = G \rtimes_\alpha T$, $\alpha : T \to Aut(G), \alpha(t)(g) = tgt^{-1}$. V 中的乘法[1]为

$$(g_1, t_1)(g_2, t_2) = (g\alpha(t_1)g_2, t_1t_2) = (g_1t_1g_2t_1^{-1}, t_1t_2).$$

若 G 是有限群，V 满足 $V \to D_\infty \to 1$，$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, 则 V 是类型 II。

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & G_0 \arrow[d] \\
G_1 \arrow[r] & G_0 *_H G_1
\end{tikzcd}$$

is a push-out square.

Definition 2.2 (Orders)**.** Let R be a Dedekind domain with quotient field F. An R-order in a F-algebra Σ is a subring Λ of Σ, having the same unity as Σ and s.t. R is contained in the center of Λ, Λ is finitely generated R-module and $F \otimes_R \Lambda = \Sigma$.

A Λ-lattice in Σ is a Λ-bisubmodule of Σ which generates Σ as a F-space.

A maximal R-order Γ in Σ is an order that is not contained in any other R-order in Σ.

Example 2.3. We give some examples:

1. G is a finite group, then RG is an R-order in FG when $ch(F) \nmid |G|$.
2. R is a maximal R-order in F.
3. $M_n(R)$ is a maximal R-order in $M_n(F)$.

Remark 2.4. Any R-order Λ is contained in at least one maximal R-order in Σ. Any semisimple F-algebra Σ contains at least one maximal R-order. However, if Σ is commutative, then Σ contains a unique maximal order, namely, the integral closure of R in Σ.

Theorem 2.5. *R, F, Λ, Σ as above, Then $K_0(\Lambda), G_0(\Lambda)$ are finitely generated abelian groups.*

2.1.2 The Farrell-Jones conjecture

Let G be a discrete group and $\mathcal{F}$ a family of subgroups of G closed under conjugation and taking subgroups, e.g., $\mathcal{VCYC}$.

Let $Or_{\mathcal{F}}(G) := \{G/H | H \in F\}$, R any ring with identity.

There exists a “Davis -Lück” functor

$$\mathbb{K}R : Or_{\mathcal{F}}(G) \longrightarrow Spectra$$

$$G/H \mapsto \mathbb{K}R(G/H) = K(RH)$$

[1]这里的 α 和文章中有些差异

where $K(RH)$ is the K-theory spectrum such that $\pi_n(K(RH)) = K_n(RH)$.

There exists a homology theory

$$H_n(-, \mathbb{K}R) : G - CWcomplexes \longrightarrow \mathbb{Z} - Mod$$

$$X \mapsto H_n(X, \mathbb{K}R)$$

Let $E_{\mathcal{F}}(G)$ be a G-CW-complex which is a model for the classifying space of $\mathcal{F}$. Note that $E_{\mathcal{F}}(G)^H$ is homotopic to the one point space (i.e., contractible) if $H \in F$ and $E_{\mathcal{F}}(G)^H = \emptyset$ if $H \notin F$ and $E_{\mathcal{F}}(G)$ is unique up to homotopy.

There exists an assembly map

$$A_{R,\mathcal{F}} : H_n(E_{\mathcal{F}}(G), \mathbb{K}R) \longrightarrow K_n(RG).$$

The Farrell-Jones isomorphism conjecture says that $A_{R,\mathcal{VCYC}} : H_n(E_{\mathcal{VCYC}}(G), \mathbb{K}R) \cong K_n(RG)$ is an isomorphism for all $n \in Z$. Note that KR is the non-connective K-theory spectrum such that $\pi_n(KR)$ is Quillen's $K_n(R), n \geq 0$, and $\pi_n(KR)$ is Bass's negetive $K_n(R)$, for $n \leq 0$.

2.1.3 Notations

- F: number field, i.e, $\mathbb{Q} \subset F$ is a finite field extension.
- R: the ring of integers in F.
- Σ: a semisimple F-algebra.
- Λ: an R-order in Σ, $\alpha : \Lambda \to \Lambda$: an R-automorphism.
- $\Gamma \in \{\alpha$-invariant R-orders in Σ containing $\Lambda\}$ is a maximal element.
- $\max(\Gamma) = \{$two-sided maximal ideals in $\Gamma\}$.
- $\max_\alpha(\Gamma) = \{$two-sided maximal α-invariant ideals in $\Gamma\}$.
- $\mathcal{C}$: exact category, $K_n(\mathcal{C}) = \pi_{n+1}(BQ\mathcal{C}), n \geq 0$. If A is a unital ring, $K_n(A) = K_n(\mathcal{P}(A)), n \geq 0$. When A is noetherian, $G_n(A) = K_n(\mathcal{M}(A))$.
- $T = \langle t \rangle$: infinite cyclic group $\cong \mathbb{Z}$, T^r: free abelian group of rank r.
- $A_\alpha[T] = A_\alpha[t, t^{-1}]$: α-twisted Laurent series ring, $A_\alpha[T] = A[T] = A[t, t^{-1}]$ additively and multiplication given by $(rt^i)(st^j) = r\alpha^i(s)t^{i+j}$.（注：这里和文章有些区别）
- $A_\alpha[t]$: the subgroup of $A_\alpha[T]$ generated by A and t, that is, $A_\alpha[t]$ is the twisted polynomial ring.
- $NK_n(A, \alpha) := \ker(K_n(A_\alpha[t]) \to K_n(A)), n \in \mathbb{Z}$ where the homomorphism is induced by the augmentation $\epsilon : A_\alpha[t] \to A$. If $\alpha = \mathrm{id}$, $NK_n(A, \mathrm{id}) = NK_n(A) = \ker(K_n(A[t]) \to K_n(A))$.

2.1.4 已知结果

Next, we focus on higher K-theory of virtually cyclic groups

Theorem 2.6 (A. Kuku). *For all $n \geq 1, K_n(\Lambda)$ and $G_n(\Lambda)$ are finitely generated Abelian groups and hence that for any finite group G, $K_n(RG)$ and $G_n(RG)$ are finitely generated.*

见 Kuku, A.O.:K_n, SK_n of integral group-ring and orders. Contemporary Mathematics Part I, 55, 333-338 (1986) 和 Kuku, A.O.:K-theory of group-rings of finite groups over maximal orders in division algebras. J. Algebra 91, 18-31 (1984).

Using the fundamental theorem for G-theory,

$$G_n(\Lambda[t]) = G_n(\Lambda)$$

$$G_n(\Lambda[t, t^{-1}]) = G_n(\Lambda) \oplus G_{n-1}(\Lambda)$$

one gets that:

Corollary 2.7. *For all $n \geq 1$, if C is a finitely generated free Abelian group or monoid, then $G_n(\Lambda[C])$ are also finitely generated.*

Remark 2.8. However we can not draw the same conclusion for $K_n(\Lambda[C])$ since for a ring A, it is known that all the $NK_n(A)$ are not finitely generated unless they are zero. 见 Weibel, C.A.: Mayer Vietoris sequences and module structures on NK_*, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 466-493,Lecture Notes in Math., 854, Springer, Berlin, 1981 的 Proposition 4.1

2.1.5 这篇文章的结果

第 1 节

Theorem 2.9 (1.1). *The set of all two-sided, α-invariant, Γ-lattices in Σ is a free Abelian group under multiplication and has* $\max_\alpha(\Gamma)$ *as a basis.*

Theorem 2.10 (1.6). *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\alpha : \Lambda \to \Lambda$ is an R-automorphism, then there exists an R-order $\Gamma \subset \Sigma$ such that*
(1) $\Lambda \subset \Gamma$,
(2) Γ is α-invariant, and
(3) Γ is a (right) regular ring. In fact, Γ is a (right) hereditary ring.

后面证明中反复用了这里的 Γ 是一个正则环。这两个定理推广了 Farrell 和 Jones 在文章 The Lower Algebraic K-Theory of Virtually Infinite Cyclic Groups.K-Theory 9, 13-30 (1995) 中的定理 1.5 和定理 1.2

Theorem 2.11 (Farrell-Jones 文章中的定理 1.5)**.** *The set of all two-sided, α-invariant, A-lattices in $\mathbb{Q}G$ is a free Abelian group under multiplication and has* $\max_\alpha(A)$ *as a basis.*

Theorem 2.12 (Farrell-Jones 文章中的定理 1.2)**.** *Given a finite group G and an automorphism $\alpha : G \to G$, then there exists a $\mathbb{Z}$-order $A \subset \mathbb{Q}G$ such that*
(1) $\mathbb{Z}G \subset A$,
(2) A is α-invariant, and
(3) A is a (right) regular ring, in fact, A is a (right) hereditary ring.

第一节的结论来源于 Farrell 和 Jones 在其文章中的结论，将 $\mathbb{Z}$ 和 $\mathbb{Q}$ 的陈述推广到数域 F 和代数整数环 R 上，并且把之前的群环 $\mathbb{Q}G$ 推广为任何半单 F 代数 Σ。

第 2 节 定理 2.1 中的方法是讲过的，关键一步是证两个范畴是自然等价。（文中有笔误:718 页第一行 mt^n 应为 xt^n, 后面所谓 m_i 应为 x_i，另有一处 Hom 所在的范畴不在 $\mathcal{B}$, 应在 $\mathcal{M}(A_\alpha[T])$）

Theorem 2.13 (2.2)**.** *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, α an automorphism of Λ. Then*

(a) For all $n \geq 0$
(i) $NK_n(\Lambda, \alpha)$ is s-torsion for some positive integer s. Hence the torsion free rank of $K_n(\Lambda_\alpha[t])$ is the torsion free rank of $K_n(\Lambda)$ and is finite.

If $n \geq 2$, then the torsion free rank of $K_n(\Lambda_\alpha[t])$ is equal to the torsion free rank of $K_n(\Sigma)$.
(ii) If G is a finite group of order r, then $NK_n(RG, \alpha)$ is r-torsion, where α is the automorphism of RG induced by that of G.

对第一类 *virtually infinite cyclic groups* 的结论*:*
(b) Let $V = G \rtimes_\alpha T$ be the semi-direct product of a finite group G of order r with an infinite cyclic group $T = \langle t \rangle$ with respect to the automorphism $\alpha : G \longrightarrow G, g \mapsto tgt^{-1}$. Then
(i) $K_n(RV) = 0$ for all $n < -1$.
(ii) The inclusion $RG \hookrightarrow RV$ induces an epimorphism $K_{-1}(RG) \twoheadrightarrow K_{-1}(RV)$. Hence $K_{-1}(RV)$ is finitely generated Abelian group.
(iii) For all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group.
(iv) $NK_n(RV)$ is r-torsion for all $n \geq 0$.

第 3 节 对第二类 virtually infinite cyclic groups 的结论:

Theorem 2.14 (3.2)**.** *If R is regular, then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$. If R is quasi-regular then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \leq 0$.*

Theorem 2.15 (3.3)**.** *Let V be a virtually infinite cylic group in the second class having the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. Then the Nil-groups*

$NK_n(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_1-H])$ defined by the triple $(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_1-H])$ are $|H|$-torsion when $n\geq 0$ and 0 when $n\leq -1$.

2.2 K-theory for the first type of virtually infinite cyclic groups

我们首先回顾 Farrell 和 Jones 在文章中的做法：

原型 G: finite group, $|G|=q$, $\mathbb{Z}G$ is a $\mathbb{Z}$-order in $\mathbb{Q}G$, then there exists a regular ring $A\subset\mathbb{Q}G$ which is a $\mathbb{Z}$-order, and we have[2] $qA\subset\mathbb{Z}G$.

Hence, we have the following Cartesian square

$$\begin{tikzcd}
\mathbb{Z}G \arrow[r] \arrow[d] & A \arrow[d] \\
\mathbb{Z}G/qA \arrow[r] & A/qA
\end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd}
\mathbb{Z}(G\rtimes_\alpha T) \arrow[r] \arrow[d] & A_\alpha[T] \arrow[d] \\
(\mathbb{Z}G/qA)_\alpha[T] \arrow[r] & (A/qA)_\alpha[T]
\end{tikzcd}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

Definition 2.16. A ring R is quasi-regular if it contains a two-sided nilpotent ideal N such that R/N is right regular.

重要的结论是

Prop1.1 If R is a (right) regular, $\alpha: R\longrightarrow R$ an automorphism, then $R_\alpha[t], R_\alpha[T]=R_\alpha[t,t^{-1}]$ are also (right) regular.

Prop1.4 $\mathbb{Z}G/qA, A/qA, (\mathbb{Z}G/qA)_\alpha[T], (A/qA)_\alpha[T]$ are all quasi-regular[3].

即得到的方块右上角是 regular ring，下方是 quasi-regular ring。于是得到 $K_n(\mathbb{Z}(G\rtimes_\alpha T))=0, n\leq 2$ 且有 $K_{-1}(\mathbb{Z}G)\twoheadrightarrow K_{-1}(\mathbb{Z}(G\rtimes_\alpha T))$ 是满射。

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[2]参考 Reiner, I.: Maximal Orders 中定理 41.1: $n=|G|$, Γ is an R-order in FG containing RG, then $RG\subset\Gamma\subset n^{-1}RG$ when $\mathrm{ch}(F)\nmid n$.

[3]If S is a quasi-regular ring, then $K_{-n}(S)=0$.(正确不？)

推广到数域 F 和代数整数环 R G: finite group, $|G| = s$, $\Lambda = RG$ is a R-order in $\Sigma = FG$, then there exists a regular ring $\Gamma \subset \Sigma = FG$ which is a R-order, and we have $s\Gamma \subset RG$.

Hence, we have the following Cartesian square

$$\begin{tikzcd} RG \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ RG/s\Gamma \arrow[r] & \Gamma/s\Gamma \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd} R(G \rtimes_\alpha T) \arrow[r] \arrow[d] & \Gamma_\alpha[T] \arrow[d] \\ (RG/s\Gamma)_\alpha[T] \arrow[r] & (\Gamma/s\Gamma)_\alpha[T] \end{tikzcd}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

对应到这里来$\Gamma, \Gamma_\alpha[T]$ 是正则环，$RG/s\Gamma, \Gamma/s\Gamma, (RG/s\Gamma)_\alpha[T], (\Gamma/s\Gamma)_\alpha[T]$ 是 quasi-regular rings.

群环推广到半单代数 考虑 $\Lambda \subset \Gamma \subset \Sigma$ 分别是 R-order, 正则环，半单 F-代数，则存在正整数 s 使得 $\Lambda \subset \Gamma \subset \Lambda(1/s)$, 令 $q = s\Gamma$

Hence, we have the following Cartesian square

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \Lambda/q \arrow[r] & \Gamma/q \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd} \Lambda_\alpha[t] \arrow[r] \arrow[d] & \Gamma_\alpha[t] \arrow[d] \\ (\Lambda/q)_\alpha[t] \arrow[r] & (\Gamma/q)_\alpha[t] \end{tikzcd}$$

写出 MV 序列后每项均 $\otimes\mathbb{Z}[1/s]$ 仍然正合[4]，再分别取核得到 Nil 群的长正合列。

$\Gamma, \Gamma_\alpha[t]$ regular $\Longrightarrow NK_n(\Gamma, \alpha) = 0$.

$\Lambda/q, \Gamma/q, (\Lambda/q)_\alpha[t], (\Gamma/q)_\alpha[t]$ are all quasi-regular.

Remark 2.17. Farrell，Jones 文章中四个环是 quasi-regular 的结论证明中用到了 Artinian 性质，从而可以推广到这篇文章所讨论的情形。

[4]文献 [16] 中是对素数 p 的陈述，对于一般的整数是否成立？

一些注记：1.A: finite, $J(A)$: its Jacobson radical, why is $A/J(A)$ regular？因为是有限环
2.720 页第四行的文献应为 [16], 引用的结论为“I is a nilpotent ideal in a $\mathbb{Z}/p^m\mathbb{Z}$-algebra Λ with unit, then $K_*(\Lambda, I)$ is a p-group”, 这个结论对一般的正整数 s 成立。同样地在 719 页得到序列 (III) 时同样参考 [16] 里的结论以及在 721 页倒数第 8 行所引用的 [16]Cor 3.3(d) 中的 p 对任何正整数成立。

原文中“By [9] the torsion free rank of $K_n(\Lambda)$ is finite and if $n \geq 2$ the torsion free rank of $K_n(\Sigma)$ is the torsion free rank of $K_n(\Lambda)$ (see [12])”引用的参考文献为

[9] van der Kallen, W.: Generators and relations in algebraic K-theory, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305-310, Acad. Sci.Fennica, Helsinki, 1980

[12] Kuku, A.O.: Ranks of K_n and G_n of orders and group rings of finite groups over integers in number fields. J. Pure Appl. Algebra 138, 39-44 (1999)
但在 [9] 中并未找到相应结论。

另外文献 [10] 在网上未找到电子文档。

Open problem: What is the rank of $K_{-1}(RV)$?

2.3 Nil-groups for the second type of virtually infinite cyclic groups

范畴 $\mathcal{T}$: 对象为 $\mathbf{R} = (R; B, C)$，其中 R 是环，B, C 是 R-双模，态射为 $(\phi, f, g) : (R, B, C) \to (S, D, E)$，其中 $\phi : R \to S$ 是环同态，$f : B \otimes_R S \to D$ 与 $g : C \otimes_R S \to E$ 是 R-S 双模同态。

$$\rho : \mathcal{T} \longrightarrow Rings$$

$$\rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(C \otimes_R B) & C \otimes_R T_R(B \otimes_R C) \\ B \otimes_R T_R(C \otimes_R B) & T_R(B \otimes_R C) \end{pmatrix}$$

If M is an R-module, then its tensor algebra $T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots$.

$$\varepsilon : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

$$NK_n(\mathbf{R}) := \ker(K_n(R_\rho) \to K_n \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix})$$

Let V be a group in the second class of the form $V = G_0 *_H G_0$ where the groups $G_i, i = 0, 1$,and H are finite and $[G_i : H] = 2$. Considering $G_i - H$ as the right coset of H in G_i which is different from H, the free $\mathbb{Z}$-module $\mathbb{Z}[Gi - H]$ with basis $G_i - H$nis a $\mathbb{Z}H$-bimodule which is isomorphic to $\mathbb{Z}H$ as a left $\mathbb{Z}H$-module, but the right action is twisted by an automorphism of $\mathbb{Z}H$ induced by an automorphism of H. Then the Waldhausen's Nil-groups are defined to be $NK_n(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_0-H])$ using the triple $(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_0-H])$. This inspires us to consider the following general case. Let R be a ring with identity and $\alpha : R \longrightarrow R$ a ring auto-morphism. We denote by R^α the R-bimodule which is R as a left R-module but with right multiplication given by $a \cdot r = a\alpha(r)$. For any automorphisms α and β of R , we consider the triple $\mathbf{R} = (R;R^\alpha,R^\beta)$. We will prove that $\rho(\mathbf{R})$ is in fact a twisted polynomial ring and this is important for later use.

Theorem 2.18 (3.1)**.** *Suppose that α and β are automorphisms of R. For the triple $\mathbf{R} = (R;R^\alpha,R^\beta)$, let R_ρ be the ring $\rho(\mathbf{R})$, and let γ be a ring automorphism of $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ defined by*

$$\gamma : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} \beta(b) & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Then there is a ring isomorphism

$$\mu : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x].$$

加法群同构是显然的，只需验证乘法同态。

利用这个结论将这种形式的 Nil 群转化为 Farrell Nil 群，利用已知的命题来证明结论。如正则环的 NK_n 为 0, 拟正则环的 NK_n 当 $n \leq 0$ 时为 0.

当我们接下来研究 $R = \mathbb{Z}H, h = |H|$ 时，取一个 regular order Γ，我们有相应的 4 triples，于是得到 4 个 twisted polynomial rings $R_\rho, \Gamma_\rho; (R/h\Gamma)_\rho, (\Gamma/h\Gamma)_\rho$.

之前第二节的方块

$$\begin{array}{ccc} RG & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ RG/s\Gamma & \longrightarrow & \Gamma/s\Gamma \end{array}$$

在这里 (之前的 R, G, s 换成 $\mathbb{Z}, H, h$) 变成了 (注意这里 $R = \mathbb{Z}H$)

$$\begin{array}{ccc} R = \mathbb{Z}H & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathbb{Z}H/h\Gamma & \longrightarrow & \Gamma/h\Gamma \end{array}$$

从而有

$$\begin{array}{ccc} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} & \longrightarrow & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix} & \longrightarrow & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix} \end{array}$$

接着有

$$
\begin{tikzcd}
\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x] \arrow[r] \arrow[d] & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}_\gamma [x] \arrow[d] \\
\begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix}_\gamma [x] \arrow[r] & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix}_\gamma [x]
\end{tikzcd}
$$

而这个方块恰好是

$$
\begin{tikzcd}
R_\rho \arrow[r] \arrow[d] & \Gamma_\rho \arrow[d] \\
(R/h\Gamma)_\rho \arrow[r] & (\Gamma/h\Gamma)_\rho
\end{tikzcd}
$$

证明中使用了 $n \leq -1$ 时 quasi-regular ring 的 NK_n 为 0.

