Notes on Algebraic K-theory

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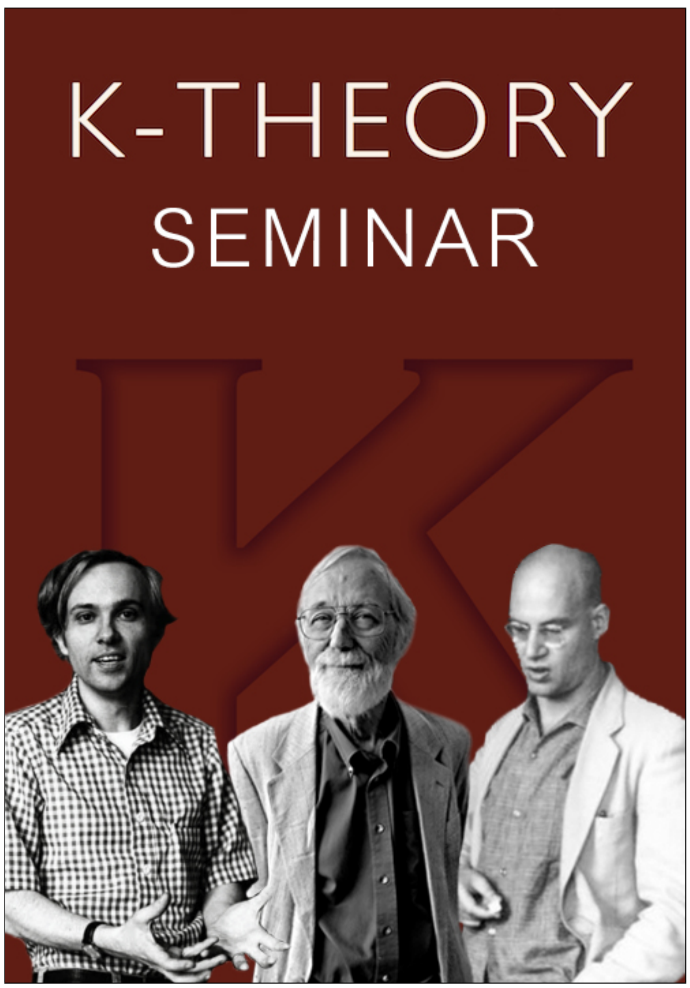

Quillen Milnor Grothendieck

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Chapter 1

Witt vectors and NK-groups

References:

part 1 J. P. Serre, Local fields.

part 1 Daniel Finkel, An overview of Witt vectors.

part 2 Hendrik Lenstra, Construction of the ring of Witt vectors.

part 2 Barry Dayton, Witt vectors, the Grothendieck Burnside ring, and Necklaces.

part 3 C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.

part 3 D. R. Grayson, Grothendieck rings and Witt vectors.

part 3 C. A. Weibel, The K-Book: An introduction to algebraic K-theory.

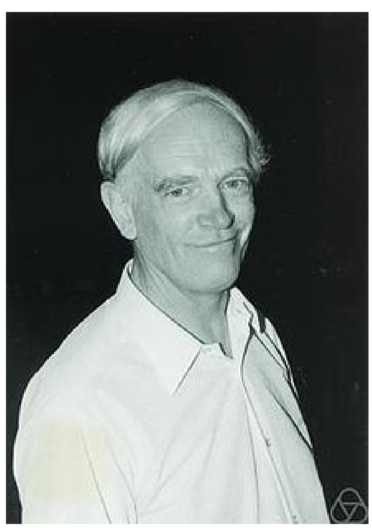

图 1.1: Ernst Witt

Ernst Witt Ernst Witt (June 26, 1911–July 3, 1991) was a German mathematician, one of the leading algebraists of his time. Witt completed his Ph.D. at the University of Göttingen in 1934 with Emmy Noether. Witt's work has been highly influential. His invention of the Witt vectors clarifies and generalizes the structure of the p-adic numbers. It has become fundamental to p-adic Hodge theory. For more information, see `https://en.wikipedia.org/wiki/Ernst_Witt` and `http://www-history.mcs.st-andrews.ac.uk/Biographies/Witt.html`.

1.1 p-Witt vectors

In this section we introduce p-Witt vectors. Witt vectors generalize the p-adics and we will see all p-Witt vectors over any commutative ring form a ring.

From now on, fix a prime number p.

Definition 1.1. A p-Witt vector over a commutative ring R is a sequence $(X_0, X_1, X_2, \cdots)$ of elements of R.

Remark 1.2. If $R = \mathbb{F}_p$, any p-Witt vector over $\mathbb{F}_p$ is just a p-adic integer $a_0 + a_1 p + a_2 p^2 + \cdots$ with $a_i \in \mathbb{F}_p$.

We introduce Witt polynomials in order to define ring structure on p-Witt vectors.

Definition 1.3. Fix a prime number p, let $(X_0, X_1, X_2, \cdots)$ be an infinite sequence of indeterminates. For every $n \geq 0$, define the n-th Witt polynomial

$$W_n(X_0, X_1, \cdots) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

For example, $W_0 = X_0,\ W_1 = X_0^p + pX_1,\ W_2 = X_0^{p^2} + pX_1^p + p^2 X_2.$

Question: how can we add and multiple Witt vectors?

Theorem 1.4. *Let $(X_0, X_1, X_2, \cdots)$, $(Y_0, Y_1, Y_2, \cdots)$ be two sequences of indeterminates. For every polynomial function $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \cdots, \varphi_n, \cdots)$ of elements of $\mathbb{Z}[X_0, \cdots, X_n, \cdots; Y_0, \cdots, Y_n, \cdots]$ such that*

$$W_n(\varphi_0, \cdots, \varphi_n, \cdots) = \Phi\big(W_n(X_0, \cdots), W_n(Y_0, \cdots)\big),\ n = 0, 1, \cdots.$$

If $\Phi = X + Y$(resp. XY), then there exist $(S_1, \cdots, S_n, \cdots)$("$S$" stands for sum) and $(P_1, \cdots, P_n, \cdots)$ ("P" stands for product) such that

$$W_n(X_0, \cdots, X_n, \cdots) + W_n(Y_0, \cdots, Y_n, \cdots) = W_n(S_1, \cdots, S_n, \cdots),$$

$$W_n(X_0,\cdots,X_n,\cdots)W_n(Y_0,\cdots,Y_n,\cdots)=W_n(P_1,\cdots,P_n,\cdots).$$

Let R be a commutative ring, if $A=(a_0,a_1,\cdots)\in R^{\mathbb{N}}$ and $B=(b_0,b_1,\cdots)\in R^{\mathbb{N}}$ are p-Witt vectors over R, we define

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots).$$

Theorem 1.5. *The p-Witt vectors over any commutative ring R form a commutative ring under the compositions defined above (called the ring of p-Witt vectors with coefficients in R, denoted by $W(R)$).*

Example 1.6. We have

$$\begin{array}{rr} S_0(A,B)=a_0+b_0 & P_0(A,B)=a_0b_0 \\ S_1(A,B)=a_1+b_1+\frac{a_0^p+b_0^p-(a_0+b_0)^p}{p} & P_1(A,B)=b_0^pa_1+a_0^pb_1+pa_1b_1 \end{array}$$

