Chapter 1

$NK_2(\mathbb{F}_2[C_4])$

$$NK_i(R) = \ker(NK_i(R[x]) \xrightarrow{x \mapsto 0} K_i(R)).$$

目标

1. 证明 $NK_2(\mathbb{F}_2[C_4])$ 中有四阶元，
2. 确定它的结构和生成元。
3. 推广到 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 和 $NK_p(\mathbb{F}_{p^f}[C_{p^n}])$。
4. 未来推广到 $NK_2(\mathbb{F}_{p^f}[G])$，其中 G 是有限交换群，$G = C_{p^n} \times H$。

1.1 思路

已经有 $NK_2(\mathbb{F}_2[C_2])$ 和 $NK_2(\mathbb{F}_p[C_p])$ 的结果 [2]。这种方法详细见下文1.3.1。

考虑 $\mathbb{F}_{p^f}[C_{p^n}]$ 时，可以将它写成截断多项式 $F_{p^f}[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$。

考虑它的 NK_2 时，可以转化成相对 K_2 群：

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1, t_2]/(t_1^{p^n}), (t_1)) \cong K_2(\mathbb{F}_{p^f}[t_1, t_2]/(t_1^{p^n})).$$

利用 van der Kallen [7] 中的方法对于这种情形的相对 K_2 群，有这样的结论 (符号说明见后文):

Theorem. $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

接下来的任务就是确定两件事情：

- 确定集合Λ^{00}，确定数值$[\alpha, i]$。
- 确定右边这个乘法群的结构。

实际上对于第一件事情，我们只需要考虑这个集合的一部分，因为 $(1+xk[x]/(x))^\times$ 是平凡的，所以只要考虑 $[\alpha,i]>1$ 所对应的 (α,i) 全体（后文详细解释）。

对于第二件事情，有一些例子是可以直接计算的，如 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$。对于一般的情况引入 big Witt vectors，

$$BigWitt_n(\mathbb{F}_q) := (1+x\mathbb{F}_q[\![x]\!])^\times/(1+x^{n+1}\mathbb{F}_q[\![x]\!])^\times \cong (1+x\mathbb{F}_q[x]/(x^{n+1}))^\times$$

由 big Witt vectors分解成 typical Witt vectors有

$$\begin{aligned}(1+x\mathbb{F}_{p^f}[x]/(x^{n+1}))^\times &\cong BigWitt_n(\mathbb{F}_{p^f})\\ &\cong \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} W_{1+\lfloor \log_p \frac{n}{m}\rfloor}(\mathbb{F}_{p^f})\\ &= \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor \log_p \frac{n}{m}\rfloor}\mathbb{Z})^f,\end{aligned}$$

其中 $\lfloor x\rfloor$ 表示不超过 x 的最大整数。

结合这个同构2.1就可以得到结论。

结论 已知的结果

Theorem. *(1)$NK_2(\mathbb{F}_2[C_2])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_2])\cong K_2(\mathbb{F}_2[t,x]/(t^2),(t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i,x\rangle \mid i\ge 0\}$ 与 $\{\langle tx^i,t\rangle \mid i\ge 1为奇数\}$ 生成的，这样的符号均为 2 阶元。

$NK_2(\mathbb{F}_2[C_4])$ 的结果

Theorem. *(1)$NK_2(\mathbb{F}_2[C_4])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_\infty\mathbb{Z}/4\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号

$$\{\langle tx^{i-1},x\rangle \mid i\ge 1\},\{\langle tx^i,t\rangle \mid i\ge 1为奇数\},\{\langle t^3x^{i-1},x\rangle \mid i\ge 1\},\{\langle t^3x^i,t\rangle \mid i\ge 1为奇数\}$$

生成的。

$NK_2(\mathbb{F}_q[C_{2^n}])$ 的结果

$$NK_2(\mathbb{F}_q[C_{2^n}])\cong\bigoplus_\infty\bigoplus_{k=1}^n\mathbb{Z}/2^k\mathbb{Z}.$$

1.2 预备知识和引理

这一节主要是介绍前文提到的如何将 NK_2 转化成相对 K_2 群，[7] 中符号说明与相对 K_2 群，以及 Witt 向量的分解。

令 k 是特征为 $p>0$ 的有限域，考虑两个变元的多项式环 $k[t_1,t_2]$，令 I 是 $k[t_1,t_2]$ 的一个真理想，满足以下条件

1. I 是由 $k[t_1]$ 中的单项式生成的，
2. 对于某个 n，$t_1^n \in I$.

实际上这样的 I 具有形式 (t_1^n), 令

$$A = k[t_1,t_2]/I,$$

M 是 A 的 nil 根 (小根)，即 $M=(t_1)$，那么有 $A/M = k[t_2]$.

Proposition 1.1. $K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \cong K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$.

Proof. 首先有下面两个相对 K 群的正合列

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k[t_2]) \longrightarrow 0$$

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k) \longrightarrow 0$$

由于 k 是有限域，它是正则环，于是有 $K_2(k)=0$, $K_2(k[t_2]) = K_2(k)\oplus NK_2(k) = 0$. 从而可以得到

$$K_2(k[t_1,t_2]/(t_1^n),(t_1)) \cong K_2(k[t_1,t_2]/(t_1^n)) \cong K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)).$$

□

当 $k=\mathbb{F}_{p^f}$ 时，$k[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$，其中 C_{p^n} 是 p^n 阶循环群。有以下可裂正合列

$$0 \longrightarrow NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow 0,$$

由于 $K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) = K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}))$，并且 $K_2(\mathbb{F}_{p^f}[C_{p^n}])=0$，从而

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)).$$

1.2.1 Dennis-Stein 符号

回到一般情形，$K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$ 可以用 Dennis-Stein 符号表示生成元 $\langle a,b\rangle$, $(a,b)\in A\times M \cup M\times A$;

关系 $\langle a,b\rangle = -\langle b,a\rangle$,

$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc,b\rangle$,

$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ 其中 $(a,b,c) \in A\times M\times A \cup M\times A\times M$.