Remark 2.19. 722 页中间参考文献 [3] 未找到 augmentation map。另外这里把 f, g 是双模同态在原文基础上进行了修改。

726 页第 8 行“(2) and (3)”应为“(3) and (4)”。

Chapter 3

Witt vectors and NK-groups

References:

part 1 J. P. Serre, Local fields.

part 1 Daniel Finkel, An overview of Witt vectors.

part 2 Hendrik Lenstra, Construction of the ring of Witt vectors.

part 2 Barry Dayton, Witt vectors, the Grothendieck Burnside ring, and Necklaces.

part 3 C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.

part 3 D. R. Grayson, Grothendieck rings and Witt vectors.

part 3 C. A. Weibel, The K-Book: An introduction to algebraic K-theory.

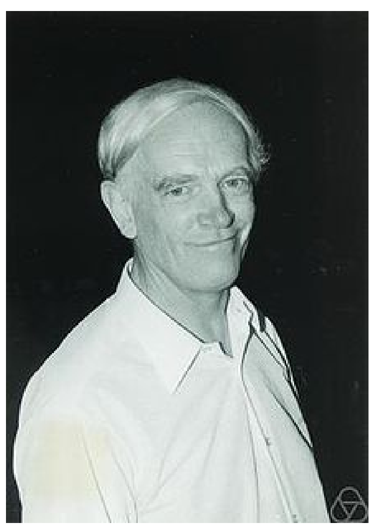

图 3.1: Ernst Witt

Ernst Witt Ernst Witt (June 26, 1911–July 3, 1991) was a German mathematician, one of the leading algebraists of his time. Witt completed his Ph.D. at the University of Göttingen in 1934 with Emmy Noether. Witt's work has been highly influential. His invention of the Witt vectors clarifies and generalizes the structure of the p-adic numbers. It has become fundamental to p-adic Hodge theory. For more information, see `https://en.wikipedia.org/wiki/Ernst_Witt` and `http://www-history.mcs.st-andrews.ac.uk/Biographies/Witt.html`.

3.1 p-Witt vectors

In this section we introduce p-Witt vectors. Witt vectors generalize the p-adics and we will see all p-Witt vectors over any commutative ring form a ring.

From now on, fix a prime number p.

Definition 3.1. A p-Witt vector over a commutative ring R is a sequence $(X_0, X_1, X_2, \cdots)$ of elements of R.

Remark 3.2. If $R = \mathbb{F}_p$, any p-Witt vector over $\mathbb{F}_p$ is just a p-adic integer $a_0 + a_1 p + a_2 p^2 + \cdots$ with $a_i \in \mathbb{F}_p$.

We introduce Witt polynomials in order to define ring structure on p-Witt vectors.

Definition 3.3. Fix a prime number p, let $(X_0, X_1, X_2, \cdots)$ be an infinite sequence of indeterminates. For every $n \geq 0$, define the n-th Witt polynomial

$$W_n(X_0, X_1, \cdots) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

For example, $W_0 = X_0,\ W_1 = X_0^p + pX_1,\ W_2 = X_0^{p^2} + pX_1^p + p^2 X_2$.

Question: how can we add and multiple Witt vectors?

Theorem 3.4. *Let $(X_0, X_1, X_2, \cdots)$, $(Y_0, Y_1, Y_2, \cdots)$ be two sequences of indeterminates. For every polynomial function $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \cdots, \varphi_n, \cdots)$ of elements of $\mathbb{Z}[X_0, \cdots, X_n, \cdots; Y_0, \cdots, Y_n, \cdots]$ such that*

$$W_n(\varphi_0, \cdots, \varphi_n, \cdots) = \Phi\big(W_n(X_0, \cdots), W_n(Y_0, \cdots)\big),\ n = 0, 1, \cdots.$$

If $\Phi = X + Y$(resp. XY), then there exist $(S_1, \cdots, S_n, \cdots)$("$S$" stands for sum) and $(P_1, \cdots, P_n, \cdots)$ ("P" stands for product) such that

$$W_n(X_0, \cdots, X_n, \cdots) + W_n(Y_0, \cdots, Y_n, \cdots) = W_n(S_1, \cdots, S_n, \cdots),$$

$$W_n(X_0,\cdots,X_n,\cdots)W_n(Y_0,\cdots,Y_n,\cdots)=W_n(P_1,\cdots,P_n,\cdots).$$

Let R be a commutative ring, if $A=(a_0,a_1,\cdots)\in R^{\mathbb{N}}$ and $B=(b_0,b_1,\cdots)\in R^{\mathbb{N}}$ are p-Witt vectors over R, we define

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots).$$

Theorem 3.5. *The p-Witt vectors over any commutative ring R form a commutative ring under the compositions defined above (called the ring of p-Witt vectors with coefficients in R, denoted by $W(R)$).*

Example 3.6. We have

$$\begin{array}{ll} S_0(A,B)=a_0+b_0 & P_0(A,B)=a_0b_0 \\ S_1(A,B)=a_1+b_1+\frac{a_0^p+b_0^p-(a_0+b_0)^p}{p} & P_1(A,B)=b_0^pa_1+a_0^pb_1+pa_1b_1 \end{array}$$

For more computations, see MO 92750

Theorem 3.7. *There is a ring homomorphism*

$$W_*\colon W(R)\longrightarrow R^{\mathbb{N}}$$
$$(X_0,X_1,\cdots,X_n,\cdots)\mapsto(W_0,W_1,\cdots,W_n,\cdots)$$

Proof. Only need to check this map is a ring homomorphism: since

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots),$$

by definition we have

$$\begin{aligned} W(A)+W(B)&=(W_0(A)+W_0(B),W_1(A)+W_1(B),\cdots)\\ &=\big(W_0(S_0(A,B),S_1(A,B),\cdots),W_1(S_0(A,B),S_1(A,B),\cdots),\cdots\big)\\ &=W(S_0(A,B),S_1(A,B),\cdots)=W(A+B). \end{aligned}$$

And similarly,

$$\begin{aligned} W(A)W(B)&=(W_0(A)W_0(B),W_1(A)W_1(B),\cdots)\\ &=\big(W_0(P_0(A,B),P_1(A,B),\cdots),W_1(P_0(A,B),P_1(A,B),\cdots),\cdots\big)\\ &=W(P_0(A,B),P_1(A,B),\cdots)=W(AB). \end{aligned}$$

Indeed, we only need to show $W_n(A)+W_n(B)=W_n(A+B)$ and $W_n(A)W_n(B)=W_n(AB)$ which are obviously true. (实际上就是为了使得这个是同态而定义出了 $A+B$ 和 AB。) □

Example 3.8. 1. If p is invertible in R, then $W(R)=R^{\mathbb{N}}$ — the product of countable number of R.(if p is invertible the homomorphism W_* is an isomorphism.)

2. $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$ — the ring of p-adic integers.
3. $W(\mathbb{F}_{p^n})$ is an unramified extension of the ring of p-adic integers.

Note that the functions P_k and S_k are actually only involve the variables of index $\leqslant k$ of A and B. In particular if we truncate all the vectors at the k-th entry, we can still add and multiply them.

Definition 3.9. Truncated p-Witt ring $W_k(R) = \{(a_0, a_1, \cdots, a_{k-1}) | a_i \in R\}$ (also called the ring of Witt vectors of length k.)

Example 3.10. $W_1(R) = R$, $W(R) = \varprojlim W_k(R)$. Since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$.

Definition 3.11. We define two special maps as follows

- The "shift" map $V\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$, this map is *additive*.
- When $char(R) = p$, the "Frobenius" map $F\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots)$, this is indeed a ring homomorphism.

Firstly, we note that $W_k(R) = W(R)/V^kW(R)$, and if we consider $V\colon W_n(R) \hookrightarrow W_{n+1}(R)$ there are exact sequences

$$0 \longrightarrow W_k(R) \xrightarrow{V^r} W_{k+r}(R) \longrightarrow W_r(R) \longrightarrow 0, \quad \forall k,\, r.$$

The map $V\colon W(R) \longrightarrow W(R)$ is additive: for it suffices to verify this when p is invertible in R, and in that case the homomorphism $W_*\colon W(R) \longrightarrow R^{\mathbb{N}}$ transforms V into the map which sends $(w_0, w_1, \cdots)$ to $(0, pw_0, pw_1, \cdots)$.

$$\begin{tikzcd}
W(R) \arrow[r, "V"] \arrow[d, "W_*"] & W(R) \arrow[d, "W_*"] \\
R^{\mathbb{N}} \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
(a_0, a_1, \cdots) \arrow[r, mapsto, "V"] \arrow[d, mapsto, "W_*"] & (0, a_0, a_1, \cdots) \arrow[d, mapsto, "W_*"] \\
(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \cdots) \arrow[r, mapsto] \arrow[d, equal] & (0, pa_0, pa_0^p + p^2a_1, \cdots) \arrow[d, equal] \\
(w_0, w_1, w_2, \cdots) \arrow[r, mapsto] & (0, pw_0, pw_1, \cdots)
\end{tikzcd}$$

If $x \in R$, define a map

$$\begin{aligned} r\colon R &\longrightarrow W(R) \\ x &\mapsto (x, 0, \cdots, 0, \cdots) \end{aligned}$$

When p is invertible in R, W_* transforms r into the mapping that $x \mapsto (x, x^p, \cdots, x^{p^n}, \cdots)$.

$$\begin{array}{ccc} R & \longrightarrow & W(R) \\ \Big\downarrow \text{id} & & \Big\downarrow W_* \\ R & \longrightarrow & R^{\mathbb{N}} \end{array}$$

$$\begin{array}{ccc} x & \longmapsto & (x, 0, \cdots, 0, \cdots) \\ \Big\| & & \Big\downarrow W_* \\ x & \longmapsto & (x, x^p, \cdots, x^{p^n}, \cdots) \end{array}$$

One deduces by the same reasoning as above the formulas:

Proposition 3.12.

$$r(xy) = r(x)r(y),\ x, y \in R$$
$$(a_0, a_1, \cdots) = \sum_{n=0}^{\infty} V^n(r(a_n)),\ a_i \in R$$
$$r(x)(a_0, \cdots) = (xa_0, x^p a_1, \cdots, x^{p^n} a_n, \cdots),\ x_i, a_i \in R.$$

Proof. The first formula: put $r(x)r(y),\ r(xy)$ to $R^{\mathbb{N}}$, we get $(x, x^p, \cdots, x^{p^n}, \cdots)(y, y^p, \cdots, y^{p^n}, \cdots)$ and $(xy, (xy)^p, \cdots, (xy)^{p^n}, \cdots)$.
The second formula: put $(a_0, a_1, \cdots)$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \cdots)$
consider $V^i(r(a_i))$: put $r(a_i)$ to $R^{\mathbb{N}}$, we get $(a_i, a_i^p, \cdots, a_i^{p^n}, \cdots) \in R^{\mathbb{N}}$, and W_* transforms V to the mapping $(w_0, w_1, \cdots, w_n, \cdots) \mapsto (0, pw_0, \cdots, pw_{n-1}, \cdots)$,
now we put $(r(a_0))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p, \cdots, a_0^{p^n}, \cdots)$
put $V^1(r(a_1))$ to $R^{\mathbb{N}}$, we get $(0, pa_1, \cdots, pa_1^{p^{n-1}}, \cdots)$
put $V^2(r(a_2))$ to $R^{\mathbb{N}}$, we get $(0, 0, p^2 a_2, \cdots, p^2 a_2^{p^{n-2}}, \cdots)$
put $V^i(r(a_i))$ to $R^{\mathbb{N}}$, we get $(\underbrace{0, 0, \cdots, 0}_{i \text{ terms}}, p^i a_i, \cdots, p^i a_i^{p^{n-i}}, \cdots)$
so put $\sum_n V^n(r(a_n))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, \cdots)$.
We leave the proof of the last formula to readers. $\square$

Proposition 3.13.

$$VF = p = FV.$$

Proof. It suffices to check this when R is perfect. Note that a ring R of characteristic p is called perfect if $x \mapsto x^p$ is an automorphism. For more details, we refer to page 44 on Serre's book *Local Fields*. $\square$

3.2 Big Witt vectors

Now we turn to the big(universal) Witt vectors. J. P, May once said "This(the theory of Witt vectors) is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once".

Take the ring of all big vectors of a commutative ring is a functor

$$\mathbf{CRing} \longrightarrow \mathbf{CRing}$$
$$R \mapsto W(R).$$

In this section, R is a commutative ring with unit.

Definition 3.14. The ring of all big Witt vectors in R which also denoted by $W(R)$ is defined as follows,
as a set: $W(R) = \{a(T) \in R[\![T]\!] | a(T) = 1 + a_1T + a_2T^2 + \cdots\} = 1 + TR[\![T]\!]$; (we note that as a set $W(R)$ is the kernel of the map $A[\![T]\!]^* \xrightarrow{T\mapsto 0} A^*$)
addition in $W(R)$: usual multiplication of formal power series, sum $a(T)b(T)$, difference $\frac{a(T)}{b(T)}$; $(W(R), +) \cong (1 + TR[\![T]\!], \times)$ which is a subgroup of the group of units $R[\![T]\!]^\times$ of the ring $R[\![T]\!]$
multiplication in $W(R)$: denoted by $*$, this is a little mysterious, we will talk the details later. For the present purposes we only define $*$ as the unique continuous functorial operation for which $(1 - aT) * (1 - bT) = (1 - abT)$.
'zero'(additive identity) of $W(R)$: 1.
'one'(multiplicative identity) of $W(R)$: $[1] = 1 - T$. Note that $[1]$ is the image of $1 \in R$ under the multiplicative (Teichmuller) map

$$R \longrightarrow W(R)$$
$$a \mapsto [a] = 1 - aT$$

functoriality: any homomorphism $f\colon R \longrightarrow S$ induces a ring homomorphism

$$W(f)\colon W(R) \longrightarrow W(S).$$

A quick way to check multiplicative formulas in $W(R)$ is to use the ghost map (indeed a ring homomorphism)

$$gh\colon W(R) \longrightarrow R^{\mathbb{N}} = \prod_i^{\infty} R.$$

It is obtained from the abelian group homomorphism

$$-T\frac{d}{dT}\log\colon (1 + TR[\![T]\!])^\times \longrightarrow (TR[\![T]\!])^+$$
$$a(T) \mapsto -T\frac{a'(T)}{a(T)}$$

the right side of gh is $R^{\mathbb{N}}$ via $\sum a_n t^n \longleftrightarrow (a_1, a_2, \cdots)$.

A basic principle of the theory of Witt vectors is that to demonstrate certain equations it suffices to check them on vectors of the form $1 - aT$.

3.3 Module structure on NK_*

Notations Λ: a ring with 1

R: commutative ring

$W(R)$: the ring of big Witt vectors of R

$\mathbf{End}(\Lambda)$: the exact category of endomorphisms of finitely generated projective right Λ-modules.

$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.

$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules.

The fundamental theorem in algebraic K-theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus NK_i(\Lambda) \cong K_i(\Lambda) \oplus \mathrm{Nil}_{i-1}(\Lambda),$$

and hence $\mathrm{Nil}(\Lambda)$ is the obstruction to K-theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (right) Λ-module has a finite projective resolution. So the resolution theorem and the fact that G-theory is homotopy invariant show that for a regular ring, $NK_*(\Lambda) = \mathrm{Nil}_{*-1}(\Lambda) = 0$. In general, one knows that the groups $\mathrm{Nil}_*(\Lambda)$, if non-zero, are infinitely generated. It is also known that the groups $\mathrm{Nil}_*(\Lambda)$ are modules over the big Witt ring $W(R)$ (just this notes want to show you).

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.
- Computations in $W(R)$ with Grothendieck rings.

3.3.1 $\mathrm{End}_0(\Lambda)$

Let $\mathbf{End}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules.

Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.

Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ M_2 & \xrightarrow{f_2} & M_2 \end{array}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —

$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.

$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathcal{C} \rightleftarrows \mathcal{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \leftarrow\!\shortmid (M, f)$$

Definition 3.15. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) \longrightarrow \mathbf{End}(\Lambda)$$
$$\otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) \longrightarrow \mathbf{Nil}(\Lambda)$$
$$(M, f) \otimes (N, g) = (M \otimes_R N, f \otimes g)$$

These induce (use "generators-and-relations" tricks on K_0)

$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) \longrightarrow K_*(\mathbf{End}(\Lambda))$$
$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) \longrightarrow K_*(\mathbf{Nil}(\Lambda))$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M, f) \in \mathbf{End}(R)$, then $(M, f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R, 1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R, 0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R, 0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R, 0) \otimes -\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. (Consider $P \in \mathbf{P}(R)$ acts on $\mathbf{End}(\Lambda)$, $(P,0) \otimes (N,g) = (P \otimes_R N, 0) \in \mathbf{P}(\Lambda)$.)

The following is immediate (and well-known):

Proposition 3.16. *If Λ is an R-algebra with* 1, $\mathrm{End}_*(\Lambda)$ *and* $\mathrm{Nil}_*(\Lambda)$ *are graded modules over the ring* $\mathrm{End}_0(R)$.

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 3.17 (Almkvist)**.** *The homomorphism (in fact it is a ring homomorphism)*

$$\chi\colon \mathrm{End}_0(R) \longrightarrow W(R) = (1 + TR[\![T]\!])^{\times}$$
$$(M,f) \mapsto \det(1 - fT)$$

is injective and $\mathrm{End}_0(R) \cong \mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1 - T$$

χ is a ring homomorphism, and $\mathrm{Im}\chi =$ the set of all rational functions in $W(R)$. Note that

$$\det(1 - fT)\det(1 - gT) = \det(1 - (f \oplus g)T), \quad \det(1 - fT) * \det(1 - gT) = \det(1 - (f \otimes g)T),$$

for more details we refer the reader to S.Lang *Algebra*, Chapter 14, Exercise 15.

Remark 3.18. when R is a algebraically closed field (for instance $\mathbb{C}$), we can use Jordan canonical forms to prove the last identity (i.e. it can be reduced to check that $\prod_i(1 - \lambda_i T) * \prod_j(1 - \mu_j T) = \prod_{i,j}(1 - \lambda_i \mu_j T)$).

Definition 3.19 (NK_*)**.** As above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in "Higher algebraic K-theory II". The map is given by

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/xy = 1) \xrightarrow{\partial} K_{n-1}(\mathbf{Nil}(\Lambda)).$$

Thus $NK_n(\Lambda)$ are $\mathrm{End}_0(R)$-modules. For $n \geq 1$, this is just 3.16; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the "contracted functor" of $NK_1(\Lambda)$.

Note that $NK_1(\Lambda) \cong K_1(\Lambda[y], (y - \lambda)), \forall \lambda \in \Lambda$, since

$$\Lambda[y] \rightleftarrows \Lambda$$
$$y \mapsto \lambda.$$

Since $[(P, \nu)] = [(P \oplus Q, \nu \oplus 0)] - [(Q, 0)] \in K_0(\mathbf{Nil}(\Lambda))$, we see $\mathrm{Nil}_0(\Lambda)$ is generated by elements of the form $[(\Lambda^n, \nu)] - n[(\Lambda, 0)]$ for some n and some nilpotent matrix ν Sign convention:

$$NK_1(\Lambda) \cong \mathrm{Nil}_0(\Lambda)$$
$$[1 - \nu y] \leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)]$$

Example 3.20. Let k be a field, $\mathbf{End}(k)$ consists pairs (V, A) with V a finite-dimensional vector space over k and A a k-endomorphism. Two pairs are isomorphic if and only if their minimal polynomials are equal. When we consider $\mathbf{Nil}(k)$, then $K_0(\mathbf{Nil}(k)) \cong \mathbb{Z}$, we conclude that $\mathrm{Nil}_0(k) = 0$. Recall that since k is a regular ring, $NK_*(k) = 0$, we have another proof of $NK_1(k) \cong \mathrm{Nil}_0(k) = 0$.

3.3.2 Grothendieck rings and Witt vectors

We refer the reader to Grayson's paper *Grothendieck rings and Witt vectors*.

Definition 3.21. A λ-ring R is a commutative ring with 1, together with an operation λ_t which assigns to each element x of R a power series

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots, \quad x \in R$$

This operation must obey $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$.

Let R is a commutative ring with unit, $K_0(R) = K_0(\mathbf{P}(R))$ becomes a λ-ring if we define

$$[M][N] = [M \otimes_R N], \quad \lambda_t^n([M]) = [\wedge_R^n M].$$

Recall $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$, $\wedge^n(M \oplus M') \cong \bigoplus_{i+j=n} \wedge^i M \otimes \wedge^j M'$, and $\wedge^n(M) := M^{\otimes n} / \langle x \otimes x \mid x \in M \rangle$, rank $\wedge^n (M) = \binom{\text{rank } M}{n}$.

For instance, if R is a field, $K_0(R) = \mathbb{Z}$ and $\lambda_t(n) = (1 + t)^n = 1 + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$, since $\dim(\wedge^i R^n) = \binom{n}{i}$.

