代数 K 理论讨论班笔记

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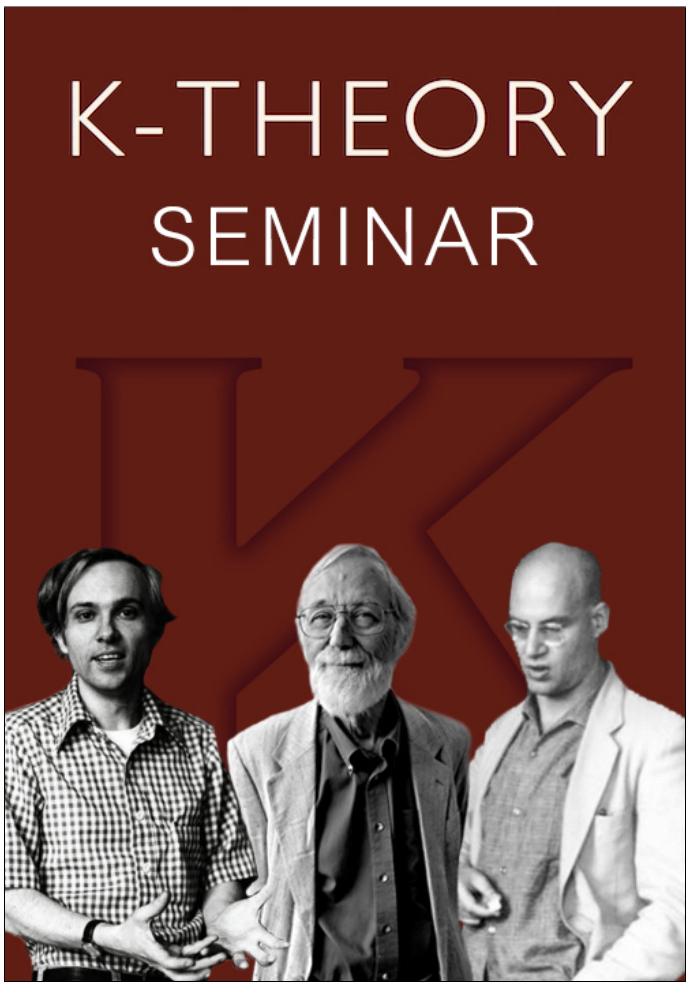

从左至右依次为
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Chapter 1

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [13].

1.1 Outline

Definition 1.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty}\mathbb{F}_2 x^i = \oplus_{i=1}^{\infty}\mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\,dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

1.2 Preliminaries

1.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)

R-mod: the category of R-modules,

$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,

$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 1.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

1.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[[x]])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$[a]\colon \alpha(x) \mapsto \alpha(ax)$$
$$V_m\colon \alpha(x) \mapsto \alpha(x^m)$$
$$F_m\colon \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}})$$
$$F_m\colon 1 - rx \mapsto 1 - r^m x$$

Remark 1.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [4].

Proposition 1.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$V_mV_n = V_{mn}$$
$$F_mF_n = F_{mn}$$
$$F_mV_n = m$$
$$[a]V_m = V_m[a^m]$$
$$F_m[a] = [a^m]F_m$$
$$[a][b] = [ab]$$
$$V_mF_k = F_kV_m, \ \textit{if } (k,m) = 1$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[[x]]$ is not.

1.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.s

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 1.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem.*

Maazen and Stienstra define the group $D(R)$ as follows:

take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,

defining relations:

(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$, (D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a,b\rangle \in K_2(R,I)$ satisfy following relations

(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$ if any of a,b,c are in I.

Theorem 1.6. *1. If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R,I) \overset{\cong}{\to} K_2(R,I)$.

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r,s \in R$ commute and $1-rs$ is a unit, that is $rs = sr$ and $1 - rs \in R^*$,

$$\langle r,s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r,s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r,s\rangle \in K_2(R,I)$

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$,

(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of r,s,t are in I).

Note that $\langle r,1\rangle = 0$ for any $r \in R$ and $\langle r,s\rangle_{version2} = \langle -r,s\rangle_{version1}$.