For more computations, see MO 92750

Theorem 1.7. *There is a ring homomorphism*

$$\begin{aligned} W_*\colon W(R)&\longrightarrow R^{\mathbb{N}} \\ (X_0,X_1,\cdots,X_n,\cdots)&\mapsto(W_0,W_1,\cdots,W_n,\cdots) \end{aligned}$$

Proof. Only need to check this map is a ring homomorphism: since

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots),$$

by definition we have

$$\begin{aligned} W(A)+W(B)&=(W_0(A)+W_0(B),W_1(A)+W_1(B),\cdots) \\ &=\big(W_0(S_0(A,B),S_1(A,B),\cdots),W_1(S_0(A,B),S_1(A,B),\cdots),\cdots\big) \\ &=W(S_0(A,B),S_1(A,B),\cdots)=W(A+B). \end{aligned}$$

And similarly,

$$\begin{aligned} W(A)W(B)&=(W_0(A)W_0(B),W_1(A)W_1(B),\cdots) \\ &=\big(W_0(P_0(A,B),P_1(A,B),\cdots),W_1(P_0(A,B),P_1(A,B),\cdots),\cdots\big) \\ &=W(P_0(A,B),P_1(A,B),\cdots)=W(AB). \end{aligned}$$

Indeed, we only need to show $W_n(A)+W_n(B)=W_n(A+B)$ and $W_n(A)W_n(B)=W_n(AB)$ which are obviously true. (实际上就是为了使得这个是同态而定义出了 $A+B$ 和 AB。) $\square$

Example 1.8. 1. If p is invertible in R, then $W(R)=R^{\mathbb{N}}$ — the product of countable number of R.(if p is invertible the homomorphism W_* is an isomorphism.)

2. $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$ — the ring of p-adic integers.
3. $W(\mathbb{F}_{p^n})$ is an unramified extension of the ring of p-adic integers.

Note that the functions P_k and S_k are actually only involve the variables of index $\leqslant k$ of A and B. In particular if we truncate all the vectors at the k-th entry, we can still add and multiply them.

Definition 1.9. Truncated p-Witt ring $W_k(R) = \{(a_0, a_1, \cdots, a_{k-1}) | a_i \in R\}$ (also called the ring of Witt vectors of length k.)

Example 1.10. $W_1(R) = R$, $W(R) = \varprojlim W_k(R)$. Since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$.

Definition 1.11. We define two special maps as follows

- The "shift" map $V\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$, this map is *additive*.
- When $char(R) = p$, the "Frobenius" map $F\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots)$, this is indeed a ring homomorphism.

Firstly, we note that $W_k(R) = W(R)/V^kW(R)$, and if we consider $V\colon W_n(R) \hookrightarrow W_{n+1}(R)$ there are exact sequences

$$0 \longrightarrow W_k(R) \xrightarrow{V^r} W_{k+r}(R) \longrightarrow W_r(R) \longrightarrow 0, \quad \forall k,\ r.$$

The map $V\colon W(R) \longrightarrow W(R)$ is additive: for it suffices to verify this when p is invertible in R, and in that case the homomorphism $W_*\colon W(R) \longrightarrow R^{\mathbb{N}}$ transforms V into the map which sends $(w_0, w_1, \cdots)$ to $(0, pw_0, pw_1, \cdots)$.

$$\begin{tikzcd}
W(R) \arrow[r, "V"] \arrow[d, "W_*"] & W(R) \arrow[d, "W_*"] \\
R^{\mathbb{N}} \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
(a_0, a_1, \cdots) \arrow[r, mapsto, "V"] \arrow[d, mapsto, "W_*"] & (0, a_0, a_1, \cdots) \arrow[d, mapsto, "W_*"] \\
(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \cdots) \arrow[r, mapsto] \arrow[d, equal] & (0, pa_0, pa_0^p + p^2a_1, \cdots) \arrow[d, equal] \\
(w_0, w_1, w_2, \cdots) \arrow[r, mapsto] & (0, pw_0, pw_1, \cdots)
\end{tikzcd}$$

If $x \in R$, define a map

$$\begin{aligned} r\colon R &\longrightarrow W(R) \\ x &\mapsto (x, 0, \cdots, 0, \cdots) \end{aligned}$$

When p is invertible in R, W_* transforms r into the mapping that $x \mapsto (x, x^p, \cdots, x^{p^n}, \cdots)$.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\mathrm{id}"] & W(R) \arrow[d, "W_*"] \\ R \arrow[r] & R^{\mathbb{N}} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, equal] & (x, 0, \cdots, 0, \cdots) \arrow[d, mapsto, "W_*"] \\ x \arrow[r, mapsto] & (x, x^p, \cdots, x^{p^n}, \cdots) \end{tikzcd}$$

One deduces by the same reasoning as above the formulas:

Proposition 1.12.

$$r(xy) = r(x)r(y),\ x, y \in R$$
$$(a_0, a_1, \cdots) = \sum_{n=0}^{\infty} V^n(r(a_n)),\ a_i \in R$$
$$r(x)(a_0, \cdots) = (xa_0, x^p a_1, \cdots, x^{p^n} a_n, \cdots),\ x_i, a_i \in R.$$

Proof. The first formula: put $r(x)r(y),\ r(xy)$ to $R^{\mathbb{N}}$, we get $(x, x^p, \cdots, x^{p^n}, \cdots)(y, y^p, \cdots, y^{p^n}, \cdots)$ and $(xy, (xy)^p, \cdots, (xy)^{p^n}, \cdots)$.
The second formula: put $(a_0, a_1, \cdots)$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \cdots)$
consider $V^i(r(a_i))$: put $r(a_i)$ to $R^{\mathbb{N}}$, we get $(a_i, a_i^p, \cdots, a_i^{p^n}, \cdots) \in R^{\mathbb{N}}$, and W_* transforms V to the mapping $(w_0, w_1, \cdots, w_n, \cdots) \mapsto (0, pw_0, \cdots, pw_{n-1}, \cdots)$,
now we put $(r(a_0))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p, \cdots, a_0^{p^n}, \cdots)$
put $V^1(r(a_1))$ to $R^{\mathbb{N}}$, we get $(0, pa_1, \cdots, pa_1^{p^{n-1}}, \cdots)$
put $V^2(r(a_2))$ to $R^{\mathbb{N}}$, we get $(0, 0, p^2a_2, \cdots, p^2a_2^{p^{n-2}}, \cdots)$
put $V^i(r(a_i))$ to $R^{\mathbb{N}}$, we get $(\underbrace{0, 0, \cdots, 0}_{i \text{ terms}}, p^i a_i, \cdots, p^i a_i^{p^{n-i}}, \cdots)$
so put $\sum_n V^n(r(a_n))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, \cdots)$.
We leave the proof of the last formula to readers. □

Proposition 1.13.

$$VF = p = FV.$$

Proof. It suffices to check this when R is perfect. Note that a ring R of characteristic p is called perfect if $x \mapsto x^p$ is an automorphism. For more details, we refer to page 44 on Serre's book *Local Fields*. □

1.2 Big Witt vectors

Now we turn to the `big`(universal) Witt vectors. J. P, May once said "This(the theory of Witt vectors) is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once".

Take the ring of all big vectors of a commutative ring is a functor

$$\mathbf{CRing} \longrightarrow \mathbf{CRing}$$
$$R \mapsto W(R).$$

In this section, R is a commutative ring with unit.