We make $K_0(\mathbf{End}(R))$ into a λ-ring by defining

$$\lambda^n([M, f]) = ([\wedge^n M, \wedge^n f]).$$

The ideal generated by the idempotent $[(R, 0)]$ is isomorphic to $K_0(R)$, the quotient $\mathrm{End}_0(R)$ is a λ-ring. It is convenient to think of End_0 as a convariant functor on the category of rings, and the functor End_0 satisfies:

1. If $R \longrightarrow S$ is surjective ring homomorphism, then $\mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(S)$ is surjective.
2. If R is an algebraically close field, then the group $\mathrm{End}_0(R)$ is generated by the elements of the form $[(R, r)]$. (This holds because any matrix over R is triagonalizable.)

Recall

$$\chi\colon \mathrm{End}_0(R) \longrightarrow W(R) = 1 + TR[\![T]\!]$$
$$(M, f) \mapsto \det(1 - fT)$$

$W(R)$ is the underlying (additive) group of the ring of Witt vectors. The λ-ring operations on $W(R)$ are the unique operations which are continuous, functorial in R, and satisfy:

$$(1 - aT) * (1 - bT) = 1 - abT$$
$$\lambda_t(1 - aT) = 1 + (1 - aT)t$$

By 3.17, χ is an injective ring homomorphism whose image consists of all Witt vectors which are quotients of polynomials. In fact χ is a λ-ring homomorphism, so we have

Theorem 3.22. $\mathrm{End}_0(R)$ *is dense sub-λ-ring of* $W(R)$.

The hard part of the theorem is the injectivity. When R is a field the injectivity follows immediately from the existence of the rational canonical form (we can see it below) for a matrix. The result is surprising when R is not a field.

Computation in $W(R)$ Computation in $W(R)$ which is tedious unless we perform it in $\mathrm{End}_0(R)$:

$$(1 - aT^2) * (1 - bT^2) = ?$$

Note that $\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) = \det\begin{pmatrix} 1 & -aT \\ -T & 1 \end{pmatrix} = 1 - aT^2$, $\chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \det\begin{pmatrix} 1 & -bT \\ -T & 1 \end{pmatrix} = 1 - bT^2$,

$$\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) * \chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Big(\begin{pmatrix} 0 & 0 & 0 & ab \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\Big)$$

$$= \det\begin{pmatrix} 1 & 0 & 0 & -aTb \\ 0 & 1 & -bT & 0 \\ 0 & -aT & 1 & 0 \\ -T & 0 & 0 & 1 \end{pmatrix} = 1 - 2abT^2 + a^2b^2T^4.$$

If we use the previous formula

$$(1 - rT^m) * (1 - sT^n) = (1 - r^{n/d}s^{m/d}T^{mn/d})^d, \; d = \gcd(m, n),$$

we obtain the same answer. Indeed we have the formula

$$\det(1 - fT) * \det(1 - gT) = \det(1 - (f \otimes g)T),$$

if a polynomial is $1 + a_1T + \cdots + a_nT^n \in W(R)$, we can write

$$f = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix} \in M_n(R).$$

Operations on $W(R)$ **and** $\mathrm{End}_0(R)$ We have already known that W and End_0 can be regarded as functors from the category of commutative rings to that of (λ-)rings. The following operations $F_n, \; V_n \colon W \Longrightarrow W$(resp. $\mathrm{End}_0 \Longrightarrow \mathrm{End}_0$) are indeed natural transformation. These auxiliary operations defined on $W(R)$ can also be computed in $\mathrm{End}_0(R)$.

1. the ghost map

$$gh \colon W(R) \xrightarrow{-T\frac{d}{dT}\log} TR[\![T]\!] \cong R^{\mathbb{N}}, \quad \alpha(T) \mapsto \frac{-T}{\alpha(T)}\frac{d\alpha}{dT}.$$

   and the n-th ghost coordinate

$$gh_n \colon W(R) \longrightarrow R$$

   it is the unique continuous natual additive map which sends $1 - aT$ to a^n.
   Remark. $gh(1 - aT) = \frac{aT}{1-aT} = \sum_{i=1} a^iT^i$.The exponential map is defined by

$$R^{\mathbb{N}} \longrightarrow W(R), \quad (r_1, \cdots) \mapsto \prod_{i=1}^{\infty} \exp(\frac{-r_iT^i}{i}).$$

2. the Frobenius endomorphism

$$F_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}}).$$

   it is the unique continuous natual additive map which sends $1 - aT$ to $1 - a^nT$.
   Remark. $F_n(1 - aT) = \sum_{\zeta^n=1}(1 - a\zeta T^{\frac{1}{n}}) = 1 - a^nT$, since "+" in $W(R)$ is the normal product.

3. the Verschiebung endomorphism

$$V_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \alpha(T^n).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - aT^n$.

ghost map $gh_n \colon W(R) \longrightarrow R$	$1 - aT \mapsto a^n$	
Frobenius endomorphism $F_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - a^n T$	$\alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}})$
Verschiebung endomorphism $V_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - aT^n$	$\alpha(T) \mapsto \alpha(T^n)$

We define similar operations on $\mathrm{End}_0(R)$ as follows:

$gh_n \colon \mathrm{End}_0(R) \longrightarrow R$	$[(M, f)] \mapsto \mathrm{tr}(f^n)$
$F_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M, f^n)]$
$V_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M^{\oplus n}, v_n f)]$

where $v_n f$ is represented by $\begin{pmatrix} 0 & & & f \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$. The matrix $v_n f$ is close to an n-th root of f.

Another equivalent description is

$$V_n \colon [(M, f)] \mapsto [(M[y]/y^n - f, y)].$$

One easily checks that the operations just defined are additive with respect to short exact sequences in $\mathbf{End}(R)$, and thus are well-defined on $\mathrm{End}_0(R)$.

Since $\mathrm{End}_0(R) \subset W(R)$ is dense and $gh_n,\ F_n,\ V_n$ are continuous, identities among them may be verified on $W(R)$ by checking them on $\mathrm{End}_0(R)$.

$W(R)$	$\longleftrightarrow$	$\mathrm{End}_0(R)$
$gh_n(v * w) = gh_n v * gh_n w$		$\mathrm{tr}((f \otimes g)^n) = \mathrm{tr}(f^n)\mathrm{tr}(g^n)$
$F_n(v * w) = F_n v * F_n w$		$(f \otimes g)^n = f^n \otimes g^n$
$F_n V_n = n$		$(v_n f)^n = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$
$gh_n V_d(v) = \begin{cases} d\, gh_{n/d}(v), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$		$\mathrm{tr}((v_d f)^n) = \begin{cases} d\, \mathrm{tr}(f^{n/d}), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$

From the last row we may recover the usual expression of the ghost coordinates in terms of the Witt coordinates . The Witt coordinates of a vector v are the coefficients in the expression

$$v = \prod_{i=1} (1 - a_i T^i) = \prod_{i=1} V_i(1 - a_i T).$$

We obtain

$$gh_n(v) = \sum_{d|n} da_d^{n/d}.$$

"Many mordern treatments of the subject of Witt vectors take this latter expression as the starting point of the theory."

The logarithmic derivative of $1 - a_d T^d$ is $\frac{d}{dT}\log(1 - a_d T^d) = -\sum_{m=1}^{\infty} da_d^m T^{dm-1}$, and $-T\frac{d}{dT}\log(1 - a_d T^d) = \sum_{n=1}^{\infty} gh_n(1 - a_d T^d)T^n$. So we obtain the formula:

$$-Tv^{-1}\frac{dv}{dT} = \sum_{n=1}^{\infty} gh_n(v)T^n$$

which yields the exponential trace formula:

$$-T\chi([M,f])^{-1}\frac{d\chi([M,f])}{dT} = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n.$$

For example, when rank $M = 2$, we have $\mathrm{tr}(f^2) = (\mathrm{tr}(f))^2 - 2\det(f)$, note that $\det(1 - fT) = 1 - \mathrm{tr}(f)T + \det(f)T^2$.

Remark 3.23. When R is a field, the exponential trace formula

$$-T\frac{d}{dT}\log\det(1 - fT) = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n$$

can be checked by $\det(1 - fT) = \prod(1 - \lambda_i T)$ where λ_i are eigenvalues. And we also have

$$\det(1 - fT) = \exp(\sum_{n=1}^{\infty} -\mathrm{tr}(f^n)\frac{T^n}{n}),$$

since $\prod(1 - \lambda_i T) = \exp\left(\ln(\prod(1 - \lambda_i T))\right) = \exp\left(\sum\ln(1 - \lambda_i T)\right)$ and recall that formally $\ln(1 - x) = -\sum\frac{x^n}{n}$.

3.3.3 $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Recall Λ is an R-algebra, where R is a commutative ring with unit. We define a map

$$\mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n, f) * [(P, \nu)] = [(P^n, f\nu)]$$

Let $\alpha_n = \alpha_n(a_1, \cdots, a_n)$ denote the $n \times n$ matrix (looks like the rational canonical form) over R:

$$\alpha_n(a_1, \cdots, a_n) = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

Recall

$$\chi\colon \mathrm{End}_0(R) \rightarrowtail W(R)$$
$$(R^n,\alpha_n) \mapsto \det(1-\alpha_n T)$$

we obtain

$$\det\begin{pmatrix} 1 & & & & a_nT \\ -T & 1 & & & a_{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2T \\ & & & -T & 1+a_1T \end{pmatrix} = 1+a_1T+\cdots+a_nT^n$$

(Computation methods: 1. the traditional computation. 2. Note that if A is invertible,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D-CA^{-1}B).$$

In this case $A^{-1} = \begin{pmatrix} 1 & & & & \\ T & 1 & & & \\ \vdots & \ddots & \ddots & & \\ T^{n-3} & \cdots & T & 1 & \\ T^{n-2} & T^{n-3} & \cdots & T & 1 \end{pmatrix}$)

Then we can also conclude that $\mathrm{Im}\chi = \left\{\frac{g(T)}{h(T)} \in W(R) \mid g(T),h(T) \in 1+TR[T]\right\}$.

Remark 3.24. Why is a general elment of the form (R^n,α_n)? Namely how to reduce an endomorphism to a rational canonical form ?

Now we want to check some identities

$$\mathrm{End}_0(R)\times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n,\alpha_n)*[(P,\nu)] = [(P^n,\alpha_n\nu)] \quad \text{by definition}$$
$$(R^{n+1},\alpha_{n+1}(a_1,\cdots,a_n,0))*[(P,\nu)] = (R^n,\alpha_n)*[(P,\nu)] \quad \text{compute under } \chi$$
$$(R^n,\alpha_n)*[(P,\nu)] = [(P^n,\beta)] \quad \text{where } \beta = \alpha_n(a_1\nu,\cdots,a_n\nu^n)$$

In fact, the last identity always holds when $R=\mathbb{Z}[a_1,\cdots,a_n]$. β is nilpotent because $\beta=\alpha_n\nu$.

We only show how to check the last equation: only need to show that

$$\alpha_n\nu = \alpha_n(a_1\nu,\cdots,a_n\nu^n)$$

$$LHS = \begin{pmatrix} 0 & & & & -a_n\nu \\ \nu & 0 & & & -a_{n-1}\nu \\ & \ddots & \ddots & & \vdots \\ & & \nu & 0 & -a_2\nu \\ & & & \nu & -a_1\nu \end{pmatrix}, \quad RHS = \begin{pmatrix} 0 & & & & -a_n\nu^n \\ 1 & 0 & & & -a_{n-1}\nu^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2\nu^2 \\ & & & 1 & -a_1\nu \end{pmatrix}$$

we can check this using the characteristic polynomial since χ is injective: check

$$\det(1 - \alpha_n \nu T) = \det(1 - \alpha_n(a_1\nu, \cdots, a_n\nu^n)T)$$

$$LHS = \det \begin{pmatrix} 1 & & & & a_n\nu T \\ -\nu T & 1 & & & a_{n-1}\nu T \\ & \ddots & \ddots & & \vdots \\ & & -\nu T & 1 & a_2\nu T \\ & & & -\nu T & 1 + a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n)$$

$$RHS = \det \begin{pmatrix} 1 & & & & a_n\nu^n T \\ -T & 1 & & & a_{n-1}\nu^{n-1} T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2\nu^2 T \\ & & & -T & 1 + a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n).$$

Note that if $\exists N$ such that $\nu^N = 0$, β is independent of the a_i for $i \geq N$. If $\nu^N = 0$ then $\alpha_n \otimes \nu$ represents 0 in $\mathrm{Nil}_0(\Lambda)$ whenever $\chi(\alpha_n) \equiv 1 \bmod t^N$.

More operations Let $F_n\mathbf{Nil}(\Lambda)$ denote the full exact subcategory of $\mathbf{Nil}(\Lambda)$ on the (P, ν) with $\nu^n = 0$. If Λ is an algebra over a commutative ring R, the kernel $F_n\mathrm{Nil}_0(\Lambda)$ of $K_0(F_n\mathbf{Nil}(\Lambda)) \longrightarrow K_0(\mathbf{P}(\Lambda))$ is an $\mathrm{End}_0(R)$-module and $F_n\mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$ is a module map.

The exact endofunctor $F_m \colon (P, \nu) \mapsto (P, \nu^m)$ on $\mathbf{Nil}(\Lambda)$ is zero on $F_m\mathbf{Ni1}(\Lambda)$. For $\alpha \in \mathrm{End}_0(R)$ and $(P, \nu) \in \mathrm{Nil}_0(\Lambda)$, nota that $(V_m\alpha) * (P, \nu) = V_m(\alpha * F_m(P, \nu))$, and we can conclude that $V_m\mathrm{End}_0(R)$ acts trivially on the image of $F_m\mathrm{Nil}_0(\lambda)$ in $\mathrm{Nil}_0(\lambda)$. For more details, see Weibel, *K*-book chapter 2, pp 155 Exercise II.7.17.

3.3.4 $W(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Once we have a nilpotent endomorphism, we can pass to infinity.

Theorem 3.25. $\mathrm{End}_0(R)$*-module structure on* $\mathrm{Nil}_0(\Lambda)$ *extends to a* $W(R)$*-module structure by the formula*

$$(1 + \sum a_i T^i) * [(P, \nu)] = (R^n, \alpha_n(a_1, \cdots, a_n)) * [(P, \nu)], \; n \gg 0.$$

3.3.5 $W(R)$-module structure on $\mathrm{Nil}_*(\Lambda)$

The induced t-adic topology on $\mathrm{End}_0(R)$ is defined by the ideals

$$I_N = \{f \in \mathrm{End}_0(R) \mid \chi(f) \equiv 1 \bmod t^N\},\ I_N \supset I_{N+1},$$

and $\mathrm{End}_0(R)$ is separated (i.e. $\cap I_N = 0$) in this topology. The key fact is:

Theorem 3.26 (Almkvist). *The map $\chi\colon \mathrm{End}_0(R) \longrightarrow W(R)$ is a ring injection, and $W(R)$ is the t-adic completion of $\mathrm{End}_0(R)$, i.e. $W(R) = \varprojlim \mathrm{End}_0(R)/I_N$.*

Theorem 3.27 (Stienstra). *For every $\gamma \in \mathrm{Nil}_*(\Lambda)$ there is an N so that γ is annihilated by the ideal*

$$I_N = \{f \mid \chi(f) \equiv 1 \bmod t^N\} \subset \mathrm{End}_0(R).$$

Consequently, $NK_*(\Lambda)$ is a module over the t-adic completion $W(R)$ of $\mathrm{End}_0(R)$.

Recall the sign convention:

$$\begin{aligned} NK_1(\Lambda) &\cong \mathrm{Nil}_0(\Lambda) \\ [1 - \nu y] &\leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)] \end{aligned}$$

The $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula

$$\alpha(t) * [1 - \nu y] = [\alpha(\nu y)].$$

And the $W(R)$-module structure on $NK_n(\Lambda)$

$$\alpha(t) * \{\gamma, 1 - \nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda), \quad \gamma \in K_{n-1}(R).$$

3.3.6 Modern version

Reference: Weibel, K-book, chapter 4, pp. 58.

3.4 Some results

Proposition 3.28. *If R is $S^{-1}\mathbb{Z}, \hat{\mathbb{Z}}_p$ or $\mathbb{Q}$-algebra, then*

$$\begin{aligned} \lambda_t\colon R &\longrightarrow W(R) \\ r &\mapsto (1-t)^r \end{aligned}$$

is a ring injection.

Corollary 3.29. *Fix an integer p and a ring Λ with 1.*
(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.
(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ)-module.
(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.
(d) If $p^m = 0$ in Λ, $NK_*(\Lambda)$ is a p-group.

Theorem 3.30 (Stienstra). *If $0 \neq n \in \mathbb{Z}$, $NK_1(R)[\frac{1}{n}] \cong NK_1(R[\frac{1}{n}])$.*

Corollary 3.31. [1] *If G is a finite group of order n, then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n. In fact, $NK_*(\mathbb{Z}[G])$ is an n-torsion group, and $Z_{(p)} \otimes NK_*(\mathbb{Z}[G]) = NK_*(\mathbb{Z}_{(p)}[G])$, where $p \mid n$.*

[1] Weibel, *K*-book chapter3, page 27.

Chapter 4

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [25].

4.1 Outline

Definition 4.1 (Bass *Nil*-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty} \mathbb{F}_2 x^i = \oplus_{i=1}^{\infty} \mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\, dx = \oplus_{i=1}^{\infty} \mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\, dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

4.2 Preliminaries

4.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 4.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

4.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[\![x]\!])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$
\begin{aligned}
[a]&: \alpha(x) \mapsto \alpha(ax) \\
V_m&: \alpha(x) \mapsto \alpha(x^m) \\
F_m&: \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m&: 1 - rx \mapsto 1 - r^m x
\end{aligned}
$$

Remark 4.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [6].

Proposition 4.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$
\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a]V_m &= V_m[a^m] \\
F_m[a] &= [a^m]F_m \\
[a][b] &= [ab] \\
V_m F_k &= F_k V_m, \textit{ if } (k, m) = 1
\end{aligned}
$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[\![x]\!]$ is not.

4.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u,v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.s

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 4.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a,b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a,b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem*.

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a,b\rangle \in K_2(R,I)$ satisfy following relations

(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$ if any of a,b,c are in I.

Theorem 4.6. *1. If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R,I) \overset{\cong}{\to} K_2(R,I)$.

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r,s \in R$ commute and $1-rs$ is a unit, that is $rs = sr$ and $1 - rs \in R^*$,

$$\langle r,s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r,s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r,s\rangle \in K_2(R,I)$

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$,

(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of r,s,t are in I).

Note that $\langle r,1\rangle = 0$ for any $r \in R$ and $\langle r,s\rangle_{version2} = \langle -r,s\rangle_{version1}$.

Theorem 4.7. *1. If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r,s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r,s\}$.*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R,I)$ is generated by $\langle r,s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively generated by its units.

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r,s\rangle$, $r,s\in R$, $1-rs\in R^*$ or by all $\{u,v\}$, $u,v\in R^*$.*

4.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 4.4.

excision 失效就是说 if $A\longrightarrow B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A,I)\longrightarrow K_n(B,I)$ need not be an isomorphism. 由于这个不是同构，没有 Mayer-Vietoris 序列

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(A/I) \arrow[r] \arrow[d] & K_i(A,I) \arrow[r] \arrow[d, shift right] & K_i(A) \arrow[r] \arrow[d] & K_i(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(B/I) \arrow[r] & K_i(B,I) \arrow[r] \arrow[u, dashed, shift right] & K_i(B) \arrow[r] & K_i(B/I) \arrow[r] & \cdots
\end{tikzcd}$$

要连接 $K_n(A,I)\longrightarrow K_n(B,I)$ 就要考虑 birelative K-groups（也称 double relative K-groups）,$K(A,B,I)$ 定义为 homotpy fiber of the map $K(A,I)\longrightarrow K(B,I)$。以下是详细的定义和性质。

Relative groups Let R be a ring (not necessarily commutative), $I\subset R$ a two-sided ideal, by definition $K_i(R)=\pi_i(BGL(R)^+)$, $i\geq 1$, there exists a map

$$BGL(R)^+\longrightarrow BGL(R/I)^+$$

Definition 4.8. $K(R,I)$ is the homotopy fibre of the map $BGL(R)^+\longrightarrow BGL(R/I)^+$. $K_i(R,I):=\pi_i(K(R,I)), i\geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots\longrightarrow K_{i+1}(R)\longrightarrow K_{i+1}(R/I)\longrightarrow K_i(R,I)\longrightarrow K_i(R)\longrightarrow K_i(R/I)\longrightarrow\cdots.$$

In particular,

$$\begin{aligned}
&K_3(R,I)\longrightarrow K_3(R)\longrightarrow K_3(R/I)\longrightarrow K_2(R,I)\longrightarrow K_2(R)\longrightarrow K_2(R/I)\longrightarrow\\
&\longrightarrow K_1(R,I)\longrightarrow K_1(R)\longrightarrow K_1(R/I)
\end{aligned}$$

Let R be any ring (not necessarily commutative), if $I,J\subset R$ are two-sided ideals, there is a map

$$K(R,I)\longrightarrow K(R/J,I+J/J).$$

If $I \cap J = 0$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R/I \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle g} \\ R/J & \xrightarrow{f} & R/I+J \end{array}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(4.8)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & K_{i+1}(R) & \xrightarrow{\alpha_*} & K_{i+1}(R/I) & \xrightarrow{\partial} & K_i(R,I) & \xrightarrow{j} & K_i(R) & \xrightarrow{\alpha_*} & K_i(R/I) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle\epsilon_i} & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & K_{i+1}(R/J) & \xrightarrow{f_*} & K_{i+1}(R/I+J) & \xrightarrow{\partial} & K_i(R/J, I+J/J) & \xrightarrow{j'} & K_i(R/J) & \xrightarrow{f_*} & K_i(R/I+J) & \longrightarrow \cdots \end{array}$$

where the induced homomorphism

$$\epsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the MayerVietoris sequence

$$\begin{aligned} K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 4.9. *1. Suppose that the excision map ϵ_i in 4.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. *If ϵ_i is an isomorphism, and in additionϵ_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is*

$$K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta}$$
$$\longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J)$$

3. *Suppose instead that ϵ_i is surjective, and let $L = ker(\epsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, and the sequence*

$$K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow$$
$$\longrightarrow K_i(R)/L \longrightarrow K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J)$$

is exact.