Theorem 1.7. *1. If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r,s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r,s\}$.*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R,I)$ is generated by $\langle r,s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively *generated by its units.*

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r, s\rangle$, $r, s \in R$, $1-rs \in R^*$ or by all $\{u, v\}$, $u, v \in R^*$.*

1.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 1.4.

excision 失效就是说 if $A \longrightarrow B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A, I) \longrightarrow K_n(B, I)$ need not be an isomorphism. 由于这个不是同构，没法有 Mayer-Vietoris 序列

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(A/I) \arrow[r] \arrow[d] & K_i(A, I) \arrow[r, green] \arrow[d, shift right] & K_i(A) \arrow[r] \arrow[d] & K_i(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(B/I) \arrow[r, green] & K_i(B, I) \arrow[r] \arrow[u, red, dashed, shift right] & K_i(B) \arrow[r] & K_i(B/I) \arrow[r] & \cdots
\end{tikzcd}$$

要连接 $K_n(A, I) \longrightarrow K_n(B, I)$ 就要考虑 birelative K-groups（也称 double relative K-groups）,$K(A, B, I)$ 定义为 homotpy fiber of the map $K(A, I) \longrightarrow K(B, I)$。以下是详细的定义和性质。

Relative groups Let R be a ring (not necessarily commutative), $I \subset R$ a two-sided ideal, by definition $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$, there exists a map

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

Definition 1.8. $K(R, I)$ is the homotopy fibre of the map $BGL(R)^+ \longrightarrow BGL(R/I)^+$. $K_i(R, I) := \pi_i(K(R, I)), i \geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots \longrightarrow K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R, I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots.$$

In particular,

$$\begin{aligned}
&K_3(R, I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R, I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow \\
\longrightarrow &K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I)
\end{aligned}$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals, there is a map

$$K(R, I) \longrightarrow K(R/J, I + J/J).$$

If $I \cap J = 0$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R/I \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle g} \\ R/J & \xrightarrow{f} & R/I+J \end{array}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(1.8)

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & K_{i+1}(R) & \xrightarrow{\alpha_*} & K_{i+1}(R/I) & \xrightarrow{\partial} & K_i(R,I) & \xrightarrow{j} & K_i(R) & \xrightarrow{\alpha_*} & K_i(R/I) & \longrightarrow \cdots \\ & \big\downarrow{\scriptstyle\beta_*} & & \big\downarrow{\scriptstyle g_*} & & \big\downarrow{\scriptstyle\epsilon_i} & & \big\downarrow & & \big\downarrow & \\ \cdots \longrightarrow & K_{i+1}(R/J) & \xrightarrow{f_*} & K_{i+1}(R/I+J) & \xrightarrow{\partial} & K_i(R/J, I+J/J) & \xrightarrow{j'} & K_i(R/J) & \xrightarrow{f_*} & K_i(R/I+J) & \longrightarrow \cdots \end{array}$$

where the induced homomorphism

$$\epsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the Mayer–Vietoris sequence

$$\begin{aligned} K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 1.9. *1. Suppose that the excision map ϵ_i in 1.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. *If ϵ_i is an isomorphism, and in additionϵ_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is*

$$K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta}$$
$$\longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J)$$

3. *Suppose instead that ϵ_i is surjective, and let $L = ker(\epsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, and the sequence*

$$K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow$$
$$\longrightarrow K_i(R)/L \longrightarrow K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J)$$

is exact.

Proof. Define $\delta_i = j\epsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. $\square$

Remark 1.10. It is known that ϵ_0 and ϵ_1 are isomorphism regardless of the specific rings. Moreover Swan [11] has shown that ϵ_2 cannot be an isomorphism in general. For more discussion, see [10].

Double relative groups

Definition 1.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I + J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd}
K(R;I,J) \arrow[d, dashed] & & \\
K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\
K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+
\end{tikzcd}$$

Remark 1.12. $K_i(R; I, J) \cong K_i(R; J, I)$, $K_i(R; I, I) = K_i(R, I)$.

We have a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R, I) \longrightarrow K_{i+1}(R/J, I+J/J) \longrightarrow K_i(R; I, J) \longrightarrow K_i(R, I) \longrightarrow K_i(R/J, I+J) \longrightarrow \cdots .$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R,I) \longrightarrow K_3(R/I, I{+}J/J) \longrightarrow I/I^2{\otimes_{R^e}}J/J^2 \xrightarrow{\psi} K_2(R,I) \longrightarrow K_2(R/I, I{+}J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [14] 3.5.10, [10], [7] or [5] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [6] theorem 2.

Remark 1.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [6].

Theorem 1.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b \rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [6] theorem 3. □

Lemma 1.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I + J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I + J/J) \longrightarrow 0$$

is split exact.