Definition 1.14. The ring of all big Witt vectors in R which also denoted by $W(R)$ is defined as follows,
as a set: $W(R) = \{a(T) \in R[\![T]\!] | a(T) = 1 + a_1 T + a_2 T^2 + \cdots\} = 1 + TR[\![T]\!]$; (we note that as a set $W(R)$ is the kernel of the map $A[\![T]\!]^* \xrightarrow{T \mapsto 0} A^*$)
addition in $W(R)$: usual multiplication of formal power series, sum $a(T)b(T)$, difference $\frac{a(T)}{b(T)}$; $(W(R), +) \cong (1 + TR[\![T]\!], \times)$ which is a subgroup of the group of units $R[\![T]\!]^\times$ of the ring $R[\![T]\!]$
multiplication in $W(R)$: denoted by $*$, this is a little mysterious, we will talk the details later. For the present purposes we only define $*$ as the unique continuous functorial operation for which $(1 - aT) * (1 - bT) = (1 - abT)$.
'zero'(additive identity) of $W(R)$: 1.
'one'(multiplicative identity) of $W(R)$: $[1] = 1 - T$. Note that $[1]$ is the image of $1 \in R$ under the multiplicative (Teichmuller) map

$$R \longrightarrow W(R)$$
$$a \mapsto [a] = 1 - aT$$

functoriality: any homomorphism $f \colon R \longrightarrow S$ induces a ring homomorphism

$$W(f) \colon W(R) \longrightarrow W(S).$$

A quick way to check multiplicative formulas in $W(R)$ is to use the `ghost` map (indeed a ring homomorphism)

$$gh \colon W(R) \longrightarrow R^{\mathbb{N}} = \prod_i^\infty R.$$

It is obtained from the abelian group homomorphism

$$-T\frac{d}{dT}\log \colon (1 + TR[\![T]\!])^\times \longrightarrow (TR[\![T]\!])^+$$
$$a(T) \mapsto -T\frac{a'(T)}{a(T)}$$

the right side of gh is $R^{\mathbb{N}}$ via $\sum a_n t^n \longleftrightarrow (a_1, a_2, \cdots)$.

A basic principle of the theory of Witt vectors is that to demonstrate certain equations it suffices to check them on vectors of the form $1 - aT$.

1.3 Module structure on NK_*

Notations Λ: a ring with 1
R: commutative ring
$W(R)$: the ring of big Witt vectors of R
$\mathbf{End}(\Lambda)$: the exact category of endomorphisms of finitely generated projective right Λ-modules.
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules.

The fundamental theorem in algebraic K-theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus NK_i(\Lambda) \cong K_i(\Lambda) \oplus \mathrm{Nil}_{i-1}(\Lambda),$$

and hence $\mathrm{Nil}(\Lambda)$ is the obstruction to K-theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (right) Λ-module has a finite projective resolution. So the resolution theorem and the fact that G-theory is homotopy invariant show that for a regular ring, $NK_*(\Lambda) = \mathrm{Nil}_{*-1}(\Lambda) = 0$. In general, one knows that the groups $\mathrm{Nil}_*(\Lambda)$, if non-zero, are infinitely generated. It is also known that the groups $\mathrm{Nil}_*(\Lambda)$ are modules over the big Witt ring $W(R)$ (just this notes want to show you).

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.
- Computations in $W(R)$ with Grothendieck rings.

1.3.1 $\mathrm{End}_0(\Lambda)$

Let $\mathbf{End}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules.
Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.
Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ M_2 & \xrightarrow{f_2} & M_2 \end{array}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathcal{C} \rightleftarrows \mathcal{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \leftarrow\!\shortmid (M, f)$$

Definition 1.15. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) \longrightarrow \mathbf{End}(\Lambda)$$
$$\otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) \longrightarrow \mathbf{Nil}(\Lambda)$$
$$(M, f) \otimes (N, g) = (M \otimes_R N, f \otimes g)$$

These induce (use "generators-and-relations" tricks on K_0)

$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) \longrightarrow K_*(\mathbf{End}(\Lambda))$$
$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) \longrightarrow K_*(\mathbf{Nil}(\Lambda))$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M, f) \in \mathbf{End}(R)$, then $(M, f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R, 1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R, 0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R, 0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R, 0) \otimes -\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. (Consider $P \in \mathbf{P}(R)$ acts on $\mathbf{End}(\Lambda)$, $(P,0) \otimes (N,g) = (P \otimes_R N, 0) \in \mathbf{P}(\Lambda)$.)

The following is immediate (and well-known):

Proposition 1.16. *If* Λ *is an* R*-algebra with* 1, $\mathrm{End}_*(\Lambda)$ *and* $\mathrm{Nil}_*(\Lambda)$ *are graded modules over the ring* $\mathrm{End}_0(R)$.

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 1.17 (Almkvist). *The homomorphism (in fact it is a ring homomorphism)*

$$\chi\colon \mathrm{End}_0(R) \longrightarrow W(R) = (1 + TR[\![T]\!])^\times$$
$$(M,f) \mapsto \det(1 - fT)$$

is injective and $\mathrm{End}_0(R) \cong \mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1 - T$$

χ is a ring homomorphism, and $\mathrm{Im}\chi =$ the set of all rational functions in $W(R)$. Note that

$$\det(1 - fT)\det(1 - gT) = \det(1 - (f \oplus g)T), \quad \det(1 - fT) * \det(1 - gT) = \det(1 - (f \otimes g)T),$$

for more details we refer the reader to S.Lang *Algebra*, Chapter 14, Exercise 15.

Remark 1.18. when R is a algebraically closed field (for instance $\mathbb{C}$), we can use Jordan canonical forms to prove the last identity (i.e. it can be reduced to check that $\prod_i (1 - \lambda_i T) * \prod_j (1 - \mu_j T) = \prod_{i,j} (1 - \lambda_i \mu_j T)$).

Definition 1.19 (NK_*)**.** As above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in “Higher algebraic K-theory II”. The map is given by

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/xy = 1) \xrightarrow{\partial} K_{n-1}(\mathbf{Nil}(\Lambda)).$$

Thus $NK_n(\Lambda)$ are $\mathrm{End}_0(R)$-modules. For $n \geq 1$, this is just 1.16; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the “contracted functor” of $NK_1(\Lambda)$.

Note that $NK_1(\Lambda) \cong K_1(\Lambda[y], (y-\lambda)), \forall \lambda \in \Lambda$, since

$$\Lambda[y] \rightleftarrows \Lambda$$
$$y \mapsto \lambda.$$

Since $[(P,\nu)] = [(P \oplus Q, \nu \oplus 0)] - [(Q,0)] \in K_0(\mathbf{Nil}(\Lambda))$, we see $\mathrm{Nil}_0(\Lambda)$ is generated by elements of the form $[(\Lambda^n, \nu)] - n[(\Lambda, 0)]$ for some n and some nilpotent matrix ν Sign convention:

$$NK_1(\Lambda) \cong \mathrm{Nil}_0(\Lambda)$$
$$[1 - \nu y] \leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)]$$

Example 1.20. Let k be a field, $\mathbf{End}(k)$ consists pairs (V, A) with V a finite-dimensional vector space over k and A a k-endomorphism. Two pairs are isomorphic if and only if their minimal polynomials are equal. When we consider $\mathbf{Nil}(k)$, then $K_0(\mathbf{Nil}(k)) \cong \mathbb{Z}$, we conclude that $\mathrm{Nil}_0(k) = 0$. Recall that since k is a regular ring, $NK_*(k) = 0$, we have another proof of $NK_1(k) \cong \mathrm{Nil}_0(k) = 0$.