Proof. Define $\delta_i = j\epsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. □

Remark 4.10. It is known that ϵ_0 and ϵ_1 are isomorphism regardless of the specific rings. Moreover Swan [20] has shown that ϵ_2 cannot be an isomorphism in general. For more discussion, see [18].

Double relative groups

Definition 4.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I+J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd}
K(R;I,J) \arrow[d, dashed] & & \\
K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\
K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+
\end{tikzcd}$$

Remark 4.12. $K_i(R; I, J) \cong K_i(R; J, I), K_i(R; I, I) = K_i(R, I)$.

We have a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R, I) \longrightarrow K_{i+1}(R/J, I+J/J) \longrightarrow K_i(R; I, J) \longrightarrow K_i(R, I) \longrightarrow K_i(R/J, I+J) \longrightarrow \cdots.$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R, I) \longrightarrow K_3(R/I, I+J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [27] 3.5.10, [18], [12] or [8] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [10] theorem 2.

Remark 4.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [10].

Theorem 4.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b \rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [10] theorem 3. ∎

Lemma 4.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/I \arrow[d] \\ R/J \arrow[r] & R/I+J \end{tikzcd}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I+J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

is split exact.

4.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [6] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$

$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$

$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$**-module structure on** $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [6]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 4.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: $ds, s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 4.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 4.18. 一个不知道有没有用的结论, see [6]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$D\colon x\mathbb{F}_2[x] \longrightarrow \Omega_{\mathbb{F}_2[x]}$$
$$x^i \mapsto ix^{i-1}dx$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty}\mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

4.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{4.18}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma + 1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$\begin{tikzcd} & NK_3(\mathbb{Z}, (2)) = 0 \arrow[d] & & \\ & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\ 0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\ & NK_2(\mathbb{Z}, (2)) = 0 & & \end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2], (\sigma+1), (\sigma-1))$, from Guin-Loday-Keune [10], $NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 4.19. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 4.20 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "\sigma\mapsto 1"', "f"] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R, I)$$
$$(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle\sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle\sigma-1, 1\rangle^p = 1$$

Also see [18].

Example 4.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

4.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $\mathbf{F}_2[\epsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{cccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\epsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\epsilon \end{array}$$

Lemma 4.22. *The map $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$ in 4.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [25] Lemma 1.2. □

Theorem 4.23.

$$NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]},$$

$$NK_0(\mathbb{Z}[D_2]) \cong V,$$

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 4.18 with $\mathbb{Z}[C_2]$. Note that $R[G_1 \times G_2] = R[G_1][G_2]$, for commutative R, $R[G_1 \times G_2] = R[G_1] \otimes R[G_2]$, $\sum_{g,h} c_g c_h g \otimes h \leftarrow \sum_{g,h} c_g g \otimes c_h h$. As for infinite product, see MO 46950.

$$\begin{tikzcd} \mathbb{Z}[D_2] = \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] \arrow[r] \arrow[d] & \mathbb{Z}[C_2] \arrow[d, two heads, "q"] \\ \mathbb{Z}[C_2] \arrow[r, two heads, "q"] & \mathbb{F}_2[C_2] \end{tikzcd} \tag{4.23}$$

Recall that $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]/(\epsilon^2)$. By [27] chapter 2 Ex 7.4.5,

$$NK_1(\mathbb{F}_2[C_2]) = NK_1(\mathbb{F}_2[\varepsilon]/(\epsilon^2)) = (1+\varepsilon x\mathbb{F}_2[\varepsilon]/(\epsilon^2)[x])^\times = (1+\varepsilon x\mathbb{F}_2[x])^\times \cong V = x\mathbb{F}_2[x]$$
$$[(P,\nu)] \mapsto \det(1-\nu x)$$

Remark 4.24. $(1+\varepsilon x\mathbb{F}_2[x])^\times \cong x\mathbb{F}_2[x]$, $1+\varepsilon x\sum a_i x^i \mapsto x\sum a_i x^i$ 的原因，左边是乘法群，右边的乘法是普通的多项式相加，左边 $(1+\varepsilon x\sum a_i x^i)(1+\varepsilon x\sum b_j x^j) = 1+\varepsilon x\sum a_i x^i + \varepsilon x\sum b_j x^j + (\varepsilon x\sum a_i x^i)(\varepsilon x\sum b_j x^j) = 1+\varepsilon x(\sum a_i x^i + \sum b_j x^j)$, 右边 $x\sum a_i x^i + x\sum b_j x^j = x(\sum a_i x^i + \sum b_j x^j)$.

As $W(\mathbb{F}_2)$-modules,

$$1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n) \mapsto a_0x+a_1x^2+\cdots+a_nx^{n+1}$$

and we can easily check that

$$V_m(1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon x^m(a_0+a_1x^m+\cdots+a_nx^{mn})$$

$$[a](1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon ax(a_0+a_1ax+\cdots+a_na^nx^n)$$

hence the module structure of $(1+\varepsilon x\mathbb{F}_2[x])^\times$ are the same as V.

By MayerVietoris sequence for the NK-functor, one has

$$\begin{tikzcd}
NK_2(\mathbb{Z}[D_2]) \arrow[r] & NK_2\mathbb{Z}[C_2]\oplus NK_2\mathbb{Z}[C_2] \arrow[r, "q\times q"] & NK_2(\mathbb{F}_2[C_2]) \arrow[dll] \\
NK_1(\mathbb{Z}[D_2]) \arrow[r] & NK_1\mathbb{Z}[C_2]\oplus NK_1\mathbb{Z}[C_2]=0 \arrow[r] & NK_1(\mathbb{F}_2[C_2]) \arrow[dll, "\cong"] \\
NK_0(\mathbb{Z}[D_2]) \arrow[r] & NK_0\mathbb{Z}[C_2]\oplus NK_0\mathbb{Z}[C_2]=0 &
\end{tikzcd}$$

Hence $NK_0(\mathbb{Z}[D_2]) \cong V$, $NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}(q\times q) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}q \cong \Omega_{\mathbb{F}_2[x]}$ since $\mathrm{Im}(q\times q) = \mathrm{Im}q$. $\square$

最后一个论断，若对则有 $(q\times q)(\Phi(V)\times V) = 0$，然而这个等式是不成立的。

4.5.1 A result from the K-book

For the convenience of the reader we copy [27] chapter 2 Ex 7.4.5 as follows.

Let R ba a commutative regular ring, $A = R[x]/(x^N)$, we claim that

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$\begin{aligned}
\mathrm{Nil}_0(A) &\cong (1+xtA[t])^\times \\
[(A,x)] &\mapsto 1-xt \\
[(P,\nu)] &\mapsto \det(1-\nu t)
\end{aligned}$$

the isomorphism $NK_1(A) \cong \mathrm{Nil}_0(A) \cong (1 + xtA[t])^{\times}$ is universal in the following sense:

Let B be a R-algebra, $(P, \nu) \in \mathbf{Nil}(B)$ with $\nu^N = 0$, regard P as an A-B-bimodule

$$\begin{aligned} \mathrm{Nil}_0(A) &\longrightarrow \mathrm{Nil}_0(B) \\ (A, x) &\mapsto (P, \nu) \end{aligned}$$

there is an $\mathrm{End}_0(R)$-module homomorphism

$$\begin{aligned} (1 + xtA[t])^{\times} &\longrightarrow \mathrm{Nil}_0(B) \\ 1 - xt &\mapsto [(P, \nu)]. \end{aligned}$$

4.5.2 About the lemma

In this subsection, we concentrate on the lemma 4.22.

For a complete proof, see [23].

4.6 NK_i of the group C_4

4.7 NK_i of the group D_4

Chapter 5

Lower Bounds for the Order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$

2016.3.19 阅读这篇文章 [17] 1976 年发表在 *Math. Ann.*。

基本假设：p: rational prime, G: elementary abelian p-group.

用的方法：Bloch; van der Kallen K_2 of truncated polynomial rings

结论：the p-rank of $K_2(\mathbb{Z}G)$[1] grows expotentially with the rank of G.

$Wh_2(G)$: "pseudo-isotopy" group is nontrivial if G has rank at least 2.

这篇文章之前已知的结论 (exact computations) Dunwoody, G cyclic of order 2 or 3, $K_2(\mathbb{Z}G)$ is an elementary abelian 2-group of rank 2 if G has order 2 and of rank 1 if G has order 3. 两者都有 $Wh_2(G)$ 平凡。

一些记号和基本结论 R commutative ring, A a subring of R. $\Omega^1_{R/A}$ the module of Kähler differentials of R considerd as an algebra over A and R^* will denote the group of units of R.

the p-rank of an abelian group G is $\dim_{\mathbb{F}_p}(G \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

elementary abelian p-groups An elementary abelian p-group is an abelian group in which every nontrivial element has order p. The number p must be prime, and the elementary abelian groups are a particular kind of p-group. The case where $p = 2$, i.e., an elementary abelian 2-group, is sometimes called a Boolean group.

结构：Every elementary abelian p-group is a vector space over the prime field $\mathbb{F}_p$ with p elements, and conversely every such vector space is an elementary abelian group.

By the classification of finitely generated abelian groups, or by the fact that every vector space has a basis, every finite elementary abelian group must be of the form $(\mathbb{Z}/p\mathbb{Z})^n$ for n a

[1] this is a finite group

non-negative integer (sometimes called the group's rank). Here, $C_p = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p.

In general, a (possibly infinite) elementary abelian p-group is a direct sum of cyclic groups of order p. (Note that in the finite case the direct product and direct sum coincide, but this is not so in the infinite case.)

5.1 第一部分

环是 $\mathbb{F}_q$ 有限域的情况。

先说结论

首先是一个奇素数的结论

Proposition 5.1. *Let $q = p^f$ be odd and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary p-group of rank $f(n-1)(p^n-1)$.*

接着是素数 2 的结论

Proposition 5.2. *Let $q = 2^f$ be odd and let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary 2-group of rank $f(n-1)(2^n-1)$.*

结论实际上是可以统一的，但是方法有些区别，因此原文中分开表述。

我们引进方法时借鉴了 van der Kallen 的方法和记号

Let R be a commutative ring. The abelian group $TD(R)$ is the universal R-module having generators $Da, Fa, a \in R$, subject to the relations

$$\begin{aligned} D(ab) &= aDb + bDa, \\ D(a+b) &= Da + Db + F(ab), \\ F(a+b) &= Fa + Fb, \\ Fa &= D(1+a) - Da. \end{aligned}$$

There is a natural surjective homomorphism of R-modules

$$TD(R) \twoheadrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 1$$

whose kernel is the submodule of $TD(R)$ generated by the $Fa, a \in R$. Relations imply

$$F(c^2 a) = cFa$$

$(F(c^2a) = F(ca \cdot c) = D(ca + c) - D(ac) - D(c) = D(c(a+1)) - D(ac) - D(c) = cD(a+1) - (a+1)D(c) - aD(c) - cD(a) - D(c) = cF(a)$, $0 = F(0) = F(a-a) = F(a) + F(-a)$, $\Rightarrow F(a) = -F(a) = F(-a)$, $\Rightarrow F(2a) = 0)$

for all $a, c \in R$ see [21]p. 1204.
Hence $F(2a) = 2F(a) = 0$, if 2 is a unit of R, $F(a) = 0$, then the kernel is trivial and $\Omega^1_{R/\mathbb{Z}} \cong TD(R)$,

$$1 \longrightarrow TD(R) \xrightarrow{\cong} \Omega^1_{R/\mathbb{Z}} \longrightarrow 1.$$

Example 5.3. $R = \mathbb{Z}$, then the kernel of the above surjection is $\mathbb{Z}/2\mathbb{Z}$.
If R is a field of characteristic $\neq 2$, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.
If R is a perfect field, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.

Definition 5.4. We define groups $\Phi_i(R)$, $i \geq 2$, by the exact sequence

$$1 \longrightarrow \Phi_i(R) \longrightarrow K_2(R[x]/(x^i)) \longrightarrow K_2(R[x]/(x^{i-1})) \longrightarrow 1. \tag{5.4}$$

This sequence is exact at the right as $SK_1(R[x]/(x^i), (x^{i-1})/(x^i)) = 1$ (cf. [14] Theorem 6.2 and [3]9.2, p. 267).

5.1.1 Remarks

我们把 Bass 书 [3] 中相关的结论 (p. 267) 放在这里以备今后查询和使用

A semi-local ring is a ring for which $R/rad(R)$ is a semisimple ring, where $rad(R)$ is the Jacobson radical of R. In commutative algebra, semi-local means "finitely many maximal ideals", for instance, all rational numbers r/s with s prime to 30 form a semi-local ring, with maximal ideals generated respectively by $2, 3$, and 5. This is a PID, as a matter of fact, any semi-local Dedekind domain is a PID. And if R is a commutative noetherian ring, the set of zero-divisors is a union of finitely many prime ideals (namely, the "associated primes" of (0)), thus its classical ring of quotients (obtained from R by inverting all of its non zero-divisors) is a semi-local ring. See [1] p.174. and [3] p. 86.

In studying the stable structure of general linear groups in algebraic K-theory, Bass proved the following basic result (ca. 1964) on the unit structure of semilocal rings.

Theorem 5.5. *If R is a semi-local ring, then R has stable range* 1, *in the sense that, whenever* $Ra + Rb = R$, *there exists* $r \in R$ *such that* $a + rb \in R^*$.

Example 5.6. Some classes of semi-local rings: left(right) artinian rings, finite direct products of local rings, matrix rings over local rings, module-finite algebras over commutative semi-local rings.
The quotient $\mathbb{Z}/m\mathbb{Z}$ is a semi-local ring. In particular, if m is a prime power, then $\mathbb{Z}/m\mathbb{Z}$ is a

local ring.
A finite direct sum of fields $\bigoplus_{i=1}^{n} F_i$ is a semi-local ring. In the case of commutative rings with unit, this example is prototypical in the following sense: the Chinese remainder theorem shows that for a semi-local commutative ring R with unit and maximal ideals $m_1, \cdots, m_n$

$$R/\bigcap_{i=1}^{n} m_i \cong \bigoplus_{i=1}^{n} R/m_i$$

(The map is the natural projection). The right hand side is a direct sum of fields. Here we note that $\cap_i m_i = rad(R)$, and we see that $R/rad(R)$ is indeed a semisimple ring.
The classical ring of quotients for any commutative Noetherian ring is a semilocal ring.
The endomorphism ring of an Artinian module is a semilocal ring.
Semi-local rings occur for example in commutative algebra when a (commutative) ring R is localized with respect to the multiplicatively closed subset $S = \cap(R - p_i)$, where the p_i are finitely many prime ideals.

Theorem 5.7. *Let I be a two-sided ideal in a ring R. Assume either that R is semi-local or that $I \subset rad(R)$. Then*

$$GL_1(R, I) \longrightarrow K_1(R, I)$$

is surjective, and, for all $m \geq 2$,

$$GL_m(R, I)/E_m(R, I) \longrightarrow K_1(R, I)$$

is an isomorphism. Moreover $[GL_m(R), GL_m(R, I)] \subset E_m(R, I)$, with equality for $m \geq 3$.

Corollary 5.8. *Suppose that R above is commutative, then $E_n(R, I) \xrightarrow{\cong} SL_n(R, I)$ is an isomorphism for all $n \geq 1$, and $SK_1(R, I) = 0$.*

Proof. The determinant induces the inverse,

$$\det\colon K_1(R, I) \longrightarrow GL_1(R, I).$$

In particular, if $\alpha \in GL_n(R, I)$ and $\det(\alpha) = 1$ then $\alpha \in E_n(R, I)$, i.e. $SL_n(R, I) \subset E_n(R, I)$. The opposite inclusion is trivial. Finally $SK_1(R, I) = SL(R, I)/E(R, I) = 0$. □

还有一个小插曲，当 k 是域时，$k[x]/(x^m)$ 是局部环的证明

Proposition 5.9. *Let I be an ideal in the ring R.*
a) If $rad(I)$ is maximal, then R/I is a local ring.
b) In particular, if m is a maximal ideal and $n \in \mathbb{Z}^+$ then R/m^n is a local ring.

Proof. a) We know that $rad(I) = \bigcap_{P \supset I} P$, so if $rad(I) = m$ is maximal it must be the only prime ideal containing I. Therefore, by correspondence R/I is a local ring. (In fact it is a ring with a unique prime ideal.)
b)$rad(m^n) = rad(m) = m$, so part a) applies. □

Example 5.10. For instance, for any prime number p, $\mathbb{Z}/(p^k)$ is a local ring, whose maximal ideal is generated by p. It is easy to see (using the Chinese Remainder Theorem) that conversely, if $\mathbb{Z}/(n)$ is a local ring then n is a prime power.

The ring $\mathbb{Z}_p$ of p-adic integers is a local ring. For any field k, the ring $k[\![t]\!]$ of formal power series with coefficients in k is a local ring. Both of these rings are also PIDs. A ring which is a local PID is called a discrete valuation ring. Note that a local ring is connected, i.e., $e^2 = e \Rightarrow e \in \{0, 1\}$.

令 R 是 $k[x]$, I 是 (x^m), 有 $rad(x^m) = (x)$ 是极大理想 (由于 $0 \to (x) \to k[x] \to k \to 0$ 正合)，从而 $k[x]/(x^i)$ 是局部环。

Remarks 到此结束

5.1.2 Theorem

The first part of the following theorem is due to van der Kallen [21] and the second to Bloch [5].

Theorem 5.11. *Let R be a commutative ring. Then*
(1) $\Phi_2(R) \cong TD(R)$*;*
(2) If R is a local $\mathbb{F}_p$*-algebra and p is odd prime, then*

$$\Phi_i(R) \cong \begin{cases} \Omega^1_{R/\mathbb{Z}}, i \not\equiv 0, 1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r}, i = mp^r, (p, m) = 1. \end{cases}$$

当 p 是 odd prime 时，这一定理 (2) 可应用于 $\mathbb{F}_p[C_p]$, 因为 $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[t]/(t^p)$

Lemma 5.12. *Let* $q = p^f$ *and let H be a finite abelian p-group. Then* $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$*is a free* $\mathbb{F}_q H$*-module of rank equal to the p-rank of H.*

Proof. In terms of polynomials, we have

$$\mathbb{F}_q H \cong \mathbb{F}_q[x_1, \cdots, x_n]/I$$

where n is the p-rank of H and I is the ideal of $\mathbb{F}_q[x_1, \cdots, x_n]$ generated by polynomials of the form $F_i = x_i^{q_i} - 1$ where q_i is a power of p. By [BoreI,A.: Linear algebraic groups. New York:

W. A. Benjamin 1969, p. 61], $\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}$ is the $\mathbb{F}_qH$-module with generators $dx_1, \cdots, dx_n$ subject to the relations

$$\sum_i \frac{dF_i}{dx_i} dx_i = 0.$$

Since the ring has characteristic p, the relations are trivial and the module is free. As $\mathbb{F}_q$ is perfect, its module of differentials is trivial. Hence $\Omega^1_{\mathbb{F}_qH/\mathbb{F}_q} = \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}$, yielding the result. □

由这个引理得到了5.1.

下面是节选一些可能用到的陈述。

- $\mathbb{F}_qG$ is a local ring, where G is an elementary abelian p-group, for example $G = (\mathbb{Z}/p\mathbb{Z})^n$.

对 odd prime 的证明如下

Proof. We begin by showing that $K_2(\mathbb{F}_qG)$ is an elementary abelian p-group even in case $p = 2$. As $\mathbb{F}_qG$ is a local ring, it follows that $K_2(\mathbb{F}_qG)$ is generated by the Steinberg symbols $\{u, v\}$, $u, v \in \mathbb{F}_qG^*$. Now $u^p, v^P \in \mathbb{F}^*$ as G is an elementary abelian p-group (p 次后 G 中的元就变成单位元了). Choose $w \in \mathbb{F}_q^*$ so that $w^p = u^p$.(这里注意之前的 u 是群环里的，这里的 w 取在域里) Then

$$\begin{aligned} \{u, v\}^p &= \{u^p, v\} \\ &= \{w^p, v\} \\ &= \{w, v^p\}. \end{aligned}$$

Thus $\{w, v^p\}$ is trivial as it lies in the image of $K_2(\mathbb{F}_q) = 1$(有限域的 K_2 是平凡的，并且这个符号是在 K_2 中). Hence $K_2(\mathbb{F}_qG)$ has exponent p.

