Chapter 1

$NK_2(\mathbb{F}_2[C_4])$

$$NK_i(R) = \ker(K_i(R[x]) \xrightarrow{x\mapsto 0} K_i(R)).$$

目标

1. 证明 $NK_2(\mathbb{F}_2[C_4])$ 中有四阶元,
2. 确定它的结构和生成元.
3. 推广到 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 和 $NK_p(\mathbb{F}_{p^f}[C_{p^n}])$.
4. 未来推广到 $NK_2(\mathbb{F}_{p^f}[G])$, 其中 G 是有限交换群, $G = C_{p^n} \times H$.

1.1 思路

已经有 $NK_2(\mathbb{F}_2[C_2])$ 和 $NK_2(\mathbb{F}_p[C_p])$ 的结果 [2]. 这种方法详细见下文1.3.1.

考虑 $\mathbb{F}_{p^f}[C_{p^n}]$ 时, 可以将它写成截断多项式 $\mathbb{F}_{p^f}[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$, $t_1 \mapsto 1-\sigma$.

$$\mathbb{F}_{p^f}[C_{p^n}] = \mathbb{F}_{p^f}[\sigma]/(1-\sigma^{p^n}) = \mathbb{F}_{p^f}[\sigma]/((1-\sigma)^{p^n}) \cong \mathbb{F}_{p^f}[t_1]/(t_1^{p^n})$$
$$1-\sigma \mapsto t_1$$

考虑它的 NK_2 时, 可以转化成相对 K_2 群:

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})).$$

利用 van der Kallen [10] 中的方法对于这种情形的相对 K_2 群, 有这样的结论 (符号说明见后文):

Theorem. $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} \left(1+xk[x]/(x^{[\alpha,i]})\right)^{\times}.$$

接下来的任务就是确定两件事情:

- 确定集合Λ^{00}, 确定数值$[\alpha,i]$.
- 确定右边这个乘法群的结构.

实际上对于第一件事情, 我们只需要考虑这个集合的一部分, 因为 $(1+xk[x]/(x))^\times$ 是平凡的, 所以只要考虑 $[\alpha,i]>1$ 所对应的 (α,i) 全体（后文详细解释）.

对于第二件事情, 有一些例子是可以直接计算的, 如 $(1+x\mathbb{F}_2[x]/(x^4))^\times\cong\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. 对于一般的情况引入 big Witt vectors,

$$BigWitt_n(\mathbb{F}_q):=(1+x\mathbb{F}_q[\![x]\!])^\times/(1+x^{n+1}\mathbb{F}_q[\![x]\!])^\times\cong(1+x\mathbb{F}_q[x]/(x^{n+1}))^\times$$

由 big Witt vectors分解成 typical Witt vectors有

$$\begin{aligned}(1+x\mathbb{F}_{p^f}[x]/(x^{n+1}))^\times &\cong BigWitt_n(\mathbb{F}_{p^f})\\ &\cong \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} W_{1+\lfloor\log_p\frac{n}{m}\rfloor}(\mathbb{F}_{p^f})\\ &= \bigoplus_{\substack{1\le m\le n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor\log_p\frac{n}{m}\rfloor}\mathbb{Z})^f,\end{aligned}$$

其中 $\lfloor x\rfloor$ 表示不超过 x 的最大整数.

结合这个同构1.3就可以得到结论.

结论 已知的结果

Theorem. *(1)$NK_2(\mathbb{F}_2[C_2])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_2])\cong K_2(\mathbb{F}_2[t,x]/(t^2),(t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i,x\rangle\mid i\ge 0\}$ 与 $\{\langle tx^i,t\rangle\mid i\ge 1$为奇数$\}$ 生成的, 这样的符号均为 2 阶元.

$NK_2(\mathbb{F}_2[C_4])$ 的结果

Theorem. *(1)$NK_2(\mathbb{F}_2[C_4])\cong\bigoplus_\infty\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_\infty\mathbb{Z}/4\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号

$$\{\langle tx^{i-1},x\rangle\mid i\ge 1\},\{\langle tx^i,t\rangle\mid i\ge 1\text{为奇数}\},\{\langle t^3x^{i-1},x\rangle\mid i\ge 1\},\{\langle t^3x^i,t\rangle\mid i\ge 1\text{为奇数}\}$$

生成的.

$NK_2(\mathbb{F}_q[C_{2^n}])$ 的结果

$$NK_2(\mathbb{F}_q[C_{2^n}])\cong\bigoplus_\infty\bigoplus_{k=1}^n\mathbb{Z}/2^k\mathbb{Z}.$$

1.2 预备知识和引理

这一节主要是介绍前文提到的如何将 NK_2 转化成相对 K_2 群, [10] 中符号说明与相对 K_2 群, 以及 Witt 向量的分解.

令 k 是特征为 $p>0$ 的有限域, 考虑两个变元的多项式环 $k[t_1,t_2]$, $I=(t_1^n)$ 是 $k[t_1,t_2]$ 的一个真理想, 令

$$A=k[t_1,t_2]/I,$$

设 M 是 A 的 nil 根 (小根), 即 $M=(t_1)$, 则有 $A/M=k[t_2]$.

Proposition 1.1. $K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \cong K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$.

Proof. 由于 $k[t_2] \overset{i_1}{\hookrightarrow} k[t_1,t_2]/(t_1^n)$, $k[t_1,t_2]/(t_1^n) \overset{p_1}{\twoheadrightarrow} k[t_2]$ 与 $k \overset{i_2}{\hookrightarrow} k[t_1,t_2]/(t_1^n)$, $k[t_1,t_2]/(t_1^n) \overset{p_2}{\twoheadrightarrow} k$ 满足 $p_1 i_1 = \mathrm{id}$, $p_2 i_2 = \mathrm{id}$, 故由 K_n 的函子性有 $K_n(p_1)K_n(i_1) = \mathrm{id}$ 与 $K_n(p_2)K_n(i_2) = \mathrm{id}$, 从而有以下相对 K 群的可裂正合列

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k[t_2]) \longrightarrow 0$$

$$0 \longrightarrow K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)) \longrightarrow K_2(k[t_1,t_2]/(t_1^n)) \longrightarrow K_2(k) \longrightarrow 0$$

由于 k 是有限域, 故 $K_2(k) = 0$, 又由于有限域都是正则环, 故 $NK_2(k) = 0$, 于是 $K_2(k[t_2]) = K_2(k) \oplus NK_2(k) = 0$. 从而得

$$K_2(k[t_1,t_2]/(t_1^n),(t_1)) \cong K_2(k[t_1,t_2]/(t_1^n)) \cong K_2(k[t_1,t_2]/(t_1^n),(t_1,t_2)).$$

□

当 $k = \mathbb{F}_{p^f}$ 时, $k[t_1]/(t_1^{p^n}) \cong \mathbb{F}_{p^f}[C_{p^n}]$, 其中 C_{p^n} 是 p^n 阶循环群. 有以下可裂正合列

$$0 \longrightarrow NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) \longrightarrow K_2(\mathbb{F}_{p^f}[C_{p^n}]) \longrightarrow 0,$$

由于 $K_2(\mathbb{F}_{p^f}[C_{p^n}][x]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}))$, 并且 $K_2(\mathbb{F}_{p^f}[C_{p^n}]) = 0$ [5], 从而

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}),(t_1)).$$

1.2.1 Dennis-Stein 符号

一般地, 通过 Dennis-Stein 符号可以给出 $K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$ 的一个表现

生成元 $\langle a,b\rangle$, 其中 $(a,b) \in A \times M \cup M \times A$;

关系 (DS1) $\langle a,b\rangle = -\langle b,a\rangle$,

(DS2) $\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc,b\rangle$,

(DS3) $\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$, 其中 $(a,b,c) \in M \times A \times A \cup A \times M \times A \cup A \times A \times M$.