1.3.2 Grothendieck rings and Witt vectors

We refer the reader to Grayson's paper *Grothendieck rings and Witt vectors*.

Definition 1.21. A λ-ring R is a commutative ring with 1, together with an operation λ_t which assigns to each element x of R a power series

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots, \quad x \in R$$

This operation must obey $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

Let R is a commutative ring with unit, $K_0(R) = K_0(\mathbf{P}(R))$ becomes a λ-ring if we define

$$[M][N] = [M \otimes_R N], \quad \lambda_t^n([M]) = [\wedge_R^n M].$$

Recall $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$, $\wedge^n(M \oplus M') \cong \bigoplus_{i+j=n} \wedge^i M \otimes \wedge^j M'$, and $\wedge^n(M) := M^{\otimes n} / \langle x \otimes x \mid x \in M \rangle$, rank $\wedge^n(M) = \binom{\text{rank } M}{n}$.

For instance, if R is a field, $K_0(R) = \mathbb{Z}$ and $\lambda_t(n) = (1+t)^n = 1 + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$, since $\dim(\wedge^i R^n) = \binom{n}{i}$.

We make $K_0(\mathbf{End}(R))$ into a λ-ring by defining

$$\lambda^n([M, f]) = ([\wedge^n M, \wedge^n f]).$$

The ideal generated by the idempotent $[(R,0)]$ is isomorphic to $K_0(R)$, the quotient $\mathrm{End}_0(R)$ is a λ-ring. It is convenient to think of End_0 as a convariant functor on the category of rings, and the functor End_0 satisfies:

1. If $R \longrightarrow S$ is surjective ring homomorphism, then $\mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(S)$ is surjective.
2. If R is an algebraically close field, then the group $\mathrm{End}_0(R)$ is generated by the elements of the form $[(R, r)]$. (This holds because any matrix over R is triagonalizable.)

Recall

$$\begin{aligned} \chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = 1 + TR[\![T]\!] \\ (M, f) &\mapsto \det(1 - fT) \end{aligned}$$

$W(R)$ is the underlying (additive) group of the ring of Witt vectors. The λ-ring operations on $W(R)$ are the unique operations which are continuous, functorial in R, and satisfy:

$$\begin{aligned} (1 - aT) * (1 - bT) &= 1 - abT \\ \lambda_t(1 - aT) &= 1 + (1 - aT)t \end{aligned}$$

By 1.17, χ is an injective ring homomorphism whose image consists of all Witt vectors which are quotients of polynomials. In fact χ is a λ-ring homomorphism, so we have

Theorem 1.22. $\mathrm{End}_0(R)$ *is dense sub-λ-ring of* $W(R)$.

The hard part of the theorem is the injectivity. When R is a field the injectivity follows immediately from the existence of the rational canonical form (we can see it below) for a matrix. The result is surprising when R is not a field.

Computation in $W(R)$ Computation in $W(R)$ which is tedious unless we perform it in $\mathrm{End}_0(R)$:

$$(1 - aT^2) * (1 - bT^2) = ?$$

Note that $\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) = \det \begin{pmatrix} 1 & -aT \\ -T & 1 \end{pmatrix} = 1 - aT^2$, $\chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \det \begin{pmatrix} 1 & -bT \\ -T & 1 \end{pmatrix} = 1 - bT^2$,

$$\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) * \chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Bigg(\begin{pmatrix} 0 & 0 & 0 & ab \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\Bigg)$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & -aTb \\ 0 & 1 & -bT & 0 \\ 0 & -aT & 1 & 0 \\ -T & 0 & 0 & 1 \end{pmatrix} = 1 - 2abT^2 + a^2b^2T^4.$$

If we use the previous formula

$$(1 - rT^m) * (1 - sT^n) = (1 - r^{n/d} s^{m/d} T^{mn/d})^d,\ d = \gcd(m, n),$$

we obtain the same answer. Indeed we have the formula

$$\det(1 - fT) * \det(1 - gT) = \det(1 - (f \otimes g)T),$$

if a polynomial is $1 + a_1 T + \cdots + a_n T^n \in W(R)$, we can write

$$f = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix} \in M_n(R).$$

Operations on $W(R)$ **and** $\mathrm{End}_0(R)$ We have already known that W and End_0 can be regarded as functors from the category of commutative rings to that of (λ-)rings. The following operations $F_n,\ V_n \colon W \Longrightarrow W$(resp. $\mathrm{End}_0 \Longrightarrow \mathrm{End}_0$) are indeed natural transformation. These auxiliary operations defined on $W(R)$ can also be computed in $\mathrm{End}_0(R)$.

1. the ghost map

$$gh \colon W(R) \xrightarrow{-T\frac{d}{dT}\log} TR[\![T]\!] \cong R^{\mathbb{N}}, \quad \alpha(T) \mapsto \frac{-T}{\alpha(T)}\frac{d\alpha}{dT}.$$

and the n-th ghost coordinate

$$gh_n \colon W(R) \longrightarrow R$$

it is the unique continuous natual additive map which sends $1 - aT$ to a^n.
Remark. $gh(1 - aT) = \frac{aT}{1-aT} = \sum_{i=1} a^i T^i$.The exponential map is defined by

$$R^{\mathbb{N}} \longrightarrow W(R), \quad (r_1, \cdots) \mapsto \prod_{i=1}^{\infty} \exp(\frac{-r_i T^i}{i}).$$

2. the Frobenius endomorphism

$$F_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}}).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - a^n T$.
Remark. $F_n(1 - aT) = \sum_{\zeta^n=1}(1 - a\zeta T^{\frac{1}{n}}) = 1 - a^n T$, since "+" in $W(R)$ is the normal product.

3. the Verschiebung endomorphism

$$V_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \alpha(T^n).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - aT^n$.

ghost map $gh_n \colon W(R) \longrightarrow R$	$1 - aT \mapsto a^n$	
Frobenius endomorphism $F_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - a^n T$	$\alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}})$
Verschiebung endomorphism $V_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - aT^n$	$\alpha(T) \mapsto \alpha(T^n)$

We define similar operations on $\mathrm{End}_0(R)$ as follows:

$gh_n \colon \mathrm{End}_0(R) \longrightarrow R$	$[(M, f)] \mapsto \mathrm{tr}(f^n)$
$F_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M, f^n)]$
$V_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M^{\oplus n}, v_n f)]$

where $v_n f$ is represented by $\begin{pmatrix} 0 & & & f \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$. The matrix $v_n f$ is close to an n-th root of f. Another equivalent description is

$$V_n \colon [(M, f)] \mapsto [(M[y]/y^n - f, y)].$$

One easily checks that the operations just defined are additive with respect to short exact sequences in $\mathbf{End}(R)$, and thus are well-defined on $\mathrm{End}_0(R)$.

Since $\mathrm{End}_0(R) \subset W(R)$ is dense and gh_n, F_n, V_n are continuous, identities among them may be verified on $W(R)$ by checking them on $\mathrm{End}_0(R)$.

