代数 *K* 理论讨论班笔记

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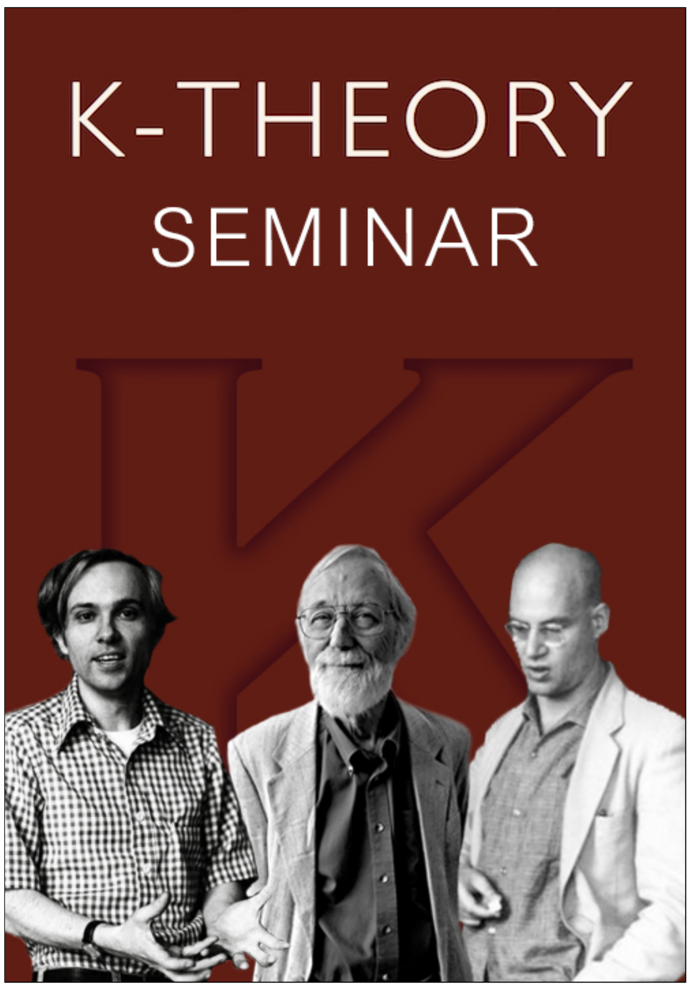

从左至右依次为
Quillen Milnor Grothendieck

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目录

Chapter 1

K 理论简介

1.1 释题

初见题目，大概最先问的问题就是“K 理论”中的“K”是什么含义，因而我们从释题开始：

“K” “K”源于德文“klassen”，中文意为“分类”。从而简略地说，“K 理论”就是分类的理论。1957 年 Grothendieck[1]在 Riemman-Roch 定理的工作中引入了函子 $K(\mathcal{A})$，这就是 K 理论的开端。他之所以用 K 而不用 C(英语“class”的首字母) 是由于 Grothendieck 在泛函分析中做的许多工作里 $C(X)$ 通常表示连续函数空间，因此用“分类”在他的母语---德语中的首字母。

对于历史感兴趣的读者请参考 C.Weibel《The development of algebraic K-theory before 1980》[50]。

[1] 1928.3.28-2014.11.13,1966 Fields Medal

分类　对于分类的思想，在数学中并不陌生，下表举了一些例子：

	例子	备注
表示论	有限群的不可约表示分类	Brauer 群，Voevodsky
代数几何	代数簇分类	Riemann-Roch-Hirzebruch-Grothendieck
代数拓扑	向量丛、拓扑空间的分类	拓扑 K-理论，Atiyah-Singer 指标定理
泛函分析	C^* 代数分类	算子 K-理论
代数数论	理想类群	Picard 群，Dedekind 环
几何拓扑	CW 复形	Whitehead 挠元
其它联系	非交换几何，上同调，谱序列等等	

1.2　历史

粗略地讲，K-理论是研究一系列函子：

$$K_n : \text{好的范畴} \longrightarrow \text{交换群范畴}, n \in \mathbb{Z}$$

$$\mathcal{C} \longrightarrow K_n(\mathcal{C})$$

- $n < 0$: 负 K-理论
- $n = 0, 1, 2$: 经典（低阶）K-理论
- $n \geq 3$: 高阶 K-理论

想法　构造环 R 的代数不变量 $K_i(R)$，称之为 K-群，这可以看作是环上的"线性代数"，更一般的看成某个空间的同伦群。构造高阶 K-群时有不同的构造方式，另外从广义上同调理论看，可以构造代数 K-理论谱 (Spectrum)，使得它的同伦群就是 K-群。

代数学分支中很多学科都可以看作线性代数的推广，如同调代数，表示论，李群李代数，矩阵分析，泛函分析等等，这里代数 K-理论某种意义上也是一门线性代数。

1.3　讲了几章 Srinivas 书后的想法

想把 K 理论推广到高阶 K 理论，并且还有类似于经典 K 理论的性质，比如正合列，MV 序列，还有基本定理。首先想得到一个长正合列，从代数上考虑是同调函子可以将复形的短正合列变成一个长正合列。换个角度思考，拓扑上得到一个长正合列除了同调函子还有一个重要的函子是同伦函子，一个 Serre 纤维化序列可以得到一个同伦群的长正合列。这是得到长正合列的方法。Quillen 了不起的想法是对于环 R，构造一个空间，

使得这个空间的同伦群就是 K 群。于是他得到了两种定义高阶 K 理论的方法，俗称为“$+$”构造和“Q”构造。首先加法构造是对环 R 的一般线性群 $GL(R)$ 做分类空间 $BGL(R)$，对于任意群都可以找到这样一个相应的拓扑空间叫做分类空间，使得群的同调就是这个拓扑空间的同调。现在有了分类空间还不够，Quillen 发明了加法构造在分类空间的基础上增加相同数目的 2-胞腔和 3-胞腔得到了 $BGL(R)^+$，从这个空间出发求其同伦群就得到了 K 群。为什么说就是 K 群呢？通过计算可以得到，K_1, K_2 的结果正是经典 K 理论里的两个函子，从而这样一次性定义的 K 群就是经典 K 理论的推广。接着 Quillen 在 1972 年的著名论文中给出了 Q 构造，并且这时普遍适用与一大类范畴---正合范畴。对于正合范畴 $\mathcal{C}$，通过做 Q 构造得到 $Q\mathcal{C}$，然后做分类空间得到 $BQ\mathcal{C}$，再然后算 n 阶同伦群也得到了新的函子。可以证明这个函子和经典 K 群是一致的！唯一有些区别在于足标，$n+1$ 阶同伦群得到的是 n 阶 K 群，于是我们对 $BQ\mathcal{C}$ 取其 loop space $\Omega BQ\mathcal{C}$ 后，n 阶同伦群就是 n 阶 K 群了。

那这样两个定义是否一致呢？著名的“$+ = Q$”定理说对于环 R 和正合范畴 $\mathcal{P}(R)$ 分别用加法构造和 Q 构造得到的两个拓扑空间是同伦等价的，于是它们取同伦群是一样的！

有了 Q 构造后，高阶 K 群自然而然想推广经典 K 理论中的结论，而恰就是这么巧，很多定理都可以推广，但都不见得是平凡的。高阶 K 群的计算首先就是非常难的一部分，Quillen 在论文里得到了四大定理：加法定理，分解定理，反旋定理和局部化序列，英文分别叫做 Additivity，Resolution，Devissage，Localization。这四大定理再加上推论可以得到一些有趣的结果。首先看加法定理是说正合函子也有类似于 Euler characteristic 的性质，即一个正合函子的短正合列，中间函子诱导的 K 群的映射等于两边函子诱导的映射之和，很容易可以把短正合列推广成长正合列，并且还可以推广到有一个 filtration。对于分解定理和 Devissage，都是通过更简单的满子范畴来替换要研究的正合范畴，并且 K 群不变。局部化序列当然是利用长正合序列从已知来得到未知的信息。

有了这些准备，对于诺特环的 K 理论就会有一个比较深刻的定理，也叫做诺特环的 G 理论，G 理论是说只研究环 R 上的有限生成模的范畴，将 K 理论中投射的要求去掉。对于诺特环的 G 理论，有著名的 homotopy invariance

$$G_n(A[t]) = G_n(A), G_n(A[t, t^{-1}]) = G_n(A) \oplus G_{n-1}(A)$$

对其进行更细致的研究和推广可以得到对于任意环的 K 理论基本定理

$$K_n(A[t, t^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus NK_n(A) \oplus NK_n(A).$$

对于诺特环 G 理论基本定理的证明就是利用局部化序列，并且反复应用四大定理来得

到，并且还详细研究了分次环和分次模的一些性质。Srinivas 的书无疑是很好的教材，Quillen 的原文也是值得一看的。

参考

1. David Eisenbud，commutative algebra with a view toward algebraic geometry.

1.4 相对 K_1 群

目标: 将环 R 与它的商环 R/I 的 K_1 群联系起来.

步骤: 定义相对 K_1 群 $K_1(R,I)$, 连接 K_0 和 K_1 的六项正合列. 由此得到计算 K 群的工具, 如 $SK_1(R,I)$ 与 Mayer-Vietoris 序列.

Definition 1.1. R: 环, $I \subset R$: 理想. $R \to R/I$ 诱导了 $GL(R) \to GL(R/I)$.

定义 $GL(I) := \ker(GL(R) \to GL(R/I))$, $GL_n(I) := \ker(GL_n(R) \to GL_n(R/I))$.

$E(R,I) :=$ 包含初等阵 $e_{ij}(x), x \in I$ 的 $E(R)$ 的最小的正规子群, 实际上是 $E(R,I) = \langle e_{ij}(x) \mid x \in I \rangle^{E(R)}$.

$E_n(R,I) :=$ 由 $e_{ij}(x), x \in I, 1 \le i \ne j \le n$ 这些矩阵生成的 $E_n(R)$ 的正规子群, 实际上是 $E_n(R,I) = \langle e_{ij}(x) \mid x \in I, 1 \le i \ne j \le n \rangle^{E_n(R)} \lhd E_n(R)$.

$E(R,I) = \bigcup_n E_n(R,I)$.

Remark 1.2. Rosenberg 的书中将 $GL(I)$ 记为 $GL(R,I)$, Weibel 在 K-book 的第三章习题 1.1.10 中说 $GL(I)$ 与 R 的选择无关, 仅与 I 视为无幺元的环的结构有关。令 I 是无幺元的环, 可以将其添加一个幺元使它成为含幺环, 记为 $I_+ = I \oplus \mathbb{Z}$,
其中的加法结构: $(x,n) + (y,m) = (x+y, n+m)$
乘法结构: $(x,n) \cdot (y,m) = (xy + ny + mx, nm)$
乘法单位元 $(0,1)$: $(x,n)(0,1) = (x,n) = (0,1)(x,n)$.
$GL_n(I) := \ker(GL_n(I \oplus \mathbb{Z}) \to GL_n(\mathbb{Z}))$, 若 $I \subset R$ 是理想, 则有 $GL_n(I) = \ker(GL_n(R) \to GL_n(R/I))$.

$E(R,I)$ 中的初等阵在模 I 后为单位矩阵, 即 $e_{ij}(x) \equiv \mathrm{id}(\mathrm{mod} I)$, 故 $E(R,I) \subset GL(I)$.

接下来想要定义 $K_1(R,I)$ 为 $GL(I)/E(R,I)$, 必须要求 $E(R,I) \lhd GL(I)$: 这就是相对 Whitehead 引理.

Lemma 1.3 (相对 Whitehead 引理)**.** $E(R,I) \lhd GL(I)$, $[GL(I), GL(I)] \subset E(R,I)$.

Proof. (i) 下面证明对于 $g \in GL_n(I), h \in E_n(G,I)$, 下式成立

$$\begin{pmatrix} ghg^{-1} & \\ & 1 \end{pmatrix} \in E(R,I).$$

实际上有

$$\begin{pmatrix} ghg^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} h & \\ & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & \\ & g \end{pmatrix}.$$

若将 g 写成 $g=1+\alpha\in GL_n(I)$, 由于 $GL(I)$ 中元映到 $GL_n(R/I)$ 为 1, 故 α 这个矩阵中的元素都属于 I,

$$\begin{aligned}\begin{pmatrix} g & 0 \\ 0 & g^{-1}\end{pmatrix} &= e_{12}(1)e_{21}(\alpha)e_{12}(-1)e_{12}(g^{-1}\alpha)e_{21}(-g\alpha) \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha & 1\end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & g^{-1}\alpha \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -g\alpha & 1\end{pmatrix},\end{aligned}$$

右边的式子前三项乘起来在 $E_{2n}(R,I)$ 中, 从而 $\begin{pmatrix} g & 0 \\ 0 & g^{-1}\end{pmatrix}\in E_{2n}(R,I)$, 故 $\begin{pmatrix} ghg^{-1} & \\ & 1\end{pmatrix}\in E(R,I)$.

(ii) 若 $g,h\in GL(I)$, 则

$$[g,h]=\begin{pmatrix} g & \\ & g^{-1}\end{pmatrix}\begin{pmatrix} h & \\ & h^{-1}\end{pmatrix}\begin{pmatrix} (hg)^{-1} & \\ & hg\end{pmatrix}\in E_{2n}(R,I)\subset E(R,I).$$

$\square$

有了上面的引理, 我们可以定义相对 K_1 群. 由于 $E(R,I)\lhd GL(I)$, 因此可以做商群, 并且由于 $[GL(I),GL(I)]\subset E(R,I)$, 因此商群是交换群, 这是因为对于任意的 $\bar{g},\bar{h}\in K_1(R,I)$, $\bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1}=\overline{ghg^{-1}h^{-1}}\in E(R,I)$, 因此 $\bar{g}\bar{h}=\bar{h}\bar{g}$.

Definition 1.4. $K_1(R,I):=GL(I)/E(R,I)$, 且为交换群.

接下来我们从 $GL(I)\rightarrowtail GL(R)$ 类比想得到 $K_1(R,I)\to K_1(R)$ 这样的映射。

若 $R\to S$ 是环同态, R 的理想 I 对应为 S 的理想 I', 则映射 $GL(I)\to GL(I')$ 和 $E(R)\to E(S)$ 诱导了映射 $K_1(R,I)\to K_1(S,I')$.

Remark 1.5. 若 $R\to S$ 是环同态, R 的理想 I 同构地对应为 S 的理想 I, 则 $K_1(R,I)\to K_1(S,I)$ 是满射, 两者都是 $GL(I)$ 的商群. 因为 $E(R,I)\subset E(S,I)$, 故 $GL(I)/E(R,I)\twoheadrightarrow GL(I)/E(S,I)$.

Chapter 2

Notes on Higher K-theory of group-rings of virtually infinite cyclic groups

2.1 Introduction

Authors: Aderemi O. Kuku and Guoping Tang

2.1.1 Preliminaries

Definition 2.1 (Virtually cyclic groups)**.** A discrete group V is called virtually cyclic if it contains a cyclic subgroup of finite index, i.e., if V is finite or virtually infinite cyclic.

Virtually infinite cyclic groups are of two types:

1 $V = G \rtimes_\alpha T$ is a semi-direct product where G is a finite group, $T = \langle t \rangle$ an infinite cyclic group generated by t, $\alpha \in Aut(G)$, and the action of T is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

2 V is a non-trivial amalgam of finite groups and has the form $V = G_0 *_H G_1$ where $[G_0 : H] = 2 = [G_1 : H]$.

We denote by $\mathcal{VCYC}$ the family of virtually cyclic subgroups of G.

$$\text{virtually cyclic groups} \begin{cases} \text{finite groups} \\ \text{virtually infinite cyclic groups} \begin{cases} \text{I.} V = G \rtimes_\alpha T, G \text{ is finite}, T = \langle t \rangle \cong \mathbb{Z} \\ \text{II.} V = G_0 *_H G_1, H \text{is finite}, [G_i : H] = 2 \end{cases} \end{cases}$$

Let G be a finite group, V be a group such that $1 \to G \to V \to T \to 1$ is exact, then V is TYPE.I, i.e., $V = G \rtimes_\alpha T$, $\alpha : T \to Aut(G), \alpha(t)(g) = tgt^{-1}$. Multiplication in V [1]:

$$(g_1, t_1)(g_2, t_2) = (g_1\alpha(t_1)(g_2), t_1t_2) = (g_1t_1g_2t_1^{-1}, t_1t_2).$$

若 G 是有限群，V 满足 $V \to D_\infty \to 1$，$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, 则 V 是类型 II。

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G_0 \arrow[d] \\ G_1 \arrow[r] & G_0 *_H G_1 \end{tikzcd}$$

is a push-out square.

Definition 2.2 (Orders)**.** Let R be a Dedekind domain with quotient field F. An R-order in a F-algebra Σ is a subring Λ of Σ, having the same unity as Σ and s.t. R is contained in the center of Λ, Λ is finitely generated R-module and $F \otimes_R \Lambda = \Sigma$.

A Λ-lattice in Σ is a Λ-bisubmodule of Σ which generates Σ as a F-space.

A maximal R-order Γ in Σ is an order that is not contained in any other R-order in Σ.

Example 2.3. We give some examples:

1. G is a finite group, then RG is an R-order in FG when $ch(F) \nmid |G|$.
2. R is a maximal R-order in F.
3. $M_n(R)$ is a maximal R-order in $M_n(F)$.

Remark 2.4. Any R-order Λ is contained in at least one maximal R-order in Σ. Any semisimple F-algebra Σ contains at least one maximal R-order. However, if Σ is commutative, then Σ contains a unique maximal order, namely, the integral closure of R in Σ.

Theorem 2.5. *R, F, Λ, Σ as above, Then $K_0(\Lambda), G_0(\Lambda)$ are finitely generated abelian groups.*

[1]there is a little difference between this note and the original paper about α

2.1.2 The Farrell-Jones conjecture

Let G be a discrete group and $\mathcal{F}$ a family of subgroups of G closed under conjugation and taking subgroups, e.g., $\mathcal{VCYC}$.

Let $Or_{\mathcal{F}}(G) := \{G/H | H \in F\}$, R any ring with identity.

There exists a "Davis -Lück"functor

$$\mathbb{K}R : Or_{\mathcal{F}}(G) \longrightarrow Spectra$$

$$G/H \mapsto \mathbb{K}R(G/H) = K(RH)$$

where $K(RH)$ is the K-theory spectrum such that $\pi_n(K(RH)) = K_n(RH)$.

There exists a homology theory

$$H_n(-, \mathbb{K}R) : G - CWcomplexes \longrightarrow \mathbb{Z} - Mod$$

$$X \mapsto H_n(X, \mathbb{K}R)$$

Let $E_{\mathcal{F}}(G)$ be a G-CW-complex which is a model for the classifying space of $\mathcal{F}$. Note that $E_{\mathcal{F}}(G)^H$ is homotopic to the one point space (i.e., contractible) if $H \in F$ and $E_{\mathcal{F}}(G)^H = \emptyset$ if $H \notin F$ and $E_{\mathcal{F}}(G)$ is unique up to homotopy.

There exists an assembly map

$$A_{R,\mathcal{F}} : H_n(E_{\mathcal{F}}(G), \mathbb{K}R) \longrightarrow K_n(RG).$$

The Farrell-Jones isomorphism conjecture says that $A_{R,\mathcal{VCYC}} : H_n(E_{\mathcal{VCYC}}(G), \mathbb{K}R) \cong K_n(RG)$ is an isomorphism for all $n \in Z$. Note that KR is the non-connective K-theory spectrum such that $\pi_n(KR)$ is Quillen's $K_n(R), n \geq 0$, and $\pi_n(KR)$ is Bass's negetive $K_n(R)$, for $n \leq 0$.

2.1.3 Notations

- F: number field, i.e, $\mathbb{Q} \subset F$ is a finite field extension.
- R: the ring of integers in F.
- Σ: a semisimple F-algebra.
- Λ: an R-order in Σ, $\alpha : \Lambda \to \Lambda$: an R-automorphism.
- $\Gamma \in \{\alpha\text{-invariant } R\text{-orders in } \Sigma \text{ containing } \Lambda\}$ is a maximal element.
- $\max(\Gamma) = \{\text{two-sided maximal ideals in } \Gamma\}$.

- $\max_\alpha(\Gamma) = \{$two-sided maximal α-invariant ideals in $\Gamma\}$.
- $\mathcal{C}$: exact category, $K_n(\mathcal{C}) = \pi_{n+1}(BQ\mathcal{C}), n \geq 0$. If A is a unital ring, $K_n(A) = K_n(\mathcal{P}(A)), n \geq 0$. When A is noetherian, $G_n(A) = K_n(\mathcal{M}(A))$.
- $T = \langle t \rangle$: infinite cyclic group $\cong \mathbb{Z}$, T^r: free abelian group of rank r.
- $A_\alpha[T] = A_\alpha[t, t^{-1}]$: α-twisted Laurent series ring, $A_\alpha[T] = A[T] = A[t, t^{-1}]$ additively and multiplication given by $(rt^i)(st^j) = r\alpha^i(s)t^{i+j}$.（注：这里和文章有些区别）
- $A_\alpha[t]$: the subgroup of $A_\alpha[T]$ generated by A and t, that is, $A_\alpha[t]$ is the twisted polynomial ring.
- $NK_n(A, \alpha) := \ker(K_n(A_\alpha[t]) \to K_n(A)), n \in \mathbb{Z}$ where the homomorphism is induced by the augmentation $\epsilon : A_\alpha[t] \to A$. If $\alpha = \mathrm{id}$, $NK_n(A, \mathrm{id}) = NK_n(A) = \ker(K_n(A[t]) \to K_n(A))$.

