内容集锦: 讨论班、课程讲义

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目录

Chapter 1

Witt rings and NK-groups

References:

- C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466–493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.
- D. R. Grayson, Grothendieck rings and witt vectors.
- C. A. Weibel, The K-Book: An Introduction to Algebraic K-theory.

1.1 Typical Witt rings

1.2 Big Witt rings

1.3 Module structure on NK_*

Notations Λ: a ring with 1
R: commutative ring
$W(R)$: Witt ring of R
End(Λ): the exact category of endomorphisms of finitely generated projective right Λ-modules.
Nil(Λ): the full exact subcategory of nilpotent endomorphisms.
P(Λ): the exact category of finitely generated projective right Λ-modules.

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.

- Computations in $W(R)$ with Grothendieck rings.

1.3.1 $\mathrm{End}_0(\Lambda)$

Let $\mathbf{End}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules.
Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.
Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & M_1 \arrow[d, "\alpha"] \\ M_2 \arrow[r, "f_2"] & M_2 \end{tikzcd}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathscr{C} \longrightarrow \mathscr{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathscr{C} \rightleftarrows \mathscr{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \leftarrow\!\shortmid (M, f)$$

Definition 1.1. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) \longrightarrow \mathbf{End}(\Lambda)$$
$$\otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) \longrightarrow \mathbf{Nil}(\Lambda)$$
$$(M, f) \otimes (N, g) = (M \otimes_R N, f \otimes g)$$

These induce (use "generators-and-relations" tricks on K_0)

$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) \longrightarrow K_*(\mathbf{End}(\Lambda))$$
$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) \longrightarrow K_*(\mathbf{Nil}(\Lambda))$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M,f) \in \mathbf{End}(R)$, then $(M,f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R,1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R,0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R,0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R,0)\otimes\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. The following is immediate (and well-known):

Proposition 1.2. *If Λ is an R-algebra with 1, $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$ are graded modules over the ring $\mathrm{End}_0(R)$.*

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 1.3 (Almkvist)**.** *The homomorphism (in fact it is a ring homomorphism)*

$$\chi\colon \mathrm{End}_0(R) \longrightarrow W(R) = (1 + TR[[T]])^\times$$
$$(M,f) \mapsto \det(1 - fT)$$

is injective and $\mathrm{End}_0(R) \cong \mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1 - T$$

χ is a ring homomorphism, and $\mathrm{Im}\chi =$ the set of all rational functions in $W(R)$.

Definition 1.4 (NK_*)**.** s above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in "Higher algebraic K-theory II".