Proposition 1.2. 对任意环 R，$q\geq 1$，$K_2(R[t]/(t^q),(t))$ 由 *Dennis-Stein* 符号 $\langle at^i,t\rangle$ 和 $\langle at^i,b\rangle$ 生成，其中 $a,b\in R, 1\leq i<q$。

Proof. 参见文献 [6]。 □

1.2.2 符号说明

为了表述方便，遵从 [7] 的符号详述如下

- $\mathbb{Z}_+$: 非负整数全体，
- $\epsilon^1 = (1,0)\in\mathbb{Z}_+^2, \epsilon^2 = (0,1)\in\mathbb{Z}_+^2$,
- 对于 $\alpha\in\mathbb{Z}_+^2$, 记 $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2}$, 于是有 $t^{\epsilon^1} = t_1, t^{\epsilon^2} = t_2$,
- $\Delta = \{\alpha\in\mathbb{Z}_+^2 \mid t^\alpha\in I\}$,
- $\Lambda = \{(\alpha,i)\in\mathbb{Z}_+^2\times\{1,2\} \mid \alpha_i\geq 1, t^\alpha\in M\}$, 若 $\delta\in\Delta$, 则有 $\delta+\epsilon^i\in\Delta, I=1,2$,
- 对于 $(\alpha,i)\in\Lambda$, 令$[\alpha,i] = \min\{m\in\mathbb{Z} \mid m\alpha-\epsilon^i\in\Delta\}$, 若 $(\alpha,i),(\alpha,j)\in\Lambda$, 有 $[\alpha,i]\leq[\alpha,j]+1$,
- 若 $gcd(p,\alpha_1,\alpha_2)=1$, 令 $[\alpha] = \max\{[\alpha,i] \mid \alpha_i\not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \{(\alpha,i)\in\Lambda \mid gcd(\alpha_1,\alpha_2)=1, i\neq\min\{j \mid \alpha_j\not\equiv 0 \bmod p, [\alpha,j]=[\alpha]\}\}$.

若 $(\alpha,i)\in\Lambda$, $f(x)\in k[x]$, 令

$$\Gamma_{\alpha,i}(1-xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i},t_i\rangle,$$

若 $g(t_1,t_2) = t_ih(t_1,t_2)\in\sqrt{I} = (t_1)$, 令

$$\Gamma_i(1-g(t_1,t_2)) = \langle h(t_1,t_2),t_i\rangle,$$

且有

$$\Gamma_{\alpha,i}(1-xf(x)) = \Gamma_i(1-t^\alpha f(t^\alpha)).$$

由于 $t_1\in\sqrt{I}, \Gamma_1$ 诱导了同态

$$\begin{aligned}(1+t_1k[t_1,t_2]/t_1I)^\times &\longrightarrow K_2(A,M)\\ 1-g(t_1,t_2) &\mapsto \langle h(t_1,t_2),t_1\rangle\end{aligned}$$

Γ_2 诱导了同态

$$\begin{aligned}(1+t_2\sqrt{I}/t_2I)^\times &\longrightarrow K_2(A,M)\\ 1-g(t_1,t_2) &\mapsto \langle h(t_1,t_2),t_2\rangle\end{aligned}$$

若 $(\alpha,i)\in\Lambda$, $\Gamma_{\alpha,i}$ 诱导了同态

$$(1+xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A,M).$$

Theorem 1.3. $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A,M)\cong\bigoplus_{(\alpha,i)\in\Lambda^{00}}(1+xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. 参见文献 [7]。 □

1.2.3 Witt 向量

令 R 是一个交换环，big Witt 环 (the ring of universal/big Witt vectors over R, 泛 Witt 环)$BigWitt(R)$ 作为 Abel 群同构于 $(1+xR[\![x]\!])^\times$，即常数项为 1 的形式幂级数全体在乘法运算下形成的交换群，

$$\begin{aligned}BigWitt(R) &\longrightarrow (1+xR[\![x]\!])^\times\\ (r_1,r_2,\cdots) &\mapsto \prod_i(1-r_ix^i)^{-1}.\end{aligned}$$

考虑子群 $(1+x^{n+1}R[\![x]\!])^\times$，定义 $BigWitt_n(R)=(1+xR[\![x]\!])^\times/(1+x^{n+1}R[\![x]\!])^\times$。显然 $BigWitt_1(R)=R$，并且当 $n\geq 3$ 时，$BigWitt_n(\mathbb{F}_2)$ 不是循环群。

Lemma 1.4. *$BigWitt_n(\mathbb{F}_q)\cong(1+x\mathbb{F}_q[x]/(x^{n+1}))^\times$.*

Proof. 由定义 $BigWitt_n(\mathbb{F}_q):=(1+x\mathbb{F}_q[\![x]\!])^\times/(1+x^{n+1}\mathbb{F}_q[\![x]\!])^\times$，且有同态

$$\begin{aligned}(1+x\mathbb{F}_q[\![x]\!])^\times &\longrightarrow (1+x\mathbb{F}_q[x]/(x^{n+1}))^\times\\ 1+\sum_{i\geq 1}a_ix^i &\mapsto 1+\sum_{i=1}^n a_ix^i\end{aligned}$$

容易看出核是 $(1+x^{n+1}\mathbb{F}_2[\![x]\!])^\times$，从而 $BigWitt_n(\mathbb{F}_q)\cong(1+x\mathbb{F}_q[x]/(x^{n+1}))^\times$。 □

Example 1.5. $BigWitt_3(\mathbb{F}_2)\cong(1+x\mathbb{F}_2[x]/(x^4))^\times\cong\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$

Proof. 考虑 $1+x \in (1+x\mathbb{F}_2[x]/(x^4))^\times$ 是 4 阶元，由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3\}$, 且 $1+x^3$ 是二阶元，令 σ,τ 分别是 $\mathbb{Z}/4\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned}\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma,\tau) &\mapsto (1+x)(1+x^3) = 1+x+x^3.\end{aligned}$$

□

Example 1.6. $BigWitt_4(\mathbb{F}_2) \cong \mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Proof. 考虑 $1+x \in BigWitt_5(\mathbb{F}_2)$, 它是 8 阶元，由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3, 1+x^4, 1+x+x^4, 1+x^2+x^4, 1+x+x^2+x^3+x^4\}$，另外 $1+x^3$ 是二阶元，令 σ,τ 分别是 $\mathbb{Z}/8\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元，则有同构

$$\begin{aligned}\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma,\tau) &\mapsto (1+x)(1+x^3) = 1+x+x^3+x^4\end{aligned}$$