Let H be generated by $x_1, \cdots, x_{n-1}$ where $x_1, \cdots, x_n$ are independent generators of G. Then (由于特征是 p 才有下面的最后一步，对于 $\mathbb{Z}$ 是不对的)

$$\mathbb{F}_qG = \mathbb{F}_qH[x_n]/(x_n^p - 1) \cong \mathbb{F}_qH[x]/(x^p).$$

Exact sequence 5.4 together with Theorem yield

$$\begin{aligned} \text{rank } K_2(\mathbb{F}_qG) &= \text{rank } K_2(\mathbb{F}_qH) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}} + \text{rank } \mathbb{F}_qH/\mathbb{F}_q \\ &= \text{rank } K_2(\mathbb{F}_qH) + f(p-1)(n-1)p^{n-1} + f(p^{n-1} - 1) \end{aligned}$$

and the result follows by induction.

上面的结论我们详细写出来是

$$
\begin{aligned}
1 \longrightarrow \Phi_p(\mathbb{F}_q H) &\longrightarrow K_2(\mathbb{F}_q G) = K_2(\mathbb{F}_q H[x]/(x^p)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow 1, \\
1 \longrightarrow \Phi_{p-1}(\mathbb{F}_q H) &\longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-2})) \longrightarrow 1, \\
&\cdots \\
1 \longrightarrow \Phi_2(\mathbb{F}_q H) &\longrightarrow K_2(\mathbb{F}_q H[x]/(x^2)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x)) \longrightarrow 1.
\end{aligned}
$$

Note that $\mathbb{F}_q H[x]/(x) = \mathbb{F}_q H$, $G = (\mathbb{Z}/p\mathbb{Z})^n$, $H = (\mathbb{Z}/p\mathbb{Z})^{n-1}$ then

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_q G) &= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H[x]/(x^{p-1})) \\
&= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } \Phi_{p-1}(\mathbb{F}_q H) + \cdots + \text{rank } \Phi_2(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H) \\
&= \text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + (p-2)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } K_2(\mathbb{F}_q H) \\
&= (p-1)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + \text{rank } K_2(\mathbb{F}_q H) \\
&= f(p-1)(n-1)p^{n-1} + f(p^{n-1}-1) + \text{rank } K_2(\mathbb{F}_q H)
\end{aligned}
$$

since

$$
\begin{aligned}
\Phi_p(\mathbb{F}_q H) &= \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} \oplus \mathbb{F}_q H/\mathbb{F}_q H^p, \\
\Phi_i(\mathbb{F}_q H) &= \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} = (\mathbb{F}_q H)^{n-1}, 2 \leq i \leq p-1, \\
\mathbb{F}_q H/\mathbb{F}_q H^p &= \mathbb{F}_q H/\mathbb{F}_q
\end{aligned}
$$

$\mathbb{F}H$ 是以 H 中元素为基的自由 F 模并且

$$
\begin{aligned}
\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} &= \text{rank } (\mathbb{F}_{p^f} H)^{n-1} = (n-1)f|H| = (n-1)fp^{n-1} \\
\text{rank } \mathbb{F}_q H/\mathbb{F}_q &= \text{rank } \mathbb{F}_q H - \text{rank } \mathbb{F}_q = f(p^{n-1}-1).
\end{aligned}
$$

接下来是归纳计算,首先我们看它截至到哪一步:最后一步应该是 $\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]$,因为 $K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]) = 0$, 这时有

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]) &= \text{rank } K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{Z}} + \text{rank } \mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{F}_q \\
&= 0 + f(p-1)(2-1)p^{2-1} + f(p^{2-1}-1)
\end{aligned}
$$

从而我们知道

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_q G) &= f(p-1)(n-1)p^{n-1} + f(p^{n-1}-1) + \cdots + f(p-1)p^1 + f(p^1-1) \\
&= \sum_{i=1}^{n-1} (f(p-1)ip^i + f(p^i-1)) \\
&= -f\frac{p-p^n}{1-p} + f(n-1)p^n + f\frac{p-p^n}{1-p} - (n-1)f \\
&= f(n-1)(p^n-1)
\end{aligned}
$$

这里的计算用到等比数列求和，记 $S = \sum_{i=1}^{n-1} ip^i$

$$pS = \sum_{i=1}^{n-1} ip^{i+1} = \sum_{i=2}^{n}(i-1)p^i$$

$$S - pS = \sum_{i=1}^{n-1} p^i - (n-1)p^n$$

因此

$$S = \frac{p-p^n}{(1-p)^2} - \frac{(n-1)p^n}{(1-p)}$$

$$\begin{aligned}
\sum_{i=1}^{n-1}(f(p-1)ip^i + f(p^i-1)) &= f(p-1)S + f\frac{p-p^n}{1-p} - (n-1)f \\
&= f(p-1)(\frac{p-p^n}{(1-p)^2} - \frac{(n-1)p^n}{(1-p)}) + f\frac{p-p^n}{1-p} - (n-1)f \\
&= -f\frac{p-p^n}{1-p} + f(n-1)p^n + f\frac{p-p^n}{1-p} - (n-1)f \\
&= f(n-1)(p^n-1)
\end{aligned}$$

□

In case $p = 2$ the details become more complicated.(暂且略过这个情形)

5.2 第二部分

第二部分是考了系数环是 $\mathbb{Z}$ 的情形，如何将上面的有限域和这里的整数环联系起来，就是用了一个相对 K 群的正合列。

We now exploit these computations of $K_2(\mathbb{F}_qG)$ to obtain lower bounds for $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. There is an exact sequence

(5.12) $$K_2(\mathbb{Z}G) \longrightarrow K_2(\mathbb{F}_pG) \longrightarrow SK_1(\mathbb{Z}G, p\mathbb{Z}G) \longrightarrow SK_1(\mathbb{Z}G) \longrightarrow 1$$

This sequence is exact on the right because $\mathbb{F}_pG$ is a local ring, which implies $SK_1(\mathbb{F}_pG) = 1$ [3], p. 267.

Theorem 5.13. *(1) Let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{Z}G)$ has 2-rank at least $(n-1)2^n + 2$ and $Wh_2(G)$ has 2-rank at least $(n-1)2^n - \frac{(n+2)(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n \geq 2$.*
(2) Let p be an odd prime and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{Z}G)$ has p-rank at least $(n-1)(p^n-1) - \binom{p+n-1}{p}$ and $Wh_2(G)$ has p-rank at least $(n-1)(p^n-1) - \binom{p+n-1}{p} - \frac{n(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n \geq 2$.

Proof. (1) Since $K_1(\mathbb{Z}G, 2\mathbb{Z}G) \longrightarrow K_1(\mathbb{Z}G)$ is injective [Keating, M.E.: On the K-theory of the quaternion group. Mathematika 20, 59–62 (1973), Remark 2.4], we see that $K_2(\mathbb{Z}G) \longrightarrow K_2(\mathbb{F}_2G)$ is surjective.

If $g_1, \cdots, g_n$, are the generators of G, then the $n+1$ symbols $\{-1,-1\}, \{-1, g_1\}, \cdots, \{-1, g_n\}$ are independent [[14], p. 65] and lie in the kernel of this map. Hence

$$\operatorname{rank} K_2(\mathbb{Z}G) \geq (n-1)(2^n-1) + (n+1) = (n-1)2^n + 2.$$

Recall that for G abelian, $Wh_2(G)$ is the quotient of $K_2(\mathbb{Z}G)$ by the subgroup generated by all symbols of the form $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ [Hatcher, A.E.: Pseudo-isotopy and K_2, pp. 328-336. Lecture Notes in Mathematics 342. Berlin, Heidelberg, New York: Springer 1973]. It is easy to see from the bimuttiplicative and anti-symmetric properties of symbols that this subgroup has rank at most $\binom{n+1}{2} + 1$. Moreover, by using the various maps $\mathbb{Z}G \longrightarrow \mathbb{Z}$ which send elements of G to ± 1, it can be shown that the rank of this subgroup is precisely $\binom{n+1}{2} + 1$. $(n-1)2^n + 2 - \binom{n+1}{2} - 1 = (n-1)2^n - \frac{(n+2)(n-1)}{2}$.

(2) 以下这一段没有完全读懂。Let B be the integral chosure of $\mathbb{Z}G$ in $\mathbb{Q}G$. Then $SK_1(B, p^{n+1}B)$ has p-rank $\frac{p^n-1}{p-1}$ [Bass, H., Milnor, J., Serre, J. P.: Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$. Publ. Math. IHES 33, 59–137 (1967), Corollary 4.3, p. 95].

But $SK_1(B, p^{n+1}B) \cong SK_1(\mathbb{Z}G, p^{n+1}B)$ [[3], p. 484] since p^nB lies in the conductor of B over $\mathbb{Z}G$, and $SK_1(\mathbb{Z}G, p^{n+1}B)$ maps onto $SK_1(\mathbb{Z}G, p\mathbb{Z}G)$ [[3], 9.3, p. 267]. Hence p-rank $SK_1(\mathbb{Z}G, p\mathbb{Z}G) \leq \frac{p^n-1}{p-1}$. The p-rank of $SK_1(\mathbb{Z}G)$ is $\frac{p^n-1}{p-1} - \binom{p+n-1}{p}$ [Alperin, R.C., Dennis, R. K., Stein, M. R.: The non-triviality of $SK_1(\mathbb{Z}\pi)$, pp. 1-7. Lecture Notes in Mathematics 353. Berlin, Heidelberg, New York: Springer t973, Theorem 2]. The result now follows from exact sequence 5.12.

And noting that the subgroup generated by the symbols $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ has p-rank at most $\frac{n(n-1)}{2}$. □

Remark 5.14. The subgroup of $K_2(\mathbb{Z}G)$ generated by elements of the form $\langle a, b\rangle, 1 + ab \in (\mathbb{Z}G)^*$ maps onto $K_2(\mathbb{F}_2G)$ for G an elementary abelian 2-group of rank ≤ 2. W. van der Kallen has shown that this subgroup maps onto in general. This follows from the rank 2 case via

Lemma (van der Kallen). Let I be a nilpotent ideal of the commutative ring R. Let $v_i \in R$ additively generate R/I and let $w_j \in I$ additively generate I. Then $K_2(I) = \ker(K_2(R) \longrightarrow K_2(R/I))$ is generated by all elements of the form $\langle v_i, w_j\rangle$ and $\langle w_j, w_i^{2^k-1} w_j\rangle$.

我的一些问题：$NK_2(\mathbb{F}_qG)$ 如何算，$NK_1(\mathbb{Z}G, p\mathbb{Z}G) =?$，最简单的可以考虑 $NK_2(\mathbb{F}_pC_p)$, 接着是 $NK_2(\mathbb{F}_{p^2}C_p)$.

5.3 推广和其它

之前考虑的是 $\mathbb{Z}G$, G elementary. 可以推广到 G finite group, $\mathcal{O}$ be the ring of integers of an algebraic number field.

If S is a Sylow p-subgroup of G, then $\mathcal{O}G$ is a free module over $\mathcal{O}S$ and the composition

$$K_2(\mathcal{O}S) \longrightarrow K_2(\mathcal{O}G) \longrightarrow K_2(\mathcal{O}S)$$

(where the second map is the transfer) is multiplication by $(G : S)$. Hence p-rank $K_2(\mathcal{O}G) \geq$ p-rank $K_2(\mathcal{O}S)$ and estimates may be obtained by restricting to the case of a p-group.

Theorem 5.15. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field which is Galois over $\mathbb{Q}$ and let G be an elementary abelian p-group of rank n. If p is unramfied in $\mathcal{O}$ with each residue field having degree f over $\mathbb{F}_p$, then $K_2(\mathcal{O}G)$ has p-rank at least*
(i) $f(n-1)(2^n-1)$ if $p=2$ and $\mathcal{O}$ has a real embedding,
(ii) $f(n-1)(2^n-l)-\binom{n+1}{2}$ if $p=2$ and $\mathcal{O}$ is totally imaginary,
(iii) $f(n-1)(p^n-l)-\binom{p+n-1}{p}$ if p is odd.

abelian p-groups which are not elementary 有以下一个结论

Proposition 5.16. *Let p be an odd prime and suppose $G = H \times C$ where C is cyclic of order p^t, $|H| = p^k$ and $s = p$-rank H. Let $\mathcal{O}$ be the ring of integers in a number field. Choose a prime $\mathfrak{p}$ of $\mathcal{O}$ lying over p and having residue degree f over $\mathbb{F}_p$. Then*

$$\begin{aligned} &\operatorname{ord}_p|K_2(\mathcal{O}G/\mathfrak{p}G)| - \operatorname{ord}_p|K_2(\mathcal{O}H/\mathfrak{p}H)| \\ \geq &f\Big(p^k\big(s(p-1)p^{t-1}+1\big) - |H^{p^t}|\Big) + p^k(p^{t-1}-1) - (p-1)\sum_{r=1}^{t-1}|H^{p^r}|p^{t-r-1}. \end{aligned}$$

Chapter 6

Some Results of Grouprings

从相关的教材和书籍中摘录与群环相关的 K-理论结果。

On the K-theory of truncated polynomial rings in non-commuting variables 中的有关结果 Vigleik Angeltveit 的文章 On the K-theory of truncated polynomial rings in non-commuting variables, Bull. London Math. Soc. 47 (2015) 731742.

Hesselholt and Madsen computed the algebraic K-theory of $k[x]/(x^a)$ when k is a perfect field of positive characteristic in terms of big Witt vectors in [L. Hesselholt and I. Madsen, ‘Cyclic polytopes and the K-theory of truncated polynomial algebras’, Invent. Math. 130 (1997) 7397.], but see [L. Hesselholt and I. Madsen, ‘On the K-theory of finite algebras over Witt vectors of perfect fields’, Topology 36 (1997) 29101.] as well. They found that

$$K_{2q-1}(k[x]/(x^a),(x)) \cong \operatorname{coker}(V_a\colon \mathbb{W}_q(k) \longrightarrow \mathbb{W}_{aq}(k)),$$

while

$$K_{2q}(k[x]/(x^a),(x)) = 0.$$

Here $\mathbb{W}_n(k)$ denotes the Witt vectors on the truncation set $\{1,2,\cdots,n\}$. 一个例子比如 $K_2(\mathbb{F}_q[x]/(x^a),(x)) = 0$

[V. Angeltveit, T. Gerhardt, M. A. Hill and A. Lindenstrauss, ‘On the algebraic K-theory of truncated polynomial algebras in several variables’, J. K-Theory 13 (2014) 5781.]

这篇文章中把上面的结果推广成 $A = k[x_1,\cdots,x_n]/(x_1^{a_1},\cdots,x_n^{a_n})$, with the big Witt vectors replaced by certain generalized Witt vectors built from truncation sets in $\mathbb{N}^n$ instead of $\mathbb{N}$, and the cokernel of V_a replaced by the iterated homotopy cofiber of an n-cube of spectra.

These calculations both use the cyclotomic trace map from algebraic K-theory to topological cyclic homology. The underlying reason for the appearance of Witt vectors is that if k is a perfect field of positive characteristic, then

$$\lim_{R, m \leq n} \pi_* THH(k)^{C_m} \cong \mathbb{W}_n(k)[\mu_0],$$

where μ_0 is a polynomial generator in degree 2.

6.1 Relative K-theory and topological cyclic homology

BJORN IAN DUNDAS

Let $f : A \to B$ be a map of rings up to homotopy. When is it possible to give a good description of the relative algebraic K-theory? Generally, K-theory is hard to calculate, so it is of special importance to be able to measure the effect of a change of input.

Special instances of the case where f induces an epimorphism $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel have been studied by several authors.

- The first general result in this direction was Goodwillie's theorem [GOODWILLIE, T. G., Relative algebraic K-theory and cyclic homology. Ann. of Math.,124 (1986), 347-402.], that in the case of simplicial rings, relative K-theory is rationally given by the corresponding relative negative cyclic homology.
- McCarthy has complemented this by giving a short and beautiful proof [MCCARTHY, R., Relative algebraic K-theory and topological cyclic homology. Acta Math., 179 (1997), 197-222.]showing that at a given prime p, the relative K-theory is given by the corresponding relative topological cyclic homology.

This paper stemmed from a desire to understand the linearization

$$A(BG) \longrightarrow K(\mathbb{Z}[G])$$

that is, the connection between the algebraic K-theory of spaces and the algebraic K-theory of rings, each of which has theorems of the desired sort. Waldhausen has shown that this map is a rational equivalence, but torsion information has so far been out of reach.

与 K, TC, THH 有关的书 GOODWILL Notes on the cyclotomic trace,
HESSELHOLT, L. MADSEN, I., On the K-theory of finite algebras over Witt vectors of perfect fields. Topology, 36 (1997), 29-101.,
BOKSTEDT,M., HSIANG, W.C. , MADSEN, I., The cyclotomic trace and algebraic K-theory of spaces. Invent. Math., 111 (1993), 463-539.
MADSEN, I., Algebraic K-theory and traces, in Current Developments in Mathematics (R. Bott, A. Jaffe and S. T. Yan, eds.), pp. 191-323. International Press, 1995. 这个是 overview，大致扫过一眼

6.2 Cyclic polytopes and the K-theory of truncated polynomial algebras

这篇文章结论比较重要。

k: perfect field of positive characteristic p. 比如 $\mathbb{F}_p$, 甚至 $\mathbb{F}_{p^n}$.

主要计算了 $K_*(k[x]/(x^n),(x))$, relative algebraic K -theory of a truncated polynomial algebra over a perfect field k of positive characteristic p.

几个观察 (x) 这个理想是 nilpotent, 因而可以用 McCarthy's theorem: the relative algebraic K -theory is isomorphic to the relative topological cyclic homology, 后者是可以算的。

最后的结果是说这个群可以用 big Witt Vectors 来表示，简单回顾 Witt vectors 的知识
Let $W_m(k)$ denote the big Witt vectors in k of length m, i.e. the multiplicative group

$$W_m(k) = (1 + xk[\![x]\!])^\times/(1 + x^{m+1}k[\![x]\!])^\times,$$

the Verschiebung map

$$\begin{aligned} V_n\colon W_m(k) &\longrightarrow W_{mn}(k) \\ f(x) &\mapsto f(x^n) \end{aligned}$$

The relative K-theory $K(k[x]/(x^n),(x))$ is given by the fibration sequence

$$K(k[x]/(x^n),(x)) \longrightarrow K(k[x]/(x^n)) \longrightarrow K(k),$$

with a corresponding exact sequence of homotopy groups

$$0 \longrightarrow K_*(k[x]/(x^n),(x)) \longrightarrow K_*(k[x]/(x^n)) \longrightarrow K_*(k) \longrightarrow 0.$$

当 k 是有限域时，Quillen 算了 $K_*(k)$. D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Ann. Math. 96 (1972), 552-586
For a general perfect field of characteristic $p > 0$ one knows that the p-adic K-groups of k vanish in positive degrees by [C. Kratzer, -structure en K -theorie algebrique, Comment. Math. Helv. t' 55 (1980), 233-254].

计算结果就是下面的定理

Theorem 6.1. *Let k be a perfect field of positive characteristic. Then*

$$K_{2m-1}(k[x]/(x^n),(x)) \cong W_{mn}(k)/V_nW_m(k)$$

and the groups in even degrees are zero.

The result extends calculations by Aisbett and Stienstra of $K_3(k[x]/(x^n),(x))$.

Chapter 7

$NK_2(\mathbb{F}_{2^f}[C_{2^n}])$

English title:NK_2-group of $\mathbb{F}_{2^f}[C_{2^n}]$

In this paper, we calculated $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ using the method from van der Kallen's paper. In particular, we also determined the explicit structure and $W(\mathbb{F}_2)$-module structure in Steinberg symbols of $NK_2(\mathbb{F}_2[C_2])$.