2.1.4 Known results

Next, we focus on higher K-theory of virtually cyclic groups

Theorem 2.6 (A. Kuku). *For all $n \geq 1$,$K_n(\Lambda)$ and $G_n(\Lambda)$ are finitely generated Abelian groups and hence that for any finite group G, $K_n(RG)$ and $G_n(RG)$ are finitely generated.*

见 Kuku, A.O.:K_n, SK_n of integral group-ring and orders. Contemporary Mathematics Part I, 55, 333-338 (1986) 和 Kuku, A.O.:K-theory of group-rings of finite groups over maximal orders in division algebras. J. Algebra 91, 18-31 (1984).

Using the fundamental theorem for G-theory,

$$G_n(\Lambda[t]) = G_n(\Lambda)$$

$$G_n(\Lambda[t, t^{-1}]) = G_n(\Lambda) \oplus G_{n-1}(\Lambda)$$

one gets that:

Corollary 2.7. *For all $n \geq 1$, if C is a finitely generated free Abelian group or monoid, then $G_n(\Lambda[C])$ are also finitely generated.*

Remark 2.8. However we can not draw the same conclusion for $K_n(\Lambda[C])$ since for a ring A, it is known that all the $NK_n(A)$ are not finitely generated unless they are zero. 见 Weibel, C.A.: Mayer Vietoris sequences and module structures on NK_*, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 466-493,Lecture Notes in Math., 854, Springer, Berlin, 1981 的 Proposition 4.1

2.1.5 Main results

Part 1

Theorem 2.9 (1.1)**.** *The set of all two-sided, α-invariant, Γ-lattices in Σ is a free Abelian group under multiplication and has* $\max_\alpha(\Gamma)$ *as a basis.*

Theorem 2.10 (1.6)**.** *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\alpha : \Lambda \to \Lambda$ is an R-automorphism, then there exists an R-order $\Gamma \subset \Sigma$ such that*
(1) $\Lambda \subset \Gamma$,
(2) Γ is α-invariant, and
(3) Γ is a (right) regular ring. In fact, Γ is a (right) hereditary ring.

后面证明中反复用了这里的 Γ 是一个正则环。这两个定理推广了 Farrell 和 Jones 在文章 The Lower Algebraic K-Theory of Virtually Infinite Cyclic Groups.K-Theory 9, 13-30 (1995) 中的定理 1.5 和定理 1.2

Theorem 2.11 (Farrell-Jones 文章中的定理 1.5)**.** *The set of all two-sided, α-invariant, A-lattices in $\mathbb{Q}G$ is a free Abelian group under multiplication and has* $\max_\alpha(A)$ *as a basis.*

Theorem 2.12 (Farrell-Jones 文章中的定理 1.2)**.** *Given a finite group G and an automorphism $\alpha : G \to G$, then there exists a $\mathbb{Z}$-order $A \subset \mathbb{Q}G$ such that*
(1) $\mathbb{Z}G \subset A$,
(2) A is α-invariant, and
(3) A is a (right) regular ring, in fact, A is a (right) hereditary ring.

第一节的结论来源于 Farrell 和 Jones 在其文章中的结论，将 $\mathbb{Z}$ 和 $\mathbb{Q}$ 的陈述推广到数域 F 和代数整数环 R 上，并且把之前的群环 $\mathbb{Q}G$ 推广为任何半单 F 代数 Σ。

Part 2 定理 2.1 中的方法是讲过的，关键一步是证两个范畴是自然等价。(文中有笔误:718 页第一行 mt^n 应为 xt^n, 后面所谓 m_i 应为 x_i，另有一处 Hom 所在的范畴不在 $\mathcal{B}$, 应在 $\mathcal{M}(A_\alpha[T])$)

Theorem 2.13 (2.2)**.** *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F -algebra Σ, α an automorphism of Λ. Then* *(a) For all $n \geq 0$*

(i) $NK_n(\Lambda, \alpha)$ is s-torsion for some positive integer s. Hence the torsion free rank of $K_n(\Lambda_\alpha[t])$ is the torsion free rank of $K_n(\Lambda)$ and is finite.

If $n \geq 2$, then the torsion free rank of $K_n(\Lambda_\alpha[t])$ is equal to the torsion free rank of $K_n(\Sigma)$.

(ii) If G is a finite group of order r, then $NK_n(RG, \alpha)$ is r-torsion, where α is the automorphism of RG induced by that of G.

对第一类 *virtually infinite cyclic groups* 的结论:

(b) Let $V = G \rtimes_\alpha T$ be the semi-direct product of a finite group G of order r with an infinite cyclic group $T = \langle t \rangle$ with respect to the automorphism $\alpha : G \longrightarrow G, g \mapsto tgt^{-1}$. Then

(i) $K_n(RV) = 0$ for all $n < -1$.

(ii) The inclusion $RG \hookrightarrow RV$ induces an epimorphism $K_{-1}(RG) \twoheadrightarrow K_{-1}(RV)$. Hence $K_{-1}(RV)$ is finitely generated Abelian group.

(iii) For all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group.

(iv) $NK_n(RV)$ is r-torsion for all $n \geq 0$.

第 3 节 对第二类 virtually infinite cyclic groups 的结论:

Theorem 2.14 (3.2)**.** *If R is regular, then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$. If R is quasi-regular then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \leq 0$.*

Theorem 2.15 (3.3)**.** *Let V be a virtually infinite cylic group in the second class having the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. Then the Nil-groups $NK_n(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ defined by the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ are $|H|$-torsion when $n \geq 0$ and 0 when $n \leq -1$.*

2.2 K-theory for the first type of virtually infinite cyclic groups

我们首先回顾 Farrell 和 Jones 在文章中的做法：

原型 G: finite group, $|G| = q$, $\mathbb{Z}G$ is a $\mathbb{Z}$-order in $\mathbb{Q}G$, then there exists a regular ring $A \subset \mathbb{Q}G$ which is a $\mathbb{Z}$-order, and we have[2] $qA \subset \mathbb{Z}G$.

[2]参考 Reiner, I.: Maximal Orders 中定理 41.1: $n = |G|$, Γ is an R-order in FG containing RG, then $RG \subset \Gamma \subset n^{-1}RG$ when $\mathrm{ch}(F) \nmid n$.

Hence, we have the following Cartesian square

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbb{Z}G/qA \arrow[r] & A/qA \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd} \mathbb{Z}(G \rtimes_\alpha T) \arrow[r] \arrow[d] & A_\alpha[T] \arrow[d] \\ (\mathbb{Z}G/qA)_\alpha[T] \arrow[r] & (A/qA)_\alpha[T] \end{tikzcd}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

Definition 2.16. A ring R is quasi-regular if it contains a two-sided nilpotent ideal N such that R/N is right regular.

重要的结论是

Prop1.1 If R is a (right) regular, $\alpha : R \longrightarrow R$ an automorphism, then $R_\alpha[t], R_\alpha[T] = R_\alpha[t, t^{-1}]$ are also (right) regular.

Prop1.4 $\mathbb{Z}G/qA, A/qA, (\mathbb{Z}G/qA)_\alpha[T], (A/qA)_\alpha[T]$ are all quasi-regular[3].

即得到的方块右上角是 regular ring，下方是 quasi-regular ring。于是得到 $K_n(\mathbb{Z}(G \rtimes_\alpha T)) = 0, n \leq 2$ 且有 $K_{-1}(\mathbb{Z}G) \twoheadrightarrow K_{-1}(\mathbb{Z}(G \rtimes_\alpha T))$ 是满射。

推广到数域 F 和代数整数环 R G: finite group, $|G| = s$, $\Lambda = RG$ is a R-order in $\Sigma = FG$, then there exists a regular ring $\Gamma \subset \Sigma = FG$ which is a R-order, and we have $s\Gamma \subset RG$.

Hence, we have the following Cartesian square

$$\begin{tikzcd} RG \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ RG/s\Gamma \arrow[r] & \Gamma/s\Gamma \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian

[3]If S is a quasi-regular ring, then $K_{-n}(S) = 0$.(正确不？)

square

$$\begin{CD} R(G \rtimes_\alpha T) @>>> \Gamma_\alpha[T] \\ @VVV @VVV \\ (RG/s\Gamma)_\alpha[T] @>>> (\Gamma/s\Gamma)_\alpha[T] \end{CD}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

对应到这里来$\Gamma, \Gamma_\alpha[T]$ 是正则环，$RG/s\Gamma, \Gamma/s\Gamma, (RG/s\Gamma)_\alpha[T], (\Gamma/s\Gamma)_\alpha[T]$ 是 quasi-regular rings.

群环推广到半单代数 考虑 $\Lambda \subset \Gamma \subset \Sigma$ 分别是 R-order, 正则环，半单 F-代数，则存在正整数 s 使得 $\Lambda \subset \Gamma \subset \Lambda(1/s)$, 令 $q = s\Gamma$

Hence, we have the following Cartesian square

$$\begin{CD} \Lambda @>>> \Gamma \\ @VVV @VVV \\ \Lambda/q @>>> \Gamma/q \end{CD}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{CD} \Lambda_\alpha[t] @>>> \Gamma_\alpha[t] \\ @VVV @VVV \\ (\Lambda/q)_\alpha[t] @>>> (\Gamma/q)_\alpha[t] \end{CD}$$

写出 MV 序列后每项均 $\otimes\mathbb{Z}[1/s]$ 仍然正合[4]，再分别取核得到 Nil 群的长正合列。

$\Gamma, \Gamma_\alpha[t]$ regular $\Longrightarrow NK_n(\Gamma, \alpha) = 0$.

$\Lambda/q, \Gamma/q, (\Lambda/q)_\alpha[t], (\Gamma/q)_\alpha[t]$ are all quasi-regular.

Remark 2.17. Farrell，Jones 文章中四个环是 quasi-regular 的结论证明中用到了 Artinian 性质，从而可以推广到这篇文章所讨论的情形。

一些注记：1.A: finite, $J(A)$: its Jacobson radical, why is $A/J(A)$ regular？因为是有限环 2.720 页第四行的文献应为 [16], 引用的结论为"I is a nilpotent ideal in a $\mathbb{Z}/p^m\mathbb{Z}$-algebra Λ with unit, then $K_*(\Lambda, I)$ is a p-group", 这个结论对一般的正整数 s 成立。同样地在 719 页得到序列 (III) 时同样参考 [16] 里的结论以及在 721 页倒数第 8 行所引用的 [16]Cor 3.3(d) 中的 p 对任何正整数成立。

[4] 文献 [16] 中是对素数 p 的陈述，对于一般的整数是否成立？

原文中“By [9] the torsion free rank of $K_n(\Lambda)$ is finite and if $n \geq 2$ the torsion free rank of $K_n(\Sigma)$ is the torsion free rank of $K_n(\Lambda)$ (see [12])”引用的参考文献为

[9] van der Kallen, W.: Generators and relations in algebraic K-theory, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305-310, Acad. Sci.Fennica, Helsinki, 1980

[12] Kuku, A.O.: Ranks of K_n and G_n of orders and group rings of finite groups over integers in number fields. J. Pure Appl. Algebra 138, 39-44 (1999)
但在 [9] 中并未找到相应结论。

另外文献 [10] 在网上未找到电子文档。

Open problem: What is the rank of $K_{-1}(RV)$?

2.3 Nil-groups for the second type of virtually infinite cyclic groups

范畴 $\mathcal{T}$: 对象为 $\mathbf{R} = (R; B, C)$，其中 R 是环，B, C 是 R-双模，态射为 (ϕ, f, g) : $(R, B, C) \to (S, D, E)$，其中 $\phi : R \to S$ 是环同态，$f : B \otimes_R S \to D$ 与 $g : C \otimes_R S \to E$ 是 R-S 双模同态。

$$\rho : \mathcal{T} \longrightarrow Rings$$

$$\rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(C \otimes_R B) & C \otimes_R T_R(B \otimes_R C) \\ B \otimes_R T_R(C \otimes_R B) & T_R(B \otimes_R C) \end{pmatrix}$$

If M is an R-module, then its tensor algebra $T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots$.

$$\varepsilon : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

$$NK_n(\mathbf{R}) := \ker(K_n(R_\rho) \to K_n \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix})$$

Let V be a group in the second class of the form $V = G_0 *_H G_0$ where the groups $G_i, i = 0, 1,$and H are finite and $[G_i : H] = 2$. Considering $G_i - H$ as the right coset of H in G_i which is different from H, the free $\mathbb{Z}$-module $\mathbb{Z}[Gi - H]$ with basis $G_i - H$nis a $\mathbb{Z}H$-bimodule which is isomorphic to $\mathbb{Z}H$ as a left $\mathbb{Z}H$-module, but the right action is twisted by an automorphism of $\mathbb{Z}H$ induced by an automorphism of H. Then the

Waldhausen's Nil-groups are defined to be $NK_n(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_0-H])$ using the triple $(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_0-H])$. This inspires us to consider the following general case. Let R be a ring with identity and $\alpha : R \longrightarrow R$ a ring auto-morphism. We denote by R^α the R-bimodule which is R as a left R-module but with right multiplication given by $a \cdot r = a\alpha(r)$. For any automorphisms α and β of R , we consider the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$. We will prove that $\rho(\mathbf{R})$ is in fact a twisted polynomial ring and this is important for later use.

Theorem 2.18 (3.1)**.** *Suppose that α and β are automorphisms of R. For the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$, let R_ρ be the ring $\rho(\mathbf{R})$, and let γ be a ring automorphism of $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ defined by*

$$\gamma : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} \beta(b) & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Then there is a ring isomorphism

$$\mu : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x].$$

加法群同构是显然的，只需验证乘法同态。

利用这个结论将这种形式的 Nil 群转化为 Farrell Nil 群，利用已知的命题来证明结论。如正则环的 NK_n 为 0, 拟正则环的 NK_n 当 $n \leq 0$ 时为 0.

当我们接下来研究 $R = \mathbb{Z}H, h = |H|$ 时，取一个 regular order Γ，我们有相应的 4 triples，于是得到 4 个 twisted polynomial rings $R_\rho, \Gamma_\rho; (R/h\Gamma)_\rho, (\Gamma/h\Gamma)_\rho$.

之前第二节的方块

$$\begin{array}{ccc} RG & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ RG/s\Gamma & \longrightarrow & \Gamma/s\Gamma \end{array}$$

在这里 (之前的 R, G, s 换成 $\mathbb{Z}, H, h$) 变成了 (注意这里 $R = \mathbb{Z}H$)

$$\begin{array}{ccc} R = \mathbb{Z}H & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathbb{Z}H/h\Gamma & \longrightarrow & \Gamma/h\Gamma \end{array}$$

从而有

$$\begin{tikzcd}
\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \arrow[r] \arrow[d] & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \arrow[d] \\
\begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix} \arrow[r] & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix}
\end{tikzcd}$$

接着有

$$\begin{tikzcd}
\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x] \arrow[r] \arrow[d] & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}_\gamma [x] \arrow[d] \\
\begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix}_\gamma [x] \arrow[r] & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix}_\gamma [x]
\end{tikzcd}$$

而这个方块恰好是

$$\begin{tikzcd}
R_\rho \arrow[r] \arrow[d] & \Gamma_\rho \arrow[d] \\
(R/h\Gamma)_\rho \arrow[r] & (\Gamma/h\Gamma)_\rho
\end{tikzcd}$$

证明中使用了 $n \leq -1$ 时 quasi-regular ring 的 NK_n 为 0.

Remark 2.19. 722 页中间参考文献 [3] 未找到 augmentation map。另外这里把 f, g 是双模同态在原文基础上进行了修改。

726 页第 8 行“(2) and (3)”应为“(3) and (4)”。

Chapter 3

Witt vectors and NK-groups

References:

part 1 J. P. Serre, Local fields.

part 1 Daniel Finkel, An overview of Witt vectors.

part 2 Hendrik Lenstra, Construction of the ring of Witt vectors.

part 2 Barry Dayton, Witt vectors, the Grothendieck Burnside ring, and Necklaces.

part 3 C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.

part 3 D. R. Grayson, Grothendieck rings and Witt vectors.

part 3 C. A. Weibel, The K-Book: An introduction to algebraic K-theory.

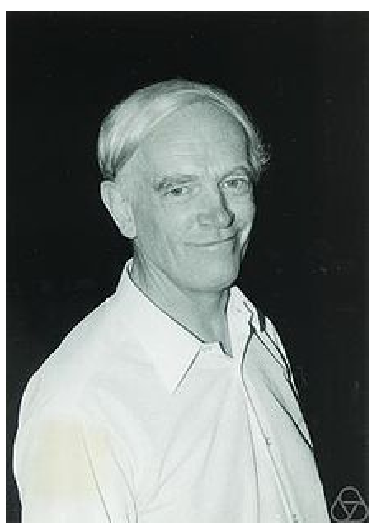

图 3.1: Ernst Witt

Ernst Witt Ernst Witt (June 26, 1911–July 3, 1991) was a German mathematician, one of the leading algebraists of his time. Witt completed his Ph.D. at the University of Göttingen in 1934 with Emmy Noether. Witt's work has been highly influential. His invention of the Witt vectors clarifies and generalizes the structure of the p-adic numbers. It has become fundamental to p-adic Hodge theory. For more information, see https://en.wikipedia.org/wiki/Ernst_Witt and http://www-history.mcs.st-andrews.ac.uk/Biographies/Witt.html.

3.1 p-Witt vectors

In this section we introduce p-Witt vectors. Witt vectors generalize the p-adics and we will see all p-Witt vectors over any commutative ring form a ring.

From now on, fix a prime number p.

Definition 3.1. A p-Witt vector over a commutative ring R is a sequence $(X_0, X_1, X_2, \cdots)$ of elements of R.

Remark 3.2. If $R = \mathbb{F}_p$, any p-Witt vector over $\mathbb{F}_p$ is just a p-adic integer $a_0 + a_1 p + a_2 p^2 + \cdots$ with $a_i \in \mathbb{F}_p$.

We introduce Witt polynomials in order to define ring structure on p-Witt vectors.

Definition 3.3. Fix a prime number p, let $(X_0, X_1, X_2, \cdots)$ be an infinite sequence of indeterminates. For every $n \geq 0$, define the n-th Witt polynomial

$$W_n(X_0, X_1, \cdots) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

For example, $W_0 = X_0,\ W_1 = X_0^p + pX_1,\ W_2 = X_0^{p^2} + pX_1^p + p^2 X_2$.

Question: how can we add and multiple Witt vectors?