1.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [4] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$

$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$

$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$-module structure on $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [4]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 1.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 1.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 1.18. 一个不知道有没有用的结论, see [4]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$D\colon x\mathbb{F}_2[x] \longrightarrow \Omega_{\mathbb{F}_2[x]}$$
$$x^i \mapsto ix^{i-1}dx$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty}\mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

1.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$
\begin{tikzcd}
\mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\
\mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2
\end{tikzcd}
\tag{1.18}
$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By Mayer–Vietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$
\begin{tikzcd}
NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\
0 & & 0 & 0 & & 0
\end{tikzcd}
$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma + 1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma + 1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma{+}1), (\sigma{-}1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma{+}1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$
\begin{tikzcd}
 & NK_3(\mathbb{Z}, (2)) = 0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\
0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\
 & NK_2(\mathbb{Z}, (2)) = 0 & &
\end{tikzcd}
$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [6], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1),\sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1-x^n(\sigma-1)(\sigma+1)=1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 1.19. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 1.20 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p] & \xrightarrow{\sigma\mapsto\zeta} & \mathbb{Z}[\zeta] \\ {\scriptstyle \sigma\mapsto 1}\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$\begin{gathered} I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R,I) \\ (\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1,1\rangle^p = 1 \end{gathered}$$

Also see [10].

Example 1.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p][x] & \longrightarrow & \mathbb{Z}[\zeta][x] \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}[x] & \longrightarrow & \mathbb{F}_p[x] \end{array}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

1.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $F_2[\epsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{cccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\epsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\epsilon \end{array}$$

Lemma 1.22. *The map $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$ in 1.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [13] Lemma 1.2. □

Theorem 1.23. *$NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]}$, $NK_0(\mathbb{Z}[D_2]) \cong V$,*

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 1.18 with $\mathbb{Z}[C_2]$

□

1.6 NK_i of the group C_4

1.7 NK_i of the group D_4

Chapter 2

Lower bounds for the order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$

2016.3.19 阅读这篇文章 [9] 1976 年发表在 *Math. Ann.*。

基本假设：p: rational prime, G: elementary abelian p-group.

用的方法：Bloch; van der Kallen K_2 of truncated polynomial rings

结论：the p-rank of $K_2(\mathbb{Z}G)$[1] grows expotentially with the rank of G.

$Wh_2(G)$: "pseudo-isotopy" group is nontrivial if G has rank at least 2.

这篇文章之前已知的结论 (exact computations) Dunwoody, G cyclic of order 2 or 3, $K_2(\mathbb{Z}G)$ is an elementary abelian 2-group of rank 2 if G has order 2 and of rank 1 if G has order 3. 两者都有 $Wh_2(G)$ 平凡。

一些记号和基本结论 R commutative ring, A a subring of R. $\Omega^1_{R/A}$ the module of Kähler differentials of R considerd as an algebra over A and R^* will denote the group of units of R.

the p-rank of an abelian group G is $\dim_{\mathbb{F}_p}(G \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

elementary abelian p-groups An elementary abelian p-group is an abelian group in which every nontrivial element has order p. The number p must be prime, and the elementary abelian groups are a particular kind of p-group. The case where $p = 2$, i.e., an elementary abelian 2-group, is sometimes called a Boolean group.

结构：Every elementary abelian p-group is a vector space over the prime field $\mathbb{F}_p$ with p elements, and conversely every such vector space is an elementary abelian group.

By the classification of finitely generated abelian groups, or by the fact that every vector space has a basis, every finite elementary abelian group must be of the form $(\mathbb{Z}/p\mathbb{Z})^n$

[1] this is a finite group

for n a non-negative integer (sometimes called the group's rank). Here, $C_p = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p.

In general, a (possibly infinite) elementary abelian p-group is a direct sum of cyclic groups of order p. (Note that in the finite case the direct product and direct sum coincide, but this is not so in the infinite case.)

2.1 第一部分

环是 $\mathbb{F}_q$ 有限域的情况。

先说结论

首先是一个奇素数的结论

Proposition 2.1. *Let $q = p^f$ be odd and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary p-group of rank $f(n-1)(p^n-1)$.*

接着是素数 2 的结论

Proposition 2.2. *Let $q = 2^f$ be odd and let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary 2-group of rank $f(n-1)(2^n-1)$.*

结论实际上是可以统一的，但是方法有些区别，因此原文中分开表述。

我们引进方法时借鉴了 van der Kallen 的方法和记号

Let R be a commutative ring. The abelian group $TD(R)$ is the universal R-module having generators $Da, Fa, a \in R$, subject to the relations

$$\begin{aligned} D(ab) &= aDb + bDa, \\ D(a+b) &= Da + Db + F(ab), \\ F(a+b) &= Fa + Fb, \\ Fa &= D(1+a) - Da. \end{aligned}$$

There is a natural surjective homomorphism of R-modules

$$TD(R) \twoheadrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 1$$

whose kernel is the submodule of $TD(R)$ generated by the $Fa, a \in R$. Relations imply

$$F(c^2a) = cFa$$

$(F(c^2a) = F(ca \cdot c) = D(ca+c) - D(ac) - D(c) = D(c(a+1)) - D(ac) - D(c) = cD(a+1) - (a+1)D(c) - aD(c) - cD(a) - D(c) = cF(a)$, $0 = F(0) = F(a-a) = F(a) + F(-a)$, $\Rightarrow F(a) = -F(a) = F(-a)$, $\Rightarrow F(2a) = 0)$

for all $a, c \in R$ see [12]p. 1204.