Keywords: NK-group, Dennis-Stein symbols, group algebra, Witt vectors

2010 Mathematics Subject Classification: 19C99, 16S34(Group rings)

7.1 引言

Bass 和 Murthy [4] 首先给出了群 G 的 Whitehead 群 $Wh(G) = K_1(\mathbb{Z}G)/\{\pm g | g \in G\}$ 不是有限生成的例子，他们的例子来源于计算某些环 R 的 NK-群 $NK_1(R)$。设 R 是含幺结合环，对于任意的 $n \in \mathbb{Z}$，定义 $NK_n(R) = \ker(K_n(R[x]) \xrightarrow{x \mapsto 0} K_n(R))$，其中 $R[x]$ 是环 R 上的一元多项式环，当 $n < 0$ 时 K_n 是 Bass 定义的负 K-理论，$n = 0, 1, 2$ 时，K_0, K_1, K_2 是由 Grothendieck, Bass, Milnor 等定义的经典 K-理论，当 $n \geq 2$ 时，K_n 是由 Quillen 等定义的高阶 K-理论。在计算环 $R[\mathbb{Z}] = R[t, t^{-1}]$ 的 K-理论中，NK-群作为 $K_n(R[t, t^{-1}])$ 的直和项自然产生。1977 年 Farrell 在文献 [7] 中证明了关于 NK-群的重要性质，即若 $NK_1(R) \neq 0$，则 $NK_1(R)$ 不是有限生成的。实际上这一结论对于任何 $NK_i(i \leq 1)$ 均成立的，并且后来 van der Kallen [22] 与 Prasolov [15] 证明对任何 $NK_i(i > 1)$ 也是成立的。Weibel [24] 在 Almkvist [2] 与 Grayson [9] 等的基础上将 NK-群与 Witt 向量联系起来。对于交换正则环 R，Weibel [27] 给出了计算 $NK_1(R[x]/(x^n))$ 的方法，而对于 $R[x]/(x^n)$ 这样的截断多项式环，van der Kallen [21] [23] 与 Stienstra [19] 等对这类环的 K_2 群做了详细的研究。本文我们根据文献 [23] 中的方法利用截断多项式的 K_2 群计算了 $NK_2(\mathbb{F}_2[C_2])$，其中 $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[t]/(t^2)$，后者称为 $\mathbb{F}_2$ 上的对偶数环，利用 Dennis-Stein 符号给出了 $NK_2(\mathbb{F}_2[C_2])$ 的一组生成元，并给出了 $W(\mathbb{F}_2)$-模结构。在不引起歧义的情况下，记 x 在 $R[x]/(x^n)$ 下的像仍为 x.

7.2 预备知识

令 k 是特征为 $p>0$ 的有限域，考虑两个变元的多项式环 $k[t_1,t_2]$，令 I 是 $k[t_1,t_2]$ 的一个真理想，满足以下条件

1. I 是由 $k[t_1]$ 中的单项式生成的，
2. 对于某个 n，$t_1^n \in I$.

实际上这样的 I 具有形式 (t_1^n), 令

$$A = k[t_1,t_2]/I,$$

M 是 A 的 nil 根 (小根)，即 $M=(t_1)$，那么有 $A/M = k[t_2]$.

Proposition 7.1. $K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) = K_2(A,(t_1,t_2)) \cong K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$

Proof. 首先有下面两个相对 K 群的正合列

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k[t_2]) \longrightarrow 0$$

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k) \longrightarrow 0$$

由于 k 是有限域，它是正则环，于是有 $K_2(k)=0, K_2(k[t_2]) = K_2(k)\oplus NK_2(k) = 0$. 从而可以得到

$$K_2(k[t_1,t_2]/(t_1^n),(t_1)) \cong K_2(k[t_1,t_2]/(t_1^n)) \cong K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)).$$

□

当 $k=\mathbb{F}_{p^f}$ 时,$k[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$,其中 C_{p^n} 是 p^n 阶循环群。定义 $NK_i(R) = \ker(NK_i(R[x]) \xrightarrow{x\mapsto 0} K_i(R))$，于是有

$$0 \longrightarrow NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow 0,$$

由于 $K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) = K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}))$，并且 $K_2(\mathbb{F}_{p^f}[C_{p^n}])=0$，从而

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)).$$

7.2.1 Dennis-Stein 符号

回到一般情形，$K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$ 可以用 Dennis-Stein 符号表示
生成元 $\langle a,b\rangle, (a,b)\in A\times M \cup M\times A$;
关系 $\langle a,b\rangle = -\langle b,a\rangle$,
$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc, b\rangle$,
$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ 其中 $(a,b,c)\in A\times M\times A \cup M\times A\times M$.

Proposition 7.2. 对任意环 R，$q \geq 1$，$K_2(R[t]/(t^q),(t))$ 由 *Dennis-Stein* 符号 $\langle at^i, t\rangle$ 和 $\langle at^i, b\rangle$ 生成，其中 $a, b \in R, 1 \leq i < q$。

Proof. 参见文献 [19]。 □

7.2.2 符号说明

为了表述方便，遵从 [23] 的符号详述如下

- $\mathbb{Z}_+$: 非负整数全体，
- $\epsilon^1 = (1,0) \in \mathbb{Z}_+^2, \epsilon^2 = (0,1) \in \mathbb{Z}_+^2$,
- 对于 $\alpha \in \mathbb{Z}_+^2$, 记 $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, 于是有 $t^{\epsilon^1} = t_1, t^{\epsilon^2} = t_2$,
- $\Delta = \{\alpha \in \mathbb{Z}_+^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{Z}_+^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$, 若 $\delta \in \Delta$, 则有 $\delta + \epsilon^i \in \Delta, I = 1,2$,
- 对于 $(\alpha, i) \in \Lambda$, 令 $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$, 若 $(\alpha, i), (\alpha, j) \in \Lambda$, 有 $[\alpha, i] \leq [\alpha, j] + 1$,
- 若 $gcd(p, \alpha_1, \alpha_2) = 1$, 令 $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$.

若 $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, 令

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha) t^{\alpha - \epsilon^i}, t_i \rangle,$$

若 $g(t_1, t_2) = t_i h(t_1, t_2) \in \sqrt{I} = (t_1)$, 令

$$\Gamma_i(1 - g(t_1, t_2)) = \langle h(t_1, t_2), t_i \rangle,$$

且有

$$\Gamma_{\alpha,i}(1 - xf(x)) = \Gamma_i(1 - t^\alpha f(t^\alpha)).$$

由于 $t_1 \in \sqrt{I}, \Gamma_1$ 诱导了同态

$$\begin{aligned}(1 + t_1 k[t_1, t_2]/t_1 I)^\times &\longrightarrow K_2(A, M)\\ 1 - g(t_1, t_2) &\mapsto \langle h(t_1, t_2), t_1 \rangle\end{aligned}$$

Γ_2 诱导了同态

$$\begin{aligned}(1 + t_2 \sqrt{I}/t_2 I)^\times &\longrightarrow K_2(A, M)\\ 1 - g(t_1, t_2) &\mapsto \langle h(t_1, t_2), t_2 \rangle\end{aligned}$$

若 $(\alpha, i) \in \Lambda, \Gamma_{\alpha,i}$ 诱导了同态

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Theorem 7.3. $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+xk[x]/(x^{[\alpha,i]}))^{\times}.$$

Proof. 参见文献 [23]。 □

7.2.3 Witt 向量

令 R 是一个交换环，big Witt 环 (the ring of universal/big Witt vectors over R, 泛 Witt 环)$BigWitt(R)$ 作为 Abel 群同构于 $(1+xR[\![x]\!])^{\times}$，即常数项为 1 的形式幂级数全体在乘法运算下形成的交换群，

$$\begin{aligned} BigWitt(R) &\longrightarrow (1+xR[\![x]\!])^{\times} \\ (r_1,r_2,\cdots) &\mapsto \prod_i (1-r_ix^i)^{-1}. \end{aligned}$$

考虑子群 $(1+x^{n+1}R[\![x]\!])^{\times}$，定义 $BigWitt_n(R) = (1+xR[\![x]\!])^{\times}/(1+x^{n+1}R[\![x]\!])^{\times}$。显然 $BigWitt_1(R)=R$，并且当 $n\geq 3$ 时，$BigWitt_n(\mathbb{F}_2)$ 不是循环群。

Example 7.4. $BigWitt_3(\mathbb{F}_2) \cong (1+x\mathbb{F}_2[x]/(x^4))^{\times} \cong \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$

Proof. 由定义 $BigWitt_3(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^{\times}/(1+x^4\mathbb{F}_2[\![x]\!])^{\times}$，且有同态

$$\begin{aligned} (1+x\mathbb{F}_2[\![x]\!])^{\times} &\longrightarrow (1+x\mathbb{F}_2[x]/(x^4))^{\times} \\ 1+\sum_{i\geq 1} a_ix^i &\mapsto 1+a_1x+a_2x^2+a_3x^3 \end{aligned}$$

它的核是 $(1+x^4\mathbb{F}_2[\![x]\!])^{\times}$. 从而 $(1+x\mathbb{F}_2[x]/(x^4))^{\times} \cong BigWitt_3(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^{\times}/(1+x^4\mathbb{F}_2[\![x]\!])^{\times}$.

考虑 $1+x\in(1+x\mathbb{F}_2[x]/(x^4))^{\times}$ 是 4 阶元，由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3\}$, 且 $1+x^3$ 是二阶元，令 σ,τ 分别是 $\mathbb{Z}/4\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2) \\ (\sigma,\tau) &\mapsto (1+x)(1+x^3) = 1+x+x^3. \end{aligned}$$

□

Example 7.5. $BigWitt_4(\mathbb{F}_2) \cong \mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Proof. 由定义 $BigWitt_4(\mathbb{F}_2) = (1 + x\mathbb{F}_2[\![x]\!])^\times / (1 + x^5\mathbb{F}_2[\![x]\!])^\times$，且有同态

$$(1 + x\mathbb{F}_2[\![x]\!])^\times \longrightarrow (1 + x\mathbb{F}_2[x]/(x^5))^\times$$

它的核是 $(1 + x^5\mathbb{F}_2[\![x]\!])^\times$. 从而 $(1 + x\mathbb{F}_2[x]/(x^5))^\times \cong BigWitt_4(\mathbb{F}_2) = (1 + x\mathbb{F}_2[\![x]\!])^\times / (1 + x^5\mathbb{F}_2[\![x]\!])^\times$.

考虑 $1 + x \in BigWitt_5(\mathbb{F}_2)$, 它是 8 阶元，由它生成的子群 $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3, 1 + x^4, 1 + x + x^4, 1 + x^2 + x^4, 1 + x + x^2 + x^3 + x^4\}$，另外 $1 + x^3$ 是二阶元，令 σ, τ 分别是 $\mathbb{Z}/8\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned}\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3 + x^4\end{aligned}$$

于是 $(\sigma^i, \tau^j), 0 \le i < 8, 0 \le j < 2$ 对应于 $(1 + x)^i(1 + x^3)^j$, 详细的对应如下

$$\begin{aligned}
(1, \tau) &\mapsto 1 + x^3, & (\sigma, \tau) &\mapsto 1 + x + x^3 + x^4,\\
(\sigma^2, \tau) &\mapsto 1 + x^2 + x^3, & (\sigma^3, \tau) &\mapsto 1 + x + x^2 + x^4,\\
(\sigma^4, \tau) &\mapsto 1 + x^3 + x^4, & (\sigma^5, \tau) &\mapsto 1 + x + x^3,\\
(\sigma^6, \tau) &\mapsto 1 + x^2 + x^3 + x^4, & (\sigma^7, \tau) &\mapsto 1 + x + x^2,\\
(1, 1) &\mapsto 1, & (\sigma, 1) &\mapsto 1 + x,\\
(\sigma^2, 1) &\mapsto 1 + x^2, & (\sigma^3, 1) &\mapsto 1 + x + x^2 + x^3,\\
(\sigma^4, 1) &\mapsto 1 + x^4, & (\sigma^5, 1) &\mapsto 1 + x + x^4,\\
(\sigma^6, 1) &\mapsto 1 + x^2 + x^4, & (\sigma^7, 1) &\mapsto 1 + x + x^2 + x^3 + x^4.
\end{aligned}$$

□

固定素数 p，考虑局部环 $\mathbb{Z}_{(p)} = \mathbb{Z}[1/\ell \mid 所有素数\ell \neq p]$，即 $\mathbb{Z}$ 在素理想 $(p) = p\mathbb{Z}$ 的局部化，于是一个 $\mathbb{Z}_{(p)}$-代数 R 就是除 p 外的素数均可逆的交换环，如 $\mathbb{F}_{p^n}$ 是 $\mathbb{Z}_{(p)}$-代数。

考虑 p-Witt 环 $W(A)$ 与截断 p-Witt 环 $W_n(A)$，p-Witt 向量为 $(a_0, a_1, \cdots)$，加法用 Witt 多项式定义，以下仅考虑用加法定义的 Abel 群结构，例如 $W(\mathbb{F}_p) = \mathbb{Z}_p$，作为 Abel 群 $W_n(\mathbb{F}_{p^f})$ 同构于 $(\mathbb{Z}/p^n\mathbb{Z})^f$。

Artin-Hasse 级数定义为

$$AH(x) = \exp(-\sum_{n \ge 0} \frac{x^{p^n}}{p^n}) = 1 - x + \cdots \in 1 + x\mathbb{Q}[\![x]\!],$$

实际上 $AH(x) \in 1 + x\mathbb{Z}_{(p)}[\![x]\!]$。对于 $BigWitt(R) = 1 + xR[\![x]\!]$ 中的任一元素 α 存在以下写成无穷乘积的表法

$$\alpha = \prod_{n \geq 1}(1 - r_n x^n),\ r_n \in R,$$

若 A 是 $\mathbb{Z}_{(p)}$-代数，$BigWitt(A) = 1 + xA[\![x]\!]$ 中的任一元素 α 还有如下表法

$$\alpha = \prod_{n \geq 1} AH(a_n x^n),\ a_n \in A.$$

将整数 n 写成 $n = mp^a$，使得 $gcd(m, p) = 1, a \geq 0$，由于 A 是 $\mathbb{Z}_{(p)}$-代数，m 可逆，从而 $[x \mapsto x^{1/m}] \in \mathrm{End}(BigWitt(A))$ 是双射，于是我们可以将 $\alpha \in BigWitt(A)$ 以如下的形式表出

$$\prod_{\substack{m \geq 1 \\ gcd(m,p)=1 \\ a \geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}.$$

另一方面对于 $\mathbb{Z}_{(p)}$-代数 A，下列映射是群同态

$$\begin{aligned} W(A) &\longrightarrow BigWitt(A) \\ (a_0, a_1, \cdots) &\mapsto \prod_{i \geq 0} AH(a_i x^i). \end{aligned}$$

$BigWitt_n(A)$ 可以分解为 p-Witt 环的直和，实际上有以下同构

$$BigWitt(A) \cong \prod_{\substack{m \geq 1 \\ gcd(m,p)=1}} W(A),$$

元素 $\prod_{\substack{m \geq 1 \\ gcd(m,p)=1 \\ a \geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}$ 对应于一个 m-分量为 $(a_m, a_{mp}, a_{mp^2}, \cdots) \in W(A)$ 的 Witt 向量。对于截断的 Witt 环，有同构

$$BigWitt_n(A) \cong \bigoplus_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} W_{\ell(m,n)}(A),$$

其中 $\ell(m, n)$ 是一个整数，定义为

$$\ell(m, n) = 1 + \text{使得 } mp^k \leq n \text{ 成立的最大整数 } k.$$

考虑特征为 p 的有限域 $\mathbb{F}_q$，有同构 [13]

$$BigWitt_n(\mathbb{F}_q) \cong \bigoplus_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} W_{\ell(m,n)}(\mathbb{F}_q),$$

注意到两边都是 q^n 阶的群，因为 $\sum_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} \ell(m, n) = n$。

Corollary 7.6. 若有限域 $\mathbb{F}_{p^f}$ 的特征 $ch(\mathbb{F}_{p^f})=p$，则作为 *Abel* 群有

$$BigWitt_n(\mathbb{F}_{p^f})\cong\bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}}W_{1+\lfloor\log_p\frac{n}{m}\rfloor}(\mathbb{F}_{p^f})=\bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}}(\mathbb{Z}/p^{1+\lfloor\log_p\frac{n}{m}\rfloor}\mathbb{Z})^f,$$

其中 $\lfloor x\rfloor$ 表示不超过 x 的最大整数。

Example 7.7. $BigWitt_3(\mathbb{F}_2)=W_{\ell(1,3)}(\mathbb{F}_2)\oplus W_{\ell(3,3)}(\mathbb{F}_2)=W_2(\mathbb{F}_2)\oplus W_1(\mathbb{F}_2)=\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$，$BigWitt_4(\mathbb{F}_2)=W_{\ell(1,4)}(\mathbb{F}_2)\oplus W_{\ell(3,4)}(\mathbb{F}_2)=W_3(\mathbb{F}_2)\oplus W_1(\mathbb{F}_2)=\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$,$BigWitt_2(\mathbb{F}_3)=W_{\ell(1,2)}(\mathbb{F}_3)\oplus W_{\ell(2,2)}(\mathbb{F}_3)=W_1(\mathbb{F}_3)\oplus W_1(\mathbb{F}_3)=\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$。

7.3 $NK_2(\mathbb{F}_2[C_2])$

这一节计算 $k=\mathbb{F}_2, p=2, n=2$ 的情形，即 $NK_2(\mathbb{F}_2[C_2])\cong K_2(\mathbb{F}_2[t_1,t_2]/(t_1^2),(t_1))$.

Theorem 7.8. *(1)*$NK_2(\mathbb{F}_2[C_2])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}$,
(2)$NK_2(\mathbb{F}_2[C_2])\cong K_2(\mathbb{F}_2[t,x]/(t^2),(t))$ 是由 *Dennis-Stein* 符号 $\{\langle tx^i,x\rangle\mid i\ge 0\}$ 与 $\{\langle tx^i,t\rangle\mid i\ge 1$为奇数$\}$ 生成的，这样的符号均为 2 阶元。

Proof. (1) 令 $A=\mathbb{F}_2[t_1,t_2]/(t_1^2)=\mathbb{F}_2[C_2][x]$, 此时 $I=(t_1^2)$, $M=(t_1)$, $A/M=\mathbb{F}_2[x]$.

$$\begin{aligned}\Delta&=\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2\mid t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1^2)\}\\&=\{(\alpha_1,\alpha_2)\mid\alpha_1\ge 2,\alpha_2\ge 0\},\end{aligned}$$

$$\begin{aligned}\Lambda&=\{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid\alpha_i\ge 1,\text{且}t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1)\}\\&=\{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid\alpha_i\ge 1,\alpha_1\ge 1,\alpha_2\ge 0\}\\&=\{((\alpha_1,\alpha_2),1)\mid\alpha_1\ge 1,\alpha_2\ge 0\}\cup\{((\alpha_1,\alpha_2),2)\mid\alpha_1\ge 1,\alpha_2\ge 1\},\end{aligned}$$

若 $(\alpha,i)\in\Lambda$, $[\alpha,i]=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^i\in\Delta\}$, 于是有

$$\begin{aligned}[\alpha,1]&=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^1\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid(m\alpha_1-1,m\alpha_2)\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid m\alpha_1\ge 3\}.\end{aligned}$$

$$\begin{aligned}[\alpha,2]&=\min\{m\in\mathbb{Z}\mid m\alpha-\epsilon^2\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid(m\alpha_1,m\alpha_2-1)\in\Delta\}\\&=\min\{m\in\mathbb{Z}\mid m\alpha_1\ge 2\}.\end{aligned}$$

此时

$$[(1,\alpha_2),1]=3,\ \alpha_2\ge 0,$$
$$[(2,\alpha_2),1]=2,\ \alpha_2\ge 0,$$
$$[(\alpha_1,\alpha_2),1]=1,\ \alpha_1\ge 3,\alpha_2\ge 0,$$
$$[(1,\alpha_2),2]=2,\ \alpha_2\ge 1,$$
$$[(\alpha_1,\alpha_2),2]=1,\ \alpha_1\ge 2,\alpha_2\ge 1.$$

若 $gcd(2,\alpha_1,\alpha_2)=1$，即 α_1,α_2 中至少一个是奇数，令 $[\alpha]=\max\{[\alpha,i]\mid \alpha_i\not\equiv 0 \bmod 2\}$，$\alpha=(\alpha_1,\alpha_2)$, 若仅 α_1 是奇数，$[\alpha]=[\alpha,1]$, 若仅 α_2 是奇数，$[\alpha]=[\alpha,2]$，若两者均为奇数，则 $[\alpha]=\max\{[\alpha,1],[\alpha,2]\}$，有

$$[(1,\alpha_2)]=\max\{[(1,\alpha_2),1],[(1,\alpha_2),2]\}=3,\alpha_2\ge 1 \text{ 是奇数}$$
$$[(1,\alpha_2)]=[(1,\alpha_2),1]=3,\alpha_2\ge 0 \text{ 是偶数}$$
$$[(3,\alpha_2)]=\max\{[(3,\alpha_2),1],[(3,\alpha_2),2]\}=1,\alpha_2\ge 1 \text{ 是奇数}$$
$$[(3,\alpha_2)]=[(3,\alpha_2),1]=1,\alpha_2\ge 0 \text{ 是偶数}$$
$$[(2,1)]=[(2,1),2]=1,$$
$$[\alpha]=1, \text{其它符合条件的 } \alpha.$$

为了方便我们把上面的计算结果列表如下

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$3,\alpha_2\ge 0$	$2,\alpha_2\ge 1$	3
$(2,\alpha_2)$	$2,\alpha_2\ge 0$	$1,\alpha_2\ge 1$	1, 当 α_2 是奇数时
$(3,\alpha_2)$	$1,\alpha_2\ge 0$	$1,\alpha_2\ge 1$	1
$(\alpha_1,0),\alpha_1\ge 3$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2),\alpha_1\ge 3,\alpha_2\ge 1$	1	1	1, 当 $(\alpha_1,\alpha_2)=1$ 时

下面我们计算 $\Lambda^{00}=\big\{(\alpha,i)\in\Lambda\mid gcd(\alpha_1,\alpha_2)=1, i\ne\min\{j\mid \alpha_j\not\equiv 0 \bmod 2,[\alpha,j]=[\alpha]\}\big\}$，分情况来讨论