于是 $(\sigma^i,\tau^j), 0\le i<8, 0\le j<2$ 对应于 $(1+x)^i(1+x^3)^j$, 详细的对应如下

$$\begin{aligned}
(1,\tau) &\mapsto 1+x^3, & (\sigma,\tau) &\mapsto 1+x+x^3+x^4,\\
(\sigma^2,\tau) &\mapsto 1+x^2+x^3, & (\sigma^3,\tau) &\mapsto 1+x+x^2+x^4,\\
(\sigma^4,\tau) &\mapsto 1+x^3+x^4, & (\sigma^5,\tau) &\mapsto 1+x+x^3,\\
(\sigma^6,\tau) &\mapsto 1+x^2+x^3+x^4, & (\sigma^7,\tau) &\mapsto 1+x+x^2,\\
(1,1) &\mapsto 1, & (\sigma,1) &\mapsto 1+x,\\
(\sigma^2,1) &\mapsto 1+x^2, & (\sigma^3,1) &\mapsto 1+x+x^2+x^3,\\
(\sigma^4,1) &\mapsto 1+x^4, & (\sigma^5,1) &\mapsto 1+x+x^4,\\
(\sigma^6,1) &\mapsto 1+x^2+x^4, & (\sigma^7,1) &\mapsto 1+x+x^2+x^3+x^4.
\end{aligned}$$

□

固定素数 p，考虑局部环 $\mathbb{Z}_{(p)} = \mathbb{Z}[1/\ell \mid 所有素数\ell \ne p]$，即 $\mathbb{Z}$ 在素理想 $(p) = p\mathbb{Z}$ 的局部化，于是一个 $\mathbb{Z}_{(p)}$-代数 R 就是除 p 外的素数均可逆的交换环，如 $\mathbb{F}_{p^n}$ 是 $\mathbb{Z}_{(p)}$-代数。

考虑 p-Witt 环 $W(A)$ 与截断 p-Witt 环 $W_n(A)$，p-Witt 向量为 $(a_0, a_1, \cdots)$，加法用 Witt 多项式定义，以下仅考虑用加法定义的 Abel 群结构，例如 $W(\mathbb{F}_p) = \mathbb{Z}_p$，作为 Abel 群 $W_n(\mathbb{F}_{p^f})$ 同构于 $(\mathbb{Z}/p^n\mathbb{Z})^f$。

Artin-Hasse 级数定义为

$$AH(x) = \exp(-\sum_{n\geq 0} \frac{x^{p^n}}{p^n}) = 1 - x + \cdots \in 1 + x\mathbb{Q}[\![x]\!],$$

实际上 $AH(x) \in 1 + x\mathbb{Z}_{(p)}[\![x]\!]$。对于 $BigWitt(R) = 1 + xR[\![x]\!]$ 中的任一元素 α 存在以下写成无穷乘积的表法

$$\alpha = \prod_{n\geq 1}(1 - r_n x^n),\ r_n \in R,$$

若 A 是 $\mathbb{Z}_{(p)}$-代数，$BigWitt(A) = 1 + xA[\![x]\!]$ 中的任一元素 α 还有如下表法

$$\alpha = \prod_{n\geq 1} AH(a_n x^n),\ a_n \in A.$$

将整数 n 写成 $n = mp^a$，使得 $gcd(m,p) = 1, a \geq 0$，由于 A 是 $\mathbb{Z}_{(p)}$-代数，m 可逆，从而 $[x \mapsto x^{1/m}] \in \mathrm{End}(BigWitt(A))$ 是双射，于是我们可以将 $\alpha \in BigWitt(A)$ 以如下的形式表出

$$\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}.$$

另一方面对于 $\mathbb{Z}_{(p)}$-代数 A，下列映射是群同态

$$\begin{aligned} W(A) &\longrightarrow BigWitt(A) \\ (a_0, a_1, \cdots) &\mapsto \prod_{i\geq 0} AH(a_i x^i). \end{aligned}$$

$BigWitt_n(A)$ 可以分解为 p-Witt 环的直和，实际上有以下同构

$$BigWitt(A) \cong \prod_{\substack{m\geq 1\\ gcd(m,p)=1}} W(A),$$

元素 $\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}$ 对应于一个 m-分量为 $(a_m, a_{mp}, a_{mp^2}, \cdots) \in W(A)$ 的 Witt 向量。对于截断的 Witt 环，有同构

$$BigWitt_n(A) \cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(A),$$

其中 $\ell(m,n)$ 是一个整数，定义为

$$\ell(m,n) = 1 + \text{使得 } mp^k \leq n \text{ 成立的最大整数 } k.$$

考虑特征为 p 的有限域 $\mathbb{F}_q$，有同构 [3]

$$BigWitt_n(\mathbb{F}_q) \cong \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} W_{\ell(m,n)}(\mathbb{F}_q),$$

注意到两边都是 q^n 阶的群，因为 $\sum\limits_{\substack{1\le m\le n\\ gcd(m,p)=1}} \ell(m,n) = n$。

Corollary 1.7. 若有限域 $\mathbb{F}_{p^f}$ 的特征 $ch(\mathbb{F}_{p^f}) = p$，则作为 *Abel* 群有

$$BigWitt_n(\mathbb{F}_{p^f}) \cong \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} W_{1+\lfloor \log_p \frac{n}{m}\rfloor}(\mathbb{F}_{p^f}) = \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor \log_p \frac{n}{m}\rfloor}\mathbb{Z})^f,$$

其中 $\lfloor x\rfloor$ 表示不超过 x 的最大整数。

Example 1.8. $BigWitt_3(\mathbb{F}_2) = W_{\ell(1,3)}(\mathbb{F}_2) \oplus W_{\ell(3,3)}(\mathbb{F}_2) = W_2(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$，

$BigWitt_4(\mathbb{F}_2) = W_{\ell(1,4)}(\mathbb{F}_2) \oplus W_{\ell(3,4)}(\mathbb{F}_2) = W_3(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$，

$BigWitt_2(\mathbb{F}_3) = W_{\ell(1,2)}(\mathbb{F}_3) \oplus W_{\ell(2,2)}(\mathbb{F}_3) = W_1(\mathbb{F}_3) \oplus W_1(\mathbb{F}_3) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$。

1.3 $NK_2(\mathbb{F}_2[C_2])$ 的计算

方法一是在讲 Weibel 文章 [9] 时讲过的。方法二是基于上面的思路给出来的详细证明。

1.3.1 方法一

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1,\sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{1.8}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma \mapsto -1} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0.$

$$\begin{tikzcd}
 & NK_3(\mathbb{Z},(2)) = 0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \arrow[d, "\cong"] & & \\
0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2],(\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\
 & NK_2(\mathbb{Z},(2)) = 0 & &
\end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2], (\sigma+1), (\sigma-1))$, from Guin-Loday-Keune [2], $NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 1.9. $NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 1.10 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd}
\mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "f"', "\sigma\mapsto 1"] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\
\mathbb{Z} \arrow[r] & \mathbb{F}_p
\end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1) \otimes (1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1) \otimes (1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R, I)$$
$$(\sigma-1) \otimes (1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1, 1\rangle^p = 1$$

Also see [5].