$W(R)$	$\longleftrightarrow$	$\mathrm{End}_0(R)$
$gh_n(v * w) = gh_n v * gh_n w$		$\mathrm{tr}((f \otimes g)^n) = \mathrm{tr}(f^n)\mathrm{tr}(g^n)$
$F_n(v * w) = F_n v * F_n w$		$(f \otimes g)^n = f^n \otimes g^n$
$F_n V_n = n$		$(v_n f)^n = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$
$gh_n V_d(v) = \begin{cases} d\, gh_{n/d}(v), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$		$\mathrm{tr}((v_d f)^n) = \begin{cases} d\, \mathrm{tr}(f^{n/d}), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$

From the last row we may recover the usual expression of the ghost coordinates in terms of the Witt coordinates . The Witt coordinates of a vector v are the coefficients in the expression

$$v = \prod_{i=1} (1 - a_i T^i) = \prod_{i=1} V_i(1 - a_i T).$$

We obtain

$$gh_n(v) = \sum_{d|n} da_d^{n/d}.$$

"Many mordern treatments of the subject of Witt vectors take this latter expression as the starting point of the theory."

The logarithmic derivative of $1 - a_d T^d$ is $\frac{d}{dT}\log(1 - a_d T^d) = -\sum_{m=1}^{\infty} da_d^m T^{dm-1}$, and $-T\frac{d}{dT}\log(1 - a_d T^d) = \sum_{n=1}^{\infty} gh_n(1 - a_d T^d)T^n$. So we obtain the formula:

$$-Tv^{-1}\frac{dv}{dT} = \sum_{n=1}^{\infty} gh_n(v)T^n$$

which yields the exponential trace formula:

$$-T\chi([M,f])^{-1}\frac{d\chi([M,f])}{dT} = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n.$$

For example, when rank $M = 2$, we have $\mathrm{tr}(f^2) = (\mathrm{tr}(f))^2 - 2\det(f)$, note that $\det(1 - fT) = 1 - \mathrm{tr}(f)T + \det(f)T^2$.

Remark 1.23. When R is a field, the exponential trace formula

$$-T\frac{d}{dT}\log\det(1 - fT) = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n$$

can be checked by $\det(1 - fT) = \prod(1 - \lambda_i T)$ where λ_i are eigenvalues. And we also have

$$\det(1 - fT) = \exp(\sum_{n=1}^{\infty} -\mathrm{tr}(f^n)\frac{T^n}{n}),$$

since $\prod(1 - \lambda_i T) = \exp\left(\ln(\prod(1 - \lambda_i T))\right) = \exp\left(\sum \ln(1 - \lambda_i T)\right)$ and recall that formally $\ln(1 - x) = -\sum \frac{x^n}{n}$.

1.3.3 $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Recall Λ is an R-algebra, where R is a commutative ring with unit. We define a map

$$\mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n, f) * [(P, \nu)] = [(P^n, f\nu)]$$

Let $\alpha_n = \alpha_n(a_1, \cdots, a_n)$ denote the $n \times n$ matrix (looks like the rational canonical form) over R:

$$\alpha_n(a_1, \cdots, a_n) = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

Recall

$$\chi\colon \mathrm{End}_0(R) \rightarrowtail W(R)$$
$$(R^n, \alpha_n) \mapsto \det(1 - \alpha_n T)$$

we obtain

$$\det \begin{pmatrix} 1 & & & & a_n T \\ -T & 1 & & & a_{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2 T \\ & & & -T & 1 + a_1 T \end{pmatrix} = 1 + a_1 T + \cdots + a_n T^n$$

(Computation methods: 1. the traditional computation. 2. Note that if A is invertible,

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

In this case $A^{-1} = \begin{pmatrix} 1 & & & & \\ T & 1 & & & \\ \vdots & \ddots & \ddots & & \\ T^{n-3} & \cdots & T & 1 & \\ T^{n-2} & T^{n-3} & \cdots & T & 1 \end{pmatrix}$)

Then we can also conclude that $\mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$.

Remark 1.24. Why is a general elment of the form (R^n, α_n)? Namely how to reduce an endomorphism to a rational canonical form ?

Now we want to check some identities

$$\mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n \nu)] \quad \text{by definition}$$
$$(R^{n+1}, \alpha_{n+1}(a_1, \cdots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)] \quad \text{compute under } \chi$$
$$(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)] \quad \text{where } \beta = \alpha_n(a_1 \nu, \cdots, a_n \nu^n)$$

In fact, the last identity always holds when $R = \mathbb{Z}[a_1, \cdots, a_n]$. β is nilpotent because $\beta = \alpha_n \nu$.

We only show how to check the last equation: only need to show that

$$\alpha_n \nu = \alpha_n(a_1 \nu, \cdots, a_n \nu^n)$$

$$LHS = \begin{pmatrix} 0 & & & & -a_n\nu \\ \nu & 0 & & & -a_{n-1}\nu \\ & \ddots & \ddots & & \vdots \\ & & \nu & 0 & -a_2\nu \\ & & & \nu & -a_1\nu \end{pmatrix}, \quad RHS = \begin{pmatrix} 0 & & & & -a_n\nu^n \\ 1 & 0 & & & -a_{n-1}\nu^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2\nu^2 \\ & & & 1 & -a_1\nu \end{pmatrix}$$

we can check this using the characteristic polynomial since χ is injective: check

$$\det(1 - \alpha_n \nu T) = \det(1 - \alpha_n(a_1\nu, \cdots, a_n\nu^n)T)$$

$$LHS = \det \begin{pmatrix} 1 & & & & a_n\nu T \\ -\nu T & 1 & & & a_{n-1}\nu T \\ & \ddots & \ddots & & \vdots \\ & & -\nu T & 1 & a_2\nu T \\ & & & -\nu T & 1 + a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n)$$

$$RHS = \det \begin{pmatrix} 1 & & & & a_n\nu^n T \\ -T & 1 & & & a_{n-1}\nu^{n-1} T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2\nu^2 T \\ & & & -T & 1 + a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n).$$

Note that if $\exists N$ such that $\nu^N = 0$, β is independent of the a_i for $i \geq N$. If $\nu^N = 0$ then $\alpha_n \otimes \nu$ represents 0 in $\mathrm{Nil}_0(\Lambda)$ whenever $\chi(\alpha_n) \equiv 1 \bmod t^N$.

More operations Let $F_n\mathbf{Nil}(\Lambda)$ denote the full exact subcategory of $\mathbf{Nil}(\Lambda)$ on the (P, ν) with $\nu^n = 0$. If Λ is an algebra over a commutative ring R, the kernel $F_n\mathrm{Nil}_0(\Lambda)$ of $K_0(F_n\mathbf{Nil}(\Lambda)) \longrightarrow K_0(\mathbf{P}(\Lambda))$ is an $\mathrm{End}_0(R)$-module and $F_n\mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$ is a module map.

The exact endofunctor $F_m\colon (P, \nu) \mapsto (P, \nu^m)$ on $\mathbf{Nil}(\Lambda)$ is zero on $F_m\mathbf{Ni1}(\Lambda)$. For $\alpha \in \mathrm{End}_0(R)$ and $(P, \nu) \in \mathrm{Nil}_0(\Lambda)$, nota that $(V_m\alpha) * (P, \nu) = V_m(\alpha * F_m(P, \nu))$, and we can conclude that $V_m\mathrm{End}_0(R)$ acts trivially on the image of $F_m\mathrm{Nil}_0(\lambda)$ in $\mathrm{Nil}_0(\lambda)$. For more details, see Weibel, *K*-book chapter 2, pp 155 Exercise II.7.17.