Hence $F(2a) = 2F(a) = 0$, if 2 is a unit of R, $F(a) = 0$, then the kernel is trivial and $\Omega^1_{R/\mathbb{Z}} \cong TD(R)$,

$$1 \longrightarrow TD(R) \xrightarrow{\cong} \Omega^1_{R/\mathbb{Z}} \longrightarrow 1.$$

Example 2.3. $R = \mathbb{Z}$, then the kernel of the above surjection is $\mathbb{Z}/2\mathbb{Z}$.
If R is a field of characteristic $\neq 2$, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.
If R is a perfect field, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.

Definition 2.4. We define groups $\Phi_i(R)$, $i \geq 2$, by the exact sequence

$$1 \longrightarrow \Phi_i(R) \longrightarrow K_2(R[x]/(x^i)) \longrightarrow K_2(R[x]/(x^{i-1})) \longrightarrow 1. \tag{2.4}$$

This sequence is exact at the right as $SK_1(R[x]/(x^i), (x^{i-1})/(x^i)) = 1$ (cf. [8] Theorem 6.2 and [2]9.2, p. 267).

2.1.1 Remarks

我们把 Bass 书 [2] 中相关的结论 (p. 267) 放在这里以备今后查询和使用

A semi-local ring is a ring for which $R/rad(R)$ is a semisimple ring, where $rad(R)$ is the Jacobson radical of R. In commutative algebra, semi-local means "finitely many maximal ideals", for instance, all rational numbers r/s with s prime to 30 form a semi-local ring, with maximal ideals generated respectively by $2, 3$, and 5. This is a PID, as a matter of fact, any semi-local Dedekind domain is a PID. And if R is a commutative noetherian ring, the set of zero-divisors is a union of finitely many prime ideals (namely, the "associated primes" of (0)), thus its classical ring of quotients (obtained from R by inverting all of its non zero-divisors) is a semi-local ring. See [1] p.174. and [2] p. 86.

In studying the stable structure of general linear groups in algebraic K-theory, Bass proved the following basic result (ca. 1964) on the unit structure of semilocal rings.

Theorem 2.5. *If R is a semi-local ring, then R has stable range 1, in the sense that, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb \in R^*$.*

Example 2.6. Some classes of semi-local rings: left(right) artinian rings, finite direct products of local rings, matrix rings over local rings, module-finite algebras over commutative semi-local rings.
The quotient $\mathbb{Z}/m\mathbb{Z}$ is a semi-local ring. In particular, if m is a prime power, then $\mathbb{Z}/m\mathbb{Z}$

is a local ring.

A finite direct sum of fields $\bigoplus_{i=1}^{n} F_i$ is a semi-local ring. In the case of commutative rings with unit, this example is prototypical in the following sense: the Chinese remainder theorem shows that for a semi-local commutative ring R with unit and maximal ideals $m_1, \cdots, m_n$

$$R/\bigcap_{i=1}^{n} m_i \cong \bigoplus_{i=1}^{n} R/m_i$$

(The map is the natural projection). The right hand side is a direct sum of fields. Here we note that $\cap_i m_i = rad(R)$, and we see that $R/rad(R)$ is indeed a semisimple ring.

The classical ring of quotients for any commutative Noetherian ring is a semilocal ring.

The endomorphism ring of an Artinian module is a semilocal ring.

Semi-local rings occur for example in commutative algebra when a (commutative) ring R is localized with respect to the multiplicatively closed subset $S = \cap(R - p_i)$, where the p_i are finitely many prime ideals.

Theorem 2.7. *Let I be a two-sided ideal in a ring R. Assume either that R is semi-local or that $I \subset rad(R)$. Then*

$$GL_1(R, I) \longrightarrow K_1(R, I)$$

is surjective, and, for all $m \geq 2$,

$$GL_m(R, I)/E_m(R, I) \longrightarrow K_1(R, I)$$

is an isomorphism. Moreover $[GL_m(R), GL_m(R, I)] \subset E_m(R, I)$, with equality for $m \geq 3$.

Corollary 2.8. *Suppose that R above is commutative, then $E_n(R, I) \overset{\cong}{\to} SL_n(R, I)$ is an isomorphism for all $n \geq 1$, and $SK_1(R, I) = 0$.*

Proof. The determinant induces the inverse,

$$\det\colon K_1(R, I) \longrightarrow GL_1(R, I).$$

In particular, if $\alpha \in GL_n(R, I)$ and $\det(\alpha) = 1$ then $\alpha \in E_n(R, I)$, i.e. $SL_n(R, I) \subset E_n(R, I)$. The opposite inclusion is trivial. Finally $SK_1(R, I) = SL(R, I)/E(R, I) = 0$. □

还有一个小插曲，当 k 是域时，$k[x]/(x^m)$ 是局部环的证明

Proposition 2.9. *Let I be an ideal in the ring R.*

a) If $rad(I)$ is maximal, then R/I is a local ring.

b) In particular, if m is a maximal ideal and $n \in \mathbb{Z}^+$ then R/m^n is a local ring.