Theorem 3.4. *Let $(X_0, X_1, X_2, \cdots)$, $(Y_0, Y_1, Y_2, \cdots)$ be two sequences of indeterminates. For every polynomial function $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \cdots, \varphi_n, \cdots)$ of elements of $\mathbb{Z}[X_0, \cdots, X_n, \cdots; Y_0, \cdots, Y_n, \cdots]$ such that*

$$W_n(\varphi_0, \cdots, \varphi_n, \cdots) = \Phi\big(W_n(X_0, \cdots), W_n(Y_0, \cdots)\big),\ n = 0, 1, \cdots.$$

If $\Phi = X+Y$(resp. XY), then there exist $(S_1, \cdots, S_n, \cdots)$("$S$" stands for sum) and $(P_1, \cdots, P_n, \cdots)$ ("P" stands for product) such that

$$W_n(X_0, \cdots, X_n, \cdots) + W_n(Y_0, \cdots, Y_n, \cdots) = W_n(S_1, \cdots, S_n, \cdots),$$

$$W_n(X_0, \cdots, X_n, \cdots) W_n(Y_0, \cdots, Y_n, \cdots) = W_n(P_1, \cdots, P_n, \cdots).$$

Let R be a commutative ring, if $A = (a_0, a_1, \cdots) \in R^{\mathbb{N}}$ and $B = (b_0, b_1, \cdots) \in R^{\mathbb{N}}$ are p-Witt vectors over R, we define

$$A + B = (S_0(A,B), S_1(A,B), \cdots), \quad AB = (P_0(A,B), P_1(A,B), \cdots).$$

Theorem 3.5. *The p-Witt vectors over any commutative ring R form a commutative ring under the compositions defined above (called the ring of p-Witt vectors with coefficients in R, denoted by $W(R)$).*

Example 3.6. We have

$$\begin{array}{cc} S_0(A,B) = a_0 + b_0 & P_0(A,B) = a_0 b_0 \\ S_1(A,B) = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p} & P_1(A,B) = b_0^p a_1 + a_0^p b_1 + p a_1 b_1 \end{array}$$

For more computations, see MO 92750

Theorem 3.7. *There is a ring homomorphism*

$$\begin{aligned} W_* \colon W(R) &\longrightarrow R^{\mathbb{N}} \\ (X_0, X_1, \cdots, X_n, \cdots) &\mapsto (W_0, W_1, \cdots, W_n, \cdots) \end{aligned}$$

Proof. Only need to check this map is a ring homomorphism: since

$$A + B = (S_0(A,B), S_1(A,B), \cdots), \quad AB = (P_0(A,B), P_1(A,B), \cdots),$$

by definition we have

$$\begin{aligned} W(A) + W(B) &= (W_0(A) + W_0(B), W_1(A) + W_1(B), \cdots) \\ &= \big(W_0(S_0(A,B), S_1(A,B), \cdots), W_1(S_0(A,B), S_1(A,B), \cdots), \cdots\big) \\ &= W(S_0(A,B), S_1(A,B), \cdots) = W(A+B). \end{aligned}$$

And similarly,

$$\begin{aligned}W(A)W(B) &= (W_0(A)W_0(B), W_1(A)W_1(B), \cdots) \\ &= \big(W_0(P_0(A,B), P_1(A,B), \cdots), W_1(P_0(A,B), P_1(A,B), \cdots), \cdots\big) \\ &= W(P_0(A,B), P_1(A,B), \cdots) = W(AB).\end{aligned}$$

Indeed, we only need to show $W_n(A) + W_n(B) = W_n(A+B)$ and $W_n(A)W_n(B) = W_n(AB)$ which are obviously true. (实际上就是为了使得这个是同态而定义出了 $A+B$ 和 AB。) □

Example 3.8. 1. If p is invertible in R, then $W(R) = R^{\mathbb{N}}$ — the product of countable number of R.(if p is invertible the homomorphism W_* is an isomorphism.)

2. $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$ — the ring of p-adic integers.
3. $W(\mathbb{F}_{p^n})$ is an unramified extension of the ring of p-adic integers.

Note that the functions P_k and S_k are actually only involve the variables of index $\leqslant k$ of A and B. In particular if we truncate all the vectors at the k-th entry, we can still add and multiply them.

Definition 3.9. Truncated p-Witt ring $W_k(R) = \{(a_0, a_1, \cdots, a_{k-1}) | a_i \in R\}$ (also called the ring of Witt vectors of length k).

Example 3.10. $W_1(R) = R$, $W(R) = \varprojlim W_k(R)$. Since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$.

Definition 3.11. We define two special maps as follows

- The "shift" map $V\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$, this map is *additive*.
- When $char(R) = p$, the "Frobenius" map $F\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots)$, this is indeed a ring homomorphism.

Firstly, we note that $W_k(R) = W(R)/V^kW(R)$, and if we consider $V\colon W_n(R) \hookrightarrow W_{n+1}(R)$ there are exact sequences

$$0 \longrightarrow W_k(R) \xrightarrow{V^r} W_{k+r}(R) \longrightarrow W_r(R) \longrightarrow 0, \quad \forall k,\, r.$$

The map $V\colon W(R) \longrightarrow W(R)$ is additive: for it suffices to verify this when p is invertible in R, and in that case the homomorphism $W_*\colon W(R) \longrightarrow R^{\mathbb{N}}$ transforms V

into the map which sends $(w_0, w_1, \cdots)$ to $(0, pw_0, pw_1, \cdots)$.

$$\begin{tikzcd}
W(R) \arrow[r, "V"] \arrow[d, "W_*"] & W(R) \arrow[d, "W_*"] \\
R^{\mathbb{N}} \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
(a_0, a_1, \cdots) \arrow[r, mapsto, "V"] \arrow[d, mapsto, "W_*"] & (0, a_0, a_1, \cdots) \arrow[d, mapsto, "W_*"] \\
(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \cdots) \arrow[r, mapsto] \arrow[d, equal] & (0, pa_0, pa_0^p + p^2a_1, \cdots) \arrow[d, equal] \\
(w_0, w_1, w_2, \cdots) \arrow[r, mapsto] & (0, pw_0, pw_1, \cdots)
\end{tikzcd}$$

If $x \in R$, define a map

$$\begin{aligned} r\colon R &\longrightarrow W(R) \\ x &\mapsto (x, 0, \cdots, 0, \cdots) \end{aligned}$$

When p is invertible in R, W_* transforms r into the mapping that $x \mapsto (x, x^p, \cdots, x^{p^n}, \cdots)$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\mathrm{id}"] & W(R) \arrow[d, "W_*"] \\
R \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, equal] & (x, 0, \cdots, 0, \cdots) \arrow[d, mapsto, "W_*"] \\
x \arrow[r, mapsto] & (x, x^p, \cdots, x^{p^n}, \cdots)
\end{tikzcd}$$

One deduces by the same reasoning as above the formulas:

Proposition 3.12.

$$r(xy) = r(x)r(y),\ x, y \in R$$

$$(a_0, a_1, \cdots) = \sum_{n=0}^{\infty} V^n(r(a_n)),\ a_i \in R$$

$$r(x)(a_0, \cdots) = (xa_0, x^p a_1, \cdots, x^{p^n} a_n, \cdots),\ x_i, a_i \in R.$$

Proof. The first formula: put $r(x)r(y)$, $r(xy)$ to $R^{\mathbb{N}}$, we get $(x, x^p, \cdots, x^{p^n}, \cdots)(y, y^p, \cdots, y^{p^n}, \cdots)$ and $(xy, (xy)^p, \cdots, (xy)^{p^n}, \cdots)$.
The second formula: put $(a_0, a_1, \cdots)$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2a_2, \cdots)$ consider $V^i(r(a_i))$: put $r(a_i)$ to $R^{\mathbb{N}}$, we get $(a_i, a_i^p, \cdots, a_i^{p^n}, \cdots) \in R^{\mathbb{N}}$, and W_* transforms V to the mapping $(w_0, w_1, \cdots, w_n, \cdots) \mapsto (0, pw_0, \cdots, pw_{n-1}, \cdots)$,
now we put $(r(a_0))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p, \cdots, a_0^{p^n}, \cdots)$
put $V^1(r(a_1))$ to $R^{\mathbb{N}}$, we get $(0, pa_1, \cdots, pa_1^{p^{n-1}}, \cdots)$
put $V^2(r(a_2))$ to $R^{\mathbb{N}}$, we get $(0, 0, p^2a_2, \cdots, p^2a_2^{p^{n-2}}, \cdots)$
put $V^i(r(a_i))$ to $R^{\mathbb{N}}$, we get $(\underbrace{0, 0, \cdots, 0}_{i \text{ terms}}, p^ia_i, \cdots, p^ia_i^{p^{n-i}}, \cdots)$
so put $\sum_n V^n(r(a_n))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, \cdots)$.
We leave the proof of the last formula to readers. □

Proposition 3.13.

$$VF = p = FV.$$

Proof. It suffices to check this when R is perfect. Note that a ring R of characteristic p is called perfect if $x \mapsto x^p$ is an automorphism. For more details, we refer to page 44 on Serre's book *Local Fields*. □

3.2 Big Witt vectors

Now we turn to the `big`(universal) Witt vectors. J. P, May once said "This(the theory of Witt vectors) is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once".

Take the ring of all big vectors of a commutative ring is a functor

$$\mathbf{CRing} \longrightarrow \mathbf{CRing}$$
$$R \mapsto W(R).$$

In this section, R is a commutative ring with unit.

Definition 3.14. The ring of all big Witt vectors in R which also denoted by $W(R)$ is defined as follows,
as a set: $W(R) = \{a(T) \in R[\![T]\!] | a(T) = 1 + a_1T + a_2T^2 + \cdots\} = 1 + TR[\![T]\!]$; (we note

that as a set $W(R)$ is the kernel of the map $A[\![T]\!]^* \xrightarrow{T\mapsto 0} A^*$)
addition in $W(R)$: usual multiplication of formal power series, sum $a(T)b(T)$, difference $\frac{a(T)}{b(T)}$; $(W(R),+) \cong (1+TR[\![T]\!],\times)$ which is a subgroup of the group of units $R[\![T]\!]^\times$ of the ring $R[\![T]\!]$
multiplication in $W(R)$: denoted by $*$, this is a little mysterious, we will talk the details later. For the present purposes we only define $*$ as the unique continuous functorial operation for which $(1-aT)*(1-bT) = (1-abT)$. Therefore $\prod_i(1-a_iT)*\prod_j(1-b_jT) = \prod_{i,j}(1-a_ib_jT)$.
'zero'(additive identity) of $W(R)$: 1.
'one'(multiplicative identity) of $W(R)$: $[1] = 1-T$. Note that $[1]$ is the image of $1 \in R$ under the multiplicative (Teichmuller) map

$$R \longrightarrow W(R) = 1+TR[\![T]\!]$$

$$a \mapsto [a] = 1-aT$$

functoriality: any homomorphism $f\colon R \longrightarrow S$ induces a ring homomorphism

$$W(f)\colon W(R) \longrightarrow W(S).$$

A quick way to check multiplicative formulas in $W(R)$ is to use the `ghost` map (indeed a ring homomorphism)

$$gh\colon W(R) \longrightarrow R^{\mathbb{N}} = \prod_i^\infty R.$$

It is obtained from the abelian group homomorphism

$$-T\frac{d}{dT}\log\colon (1+TR[\![T]\!])^\times \longrightarrow (TR[\![T]\!])^+$$
$$a(T) \mapsto -T\frac{a'(T)}{a(T)}$$

the right side of gh is $R^{\mathbb{N}}$ via $\sum a_nt^n \longleftrightarrow (a_1,a_2,\cdots)$.

A basic principle of the theory of Witt vectors is that to demonstrate certain equations it suffices to check them on vectors of the form $1-aT$.

3.3 Module structure on NK_*

Notations Λ: a ring with 1
R: commutative ring
$W(R)$: the ring of big Witt vectors of R
$\mathbf{End}(\Lambda)$: the exact category of endomorphisms of finitely generated projective right Λ-modules.
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules.

The fundamental theorem in algebraic K-theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus NK_i(\Lambda) \cong K_i(\Lambda) \oplus \mathrm{Nil}_{i-1}(\Lambda),$$

and hence $\mathrm{Nil}(\Lambda)$ is the obstruction to K-theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (right) Λ-module has a finite projective resolution. So the resolution theorem and the fact that G-theory is homotopy invariant show that for a regular ring, $NK_*(\Lambda) = \mathrm{Nil}_{*-1}(\Lambda) = 0$. In general, one knows that the groups $\mathrm{Nil}_*(\Lambda)$, if non-zero, are infinitely generated. It is also known that the groups $\mathrm{Nil}_*(\Lambda)$ are modules over the big Witt ring $W(R)$ (just this notes want to show you).

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.
- Computations in $W(R)$ with Grothendieck rings.

3.3.1 $\mathrm{End}_0(\Lambda)$

Let $\mathbf{End}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules.
Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.
Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ M_2 & \xrightarrow{f_2} & M_2 \end{array}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathcal{C} \rightleftarrows \mathcal{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \leftarrow\!\shortmid (M, f)$$

Definition 3.15. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\begin{aligned} \otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) &\longrightarrow \mathbf{End}(\Lambda) \\ \otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) &\longrightarrow \mathbf{Nil}(\Lambda) \\ (M, f) \otimes (N, g) &= (M \otimes_R N, f \otimes g) \end{aligned}$$

These induce (use "generators-and-relations" tricks on K_0)

$$\begin{aligned} K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) &\longrightarrow K_*(\mathbf{End}(\Lambda)) \\ K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) &\longrightarrow K_*(\mathbf{Nil}(\Lambda)) \end{aligned}$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M, f) \in \mathbf{End}(R)$, then $(M, f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R,1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R,0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R,0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R,0)\otimes -\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. (Consider $P \in \mathbf{P}(R)$ acts on $\mathbf{End}(\Lambda)$, $(P,0)\otimes(N,g) = (P\otimes_R N, 0) \in \mathbf{P}(\Lambda)$.)

The following is immediate (and well-known):

Proposition 3.16. *If Λ is an R-algebra with 1, $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$ are graded modules over the ring $\mathrm{End}_0(R)$.*

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 3.17 (Almkvist)**.** *The homomorphism (in fact it is a ring homomorphism)*

$$\begin{aligned}\chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = (1+TR[\![T]\!])^\times\\ (M,f) &\mapsto \det(1-fT)\end{aligned}$$

is injective and $\mathrm{End}_0(R) \cong \operatorname{Im}\chi = \left\{\frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1+TR[T]\right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1-T$$

χ is a ring homomorphism, and $\operatorname{Im}\chi =$ the set of all rational functions in $W(R)$. Note that

$$\det(1-fT)\det(1-gT) = \det(1-(f\oplus g)T), \quad \det(1-fT)*\det(1-gT) = \det(1-(f\otimes g)T),$$

for more details we refer the reader to S.Lang *Algebra*, Chapter 14, Exercise 15.

Remark 3.18. when R is a algebraically closed field (for instance $\mathbb{C}$), we can use Jordan canonical forms to prove the last identity (i.e. it can be reduced to check that $\prod_i(1-\lambda_iT)*\prod_j(1-\mu_jT) = \prod_{i,j}(1-\lambda_i\mu_jT)$).

Definition 3.19 (NK_*)**.** As above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in "Higher algebraic K-theory II". The map is given by

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/xy = 1) \xrightarrow{\partial} K_{n-1}(\mathbf{Nil}(\Lambda)).$$

Thus $NK_n(\Lambda)$ are $\mathrm{End}_0(R)$-modules. For $n \geq 1$, this is just 3.16; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the "contracted functor" of $NK_1(\Lambda)$.

Note that $NK_1(\Lambda) \cong K_1(\Lambda[y], (y - \lambda)), \forall \lambda \in \Lambda$, since

$$\begin{gathered} \Lambda[y] \rightleftarrows \Lambda \\ y \mapsto \lambda. \end{gathered}$$

Since $[(P,\nu)] = [(P \oplus Q, \nu \oplus 0)] - [(Q,0)] \in K_0(\mathbf{Nil}(\Lambda))$, we see $\mathrm{Nil}_0(\Lambda)$ is generated by elements of the form $[(\Lambda^n, \nu)] - n[(\Lambda, 0)]$ for some n and some nilpotent matrix ν

Sign convention:

$$\begin{aligned} NK_1(\Lambda) &\cong \mathrm{Nil}_0(\Lambda) \\ [1 - \nu y] &\leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)] \end{aligned}$$

Example 3.20. Let k be a field, $\mathbf{End}(k)$ consists pairs (V, A) with V a finite-dimensional vector space over k and A a k-endomorphism. Two pairs are isomorphic if and only if their minimal polynomials are equal. When we consider $\mathbf{Nil}(k)$, then $K_0(\mathbf{Nil}(k)) \cong \mathbb{Z}$, we conclude that $\mathrm{Nil}_0(k) = 0$. Recall that since k is a regular ring, $NK_*(k) = 0$, we have another proof of $NK_1(k) \cong \mathrm{Nil}_0(k) = 0$.

3.3.2 Grothendieck rings and Witt vectors

We refer the reader to Grayson's paper *Grothendieck rings and Witt vectors*.

Definition 3.21. A λ-ring R is a commutative ring with 1, together with an operation λ_t which assigns to each element x of R a power series

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots, \quad x \in R$$

This operation must obey $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$.

Let R is a commutative ring with unit, $K_0(R) = K_0(\mathbf{P}(R))$ becomes a λ-ring if we define

$$[M][N] = [M \otimes_R N], \quad \lambda_t^n([M]) = [\wedge_R^n M].$$

Recall $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$, $\wedge^n(M \oplus M') \cong \bigoplus_{i+j=n} \wedge^i M \otimes \wedge^j M'$, and $\wedge^n(M) := M^{\otimes n} / \langle x \otimes x \mid x \in M \rangle$, rank $\wedge^n(M) = \binom{\text{rank } M}{n}$.

For instance, if R is a field, $K_0(R) = \mathbb{Z}$ and $\lambda_t(n) = (1+t)^n = 1 + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$, since $\dim(\wedge^i R^n) = \binom{n}{i}$.

We make $K_0(\mathbf{End}(R))$ into a λ-ring by defining

$$\lambda^n([M, f]) = ([\wedge^n M, \wedge^n f]).$$

The ideal generated by the idempotent $[(R, 0)]$ is isomorphic to $K_0(R)$, the quotient $\mathrm{End}_0(R)$ is a λ-ring. It is convenient to think of End_0 as a convariant functor on the category of rings, and the functor End_0 satisfies:

1. If $R \longrightarrow S$ is surjective ring homomorphism, then $\mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(S)$ is surjective.
2. If R is an algebraically close field, then the group $\mathrm{End}_0(R)$ is generated by the elements of the form $[(R, r)]$. (This holds because any matrix over R is triagonalizable.)

Recall

$$\begin{aligned} \chi \colon \mathrm{End}_0(R) &\longrightarrow W(R) = 1 + TR[\![T]\!] \\ (M, f) &\mapsto \det(1 - fT) \end{aligned}$$

$W(R)$ is the underlying (additive) group of the ring of Witt vectors. The λ-ring operations on $W(R)$ are the unique operations which are continuous, functorial in R, and satisfy:

$$\begin{aligned} (1 - aT) * (1 - bT) &= 1 - abT \\ \lambda_t(1 - aT) &= 1 + (1 - aT)t \end{aligned}$$

By 3.17, χ is an injective ring homomorphism whose image consists of all Witt vectors which are quotients of polynomials. In fact χ is a λ-ring homomorphism, so we have

Theorem 3.22. $\mathrm{End}_0(R)$ *is dense sub-λ-ring of* $W(R)$.

The hard part of the theorem is the injectivity. When R is a field the injectivity follows immediately from the existence of the rational canonical form (we can see it below) for a matrix. The result is surprising when R is not a field.

Computation in $W(R)$ Computation in $W(R)$ which is tedious unless we perform it in $\mathrm{End}_0(R)$:

$$(1-aT^2)*(1-bT^2)=?$$

Note that $\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\Big)=\det\begin{pmatrix}1 & -aT\\ -T & 1\end{pmatrix}=1-aT^2, \chi\Big(\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\det\begin{pmatrix}1 & -bT\\ -T & 1\end{pmatrix}=1-bT^2$,

$$\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\Big)*\chi\Big(\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\chi\Big(\begin{pmatrix}0 & 0 & 0 & ab\\ 0 & 0 & b & 0\\ 0 & a & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\Big)$$

$$=\det\begin{pmatrix}1 & 0 & 0 & -aTb\\ 0 & 1 & -bT & 0\\ 0 & -aT & 1 & 0\\ -T & 0 & 0 & 1\end{pmatrix}=1-2abT^2+a^2b^2T^4.$$

If we use the previous formula

$$(1-rT^m)*(1-sT^n)=(1-r^{n/d}s^{m/d}T^{mn/d})^d,\ d=\gcd(m,n),$$

we obtain the same answer. Indeed we have the formula

$$\det(1-fT)*\det(1-gT)=\det(1-(f\otimes g)T),$$

if a polynomial is $1+a_1T+\cdots+a_nT^n\in W(R)$, we can write

$$f=\begin{pmatrix}0 & & & & -a_n\\ 1 & 0 & & & -a_{n-1}\\ & \ddots & \ddots & & \vdots\\ & & 1 & 0 & -a_2\\ & & & 1 & -a_1\end{pmatrix}\in M_n(R).$$

Operations on $W(R)$ and $\mathrm{End}_0(R)$ We have already known that W and End_0 can be regarded as functors from the category of commutative rings to that of (λ-)rings. The following operations $F_n,\ V_n\colon W \Longrightarrow W$(resp. $\mathrm{End}_0 \Longrightarrow \mathrm{End}_0$) are indeed natural transformation. These auxiliary operations defined on $W(R)$ can also be computed in $\mathrm{End}_0(R)$.

1. the ghost map

$$gh\colon W(R) \xrightarrow{-T\frac{d}{dT}\log} TR[\![T]\!] \cong R^{\mathbb{N}}, \quad \alpha(T) \mapsto \frac{-T}{\alpha(T)}\frac{d\alpha}{dT}.$$

and the n-th ghost coordinate

$$gh_n\colon W(R) \longrightarrow R$$

it is the unique continuous natual additive map which sends $1 - aT$ to a^n.
Remark. $gh(1-aT) = \frac{aT}{1-aT} = \sum_{i=1} a^i T^i$.The exponential map is defined by

$$R^{\mathbb{N}} \longrightarrow W(R), \quad (r_1, \cdots) \mapsto \prod_{i=1}^{\infty} \exp(\frac{-r_i T^i}{i}).$$

2. the Frobenius endomorphism

$$F_n\colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}}).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - a^n T$.
Remark. $F_n(1-aT) = \sum_{\zeta^n=1}(1 - a\zeta T^{\frac{1}{n}}) = 1 - a^n T$, since "$+$" in $W(R)$ is the normal product.