Example 1.11 ($\mathbb{Z}[C_p][x]$). There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

1.3.2 方法二

计算 $k = \mathbb{F}_2, p = 2, n = 2$ 的情形，即 $NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^2), (t_1))$.

Theorem 1.12. *(1)$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t, x]/(t^2), (t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1 \text{为奇数}\}$ 生成的，这样的符号均为 2 阶元。

Proof. (1) 令 $A = \mathbb{F}_2[t_1, t_2]/(t_1^2) = \mathbb{F}_2[C_2][x]$, 此时 $I = (t_1^2)$, $M = (t_1)$, $A/M = \mathbb{F}_2[x]$.

$$\begin{aligned} \Delta &= \{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \mid t_1^{\alpha_1} t_2^{\alpha_2} \in (t_1^2)\} \\ &= \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 2, \alpha_2 \geq 0\}, \end{aligned}$$

$$\begin{aligned} \Lambda &= \{((\alpha_1, \alpha_2), i) \in \mathbb{Z}_+^2 \times \{1, 2\} \mid \alpha_i \geq 1, \text{且} t_1^{\alpha_1} t_2^{\alpha_2} \in (t_1)\} \\ &= \{((\alpha_1, \alpha_2), i) \in \mathbb{Z}_+^2 \times \{1, 2\} \mid \alpha_i \geq 1, \alpha_1 \geq 1, \alpha_2 \geq 0\} \\ &= \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1, \alpha_2 \geq 0\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\}, \end{aligned}$$

若 $(\alpha, i) \in \Lambda$, $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$, 于是有

$$\begin{aligned} [\alpha, 1] &= \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^1 \in \Delta\} \\ &= \min\{m \in \mathbb{Z} \mid (m\alpha_1 - 1, m\alpha_2) \in \Delta\} \\ &= \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 3\}. \end{aligned}$$

$$\begin{aligned}[\alpha,2] &= \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^2 \in \Delta\} \\ &= \min\{m \in \mathbb{Z} \mid (m\alpha_1, m\alpha_2 - 1) \in \Delta\} \\ &= \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 2\}.\end{aligned}$$

此时

$$\begin{aligned}[(1,\alpha_2),1] &= 3,\ \alpha_2 \geq 0, \\ [(2,\alpha_2),1] &= 2,\ \alpha_2 \geq 0, \\ [(\alpha_1,\alpha_2),1] &= 1,\ \alpha_1 \geq 3, \alpha_2 \geq 0, \\ [(1,\alpha_2),2] &= 2,\ \alpha_2 \geq 1, \\ [(\alpha_1,\alpha_2),2] &= 1,\ \alpha_1 \geq 2, \alpha_2 \geq 1.\end{aligned}$$

若 $gcd(2,\alpha_1,\alpha_2) = 1$，即 α_1, α_2 中至少一个是奇数，令 $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod 2\}, \alpha = (\alpha_1, \alpha_2)$, 若仅 α_1 是奇数，$[\alpha] = [\alpha, 1]$, 若仅 α_2 是奇数，$[\alpha] = [\alpha, 2]$，若两者均为奇数，则 $[\alpha] = \max\{[\alpha,1],[\alpha,2]\}$，有

$$\begin{aligned}[(1,\alpha_2)] &= \max\{[(1,\alpha_2),1],[(1,\alpha_2),2]\} = 3, \alpha_2 \geq 1 \text{ 是奇数} \\ [(1,\alpha_2)] &= [(1,\alpha_2),1] = 3, \alpha_2 \geq 0 \text{ 是偶数} \\ [(3,\alpha_2)] &= \max\{[(3,\alpha_2),1],[(3,\alpha_2),2]\} = 1, \alpha_2 \geq 1 \text{ 是奇数} \\ [(3,\alpha_2)] &= [(3,\alpha_2),1] = 1, \alpha_2 \geq 0 \text{ 是偶数} \\ [(2,1)] &= [(2,1),2] = 1, \\ [\alpha] &= 1, \text{其它符合条件的 } \alpha.\end{aligned}$$

为了方便我们把上面的计算结果列表如下

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$3, \alpha_2 \geq 0$	$2, \alpha_2 \geq 1$	3
$(2,\alpha_2)$	$2, \alpha_2 \geq 0$	$1, \alpha_2 \geq 1$	1, 当 α_2 是奇数时
$(3,\alpha_2)$	$1, \alpha_2 \geq 0$	$1, \alpha_2 \geq 1$	1
$(\alpha_1,0), \alpha_1 \geq 3$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2), \alpha_1 \geq 3, \alpha_2 \geq 1$	1	1	1, 当 $(\alpha_1,\alpha_2) = 1$ 时

下面我们计算 $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1,\alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha, j] = [\alpha]\}\big\}$，

