代数 K 理论讨论班笔记

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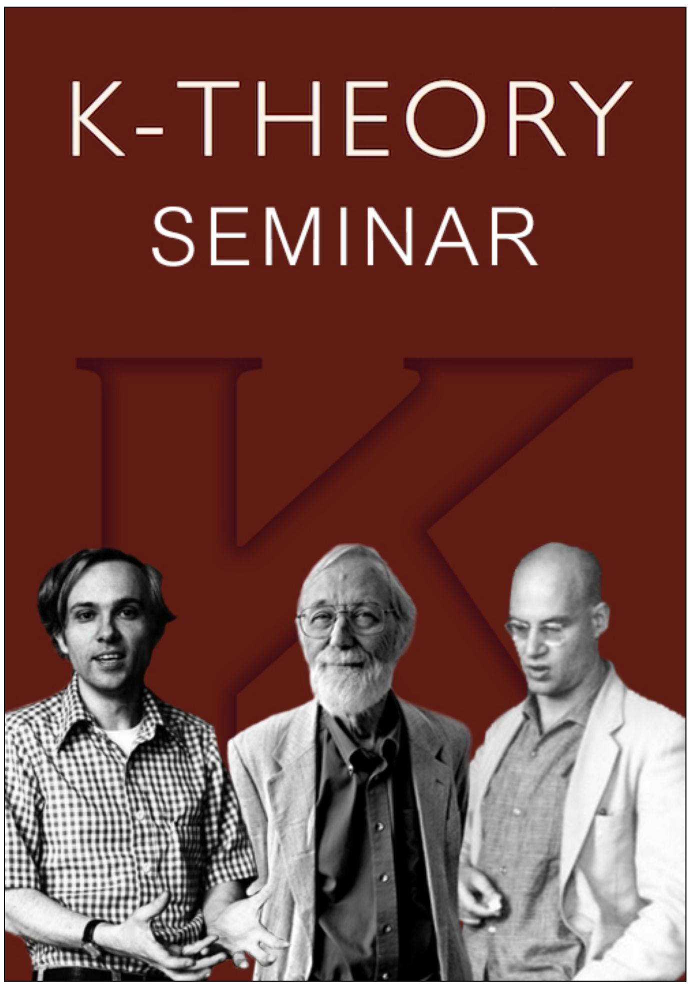

从左至右依次为
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2016 年 3 月 15 日

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Chapter 1

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [7].

1.1 Outline

Definition 1.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty}\mathbb{F}_2 x^i = \oplus_{i=1}^{\infty}\mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\,dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

1.2 Preliminaries

1.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 1.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

1.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[[x]])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$[a]\colon \alpha(x) \mapsto \alpha(ax)$$
$$V_m\colon \alpha(x) \mapsto \alpha(x^m)$$
$$F_m\colon \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}})$$
$$F_m\colon 1 - rx \mapsto 1 - r^m x$$

Remark 1.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [1].

Proposition 1.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$V_mV_n = V_{mn}$$
$$F_mF_n = F_{mn}$$
$$F_mV_n = m$$
$$[a]V_m = V_m[a^m]$$
$$F_m[a] = [a^m]F_m$$
$$[a][b] = [ab]$$
$$V_mF_k = F_kV_m, \text{ if } (k,m) = 1$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[[x]]$ is not.

1.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_12(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 1.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem.*

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a,b\rangle \in K_2(R,I)$ satisfy following relations

(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$ if any of a,b,c are in I.

Theorem 1.6. *1. If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R,I) \overset{\cong}{\to} K_2(R,I)$.

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r,s \in R$ commute and $1-rs$ is a unit, that is $rs = sr$ and $1-rs \in R^*$,

$$\langle r,s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r,s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r,s\rangle \in K_2(R,I)$

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$,

(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of r,s,t are in I).

Note that $\langle r,1\rangle = 0$ for any $r \in R$ and $\langle r,s\rangle_{version2} = \langle -r,s\rangle_{version1}$.

Theorem 1.7. *1. If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r,s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r,s\}$.*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R,I)$ is generated by $\langle r,s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively *generated by its units.*

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r, s\rangle$, $r, s \in R$, $1-rs \in R^*$ or by all $\{u, v\}$, $u, v \in R^*$.*

1.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 1.4.