3. the Verschiebung endomorphism

$$V_n\colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \alpha(T^n).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - aT^n$.

ghost map $gh_n\colon W(R) \longrightarrow R$	$1 - aT \mapsto a^n$	
Frobenius endomorphism $F_n\colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - a^n T$	$\alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}})$
Verschiebung endomorphism $V_n\colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - aT^n$	$\alpha(T) \mapsto \alpha(T^n)$

We define similar operations on $\mathrm{End}_0(R)$ as follows:

$gh_n\colon \mathrm{End}_0(R) \longrightarrow R$	$[(M,f)] \mapsto \mathrm{tr}(f^n)$
$F_n\colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M,f)] \mapsto [(M,f^n)]$
$V_n\colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M,f)] \mapsto [(M^{\oplus n}, v_n f)]$

where $v_n f$ is represented by $\begin{pmatrix} 0 & & & f \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$. The matrix $v_n f$ is close to an n-th root of f. Another equivalent description is

$$V_n\colon [(M, f)] \mapsto [(M[y]/y^n - f, y)].$$

One easily checks that the operations just defined are additive with respect to short exact sequences in $\mathbf{End}(R)$, and thus are well-defined on $\mathrm{End}_0(R)$.

Since $\mathrm{End}_0(R) \subset W(R)$ is dense and $gh_n,\ F_n,\ V_n$ are continuous, identities among them may be verified on $W(R)$ by checking them on $\mathrm{End}_0(R)$.

$W(R)$	$\longleftrightarrow$	$\mathrm{End}_0(R)$
$gh_n(v * w) = gh_n v * gh_n w$		$\mathrm{tr}((f \otimes g)^n) = \mathrm{tr}(f^n)\mathrm{tr}(g^n)$
$F_n(v * w) = F_n v * F_n w$		$(f \otimes g)^n = f^n \otimes g^n$
$F_n V_n = n$		$(v_n f)^n = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$
$gh_n V_d(v) = \begin{cases} d\, gh_{n/d}(v), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$		$\mathrm{tr}((v_d f)^n) = \begin{cases} d\, \mathrm{tr}(f^{n/d}), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$

From the last row we may recover the usual expression of the ghost coordinates in terms of the Witt coordinates . The Witt coordinates of a vector v are the coefficients in the expression

$$v = \prod_{i=1}(1 - a_i T^i) = \prod_{i=1} V_i(1 - a_i T).$$

We obtain

$$gh_n(v) = \sum_{d|n} d a_d^{n/d}.$$

"Many mordern treatments of the subject of Witt vectors take this latter expression as the starting point of the theory."

The logarithmic derivative of $1 - a_d T^d$ is $\frac{d}{dT}\log(1 - a_d T^d) = -\sum_{m=1}^{\infty} d a_d^m T^{dm-1}$, and $-T\frac{d}{dT}\log(1 - a_d T^d) = \sum_{n=1}^{\infty} gh_n(1 - a_d T^d)T^n$. So we obtain the formula:

$$-Tv^{-1}\frac{dv}{dT} = \sum_{n=1}^{\infty} gh_n(v)T^n$$

which yields the exponential trace formula:

$$-T\chi([M,f])^{-1}\frac{d\chi([M,f])}{dT} = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n.$$

For example, when rank $M = 2$, we have $\mathrm{tr}(f^2) = (\mathrm{tr}(f))^2 - 2\det(f)$, note that $\det(1 - fT) = 1 - \mathrm{tr}(f)T + \det(f)T^2$.

Remark 3.23. When R is a field, the exponential trace formula

$$-T\frac{d}{dT}\log\det(1 - fT) = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n$$

can be checked by $\det(1 - fT) = \prod(1 - \lambda_i T)$ where λ_i are eigenvalues. And we also have

$$\det(1 - fT) = \exp(\sum_{n=1}^{\infty} -\mathrm{tr}(f^n)\frac{T^n}{n}),$$

since $\prod(1 - \lambda_i T) = \exp\big(\ln(\prod(1 - \lambda_i T))\big) = \exp\big(\sum \ln(1 - \lambda_i T)\big)$ and recall that formally $\ln(1 - x) = -\sum \frac{x^n}{n}$.

3.3.3 $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Recall Λ is an R-algebra, where R is a commutative ring with unit. We define a map

$$\begin{aligned} \mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) &\longrightarrow \mathrm{Nil}_0(\Lambda) \\ (R^n, f) * [(P, \nu)] &= [(P^n, f\nu)] \end{aligned}$$

Let $\alpha_n = \alpha_n(a_1, \cdots, a_n)$ denote the $n \times n$ matrix (looks like the rational canonical form) over R:

$$\alpha_n(a_1, \cdots, a_n) = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

Recall

$$\begin{aligned} \chi\colon \mathrm{End}_0(R) &\rightarrowtail W(R) \\ (R^n, \alpha_n) &\mapsto \det(1 - \alpha_n T) \end{aligned}$$

we obtain

$$\det\begin{pmatrix} 1 & & & & a_nT \\ -T & 1 & & & a_{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2T \\ & & & -T & 1+a_1T \end{pmatrix} = 1 + a_1T + \cdots + a_nT^n$$

(Computation methods: 1. the traditional computation. 2. Note that if A is invertible,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

In this case $A^{-1} = \begin{pmatrix} 1 & & & & \\ T & 1 & & & \\ \vdots & \ddots & \ddots & & \\ T^{n-3} & \cdots & T & 1 & \\ T^{n-2} & T^{n-3} & \cdots & T & 1 \end{pmatrix}$)

Then we can also conclude that $\operatorname{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$.

Remark 3.24. Why is a general elment of the form (R^n, α_n)? Namely how to reduce an endomorphism to a rational canonical form ?

Now we want to check some identities

$$\begin{gathered} \mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda) \\ (R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n \nu)] \quad \text{by definition} \\ (R^{n+1}, \alpha_{n+1}(a_1, \cdots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)] \quad \text{compute under } \chi \\ (R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)] \quad \text{where } \beta = \alpha_n(a_1\nu, \cdots, a_n\nu^n) \end{gathered}$$

In fact, the last identity always holds when $R = \mathbb{Z}[a_1, \cdots, a_n]$. β is nilpotent because $\beta = \alpha_n \nu$.

We only show how to check the last equation: only need to show that

$$\alpha_n \nu = \alpha_n(a_1\nu, \cdots, a_n\nu^n)$$

$$LHS = \begin{pmatrix} 0 & & & & -a_n\nu \\ \nu & 0 & & & -a_{n-1}\nu \\ & \ddots & \ddots & & \vdots \\ & & \nu & 0 & -a_2\nu \\ & & & \nu & -a_1\nu \end{pmatrix}, \quad RHS = \begin{pmatrix} 0 & & & & -a_n\nu^n \\ 1 & 0 & & & -a_{n-1}\nu^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2\nu^2 \\ & & & 1 & -a_1\nu \end{pmatrix}$$

we can check this using the characteristic polynomial since χ is injective: check

$$\det(1-\alpha_n\nu T) = \det(1-\alpha_n(a_1\nu,\cdots,a_n\nu^n)T)$$

$$LHS = \det\begin{pmatrix} 1 & & & & a_n\nu T \\ -\nu T & 1 & & & a_{n-1}\nu T \\ & \ddots & \ddots & & \vdots \\ & & -\nu T & 1 & a_2\nu T \\ & & & -\nu T & 1+a_1\nu T \end{pmatrix} = \det(1+a_1\nu T+\cdots+a_n\nu^nT^n)$$

$$RHS = \det\begin{pmatrix} 1 & & & & a_n\nu^n T \\ -T & 1 & & & a_{n-1}\nu^{n-1} T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2\nu^2 T \\ & & & -T & 1+a_1\nu T \end{pmatrix} = \det(1+a_1\nu T+\cdots+a_n\nu^nT^n).$$

Note that if $\exists N$ such that $\nu^N = 0$, β is independent of the a_i for $i \geq N$. If $\nu^N = 0$ then $\alpha_n \otimes \nu$ represents 0 in $\mathrm{Nil}_0(\Lambda)$ whenever $\chi(\alpha_n) \equiv 1 \bmod t^N$.

More operations Let $F_n\mathbf{Nil}(\Lambda)$ denote the full exact subcategory of $\mathbf{Nil}(\Lambda)$ on the (P,ν) with $\nu^n = 0$. If Λ is an algebra over a commutative ring R, the kernel $F_n\mathrm{Nil}_0(\Lambda)$ of $K_0(F_n\mathbf{Nil}(\Lambda)) \longrightarrow K_0(\mathbf{P}(\Lambda))$ is an $\mathrm{End}_0(R)$-module and $F_n\mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$ is a module map.

The exact endofunctor $F_m\colon (P,\nu) \mapsto (P,\nu^m)$ on $\mathbf{Nil}(\Lambda)$ is zero on $F_m\mathbf{Ni1}(\Lambda)$. For $\alpha \in \mathrm{End}_0(R)$ and $(P,\nu) \in \mathrm{Nil}_0(\Lambda)$, note that $(V_m\alpha) * (P,\nu) = V_m(\alpha * F_m(P,\nu))$, and we can conclude that $V_m\mathrm{End}_0(R)$ acts trivially on the image of $F_m\mathrm{Nil}_0(\lambda)$ in $\mathrm{Nil}_0(\lambda)$. For more details, see Weibel, K-book chapter 2, pp 155 Exercise II.7.17.

3.3.4 $W(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Once we have a nilpotent endomorphism, we can pass to infinity.

Theorem 3.25. $\mathrm{End}_0(R)$*-module structure on* $\mathrm{Nil}_0(\Lambda)$ *extends to a* $W(R)$*-module structure by the formula*

$$(1+\sum a_i T^i) * [(P,\nu)] = (R^n, \alpha_n(a_1,\cdots,a_n)) * [(P,\nu)],\ n \gg 0.$$

3.3.5 $W(R)$-module structure on $\mathrm{Nil}_*(\Lambda)$

The induced t-adic topology on $\mathrm{End}_0(R)$ is defined by the ideals

$$I_N = \{f \in \mathrm{End}_0(R) \mid \chi(f) \equiv 1 \bmod t^N\},\ I_N \supset I_{N+1},$$

and $\mathrm{End}_0(R)$ is separated (i.e. $\cap I_N = 0$) in this topology. The key fact is:

Theorem 3.26 (Almkvist)**.** *The map* $\chi\colon \mathrm{End}_0(R) \longrightarrow W(R)$ *is a ring injection, and* $W(R)$ *is the t-adic completion of* $\mathrm{End}_0(R)$*, i.e.* $W(R) = \varprojlim \mathrm{End}_0(R)/I_N$*.*

Theorem 3.27 (Stienstra)**.** *For every* $\gamma \in \mathrm{Nil}_*(\Lambda)$ *there is an* N *so that* γ *is annihilated by the ideal*

$$I_N = \{f \mid \chi(f) \equiv 1 \bmod t^N\} \subset \mathrm{End}_0(R).$$

Consequently, $NK_*(\Lambda)$ *is a module over the t-adic completion* $W(R)$ *of* $\mathrm{End}_0(R)$*.*

Recall the sign convention:

$$\begin{aligned} NK_1(\Lambda) &\cong \mathrm{Nil}_0(\Lambda) \\ [1-\nu y] &\leftrightarrow [(\Lambda^n,\nu)] - n[(\Lambda,0)] \end{aligned}$$

The $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula

$$\alpha(t) * [1-\nu y] = [\alpha(\nu y)].$$

And the $W(R)$-module structure on $NK_n(\Lambda)$

$$\alpha(t) * \{\gamma, 1-\nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda), \quad \gamma \in K_{n-1}(R).$$

3.3.6 Modern version

Reference: Weibel, K-book, chapter 4, pp. 58.

3.4 Some results

Proposition 3.28. *If R is $S^{-1}\mathbb{Z}, \hat{\mathbb{Z}}_p$ or $\mathbb{Q}$-algebra, then*

$$\begin{aligned} \lambda_t \colon R &\longrightarrow W(R) \\ r &\mapsto (1-t)^r \end{aligned}$$

is a ring injection.

Corollary 3.29. *Fix an integer p and a ring Λ with 1.*
(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.
(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ)-module.
(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.
(d) If $p^m = 0$ in Λ, $NK_*(\Lambda)$ is a p-group.

Theorem 3.30 (Stienstra)**.** *If $0 \neq n \in \mathbb{Z}$, $NK_1(R)[\frac{1}{n}] \cong NK_1(R[\frac{1}{n}])$.*

Corollary 3.31. [1] *If G is a finite group of order n, then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n. In fact, $NK_*(\mathbb{Z}[G])$ is an n-torsion group, and $Z_{(p)} \otimes NK_*(\mathbb{Z}[G]) = NK_*(\mathbb{Z}_{(p)}[G])$, where $p \mid n$.*

[1] Weibel, K-book chapter3, page 27.

Chapter 4

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [51].

4.1 Outline

Definition 4.1 (Bass *Nil*-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \xrightarrow{x\mapsto 0} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty} \mathbb{F}_2 x^i = \oplus_{i=1}^{\infty} \mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\, dx = \oplus_{i=1}^{\infty} \mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\, dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

4.2 Preliminaries

4.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 4.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

4.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[\![x]\!])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$\begin{aligned}
[a]&\colon \alpha(x) \mapsto \alpha(ax) \\
V_m&\colon \alpha(x) \mapsto \alpha(x^m) \\
F_m&\colon \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m&\colon 1 - rx \mapsto 1 - r^m x
\end{aligned}$$

Remark 4.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [7].

Proposition 4.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$\begin{aligned} V_m V_n &= V_{mn} \\ F_m F_n &= F_{mn} \\ F_m V_n &= m \\ [a] V_m &= V_m [a^m] \\ F_m [a] &= [a^m] F_m \\ [a][b] &= [ab] \\ V_m F_k &= F_k V_m, \textit{ if } (k, m) = 1 \end{aligned}$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1 - rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[\![x]\!]$ is not.

4.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv) h_{12}(u)^{-1} h_{12}(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u) w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u) x_{ji}(-u^{-1}) x_{ij}(u)$.s

These symbols satisfy

(a) $\{u_1 u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1 - u\} = 1$ for $u, 1 - u \in R^*$.

Theorem 4.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol **version 1** If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem*.

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b + c + abc\rangle$,

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$.
If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a, b\rangle \in K_2(R, I)$ satisfy following relations
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b + c + abc\rangle$,

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$ if any of a, b, c are in I.

Theorem 4.6. *1. If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R, I) \overset{\cong}{\to} K_2(R, I)$.

Dennis-Stein symbol **version 2** In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r, s \in R$ commute and $1 - rs$ is a unit, that is $rs = sr$ and $1 - rs \in R^*$,

$$\langle r, s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r, s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r, s\rangle \in K_2(R, I)$

(D1) $\langle r, s\rangle\langle s, r\rangle = 1$,

(D2) $\langle r, s\rangle\langle r, t\rangle = \langle r, s + t - rst\rangle$,

(D3) $\langle r, st\rangle = \langle rs, t\rangle\langle tr, s\rangle$ (this holds in $K_2(R, I)$ if any of r, s, t are in I).

Note that $\langle r, 1\rangle = 0$ for any $r \in R$ and $\langle r, s\rangle_{version2} = \langle -r, s\rangle_{version1}$.

Theorem 4.7. 1. *If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r, s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r, s\}$.*

2. *Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R, I)$ is generated by $\langle r, s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively generated by its units.*

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r, s\rangle$, $r, s \in R, 1 - rs \in R^*$ or by all $\{u, v\}$, $u, v \in R^*$.*

4.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 4.4.

excision 失效就是说 if $A \longrightarrow B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A, I) \longrightarrow K_n(B, I)$ need not be an isomorphism. 由于这个不是同构，没法有 Mayer-Vietoris 序列

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(A/I) \arrow[r] \arrow[d] & K_i(A, I) \arrow[r] \arrow[d] & K_i(A) \arrow[r] \arrow[d] & K_i(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(B/I) \arrow[r] & K_i(B, I) \arrow[r] \arrow[u, dashed, shift right] & K_i(B) \arrow[r] & K_i(B/I) \arrow[r] & \cdots
\end{tikzcd}$$

要连接 $K_n(A,I) \longrightarrow K_n(B,I)$ 就要考虑 birelative K-groups (也称 double relative K-groups) ,$K(A,B,I)$ 定义为 homotpy fiber of the map $K(A,I) \longrightarrow K(B,I)$。以下是详细的定义和性质。

Relative groups Let R be a ring (not necessarily commutative), $I \subset R$ a two-sided ideal, by definition $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$, there exists a map

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

Definition 4.8. $K(R,I)$ is the homotopy fibre of the map $BGL(R)^+ \longrightarrow BGL(R/I)^+$. $K_i(R,I) := \pi_i(K(R,I)), i \geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots \longrightarrow K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R,I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots .$$

In particular,

$$\begin{aligned} K_3(R,I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R,I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow \\ \longrightarrow K_1(R,I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \end{aligned}$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals, there is a map

$$K(R,I) \longrightarrow K(R/J, I+J/J).$$

If $I \cap J = 0$, then $I + J/J \cong J/I \cap J = J$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"] & R/I \arrow[d, "g"] \\ R/J \arrow[r, "f"] & R/I+J \end{tikzcd}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(4.8)

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(R) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & K_{i+1}(R/I) \arrow[r, "\partial"] \arrow[d, "g_*"] & K_i(R,I) \arrow[r, "j"] \arrow[d, "\varepsilon_i"] & K_i(R) \arrow[r, "\alpha_*"] \arrow[d] & K_i(R/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(R/J) \arrow[r, "f_*"] & K_{i+1}(R/I+J) \arrow[r, "\partial"] & K_i(R/J, I+J/J) \arrow[r, "j'"] & K_i(R/J) \arrow[r, "f_*"] & K_i(R/I+J) \arrow[r] & \cdots
\end{tikzcd}$$

where the induced homomorphism

$$\varepsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the MayerVietoris sequence

$$\begin{aligned} K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 4.9. 1. *Suppose that the excision map ε_i in 4.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. *If ε_i is an isomorphism, and in addition ε_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is*

$$\begin{aligned} K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

3. *Suppose instead that ε_i is surjective, and let $L = ker(\varepsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, and the sequence*

$$K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow$$
$$\longrightarrow K_i(R)/L \longrightarrow K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J)$$

is exact.

Proof. Define $\delta_i = j\varepsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. □

Remark 4.10. It is known that ε_0 and ε_1 are isomorphism regardless of the specific rings. Moreover Swan [42] has shown that ε_2 cannot be an isomorphism in general. For more discussion, see [40].

For $R = \mathbb{Z}[C_n]$, $I = (\sigma - 1)$, $J = (1 + \sigma + \cdots + \sigma^{n-1})$, $I \cap J = 0$, $I + J = (n, \sigma - 1)$, $\varepsilon_2 \colon K_2(R, I) \longrightarrow K_2(R/J, I + J/J)$ is isomophic.

Double relative groups

Definition 4.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I + J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd}
K(R;I,J) \arrow[d, dashed] & & \\
K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\
K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+
\end{tikzcd}$$

Remark 4.12. $K_i(R; I, J) \cong K_i(R; J, I)$, $K_i(R; I, I) = K_i(R, I)$.

We have a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R, I) \longrightarrow K_{i+1}(R/J, I + J/J) \longrightarrow K_i(R; I, J) \longrightarrow K_i(R, I) \longrightarrow K_i(R/J, I + J) \longrightarrow \cdots.$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R, I) \longrightarrow K_3(R/I, I + J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R, I) \longrightarrow K_2(R/I, I + J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [52] 3.5.10, [40], [25] or [10] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [17] theorem 2.

Remark 4.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [17].

Theorem 4.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b \rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [17] theorem 3. □

Lemma 4.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I + J \end{array}$$

suppose $f \colon (R, I) \longrightarrow (R/J, I + J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I + J/J) \longrightarrow 0$$

is split exact.