分情况来讨论

1. 对于任何的 $\alpha_2 \geq 0$，$((1,\alpha_2),1) \notin \Lambda^{00}$，这是因为 $1 \not\equiv 0 \bmod 2$ 且 $[(1,\alpha_2),1] = 3 = [(1,\alpha_2)]$，从而 $\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(1,\alpha_2),j] = [(1,\alpha_2)]\} = 1$；
2. 对于任何的奇数 $\alpha_2 \geq 0$，$((2,\alpha_2),1) \in \Lambda^{00}$，偶数 $\alpha_2 \geq 0$，$((2,\alpha_2),1) \notin \Lambda^{00}$，因为 $\alpha_2 \not\equiv 0 \bmod 2$ 并且 $[(2,\alpha_2),2] = 1 = [(2,\alpha_2)]$，故 $\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(2,\alpha_2),j] = [(2,\alpha_2)]\} = 2 \neq 1$，此时 $[(2,\alpha_2),1] = 2$；
3. 对于偶数 $\alpha_1 \geq 3$ 和奇数 $\alpha_2 \geq 1$,$((\alpha_1,\alpha_2),1) \in \Lambda^{00}$,其余情况当 $\alpha_1 \geq 3$ 为奇数或 α_1,α_2 均为偶数时 $((\alpha_1,\alpha_2),1) \notin \Lambda^{00}$。由于要求 $1 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$，当 $\alpha_1 \geq 3$ 为奇数时上式不成立，$2 = \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$ 当且仅当 $\alpha_1 \geq 3$ 为偶数且 $\alpha_2 \geq 1$ 为奇数，此时 $[(\alpha_1,\alpha_2),1] = 1$；
4. 对于任何的 $\alpha_2 \geq 1$，$((1,\alpha_2),2) \in \Lambda^{00}$，由于此时 $[(1,\alpha_2),1] = 3 = [(1,\alpha_2)]$，$\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\} = 1$，此时 $[(1,\alpha_2),2] = 2$；
5. 对于任何的奇数 $\alpha_2 \geq 1$，$((2,\alpha_2),2) \notin \Lambda^{00}$，由于 $[(2,\alpha_2),2] = 1 = [(2,\alpha_2)]$，与 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\}$ 矛盾；
6. 对于奇数 $\alpha_1 \geq 3$ 和任意 $\alpha_2 \geq 1$，$((\alpha_1,\alpha_2),2) \in \Lambda^{00}$，其余情况只要当 $\alpha_1 \geq 3$ 为偶数时 $((\alpha_1,\alpha_2),2) \notin \Lambda^{00}$。要求 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$，当 α_1 为偶数时上式不成立，而当 α_1 为奇数时，任意 $\alpha_2 \geq 1$，$[(\alpha_1,\alpha_2),1] = 1 = [(\alpha_1,\alpha_2)]$，此时 $[(\alpha_1,\alpha_2),2] = 1$。

   从而

$$\begin{aligned}\Lambda^{00} =& \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \\ & \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\} \\ & \cup \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3\text{为偶数}, \alpha_2 \geq 1\text{为奇数}\} \\ & \cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3\text{为奇数}, \alpha_2 \geq 1\}.\end{aligned}$$

记 $\Lambda_1^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 1\}$，$\Lambda_2^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 2\}$，我们有

$$\Lambda_1^{00} = \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3\text{为偶数}, \alpha_2 \geq 1\text{为奇数}\} \cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3\text{为奇数}, \alpha_2 \geq 1\}$$

$$\Lambda_2^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$$

$$\Lambda^{00} = \Lambda_1^{00} \sqcup \Lambda_2^{00}.$$

若 $[\alpha,i] = 1$ 时，$(1 + x\mathbb{F}_2[x]/(x))^\times$ 是平凡的，$[\alpha,i] = 2$ 时，$(1 + x\mathbb{F}_2[x]/(x^2))^\times \cong \mathbb{Z}/2\mathbb{Z}$，从而由定理2.1得

$$
\begin{aligned}
NK_2(\mathbb{F}_2[C_2]) \cong K_2(A,M) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{(\alpha,i)\in\Lambda_2^{00}} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
&= \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
&= \bigoplus_{\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z},
\end{aligned}
$$

作为 Abel 群，

$$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z}.$$

(2) 由2.1，对于任意 $(\alpha,i)\in\Lambda^{00}$，$\Gamma_{\alpha,i}$ 诱导了同态

$$
\begin{aligned}
\Gamma_{\alpha,i}\colon (1+xk[x]/(x^{[\alpha,i]}))^\times &\longrightarrow K_2(A,M) \\
1-xf(x) &\mapsto \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i\rangle.
\end{aligned}
$$

此时只需考虑 $\Lambda_2^{00}=\{((2,\alpha_2),1)\mid \alpha_2\geq 1\text{为奇数}\}\cup\{((1,\alpha_2),2)\mid \alpha_2\geq 1\}$，对于任意 $(\alpha,i)\in\Lambda_2^{00}$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2\geq 1$，

$$
\begin{aligned}
\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle,
\end{aligned}
$$

对任意 $\alpha_2\geq 1$ 为奇数，

$$
\begin{aligned}
\Gamma_{(2,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M) \\
1+x &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle,
\end{aligned}
$$

我们作简单的替换令 $t=t_1, x=t_2$，于是 $\langle t_1t_2^{\alpha_2-1},t_2\rangle=\langle tx^{\alpha_2-1},x\rangle$，$\langle t_1t_2^{\alpha_2},t_1\rangle=\langle tx^{\alpha_2},t\rangle$。由同构2.1可知 $NK_2(\mathbb{F}_2[C_2])$ 是由 Dennis-Stein 符号 $\{\langle tx^i,x\rangle\mid i\geq 0\}$ 与 $\{\langle tx^i,t\rangle\mid i\geq 1\text{为奇数}\}$ 生成的，由于 $t^2=0$ 故 $\langle tx^i,x\rangle+\langle tx^i,x\rangle=\langle tx^i+tx^i-t^2x^{2i+1},x\rangle=0$，$\langle tx^i,t\rangle+\langle tx^i,t\rangle=\langle tx^i+tx^i-t^3x^{2i},t\rangle=0$。 □

Remark 1.13. 对于 $i\geq 1$为偶数，$\langle tx^i,t\rangle=\langle x^{i/2},t\rangle+\langle x^{i/2},t\rangle=\langle x^{i/2}+x^{i/2}+tx^i,t\rangle=0$。

Weibel 在文献 [9] 中给出了以下可裂正合列

$$0\longrightarrow V/\Phi(V)\xrightarrow{F} NK_2(\mathbb{F}_2[C_2])\xrightarrow{D}\Omega_{\mathbb{F}_2[x]}\longrightarrow 0,$$

其中 $V = x\mathbb{F}_2[x]$，$\Phi(V) = x^2\mathbb{F}_2[x^2]$ 是 V 的子群，$\Omega_{\mathbb{F}_2[x]} \cong \mathbb{F}_2[x]\,dx$ 是绝对 Kähler 微分模，$F(x^n) = \langle tx^n, t\rangle$，$D(\langle ft, g+g't\rangle) = f\,dg$。显然 $D(\langle tx^i, t\rangle) = 0$，$D(\langle tx^i, x\rangle) = x^i\,dx$，可以看出 $NK_2(\mathbb{F}_2[C_2])$ 的直和项 $\bigoplus_{((2,\alpha_2),1),\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z} \cong V/\Phi(V)$，直和项 $\bigoplus_{((1,\alpha_2),2),\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[x]\,dx$。

V 和 $\Omega_{\mathbb{F}_2[x]}$ 作为 Abel 群是同构的，但作为 $W(\mathbb{F}_2)$-模是不同的。$V = x\mathbb{F}_2[x]$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$\begin{aligned}
V_m(x^n) &= x^{mn},\\
F_d(x^n) &= \begin{cases} dx^{n/d}, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},\\
[a]x^n &= a^n x^n.
\end{aligned}$$