Proposition 1.2. *对任意环 R, 任意自然数 $q > 1$, $K_2(R[t]/(t^q),(t))$ 由满足上述关系的 Dennis-Stein 符号 $\langle at^i,t\rangle$ 和 $\langle at^i,b\rangle$ 生成, 其中 $a,b \in R, 1 \le i < q$.*

Proof. 参见 [9] Proposition 1.7. □

1.2.2 符号说明

为了陈述定理, 将文中常用的记号详述如下 (参考 [10])

- $\mathbb{Z}_+$ 表示非负整数全体.
- $\varepsilon^1 = (1,0) \in \mathbb{Z}_+^2$, $\varepsilon^2 = (0,1) \in \mathbb{Z}_+^2$.
- 对于 $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2$, 记 $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, 于是有 $t^{\varepsilon^1} = t_1$, $t^{\varepsilon^2} = t_2$.
- $\Delta = \{\alpha \in \mathbb{Z}_+^2 \mid t^\alpha \in I\}$, 若 $\delta \in \Delta$, 则 $\delta + \varepsilon^i \in \Delta, i = 1,2$.
- $\Lambda = \{(\alpha,i) \in \mathbb{Z}_+^2 \times \{1,2\} \mid \alpha_i \ge 1, t^\alpha \in M\}$.
- 对于 $(\alpha,i) \in \Lambda$, 令 $[\alpha,i] = \min\{m \in \mathbb{Z} \mid m\alpha - \varepsilon^i \in \Delta\}$. 若 $(\alpha,i),(\alpha,j) \in \Lambda$, 有 $[\alpha,i] \le [\alpha,j] + 1$.
- 若 $gcd(p,\alpha_1,\alpha_2) = 1$, 令 $[\alpha] = \max\{[\alpha,i] \mid \alpha_i \not\equiv 0 \bmod p\}$, 其中 p 是域 k 的特征.
- $\Lambda^{00} = \{(\alpha,i) \in \Lambda \mid gcd(\alpha_1,\alpha_2) = 1, i \ne \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha,j] = [\alpha]\}\}$.

若 $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, 令

$$\begin{aligned}\Gamma_{\alpha,i}\colon (1+xk[x])^{\times} &\longrightarrow K_2(A,M)\\ 1-xf(x) &\mapsto \langle f(t^{\alpha})t^{\alpha-\varepsilon^i}, t_i\rangle.\end{aligned}$$

$\Gamma_{\alpha,i}$ 是群同态: $(1-xf_1(x))(1-xf_2(x)) = 1-x(f_1(x)+f_2(x)-xf_1(x)f_2(x))$,

$$\begin{aligned}\Gamma_{\alpha,i}((1-xf_1(x))(1-xf_2(x))) &= \Gamma_{\alpha,i}(1-x(f_1(x)+f_2(x)-xf_1(x)f_2(x)))\\ &= \langle f_1(t^{\alpha})t^{\alpha-\varepsilon^i}+f_2(t^{\alpha})t^{\alpha-\varepsilon^i}-t^{\alpha}f_1(t^{\alpha})f_2(t^{\alpha})t^{\alpha-\varepsilon^i}, t_i\rangle\\ \Gamma_{\alpha,i}(1-xf_1(x))\Gamma_{\alpha,i}(1-xf_2(x)) &= \langle f_1(t^{\alpha})t^{\alpha-\varepsilon^i}, t_i\rangle + \langle f_2(t^{\alpha})t^{\alpha-\varepsilon^i}, t_i\rangle\\ &= \langle f_1(t^{\alpha})t^{\alpha-\varepsilon^i}+f_2(t^{\alpha})t^{\alpha-\varepsilon^i}-f_1(t^{\alpha})t^{\alpha-\varepsilon^i}f_2(t^{\alpha})t^{\alpha-\varepsilon^i}t_i, t_i\rangle\end{aligned}$$

若 $(\alpha, i) \in \Lambda$, $\Gamma_{\alpha,i}$ 诱导了同态

$$(1+xk[x]/(x^{[\alpha,i]}))^{\times} \longrightarrow K_2(A,M).$$

Theorem 1.3. $\Gamma_{\alpha,i}$ 诱导了同构

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+xk[x]/(x^{[\alpha,i]}))^{\times}.$$

Proof. 参见 [10] Corollary 2.6.. □

1.2.3 Witt 向量

令 R 是交换环, R 上的泛 Witt 向量环 (the ring of universal/big Witt vectors over R)$BigWitt(R)$ 作为交换群同构于 $(1+xR[\![x]\!])^{\times}$, 即常数项为 1 的形式幂级数全体在乘法运算下形成的交换群,

$$\begin{aligned}BigWitt(R) &\xrightarrow{\sim} (1+xR[\![x]\!])^{\times}\\ (r_1, r_2, \cdots) &\mapsto \prod_{n=1}^{\infty}(1-r_nx^n).\end{aligned}$$

考虑 $(1+xR[\![x]\!])^{\times}$ 的子群 $(1+x^{n+1}R[\![x]\!])^{\times}$, 记交换群 $BigWitt_n(R) = (1+xR[\![x]\!])^{\times}/(1+x^{n+1}R[\![x]\!])^{\times}$. 显然 $BigWitt_1(R) = R$, 并且注意当 $n \geq 3$ 时, $BigWitt_n(\mathbb{F}_2)$ 不是循环群.