4.3 $W(R)$-module structure

$W(\mathbb{F}_2)$-module structure on $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [7] example 2.6, 2.9.

$$V_m(x^n) = x^{mn}$$

$$F_d(x^n) = \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases}$$

$$[a]x^n = a^n x^n$$

$W(\mathbb{F}_2)$**-module structure on** $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [7]example 2.10

$$
\begin{aligned}
V_m(x^{n-1}\,dx) &= m x^{mn-1}\,dx \\
F_d(x^{n-1}\,dx) &= \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases} \\
[a]x^{n-1}\,dx &= a^n x^{n-1}\,dx
\end{aligned}
$$

Remark 4.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 4.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$
\begin{aligned}
V_m(x^i) &= x^{im}, \\
V_m(x^{i-1}dx) &= m x^{im-1}dx
\end{aligned}
$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 4.18. 一个不知道有没有用的结论, see [7]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$
\begin{aligned}
D\colon x\mathbb{F}_2[x] &\longrightarrow \Omega_{\mathbb{F}_2[x]} \\
x^i &\mapsto i x^{i-1}dx
\end{aligned}
$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\operatorname{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty}\mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

暂时没用：$D_{1,R} = \langle da \mid a \in \mathbb{F}_2[x]\rangle$. $a = \sum a_i x^i$, $da = \sum i a_i x^{i-1}dx$

4.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2 \end{tikzcd} \tag{4.18}$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$\begin{tikzcd} NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\ 0 & & 0 & 0 & & 0 \end{tikzcd}$$

$$\ker(\mathbb{Z}[C_2] \xrightarrow{\sigma\mapsto -1} \mathbb{Z}) = (\sigma + 1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \xrightarrow{\cong} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$\begin{tikzcd}
 & NK_3(\mathbb{Z},(2))=0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1)) \arrow[d, "\cong"] & & \\
0=NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2],(\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z})=0 \\
 & NK_2(\mathbb{Z},(2))=0 & &
\end{tikzcd}$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [17], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1), \sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1 - x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 4.19. $NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 4.20 ($\mathbb{Z}[C_p]$). $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd}
\mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "\sigma\mapsto 1"', "f"] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\
\mathbb{Z} \arrow[r] & \mathbb{F}_p
\end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \longrightarrow K_2(R,I)$$
$$(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) \mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1, 1\rangle^p = 1$$

Also see [40].

Example 4.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

4.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $\mathbf{F}_2[\varepsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{ccccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\varepsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\varepsilon \end{array}$$

Lemma 4.22. *The map $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]$ in 4.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\varepsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [51] Lemma 1.2. □

Theorem 4.23.

$$NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]},$$

$$NK_0(\mathbb{Z}[D_2]) \cong V,$$

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 4.18 with $\mathbb{Z}[C_2]$. Note that $R[G_1 \times G_2] = R[G_1][G_2]$, for commutative R, $R[G_1 \times G_2] = R[G_1] \otimes R[G_2]$, $\sum_{g,h} c_g c_h g \otimes h \leftarrow \sum_{g,h} c_g g \otimes c_h h$. As for infinite product, see MO 46950.

$$\begin{tikzcd} \mathbb{Z}[D_2] = \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] \arrow[r] \arrow[d] & \mathbb{Z}[C_2] \arrow[d, two heads, "q"] \\ \mathbb{Z}[C_2] \arrow[r, two heads, "q"] & \mathbb{F}_2[C_2] \end{tikzcd} \tag{4.23}$$

Recall that $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]/(\varepsilon^2)$. By [52] chapter 2 Ex 7.4.5,

$$NK_1(\mathbb{F}_2[C_2]) = NK_1(\mathbb{F}_2[\varepsilon]/(\varepsilon^2)) = (1 + \varepsilon x \mathbb{F}_2[\varepsilon]/(\varepsilon^2)[x])^\times = (1 + \varepsilon x \mathbb{F}_2[x])^\times \cong V = x\mathbb{F}_2[x]$$
$$[(P, \nu)] \mapsto \det(1 - \nu x)$$

Remark 4.24. $(1 + \varepsilon x \mathbb{F}_2[x])^\times \cong x\mathbb{F}_2[x]$, $1 + \varepsilon x \sum a_i x^i \mapsto x \sum a_i x^i$ 的原因，左边是乘法群，右边的乘法是普通的多项式相加，左边 $(1 + \varepsilon x \sum a_i x^i)(1 + \varepsilon x \sum b_j x^j) = 1 + \varepsilon x \sum a_i x^i + \varepsilon x \sum b_j x^j + (\varepsilon x \sum a_i x^i)(\varepsilon x \sum b_j x^j) = 1 + \varepsilon x(\sum a_i x^i + \sum b_j x^j)$，右边 $x \sum a_i x^i + x \sum b_j x^j = x(\sum a_i x^i + \sum b_j x^j)$.

As $W(\mathbb{F}_2)$-modules,

$$1 + \varepsilon x(a_0 + a_1 x + \cdots + a_n x^n) \mapsto a_0 x + a_1 x^2 + \cdots + a_n x^{n+1}$$

and we can easily check that

$$V_m(1 + \varepsilon x(a_0 + a_1 x + \cdots + a_n x^n)) = 1 + \varepsilon x^m(a_0 + a_1 x^m + \cdots + a_n x^{mn})$$

$$[a](1 + \varepsilon x(a_0 + a_1 x + \cdots + a_n x^n)) = 1 + \varepsilon a x(a_0 + a_1 a x + \cdots + a_n a^n x^n)$$

hence the module structure of $(1 + \varepsilon x \mathbb{F}_2[x])^\times$ are the same as V.

By MayerVietoris sequence for the NK-functor, one has

$$
\begin{tikzcd}
& NK_2(\mathbb{Z}[D_2]) \arrow[r] & NK_2\mathbb{Z}[C_2] \oplus NK_2\mathbb{Z}[C_2] \arrow[r, "q\times q"] & NK_2(\mathbb{F}_2[C_2]) \arrow[dlll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\arrow[r] & NK_1(\mathbb{Z}[D_2]) \arrow[r] & NK_1\mathbb{Z}[C_2] \oplus NK_1\mathbb{Z}[C_2] = 0 \arrow[r] & NK_1(\mathbb{F}_2[C_2]) \arrow[dlll, "\cong"] \\
\arrow[r] & NK_0(\mathbb{Z}[D_2]) \arrow[r] & NK_0\mathbb{Z}[C_2] \oplus NK_0\mathbb{Z}[C_2] = 0 &
\end{tikzcd}
$$

Hence $NK_0(\mathbb{Z}[D_2]) \cong V$, $NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}\,(q\times q) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}\,q \cong \Omega_{\mathbb{F}_2[x]}$ since $\mathrm{Im}\,(q\times q) = \mathrm{Im}\,q$. $\square$

最后一个论断，若对则有 $(q\times q)(\Phi(V)\times V) = 0$，然而这个等式是不成立的。

4.5.1 A result from the K-book

For the convenience of the reader we copy [52] chapter 2 Ex 7.4.5 as follows.

Let R ba a commutative regular ring, $A = R[x]/(x^N)$, we claim that

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$
\begin{aligned}
\mathrm{Nil}_0(A) &\cong (1 + xtA[t])^\times \\
[(A,x)] &\mapsto 1 - xt \\
[(P,\nu)] &\mapsto \det(1 - \nu t)
\end{aligned}
$$

the isomorphism $NK_1(A) \cong \mathrm{Nil}_0(A) \cong (1 + xtA[t])^\times$ is universal in the following sense:

Let B be a R-algebra, $(P,\nu) \in \mathbf{Nil}(B)$ with $\nu^N = 0$, regard P as an A-B-bimodule

$$
\begin{aligned}
\mathrm{Nil}_0(A) &\longrightarrow \mathrm{Nil}_0(B) \\
(A,x) &\mapsto (P,\nu)
\end{aligned}
$$

there is an $\mathrm{End}_0(R)$-module homomorphism

$$
\begin{aligned}
(1 + xtA[t])^\times &\longrightarrow \mathrm{Nil}_0(B) \\
1 - xt &\mapsto [(P,\nu)].
\end{aligned}
$$

4.5.2 About the lemma

In this subsection, we concentrate on the lemma 4.22.

For a complete proof, see [46].

4.6 NK_i of the group C_4

4.7 NK_i of the group D_4

Chapter 5

Lower Bounds for the Order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$

2016.3.19 阅读这篇文章 [39] 1976 年发表在 *Math. Ann.*。

基本假设：p: rational prime, G: elementary abelian p-group.

用的方法：Bloch; van der Kallen K_2 of truncated polynomial rings

结论：the p-rank of $K_2(\mathbb{Z}G)$[1] grows expotentially with the rank of G.

$Wh_2(G)$: "pseudo-isotopy" group is nontrivial if G has rank at least 2.

这篇文章之前已知的结论 (exact computations) Dunwoody, G cyclic of order 2 or 3, $K_2(\mathbb{Z}G)$ is an elementary abelian 2-group of rank 2 if G has order 2 and of rank 1 if G has order 3. 两者都有 $Wh_2(G)$ 平凡。

一些记号和基本结论 R commutative ring, A a subring of R. $\Omega^1_{R/A}$ the module of Kähler differentials of R considerd as an algebra over A and R^* will denote the group of units of R.

the p-rank of an abelian group G is $\dim_{\mathbb{F}_p}(G \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

elementary abelian p-groups An elementary abelian p-group is an abelian group in which every nontrivial element has order p. The number p must be prime, and the elementary abelian groups are a particular kind of p-group. The case where $p = 2$, i.e., an elementary abelian 2-group, is sometimes called a Boolean group.

结构：Every elementary abelian p-group is a vector space over the prime field $\mathbb{F}_p$ with p elements, and conversely every such vector space is an elementary abelian

[1] this is a finite group

group.

By the classification of finitely generated abelian groups, or by the fact that every vector space has a basis, every finite elementary abelian group must be of the form $(\mathbb{Z}/p\mathbb{Z})^n$ for n a non-negative integer (sometimes called the group's rank). Here, $C_p = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p.

In general, a (possibly infinite) elementary abelian p-group is a direct sum of cyclic groups of order p. (Note that in the finite case the direct product and direct sum coincide, but this is not so in the infinite case.)

5.1 Part 1

环是 $\mathbb{F}_q$ 有限域的情况。

先说结论

首先是一个奇素数的结论

Proposition 5.1. *Let $q = p^f$ be odd and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary p-group of rank $f(n-1)(p^n-1)$.*

接着是素数 2 的结论

Proposition 5.2. *Let $q = 2^f$ be odd and let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary 2-group of rank $f(n-1)(2^n-1)$.*

结论实际上是可以统一的，但是方法有些区别，因此原文中分开表述。

我们引进方法时借鉴了 van der Kallen 的方法和记号

Let R be a commutative ring. The abelian group $TD(R)$ is the universal R-module having generators $Da, Fa, a \in R$, subject to the relations

$$\begin{aligned} D(ab) &= aDb + bDa, \\ D(a+b) &= Da + Db + F(ab), \\ F(a+b) &= Fa + Fb, \\ Fa &= D(1+a) - Da. \end{aligned}$$

There is a natural surjective homomorphism of R-modules

$$TD(R) \twoheadrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 1$$

whose kernel is the submodule of $TD(R)$ generated by the $Fa, a \in R$. Relations imply

$$F(c^2a) = cFa$$

$(F(c^2a) = F(ca \cdot c) = D(ca + c) - D(ac) - D(c) = D(c(a+1)) - D(ac) - D(c) = cD(a+1) - (a+1)D(c) - aD(c) - cD(a) - D(c) = cF(a), 0 = F(0) = F(a-a) = F(a) + F(-a), \Rightarrow F(a) = -F(a) = F(-a), \Rightarrow F(2a) = 0)$

for all $a, c \in R$ see [44]p. 1204.

Hence $F(2a) = 2F(a) = 0$, if 2 is a unit of R, $F(a) = 0$, then the kernel is trivial and $\Omega^1_{R/\mathbb{Z}} \cong TD(R)$,

$$1 \longrightarrow TD(R) \xrightarrow{\cong} \Omega^1_{R/\mathbb{Z}} \longrightarrow 1.$$

Example 5.3. $R = \mathbb{Z}$, then the kernel of the above surjection is $\mathbb{Z}/2\mathbb{Z}$.
If R is a field of characteristic $\neq 2$, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.
If R is a perfect field, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.

Definition 5.4. We define groups $\Phi_i(R)$, $i \geq 2$, by the exact sequence

$$1 \longrightarrow \Phi_i(R) \longrightarrow K_2(R[x]/(x^i)) \longrightarrow K_2(R[x]/(x^{i-1})) \longrightarrow 1. \tag{5.4}$$

This sequence is exact at the right as $SK_1(R[x]/(x^i), (x^{i-1})/(x^i)) = 1$ (cf. [29] Theorem 6.2 and [3]9.2, p. 267).

5.1.1 Remarks

我们把 Bass 书 [3] 中相关的结论 (p. 267) 放在这里以备今后查询和使用

A semi-local ring is a ring for which $R/rad(R)$ is a semisimple ring, where $rad(R)$ is the Jacobson radical of R. In commutative algebra, semi-local means "finitely many maximal ideals", for instance, all rational numbers r/s with s prime to 30 form a semi-local ring, with maximal ideals generated respectively by $2, 3$, and 5. This is a PID, as a matter of fact, any semi-local Dedekind domain is a PID. And if R is a commutative noetherian ring, the set of zero-divisors is a union of finitely many prime ideals (namely, the "associated primes" of (0)), thus its classical ring of quotients (obtained from R by inverting all of its non zero-divisors) is a semi-local ring. See [1] p.174. and [3] p. 86.

In studying the stable structure of general linear groups in algebraic K-theory, Bass proved the following basic result (ca. 1964) on the unit structure of semilocal rings.

Theorem 5.5. *If R is a semi-local ring, then R has stable range 1, in the sense that, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb \in R^*$.*

Example 5.6. Some classes of semi-local rings: left(right) artinian rings, finite direct products of local rings, matrix rings over local rings, module-finite algebras over commutative semi-local rings.
The quotient $\mathbb{Z}/m\mathbb{Z}$ is a semi-local ring. In particular, if m is a prime power, then $\mathbb{Z}/m\mathbb{Z}$ is a local ring.
A finite direct sum of fields $\bigoplus_{i=1}^{n} F_i$ is a semi-local ring. In the case of commutative rings with unit, this example is prototypical in the following sense: the Chinese remainder theorem shows that for a semi-local commutative ring R with unit and maximal ideals $m_1, \cdots, m_n$

$$R/\bigcap_{i=1}^{n} m_i \cong \bigoplus_{i=1}^{n} R/m_i$$

(The map is the natural projection). The right hand side is a direct sum of fields. Here we note that $\cap_i m_i = rad(R)$, and we see that $R/rad(R)$ is indeed a semisimple ring.
The classical ring of quotients for any commutative Noetherian ring is a semilocal ring.
The endomorphism ring of an Artinian module is a semilocal ring.
Semi-local rings occur for example in commutative algebra when a (commutative) ring R is localized with respect to the multiplicatively closed subset $S = \cap(R - p_i)$, where the p_i are finitely many prime ideals.

Theorem 5.7. *Let I be a two-sided ideal in a ring R. Assume either that R is semi-local or that $I \subset rad(R)$. Then*

$$GL_1(R, I) \longrightarrow K_1(R, I)$$

is surjective, and, for all $m \geq 2$,

$$GL_m(R, I)/E_m(R, I) \longrightarrow K_1(R, I)$$

is an isomorphism. Moreover $[GL_m(R), GL_m(R, I)] \subset E_m(R, I)$, with equality for $m \geq 3$.

Corollary 5.8. *Suppose that R above is commutative, then $E_n(R, I) \xrightarrow{\cong} SL_n(R, I)$ is an isomorphism for all $n \geq 1$, and $SK_1(R, I) = 0$.*

Proof. The determinant induces the inverse,

$$\det\colon K_1(R,I) \longrightarrow GL_1(R,I).$$

In particular, if $\alpha \in GL_n(R,I)$ and $\det(\alpha) = 1$ then $\alpha \in E_n(R,I)$, i.e. $SL_n(R,I) \subset E_n(R,I)$. The opposite inclusion is trivial. Finally $SK_1(R,I) = SL(R,I)/E(R,I) = 0$. □

还有一个小插曲，当 k 是域时，$k[x]/(x^m)$ 是局部环的证明

Proposition 5.9. *Let I be an ideal in the ring R.*
a) If $rad(I)$ is maximal, then R/I is a local ring.
b) In particular, if m is a maximal ideal and $n \in \mathbb{Z}^+$ then R/m^n is a local ring.

Proof. a) We know that $rad(I) = \bigcap_{P \supset I} P$, so if $rad(I) = m$ is maximal it must be the only prime ideal containing I. Therefore, by correspondence R/I is a local ring. (In fact it is a ring with a unique prime ideal.)
b)$rad(m^n) = rad(m) = m$, so part a) applies. □

Example 5.10. For instance, for any prime number p, $\mathbb{Z}/(p^k)$ is a local ring, whose maximal ideal is generated by p. It is easy to see (using the Chinese Remainder Theorem) that conversely, if $\mathbb{Z}/(n)$ is a local ring then n is a prime power.

The ring $\mathbb{Z}_p$ of p-adic integers is a local ring. For any field k, the ring $k[\![t]\!]$ of formal power series with coefficients in k is a local ring. Both of these rings are also PIDs. A ring which is a local PID is called a discrete valuation ring. Note that a local ring is connected, i.e., $e^2 = e \Rightarrow e \in \{0,1\}$.

令 R 是 $k[x]$, I 是 (x^m), 有 $rad(x^m) = (x)$ 是极大理想 (由于 $0 \to (x) \to k[x] \to k \to 0$ 正合)，从而 $k[x]/(x^i)$ 是局部环。

Remarks 到此结束

5.1.2 Theorem

The first part of the following theorem is due to van der Kallen [44] and the second to Bloch [5].

Theorem 5.11. *Let R be a commutative ring. Then*
(1) $\Phi_2(R) \cong TD(R)$*;*
(2) If R is a local $\mathbb{F}_p$*-algebra and p is odd prime, then*

$$\Phi_i(R) \cong \begin{cases} \Omega^1_{R/\mathbb{Z}}, i \not\equiv 0, 1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r}, i = mp^r, (p, m) = 1. \end{cases}$$

当 p 是 odd prime 时，这一定理 (2) 可应用于 $\mathbb{F}_p[C_p]$, 因为 $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[t]/(t^p)$

Lemma 5.12. *Let* $q = p^f$ *and let H be a finite abelian p-group. Then* $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$*is a free* $\mathbb{F}_q H$*-module of rank equal to the p-rank of H.*

Proof. In terms of polynomials, we have

$$\mathbb{F}_q H \cong \mathbb{F}_q[x_1, \cdots, x_n]/I$$

where n is the p-rank of H and I is the ideal of $\mathbb{F}_q[x_1, \cdots, x_n]$ generated by polynomials of the form $F_i = x_i^{q_i} - 1$ where q_i is a power of p. By [BoreI,A.: Linear algebraic groups. New York: W. A. Benjamin 1969, p. 61], $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$ is the $\mathbb{F}_q H$-module with generators $dx_1, \cdots, dx_n$ subject to the relations

$$\sum_i \frac{dF_i}{dx_i} dx_i = 0.$$

Since the ring has characteristic p, the relations are trivial and the module is free. As $\mathbb{F}_q$ is perfect, its module of differentials is trivial. Hence $\Omega^1_{\mathbb{F}_q H/\mathbb{F}_q} = \Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$, yielding the result. □

由这个引理得到了5.1.

下面是节选一些可能用到的陈述。

- $\mathbb{F}_q G$ is a local ring, where G is an elementary abelian p-group, for example $G = (\mathbb{Z}/p\mathbb{Z})^n$.

对 odd prime 的证明如下

Proof. We begin by showing that $K_2(\mathbb{F}_q G)$ is an elementary abelian p-group even in case $p = 2$. As $\mathbb{F}_q G$ is a local ring, it follows that $K_2(\mathbb{F}_q G)$ is generated by the Steinberg symbols $\{u, v\}$, $u, v \in \mathbb{F}_q G^*$. Now $u^p, v^P \in \mathbb{F}^*$ as G is an elementary abelian p-group

(p 次后 G 中的元就变成单位元了). Choose $w \in \mathbb{F}_q^*$ so that $w^p = u^p$.(这里注意之前的 u 是群环里的，这里的 w 取在域里) Then

$$\begin{aligned}\{u, v\}^p &= \{u^p, v\} \\ &= \{w^p, v\} \\ &= \{w, v^p\}.\end{aligned}$$

Thus $\{w, v^p\}$ is trivial as it lies in the image of $K_2(\mathbb{F}_q) = 1$(有限域的 K_2 是平凡的，并且这个符号是在 K_2 中). Hence $K_2(\mathbb{F}_q G)$ has exponent p.

Let H be generated by $x_1, \cdots, x_{n-1}$ where $x_1, \cdots, x_n$ are independent generators of G. Then (由于特征是 p 才有下面的最后一步，对于 $\mathbb{Z}$ 是不对的)

$$\mathbb{F}_q G = \mathbb{F}_q H[x_n]/(x_n^p - 1) \cong \mathbb{F}_q H[x]/(x^p).$$

Exact sequence 5.4 together with Theorem yield

$$\begin{aligned}\text{rank } K_2(\mathbb{F}_q G) &= \text{rank } K_2(\mathbb{F}_q H) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q \\ &= \text{rank } K_2(\mathbb{F}_q H) + f(p-1)(n-1)p^{n-1} + f(p^{n-1} - 1)\end{aligned}$$

and the result follows by induction.