$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$\begin{aligned}
V_m(x^{n-1}\,dx) &= mx^{mn-1}\,dx,\\
F_d(x^{n-1}\,dx) &= \begin{cases} x^{n/d-1}\,dx, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},\\
[a]x^{n-1}\,dx &= a^n x^{n-1}\,dx.
\end{aligned}$$

结合两者我们可以得到 $NK_2(\mathbb{F}_2[C_2])$ 的 $W(\mathbb{F}_2)$-模结构为

$$\begin{aligned}
V_m(\langle tx^n, t\rangle) &= \begin{cases} \langle tx^{mn}, t\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1 \text{ 为奇数}\\
V_m(\langle tx^{n-1}, x\rangle) &= \begin{cases} \langle tx^{mn-1}, x\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1\\
F_d(\langle tx^n, t\rangle) &= \begin{cases} \langle tx^{n/d}, t\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1 \text{ 为奇数}\\
F_d(\langle tx^{n-1}, x\rangle) &= \begin{cases} \langle tx^{n/d-1}, x\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1\\
[1]\langle tx^n, t\rangle &= \langle tx^n, t\rangle, \quad n \geq 1 \text{ 为奇数}\\
[1]\langle tx^{n-1}, x\rangle &= \langle tx^{n-1}, x\rangle, \quad n \geq 1.
\end{aligned}$$

1.4 $NK_2(\mathbb{F}_2[C_4])$ 的结构

用同样的方法计算 $NK_2(\mathbb{F}_2[C_{2^2}])$，继而对于任意 n 可以得到类似的结果。

Theorem 1.14. $NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$。

Proof. $\mathbb{F}_2[t_1,t_2]/(t_1^4) = \mathbb{F}_2[C_4][t_2]$，此时 $I=(t_1^4)$，$M=(t_1)$ 不变，我们直接写出以下集合

$$\Delta = \{(\alpha_1,\alpha_2) \mid \alpha_1 \geq 4, \alpha_2 \geq 0\},$$
$$\Lambda = \{((\alpha_1,\alpha_2),1) \mid \alpha_1 \geq 1\} \cup \{((\alpha_1,\alpha_2),2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\},$$

用 $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 x 的最小整数，

$$[\alpha,1] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 5\} = \lceil 5/\alpha_1 \rceil,$$
$$[\alpha,2] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 4\} = \lceil 4/\alpha_1 \rceil.$$

例如

$$[(1,\alpha_2),1] = 5,\ \alpha_2 \geq 0,$$
$$[(2,\alpha_2),1] = 3,\ \alpha_2 \geq 0,$$
$$[(3,\alpha_2),1] = 2,\ \alpha_2 \geq 0,$$
$$[(4,\alpha_2),1] = 2,\ \alpha_2 \geq 0,$$
$$[(\alpha_1,\alpha_2),1] = 1,\ \alpha_1 \geq 5, \alpha_2 \geq 0,$$
$$[(1,\alpha_2),2] = 4,\ \alpha_2 \geq 1,$$
$$[(2,\alpha_2),2] = 2,\ \alpha_2 \geq 1,$$
$$[(3,\alpha_2),2] = 2,\ \alpha_2 \geq 1,$$
$$[(\alpha_1,\alpha_2),2] = 1,\ \alpha_1 \geq 4, \alpha_2 \geq 1.$$

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$5,\alpha_2 \geq 0$	$4,\alpha_2 \geq 1$	5
$(2,\alpha_2)$	$3,\alpha_2 \geq 0$	$2,\alpha_2 \geq 1$	2, 当 α_2 是奇数时
$(3,\alpha_2)$	$2,\alpha_2 \geq 0$	$2,\alpha_2 \geq 1$	2
$(4,\alpha_2)$	$2,\alpha_2 \geq 0$	$1,\alpha_2 \geq 1$	1当 α_2 是奇数时
$(\alpha_1,0),\alpha_1 \geq 5$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2),\alpha_1 \geq 5,\alpha_2 \geq 1$	1	1	1, 当 $(\alpha_1,\alpha_2)=1$ 时

记 $\Lambda^{00}_d=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]=d\}$，$\Lambda^{00}_{>1}=\{(\alpha,i)\in\Lambda^{00}|[(\alpha,i)]>1\}$

由于 $(\alpha,i)\in\Lambda^{00}_1$ 均有 $[(\alpha,i)]=1$，实际上要计算 $(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ 只需确定 $\Lambda^{00}_{>1}$。由同样的方法可得 $\Lambda^{00}_4=\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}$，$\Lambda^{00}_3=\{((2,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}$，$\Lambda^{00}_2=\{((3,\alpha_2),2)\mid gcd(3,\alpha_2)=1,\alpha_2\geq 1\}\cup\{((4,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}$，

$$\begin{aligned}\Lambda^{00}_{>1}=&\{((1,\alpha_2),2)\mid\alpha_2\geq 1\}\cup\{((3,\alpha_2),2)\mid gcd(3,\alpha_2)=1,\alpha_2\geq 1\}\\&\cup\{((2,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}\cup\{((4,\alpha_2),1)\mid\alpha_2\geq 1\text{为奇数}\}.\end{aligned}$$

由定理2.1，

$$\begin{aligned}NK_2(\mathbb{F}_2[C_4])\cong K_2(A,M)\cong&\bigoplus_{(\alpha,i)\in\Lambda^{00}}(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\=&\bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}}(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\=&\bigoplus_{\substack{((3,\alpha_2),2)\\gcd(3,\alpha_2)=1\\\alpha_2\geq 1}}(1+x\mathbb{F}_2[x]/(x^2))^\times\oplus\bigoplus_{\substack{((4,\alpha_2),1)\\\alpha_2\geq 1\text{为奇数}}}(1+x\mathbb{F}_2[x]/(x^2))^\times\\&\oplus\bigoplus_{\substack{((2,\alpha_2),1)\\\alpha_2\geq 1\text{为奇数}}}(1+x\mathbb{F}_2[x]/(x^3))^\times\oplus\bigoplus_{\substack{((1,\alpha_2),2)\\\alpha_2\geq 1}}(1+x\mathbb{F}_2[x]/(x^4))^\times.\end{aligned}$$