Lemma 1.4. *$BigWitt_n(\mathbb{F}_q) \cong (1+x\mathbb{F}_q[x]/(x^{n+1}))^{\times}$.*

Proof. 考虑群的满同态

$$\begin{aligned}(1+x\mathbb{F}_q[\![x]\!])^{\times} &\longrightarrow (1+x\mathbb{F}_q[x]/(x^{n+1}))^{\times}\\ 1+\sum_{i\geq 1}a_ix^i &\mapsto 1+\sum_{i=1}^{n}a_ix^i+(x^{n+1})\end{aligned}$$

其中 $a_i \in \mathbb{F}_q$, 易知同态核为 $(1+x^{n+1}\mathbb{F}_q[\![x]\!])^{\times}$, 从而 $BigWitt_n(\mathbb{F}_q) = (1+x\mathbb{F}_q[\![x]\!])^{\times}/(1+x^{n+1}\mathbb{F}_q[\![x]\!])^{\times} \cong (1+x\mathbb{F}_q[x]/(x^{n+1}))^{\times}$. □

Example 1.5. $BigWitt_3(\mathbb{F}_2) \cong (1+x\mathbb{F}_2[x]/(x^4))^{\times} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. $1+x \in (1+x\mathbb{F}_2[x]/(x^4))^\times$ 是 4 阶元, 由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3\}$, 且 $1+x^3$ 是二阶元, $\langle 1+x^3\rangle = \{1, 1+x^3\}$. 令 σ, τ 分别是 $\mathbb{Z}/4\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元, 则有同构

$$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow BigWitt_4(\mathbb{F}_2)$$
$$(\sigma, \tau) \mapsto (1+x)(1+x^3) = 1+x+x^3.$$

□

Example 1.6. $BigWitt_4(\mathbb{F}_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. $1+x \in BigWitt_5(\mathbb{F}_2)$ 是 8 阶元, 由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3, 1+x^4, 1+x+x^4, 1+x^2+x^4, 1+x+x^2+x^3+x^4\}$, 另外 $1+x^3$ 是二阶元, $\langle 1+x^3\rangle = \{1, 1+x^3\}$. 令 σ, τ 分别是 $\mathbb{Z}/8\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元, 则有同构

$$\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow BigWitt_4(\mathbb{F}_2)$$
$$(\sigma, \tau) \mapsto (1+x)(1+x^3) = 1+x+x^3+x^4$$

于是 $(\sigma^i, \tau^j), 0 \le i < 8, 0 \le j < 2$ 对应于 $(1+x)^i(1+x^3)^j$, 详细的对应如下

$$\begin{aligned}
(1,\tau) &\mapsto 1+x^3, & (\sigma,\tau) &\mapsto 1+x+x^3+x^4,\\
(\sigma^2,\tau) &\mapsto 1+x^2+x^3, & (\sigma^3,\tau) &\mapsto 1+x+x^2+x^4,\\
(\sigma^4,\tau) &\mapsto 1+x^3+x^4, & (\sigma^5,\tau) &\mapsto 1+x+x^3,\\
(\sigma^6,\tau) &\mapsto 1+x^2+x^3+x^4, & (\sigma^7,\tau) &\mapsto 1+x+x^2,\\
(1,1) &\mapsto 1, & (\sigma,1) &\mapsto 1+x,\\
(\sigma^2,1) &\mapsto 1+x^2, & (\sigma^3,1) &\mapsto 1+x+x^2+x^3,\\
(\sigma^4,1) &\mapsto 1+x^4, & (\sigma^5,1) &\mapsto 1+x+x^4,\\
(\sigma^6,1) &\mapsto 1+x^2+x^4, & (\sigma^7,1) &\mapsto 1+x+x^2+x^3+x^4.
\end{aligned}$$

□

固定素数 p, 考虑局部环 $\mathbb{Z}_{(p)} = \mathbb{Z}[1/\ell \mid \text{所有素数}\ell \neq p]$, 即 $\mathbb{Z}$ 在非零素理想 $(p) = p\mathbb{Z}$ 处的局部化, 于是 $\mathbb{Z}_{(p)}$-代数 R 就是所有除 p 外的素数均在其中可逆的交换环, 如 $\mathbb{F}_{p^n}$ 是一个 $\mathbb{Z}_{(p)}$-代数.

下面考虑 p-Witt 向量环 $W(A)$ 与截断 p-Witt 向量环 $W_n(A)$, p-Witt 向量为 $(a_0, a_1, \cdots)$, 加法用 Witt 多项式定义, 本文仅考虑用加法定义的交换群结构, 例如作为交换群 $W_n(\mathbb{F}_{p^f})$ 同构于 $(\mathbb{Z}/p^n\mathbb{Z})^f$.

Artin-Hasse 级数定义为

$$AH(x) = \exp(\sum_{n\ge 0} \frac{x^{p^n}}{p^n}) = 1 + x + \cdots \in 1 + x\mathbb{Q}[\![x]\!],$$

实际上 $AH(x) \in 1 + x\mathbb{Z}_{(p)}[\![x]\!]$(参考 [6] Theorem 7.2). 对于 $BigWitt(R) = (1+xR[\![x]\!])^\times$ 中的任一元素 α 可以写成无穷乘积

$$\alpha = \prod_{n=1}^{\infty}(1 - r_n x^n),$$

其中 $r_n \in R$ 是唯一的. 若 A 是 $\mathbb{Z}_{(p)}$-代数, $BigWitt(A) = (1+xA[\![x]\!])^\times$ 中的任一元素 α 还有如下表法 [3]

$$\alpha = \prod_{n\ge 1} AH(a_n x^n),\ a_n \in A.$$

将整数 n 写成 $n=mp^a$, 使得 $gcd(m,p)=1, a\geq 0$, 由于 A 是 $\mathbb{Z}_{(p)}$-代数, m 可逆, 从而 $[x\mapsto x^{1/m}]\in \mathrm{End}(BigWitt(A))$ 是双射, 于是我们可以将 $\alpha\in BigWitt(A)$ 以如下的形式表出

$$\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a}x^{mp^a})^{1/m}.$$

另一方面对于 $\mathbb{Z}_{(p)}$-代数 A, 下列映射是群同态

$$W(A)\longrightarrow BigWitt(A)$$
$$(a_0,a_1,\cdots)\mapsto \prod_{i\geq 0} AH(a_ix^i).$$

作为交换群, $BigWitt_n(A)$ 可以分解为 p-Witt 向量环的直和, 实际上有以下同构 [4]

$$BigWitt(A)\cong \prod_{\substack{m\geq 1\\ gcd(m,p)=1}} W(A),$$

元素 $\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a}x^{mp^a})^{1/m}$ 对应于一个 m-分量为 $(a_m,a_{mp},a_{mp^2},\cdots)\in W(A)$ 的 Witt 向量. 对于截断的 Witt 向量环, 有同构

$$BigWitt_n(A)\cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(A),$$

其中 $\ell(m,n)$ 是一个整数, 定义为

$$\ell(m,n)=1+\left\lfloor \log_p\frac{n}{m}\right\rfloor,$$

即 $\ell(m,n)=1+$ 使得 $mp^k\leq n$ 成立的最大整数 k.

考虑特征为 p 的有限域 $\mathbb{F}_q$, 有同构 [4]

$$BigWitt_n(\mathbb{F}_q)\cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(\mathbb{F}_q),$$

注意到 $\sum_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} \ell(m,n)=n$, 因此两边都是 q^n 阶交换群.

Corollary 1.7. 若有限域 $\mathbb{F}_{p^f}$ 的特征 $ch(\mathbb{F}_{p^f})=p$, 则作为交换群有

$$BigWitt_n(\mathbb{F}_{p^f})\cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{1+\lfloor \log_p\frac{n}{m}\rfloor}(\mathbb{F}_{p^f})=\bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\lfloor \log_p\frac{n}{m}\rfloor}\mathbb{Z})^f,$$

其中 $\lfloor x\rfloor$ 表示不超过 x 的最大整数.