1.3.4 $W(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Once we have a nilpotent endomorphism, we can pass to infinity.

Theorem 1.25. $\mathrm{End}_0(R)$*-module structure on* $\mathrm{Nil}_0(\Lambda)$ *extends to a* $W(R)$*-module structure by the formula*

$$(1 + \sum a_i T^i) * [(P, \nu)] = (R^n, \alpha_n(a_1, \cdots, a_n)) * [(P, \nu)], \ n \gg 0.$$

1.3.5 $W(R)$-module structure on $\mathrm{Nil}_*(\Lambda)$

The induced t-adic topology on $\mathrm{End}_0(R)$ is defined by the ideals

$$I_N = \{f \in \mathrm{End}_0(R) \mid \chi(f) \equiv 1 \bmod t^N\},\ I_N \supset I_{N+1},$$

and $\mathrm{End}_0(R)$ is separated (i.e. $\cap I_N = 0$) in this topology. The key fact is:

Theorem 1.26 (Almkvist)**.** *The map $\chi\colon \mathrm{End}_0(R) \longrightarrow W(R)$ is a ring injection, and $W(R)$ is the t-adic completion of $\mathrm{End}_0(R)$, i.e. $W(R) = \varprojlim \mathrm{End}_0(R)/I_N$.*

Theorem 1.27 (Stienstra)**.** *For every $\gamma \in \mathrm{Nil}_*(\Lambda)$ there is an N so that γ is annihilated by the ideal*

$$I_N = \{f \mid \chi(f) \equiv 1 \bmod t^N\} \subset \mathrm{End}_0(R).$$

Consequently, $NK_*(\Lambda)$ is a module over the t-adic completion $W(R)$ of $\mathrm{End}_0(R)$.

Recall the sign convention:

$$\begin{aligned} NK_1(\Lambda) &\cong \mathrm{Nil}_0(\Lambda) \\ [1 - \nu y] &\leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)] \end{aligned}$$

The $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula

$$\alpha(t) * [1 - \nu y] = [\alpha(\nu y)].$$

And the $W(R)$-module structure on $NK_n(\Lambda)$

$$\alpha(t) * \{\gamma, 1 - \nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda), \quad \gamma \in K_{n-1}(R).$$

1.3.6 Modern version

Reference: Weibel, K-book, chapter 4, pp. 58.

1.4 Some results

Proposition 1.28. *If R is $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$ or $\mathbb{Q}$-algebra, then*

$$\begin{aligned} \lambda_t\colon R &\longrightarrow W(R) \\ r &\mapsto (1-t)^r \end{aligned}$$

is a ring injection.

Corollary 1.29. *Fix an integer p and a ring Λ with 1.*
(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.
(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ)-module.
(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.
(d) If $p^m = 0$ in Λ, $NK_*(\Lambda)$ is a p-group.

Theorem 1.30 (Stienstra). *If $0 \neq n \in \mathbb{Z}$, $NK_1(R)[\frac{1}{n}] \cong NK_1(R[\frac{1}{n}])$.*

Corollary 1.31. [1] *If G is a finite group of order n, then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n. In fact, $NK_*(\mathbb{Z}[G])$ is an n-torsion group, and $Z_{(p)} \otimes NK_*(\mathbb{Z}[G]) = NK_*(\mathbb{Z}_{(p)}[G])$, where $p \mid n$.*

[1] Weibel, *K*-book chapter3, page 27.

Chapter 2

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [8].

2.1 Outline

Definition 2.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x \mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty} \mathbb{F}_2 x^i = \oplus_{i=1}^{\infty} \mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\, dx = \oplus_{i=1}^{\infty} \mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\, dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

2.2 Preliminaries

2.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 2.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

2.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[\![x]\!])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$\begin{aligned}
[a]&: \alpha(x) \mapsto \alpha(ax) \\
V_m&: \alpha(x) \mapsto \alpha(x^m) \\
F_m&: \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m&: 1 - rx \mapsto 1 - r^m x
\end{aligned}$$

Remark 2.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [1].

Proposition 2.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a]V_m &= V_m[a^m] \\
F_m[a] &= [a^m]F_m \\
[a][b] &= [ab] \\
V_m F_k &= F_k V_m, \textit{ if } (k, m) = 1
\end{aligned}$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[\![x]\!]$ is not.

2.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.s

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 2.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem*.

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a, b\rangle \in K_2(R, I)$ satisfy following relations

(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$ if any of a, b, c are in I.

Theorem 2.6. *1. If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R, I) \overset{\cong}{\to} K_2(R, I)$.

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r, s \in R$ commute and $1 - rs$ is a unit, that is $rs = sr$ and $1 - rs \in R^*$,

$$\langle r, s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r, s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r, s\rangle \in K_2(R, I)$

(D1) $\langle r, s\rangle\langle s, r\rangle = 1$,

(D2) $\langle r, s\rangle\langle r, t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r, st\rangle = \langle rs, t\rangle\langle tr, s\rangle$ (this holds in $K_2(R, I)$ if any of r, s, t are in I).

Note that $\langle r, 1\rangle = 0$ for any $r \in R$ and $\langle r, s\rangle_{version2} = \langle -r, s\rangle_{version1}$.

Theorem 2.7. *1. If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r, s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r, s\}$.*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R, I)$ is generated by $\langle r, s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively generated by its units.

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r,s\rangle$, $r,s\in R$, $1-rs\in R^*$ or by all $\{u,v\}$, $u,v\in R^*$.*

2.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 2.4.

excision 失效就是说 if $A\longrightarrow B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A,I)\longrightarrow K_n(B,I)$ need not be an isomorphism. 由于这个不是同构，没法有 Mayer-Vietoris 序列

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(A/I) \arrow[r] \arrow[d] & K_i(A,I) \arrow[r] \arrow[d, shift right] & K_i(A) \arrow[r] \arrow[d] & K_i(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(B/I) \arrow[r] & K_i(B,I) \arrow[r] \arrow[u, dashed, shift right] & K_i(B) \arrow[r] & K_i(B/I) \arrow[r] & \cdots
\end{tikzcd}$$

要连接 $K_n(A,I)\longrightarrow K_n(B,I)$ 就要考虑 birelative K-groups（也称 double relative K-groups）,$K(A,B,I)$ 定义为 homotpy fiber of the map $K(A,I)\longrightarrow K(B,I)$。以下是详细的定义和性质。

Relative groups Let R be a ring (not necessarily commutative), $I\subset R$ a two-sided ideal, by definition $K_i(R)=\pi_i(BGL(R)^+)$, $i\geq 1$, there exists a map

$$BGL(R)^+\longrightarrow BGL(R/I)^+$$

Definition 2.8. $K(R,I)$ is the homotopy fibre of the map $BGL(R)^+\longrightarrow BGL(R/I)^+$. $K_i(R,I):=\pi_i(K(R,I)), i\geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots\longrightarrow K_{i+1}(R)\longrightarrow K_{i+1}(R/I)\longrightarrow K_i(R,I)\longrightarrow K_i(R)\longrightarrow K_i(R/I)\longrightarrow\cdots.$$