1. 对于任何的 $\alpha_2\ge 0$, $((1,\alpha_2),1)\notin\Lambda^{00}$, 这是因为 $1\not\equiv 0 \bmod 2$ 且 $[(1,\alpha_2),1]=3=[(1,\alpha_2)]$，从而 $\min\{j\mid \alpha_j\not\equiv 0 \bmod 2,[(1,\alpha_2),j]=[(1,\alpha_2)]\}=1$；
2. 对于任何的奇数 $\alpha_2\ge 0$，$((2,\alpha_2),1)\in\Lambda^{00}$，偶数 $\alpha_2\ge 0$，$((2,\alpha_2),1)\notin\Lambda^{00}$，因为 $\alpha_2\not\equiv 0 \bmod 2$ 并且 $[(2,\alpha_2),2]=1=[(2,\alpha_2)]$，故 $\{j\mid \alpha_j\not\equiv 0 \bmod 2,[(2,\alpha_2),j]=[(2,\alpha_2)]\}=2\ne 1$，此时 $[(2,\alpha_2),1]=2$；
3. 对于偶数 $\alpha_1\ge 3$ 和奇数 $\alpha_2\ge 1$, $((\alpha_1,\alpha_2),1)\in\Lambda^{00}$, 其余情况当 $\alpha_1\ge 3$ 为奇数或 α_1,α_2 均为偶数时 $((\alpha_1,\alpha_2),1)\notin\Lambda^{00}$。由于要求 $1\ne\min\{j\mid \alpha_j\not\equiv 0 \bmod 2,[(\alpha_1,\alpha_2),j]=[(\alpha_1,\alpha_2)]\}$，

当 $\alpha_1 \geq 3$ 为奇数时上式不成立，$2 = \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$ 当且仅当 $\alpha_1 \geq 3$ 为偶数且 $\alpha_2 \geq 1$ 为奇数，此时 $[(\alpha_1,\alpha_2),1] = 1$；

4. 对于任何的 $\alpha_2 \geq 1$，$((1,\alpha_2),2) \in \Lambda^{00}$，由于此时 $[(1,\alpha_2),1] = 3 = [(1,\alpha_2)]$，$\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\} = 1$，此时 $[(1,\alpha_2),2] = 2$；
5. 对于任何的奇数 $\alpha_2 \geq 1$，$((2,\alpha_2),2) \notin \Lambda^{00}$，由于 $[(2,\alpha_2),2] = 1 = [(2,\alpha_2)]$，与 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\}$ 矛盾；
6. 对于奇数 $\alpha_1 \geq 3$ 和任意 $\alpha_2 \geq 1$，$((\alpha_1,\alpha_2),2) \in \Lambda^{00}$，其余情况只要当 $\alpha_1 \geq 3$ 为偶数时 $((\alpha_1,\alpha_2),2) \notin \Lambda^{00}$。要求 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$，当 α_1 为偶数时上式不成立，而当 α_1 为奇数时，任意 $\alpha_2 \geq 1$，$[(\alpha_1,\alpha_2),1] = 1 = [(\alpha_1,\alpha_2)]$，此时 $[(\alpha_1,\alpha_2),2] = 1$。

从而

$$\begin{aligned}\Lambda^{00} =& \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}\\ &\cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}\\ &\cup \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3\text{为偶数}, \alpha_2 \geq 1\text{为奇数}\}\\ &\cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3\text{为奇数}, \alpha_2 \geq 1\}.\end{aligned}$$

记 $\Lambda_1^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 1\}$，$\Lambda_2^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 2\}$，我们有

$$\Lambda_1^{00} = \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3\text{为偶数}, \alpha_2 \geq 1\text{为奇数}\} \cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3\text{为奇数}, \alpha_2 \geq 1\}$$

$$\Lambda_2^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$$

$$\Lambda^{00} = \Lambda_1^{00} \sqcup \Lambda_2^{00}$$

若 $[\alpha,i] = 1$ 时，$(1 + x\mathbb{F}_2[x]/(x))^\times$ 是平凡的，$[\alpha,i] = 2$ 时，$(1 + x\mathbb{F}_2[x]/(x^2))^\times \cong \mathbb{Z}/2\mathbb{Z}$，从而由定理2.1得

$$\begin{aligned}NK_2(\mathbb{F}_2[C_2]) \cong K_2(A,M) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\ &= \bigoplus_{(\alpha,i)\in\Lambda_2^{00}} (1 + x\mathbb{F}_2[x]/(x^2))^\times\\ &= \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1 + x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1 + x\mathbb{F}_2[x]/(x^2))^\times\\ &= \bigoplus_{\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z},\end{aligned}$$

作为 Abel 群，

$$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z}.$$

(2) 由2.1，对于任意 $(\alpha, i) \in \Lambda^{00}$，$\Gamma_{\alpha,i}$ 诱导了同态

$$\begin{aligned}\Gamma_{\alpha,i}\colon (1+xk[x]/(x^{[\alpha,i]}))^{\times} &\longrightarrow K_2(A,M)\\ 1-xf(x) &\mapsto \langle f(t^{\alpha})t^{\alpha-\epsilon^i}, t_i\rangle.\end{aligned}$$

此时只需考虑 $\Lambda_2^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$，对于任意 $(\alpha, i) \in \Lambda_2^{00}$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2 \geq 1$，

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^{\times} &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle,\end{aligned}$$

对任意 $\alpha_2 \geq 1$ 为奇数，

$$\begin{aligned}\Gamma_{(2,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^{\times} &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle,\end{aligned}$$

我们作简单的替换令 $t = t_1, x = t_2$，于是 $\langle t_1t_2^{\alpha_2-1}, t_2\rangle = \langle tx^{\alpha_2-1}, x\rangle$，$\langle t_1t_2^{\alpha_2}, t_1\rangle = \langle tx^{\alpha_2}, t\rangle$。由同构2.1可知 $NK_2(\mathbb{F}_2[C_2])$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1\text{为奇数}\}$ 生成的，由于 $t^2 = 0$ 故 $\langle tx^i, x\rangle + \langle tx^i, x\rangle = \langle tx^i + tx^i - t^2x^{2i+1}, x\rangle = 0$，$\langle tx^i, t\rangle + \langle tx^i, t\rangle = \langle tx^i + tx^i - t^3x^{2i}, t\rangle = 0$。 □

Remark 7.9. 对于 $i \geq 1$为偶数，$\langle tx^i, t\rangle = \langle x^{i/2}, t\rangle + \langle x^{i/2}, t\rangle = \langle x^{i/2} + x^{i/2} + tx^i, t\rangle = 0$。

Weibel 在文献 [25] 中给出了以下可裂正合列

$$0 \longrightarrow V/\Phi(V) \xrightarrow{F} NK_2(\mathbb{F}_2[C_2]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0,$$

其中 $V = x\mathbb{F}_2[x]$，$\Phi(V) = x^2\mathbb{F}_2[x^2]$ 是 V 的子群，$\Omega_{\mathbb{F}_2[x]} \cong \mathbb{F}_2[x]\,dx$ 是绝对 Kähler 微分模，$F(x^n) = \langle tx^n, t\rangle$，$D(\langle ft, g+g't\rangle) = f\,dg$。显然 $D(\langle tx^i, t\rangle) = 0$，$D(\langle tx^i, x\rangle) = x^i\,dx$，可以看出 $NK_2(\mathbb{F}_2[C_2])$ 的直和项 $\bigoplus_{((2,\alpha_2),1),\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z} \cong V/\Phi(V)$，直和项 $\bigoplus_{((1,\alpha_2),2),\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[x]\,dx$。

V 和 $\Omega_{\mathbb{F}_2[x]}$ 作为 Abel 群是同构的，但作为 $W(\mathbb{F}_2)$-模是不同的。$V = x\mathbb{F}_2[x]$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [6]) 为

$$\begin{aligned}V_m(x^n) &= x^{mn},\\ F_d(x^n) &= \begin{cases} dx^{n/d}, & \text{若 } d|n \\ 0, & \text{其它}\end{cases},\\ [a]x^n &= a^nx^n.\end{aligned}$$

$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [6]) 为

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx,$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx.$$

结合两者我们可以得到 $NK_2(\mathbb{F}_2[C_2])$ 的 $W(\mathbb{F}_2)$-模结构为

$$V_m(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{mn}, t\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1 \text{ 为奇数}$$
$$V_m(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{mn-1}, x\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1$$
$$F_d(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{n/d}, t\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1 \text{ 为奇数}$$
$$F_d(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{n/d-1}, x\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1$$
$$[1]\langle tx^n, t\rangle = \langle tx^n, t\rangle, \quad n \geq 1 \text{ 为奇数}$$
$$[1]\langle tx^{n-1}, x\rangle = \langle tx^{n-1}, x\rangle, \quad n \geq 1.$$

7.4 $NK_2(\mathbb{F}_2[C_4])$

这一节首先用同样的方法计算 $NK_2(\mathbb{F}_2[C_{2^2}])$，继而对于任意 n 可以得到类似的结果。

Theorem 7.10. $NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$。

Proof. $\mathbb{F}_2[t_1, t_2]/(t_1^4) = \mathbb{F}_2[C_4][t_2]$，此时 $I = (t_1^4)$，$M = (t_1)$ 不变，我们直接写出以下集合

$$\Delta = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 4, \alpha_2 \geq 0\},$$
$$\Lambda = \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\},$$

用 $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 x 的最小整数，

$$[\alpha, 1] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 5\} = \lceil 5/\alpha_1 \rceil,$$
$$[\alpha, 2] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 4\} = \lceil 4/\alpha_1 \rceil.$$

例如

$$
\begin{aligned}
&[(1,\alpha_2),1]=5,\ \alpha_2\geq 0,\\
&[(2,\alpha_2),1]=3,\ \alpha_2\geq 0,\\
&[(3,\alpha_2),1]=2,\ \alpha_2\geq 0,\\
&[(4,\alpha_2),1]=2,\ \alpha_2\geq 0,\\
&[(\alpha_1,\alpha_2),1]=1,\ \alpha_1\geq 5,\alpha_2\geq 0,\\
&[(1,\alpha_2),2]=4,\ \alpha_2\geq 1,\\
&[(2,\alpha_2),2]=2,\ \alpha_2\geq 1,\\
&[(3,\alpha_2),2]=2,\ \alpha_2\geq 1,\\
&[(\alpha_1,\alpha_2),2]=1,\ \alpha_1\geq 4,\alpha_2\geq 1.
\end{aligned}
$$

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$5,\alpha_2\geq 0$	$4,\alpha_2\geq 1$	5
$(2,\alpha_2)$	$3,\alpha_2\geq 0$	$2,\alpha_2\geq 1$	2, 当 α_2 是奇数时
$(3,\alpha_2)$	$2,\alpha_2\geq 0$	$2,\alpha_2\geq 1$	2
$(4,\alpha_2)$	$2,\alpha_2\geq 0$	$1,\alpha_2\geq 1$	1当 α_2 是奇数时
$(\alpha_1,0),\alpha_1\geq 5$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2),\alpha_1\geq 5,\alpha_2\geq 1$	1	1	1, 当 $(\alpha_1,\alpha_2)=1$ 时

记 $\Lambda_d^{00}=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]=d\}$，$\Lambda_{>1}^{00}=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]>1\}$

由于 $(\alpha,i)\in\Lambda_1^{00}$ 均有 $[(\alpha,i)]=1$，实际上要计算 $(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ 只需确定 $\Lambda_{>1}^{00}$。由同样的方法可得 $\Lambda_4^{00}=\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}$，$\Lambda_3^{00}=\{((2,\alpha_2),1)\mid\alpha_2\geq 1$为奇数$\}$，$\Lambda_2^{00}=\{((3,\alpha_2),2)\mid gcd(3,\alpha_2)=1,\alpha_2\geq 1\}\cup\{((4,\alpha_2),1)\mid\alpha_2\geq 1$为奇数$\}$，

$$
\begin{aligned}
\Lambda_{>1}^{00}=&\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}\cup\{((3,\alpha_2),2)\mid gcd(3,\alpha_2)=1,\alpha_2\geq 1\}\\
&\cup\{((2,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}\cup\{((4,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}
\end{aligned}
$$

由定理2.1, 从而

$$
\begin{aligned}
NK_2(\mathbb{F}_2[C_4]) \cong K_2(A,M) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{\substack{((3,\alpha_2),2)\\ gcd(3,\alpha_2)=1\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((4,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
&\oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^3))^\times \oplus \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^4))^\times
\end{aligned}
$$

由7.4有 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$,$(1+x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$,于是 $NK_2(\mathbb{F}_2[C_4])$ 作为 Abel 群有

$$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}.$$

对于任意 $(\alpha,i)\in\Lambda^{00}_{>1}$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2\geq 1$，$gcd(3,\alpha_2)=1$

$$
\begin{aligned}
\Gamma_{(3,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1^3t_2^{\alpha_2-1}, t_2\rangle,
\end{aligned}
$$

对任意 $\alpha_2\geq 1$，

$$
\begin{aligned}
\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^4))^\times &\rightarrowtail K_2(A,M) \\
1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle, \\
1+x^3(\text{二阶元}) &\mapsto \langle t_1^3t_2^{3\alpha_2-1}, t_2\rangle,
\end{aligned}
$$

对任意 $\alpha_2\geq 1$ 为奇数，

$$
\begin{aligned}
\Gamma_{(4,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1^3t_2^{\alpha_2}, t_1\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{(2,\alpha_1),1}\colon (1+x\mathbb{F}_2[x]/(x^3))^\times &\rightarrowtail K_2(A,M) \\
1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle.
\end{aligned}
$$

我们作简单的替换令 $t=t_1, x=t_2$，由同构2.1可知 $NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号 $\{\langle tx^{i-1},x\rangle \mid i\geq 1\}$, $\{\langle t^3x^{3i-1},x\rangle \mid i\geq 1\}$, $\{\langle t^3x^{i-1},x\rangle \mid i\geq 1, gcd(i,3)=1\}$, $\{\langle tx^i,t\rangle \mid i\geq 1\text{为奇数}\}$, $\{\langle t^3x^i,t\rangle \mid i\geq 1\text{为奇数}\}$ 生成的。

□

Remark 7.11. $\langle t^3x^{2i}, t\rangle = \langle tx^i, t\rangle + \langle tx^i, t\rangle$ 是二阶元。根据 [16]，存在同态

$$\begin{aligned} \rho_1 \colon \mathbb{F}_2[x]dx &\longrightarrow NK_2(\mathbb{F}_2[C_4]) \\ x^i dx &\mapsto \langle t^3x^i, x\rangle \\ \rho_2 \colon x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2] &\longrightarrow NK_2(\mathbb{F}_2[C_4]) \\ x^i &\mapsto \langle t^3x^i, t\rangle \end{aligned}$$

$\{\langle t^3x^{i-1}, x\rangle \mid i \geq 1\} = \{\langle t^3x^{3i-1}, x\rangle \mid i \geq 1\} \cup \{\langle t^3x^{i-1}, x\rangle \mid i \geq 1, gcd(i,3) = 1\}$，从而 $\Omega_{\mathbb{F}_2[x]} \oplus x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2]$ 是 $NK_2(\mathbb{F}_2[C_4])$ 的直和项。

7.5 $NK_2(\mathbb{F}_q[C_{2^n}])$

设 $\mathbb{F}_q$ 是特征为 2 的有限域，$q = 2^f$，C_{2^n} 是 2^n 阶循环群，这一节计算 $NK_2(\mathbb{F}_q[C_{2^n}])$。假设 $A = \mathbb{F}_q[t_1, t_2]/(t_1^{2^n}) = \mathbb{F}_q[C_{2^n}][x]$, 此时 $I = (t_1^{2^n})$, $M = (t_1)$, $A/M = \mathbb{F}_q[x]$。

Lemma 7.12. $\Delta = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 2^n, \alpha_2 \geq 0\}$，$\Lambda = \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1, \alpha_2 \geq 0\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\}$，对任意 $(\alpha, i) \in \Lambda$，$[\alpha, 1] = \lceil (2^n+1)/\alpha_1 \rceil$，$[\alpha, 2] = \lceil 2^n/\alpha_1 \rceil$，其中 $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 x 的最小整数,。

Lemma 7.13. 令 $I_1 = \{((\alpha_1, \alpha_2), 1) \mid gcd(\alpha_1, \alpha_2) = 1, 1 < \alpha_1 \leq 2^n$为偶数$, \alpha_2 \geq 1$为奇数$\}$，$I_2 = \{((\alpha_1, \alpha_2), 2) \mid gcd(\alpha_1, \alpha_2) = 1, 1 \leq \alpha_1 < 2^n$为奇数$, \alpha_2 \geq 1\}$，则 $\Lambda^{00}_{>1} = I_1 \sqcup I_2$。

由定理2.1，

$$\begin{aligned} NK_2(\mathbb{F}_q[C_{2^n}]) \cong K_2(A, M) \cong &\bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ = &\bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ = &\bigoplus_{(\alpha,1)\in I_1} (1 + x\mathbb{F}_q[x]/(x^{\lceil (2^n+1)/\alpha_1 \rceil}))^\times \\ \oplus &\bigoplus_{(\alpha,2)\in I_2} (1 + x\mathbb{F}_q[x]/(x^{\lceil 2^n/\alpha_1 \rceil}))^\times. \end{aligned}$$

注意到 $BigWitt_k(R) = (1 + xR[\![x]\!])^\times/(1 + x^{k+1}R[\![x]\!])^\times \cong (1 + xR[x]/(x^{k+1}))^\times$，根据公式7.6，

$$
\begin{aligned}
NK_2(\mathbb{F}_q[C_{2^n}]) \cong & \bigoplus_{(\alpha,1)\in I_1} \bigoplus_{\substack{1\le m\le \lceil (2^n+1)/\alpha_1\rceil -1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil (2^n+1)/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f \\
& \oplus \bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le \lceil 2^n/\alpha_1\rceil -1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil 2^n/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f.
\end{aligned}
$$

接下来我们证明对于任意 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 都在 $NK_2(\mathbb{F}_q[C_{p^n}])$ 出现无限多次

Lemma 7.14. 对于任意的 $1\le k<n$，$1+\left\lfloor \log_2(\frac{2^n-1}{2^k+1})\right\rfloor = n-k$。

Proof. 当 $1\le k<n$ 时，$2^k-1\ge 1\ge \frac{1}{2^{n-k-1}}$，即

$$2^{n-1}-2^{n-k-1}\ge 1$$

上式等价于 $2^n-1\ge 2^{n-k-1}(2^k+1)$，且 $2^n-1<2^{n-k}(2^k+1)$，于是

$$2^{n-k}>\frac{2^n-1}{2^k+1}\ge 2^{n-k-1}$$

取对数得 $\left\lfloor \log_2(\frac{2^n-1}{2^k+1})\right\rfloor = n-k-1$。 □

考虑 $((1,\alpha_2),2)\in I_2$，

$$\bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le 2^n-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{2^n-1}{m}\right\rfloor}\mathbb{Z})^f$$

是 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 的直和项，当 $m=1$ 时 $1+\lfloor \log_2(2^n-1)\rfloor = n$，当 $m=2^k+1(1\le k<n)$ 为奇数时，由7.14，$1+\left\lfloor \log_2\frac{2^n-1}{m}\right\rfloor = n-k$，于是对于任何的 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 均出现在直和项中，且对于任意 $\alpha_2\ge 1$，这样的项总会出现，于是

$$NK_2(\mathbb{F}_q[C_{2^n}])\cong \bigoplus_{\infty}\bigoplus_{k=1}^{n}\mathbb{Z}/2^k\mathbb{Z}.$$

接下来给出一些 $NK_2(\mathbb{F}_q[C_{2^n}])$ 中的 $2^k(1\le k\le n)$ 阶元素。

对任意 $\alpha_2\ge 1, a\in\mathbb{F}_q$，

$$
\begin{aligned}
\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_q[x]/(x^{2^n}))^\times &\rightarrowtail K_2(A,M) \\
1+ax(2^n\text{ 阶元}) &\mapsto \langle atx^{\alpha_2-1},x\rangle, \\
1+ax^3(2^{n-1}\text{ 阶元}) &\mapsto \langle at^3x^{3\alpha_2-1},x\rangle, \\
1+ax^{2^k+1}(2^{n-k}\text{ 阶元}) &\mapsto \langle at^{2k+1}x^{(2k+1)\alpha_2-1},x\rangle.
\end{aligned}
$$

7.6 NK_2 of finite abelian p-groups

7.7 其他可以考虑的问题

$NK_2(\mathbb{F}_{p^m}[C_{p^n}]) =?$

$\mathbb{F}_2[C_2 \times C_2] \cong \mathbb{F}_2[C_2] \otimes \mathbb{F}_2[C_2] \cong \mathbb{F}_2[x,y]/(x^2,y^2)$，看看能否用同样的方法得到一些结果。

$$0 \longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n)) \longrightarrow K_2(k[t_3]) \longrightarrow 0$$

对于有限域 k 来讲 $K_2(k[t_3]) = 0$,

$$0 \longrightarrow NK_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2][x]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow 0,$$

中间那项就是可以用这篇文章里的方法确定，又 $K_2(\mathbb{F}_2[C_2 \times C_2])$ 可以通过 Gao Yubin 等文章得到，应该是 C_2^3, 于是可以得到 $NK_2(\mathbb{F}_2[C_2 \times C_2])$，猜测也是 $\oplus_\infty \mathbb{Z}/2\mathbb{Z}$.