Example 1.8. 作为交换群, $BigWitt_3(\mathbb{F}_2)=W_{\ell(1,3)}(\mathbb{F}_2)\oplus W_{\ell(3,3)}(\mathbb{F}_2)=W_2(\mathbb{F}_2)\oplus W_1(\mathbb{F}_2)=\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$,
$BigWitt_4(\mathbb{F}_2)=W_{\ell(1,4)}(\mathbb{F}_2)\oplus W_{\ell(3,4)}(\mathbb{F}_2)=W_3(\mathbb{F}_2)\oplus W_1(\mathbb{F}_2)=\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$,
$BigWitt_2(\mathbb{F}_3)=W_{\ell(1,2)}(\mathbb{F}_3)\oplus W_{\ell(2,2)}(\mathbb{F}_3)=W_1(\mathbb{F}_3)\oplus W_1(\mathbb{F}_3)=\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$.

1.3 $NK_2(\mathbb{F}_2[C_2])$ 的计算

方法一是在讲 Weibel 文章 [12] 时讲过的. 方法二是基于上面的思路给出来的详细证明.

1.3.1 方法一

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{1.8}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$\begin{tikzcd} & NK_3(\mathbb{Z}, (2)) = 0 \arrow[d] & & \\ & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\ 0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\ & NK_2(\mathbb{Z}, (2)) = 0 & & \end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2], (\sigma+1), (\sigma-1))$, from Guin-Loday-Keune [2], $NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 1.9. $NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 1.10 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma - 1)$, $J = (1 + \sigma + \cdots + \sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "f", "\sigma\mapsto 1"'] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1})$. Note that $p(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1}) = 0$ since $(1 + \sigma + \cdots + \sigma^{p-1})^2 = p(1 + \sigma + \cdots + \sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R, I)$$
$$(\sigma - 1) \otimes (1 + \sigma + \cdots + \sigma^{p-1}) \mapsto \langle \sigma - 1, 1 + \sigma + \cdots + \sigma^{p-1} \rangle = \langle \sigma - 1, 1 \rangle^p = 1$$

Also see [8].

Example 1.11 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i. e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

1.3.2 方法二

计算 $k = \mathbb{F}_2$, $p = 2$, $n = 2$ 的情形, 即 $NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^2), (t_1))$.

Theorem 1.12. *(1)$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$,*
(2)$NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t, x]/(t^2), (t))$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1$为奇数$\}$ 生成的, 这样的符号均为 2 阶元.

Proof. (1) 令 $A=\mathbb{F}_2[t_1,t_2]/(t_1^2)\cong\mathbb{F}_2[C_2][x]$, 此时 $I=(t_1^2)$, $M=(t_1)$, $A/M\cong\mathbb{F}_2[x]$. 记号如1.2.2所述.

$$\begin{aligned}
\Delta &= \{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2 \mid t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1^2)\}\\
&= \{(\alpha_1,\alpha_2)\mid \alpha_1\geq 2,\alpha_2\geq 0\},\\
\Lambda &= \{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid \alpha_i\geq 1,\text{且} t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1)\}\\
&= \{((\alpha_1,\alpha_2),i)\in\mathbb{Z}_+^2\times\{1,2\}\mid \alpha_i\geq 1,\alpha_1\geq 1,\alpha_2\geq 0\}\\
&= \{((\alpha_1,\alpha_2),1)\mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2)\mid \alpha_1\geq 1,\alpha_2\geq 1\}.
\end{aligned}$$

若 $(\alpha,i)\in\Lambda$, $[\alpha,i]=\min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^i\in\Delta\}$, 于是有

$$\begin{aligned}
[\alpha,1] &= \min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^1\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid (m\alpha_1-1,m\alpha_2)\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid m\alpha_1\geq 3\}.\\
[\alpha,2] &= \min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^2\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid (m\alpha_1,m\alpha_2-1)\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid m\alpha_1\geq 2\}.
\end{aligned}$$

此时

$$\begin{aligned}
&[(1,\alpha_2),1]=3,\ \alpha_2\geq 0,\\
&[(2,\alpha_2),1]=2,\ \alpha_2\geq 0,\\
&[(\alpha_1,\alpha_2),1]=1,\ \alpha_1\geq 3,\alpha_2\geq 0,\\
&[(1,\alpha_2),2]=2,\ \alpha_2\geq 1,\\
&[(\alpha_1,\alpha_2),2]=1,\ \alpha_1\geq 2,\alpha_2\geq 1.
\end{aligned}$$

若 $gcd(2,\alpha_1,\alpha_2)=1$, 即 α_1,α_2 中至少一个是奇数, 令 $[\alpha]=\max\{[\alpha,i]\mid \alpha_i\not\equiv 0 \bmod 2\}$. 若仅 α_1 是奇数, $[\alpha]=[\alpha,1]$, 若仅 α_2 是奇数, $[\alpha]=[\alpha,2]$, 若两者均为奇数, 则 $[\alpha]=\max\{[\alpha,1],[\alpha,2]\}$, 即有

$$\begin{aligned}
&[(1,\alpha_2)]=\max\{[(1,\alpha_2),1],[(1,\alpha_2),2]\}=3,\alpha_2\geq 1 \text{ 是奇数}\\
&[(1,\alpha_2)]=[(1,\alpha_2),1]=3,\alpha_2\geq 0 \text{ 是偶数}\\
&[(3,\alpha_2)]=\max\{[(3,\alpha_2),1],[(3,\alpha_2),2]\}=1,\alpha_2\geq 1 \text{ 是奇数}\\
&[(3,\alpha_2)]=[(3,\alpha_2),1]=1,\alpha_2\geq 0 \text{ 是偶数}\\
&[(2,1)]=[(2,1),2]=1,\\
&[\alpha]=1, \text{其它符合条件的 } \alpha.
\end{aligned}$$

为了方便我们把上面的计算结果列表如下

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$3,\alpha_2\geq 0$	$2,\alpha_2\geq 1$	3
$(2,\alpha_2)$	$2,\alpha_2\geq 0$	$1,\alpha_2\geq 1$	1, 当 α_2 是奇数时
$(3,\alpha_2)$	$1,\alpha_2\geq 0$	$1,\alpha_2\geq 1$	1
$(\alpha_1,0),\alpha_1\geq 3$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2),\alpha_1\geq 3,\alpha_2\geq 1$	1	1	1, 当 $gcd(2,\alpha_1,\alpha_2)=1$ 时

下面计算 $\Lambda^{00} = \{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha, j] = [\alpha]\}\}$.