In particular,

$$\begin{aligned}&K_3(R,I)\longrightarrow K_3(R)\longrightarrow K_3(R/I)\longrightarrow K_2(R,I)\longrightarrow K_2(R)\longrightarrow K_2(R/I)\longrightarrow\\ \longrightarrow &K_1(R,I)\longrightarrow K_1(R)\longrightarrow K_1(R/I)\end{aligned}$$

Let R be any ring (not necessarily commutative), if $I,J\subset R$ are two-sided ideals, there is a map

$$K(R,I)\longrightarrow K(R/J,I+J/J).$$

If $I \cap J = 0$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R/I \\ \Big\downarrow \beta & & \Big\downarrow g \\ R/J & \xrightarrow{f} & R/I+J \end{array}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(2.8)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow K_{i+1}(R) & \xrightarrow{\alpha_*} & K_{i+1}(R/I) & \xrightarrow{\partial} & K_i(R,I) & \xrightarrow{j} & K_i(R) & \xrightarrow{\alpha_*} & K_i(R/I) & \longrightarrow \cdots \\ \Big\downarrow \beta_* & & \Big\downarrow g_* & & \Big\downarrow \epsilon_i & & \Big\downarrow & & \Big\downarrow & \\ \cdots \longrightarrow K_{i+1}(R/J) & \xrightarrow{f_*} & K_{i+1}(R/I+J) & \xrightarrow{\partial} & K_i(R/J, I+J/J) & \xrightarrow{j'} & K_i(R/J) & \xrightarrow{f_*} & K_i(R/I+J) & \longrightarrow \cdots \end{array}$$

where the induced homomorphism

$$\epsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the MayerVietoris sequence

$$\begin{aligned} K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 2.9. *1. Suppose that the excision map ϵ_i in 2.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. *If ϵ_i is an isomorphism, and in additionϵ_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is*

$$\begin{aligned} K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

3. *Suppose instead that ϵ_i is surjective, and let $L = ker(\epsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, and the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow \\ \longrightarrow K_i(R)/L \longrightarrow K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

   is exact.

Proof. Define $\delta_i = j\epsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. □

Remark 2.10. It is known that ϵ_0 and ϵ_1 are isomorphism regardless of the specific rings. Moreover Swan [6] has shown that ϵ_2 cannot be an isomorphism in general. For more discussion, see [5].

Double relative groups

Definition 2.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I+J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd} K(R;I,J) \arrow[d, dashed] & & \\ K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\ K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+ \end{tikzcd}$$

Remark 2.12. $K_i(R; I, J) \cong K_i(R; J, I), K_i(R; I, I) = K_i(R, I)$.

We have a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R, I) \longrightarrow K_{i+1}(R/J, I+J/J) \longrightarrow K_i(R; I, J) \longrightarrow K_i(R, I) \longrightarrow K_i(R/J, I+J) \longrightarrow \cdots.$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R, I) \longrightarrow K_3(R/I, I+J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b\rangle$, see [9] 3.5.10, [5], [4] or [2] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [3] theorem 2.

Remark 2.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [3].

Theorem 2.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b\rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [3] theorem 3. □

Lemma 2.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I+J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

is split exact.

2.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [1] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$
$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$**-module structure on** $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [1]example 2.10

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx$$
$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$
$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx$$

Remark 2.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 2.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$V_m(x^i) = x^{im},$$
$$V_m(x^{i-1}dx) = mx^{im-1}dx$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 2.18. 一个不知道有没有用的结论, see [1]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$D\colon x\mathbb{F}_2[x] \longrightarrow \Omega_{\mathbb{F}_2[x]}$$
$$x^i \mapsto ix^{i-1}dx$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty}\mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

2.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{2.18}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$\begin{tikzcd} & NK_3(\mathbb{Z},(2)) = 0 \arrow[d] & & \\ & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\ 0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\ & NK_2(\mathbb{Z},(2)) = 0 & & \end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [3], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1),\sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1-x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 2.19. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 2.20 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p] & \xrightarrow{\sigma\mapsto\zeta} & \mathbb{Z}[\zeta] \\ {\scriptstyle \sigma\mapsto 1}\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R,I)$$
$$(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1, 1\rangle^p = 1$$

Also see [5].

Example 2.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{array}{ccc} \mathbb{Z}[C_p][x] & \longrightarrow & \mathbb{Z}[\zeta][x] \\ \downarrow & & \downarrow \\ \mathbb{Z}[x] & \longrightarrow & \mathbb{F}_p[x] \end{array}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

2.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $\mathbf{F}_2[\epsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{cccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\epsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\epsilon \end{array}$$

Lemma 2.22. *The map $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$ in 2.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [8] Lemma 1.2. □

Theorem 2.23.

$$NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]},$$

$$NK_0(\mathbb{Z}[D_2]) \cong V,$$

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 2.18 with $\mathbb{Z}[C_2]$. Note that $R[G_1 \times G_2] = R[G_1][G_2]$, for commutative R, $R[G_1 \times G_2] = R[G_1] \otimes R[G_2]$, $\sum_{g,h} c_g c_h g \otimes h \leftarrow \sum_{g,h} c_g g \otimes c_h h$. As for infinite product, see MO 46950.

$$\begin{tikzcd} \mathbb{Z}[D_2] = \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] \arrow[r] \arrow[d] & \mathbb{Z}[C_2] \arrow[d, two heads, "q"] \\ \mathbb{Z}[C_2] \arrow[r, two heads, "q"] & \mathbb{F}_2[C_2] \end{tikzcd} \tag{2.23}$$

Recall that $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]/(\epsilon^2)$. By [9] chapter 2 Ex 7.4.5,

$$NK_1(\mathbb{F}_2[C_2]) = NK_1(\mathbb{F}_2[\varepsilon]/(\epsilon^2)) = (1 + \varepsilon x \mathbb{F}_2[\varepsilon]/(\epsilon^2)[x])^\times = (1 + \varepsilon x \mathbb{F}_2[x])^\times \cong V = x\mathbb{F}_2[x]$$
$$[(P, \nu)] \mapsto \det(1 - \nu x)$$

Remark 2.24. $(1+\varepsilon x\mathbb{F}_2[x])^\times \cong x\mathbb{F}_2[x]$, $1+\varepsilon x\sum a_i x^i \mapsto x\sum a_i x^i$ 的原因，左边是乘法群，右边的乘法是普通的多项式相加，左边 $(1+\varepsilon x\sum a_i x^i)(1+\varepsilon x\sum b_j x^j) = 1+\varepsilon x\sum a_i x^i + \varepsilon x\sum b_j x^j + (\varepsilon x\sum a_i x^i)(\varepsilon x\sum b_j x^j) = 1+\varepsilon x(\sum a_i x^i + \sum b_j x^j)$, 右边 $x\sum a_i x^i + x\sum b_j x^j = x(\sum a_i x^i + \sum b_j x^j)$.

As $W(\mathbb{F}_2)$-modules,

$$1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n) \mapsto a_0x+a_1x^2+\cdots+a_nx^{n+1}$$

and we can easily check that

$$V_m(1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon x^m(a_0+a_1x^m+\cdots+a_nx^{mn})$$

$$[a](1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon ax(a_0+a_1ax+\cdots+a_na^nx^n)$$

hence the module structure of $(1+\varepsilon x\mathbb{F}_2[x])^\times$ are the same as V.