Proof. a) We know that $rad(I) = \bigcap_{P \supset I} P$, so if $rad(I) = m$ is maximal it must be the only prime ideal containing I. Therefore, by correspondence R/I is a local ring. (In fact it is a ring with a unique prime ideal.)
b)$rad(m^n) = rad(m) = m$, so part a) applies. □

Example 2.10. For instance, for any prime number p, $\mathbb{Z}/(p^k)$ is a local ring, whose maximal ideal is generated by p. It is easy to see (using the Chinese Remainder Theorem) that conversely, if $\mathbb{Z}/(n)$ is a local ring then n is a prime power.

The ring $\mathbb{Z}_p$ of p-adic integers is a local ring. For any field k, the ring $k[[t]]$ of formal power series with coefficients in k is a local ring. Both of these rings are also PIDs. A ring which is a local PID is called a discrete valuation ring. Note that a local ring is connected, i.e., $e^2 = e \Rightarrow e \in \{0, 1\}$.

令 R 是 $k[x]$, I 是 (x^m), 有 $rad(x^m) = (x)$ 是极大理想 (由于 $0 \to (x) \to k[x] \to k \to 0$ 正合)，从而 $k[x]/(x^i)$ 是局部环。

Remarks 到此结束

2.1.2 Theorem

The first part of the following theorem is due to van der Kallen [12] and the second to Bloch [3].

Theorem 2.11. *Let R be a commutative ring. Then*
(1) $\Phi_2(R) \cong TD(R)$;
(2) If R is a local $\mathbb{F}_p$-algebra and p is odd prime, then

$$\Phi_i(R) \cong \begin{cases} \Omega^1_{R/\mathbb{Z}}, i \not\equiv 0, 1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r}, i = mp^r, (p, m) = 1. \end{cases}$$

当 p 是 odd prime 时，这一定理 (2) 可应用于 $\mathbb{F}_p[C_p]$, 因为 $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[$

Lemma 2.12. *Let $q = p^f$ and let H be a finite abelian p-group. Then $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$is a free $\mathbb{F}_q H$-module of rank equal to the p-rank of H.*

Proof. In terms of polynomials, we have

$$\mathbb{F}_q H \cong \mathbb{F}_q[x_1, \cdots, x_n]/I$$

where n is the p-rank of H and I is the ideal of $\mathbb{F}_q[x_1, \cdots, x_n]$ generated by polynomials of the form $F_i = x_i^{q_i} - 1$ where q_i is a power of p. By [BoreI,A.: Linear algebraic groups.

New York: W. A. Benjamin 1969, p. 61], $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$ is the $\mathbb{F}_q H$-module with generators $dx_1, \cdots, dx_n$ subject to the relations

$$\sum_i \frac{dF_i}{dx_i} dx_i = 0.$$

Since the ring has characteristic p, the relations are trivial and the module is free. As $\mathbb{F}_q$ is perfect, its module of differentials is trivial. Hence $\Omega^1_{\mathbb{F}_q H/\mathbb{F}_q} = \Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$, yielding the result. □

由这个引理得到了2.1.

下面是节选一些可能用到的陈述。

- $\mathbb{F}_q G$ is a local ring, where G is an elementary abelian p-group, for example $G = (\mathbb{Z}/p\mathbb{Z})^n$.

对 odd prime 的证明如下

Proof. We begin by showing that $K_2(\mathbb{F}_q G)$ is an elementary abelian p-group even in case $p = 2$. As $\mathbb{F}_q G$ is a local ring, it follows that $K_2(\mathbb{F}_q G)$ is generated by the Steinberg symbols $\{u, v\}$, $u, v \in \mathbb{F}_q G^*$. Now $u^p, v^P \in \mathbb{F}^*$ as G is an elementary abelian p-group (p 次后 G 中的元就变成单位元了). Choose $w \in \mathbb{F}_q^*$ so that $w^p = u^p$.(这里注意之前的 u 是群环里的，这里的 w 取在域里) Then

$$\begin{aligned} \{u, v\}^p &= \{u^p, v\} \\ &= \{w^p, v\} \\ &= \{w, v^p\}. \end{aligned}$$

Thus $\{w, v^p\}$ is trivial as it lies in the image of $K_2(\mathbb{F}_q) = 1$(有限域的 K_2 是平凡的，并且这个符号是在 K_2 中). Hence $K_2(\mathbb{F}_q G)$ has exponent p.