1. 对于任何的 $\alpha_2 \geq 0$, $((1, \alpha_2), 1) \notin \Lambda^{00}$, 这是因为 $1 \not\equiv 0 \bmod 2$ 且 $[(1, \alpha_2), 1] = 3 = [(1, \alpha_2)]$, 从而 $\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(1, \alpha_2), j] = [(1, \alpha_2)]\} = 1$.
2. 对于任何的 $\alpha_2 \geq 1$, $((1, \alpha_2), 2) \in \Lambda^{00}$, 且 $[(1, \alpha_2), 2] = 2$. 这是因为此时 $[(1, \alpha_2), 1] = 3 = [(1, \alpha_2)]$, $\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha, j] = [\alpha]\} = 1$, .
3. 对于任何的奇数 $\alpha_2 \geq 1$, $((2, \alpha_2), 1) \in \Lambda^{00}$, 且 $[(2, \alpha_2), 1] = 2$. 因为 $\alpha_2 \not\equiv 0 \bmod 2$ 并且 $[(2, \alpha_2), 2] = 1 = [(2, \alpha_2)]$, 故 $\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(2, \alpha_2), j] = [(2, \alpha_2)]\} = 2 \neq 1$. 注意若 $\alpha_2 \geq 0$ 为偶数时, $[2, \alpha_2]$ 无定义, 因此 $((2, \alpha_2), 1) \notin \Lambda^{00}$, 同理 $((2, \alpha_2), 2) \notin \Lambda^{00}$.
4. 对于任何的奇数 $\alpha_2 \geq 1$, $((2, \alpha_2), 2) \notin \Lambda^{00}$. 由于 $[(2, \alpha_2), 2] = 1 = [(2, \alpha_2)]$, 与 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha, j] = [\alpha]\}$ 矛盾.
5. 对于偶数 $\alpha_1 \geq 3$ 和奇数 $\alpha_2 \geq 1$, $((\alpha_1, \alpha_2), 1) \in \Lambda^{00}$, 且 $[(\alpha_1, \alpha_2), 1] = 1$. 而当 $\alpha_1 \geq 3$ 为奇数 $\alpha_2 \geq 1$ 时, 或 α_1, α_2 均为偶数时, $((\alpha_1, \alpha_2), 1) \notin \Lambda^{00}$. 由于 $((\alpha_1, \alpha_2), 1) \in \Lambda^{00}$ 要求 $1 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1, \alpha_2), j] = [(\alpha_1, \alpha_2)]\}$, 当 $\alpha_1 \geq 3$ 为奇数时上式不成立, $2 = \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1, \alpha_2), j] = [(\alpha_1, \alpha_2)]\}$ 当且仅当 $\alpha_1 \geq 3$ 为偶数且 $\alpha_2 \geq 1$ 为奇数.
6. 对于奇数 $\alpha_1 \geq 3$ 和任意 $\alpha_2 \geq 1$, 且 $[(\alpha_1, \alpha_2), 2] = 1$. $((\alpha_1, \alpha_2), 2) \in \Lambda^{00}$, 其余情况只要当 $\alpha_1 \geq 3$ 为偶数时 $((\alpha_1, \alpha_2), 2) \notin \Lambda^{00}$. 由于 $((\alpha_1, \alpha_2), 2) \in \Lambda^{00}$ 要求 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1, \alpha_2), j] = [(\alpha_1, \alpha_2)]\}$, 当 α_1 为偶数时上式不成立, 而当 α_1 为奇数时, 任意 $\alpha_2 \geq 1$, $[(\alpha_1, \alpha_2), 1] = 1 = [(\alpha_1, \alpha_2)]$.

从而

$$\begin{aligned}
\Lambda^{00} =& \{((1, \alpha_2), 2) \mid \alpha_2 \geq 1\} \\
& \cup \{((2, \alpha_2), 1) \mid \alpha_2 \geq 1 \text{为奇数}\} \\
& \cup \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 3 \text{为偶数}, \alpha_2 \geq 1 \text{为奇数}\} \\
& \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 3 \text{为奇数}, \alpha_2 \geq 1\}.
\end{aligned}$$

记 $\Lambda^{00}_d = \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] = d\}$, $\Lambda^{00}_{>1} = \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] > 1\}$, 则有 $\Lambda^{00}_1 = \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] = 1\}$, $\Lambda^{00}_{>1} = \Lambda^{00}_2 = \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] = 2\}$, 于是有

$$\begin{aligned}
\Lambda^{00}_1 &= \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 3 \text{为偶数}, \alpha_2 \geq 1 \text{为奇数}\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 3 \text{为奇数}, \alpha_2 \geq 1\} \\
\Lambda^{00}_2 &= \{((2, \alpha_2), 1) \mid \alpha_2 \geq 1 \text{为奇数}\} \cup \{((1, \alpha_2), 2) \mid \alpha_2 \geq 1\}
\end{aligned}$$

$$\Lambda^{00} = \Lambda^{00}_1 \sqcup \Lambda^{00}_2.$$

若 $[\alpha, i] = 1$ 时, $(1 + x\mathbb{F}_2[x]/(x))^\times$ 是平凡的, $[\alpha, i] = 2$ 时, $(1 + x\mathbb{F}_2[x]/(x^2))^\times \cong \mathbb{Z}/2\mathbb{Z}$, 从而由定理1.3得

$$\begin{aligned}
NK_2(\mathbb{F}_2[C_2]) \cong K_2(A, M) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{(\alpha,i)\in\Lambda^{00}_2} (1 + x\mathbb{F}_2[x]/(x^2))^\times \\
&= \bigoplus_{\substack{((1,\alpha_2),2) \\ \alpha_2\geq 1}} (1 + x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((2,\alpha_2),1) \\ \alpha_2\geq 1\text{为奇数}}} (1 + x\mathbb{F}_2[x]/(x^2))^\times \\
&= \bigoplus_{\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z},
\end{aligned}$$

作为交换群,

$$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z}.$$

(2) 由1.3, 对于任意 $(\alpha, i) \in \Lambda^{00}$, $\Gamma_{\alpha,i}$ 诱导了同态

$$\begin{aligned}\Gamma_{\alpha,i} \colon \left(1 + xk[x]/(x^{[\alpha,i]})\right)^{\times} &\longrightarrow K_2(A, M)\\ 1 - xf(x) &\mapsto \langle f(t^{\alpha})t^{\alpha-\varepsilon^i}, t_i\rangle.\end{aligned}$$

此时只需考虑 $\Lambda_2^{00} = \{((2, \alpha_2), 1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((1, \alpha_2), 2) \mid \alpha_2 \geq 1\}$, 对于任意 $(\alpha, i) \in \Lambda_2^{00}$, $\Gamma_{\alpha,i}$ 均诱导了单射, 对任意 $\alpha_2 \geq 1$,

$$\begin{aligned}\Gamma_{(1,\alpha_2),2} \colon (1 + x\mathbb{F}_2[x]/(x^2))^{\times} &\rightarrowtail K_2(A, M)\\ 1 + x &\mapsto \langle t_1 t_2^{\alpha_2 - 1}, t_2\rangle,\end{aligned}$$

对任意 $\alpha_2 \geq 1$ 为奇数,

$$\begin{aligned}\Gamma_{(2,\alpha_2),1} \colon (1 + x\mathbb{F}_2[x]/(x^2))^{\times} &\rightarrowtail K_2(A, M)\\ 1 + x &\mapsto \langle t_1 t_2^{\alpha_2}, t_1\rangle,\end{aligned}$$

我们作简单的替换令 $t = t_1, x = t_2$, 于是 $\langle t_1 t_2^{\alpha_2-1}, t_2\rangle = \langle tx^{\alpha_2-1}, x\rangle$, $\langle t_1 t_2^{\alpha_2}, t_1\rangle = \langle tx^{\alpha_2}, t\rangle$. 由同构1.3可知 $NK_2(\mathbb{F}_2[C_2])$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1\text{为奇数}\}$ 生成的, 由于 $t^2 = 0$ 故 $\langle tx^i, x\rangle + \langle tx^i, x\rangle = \langle tx^i + tx^i - t^2x^{2i+1}, x\rangle = 0$, $\langle tx^i, t\rangle + \langle tx^i, t\rangle = \langle tx^i + tx^i - t^3x^{2i}, t\rangle = 0$. □

Remark 1.13. 对于 $i \geq 1$为偶数, $\langle tx^i, t\rangle = \langle x^{i/2}, t\rangle + \langle x^{i/2}, t\rangle = \langle x^{i/2} + x^{i/2} + tx^i, t\rangle = 0$.