By MayerVietoris sequence for the NK-functor, one has

$$\begin{tikzcd}
NK_2(\mathbb{Z}[D_2]) \arrow[r] & NK_2\mathbb{Z}[C_2]\oplus NK_2\mathbb{Z}[C_2] \arrow[r, "q\times q"] & NK_2(\mathbb{F}_2[C_2]) \arrow[dll] \\
NK_1(\mathbb{Z}[D_2]) \arrow[r] & NK_1\mathbb{Z}[C_2]\oplus NK_1\mathbb{Z}[C_2] = 0 \arrow[r] & NK_1(\mathbb{F}_2[C_2]) \arrow[dll, "\cong"] \\
NK_0(\mathbb{Z}[D_2]) \arrow[r] & NK_0\mathbb{Z}[C_2]\oplus NK_0\mathbb{Z}[C_2] = 0 &
\end{tikzcd}$$

Hence $NK_0(\mathbb{Z}[D_2]) \cong V$, $NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}(q\times q) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}q \cong \Omega_{\mathbb{F}_2[x]}$ since $\mathrm{Im}(q\times q) = \mathrm{Im}q$. □

最后一个论断，若对则有 $(q\times q)(\Phi(V)\times V) = 0$，然而这个等式是不成立的。

2.5.1 A result from the K-book

For the convenience of the reader we copy [9] chapter 2 Ex 7.4.5 as follows.

Let R ba a commutative regular ring, $A = R[x]/(x^N)$, we claim that

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$\begin{aligned}
\mathrm{Nil}_0(A) &\cong (1+xtA[t])^\times \\
[(A,x)] &\mapsto 1-xt \\
[(P,\nu)] &\mapsto \det(1-\nu t)
\end{aligned}$$

the isomorphism $NK_1(A) \cong \mathrm{Nil}_0(A) \cong (1 + xtA[t])^\times$ is universal in the following sense:

Let B be a R-algebra, $(P, \nu) \in \mathbf{Nil}(B)$ with $\nu^N = 0$, regard P as an A-B-bimodule

$$\begin{aligned} \mathrm{Nil}_0(A) &\longrightarrow \mathrm{Nil}_0(B) \\ (A, x) &\mapsto (P, \nu) \end{aligned}$$

there is an $\mathrm{End}_0(R)$-module homomorphism

$$\begin{aligned} (1 + xtA[t])^\times &\longrightarrow \mathrm{Nil}_0(B) \\ 1 - xt &\mapsto [(P, \nu)]. \end{aligned}$$

2.5.2 About the lemma

In this subsection, we concentrate on the lemma 2.22.

For a complete proof, see [7].

2.6 NK_i of the group C_4

2.7 NK_i of the group D_4

Chapter 3

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

3.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

3.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [10] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2 C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2 C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

3.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1, t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:

generators: $\langle a, b \rangle$, $(a, b) \in A \times M \cup M \times A$;

relations: $\langle a, b \rangle = -\langle b, a \rangle$,

$\langle a, b \rangle + \langle c, b \rangle = \langle a + c - abc, b \rangle$,

$\langle a, bc \rangle = \langle ab, c \rangle + \langle ac, b \rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [7]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1, 0) \in \mathbb{N}^2$, $\epsilon^2 = (0, 1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,

- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha - \epsilon^i}, t_i \rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 3.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [7]。 □

Lemma 3.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3 \rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

3.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 3.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4),(t_1,t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. □

Theorem 3.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)$, then $I = (t_1^4,t_2^4)$, $M = (t_1,t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1,\alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha,i) \mid \alpha_i \geq 1\}.$$

For $(\alpha,i) \in \Lambda$,

$$[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha,i) \in \Lambda^{00} \mid [(\alpha,i)] > 1\}$, and then

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha,1) \in \Lambda^{00}_{>1}$ and $[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha,2) \in \Lambda^{00}_{>1}$ and $[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1,\alpha_2 \leq 4$ are both odd and $gcd(\alpha_1,\alpha_2) = 1$, then $(\alpha,2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha,1]$.

By the computation 3.2, we can get the following table

$(\alpha, i) \in \Lambda_{>1}^{00}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below, the generators of order 4:

$$\langle t_1t_2, t_1\rangle, \langle t_1t_2, t_2\rangle, \langle t_1, t_2\rangle,$$

the generators of order 2:

$$\langle t_1t_2^3, t_1\rangle, \langle t_1^3t_2, t_1\rangle, \langle t_1^3t_2^3, t_1\rangle, \langle t_1t_2^3, t_2\rangle, \langle t_1^3t_2^2, t_2\rangle, \langle t_1t_2^2, t_2\rangle, \langle t_1^3t_2, t_2\rangle, \langle t_1^3t_2^3, t_2\rangle, \langle t_1^3, t_2\rangle.$$

□

Remark 3.5. Compared with [10], note that $\langle t_1^3, t_2\rangle = \langle t_1^2t_2, t_1\rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1t_2\rangle - \langle t_1^2t_2, t_1\rangle \\
&= \langle t_1, t_1^2t_2\rangle - \langle t_1^2t_2, t_1\rangle - \langle t_1^2t_2, t_1\rangle \\
&= -3\langle t_1^2t_2, t_1\rangle \\
&= -\langle t_1^2t_2, t_1\rangle \\
&= \langle t_1^2t_2, t_1\rangle,
\end{aligned}$$

since $\langle t_1^2t_2, t_1\rangle + \langle t_1^2t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

References

[1] B. H. Dayton and Charles A. Weibel. Module structures on the Hochschild and cyclic homology of graded rings. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, pages 63–90. Kluwer Acad. Publ., Dordrecht, 1993.

[2] Eric Friedlander and MR Stein. *Algebraic K-theory. Proc. conf. Evanston, 1980*. Springer, 1981.

[3] Dominique Guin-Waléry and Jean-Louis Loday. *Algebraic K-Theory Evanston 1980: Proceedings of the Conference Held at Northwestern University Evanston, March 24–27, 1980*, chapter Obstruction a l'Excision En K-Theorie Algebrique, pages 179–216. Springer Berlin Heidelberg, Berlin, Heidelberg, 1981.

[4] Frans Keune. The relativization of K_2. *Journal of Algebra*, 54(1):159–177, 1978.

[5] Michael R. Stein. Excision and K_2 of group rings. *Journal of Pure and Applied Algebra*, 18(2):213 – 224, 1980.

[6] Richard G. Swan. Excision in algebraic K-theory. *Journal of Pure and Applied Algebra*, 1(3):221 – 252, 1971.

[7] Wilberd van der Kallen and Jan Stienstra. The relative K_2 of truncated polynomial rings. In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pages 277–289, 1984.

[8] Charles A. Weibel. NK_0 and NK_1 of the groups C_4 and D_4. *Comment. Math. Helv*, 84:339–349, 2009.

[9] Charles A. Weibel. *The K-book: An introduction to algebraic K-theory*. American Mathematical Society Providence (RI), 2013.

[10] 陈虹，高玉彬，唐国平. $K_2(\mathbb{F}_2[C_4 \times C_4])$ 的计算. 中国科学院大学学报, 28(4):419, 2011.