Let H be generated by $x_1, \cdots, x_{n-1}$ where $x_1, \cdots, x_n$ are independent generators of G. Then (由于特征是 p 才有下面的最后一步，对于 $\mathbb{Z}$ 是不对的)

$$\mathbb{F}_q G = \mathbb{F}_q H[x_n]/(x_n^p - 1) \cong \mathbb{F}_q H[x]/(x^p).$$

Exact sequence 2.4 together with Theorem yield

$$\begin{aligned} \text{rank } K_2(\mathbb{F}_q G) &= \text{rank } K_2(\mathbb{F}_q H) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q \\ &= \text{rank } K_2(\mathbb{F}_q H) + f(p-1)(n-1)p^{n-1} + f(p^{n-1} - 1) \end{aligned}$$

and the result follows by induction.

上面的结论我们详细写出来是

$$
\begin{aligned}
1 \longrightarrow \Phi_p(\mathbb{F}_qH) &\longrightarrow K_2(\mathbb{F}_qG)=K_2(\mathbb{F}_qH[x]/(x^p)) \longrightarrow K_2(\mathbb{F}_qH[x]/(x^{p-1})) \longrightarrow 1,\\
1 \longrightarrow \Phi_{p-1}(\mathbb{F}_qH) &\longrightarrow K_2(\mathbb{F}_qH[x]/(x^{p-1})) \longrightarrow K_2(\mathbb{F}_qH[x]/(x^{p-2})) \longrightarrow 1,\\
&\cdots\\
1 \longrightarrow \Phi_2(\mathbb{F}_qH) &\longrightarrow K_2(\mathbb{F}_qH[x]/(x^2)) \longrightarrow K_2(\mathbb{F}_qH[x]/(x)) \longrightarrow 1.
\end{aligned}
$$

Note that $\mathbb{F}_qH[x]/(x)=\mathbb{F}_qH$, $G=(\mathbb{Z}/p\mathbb{Z})^n$, $H=(\mathbb{Z}/p\mathbb{Z})^{n-1}$ then

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_qG) &= \text{rank } \Phi_p(\mathbb{F}_qH)+\text{rank } K_2(\mathbb{F}_qH[x]/(x^{p-1}))\\
&= \text{rank } \Phi_p(\mathbb{F}_qH)+\text{rank } \Phi_{p-1}(\mathbb{F}_qH)+\cdots+\text{rank } \Phi_2(\mathbb{F}_qH)+\text{rank } K_2(\mathbb{F}_qH)\\
&= \text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}+\text{rank } \mathbb{F}_qH/\mathbb{F}_q+(p-2)\text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}+\text{rank } K_2(\mathbb{F}_qH)\\
&= (p-1)\text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}+\text{rank } \mathbb{F}_qH/\mathbb{F}_q+\text{rank } K_2(\mathbb{F}_qH)\\
&= f(p-1)(n-1)p^{n-1}+f(p^{n-1}-1)+\text{rank } K_2(\mathbb{F}_qH)
\end{aligned}
$$

since

$$
\begin{gathered}
\Phi_p(\mathbb{F}_qH)=\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}\oplus\mathbb{F}_qH/\mathbb{F}_qH^p,\\
\Phi_i(\mathbb{F}_qH)=\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}=(\mathbb{F}_qH)^{n-1}, 2\le i\le p-1,\\
\mathbb{F}_qH/\mathbb{F}_qH^p=\mathbb{F}_qH/\mathbb{F}_q
\end{gathered}
$$

$\mathbb{F}H$ 是以 H 中元素为基的自由 F 模并且

$$
\begin{gathered}
\text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}=\text{rank } (\mathbb{F}_{p^f}H)^{n-1}=(n-1)f|H|=(n-1)fp^{n-1}\\
\text{rank } \mathbb{F}_qH/\mathbb{F}_q=\text{rank } \mathbb{F}_qH-\text{rank } \mathbb{F}_q=f(p^{n-1}-1).
\end{gathered}
$$

接下来是归纳计算，首先我们看它截至到哪一步：最后一步应该是 $\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]$，因为 $K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}])+++=0$, 这时有

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]) &= \text{rank } K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}])+(p-1)\text{rank } \Omega^1_{\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{Z}}+\text{rank } \mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{F}_q\\
&= 0+f(p-1)(2-1)p^{2-1}+f(p^{2-1}-1)
\end{aligned}
$$