Weibel 在 [12] 中给出了以下可裂正合列

$$0 \longrightarrow V/\Phi(V) \xrightarrow{F} NK_2(\mathbb{F}_2[C_2]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0,$$

其中 $V = x\mathbb{F}_2[x]$, $\Phi(V) = x^2\mathbb{F}_2[x^2]$ 是 V 的子群, $\Omega_{\mathbb{F}_2[x]} \cong \mathbb{F}_2[x]\,\mathrm{d}x$ 是绝对 Kähler 微分模, $F(x^n) = \langle tx^n, t\rangle$, $D(\langle ft, g + g't\rangle) = f\,dg$. 显然 $D(\langle tx^i, t\rangle) = 0$, $D(\langle tx^i, x\rangle) = x^i\,\mathrm{d}x$, 可以看出 $NK_2(\mathbb{F}_2[C_2])$ 的直和项

$$\bigoplus_{((2,\alpha_2),1),\alpha_2\geq 1\text{为奇数}} \mathbb{Z}/2\mathbb{Z} \cong V/\Phi(V),$$

直和项

$$\bigoplus_{((1,\alpha_2),2),\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[x]\,\mathrm{d}x.$$

V 和 $\Omega_{\mathbb{F}_2[x]}$ 作为交换群是同构的, 但作为 $W(\mathbb{F}_2)$-模是不同的. $V = x\mathbb{F}_2[x]$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$V_m(x^n) = x^{mn},$$

$$F_d(x^n) = \begin{cases} \mathrm{d}x^{n/d}, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},$$

$$[a]x^n = a^n x^n.$$

$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,\mathrm{d}x$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [1]) 为

$$\begin{aligned}
V_m(x^{n-1}\,\mathrm{d}x) &= mx^{mn-1}\,\mathrm{d}x,\\
F_d(x^{n-1}\,\mathrm{d}x) &= \begin{cases} x^{n/d-1}\,\mathrm{d}x, & \text{若 } d|n \\ 0, & \text{其它} \end{cases},\\
[a]x^{n-1}\,\mathrm{d}x &= a^n x^{n-1}\,\mathrm{d}x.
\end{aligned}$$

结合两者我们可以得到 $NK_2(\mathbb{F}_2[C_2])$ 的 $W(\mathbb{F}_2)$-模结构为

$$\begin{aligned}
V_m(\langle tx^n, t\rangle) &= \begin{cases} \langle tx^{mn}, t\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1 \text{ 为奇数}\\
V_m(\langle tx^{n-1}, x\rangle) &= \begin{cases} \langle tx^{mn-1}, x\rangle, & \text{若 } m \text{ 是奇数} \\ 0, & \text{若 } m \text{ 是偶数} \end{cases}, \quad n \geq 1\\
F_d(\langle tx^n, t\rangle) &= \begin{cases} \langle tx^{n/d}, t\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1 \text{ 为奇数}\\
F_d(\langle tx^{n-1}, x\rangle) &= \begin{cases} \langle tx^{n/d-1}, x\rangle, & \text{若 } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1\\
[1]\langle tx^n, t\rangle &= \langle tx^n, t\rangle, \quad n \geq 1 \text{ 为奇数}\\
[1]\langle tx^{n-1}, x\rangle &= \langle tx^{n-1}, x\rangle, \quad n \geq 1.
\end{aligned}$$

1.4 $NK_2(\mathbb{F}_2[C_4])$ 的结构

用同样的方法计算 $NK_2(\mathbb{F}_2[C_{2^2}])$, 继而对于任意 n 可以得到类似的结果.

Theorem 1.14. $NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$.

Proof. $\mathbb{F}_2[t_1, t_2]/(t_1^4) = \mathbb{F}_2[C_4][t_2]$, 此时 $I = (t_1^4)$, $M = (t_1)$ 不变, 我们直接写出以下集合

$$\begin{aligned}
\Delta &= \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 4, \alpha_2 \geq 0\},\\
\Lambda &= \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\},
\end{aligned}$$

用 $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$ 表示不小于 x 的最小整数,

$$\begin{aligned}
[\alpha, 1] &= \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 5\} = \lceil 5/\alpha_1 \rceil,\\
[\alpha, 2] &= \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 4\} = \lceil 4/\alpha_1 \rceil.
\end{aligned}$$

例如

$$
\begin{aligned}
[(1,\alpha_2),1] &= 5,\ \alpha_2 \geq 0,\\
[(2,\alpha_2),1] &= 3,\ \alpha_2 \geq 0,\\
[(3,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(4,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(\alpha_1,\alpha_2),1] &= 1,\ \alpha_1 \geq 5, \alpha_2 \geq 0,\\
[(1,\alpha_2),2] &= 4,\ \alpha_2 \geq 1,\\
[(2,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(3,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(\alpha_1,\alpha_2),2] &= 1,\ \alpha_1 \geq 4, \alpha_2 \geq 1.
\end{aligned}
$$

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$5,\alpha_2 \geq 0$	$4,\alpha_2 \geq 1$	5
$(2,\alpha_2)$	$3,\alpha_2 \geq 0$	$2,\alpha_2 \geq 1$	2, 当 α_2 是奇数时
$(3,\alpha_2)$	$2,\alpha_2 \geq 0$	$2,\alpha_2 \geq 1$	2
$(4,\alpha_2)$	$2,\alpha_2 \geq 0$	$1,\alpha_2 \geq 1$	1当 α_2 是奇数时
$(\alpha_1,0),\alpha_1 \geq 5$	1	无定义	1, 当 α_1 是奇数时
$(\alpha_1,\alpha_2),\alpha_1 \geq 5,\alpha_2 \geq 1$	1	1	1, 当 $(\alpha_1,\alpha_2)=1$ 时

记 $\Lambda_d^{00} = \{(\alpha,i) \in \Lambda^{00} \mid [(\alpha,i)] = d\}$, $\Lambda_{>1}^{00} = \{(\alpha,i) \in \Lambda^{00} \mid [(\alpha,i)] > 1\}$