由1.5有 $(1+x\mathbb{F}_2[x]/(x^4))^\times\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$，$(1+x\mathbb{F}_2[x]/(x^3))^\times\cong\mathbb{Z}/4\mathbb{Z}$，于是 $NK_2(\mathbb{F}_2[C_4])$ 作为 Abel 群有

$$NK_2(\mathbb{F}_2[C_4])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_\infty\mathbb{Z}/4\mathbb{Z}.$$

对于任意 $(\alpha,i)\in\Lambda^{00}_{>1}$，$\Gamma_{\alpha,i}$ 均诱导了单射，对任意 $\alpha_2\geq 1$，$gcd(3,\alpha_2)=1$

$$\begin{aligned}\Gamma_{(3,\alpha_2),2}\colon(1+x\mathbb{F}_2[x]/(x^2))^\times&\rightarrowtail K_2(A,M)\\1+x&\mapsto\langle t_1^3t_2^{\alpha_2-1},t_2\rangle,\end{aligned}$$

对任意 $\alpha_2\geq 1$，

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon(1+x\mathbb{F}_2[x]/(x^4))^\times&\rightarrowtail K_2(A,M)\\1+x(\text{四阶元})&\mapsto\langle t_1t_2^{\alpha_2-1},t_2\rangle,\\1+x^3(\text{二阶元})&\mapsto\langle t_1^3t_2^{3\alpha_2-1},t_2\rangle,\end{aligned}$$

对任意 $\alpha_2 \geq 1$ 为奇数，

$$\begin{aligned}\Gamma_{(4,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1^3 t_2^{\alpha_2}, t_1\rangle,\\ \Gamma_{(2,\alpha_1),1}\colon (1+x\mathbb{F}_2[x]/(x^3))^\times &\rightarrowtail K_2(A,M)\\ 1+x(\text{四阶元}) &\mapsto \langle t_1 t_2^{\alpha_2}, t_1\rangle.\end{aligned}$$

我们作简单的替换令 $t=t_1, x=t_2$，由同构2.1可知 $NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号 $\{\langle tx^{i-1},x\rangle \mid i\geq 1\},\{\langle tx^i,t\rangle \mid i\geq 1\text{为奇数}\},\{\langle t^3x^{3i-1},x\rangle \mid i\geq 1\},\{\langle t^3x^{i-1},x\rangle \mid i\geq 1, gcd(i,3)=1\},\{\langle t^3x^i,t\rangle \mid i\geq 1\text{为奇数}\}$ 生成的。

□

Remark 1.15. $\langle t^3x^{2i},t\rangle = \langle tx^i,t\rangle + \langle tx^i,t\rangle$ 是二阶元。根据 [4]，存在同态

$$\begin{aligned}\rho_1\colon \mathbb{F}_2[x]dx &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^i dx &\mapsto \langle t^3x^i,x\rangle\\ \rho_2\colon x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2] &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^i &\mapsto \langle t^3x^i,t\rangle\end{aligned}$$

$\{\langle t^3x^{i-1},x\rangle \mid i\geq 1\} = \{\langle t^3x^{3i-1},x\rangle \mid i\geq 1\} \cup \{\langle t^3x^{i-1},x\rangle \mid i\geq 1, gcd(i,3)=1\}$，从而 $\Omega_{\mathbb{F}_2[x]} \oplus x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2]$ 是 $NK_2(\mathbb{F}_2[C_4])$ 的直和项。

1.5 $NK_2(\mathbb{F}_q[C_{2^n}])$

设 $\mathbb{F}_q$ 是特征为 2 的有限域，$q=2^f$，C_{2^n} 是 2^n 阶循环群，这一节计算 $NK_2(\mathbb{F}_q[C_{2^n}])$。假设 $A=\mathbb{F}_q[t_1,t_2]/(t_1^{2^n})=\mathbb{F}_q[C_{2^n}][x]$, 此时 $I=(t_1^{2^n})$, $M=(t_1)$, $A/M=\mathbb{F}_q[x]$。

Lemma 1.16. $\Delta=\{(\alpha_1,\alpha_2) \mid \alpha_1\geq 2^n, \alpha_2\geq 0\}$，$\Lambda=\{((\alpha_1,\alpha_2),1) \mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2) \mid \alpha_1\geq 1,\alpha_2\geq 1\}$,对任意 $(\alpha,i)\in\Lambda, [\alpha,1]=\lceil(2^n+1)/\alpha_1\rceil, [\alpha,2]=\lceil 2^n/\alpha_1\rceil$, 其中 $\lceil x\rceil=\min\{m\in\mathbb{Z} | m\geq x\}$ 表示不小于 x 的最小整数。

Lemma 1.17. 令 $I_1=\{((\alpha_1,\alpha_2),1) \mid gcd(\alpha_1,\alpha_2)=1, 1<\alpha_1\leq 2^n\text{为偶数}, \alpha_2\geq 1\text{为奇数}\}$, $I_2=\{((\alpha_1,\alpha_2),2) \mid gcd(\alpha_1,\alpha_2)=1, 1\leq\alpha_1<2^n\text{为奇数}, \alpha_2\geq 1\}$，则 $\Lambda^{00}_{>1}=I_1\sqcup I_2$。

由定理2.1，

$$
\begin{aligned}
NK_2(\mathbb{F}_q[C_{2^n}]) \cong K_2(A,M) \cong & \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\
= & \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\
= & \bigoplus_{(\alpha,1)\in I_1} (1+x\mathbb{F}_q[x]/(x^{\lceil (2^n+1)/\alpha_1 \rceil}))^\times \\
\oplus & \bigoplus_{(\alpha,2)\in I_2} (1+x\mathbb{F}_q[x]/(x^{\lceil 2^n/\alpha_1 \rceil}))^\times .
\end{aligned}
$$

注意到 $BigWitt_k(R) = (1+xR[\![x]\!])^\times/(1+x^{k+1}R[\![x]\!])^\times \cong (1+xR[x]/(x^{k+1}))^\times$，根据公式1.7，

$$
\begin{aligned}
NK_2(\mathbb{F}_q[C_{2^n}]) \cong & \bigoplus_{(\alpha,1)\in I_1} \bigoplus_{\substack{1\le m\le \lceil (2^n+1)/\alpha_1\rceil -1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil (2^n+1)/\alpha_1\rceil -1}{m} \right\rfloor}\mathbb{Z})^f \\
\oplus & \bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le \lceil 2^n/\alpha_1\rceil -1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil 2^n/\alpha_1\rceil -1}{m} \right\rfloor}\mathbb{Z})^f .
\end{aligned}
$$