从而我们知道

$$
\begin{aligned}
\text{rank } K_2(\mathbb{F}_qG) &= f(p-1)(n-1)p^{n-1}+f(p^{n-1}-1)+\cdots+f(p-1)p^1+f(p^1-1)\\
&= \sum_{i=1}^{n-1}(f(p-1)ip^i+f(p^i-1))\\
&= -f\frac{p-p^n}{1-p}+f(n-1)p^n+f\frac{p-p^n}{1-p}-(n-1)f\\
&= f(n-1)(p^n-1)
\end{aligned}
$$

这里的计算用到等比数列求和，记 $S=\sum_{i=1}^{n-1} ip^i$

$$pS=\sum_{i=1}^{n-1} ip^{i+1}=\sum_{i=2}^{n}(i-1)p^i$$

$$S-pS=\sum_{i=1}^{n-1} p^i-(n-1)p^n$$

因此

$$S=\frac{p-p^n}{(1-p)^2}-\frac{(n-1)p^n}{(1-p)}$$

$$\begin{aligned}\sum_{i=1}^{n-1}(f(p-1)ip^i+f(p^i-1)) &= f(p-1)S+f\frac{p-p^n}{1-p}-(n-1)f\\ &= f(p-1)(\frac{p-p^n}{(1-p)^2}-\frac{(n-1)p^n}{(1-p)})+f\frac{p-p^n}{1-p}-(n-1)f\\ &= -f\frac{p-p^n}{1-p}+f(n-1)p^n+f\frac{p-p^n}{1-p}-(n-1)f\\ &= f(n-1)(p^n-1)\end{aligned}$$

□

In case p = 2 the details become more complicated.(暂且略过这个情形)

2.2 第二部分

第二部分是考了系数环是 $\mathbb{Z}$ 的情形，如何将上面的有限域和这里的整数环联系起来，就是用了一个相对 K 群的正合列。

We now exploit these computations of $K_2(\mathbb{F}_qG)$ to obtain lower bounds for $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. There is an exact sequence

$$K_2(\mathbb{Z}G)\longrightarrow K_2(\mathbb{F}_pG)\longrightarrow SK_1(\mathbb{Z}G,p\mathbb{Z}G)\longrightarrow SK_1(\mathbb{Z}G)\longrightarrow 1 \tag{2.12}$$

This sequence is exact on the right because $\mathbb{F}_pG$ is a local ring, which implies $SK_1(\mathbb{F}_pG)=1$ [2], p. 267.

Theorem 2.13. *(1) Let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{Z}G)$ has 2-rank at least $(n-1)2^n+2$ and $Wh_2(G)$ has 2-rank at least $(n-1)2^n-\frac{(n+2)(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n\geq 2$.*
(2) Let p be an odd prime and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{Z}G)$ has p-rank at least $(n-1)(p^n-1)-\binom{p+n-1}{p}$ and $Wh_2(G)$ has p-rank at least $(n-1)(p^n-1)-\binom{p+n-1}{p}-\frac{n(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n\geq 2$.

Proof. (1) Since $K_1(\mathbb{Z}G, 2\mathbb{Z}G) \longrightarrow K_1(\mathbb{Z}G)$ is injective [Keating, M.E.: On the K-theory of the quaternion group. Mathematika 20, 59–62 (1973), Remark 2.4], we see that $K_2(\mathbb{Z}G) \longrightarrow K_2(\mathbb{F}_2 G)$ is surjective.

If $g_1, \cdots, g_n$, are the generators of G, then the $n+1$ symbols $\{-1,-1\}, \{-1, g_1\}, \cdots, \{-1, g_n\}$ are independent [[8], p. 65] and lie in the kernel of this map. Hence

$$\text{rank } K_2(\mathbb{Z}G) \geq (n-1)(2^n - 1) + (n+1) = (n-1)2^n + 2.$$

Recall that for G abelian, $Wh_2(G)$ is the quotient of $K_2(\mathbb{Z}G)$ by the subgroup generated by all symbols of the form $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ [Hatcher, A.E.: Pseudo-isotopy and K_2, pp. 328-336. Lecture Notes in Mathematics 342. Berlin, Heidelberg, New York: Springer 1973]. It is easy to see from the bimuttiplicative and anti-symmetric properties of symbols that this subgroup has rank at most $\binom{n+1}{2} + 1$. Moreover, by using the various maps $\mathbb{Z}G \longrightarrow \mathbb{Z}$ which send elements of G to ± 1, it can be shown that the rank of this subgroup is precisely $\binom{n+1}{2} + 1$. $(n-1)2^n + 2 - \binom{n+1}{2} - 1 = (n-1)2^n - \frac{(n+2)(n-1)}{2}$.