由于 $(\alpha,i) \in \Lambda_1^{00}$ 均有 $[(\alpha,i)] = 1$, 实际上要计算 $(1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ 只需确定 $\Lambda_{>1}^{00}$. 由同样的方法可得 $\Lambda_4^{00} = \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$, $\Lambda_3^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}$, $\Lambda_2^{00} = \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}$,

$$
\begin{aligned}
\Lambda_{>1}^{00} = &\{((1,\alpha_2),2) \mid \alpha_2 \geq 1\} \cup \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\}\\
&\cup \{((2,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1\text{为奇数}\}.
\end{aligned}
$$

由定理1.3,

$$
\begin{aligned}
NK_2(\mathbb{F}_2[C_4]) \cong K_2(A,M) \cong &\bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\
= &\bigoplus_{(\alpha,i)\in\Lambda_{>1}^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\
= &\bigoplus_{\substack{((3,\alpha_2),2)\\ gcd(3,\alpha_2)=1\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((4,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^2))^\times\\
&\oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1\text{为奇数}}} (1+x\mathbb{F}_2[x]/(x^3))^\times \oplus \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^4))^\times.
\end{aligned}
$$

由1.5有 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $(1+x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, 于是 $NK_2(\mathbb{F}_2[C_4])$ 作为交换群有

$$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}.$$

对于任意 $(\alpha,i)\in\Lambda^{00}_{>1}$, $\Gamma_{\alpha,i}$ 均诱导了单射, 对任意 $\alpha_2\geq 1, gcd(3,\alpha_2)=1$

$$\begin{aligned}\Gamma_{(3,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1^3t_2^{\alpha_2-1}, t_2\rangle,\end{aligned}$$

对任意 $\alpha_2\geq 1$,

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^4))^\times &\rightarrowtail K_2(A,M)\\ 1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle,\\ 1+x^3(\text{二阶元}) &\mapsto \langle t_1^3t_2^{3\alpha_2-1}, t_2\rangle,\end{aligned}$$

对任意 $\alpha_2\geq 1$ 为奇数,

$$\begin{aligned}\Gamma_{(4,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1^3t_2^{\alpha_2}, t_1\rangle,\\ \Gamma_{(2,\alpha_1),1}\colon (1+x\mathbb{F}_2[x]/(x^3))^\times &\rightarrowtail K_2(A,M)\\ 1+x(\text{四阶元}) &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle.\end{aligned}$$

我们作简单的替换令 $t=t_1, x=t_2$, 由同构1.3可知 $NK_2(\mathbb{F}_2[C_4])$ 是由 Dennis-Stein 符号 $\{\langle tx^{i-1},x\rangle \mid i\geq 1\}$, $\{\langle tx^i,t\rangle \mid i\geq 1\text{为奇数}\}$, $\{\langle t^3x^{3i-1},x\rangle \mid i\geq 1\}$, $\{\langle t^3x^{i-1},x\rangle \mid i\geq 1, gcd(i,3)=1\}$, $\{\langle t^3x^i,t\rangle \mid i\geq 1\text{为奇数}\}$ 生成的.

□

Remark 1.15. $\langle t^3x^{2i},t\rangle = \langle tx^i,t\rangle + \langle tx^i,t\rangle$ 是二阶元. 根据 [7], 存在同态

$$\begin{aligned}\rho_1\colon \mathbb{F}_2[x]\,\mathrm{d}x &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^i\,\mathrm{d}x &\mapsto \langle t^3x^i, x\rangle\\ \rho_2\colon x\mathbb{F}_2[x]/x^4\mathbb{F}_2[x^4] &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ x^i &\mapsto \langle t^3x^i, t\rangle\end{aligned}$$

$\{\langle t^3x^{i-1},x\rangle \mid i\geq 1\} = \{\langle t^3x^{3i-1},x\rangle \mid i\geq 1\}\cup\{\langle t^3x^{i-1},x\rangle \mid i\geq 1, gcd(i,3)=1\}$.

1.5 $NK_2(\mathbb{F}_q[C_{2^n}])$

设 $\mathbb{F}_q$ 是特征为 2 的有限域, $q=2^f$, C_{2^n} 是 2^n 阶循环群, 这一节计算 $NK_2(\mathbb{F}_q[C_{2^n}])$. 假设 $A=\mathbb{F}_q[t_1,t_2]/(t_1^{2^n})=\mathbb{F}_q[C_{2^n}][x]$, 此时 $I=(t_1^{2^n})$, $M=(t_1)$, $A/M=\mathbb{F}_q[x]$.

Lemma 1.16. $\Delta=\{(\alpha_1,\alpha_2)\mid \alpha_1\geq 2^n, \alpha_2\geq 0\}$, $\Lambda=\{((\alpha_1,\alpha_2),1)\mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2)\mid \alpha_1\geq 1,\alpha_2\geq 1\}$, 对任意 $(\alpha,i)\in\Lambda$, $[\alpha,1]=\lceil(2^n+1)/\alpha_1\rceil$, $[\alpha,2]=\lceil 2^n/\alpha_1\rceil$, 其中 $\lceil x\rceil=\min\{m\in\mathbb{Z}\mid m\geq x\}$ 表示不小于 x 的最小整数.

Lemma 1.17. 令 $I_1 = \{((\alpha_1,\alpha_2),1) \mid gcd(\alpha_1,\alpha_2) = 1, 1 < \alpha_1 \le 2^n$为偶数$, \alpha_2 \ge 1$为奇数$\}$, $I_2 = \{((\alpha_1,\alpha_2),2) \mid gcd(\alpha_1,\alpha_2) = 1, 1 \le \alpha_1 < 2^n$为奇数$, \alpha_2 \ge 1\}$, 则 $\Lambda^{00}_{>1} = I_1 \sqcup I_2$.

由定理1.3,

$$\begin{aligned} NK_2(\mathbb{F}_q[C_{2^n}]) \cong K_2(A,M) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ &= \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1+x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ &= \bigoplus_{(\alpha,1)\in I_1} (1+x\mathbb{F}_q[x]/(x^{\lceil (2^n+1)/\alpha_1\rceil}))^\times \\ &\oplus \bigoplus_{(\alpha,2)\in I_2} (1+x\mathbb{F}_q[x]/(x^{\lceil 2^n/\alpha_1\rceil}))^\times. \end{aligned}$$

注意到 $BigWitt_k(R) = (1+xR[\![x]\!])^\times/(1+x^{k+1}R[\![x]\!])^\times \cong (1+xR[x]/(x^{k+1}))^\times$, 根据公式1.7,

$$\begin{aligned} NK_2(\mathbb{F}_q[C_{2^n}]) \cong & \bigoplus_{(\alpha,1)\in I_1} \bigoplus_{\substack{1\le m\le\lceil (2^n+1)/\alpha_1\rceil-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil (2^n+1)/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f \\ \oplus & \bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le\lceil 2^n/\alpha_1\rceil-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{\lceil 2^n/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f. \end{aligned}$$

接下来我们证明对于任意 $1 \le k \le n$, $\mathbb{Z}/2^k\mathbb{Z}$ 都在 $NK_2(\mathbb{F}_q[C_{p^n}])$ 出现无限多次

Lemma 1.18. 对于任意的 $1 \le k < n$, $1 + \left\lfloor \log_2(\frac{2^n-1}{2^k+1})\right\rfloor = n-k$.