上面的结论我们详细写出来是

$$1 \longrightarrow \Phi_p(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q G) = K_2(\mathbb{F}_q H[x]/(x^p)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow 1,$$

$$1 \longrightarrow \Phi_{p-1}(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-2})) \longrightarrow 1,$$

$$\cdots$$

$$1 \longrightarrow \Phi_2(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^2)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x)) \longrightarrow 1.$$

Note that $\mathbb{F}_q H[x]/(x) = \mathbb{F}_q H$, $G = (\mathbb{Z}/p\mathbb{Z})^n$, $H = (\mathbb{Z}/p\mathbb{Z})^{n-1}$ then

$$\begin{aligned}\text{rank } K_2(\mathbb{F}_q G) &= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H[x]/(x^{p-1})) \\ &= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } \Phi_{p-1}(\mathbb{F}_q H) + \cdots + \text{rank } \Phi_2(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H) \\ &= \text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + (p-2)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } K_2(\mathbb{F}_q H) \\ &= (p-1)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + \text{rank } K_2(\mathbb{F}_q H) \\ &= f(p-1)(n-1)p^{n-1} + f(p^{n-1} - 1) + \text{rank } K_2(\mathbb{F}_q H)\end{aligned}$$

since

$$\Phi_p(\mathbb{F}_qH)=\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}\oplus\mathbb{F}_qH/\mathbb{F}_qH^p,$$

$$\Phi_i(\mathbb{F}_qH)=\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}=(\mathbb{F}_qH)^{n-1},2\leq i\leq p-1,$$

$$\mathbb{F}_qH/\mathbb{F}_qH^p=\mathbb{F}_qH/\mathbb{F}_q$$

$\mathbb{F}H$ 是以 H 中元素为基的自由 F 模并且

$$\text{rank }\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}=\text{rank }(\mathbb{F}_{p^f}H)^{n-1}=(n-1)f|H|=(n-1)fp^{n-1}$$

$$\text{rank }\mathbb{F}_qH/\mathbb{F}_q=\text{rank }\mathbb{F}_qH-\text{rank }\mathbb{F}_q=f(p^{n-1}-1).$$

接下来是归纳计算，首先我们看它截至到哪一步：最后一步应该是 $\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]$，因为 $K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}])=0$, 这时有

$$\begin{aligned}\text{rank }K_2(\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2])&=\text{rank }K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}])+(p-1)\text{rank }\Omega^1_{\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{Z}}+\text{rank }\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{F}_q\\&=0+f(p-1)(2-1)p^{2-1}+f(p^{2-1}-1)\end{aligned}$$

从而我们知道

$$\begin{aligned}\text{rank }K_2(\mathbb{F}_qG)&=f(p-1)(n-1)p^{n-1}+f(p^{n-1}-1)+\cdots+f(p-1)p^1+f(p^1-1)\\&=\sum_{i=1}^{n-1}(f(p-1)ip^i+f(p^i-1))\\&=-f\frac{p-p^n}{1-p}+f(n-1)p^n+f\frac{p-p^n}{1-p}-(n-1)f\\&=f(n-1)(p^n-1)\end{aligned}$$

这里的计算用到等比数列求和，记 $S=\sum_{i=1}^{n-1}ip^i$

$$pS=\sum_{i=1}^{n-1}ip^{i+1}=\sum_{i=2}^{n}(i-1)p^i$$

$$S-pS=\sum_{i=1}^{n-1}p^i-(n-1)p^n$$

因此

$$S=\frac{p-p^n}{(1-p)^2}-\frac{(n-1)p^n}{(1-p)}$$

$$
\begin{aligned}
\sum_{i=1}^{n-1}(f(p-1)ip^i+f(p^i-1)) &= f(p-1)S+f\frac{p-p^n}{1-p}-(n-1)f \\
&= f(p-1)(\frac{p-p^n}{(1-p)^2}-\frac{(n-1)p^n}{(1-p)})+f\frac{p-p^n}{1-p}-(n-1)f \\
&= -f\frac{p-p^n}{1-p}+f(n-1)p^n+f\frac{p-p^n}{1-p}-(n-1)f \\
&= f(n-1)(p^n-1) \qquad \square
\end{aligned}
$$

In case $p=2$ the details become more complicated.(暂且略过这个情形)

5.2 Part 2

第二部分是考了系数环是 $\mathbb{Z}$ 的情形，如何将上面的有限域和这里的整数环联系起来，就是用了一个相对 K 群的正合列。

We now exploit these computations of $K_2(\mathbb{F}_qG)$ to obtain lower bounds for $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. There is an exact sequence

(5.12) $$K_2(\mathbb{Z}G)\longrightarrow K_2(\mathbb{F}_pG)\longrightarrow SK_1(\mathbb{Z}G,p\mathbb{Z}G)\longrightarrow SK_1(\mathbb{Z}G)\longrightarrow 1$$

This sequence is exact on the right because $\mathbb{F}_pG$ is a local ring, which implies $SK_1(\mathbb{F}_pG)=1$ [3], p. 267.

Theorem 5.13. *(1) Let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{Z}G)$ has 2-rank at least $(n-1)2^n+2$ and $Wh_2(G)$ has 2-rank at least $(n-1)2^n-\frac{(n+2)(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n\geq 2$.*
(2) Let p be an odd prime and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{Z}G)$ has p-rank at least $(n-1)(p^n-1)-\binom{p+n-1}{p}$ and $Wh_2(G)$ has p-rank at least $(n-1)(p^n-1)-\binom{p+n-1}{p}-\frac{n(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n\geq 2$.

Proof. (1) Since $K_1(\mathbb{Z}G,2\mathbb{Z}G)\longrightarrow K_1(\mathbb{Z}G)$ is injective [Keating, M.E.: On the K-theory of the quaternion group. Mathematika 20, 59–62 (1973), Remark 2.4], we see that $K_2(\mathbb{Z}G)\longrightarrow K_2(\mathbb{F}_2G)$ is surjective.

If $g_1,\cdots,g_n$, are the generators of G, then the $n+1$ symbols $\{-1,-1\},\{-1,g_1\},\cdots,\{-1,g_n\}$ are independent [[29], p. 65] and lie in the kernel of this map. Hence

$$\text{rank } K_2(\mathbb{Z}G)\geq (n-1)(2^n-1)+(n+1)=(n-1)2^n+2.$$

Recall that for G abelian, $Wh_2(G)$ is the quotient of $K_2(\mathbb{Z}G)$ by the subgroup generated by all symbols of the form $\{\sigma, \tau\}, \sigma, \tau \in \pm G$ [Hatcher, A.E.: Pseudo-isotopy and K_2, pp. 328-336. Lecture Notes in Mathematics 342. Berlin, Heidelberg, New York: Springer 1973]. It is easy to see from the bimuttiplicative and anti-symmetric properties of symbols that this subgroup has rank at most $\binom{n+1}{2}+1$. Moreover, by using the various maps $\mathbb{Z}G \longrightarrow \mathbb{Z}$ which send elements of G to ± 1, it can be shown that the rank of this subgroup is precisely $\binom{n+1}{2}+1$. $(n-1)2^n+2-\binom{n+1}{2}-1=(n-1)2^n-\frac{(n+2)(n-1)}{2}$.

(2) 以下这一段没有完全读懂。Let B be the integral chosure of $\mathbb{Z}G$ in $\mathbb{Q}G$. Then $SK_1(B, p^{n+1}B)$ has p-rank $\frac{p^n-1}{p-1}$ [Bass, H., Milnor, J., Serre, J. P.: Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$. Publ. Math. IHES 33, 59–137 (1967), Corollary 4.3, p. 95].

But $SK_1(B, p^{n+1}B) \cong SK_1(\mathbb{Z}G, p^{n+1}B)$ [[3], p. 484] since p^nB lies in the conductor of B over $\mathbb{Z}G$, and $SK_1(\mathbb{Z}G, p^{n+1}B)$ maps onto $SK_1(\mathbb{Z}G, p\mathbb{Z}G)$ [[3], 9.3, p. 267]. Hence p-rank $SK_1(\mathbb{Z}G, p\mathbb{Z}G) \leq \frac{p^n-1}{p-1}$. The p-rank of $SK_1(\mathbb{Z}G)$ is $\frac{p^n-1}{p-1}-\binom{p+n-1}{p}$ [Alperin, R.C., Dennis, R. K., Stein, M. R.: The non-triviality of $SK_1(\mathbb{Z}\pi)$, pp. 1-7. Lecture Notes in Mathematics 353. Berlin, Heidelberg, New York: Springer t973, Theorem 2]. The result now follows from exact sequence 5.12.

And noting that the subgroup generated by the symbols $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ has p-rank at most $\frac{n(n-1)}{2}$. □

Remark 5.14. The subgroup of $K_2(\mathbb{Z}G)$ generated by elements of the form $\langle a, b\rangle$, $1+ab \in (\mathbb{Z}G)^*$ maps onto $K_2(\mathbb{F}_2G)$ for G an elementary abelian 2-group of rank ≤ 2. W. van der Kallen has shown that this subgroup maps onto in general. This follows from the rank 2 case via

Lemma (van der Kallen). Let I be a nilpotent ideal of the commutative ring R. Let $v_i \in R$ additively generate R/I and let $w_j \in I$ additively generate I. Then $K_2(I) = \ker(K_2(R) \longrightarrow K_2(R/I))$ is generated by all elements of the form $\langle v_i, w_j\rangle$ and $\langle w_j, w_i^{2^k-1}w_j\rangle$.

我的一些问题：$NK_2(\mathbb{F}_qG)$ 如何算,$NK_1(\mathbb{Z}G, p\mathbb{Z}G)$ =?,最简单的可以考虑 $NK_2(\mathbb{F}_pC_p)$, 接着是 $NK_2(\mathbb{F}_{p^2}C_p)$.

5.3 Generalizations

之前考虑的是 $\mathbb{Z}G$, G elementary. 可以推广到 G finite group, $\mathcal{O}$ be the ring of integers of an algebraic number field.

If S is a Sylow p-subgroup of G, then $\mathcal{O}G$ is a free module over $\mathcal{O}S$ and the composition

$$K_2(\mathcal{O}S) \longrightarrow K_2(\mathcal{O}G) \longrightarrow K_2(\mathcal{O}S)$$

(where the second map is the transfer) is multiplication by $(G : S)$. Hence p-rank $K_2(\mathcal{O}G) \geq p$-rank $K_2(\mathcal{O}S)$ and estimates may be obtained by restricting to the case of a p-group.

Theorem 5.15. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field which is Galois over $\mathbb{Q}$ and let G be an elementary abelian p-group of rank n. If p is unramfied in $\mathcal{O}$ with each residue field having degree f over $\mathbb{F}_p$, then $K_2(\mathcal{O}G)$ has p-rank at least*
(i) $f(n-1)(2^n-1)$ if $p=2$ and $\mathcal{O}$ has a real embedding,
(ii) $f(n-1)(2^n-l)-\binom{n+1}{2}$ if $p=2$ and $\mathcal{O}$ is totally imaginary,
(iii) $f(n-1)(p^n-l)-\binom{p+n-1}{p}$ if p is odd.

abelian p-groups which are not elementary 有以下一个结论

Proposition 5.16. *Let p be an odd prime and suppose $G = H \times C$ where C is cyclic of order p^t, $|H| = p^k$ and $s = p$-rank H. Let $\mathcal{O}$ be the ring of integers in a number field. Choose a prime $\mathfrak{p}$ of $\mathcal{O}$ lying over p and having residue degree f over $\mathbb{F}_p$. Then*

$$\begin{aligned}
&\mathrm{ord}_p|K_2(\mathcal{O}G/\mathfrak{p}G)| - \mathrm{ord}_p|K_2(\mathcal{O}H/\mathfrak{p}H)| \\
\geq &f\left(p^k\big(s(p-1)p^{t-1}+1\big) - |H^{p^t}|\right) + p^k(p^{t-1}-1) - (p-1)\sum_{r=1}^{t-1}|H^{p^r}|p^{t-r-1}.
\end{aligned}$$

Chapter 6

Some Results of Grouprings

从相关的教材和书籍中摘录与群环相关的 K-理论结果。

On the K-theory of truncated polynomial rings in non-commuting variables 中的有关结果 Vigleik Angeltveit 的文章 On the K-theory of truncated polynomial rings in non-commuting variables, Bull. London Math. Soc. 47 (2015) 731742.

Hesselholt and Madsen computed the algebraic K-theory of $k[x]/(x^a)$ when k is a perfect field of positive characteristic in terms of big Witt vectors in [L. Hesselholt and I. Madsen, 'Cyclic polytopes and the K-theory of truncated polynomial algebras', Invent. Math. 130 (1997) 7397.], but see [L. Hesselholt and I. Madsen, 'On the K-theory of finite algebras over Witt vectors of perfect fields', Topology 36 (1997) 29101.] as well. They found that

$$K_{2q-1}(k[x]/(x^a),(x)) \cong \operatorname{coker}(V_a\colon \mathbb{W}_q(k) \longrightarrow \mathbb{W}_{aq}(k)),$$

while

$$K_{2q}(k[x]/(x^a),(x)) = 0.$$

Here $\mathbb{W}_n(k)$ denotes the Witt vectors on the truncation set $\{1,2,\cdots,n\}$. 一个例子比如 $K_2(\mathbb{F}_q[x]/(x^a),(x)) = 0$

[V. Angeltveit, T. Gerhardt, M. A. Hill and A. Lindenstrauss, 'On the algebraic K-theory of truncated polynomial algebras in several variables', J. K-Theory 13 (2014) 5781.]

这篇文章中把上面的结果推广成 $A = k[x_1,\cdots,x_n]/(x_1^{a_1},\cdots,x_n^{a_n})$, with the big Witt vectors replaced by certain generalized Witt vectors built from truncation sets in

$\mathbb{N}^n$ instead of $\mathbb{N}$, and the cokernel of V_a replaced by the iterated homotopy cofiber of an n-cube of spectra.

These calculations both use the cyclotomic trace map from algebraic K-theory to topological cyclic homology. The underlying reason for the appearance of Witt vectors is that if k is a perfect field of positive characteristic, then

$$\lim_{R,m \leq n} \pi_* THH(k)^{C_m} \cong \mathbb{W}_n(k)[\mu_0],$$

where μ_0 is a polynomial generator in degree 2.

6.1 Relative K-theory and topological cyclic homology

BJORN IAN DUNDAS

Let $f : A \to B$ be a map of rings up to homotopy. When is it possible to give a good description of the relative algebraic K-theory? Generally, K-theory is hard to calculate, so it is of special importance to be able to measure the effect of a change of input.

Special instances of the case where f induces an epimorphism $\pi_0(A) \to \pi_0(B)$ with nilpotent kernel have been studied by several authors.

- The first general result in this direction was Goodwillie's theorem [GOODWILLIE, T. G., Relative algebraic K-theory and cyclic homology. Ann. of Math.,124 (1986), 347-402.], that in the case of simplicial rings, relative K-theory is rationally given by the corresponding relative negative cyclic homology.
- McCarthy has complemented this by giving a short and beautiful proof [MCCARTHY, R., Relative algebraic K-theory and topological cyclic homology. Acta Math., 179 (1997), 197-222.]showing that at a given prime p, the relative K-theory is given by the corresponding relative topological cyclic homology.

This paper stemmed from a desire to understand the linearization

$$A(BG) \longrightarrow K(\mathbb{Z}[G])$$

that is, the connection between the algebraic K-theory of spaces and the algebraic K-theory of rings, each of which has theorems of the desired sort. Waldhausen has shown that this map is a rational equivalence, but torsion information has so far been out of reach.

与 K, TC, THH 有关的书 GOODWILL Notes on the cyclotomic trace,
HESSELHOLT, L. MADSEN, I., On the K-theory of finite algebras over Witt vectors of perfect fields. Topology, 36 (1997), 29-101.,
BOKSTEDT,M., HSIANG, W.C. , MADSEN, I., The cyclotomic trace and algebraic K-theory of spaces. Invent. Math., 111 (1993), 463-539.
MADSEN, I., Algebraic K-theory and traces, in Current Developments in Mathematics (R. Bott, A. Jaffe and S. T. Yan, eds.), pp. 191-323. International Press, 1995. 这个是 overview，大致扫过一眼

6.2 Cyclic polytopes and the K-theory of truncated polynomial algebras

这篇文章结论比较重要。

k: perfect field of positive characteristic p. 比如 $\mathbb{F}_p$, 甚至 $\mathbb{F}_{p^n}$.

主要计算了 $K_*(k[x]/(x^n),(x))$, relative algebraic K -theory of a truncated polynomial algebra over a perfect field k of positive characteristic p.

几个观察 (x) 这个理想是 nilpotent, 因而可以用 McCarthy's theorem: the relative algebraic K -theory is isomorphic to the relative topological cyclic homology, 后者是可以算的。

最后的结果是说这个群可以用 big Witt Vectors 来表示，简单回顾 Witt vectors 的知识

Let $W_m(k)$ denote the big Witt vectors in k of length m, i.e. the multiplicative group

$$W_m(k) = (1 + xk[\![x]\!])^\times / (1 + x^{m+1}k[\![x]\!])^\times,$$

the Verschiebung map

$$\begin{aligned} V_n \colon W_m(k) &\longrightarrow W_{mn}(k) \\ f(x) &\mapsto f(x^n) \end{aligned}$$

The relative K-theory $K(k[x]/(x^n),(x))$ is given by the fibration sequence

$$K(k[x]/(x^n),(x)) \longrightarrow K(k[x]/(x^n)) \longrightarrow K(k),$$

with a corresponding exact sequence of homotopy groups

$$0 \longrightarrow K_*(k[x]/(x^n),(x)) \longrightarrow K_*(k[x]/(x^n)) \longrightarrow K_*(k) \longrightarrow 0.$$

当 k 是有限域时, Quillen 算了 $K_*(k)$. D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Ann. Math. 96 (1972), 552-586
For a general perfect field of characteristic $p > 0$ one knows that the p-adic K-groups of k vanish in positive degrees by [C. Kratzer, -structure en K -theorie algebrique, Comment. Math. Helv. t′ 55 (1980), 233-254].

计算结果就是下面的定理

Theorem 6.1. *Let k be a perfect field of positive characteristic. Then*

$$K_{2m-1}(k[x]/(x^n),(x)) \cong W_{mn}(k)/V_nW_m(k)$$

and the groups in even degrees are zero.

The result extends calculations by Aisbett and Stienstra of $K_3(k[x]/(x^n),(x))$.

Chapter 7

An introduction to NK-groups

There are functors

$$K_n\colon \textbf{Ring} \longrightarrow \textbf{Ab}$$
$$R \mapsto K_n(R)$$
$$[f : R \to S] \mapsto [K_n(f) : K_n(R) \to K_n(S)]$$

- $n < 0$: negative K-theory.
- K_0, K_1, K_2: classical K-theory:
  - (Grothendieck) $K_0(R) = G(\mathbf{P}(R))$. If R is a field or the ring of integers $\mathbb{Z}$, then $K_0(R) = \mathbb{Z}$.
  - (Bass) $K_1(R) = GL(R)/E(R)$. $K_1(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$.
  - (Milnor) $K_2(R) = \ker(St(R) \longrightarrow E(R))$.
- $n \geq 3$: (Quillen) higher K-theory, $K_n(R) = \pi_n\big(BGL(R)^+\big) \cong \pi_n\big(\Omega BQ\mathbf{P}(R)\big)$.

Definition 7.1. Whitehead group $Wh(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

A question arise $K_n(RG) =?$ where RG is a group ring.

For example, one want to compute $K_n(R[\mathbb{Z}])$, first note that $R[\mathbb{Z}] = R[x, x^{-1}]$.

7.1 Nil-groups

Definition 7.2 (Bass NK-groups)**.** Let $R[x]$ be a polynomial ring over R, the map i the injection $R \overset{i}{\hookrightarrow} R[x]$, then there is a surjection ε split by i

$$\varepsilon\colon R[x] \xrightarrow{x \mapsto 0} R.$$

The Bass NK-groups(Nil-groups) are defined by

$$NK_n(R) := \ker\left(K_n(R[x]) \longrightarrow K_n(R)\right), \forall n \in \mathbb{Z}.$$

Hence $K_n(R[x]) \cong K_n(R) \oplus NK_n(R)$.

Theorem 7.3 (Fundamental theorem of algebraic K-theory, Bass-Heller-Swan formula)**.** *The following exact sequence is split*

$$0 \longrightarrow K_n(R) \xrightarrow{\Delta} K_n(R[x]) \oplus K_n(R[x^{-1}]) \xrightarrow{\pm} K_n(R[x,x^{-1}]) \xrightarrow{\partial} K_{n-1}(R) \longrightarrow 0.$$

And we have

$$K_n(R[x,x^{-1}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R).$$

Remark 7.4. History: H. Bass defined negative K-theory

$$K_n(R) := \operatorname{coker}\left(K_n(R[x]) \oplus K_n(R[x^{-1}]) \longrightarrow K_n(R[x,x^{-1}])\right)$$

and NK_i for $i \leq 1$

$$NK_n(R) := \ker\left(K_n(R[x]) \longrightarrow K_n(R)\right).$$

D. Quillen defined higher K-theory in 1970s (especially the famous paper *Higher Algebraic K-theory I* [33] in 1972).

The two copies of $NK_i(R)$ come from the embeddings $R[x] \hookrightarrow R[x,x^{-1}]$ and $R[x^{-1}] \hookrightarrow R[x,x^{-1}]$.