(2) 以下这一段没有完全读懂。Let B be the integral chosure of $\mathbb{Z}G$ in $\mathbb{Q}G$. Then $SK_1(B, p^{n+1}B)$ has p-rank $\frac{p^n-1}{p-1}$ [Bass, H., Milnor, J., Serre, J. P.: Solution of the congruence subgroup problem for $SL_n (n \geq 3)$ and $Sp_{2n} (n \geq 2)$. Publ. Math. IHES 33, 59–137 (1967), Corollary 4.3, p. 95].

But $SK_1(B, p^{n+1}B) \cong SK_1(\mathbb{Z}G, p^{n+1}B)$ [[2], p. 484] since $p^n B$ lies in the conductor of B over $\mathbb{Z}G$, and $SK_1(\mathbb{Z}G, p^{n+1}B)$ maps onto $SK_1(\mathbb{Z}G, p\mathbb{Z}G)$ [[2], 9.3, p. 267]. Hence p-rank $SK_1(\mathbb{Z}G, p\mathbb{Z}G) \leq \frac{p^n-1}{p-1}$. The p-rank of $SK_1(\mathbb{Z}G)$ is $\frac{p^n-1}{p-1} - \binom{p+n-1}{p}$ [Alperin, R.C., Dennis, R. K., Stein, M. R.: The non-triviality of $SK_1(\mathbb{Z}\pi)$, pp. 1-7. Lecture Notes in Mathematics 353. Berlin, Heidelberg, New York: Springer t973, Theorem 2]. The result now follows from exact sequence 2.12.

And noting that the subgroup generated by the symbols $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ has p-rank at most $\frac{n(n-1)}{2}$. □

Remark 2.14. The subgroup of $K_2(\mathbb{Z}G)$ generated by elements of the form $\langle a, b\rangle$, $1+ab \in (\mathbb{Z}G)^*$ maps onto $K_2(\mathbb{F}_2 G)$ for G an elementary abelian 2-group of rank ≤ 2. W. van der Kallen has shown that this subgroup maps onto in general. This follows from the rank 2 case via

Lemma (van der Kallen). Let I be a nilpotent ideal of the commutative ring R. Let $v_i \in R$ additively generate R/I and let $w_j \in I$ additively generate I. Then $K_2(I) = \ker(K_2(R) \longrightarrow K_2(R/I))$ is generated by all elements of the form $\langle v_i, w_j\rangle$ and $\langle w_j, w_i^{2^k-1} w_j\rangle$.

我的一些问题：$NK_2(\mathbb{F}_q G)$ 如何算，$NK_1(\mathbb{Z}G, p\mathbb{Z}G) =?$

2.3 推广和其它

之前考虑的是 $\mathbb{Z}G$, G elementary. 可以推广到 G finite group, $\mathcal{O}$ be the ring of integers of an algebraic number field.

If S is a Sylow p-subgroup of G, then $\mathcal{O}G$ is a free module over $\mathcal{O}S$ and the composition

$$K_2(\mathcal{O}S) \longrightarrow K_2(\mathcal{O}G) \longrightarrow K_2(\mathcal{O}S)$$

(where the second map is the transfer) is multiplication by $(G : S)$. Hence p-rank $K_2(\mathcal{O}G) \geq p$-rank $K_2(\mathcal{O}S)$ and estimates may be obtained by restricting to the case of a p-group.

Theorem 2.15. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field which is Galois over $\mathbb{Q}$ and let G be an elementary abelian p-group of rank n. If p is unramfied in $\mathcal{O}$ with each residue field having degree f over $\mathbb{F}_p$, then $K_2(\mathcal{O}G)$ has p-rank at least*
(i) $f(n-1)(2^n-1)$ if $p=2$ and $\mathcal{O}$ has a real embedding,
(ii) $f(n-1)(2^n-l)-\binom{n+1}{2}$ if $p=2$ and $\mathcal{O}$ is totally imaginary,
(iii) $f(n-1)(p^n-l)-\binom{p+n-1}{p}$ if p is odd.

abelian p-groups which are not elementary 有以下一个结论

Proposition 2.16. *Let p be an odd prime and suppose $G = H \times C$ where C is cyclic of order p^t, $|H| = p^k$ and $s = p\text{-rank } H$. Let $\mathcal{O}$ be the ring of integers in a number field. Choose a prime $\mathfrak{p}$ of $\mathcal{O}$ lying over p and having residue degree f over $\mathbb{F}_p$. Then*

$$\begin{aligned} &\operatorname{ord}_p|K_2(\mathcal{O}G/\mathfrak{p}G)| - \operatorname{ord}_p|K_2(\mathcal{O}H/\mathfrak{p}H)| \\ \geq & f\Big(p^k\big(s(p-1)p^{t-1}+1\big) - |H^{p^t}|\Big) + p^k(p^{t-1}-1) - (p-1)\sum_{r=1}^{t-1}|H^{p^r}|p^{t-r-1}. \end{aligned}$$

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