内容集锦: 讨论班、课程讲义

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目录

Chapter 1

Notes on NK_0 and NK_1 of the groups C_4 and D_4

1.1 Outline

Definition 1.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty}\mathbb{F}_2 x^i = \oplus_{i=1}^{\infty}\mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\,dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has different $W(\mathbb{F}_2)$-module structure.

1.2 Preliminaries

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[[x]])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$
\begin{aligned}
[a]\colon \alpha(x) &\mapsto \alpha(ax) \\
V_m\colon \alpha(x) &\mapsto \alpha(x^m) \\
F_m\colon \alpha(x) &\mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m\colon 1-rx &\mapsto 1-r^m x
\end{aligned}
$$

Remark 1.2. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1-r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See Dayton& Weibel.

Proposition 1.3. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$
\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a]V_m &= V_m[a^m] \\
F_m[a] &= [a^m]F_m \\
[a][b] &= [ab] \\
\textit{if } (k,m) = 1, V_m F_k &= F_k V_m
\end{aligned}
$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[[x]]$ is not.

1.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel example 2.6, 2.9.

$$
\begin{aligned}
V_m(x^n) &= x^{mn} \\
F_d(x^n) &= \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases} \\
[a]x^n &= a^n x^n
\end{aligned}
$$

$W(\mathbb{F}_2)$**-module structure on** $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$

$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$

$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 1.4. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.