另外可以考虑直接用本章里的方式重新计算高玉彬师兄文章里的结果，看是否更简洁，或者是否更繁复，复杂在哪里，哪里可以进行简化，简化后是否可以用到算 NK 的内容中。

一个关于模结构的问题，在 Weibel 的文章 [26] 中 5.5 和 5.7 给出的模结构和本文上面的模结构并不一致，用 V_m 作用差一个 t^m。

Chapter 8

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

8.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

8.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [28] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

8.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1, t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:

generators: $\langle a, b\rangle$, $(a, b) \in A \times M \cup M \times A$;

relations: $\langle a, b\rangle = -\langle b, a\rangle$,

$\langle a, b\rangle + \langle c, b\rangle = \langle a + c - abc, b\rangle$,

$\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [23]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1, 0) \in \mathbb{N}^2$, $\epsilon^2 = (0, 1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,

- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i\rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 8.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [23]。 □

Lemma 8.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$*,* $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$*.*

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x\rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3\rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

8.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 8.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$*.*

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4),(t_1,t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. □

Theorem 8.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)$, then $I = (t_1^4,t_2^4)$, $M = (t_1,t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1,\alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha,i) \mid \alpha_i \geq 1\}.$$

For $(\alpha,i) \in \Lambda$,

$$[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha,i) \in \Lambda^{00} \mid [(\alpha,i)] > 1\}$, and then

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha,1) \in \Lambda^{00}_{>1}$ and $[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha,2) \in \Lambda^{00}_{>1}$ and $[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1,\alpha_2 \leq 4$ are both odd and $gcd(\alpha_1,\alpha_2) = 1$, then $(\alpha,2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha,1]$.

By the computation 2.2, we can get the following table

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below, the generators of order 4:

$$\langle t_1t_2, t_1\rangle, \langle t_1t_2, t_2\rangle, \langle t_1, t_2\rangle,$$

the generators of order 2:

$$\langle t_1t_2^3, t_1\rangle, \langle t_1^3t_2, t_1\rangle, \langle t_1^3t_2^3, t_1\rangle, \langle t_1t_2^3, t_2\rangle, \langle t_1^3t_2^2, t_2\rangle, \langle t_1t_2^2, t_2\rangle, \langle t_1^3t_2, t_2\rangle, \langle t_1^3t_2^3, t_2\rangle, \langle t_1^3, t_2\rangle.$$

$\square$

Remark 8.5. Compared with [28], note that $\langle t_1^3, t_2\rangle = \langle t_1^2t_2, t_1\rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1t_2\rangle - \langle t_1^2t_2, t_1\rangle \\
&= \langle t_1, t_1^2t_2\rangle - \langle t_1^2t_2, t_1\rangle - \langle t_1^2t_2, t_1\rangle \\
&= -3\langle t_1^2t_2, t_1\rangle \\
&= -\langle t_1^2t_2, t_1\rangle \\
&= \langle t_1^2t_2, t_1\rangle,
\end{aligned}$$

since $\langle t_1^2t_2, t_1\rangle + \langle t_1^2t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

Chapter 9

$NK_i(FG)$

9.1 Preliminaries

Definition 9.1. If F is any functor from the category of rings to the category of abelian groups,we write $NF(R)$ for the cokernel of the natural map $F(R) \xrightarrow{x\mapsto 1} F(R[x])$; NF is also a functor on rings. Moreover, the ring map $R[x] \longrightarrow R$ provides a splitting $F(R[x]) \longrightarrow F(R)$ of the natural map, so we have a decomposition $F(R[x]) \cong F(R) \oplus NF(R)$.

In particular, when F is K_i we have functors NK_i and a decomposition $K_i(R[x]) = K_i(R) \oplus NK_i(R)$. Since the ring maps $R[x] \xrightarrow{x\mapsto r} R$ are split surjections for every $r \in R$, hence for every r we also have $NK_0(R) \cong K_0(R[x], (x-r))$ and $NK_1(R) \cong K_1(R[x], (x-r))$.

Let U be a functor from the category of commutative rings to the category of abelian groups,

$$\begin{aligned} U\colon \mathbf{CRing} &\longrightarrow \mathbf{Ab} \\ R &\mapsto U(R) = R^{\times}. \end{aligned}$$

We can also define the functor NU.

Lemma 9.2. *Let R be a commutative ring with nilradical $\mathfrak{N}$. If $r_0 + r_1x + \cdots + r_nx^n$ is a unit of $R[x]$ then $r_0 \in R^{\times}$ and $r_1, \cdots, r_n$ are nilpotent. We have*

1. *$NU(R)$ is the subgroup $1 + x\mathfrak{N}[x]$ of $R[x]^{\times}$;*
2. *$R[x]^{\times} = R^{\times} \oplus NU(R)$;*
3. *R is reduced if and only if $R^{\times} = R[x]^{\times}$;*
4. *Suppose that R is an algebra over a field k. If $char(k) = p$, $NU(R)$ is a p-group. If $char(k) = 0$, $NU(R)$ is a uniquely divisible abelian group (= a $\mathbb{Q}$-module).*

Definition 9.3. Let R be a commutative ring, the determinant of a matrix provides a group homomorphism from $GL(R)$ onto the group $R^\times$ of units of R. Define $SK_1(R)$ to be the kernel of the induced surjection $\det\colon K_1(R) \longrightarrow R^\times$. Moreover, there is a direct sum decomposition $K_1(R) = R^\times \oplus SK_1(R)$.

Example 9.4. If F is a field then $SK_1(F) = 0$. Similarly, if R is a Euclidean domain such as $\mathbb{Z}$ or $F[x]$ then $SK_1(R) = 0$ and hence $K_1(R) = R^\times$. In particular, $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ and $K_1(F[x]) = F^\times$. If F is a finite field extension of $\mathbb{Q}$ (a number field) and R is an integrally closed subring of F , then Bass, Milnor and Serre proved that $SK_1(R) = 0$.

Lemma 9.5. *If R is a commutative semilocal ring, then $SK_1(R) = 0$ and $K_1(R) = R^\times$. In particular, if $(R, \mathfrak{m})$ is a commutative local ring, then $SK_1(R) = 0$.*

Proof. See [27] lemma 3.1.4. □

Lemma 9.6. *Let R ba a commutative regular ring, $A = R[t]/(t^N)$, then*

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$\begin{aligned} NK_1(A) \cong \mathrm{Nil}_0(A) &\cong (1 + txA[x])^\times = (1 + xA[x])^\times \\ [(A,t)] &\mapsto 1 - tx \\ [(P,\nu)] &\mapsto \det(1 - \nu x) \end{aligned}$$

Proof. See [27] chapter 2 example 7.4.5 and chapter 3 example 3.8.1. □

9.2 Main Theorem

Theorem 9.7. *(1) Let R be a commutative semilocal ring, if $NK_1(R) = 0$, then $SK_1(R[x]) = 0$ and $R[x]^\times = R^\times$ (which means R is reduced);*
(2) If R is a reduced commutative semilocal ring, then $NK_1(R) = SK_1(R[x])$;
(3) If R is a commutative semilocal ring but not reduced, then $NK_1(R) \neq 0$, this is equivalent to say that $NK_1(R)$ is not finitely generated.

Proof. Let R be a commutative semilocal ring. There are two split exact sequences

$$0 \longrightarrow SK_1(R[x]) \longrightarrow K_1(R[x]) \longrightarrow R[x]^\times \longrightarrow 0,$$

$$0 \longrightarrow NK_1(R) \longrightarrow K_1(R[x]) \longrightarrow K_1(R) \longrightarrow 0.$$

Recall that $SK_1(R) = 0$ for any commutative semilocal ring R, hence $K_1(R) \cong R^\times$. We have

$$\begin{aligned} K_1(R[x]) &= NK_1(R) \oplus R^\times \\ &= SK_1(R[x]) \oplus R[x]^\times \\ &= SK_1(R[x]) \oplus R^\times \oplus NU(R) \\ &= SK_1(R[x]) \oplus R^\times \oplus (1 + x\mathfrak{N}[x]), \end{aligned}$$

hence $NK_1(R) \cong SK_1(R[x]) \oplus (1 + x\mathfrak{N}[x])$.

(1) If $NK_1(R) = 0$, one can obtain $SK_1(R[x]) = 0$ and $NU(R) = 0$;

(2) Since $NU(R) = 0$;

(3) In this case $NK_1(R) \supset 1 + x\mathfrak{N}[x] \neq 0$, and it is known that NK_i is either 0 or is not finitely generated. □

Theorem 9.8. *If F is a field of characteristic $p > 0$ and G is a finite abelian p-group, then $NK_1(FG) \neq 0$. In particular $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \neq 0$.*

Proof. First we claim that $R = FG$ is a commutative local ring. In fact suppose F is a field of characteristic p and G is a finite group, then FG is a local ring if and only if G is a finite p-group, in this case the maximal ideal is the augmentation ideal $\mathfrak{m} = IG$ which is also nilpotent, see [11]. And it is easy to see that R is not reduced: take an element a of the maximal order $o(a)$, $o(a) = p^m \leq |G|$ for some m, then $a^{o(a)/p} - 1$ is a nilpotent element since $(a^{o(a)/p} - 1)^p = 0$.

Since local rings are semilocal, we conclude that $NK_1(R) \neq 0$ by Theorem 9.7(3). □

Theorem 9.9. *Let $\mathbb{F}$ be a finite field of characteristic $p > 0$ and H is a finite group such that $(p, |H|) = 1$, then $NK_i(\mathbb{F}H) = 0$ for all i.*

Proof. By Maschke's theorem, $\mathbb{F}H$ is semisimple. Since it is finite, by Wedderburn-Artin theorem $\mathbb{F}H$ is a direct product of matrix rings over finite fields F of characteristic p, $\mathbb{F}H \cong \prod_m M_m(F)$ and $\mathbb{F}H[x] \cong \prod_m M_m(F[x])$. Hence $K_i(\mathbb{F}H)$ is isomorphic to a direct product of groups $K_i(F)$, and $NK_i(\mathbb{F}H)$ is isomorphic to a direct product of groups $NK_i(F)$for all i. Since finite fields are regular, we obtain $NK_i(\mathbb{F}H) = 0$. □

Since $\mathbb{F}_{p^m}[C_{p^n}] = \mathbb{F}_{p^m}[t]/(t^{p^n})$ and $\mathbb{F}_{p^m}$ is regular, we can use Lemma 9.6 to describe $NK_1(\mathbb{F}_{p^m}[C_{p^n}])$ as follows.

Theorem 9.10. $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \cong$. 还没算完

这个证明还有问题

Proof. By lemma 9.6, $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \cong (1+tx\mathbb{F}_{p^m}[t]/(t^{p^n})[x])^\times$. As an abelian group, we have

$$(1+tx\mathbb{F}_{p^m}[t]/(t^{p^n})[x])^\times = (1+tx\mathbb{F}_{p^m}[x][\![t]\!])^\times/(1+t^{p^n}x\mathbb{F}_{p^m}[x][\![t]\!])^\times = W_{p^n-1}(x\mathbb{F}_{p^m}[x]).$$

□

由同态

$$(1+t\mathbb{F}_3[\![t]\!])^\times \longrightarrow (1+t\mathbb{F}_3[t]/(t^3))^\times$$

它的核是 $(1+t^3\mathbb{F}_3[\![t]\!])^\times$. 从而 $(1+t\mathbb{F}_3[t]/(t^3))^\times \cong (1+t\mathbb{F}_3[\![t]\!])^\times/(1+t^3\mathbb{F}_3[\![t]\!])^\times$, 又有 $(1+t\mathbb{F}_3[\![t]\!])^\times/(1+t^3\mathbb{F}_3[\![t]\!])^\times = W_2(\mathbb{F}_3)$. 注意这里的 $W(R)$ 是 big Witt 向量，$W_2(\mathbb{F}_3) \neq \mathbb{Z}/\mathbb{Z}^9$

算 $(1+t\mathbb{F}_3[t]/(t^3))^\times$ 中的元素，只有 3 阶元:

$(1+t(a_0+a_1t))(1+t(b_0+b_1t)) = 1+t(a_0+b_0+(a_1+b_1+a_0b_0)t)$, 考虑 pair$(a_0,a_1)$，想找到它和 $\mathbb{Z}/9\mathbb{Z}$ 之间的对应，但是经过计算这样的 pair 没有 9 阶元，比如 $(2,1)(2,1)(2,1) = (1,0)(2,1) = (0,0)$.

最终算出来 $(1+t\mathbb{F}_3[t]/(t^3))^\times = \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

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致谢

Chapter 10

有限域总结

基本的结果：

- Every finite field has prime power order. Every finite field must have characteristic p for some prime p.
- For every prime power $q = p^n$, there is a finite field of that order. Any finite field with $q = p^n$ elements is isomorphic to the splitting field of $x^q - x$ over $\mathbb{F}_p$.
- Any two finite fields of the same size are isomorphic (usually not in just one way).
- A subfield of $\mathbb{F}_{p^n}$ has order p^d where $d|n$, and there is one such subfield for each d.
- Let F be a finite field containing a subfield K with q elements. Then F has q^m elements, where $m = [F : K]$.
- Let F be a finite field. Then F has p^n elements, where the prime p is the characteristic of F and n is the degree of F over its prime subfield.
- For a prime p and positive integer n, there is an irreducible $g(x)$ of degree n in $\mathbb{F}_p[x]$, and $\mathbb{F}_p[x]/(g(x))$ is a field of order p^n. 这样构造的域也称为 Galois 域，the term Galois field lives on today among coding theorists in computer science and electrical engineering as a synonym for finite field and Moore’s notation $GF(q)$ is often used in place of $\mathbb{F}_q$.
- If $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = d$, the $\mathbb{F}_p$-conjugates of α are $\alpha, \alpha^p, \alpha^{p^2}, \cdots, \alpha^{p^{d-1}}$.
- Every finite extension of $\mathbb{F}_p$ is a Galois extension whose Galois group over $\mathbb{F}_p$ is generated by the pth power map. The Galois group $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic and a generator is the p-th power map $\phi_p \colon t \mapsto t^p$ on $\mathbb{F}_{p^n}$.
- If F is a finite field with q elements, then every $a \in F$ satisfies $a^q = a$.
- If F is a finite field with q elements and K is a subfield of F, then the polynomial $x^q - x$ in $K[x]$ factors in $F[x]$ as $x^q - x = \prod_{a \in F}(x - a)$ and F is a splitting field of $x^q - x$ over K.
- If F is a finite field of order q, the group $F^\times$ is cyclic of order $q - 1$.

例子：

- $\mathbb{F}_4$ as $\mathbb{F}_2(\theta) = \{0, 1, \theta, \theta+1\}$, where $\theta^2+\theta+1 = 0$, we find that both θ and $\theta+1$ are primitive elements.
- Two fields of order 8 are $\mathbb{F}_2[x]/(x^3+x+1)$ and $\mathbb{F}_2[x]/(x^3+x^2+1)$.
- Two fields of order 9 are $\mathbb{F}_3[x]/(x^2+1)$ and $\mathbb{F}_3[x]/(x^2+x+2)$.
- The polynomial x^3-2 is irreducible in $\mathbb{F}_7[x]$, so $\mathbb{F}_7[x]/(x^3-2)$ is a field of order $7^3 = 343$.

警告：The ring $\mathbb{Z}/(m)$ is a field only when m is a prime number. In order to create fields of non-prime size we must do something other than look at $\mathbb{Z}/(m)$. Every finite field is isomorphic to a field of the form $\mathbb{F}_p[x]/(f(x))$

In the field $\mathbb{F}_3[x]/(x^2+1)$, the nonzero numbers are a group of order 8. The powers of x only take on 4 values, so x is not a generator. The element $x+1$ is a generator: its successive powers are exhaust all the nonzero elements of $\mathbb{F}_3[x]/(x^2+1)$.

k	1	2	3	4	5	6	7	8
x^k	$x+1$	$2x$	$2x+1$	2	$2x+2$	x	$x+2$	1

计算 For every $f(x) \in \mathbb{F}_p[x]$, $f(x)^{p^m} = f(x^{p^m})$ for $m \geq 0$.

Combinatorics. An important theme in combinatorics is q-analogues, which are algebraic expressions in a variable q that become classical objects when $q = 1$, or when $q \mapsto 1$. For example, the q-binomial coefficient is

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})},$$

which for $n \geq k$ is a polynomial in q with integer coefficients. When $q \mapsto 1$ this has the value $\binom{n}{k}$. While $\binom{n}{k}$ counts the number of k-element subsets of a finite set, when q is a prime power the number $\binom{n}{k}_q$ counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$. Identities involving q-binomial coefficients can be proved by checking them when q runs through prime powers, using linear algebra over the fields $\mathbb{F}_q$.

参考文献

[1] Günter F. Pilz (auth.) Alexander V. Mikhalev. *The Concise Handbook of Algebra*. Springer Netherlands, 1 edition, 2002.

[2] Gert Almkvist. The Grothendieck ring of the category of endomorphisms. *J. Algebra*, 28:375–388, 1974.

[3] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[4] Hyman Bass and M. Pavaman Murthy. Grothendieck groups and Picard groups of abelian group rings. *Ann. of Math. (2)*, 86:16–73, 1967.

[5] Spencer Bloch. Algebraic K-theory and crystalline cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (47):187–268 (1978), 1977.

[6] B. H. Dayton and C. A. Weibel. Module structures on the Hochschild and cyclic homology of graded rings. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, pages 63–90. Kluwer Acad. Publ., Dordrecht, 1993.

[7] F. T. Farrell. The nonfiniteness of Nil. *Proc. Amer. Math. Soc.*, 65(2):215–216, 1977.

[8] Eric Friedlander and MR Stein. *Algebraic K-theory. Proc. conf. Evanston, 1980*. Springer, 1981.

[9] Daniel R. Grayson. Grothendieck rings and Witt vectors. *Comm. Algebra*, 6(3):249–255, 1978.

[10] Dominique Guin-Waléry and Jean-Louis Loday. *Algebraic K-Theory Evanston 1980: Proceedings of the Conference Held at Northwestern University Evanston, March 24–27, 1980*, chapter Obstruction a l'Excision En K-Theorie Algebrique, pages 179–216. Springer Berlin Heidelberg, Berlin, Heidelberg, 1981.

[11] Srikanth Iyengar. Modules and cohomology over group algebras: One commutative algebraist's perspective. *Math.sci.res.inst.publ*, pages 51–86, 2010.

[12] Frans Keune. The relativization of K_2. *Journal of Algebra*, 54(1):159–177, 1978.

[13] Kristin Lauter. A formula for constructing curves over finite fields with many rational points. *Journal of Number Theory*, 74(1):5672, 1999.

[14] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[15] A. V. Prasolov. The nonfiniteness of the group Nil. *Mat. Zametki*, 32(1):9–12, 124, 1982.

[16] Leslie G. Roberts. K_2 of some truncated polynomial rings. In *Ring theory (Proc. Conf., Univ. Waterloo, Waterloo, 1978)*, pages 249–278. Springer, Berlin, 1979. With a section written jointly with S. Geller.

[17] Dennis R.Keith Keating Michael E. Stein, Michael R. Lower bounds for the order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. *Mathematische Annalen*, 223:97–104, 1976.

[18] Michael R. Stein. Excision and K_2 of group rings. *Journal of Pure and Applied Algebra*, 18(2):213 – 224, 1980.

[19] Jan Stienstra. On K_2 and K_3 of truncated polynomial rings. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, pages 409–455. Springer, Berlin, 1981.

[20] Richard G. Swan. Excision in algebraic K-theory. *Journal of Pure and Applied Algebra*, 1(3):221 – 252, 1971.

[21] Wilberd van der Kallen. Le K_2 des nombres duaux. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A1204–A1207, 1971.

[22] Wilberd van der Kallen. Generators and relations in algebraic K-theory. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 305–310, Helsinki, 1980. Acad. Sci. Fennica.

[23] Wilberd van der Kallen and Jan Stienstra. The relative K_2 of truncated polynomial rings. In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pages 277–289, 1984.

[24] C. A. Weibel. Mayer-Vietoris sequences and module structures on NK_*. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, pages 466–493. Springer, Berlin, 1981.

[25] Charles Weibel. NK_0 and NK_1 of the groups C_4 and D_4. *Comment. Math. Helv*, 84:339–349, 2009.

[26] Charles A. Weibel. Module structures on the K-theory of graded rings. *J. Algebra*, 105(2):465–483, 1987.

[27] Charles A Weibel. *The K-book: An introduction to algebraic K-theory*. American Mathematical Society Providence (RI), 2013.

[28] 唐国平陈虹, 高玉彬. $K_2(\mathbb{F}_2[C_4 \times C_4])$ 的计算. 中国科学院大学学报, 28(4):419, 2011.

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