Relative groups Let R be a ring (not necessarily commutative), $I \subset R$ a two-sided ideal, by definition $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$, there exists a map

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

Definition 1.8. $K(R, I)$ is the homotopy fibre of the map $BGL(R)^+ \longrightarrow BGL(R/I)^+$. $K_i(R, I) := \pi_i(K(R, I)), i \geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots \longrightarrow K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R, I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots .$$

In particular,

$$\begin{aligned} &K_3(R, I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R, I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow \\ \longrightarrow &K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \end{aligned}$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals, there is a map

$$K(R, I) \longrightarrow K(R/J, I + J/J).$$

If $I \cap J = 0$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R/I \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle g} \\ R/J & \xrightarrow{f} & R/I + J \end{array}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(1.8)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & K_{i+1}(R) & \xrightarrow{\alpha_*} & K_{i+1}(R/I) & \xrightarrow{\partial} & K_i(R,I) & \xrightarrow{j} & K_i(R) & \xrightarrow{\alpha_*} & K_i(R/I) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \epsilon_i} & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & K_{i+1}(R/J) & \xrightarrow{f_*} & K_{i+1}(R/I+J) & \xrightarrow{\partial} & K_i(R/J, I+J/J) & \xrightarrow{j'} & K_i(R/J) & \xrightarrow{f_*} & K_i(R/I+J) & \longrightarrow \cdots \end{array}$$

where the induced homomorphism

$$\epsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the Mayer–Vietoris sequence

$$\begin{aligned} K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 1.9. *1. Suppose that the excision map ϵ_i in 1.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. If ϵ_i is an isomorphism, and in additionϵ_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is

$$\begin{aligned} K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

3. Suppose instead that ϵ_i is surjective, and let $L = ker(\epsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, *and the sequence*

$$\begin{aligned} &K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow \\ \longrightarrow K_i(R)/L \longrightarrow &K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

is exact.

Proof. Define $\delta_i = j\epsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. □

Remark 1.10. It is known that ϵ_0 and ϵ_1 are isomorphism regardless of the specific rings. Moreover Swan [6] has shown that ϵ_2 cannot be an isomorphism in general. For more discussion, see [5].

Double relative groups

Definition 1.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I + J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd}
K(R;I,J) \arrow[d, dashed] & & \\
K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\
K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+
\end{tikzcd}$$

Remark 1.12. $K_i(R; I, J) \cong K_i(R; J, I)$, $K_i(R; I, I) = K_i(R, I)$.

We have long exact sequence

$$\cdots \longrightarrow K_{i+1}(R,I) \longrightarrow K_{i+1}(R/J, I+J/J) \longrightarrow K_i(R;I,J) \longrightarrow K_i(R,I) \longrightarrow K_i(R/J, I+J) \longrightarrow \cdots .$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R,I) \longrightarrow K_3(R/I, I+J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R,I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [8] 3.5.10, [5], [4] or [2] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [3] theorem 2.

Remark 1.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [3].

Theorem 1.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b\rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [3] theorem 3. $\square$

Lemma 1.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I + J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I + J/J) \longrightarrow 0$$

is split exact.

1.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [1] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$
$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$-module structure on $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [1]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 1.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 1.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 1.18. 一个不知道有没有用的结论, see [1]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$\begin{aligned} D\colon x\mathbb{F}_2[x] &\longrightarrow \Omega_{\mathbb{F}_2[x]} \\ x^i &\mapsto ix^{i-1}dx \end{aligned}$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty} \mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

1.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{1.18}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By Mayer–Vietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd}
NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\
0 & & 0 & 0 & & 0
\end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z},(2)) \longrightarrow NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2],(\sigma+1)) \longrightarrow NK_2(\mathbb{Z},(2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z},(2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$\begin{tikzcd}
 & NK_3(\mathbb{Z},(2)) = 0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \arrow[d, "\cong"] & & \\
0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2],(\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\
 & NK_2(\mathbb{Z},(2)) = 0 & &
\end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [3], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 1.19. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 1.20 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma - 1)$, $J = (1 + \sigma + \cdots + \sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "\sigma\mapsto 1"', "f"] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1})$. Note that $p(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1}) = 0$ since $(1 + \sigma + \cdots + \sigma^{p-1})^2 = p(1 + \sigma + \cdots + \sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R, I)$$
$$(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1}) \mapsto \langle \sigma - 1, 1 + \sigma + \cdots + \sigma^{p-1} \rangle = \langle \sigma - 1, 1 \rangle^p = 1$$

Also see [5].

Example 1.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

1.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $F_2[\epsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{cccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\epsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\epsilon \end{array}$$

Lemma 1.22. *The map* $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$ *in 1.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [7] Lemma 1.2. □

Theorem 1.23. $NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]}$, $NK_0(\mathbb{Z}[D_2]) \cong V$,

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 1.18 with $\mathbb{Z}[C_2]$

□

1.6 NK_i of the group C_4

1.7 NK_i of the group D_4

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