Proof. 当 $1 \le k < n$ 时, $2^k - 1 \ge 1 \ge \frac{1}{2^{n-k-1}}$, 即

$$2^{n-1} - 2^{n-k-1} \ge 1,$$

上式等价于 $2^n - 1 \ge 2^{n-k-1}(2^k+1)$, 且 $2^n - 1 < 2^{n-k}(2^k+1)$, 于是

$$2^{n-k} > \frac{2^n-1}{2^k+1} \ge 2^{n-k-1},$$

取对数得 $\left\lfloor \log_2(\frac{2^n-1}{2^k+1})\right\rfloor = n-k-1$. □

考虑 $((1,\alpha_2),2) \in I_2$,

$$\bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\le m\le 2^n-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\lfloor \log_2 \frac{2^n-1}{m}\rfloor}\mathbb{Z})^f$$

是 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 的直和项, 当 $m = 1$ 时 $1 + \lfloor \log_2(2^n-1)\rfloor = n$, 当 $m = 2^k+1 (1 \le k < n)$ 为奇数时, 由1.18, $1 + \left\lfloor \log_2 \frac{2^n-1}{m}\right\rfloor = n-k$, 于是对于任何的 $1 \le k \le n$, $\mathbb{Z}/2^k\mathbb{Z}$ 均出现在直和项中, 且对于任意 $\alpha_2 \ge 1$, 这样的项总会出现, 于是

$$NK_2(\mathbb{F}_q[C_{2^n}]) \cong \bigoplus_\infty \bigoplus_{k=1}^n \mathbb{Z}/2^k\mathbb{Z}.$$

接下来给出一些 $NK_2(\mathbb{F}_q[C_{2^n}])$ 中的 $2^k(1 \le k \le n)$ 阶元素.

对任意 $\alpha_2 \ge 1, a \in \mathbb{F}_q$,

$$\begin{aligned}\Gamma_{(1,\alpha_2),2}\colon (1 + x\mathbb{F}_q[x]/(x^{2^n}))^{\times} &\rightarrowtail K_2(A, M)\\ 1 + ax(2^n \text{ 阶元}) &\mapsto \langle atx^{\alpha_2 - 1}, x\rangle,\\ 1 + ax^3(2^{n-1} \text{ 阶元}) &\mapsto \langle at^3x^{3\alpha_2 - 1}, x\rangle,\\ 1 + ax^{2^k+1}(2^{n-k} \text{ 阶元}) &\mapsto \langle at^{2k+1}x^{(2k+1)\alpha_2 - 1}, x\rangle.\end{aligned}$$

1.6 其他问题和说明

$NK_2(\mathbb{F}_{p^m}[C_{p^n}]) = ?$

$\mathbb{F}_2[C_2 \times C_2] \cong \mathbb{F}_2[C_2] \otimes \mathbb{F}_2[C_2] \cong \mathbb{F}_2[x, y]/(x^2, y^2)$, 可以用同样的方法得到一些结果.

$$0 \longrightarrow K_2(k[t_1, t_2, t_3]/(t_1^n, t_2^n), (t_1, t_2)) \longrightarrow K_2(k[t_1, t_2, t_3]/(t_1^n, t_2^n)) \longrightarrow K_2(k[t_3]) \longrightarrow 0$$

对于有限域 k 来讲 $K_2(k[t_3]) = 0$,

$$0 \longrightarrow NK_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2][x]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow 0,$$

中间那项可以用这篇文章里的方法确定, 又 $K_2(\mathbb{F}_2[C_2 \times C_2]) = C_2^3$, 于是可以得到 $NK_2(\mathbb{F}_2[C_2 \times C_2])$, 是 $\oplus_\infty \mathbb{Z}/2\mathbb{Z}$.

另外可以直接用这种方式重新计算 $K_2(\mathbb{F}_2[C_4 \times C_4])$, 见下一篇笔记.

一个关于模结构的问题, 在 Weibel 的文章 [11] 中 5. 5 和 5. 7 给出的模结构和本文上面的模结构并不一致, 用 V_m 作用差一个 t^m.

Chapter 2

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

2.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4),(t_1,t_2))$.

2.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [13] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4),(t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

2.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1,t_2]$. Put $A = k[t_1,t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1,t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A,M)$ in terms of Dennis-Stein symbols:

generators: $\langle a,b\rangle$, $(a,b) \in A \times M \cup M \times A$;

relations: $\langle a,b\rangle = -\langle b,a\rangle$,

$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc, b\rangle$,

$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ for $(a,b,c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [10]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1,0) \in \mathbb{N}^2$, $\epsilon^2 = (0,1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,
- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha,i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha,i) \in \Lambda$, set $[\alpha,i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p,\alpha_1,\alpha_2) = 1$, let $[\alpha] = \max\{[\alpha,i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \{(\alpha,i) \in \Lambda \mid gcd(\alpha_1,\alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha,j] = [\alpha]\}\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i \rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 2.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [10]。 □

Lemma 2.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3 \rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

2.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 2.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. □

Theorem 2.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$, then $I = (t_1^4, t_2^4)$, $M = (t_1, t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] > 1\}$, and then

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha, 1) \in \Lambda^{00}_{>1}$ and $[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ and $[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1, \alpha_2 \leq 4$ are both odd and $gcd(\alpha_1, \alpha_2) = 1$, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha, 1]$.

By the computation 2.2, we can get the following table

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below, the generators of order 4:

$$\langle t_1 t_2, t_1 \rangle, \langle t_1 t_2, t_2 \rangle, \langle t_1, t_2 \rangle,$$

the generators of order 2:

$$\langle t_1 t_2^3, t_1 \rangle, \langle t_1^3 t_2, t_1 \rangle, \langle t_1^3 t_2^3, t_1 \rangle, \langle t_1 t_2^3, t_2 \rangle, \langle t_1^3 t_2^2, t_2 \rangle, \langle t_1 t_2^2, t_2 \rangle, \langle t_1^3 t_2, t_2 \rangle, \langle t_1^3 t_2^3, t_2 \rangle, \langle t_1^3, t_2 \rangle.$$

□

Remark 2.5. Compared with [13], note that $\langle t_1^3, t_2\rangle = \langle t_1^2 t_2, t_1\rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1 t_2\rangle - \langle t_1^2 t_2, t_1\rangle \\
&= \langle t_1, t_1^2 t_2\rangle - \langle t_1^2 t_2, t_1\rangle - \langle t_1^2 t_2, t_1\rangle \\
&= -3\langle t_1^2 t_2, t_1\rangle \\
&= -\langle t_1^2 t_2, t_1\rangle \\
&= \langle t_1^2 t_2, t_1\rangle,
\end{aligned}$$

since $\langle t_1^2 t_2, t_1\rangle + \langle t_1^2 t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

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