接下来我们证明对于任意 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 都在 $NK_2(\mathbb{F}_q[C_{p^n}])$ 出现无限多次

Lemma 1.18. 对于任意的 $1\le k<n$，$1+\left\lfloor \log_2(\frac{2^n-1}{2^k+1}) \right\rfloor = n-k$。

Proof. 当 $1\le k<n$ 时，$2^k-1\ge 1\ge \frac{1}{2^{n-k-1}}$，即

$$2^{n-1}-2^{n-k-1}\ge 1,$$

上式等价于 $2^n-1\ge 2^{n-k-1}(2^k+1)$，且 $2^n-1<2^{n-k}(2^k+1)$，于是

$$2^{n-k} > \frac{2^n-1}{2^k+1} \ge 2^{n-k-1},$$

取对数得 $\left\lfloor \log_2(\frac{2^n-1}{2^k+1}) \right\rfloor = n-k-1$。 □

考虑 $((1,\alpha_2),2)\in I_2$，

$$\bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le 2^n-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{2^n-1}{m} \right\rfloor}\mathbb{Z})^f$$

是 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 的直和项，当 $m=1$ 时 $1+\lfloor\log_2(2^n-1)\rfloor=n$，当 $m=2^k+1(1\le k<n)$ 为奇数时，由1.18，$1+\left\lfloor\log_2\frac{2^n-1}{m}\right\rfloor=n-k$，于是对于任何的 $1\le k\le n$，$\mathbb{Z}/2^k\mathbb{Z}$ 均出现在直和项中，且对于任意 $\alpha_2\ge 1$，这样的项总会出现，于是

$$NK_2(\mathbb{F}_q[C_{2^n}])\cong\bigoplus_{\infty}\bigoplus_{k=1}^{n}\mathbb{Z}/2^k\mathbb{Z}.$$

接下来给出一些 $NK_2(\mathbb{F}_q[C_{2^n}])$ 中的 $2^k(1\le k\le n)$ 阶元素。

对任意 $\alpha_2\ge 1, a\in\mathbb{F}_q$，

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_q[x]/(x^{2^n}))^\times &\rightarrowtail K_2(A,M)\\ 1+ax(2^n\text{ 阶元})&\mapsto\langle atx^{\alpha_2-1},x\rangle,\\ 1+ax^3(2^{n-1}\text{ 阶元})&\mapsto\langle at^3x^{3\alpha_2-1},x\rangle,\\ 1+ax^{2^k+1}(2^{n-k}\text{ 阶元})&\mapsto\langle at^{2k+1}x^{(2k+1)\alpha_2-1},x\rangle.\end{aligned}$$

1.6 其他问题和说明

$NK_2(\mathbb{F}_{p^m}[C_{p^n}])=?$

$\mathbb{F}_2[C_2\times C_2]\cong\mathbb{F}_2[C_2]\otimes\mathbb{F}_2[C_2]\cong\mathbb{F}_2[x,y]/(x^2,y^2)$，可以用同样的方法得到一些结果。

$$0\longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n),(t_1,t_2))\longrightarrow K_2(k[t_1,t_2,t_3]/(t_1^n,t_2^n))\longrightarrow K_2(k[t_3])\longrightarrow 0$$

对于有限域 k 来讲 $K_2(k[t_3])=0$,

$$0\longrightarrow NK_2(\mathbb{F}_2[C_2\times C_2])\longrightarrow K_2(\mathbb{F}_2[C_2\times C_2][x])\longrightarrow K_2(\mathbb{F}_2[C_2\times C_2])\longrightarrow 0,$$

中间那项可以用这篇文章里的方法确定,又 $K_2(\mathbb{F}_2[C_2\times C_2])=C_2^3$,于是可以得到 $NK_2(\mathbb{F}_2[C_2\times C_2])$，是 $\oplus_\infty\mathbb{Z}/2\mathbb{Z}$.

另外可以直接用这种方式重新计算 $K_2(\mathbb{F}_2[C_4\times C_4])$，见下一篇笔记。

一个关于模结构的问题，在 Weibel 的文章 [8] 中 5.5 和 5.7 给出的模结构和本文上面的模结构并不一致，用 V_m 作用差一个 t^m。

Chapter 2

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

2.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

2.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [10] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

2.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1, t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:

generators: $\langle a, b\rangle$, $(a, b) \in A \times M \cup M \times A$;

relations: $\langle a, b\rangle = -\langle b, a\rangle$,

$\langle a, b\rangle + \langle c, b\rangle = \langle a + c - abc, b\rangle$,

$\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [7]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1, 0) \in \mathbb{N}^2$, $\epsilon^2 = (0, 1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,

- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha - \epsilon^i}, t_i \rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 2.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [7]。 □

Lemma 2.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3 \rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

2.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 2.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. □

Theorem 2.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$, then $I = (t_1^4, t_2^4)$, $M = (t_1, t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] > 1\}$, and then

$$K_2(A, M) \cong \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i) \in \Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha, 1) \in \Lambda^{00}_{>1}$ and $[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ and $[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1, \alpha_2 \leq 4$ are both odd and $gcd(\alpha_1, \alpha_2) = 1$, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha, 1]$.

By the computation 2.2, we can get the following table

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below,
the generators of order 4:

$$\langle t_1t_2, t_1\rangle, \langle t_1t_2, t_2\rangle, \langle t_1, t_2\rangle,$$

the generators of order 2:

$$\langle t_1t_2^3, t_1\rangle, \langle t_1^3t_2, t_1\rangle, \langle t_1^3t_2^3, t_1\rangle, \langle t_1t_2^3, t_2\rangle, \langle t_1^3t_2^2, t_2\rangle, \langle t_1t_2^2, t_2\rangle, \langle t_1^3t_2, t_2\rangle, \langle t_1^3t_2^3, t_2\rangle, \langle t_1^3, t_2\rangle.$$

□

Remark 2.5. Compared with [10], note that $\langle t_1^3, t_2\rangle = \langle t_1^2t_2, t_1\rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1t_2\rangle - \langle t_1^2t_2, t_1\rangle \\
&= \langle t_1, t_1^2t_2\rangle - \langle t_1^2t_2, t_1\rangle - \langle t_1^2t_2, t_1\rangle \\
&= -3\langle t_1^2t_2, t_1\rangle \\
&= -\langle t_1^2t_2, t_1\rangle \\
&= \langle t_1^2t_2, t_1\rangle,
\end{aligned}$$

since $\langle t_1^2t_2, t_1\rangle + \langle t_1^2t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

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