注意这里有遗留的符号问题，就是 α 的符号

Definition 7.5 (α-twisted polynomial rings and α-twisted Laurent polynomial rings)**.** Let $\alpha\colon R \longrightarrow R$ be an automorphism, $R_\alpha[x,x^{-1}]$ is called the α-twisted Laurent polynomial ring. Addtively $R_\alpha[x,x^{-1}] = R[x,x^{-1}]$. Multiplication in $R_\alpha[x]$is defined by $(rx^i)(sx^j) = r\alpha^{-i}(s)x^{i+j}$. There is an automorphism of $R_\alpha[x,x^{-1}]$ induced by α and which we also denote by α, defined by the following condition

$$\alpha(rx^i) = \alpha(r)x^i, r \in R.$$

The subrings $R_\alpha[x], R_\alpha[x^{-1}]$ of $R_\alpha[x,x^{-1}]$ generated by R and x, and R and x^{-1} respectively are called α-twisted polynomial rings.

Some results about twisted plynomial rings and twisted Laurent polynomial rings
According to [9], we list some useful results here, all modules are considered as right R-modules:

- *Twisted Hilbert syzygy theorem:* If the right global dimension $r.gl.dim(R) = n$, then $r.gl.dim(R_\alpha[x]) = n+1$ and $r.gl.dim(R_\alpha[x, x^{-1}]) \leq n+1$.
- *Twisted Hilbert basis theorem:* If R is (right) noetheridan, then both $R_\alpha[x]$ and $R_\alpha[x, x^{-1}]$ are (right) noetherian.
- If R is (right) regular, then $R_\alpha[x]$ and $R_\alpha[x, x^{-1}]$ are (right) regular.
- *Twisted Grothendieck theorem:* If R is (right) regular, then $K_0(R) \longrightarrow K_0(R_\alpha[x])$ is an isomorphism and $K_0(R) \longrightarrow K_0(R_\alpha[x, x^{-1}])$ is an epimorphism. 看下这个能不能推广到一般 n.

Definition 7.6 (Farrell Nil-groups)**.** Let ε be the surjection $R_\alpha[x] \xrightarrow{x \mapsto 0} R$, the Farrell twisted Nil-group $NK_n(R, \alpha)$ is defined as the kernel of $K_n(\varepsilon) \colon K_n(R_\alpha[x]) \longrightarrow K_n(R)$ for any $n \in \mathbb{Z}$, then $K_n(R_\alpha[x]) \cong K_n(R) \oplus NK_n(R, \alpha)$.

Remark 7.7. If $\alpha = \mathrm{id}$, then $NK_n(R, \mathrm{id}) = NK_n(R)$ are Bass Nil-groups.

Let $\mathbf{P}(R)$ be the exact category of finitely generated projective R-modules, define $\mathbf{Nil}(R)$ to be the exact category of nilpotent endomorphisms, objects in $\mathbf{Nil}(R)$ are pairs (P, f) with P finitely generated projective R-module and $f \in \mathrm{End}(P)$ nilpotent, morphisms are $(P_1, f_1) \xrightarrow{\alpha} (P_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{tikzcd} P_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & P_1 \arrow[d, "\alpha"] \\ P_2 \arrow[r, "f_2"] & P_2 \end{tikzcd} .$$

D. Grayson defined $\mathbf{Nil}(R, \alpha)$ to be the exact category of α-semilinear nilpotent endomorphisms [15], objects are pairs (P, f) with P finitely generated projective and $f \in \mathrm{End}(P)$ nilpotent and α-semilinear (i.e. $f(mr) = f(m)\alpha(r)$).

Since the inclusion $i \colon \mathbf{P}(R) \longrightarrow \mathbf{Nil}(R)$ and $i \colon \mathbf{P}(R) \longrightarrow \mathbf{Nil}(R, \alpha)$ are split and the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(R) \rightleftarrows \mathbf{Nil}(R, \alpha)$$

$$\mathbf{P}(R) \rightleftarrows \mathbf{Nil}(R)$$

$$P \mapsto (P,0)$$
$$P \leftarrow\!\shortmid (P,f)$$

Definition 7.8. For $n \in \mathbb{Z}$, define $\mathrm{Nil}_n(R) := \ker\big(K_n(\mathbf{Nil}(R)) \longrightarrow K_n(R)\big)$ and $\mathrm{Nil}_n(R,\alpha) := \ker\big(K_n(\mathbf{Nil}(R,\alpha)) \longrightarrow K_n(R)\big)$ where $K_n(R) = K_n(\mathbf{P}(R))$ and K_n is defined by Quillen's Q-construction, then $K_n(\mathbf{Nil}(R)) = K_n(R) \oplus \mathrm{Nil}_n(R)$, $K_n(\mathbf{Nil}(R,\alpha)) = K_n(R) \oplus \mathrm{Nil}_n(R,\alpha)$.

注意这里有遗留的符号问题，就是 α 的符号，因为 [9] 和 [15] 记法不同，需要读这两篇文章确定符号！

Theorem 7.9 ([15]). *There is a localization exact sequence*

$$\cdots \longrightarrow K_{n+1}(R_\alpha[x,x^{-1}]) \longrightarrow K_n(\mathbf{Nil}(R,\alpha)) \longrightarrow K_n(R_\alpha[x]) \longrightarrow K_n(R_\alpha[x,x^{-1}]) \longrightarrow \cdots,$$

and $K_n(R_\alpha[x,x^{-1}]) \cong F_{n-1}(\alpha) \oplus \mathrm{Nil}_{n-1}(R,\alpha) \oplus \mathrm{Nil}^{\alpha^{-1}}_{n-1}(R)$.

Proposition 7.10 (Grayson [13], [15]). *For any* $n \geq 1$, $NK_n(R) \cong \mathrm{Nil}_{n-1}(R)$, $NK_n^{\alpha^{-1}}(R) \cong \mathrm{Nil}_{n-1}(R,\alpha)$.

For $n = 1$, F. T. Farrell and W.-C. Hsiang first got the result $NK_1^{\alpha^{-1}}(R) \cong \mathrm{Nil}_0(R,\alpha)$ in [9].

Notice that $NK_1(R) \cong K_1(R[x],(x-r)), \forall r \in R$. Since $[(P,\nu)] = [(P\oplus Q, \nu\oplus 0)] - [(Q,0)] \in K_0(\mathbf{Nil}(R))$, we see $\mathrm{Nil}_0(R)$ is generated by elements of the form $[(R^n,\nu)] - n[(R,0)]$ for some n and some nilpotent matrix ν, in fact one has

$$NK_1(R) \cong \mathrm{Nil}_0(R),$$
$$[1-\nu x] \leftrightarrow [(R^n,\nu)] - n[(R,0)].$$

Definition 7.11 (Virtually cyclic groups). A discrete group V is called virtually cyclic if it contains a cyclic subgroup of finite index, i.e., if V is finite or virtually infinite cyclic.

Virtually infinite cyclic groups are of two types:

1 $V = G \rtimes_\alpha T$ is a semi-direct product where G is a finite group, $T = \langle x\rangle$ an infinite cyclic group generated by x, $\alpha \in Aut(G)$, and the multiplication of V is given by $(g_1,x^i)(g_2,x^j) = (g_1\alpha^{-i}(g_2), x^{i+j})$.

2 V is a non-trivial amalgam of finite groups and has the form $V = G_0 *_H G_1$ where $[G_0 : H] = 2 = [G_1 : H]$.

What we are really interested in here is $G \rtimes_\alpha T$ — the first type of infinite virtually cyclic groups. Note that $R[G \rtimes_\alpha T] \cong (R[G])_\alpha[x,x^{-1}]$.

7.2 Regular rings

Definition 7.12. A noetherian ring R is regular if every finitely generated R-module M has a finite resolution by finite generated projective R-modules, i.e., an acyclic complex of finitely generated R-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

with P_i projective.

Theorem 7.13 (Fundamental theorem of regular rings)**.** *Assume R is a (left, noetherian) regular ring, then*

(1) (homotopy invariance) $K_i(R[x]) \cong K_i(R)$,

(2) $K_i(R[x, x^{-1}]) \cong K_i(R) \oplus K_{i-1}(R)$, for $i \in \mathbb{Z}$.

Hence $NK_i(R) = 0$ for $i \in \mathbb{Z}$, and $K_n(R) = 0$ for $n < 0$.

We list some examples here: PIDs (principal ideal domains) and Dedekind domains are regular, such as $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{F}_q$, $\mathbb{Z}[\zeta_n]$, etc. Localization of a regular ring is again regular. Hilbert's Syzygy Theorem simply says that polynomial rings over a field are regular. Note that if a ring is (left) noetherian and has finite global dimension, then it is (left) regular. So noetherian hereditary rings are regular.

Set $R_n = \mathbb{C}[x_0, \cdots, x_n]/(\sum x_i^2 = 1)$. This is the complex coordinate ring of the n-sphere; it is a regular ring for every n and $R_1 \cong \mathbb{C}[z, z^{-1}]$.

However, the integral group rings of non-trivial finite groups are not regular. If H is a non-trivial finite cyclic group, then $H_n(H, \mathbb{Z}) \neq 0$ for all odd n. Therefore the finitely generated $\mathbb{Z}H$-module $\mathbb{Z}$ CANNOT have a finite projective resolution, hence $\mathbb{Z}H$ is not regular. Let H be a cyclic group of order n, t a generator of H, then $\mathbb{Z}H \cong \mathbb{Z}[t]/(t^n - 1)$. Put $N = \sum_{i=0}^{n-1} t^i = 1 + t + t^2 + \cdots + t^{n-1}$ (norm element of $\mathbb{Z}H$), then $(t-1)N = 0$ (in the polynomial ring $\mathbb{Z}[t]$, $t^n - 1 = (t-1)N$). There is a resolution $P.$ of $\mathbb{Z}$

$$\cdots \xrightarrow{N} \mathbb{Z}H \xrightarrow{t-1} \mathbb{Z}H \xrightarrow{N} \mathbb{Z}H \xrightarrow{t-1} \mathbb{Z}H \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

then tensor $P.$ with $\mathbb{Z}$ over $\mathbb{Z}H$, one has

$$\cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

So we have

$$H_i(H,\mathbb{Z}) = \mathrm{Tor}_i^{\mathbb{Z}H}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0, \\ \mathbb{Z}/n\mathbb{Z}, & i \text{ odd}, \\ 0, & i \text{ even}. \end{cases}$$

For a finite group G, consider $0 \neq t \in G$, let H be the subgroup generated by t, it follows from Shapiro's lemma that $H_n(H,\mathbb{Z}) = H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z})$.

Definition 7.14 (F-regular rings)**.** If F is any functor from rings to abelian groups, write $NF(R)$ for the cokernel of the natural map $F(R) \longrightarrow F(R[x])$; NF is also a functor on rings. Moreover, the ring map $R[x] \xrightarrow{x \mapsto 1} R$ provides a splitting $F(R[x]) \longrightarrow F(R)$ of the natural map, so we have a decomposition $F(R[x]) \cong F(R) \oplus NF(R)$.

We say that a ring R is F-regular if $F(R) = F(R[x_1, \cdots, x_n])$ for all n. Since $NF(R[t]) = NF(R) \oplus N^2F(R)$, we see by induction on p that R is F-regular if and only if $N^pF(R) = 0$ for all $p \geq 1$.

For example,

- Regular rings are K_n-regular for all $n \in \mathbb{Z}$.
- (Rosenberg) Commutative C^*-algebra are K_n-regular for all n.
- If R is K_0-regular, then so is $R[x, x^{-1}]$.
- If R is K_n-regular for some $n \leq 0$, then R is also K_{n-1}-regular.
- If R is an artinian ring, then R is K_0-regular and $K_n(R) = 0$ for all $n < 0$.
- If R is regular, then the graded rings $R[t_0, \cdots, t_n]/(t_0 \cdots t_n)$ are K_i-regular for $i \leq 1$.

Lemma 7.15 ([52] chapter III, 3.4.1)**.** *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Then the kernel of $F(R) \longrightarrow F(R_0)$ embeds in $NF(R)$ and even in the kernel of $NF(R) \longrightarrow NF(R_0)$. In particular, if $NF(R) = 0$ then $F(R) \cong F(R_0)$.*

It follows that for any functor F, $NF(R)$ is a summand of $N^2F(R)$ and hence $N^pF(R)$ for all $p > 0$. For every F, $F(R[t_1, \cdots, t_n]) \cong (1+N)^nF(R)$. If F is a contracted functor, then $F(R[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]) \cong (1+2N+L)^nF(R)$.

Corollary 7.16. *If $N^nF(R) = 0$, then $F(R[t_1, \cdots, t_n]) = F(R)$.*

Proof. Since $N^nF(R) = 0$ implies $N^jF(R) = 0$ for any $1 \leq j \leq n$, then

$$F(R[t_1, \cdots, t_n]) \cong (1+N)^nF(R) = F(R) \oplus \bigoplus_{j=1}^{n} \binom{n}{j} N^jF(R) = F(R).$$

□

Example 7.17 ([52] chapter III, 3.8.1)**.** Let R be a commutative regular ring and $A = R[x]/(x^n)$, then $NK_1(A) \cong (1 + tA[t])^\times = (1 + xtA[t])^\times$.

There is a conjecture for group rings whose coefficient ring are regular:

Conjecture 7.18 (Isomorphism Conjecture for torsion-free groups)**.** *If G is a torsion-free group, then the assembly map*

$$H_n(BG; K(R)) = \pi_n(BGL(R)^+ \wedge BG_+) \longrightarrow K_n(R[G])$$

should be an isomorphism for any regular ring R where $H_n(BG; K(R))$ is the generalized homology of BG with coefficients in $K(R)$.

7.3 The nonfiniteness of Nil

Bass Nil-groups

Theorem 7.19. *For $i \in \mathbb{Z}$, if $NK_i(R) \neq 0$, then $NK_i(R)$ is not finitely generated.*

In 1977, F. T. Farrell [8] stated that $NK_1(R)$ is finitely generated only when it vanishes. In fact, for $i \leq 1$, $NK_i(R)$ are always either trivial or infinitely generated. This result was subsequently extended to the higher Bass Nil-groups $NK_i(R)$ with $i \geq 1$ by van der Kallen(1978) [45]) and Prasolov(1982) [31] .

证明步骤可见 papers，GTM147 习题，苏阳书中有详细的步骤。

For Bass Nil-groups, the proof is based on the following lemmas

Lemma 7.20. *The composition $t_n \circ s_n$ is multiplication by n in the group* $\mathrm{Nil}_i(R)$*.*

Lemma 7.21. *For any element $x \in \mathrm{Nil}_i(R)$ there exists a natural number N such that $t_n(x) = 0$ for all $n \geq N$.*

To prove the theorem we now assume that the group $\mathrm{Nil}_i(R) \neq 0$ is finitely generated as an abelian group. By Lemma 7.21, there exists a natural number N such that $t_n = 0$ for $n \geq N$.

Since $\mathrm{Nil}_i(R) \neq 0$ is finitely generated, choose a prime p such that p does not appear in the decomposition of $\mathrm{Nil}_i(R)$, then multiplication by p is a monomorphism of $\mathrm{Nil}_i(R)$ into itself, and choose an integer k such that $n = p^k \geq N$. Then $t_n \circ s_n = 0$, and by Lemma 7.20, $t_n \circ s_n$ is multiplication by p^k, i.e., a monomorphism. Thus the group $\mathrm{Nil}_i(R)$ cannot be finitely generated.

Farrell Nil-groups

Theorem 7.22. *Let R be a ring, $\alpha\colon R \longrightarrow R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. Then $NK_i(R, \alpha)$ is either trivial, or infinitely generated as an abelian group.*

For Farrell's twisted Nils, when the automorphism α has finite order, Grunewald(2007) [16] and Ramos(2007) [35] independently proved the result for $i \leq 1$. And in 2014, Lafont, Prassidis and Wang [27] proved the general case.

Khoroshevskii [26] proved that every automorphism ϕ has order bounded by $|G| - 1$, provided G is not the trivial group, i.e. $|\phi| \leq |G| - 1$ and the equality is achieved only for elementary abelian groups. So we have the following

Corollary 7.23. *Let R be a ring, G a finite group, $\alpha\colon G \longrightarrow G$ a group automorphism, then $NK_n(RG, \alpha)$ is either trivial, or infinitely generated.*

7.4 Vanishing results

Since Nil-groups are either zero or infinitely generated, we firstly sumarize some vanishing results, and then give some non-vanishing results.

- If R is regular, then $NK_n(R) = 0$ for all n and $K_n(R) = 0$ for $n < 0$ (Quillen). Moreover for any ring automorphism α, $NK_n(R, \alpha) = 0$ for all n (Farrell and Hsiang).
- (Connolly and Prassidis [6]) Let $\alpha : R \longrightarrow R$ be an automorphism, if R is α-quasi-regular (that is, there exist a two-sided nilpotent ideal J of R which is α-invariant such that R/J is regular), then $NK_n(R, \alpha) = 0$ for any $n \leq 0$.
- $NK_1(\mathbb{Z}F) = 0$ for any free group F of finite rank.
- Group rings of nontrivial finite groups:
  - Let G be a finite group , then $NK_n(\mathbb{Z}G) = 0$ for $n \leq -1$.
  - (Harmon [19]) If G is a finite group of square-free order, then $NK_n(\mathbb{Z}G) = 0$ for $n \leq 1$. Moreover, if $n \leq 1$, then $N^j K_n(\mathbb{Z}G) = 0$ for $j \geq 1$.
  - (Juan-Pineda and Ramos [22]) If G is a non-trivial cyclic group of square-free order, then $NK_n(\mathbb{Z}G, \alpha) = 0$ for $n \leq 1$ and any group automorphism $\alpha : G \longrightarrow G$.

7.5 Non-vanishing results

Note that very few Bass Nil-groups are explicitly known and virtually nothing is known about the NK_i for $i \geq 3$.

C. Weibel [51] computed some explicit examples and gave $W(\mathbb{F}_2)$-module structure on some of them (omitted here):

- $NK_0(\mathbb{Z}[C_4]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, $NK_1(\mathbb{Z}[C_4]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.
- $NK_0(\mathbb{Z}[C_2 \times C_2]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, $NK_1(\mathbb{Z}[C_2 \times C_2]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.
- $NK_2(\mathbb{Z}[C_2]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$.
- $NK_i(\mathbb{Z}[D_4]) \neq 0$ for $i = 0, 1$.

Guin-Walery and Loday [17] proved that $NK_2(\mathbb{Z}[C_p]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/p\mathbb{Z}$, and is generated by Dennis-Stein symbols $\langle (1-\sigma)x^j, (1+\sigma+\cdots+\sigma^{p-1})\rangle$, where C_p is the cyclic group of prime order p and generated by σ.

D. J.-Pineda [21] gave some non-vanishing results:

- Let C_n be a nontrivial finite cyclic group, then $NK_2(\mathbb{Z}[C_n]) \neq 0$. C. Weibel proved ([52] chapter III, 3.4.2]) that for any ring if $N^sK_i(R) = 0$, it follows that $N^jK_i(R) = 0$ for $j = 1, 2, \cdots, s-1$. Therefore $N^jK_2(\mathbb{Z}[C_n]) \neq 0$, for all $j \geq 1$.
- Let G be a nontrivial finite cyclic group or a split extension of a nontrivial finite cyclic group. Then $N^jK_n(\mathbb{Z}[G]) \neq 0$ for all $n \geq 2$ and $j \geq n-1$.
- By lemma 7.25, if $NK_2(R) \neq 0$, then $N^jK_i(R) \neq 0$ for all $i \geq 2$ and $j \geq i-1$.

Proposition 7.24 (Vorst [47], Corollary 2.1). *For all $n \geq 1$ we have*
(1) $NK_n(R[x]) = 0$ implies $NK_{n-1}(R) = 0$.
(2) K_n-regularity implies K_{n-1}-regularity.

Lemma 7.25. *Let R be a commutative ring with identity.*
(1)If $N^2K_n(R) = 0$, then $NK_n(R) = 0$ and $NK_{n-1}(R) = 0$.
(2)Moreover, let $j \geq 2$ be an integer, if $N^jK_n(R) = 0$, then $N^iK_n(R) = 0$ for $1 \leq i \leq j$ and $N^{j-1}K_{n-1}(R) = 0$. Equivalently, $N^{j-1}K_{n-1}(R) \neq 0$ implies $N^jK_n(R) \neq 0$, and $N^iK_n(R) \neq 0$ implies $N^kK_n(R) \neq 0$ for $k \geq i \geq 1$.

Proof. For any functor F, $NF(R)$ is a summand of $N^pF(R)$. If $N^sK_i(R) = 0$, it follows that $N^jK_i(R) = 0$ for $j = 1, 2, \cdots, s-1$ ([52] chapter III, 3.4.2]). So $N^2K_n(R) = 0$ implies $NK_n(R) = 0$, hence $NK_n(R[x]) \cong NK_n(R) \oplus N^2K_n(R) = 0$. By proposition 7.24, $NK_{n-1}(R) = 0$.

Next, replace R by $R[x]$, one has $N^2K_n(R[x]) = 0$ implies $NK_{n-1}(R[x]) = 0$. $N^2K_n(R[x]) \cong N^2K_n(R) \oplus N^3K_n(R) = 0$ and $NK_{n-1}(R[x]) = NK_{n-1}(R) \oplus N^2K_{n-1}(R) = 0$, so $N^3K_n(R) = 0$ implies $N^2K_{n-1}(R) = 0$. Hence the second part is obtained by induction. $\square$

Chapter 8

$NK_2(\mathbb{F}_{2^f}[C_{2^n}])$

English title:NK_2-group of $\mathbb{F}_{2^f}[C_{2^n}]$

In this paper, we calculated $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ using the method from van der Kallen's paper. In particular, we also determined the explicit structure and $W(\mathbb{F}_2)$-module structure in Steinberg symbols of $NK_2(\mathbb{F}_2[C_2])$.

Keywords: NK-group, Dennis-Stein symbols, group algebra, Witt vectors

2010 Mathematics Subject Classification: 19C99, 16S34(Group rings)

8.1 Introduction

- history. The properties of NK-group(just intro here), compute NK_1, and the computation of $NK_2(\mathbb{Z}[C_p])$, the computation of dual numbers $\mathbb{F}_2[C_2]$. (计算 $\mathbb{F}_p[C_p]$, use the relative sequences by steps, maybe some of the sequence is not spilt exact.)
- introduce the main results

Bass 和 Murthy [4] 首先给出了群 G 的 Whitehead 群 $Wh(G) = K_1(\mathbb{Z}G)/\{\pm g | g \in G\}$ 不是有限生成的例子, 他们的例子来源于计算某些环 R 的 NK-群 $NK_1(R)$. For any $n \in \mathbb{Z}$, the Bass Nil-group $NK_n(R)$ of a unital ring R is defined to be the kernel of the map $K_n(R[x]) \longrightarrow K_n(R)$ which is induced by $R[x] \longrightarrow R, t \mapsto 0$, that is,

$$NK_n(R) = \ker(K_n(R[x]) \xrightarrow{x \mapsto 0} K_n(R)).$$

当 $n < 0$ 时 K_n 是 Bass 定义的负 K-理论, $n = 0, 1, 2$ 时, K_0, K_1, K_2 是由 Grothendieck, Bass, Milnor 等定义的经典 K-理论, 当 $n \geq 2$ 时, K_n 是由 Quillen 等定义的高阶 K-理

论. 在计算环 $R[\mathbb{Z}] = R[t, t^{-1}]$ 的 K-理论中, NK-群作为 $K_n(R[t, t^{-1}])$ 的直和项自然产生. 1977 年 Farrell 在文献 [8] 中证明了关于 NK-群的重要性质, 即 if $NK_1(R) \neq 0$, 则 $NK_1(R)$ 不是有限生成的. 实际上这一结论对于任何 $NK_i(i \leq 1)$ 均成立的, 并 and 后来 van der Kallen [45] 与 Prasolov [32] 证明对任何 $NK_i(i > 1)$ 也是成立的. Weibel [48] 在 Almkvist [2] 与 Grayson [14] 等的基础上将 NK-群与 Witt 向量联系起来. 对于交换正则环 R, Weibel [52] 给出了计算 $NK_1(R[x]/(x^n))$ 的方法, 而对于 $R[x]/(x^n)$ 这样的截断多项式环, van der Kallen [44] [46] 与 Stienstra [41] 等对这类环的 K_2 群做了详细的研究. 本文我们根据文献 [46] 中的方法利用截断多项式的 K_2 群计算了 $NK_2(\mathbb{F}_2[C_2])$, 其中 $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[t]/(t^2)$, 后者称为 $\mathbb{F}_2$ 上的对偶数环, 利用 Dennis-Stein 符号给出了 $NK_2(\mathbb{F}_2[C_2])$ 的一组生成元, 并给出了 $W(\mathbb{F}_2)$-模结构. 在不引起歧义的情况下, 记 x 在 $R[x]/(x^n)$ 下的像仍为 x.

8.2 Preliminaries

- relative sequence, Definition of NK-groups, regular rings(K_i-regular rings chapter 3 P22 in Weibel's K-book) Bass-Heller-Swan formula(the fundamental theorem of K-theory.)
- farrell NK-groups are either 0 or not finitely generated as abelian groups.
- Witt vectors and Witt decompositions
- the notations and the theorem of van der Kallen and Stiensta.

8.2.1 Relative K-groups

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^{n_1}, t_2^{n_2}, \cdots, t_r^{n_r})$ be a proper ideal in the polynomial ring $k[t_1, t_2, \cdots, t_s]$ where $1 \leq r \leq s$. Put $A = k[t_1, t_2, \cdots, t_s]/I$, let $M = (t_1, \cdots, t_r)$ be the nilradical of A, then $A/M = k[t_{r+1}, \cdots, t_s]$.

Proposition 8.1. *Given A, M as above, $K_2(A) \cong K_2(A, M) \cong K_2(A, (t_1, t_2, \cdots, t_s))$.*

Proof. Since $k[t_{r+1}, \cdots, t_s] \overset{i_1}{\hookrightarrow} k[t_1, t_2, \cdots, t_s]/I$, $k[t_1, t_2, \cdots, t_s]/I \overset{p_1}{\twoheadrightarrow} k[t_{r+1}, \cdots, t_s]$ and $k \overset{i_2}{\hookrightarrow} k[t_1, t_2, \cdots, t_s]/I$, $k[t_1, t_2, \cdots, t_s]/I \overset{p_2}{\twoheadrightarrow} k$ satisfying $p_1 i_1 = \mathrm{id}$, $p_2 i_2 = \mathrm{id}$, one has $K_n(p_1)K_n(i_1) = \mathrm{id}$ and $K_n(p_2)K_n(i_2) = \mathrm{id}$ by the functority of K_n. Then there are two split exact sequences of relative K-groups

$$0 \longrightarrow K_2(A, M) \longrightarrow K_2(A) \longrightarrow K_2(k[t_{r+1}, \cdots, t_s]) \longrightarrow 0$$

$$0 \longrightarrow K_2(A,(t_1,t_2,\cdots,t_s)) \longrightarrow K_2(A) \longrightarrow K_2(k) \longrightarrow 0$$

Since k is a finite field, $K_2(k)=0$, and every finite field is regular, hence K_i-regular, that is $K_2(k[t_{r+1},\cdots,t_s]) = K_2(k) = 0$. Therefore

$$K_2(A,(t_1,\cdots,t_s)) \cong K_2(A) \cong K_2(A,M). \qquad \square$$

8.2.2 Dennis-Stein symbols

In general, one has a presentation for $K_2(A,M) = K_2(k[t_1,t_2]/(t_1^n),(t_1))$ in terms of Dennis-Stein symbols:

generators: $\langle a,b\rangle$, for $(a,b) \in A \times M \cup M \times A$;

relations: $\langle a,b\rangle = -\langle b,a\rangle$,

$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc,b\rangle$,

$\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ for $(a,b,c) \in A \times M \times A \cup M \times A \times M$.

Proposition 8.2. *For every ring R and all integers $q>1$, the relative group $K_2(R[t]/(t^q),(t))$ is generated by the elements $\langle at^i,t\rangle$ and $\langle at^i,b\rangle$ with $a,b \in R$ and $1 \le i < q$.*

Proof. See [41] Proposition 1.7. $\square$

8.2.3 Notations

The notations and results below are taken from van der Kallen and Stienstra [46].

- Let $\mathbb{N}$ be the monoid of natural numbers.
- Let $\varepsilon^1 = (1,0) \in \mathbb{N}^2$, $\varepsilon^2 = (0,1) \in \mathbb{N}^2$.
- For $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\varepsilon^1} = t_1$, $t^{\varepsilon^2} = t_2$.
- Put $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$.
- Put $\Lambda = \{(\alpha,i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \ge 1, t^\alpha \in M\}$.
- For $(\alpha,i) \in \Lambda$, set $[\alpha,i] = \min\{m \in \mathbb{Z} \mid m\alpha - \varepsilon^i \in \Delta\}$.
- If $gcd(p,\alpha_1,\alpha_2)=1$, set $[\alpha] = \max\{[\alpha,i] \mid \alpha_i \not\equiv 0 \bmod p\}$.
- Put $\Lambda^{00} = \{(\alpha,i) \in \Lambda \mid gcd(\alpha_1,\alpha_2)=1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha,j]=[\alpha]\}\}$.

If $(\alpha,i) \in \Lambda$ and $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1-xf(x)) = \langle f(t^\alpha)t^{\alpha-\varepsilon^i}, t_i\rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1+xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A,M).$$

Theorem 8.3. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See [46] Corollary 2.6. $\square$

Corollary 8.4. *Let $\mathcal{B}$ be a basis of k as a vector space over $\mathbb{F}_p$. Then $K_2(A,M)$ has a presentation, as an abelian group, with*
generators: $\langle bt^{\alpha-\varepsilon^i}, t_i, \rangle$ where $b \in \mathcal{B}$, $(a,i) \in \Lambda^0 = \{(m\alpha, i) \in \Lambda \mid gcd(m,p) = 1, (\alpha,i) \in \Lambda^{00}\}$;
relations: $p^{w(\alpha,i)}\langle bt^{\alpha-\varepsilon^i}, t_i\rangle = 0$ where $w(\alpha,i) = \min\{w \in \mathbb{N} \mid p^w \geq [\alpha,i]\}$.
Thus $K_2(A,M)$ is a p-group and is a module over $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \mid gcd(p,n) = 1\}$.

8.2.4 Big Witt vectors and Witt decomposition

令 R 是一个交换环, big Witt 环 (the ring of universal/big Witt vectors over R, 泛 Witt 环)$BigWitt(R)$ 作为 abelian group 同构于 $(1+xR[\![x]\!])^\times$, 即常数项为 1 的形式幂级数全体在乘法运算下形成的交换群,

$$\begin{aligned} BigWitt(R) &\longrightarrow (1+xR[\![x]\!])^\times \\ (r_1, r_2, \cdots) &\mapsto \prod_i (1-r_i x^i)^{-1}. \end{aligned}$$

考虑子群 $(1+x^{n+1}R[\![x]\!])^\times$, 定义 $BigWitt_n(R) = (1+xR[\![x]\!])^\times/(1+x^{n+1}R[\![x]\!])^\times$. 显然 $BigWitt_1(R) = R$, 并 and 当 $n \geq 3$ 时, $BigWitt_n(\mathbb{F}_2)$ 不是循环群.

Example 8.5. $BigWitt_3(\mathbb{F}_2) \cong (1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Proof. 由定义 $BigWitt_3(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^\times/(1+x^4\mathbb{F}_2[\![x]\!])^\times$, and 有同态

$$\begin{aligned} (1+x\mathbb{F}_2[\![x]\!])^\times &\longrightarrow (1+x\mathbb{F}_2[x]/(x^4))^\times \\ 1+\sum_{i\geq 1} a_i x^i &\mapsto 1+a_1x+a_2x^2+a_3x^3 \end{aligned}$$

它的核是 $(1+x^4\mathbb{F}_2[\![x]\!])^\times$. 从而 $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong BigWitt_3(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^\times/(1+x^4\mathbb{F}_2[\![x]\!])^\times$.

考虑 $1+x \in (1+x\mathbb{F}_2[x]/(x^4))^\times$ 是 4 阶元, 由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3\}$, and $1+x^3$ 是二阶元, 令 σ, τ 分别是 $\mathbb{Z}/4\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元, 则有

同构

$$\begin{aligned}\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma, \tau) &\mapsto (1+x)(1+x^3) = 1+x+x^3.\end{aligned}$$

□

Example 8.6. $BigWitt_4(\mathbb{F}_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. 由定义 $BigWitt_4(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^\times/(1+x^5\mathbb{F}_2[\![x]\!])^\times$, and 有同态

$$(1+x\mathbb{F}_2[\![x]\!])^\times \longrightarrow (1+x\mathbb{F}_2[x]/(x^5))^\times$$

它的核是 $(1+x^5\mathbb{F}_2[\![x]\!])^\times$. 从而 $(1+x\mathbb{F}_2[x]/(x^5))^\times \cong BigWitt_4(\mathbb{F}_2) = (1+x\mathbb{F}_2[\![x]\!])^\times/(1+x^5\mathbb{F}_2[\![x]\!])^\times$.

考虑 $1+x \in BigWitt_5(\mathbb{F}_2)$, 它是 8 阶元, 由它生成的子群 $\langle 1+x\rangle = \{1, 1+x, 1+x^2, 1+x+x^2+x^3, 1+x^4, 1+x+x^4, 1+x^2+x^4, 1+x+x^2+x^3+x^4\}$, 另外 $1+x^3$ 是二阶元, 令 σ, τ 分别是 $\mathbb{Z}/8\mathbb{Z}$ 和 $\mathbb{Z}/2\mathbb{Z}$ 的生成元, 则有同构

$$\begin{aligned}\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow BigWitt_4(\mathbb{F}_2)\\ (\sigma, \tau) &\mapsto (1+x)(1+x^3) = 1+x+x^3+x^4\end{aligned}$$

于是 $(\sigma^i, \tau^j), 0 \le i < 8, 0 \le j < 2$ 对应于 $(1+x)^i(1+x^3)^j$, 详细的对应如下

$$\begin{aligned}
(1,\tau) &\mapsto 1+x^3, & (\sigma,\tau) &\mapsto 1+x+x^3+x^4,\\
(\sigma^2,\tau) &\mapsto 1+x^2+x^3, & (\sigma^3,\tau) &\mapsto 1+x+x^2+x^4,\\
(\sigma^4,\tau) &\mapsto 1+x^3+x^4, & (\sigma^5,\tau) &\mapsto 1+x+x^3,\\
(\sigma^6,\tau) &\mapsto 1+x^2+x^3+x^4, & (\sigma^7,\tau) &\mapsto 1+x+x^2,\\
(1,1) &\mapsto 1, & (\sigma,1) &\mapsto 1+x,\\
(\sigma^2,1) &\mapsto 1+x^2, & (\sigma^3,1) &\mapsto 1+x+x^2+x^3,\\
(\sigma^4,1) &\mapsto 1+x^4, & (\sigma^5,1) &\mapsto 1+x+x^4,\\
(\sigma^6,1) &\mapsto 1+x^2+x^4, & (\sigma^7,1) &\mapsto 1+x+x^2+x^3+x^4.
\end{aligned}$$

□

Fix a prime p, consider the local ring $\mathbb{Z}_{(p)} = \mathbb{Z}[1/\ell \mid \text{all primes } \ell \neq p]$, it is the localization of $\mathbb{Z}$ at prime ideal $(p) = p\mathbb{Z}$. Consequently, a $\mathbb{Z}_{(p)}$-algebra R is a ring which all primes other than p are invertible. For example, $\mathbb{F}_{p^n}$ is a $\mathbb{Z}_{(p)}$-algebra.

考虑 p-Witt 环 $W(A)$ 与截断 p-Witt 环 $W_n(A)$, p-Witt 向量为 $(a_0, a_1, \cdots)$, 加法用 Witt 多项式定义, 以下仅考虑用加法定义的 abelian group 结构, 例如 $W(\mathbb{F}_p) = \mathbb{Z}_p$, 作为 abelian group$W_n(\mathbb{F}_{p^f})$ 同构于 $(\mathbb{Z}/p^n\mathbb{Z})^f$.

The Artin-Hasse exponential is the formal series

$$AH(x) = \exp(-\sum_{n\geq 0} \frac{x^{p^n}}{p^n}) = 1 - x + \cdots \in 1 + x\mathbb{Q}[\![x]\!],$$

In fact, $AH(x) \in 1 + x\mathbb{Z}_{(p)}[\![x]\!]$. For any ring R, any element $\alpha \in BigWitt(R) = 1 + xR[\![x]\!]$ has a unique expression as an infinite product

$$\alpha = \prod_{n\geq 1}(1 - r_n x^n),\ r_n \in R.$$

Moreover, if A is a $\mathbb{Z}_{(p)}$-algebra, $\alpha \in BigWitt(A) = 1 + xA[\![x]\!]$ has a unique expression as an infinite product

$$\alpha = \prod_{n\geq 1} AH(a_n x^n),\ a_n \in A.$$

Write $n = mp^a$ such that $gcd(m,p) = 1$ and $a \geq 0$, if A is a $\mathbb{Z}_{(p)}$-algebra and m is invertible, then $[x \mapsto x^{1/m}] \in \mathrm{End}(BigWitt(A))$ is a bijection, we may write $\alpha \in BigWitt(A)$ uniquely as an infinite product

$$\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}.$$

On the other hand, for a $\mathbb{Z}_{(p)}$-algebra A, the following map is a group homomorphism

$$\begin{aligned} W(A) &\longrightarrow BigWitt(A)\\ (a_0, a_1, \cdots) &\mapsto \prod_{i\geq 0} AH(a_i x^i). \end{aligned}$$

$BigWitt_n(A)$ 可以分解为 p-Witt 环的直和, 实际上有以下同构

$$BigWitt(A) \cong \prod_{\substack{m\geq 1\\ gcd(m,p)=1}} W(A),$$

元素 $\prod_{\substack{m\geq 1\\ gcd(m,p)=1\\ a\geq 0}} AH(a_{mp^a} x^{mp^a})^{1/m}$ 对应于一个 m-分量为 $(a_m, a_{mp}, a_{mp^2}, \cdots) \in W(A)$ 的 Witt 向量. 对于截断的 Witt 环, 有同构

$$BigWitt_n(A) \cong \bigoplus_{\substack{1\leq m\leq n\\ gcd(m,p)=1}} W_{\ell(m,n)}(A),$$

where $\ell(m,n)$ is an integer defined by $\ell(m,n) = 1 + \left\lfloor \log_p \frac{n}{m} \right\rfloor$, i.e.

$$\ell(m,n) = 1 + \text{the largest integer } k \text{ such that } mp^k \leq n.$$

Now let $\mathbb{F}_q$ be a finite field of characteristic p, then [28]

$$BigWitt_n(\mathbb{F}_q) \cong \bigoplus_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} W_{\ell(m,n)}(\mathbb{F}_q).$$

Note that both sides are groups of order q^n, because of $\sum_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} \ell(m,n) = n$.

Corollary 8.7. *Let $\mathbb{F}_{p^f}$ be the finite field of characteristic p, then as an abelian group*

$$BigWitt_n(\mathbb{F}_{p^f}) \cong \bigoplus_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} W_{1+\left\lfloor \log_p \frac{n}{m} \right\rfloor}(\mathbb{F}_{p^f}) = \bigoplus_{\substack{1 \leq m \leq n \\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\left\lfloor \log_p \frac{n}{m} \right\rfloor}\mathbb{Z})^f,$$

where $\lfloor x \rfloor$ denotes the largest integer no more than x.

Example 8.8. $BigWitt_3(\mathbb{F}_2) = W_{\ell(1,3)}(\mathbb{F}_2) \oplus W_{\ell(3,3)}(\mathbb{F}_2) = W_2(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $BigWitt_4(\mathbb{F}_2) = W_{\ell(1,4)}(\mathbb{F}_2) \oplus W_{\ell(3,4)}(\mathbb{F}_2) = W_3(\mathbb{F}_2) \oplus W_1(\mathbb{F}_2) = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $BigWitt_2(\mathbb{F}_3) = W_{\ell(1,2)}(\mathbb{F}_3) \oplus W_{\ell(2,2)}(\mathbb{F}_3) = W_1(\mathbb{F}_3) \oplus W_1(\mathbb{F}_3) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

8.2.5 Artin-Hasse exponential

Fix a prime p.

$$AH(x) = \exp(\sum_{n \geq 0} \frac{x^{p^n}}{p^n}) = \exp(x + \frac{x^p}{p} + \cdots)$$

$$\begin{aligned}
1 - x &= \exp(\ln(1-x)) = \exp(\sum_{i \geq 1} \frac{x^i}{i}) \\
&= \exp(\sum_{m \geq 1, gcd(m,p)=1} \sum_{n \geq 0} \frac{x^{mp^n}}{mp^n}) = \prod_{m \geq 1, gcd(m,p)=1} \exp(\sum_{n \geq 0} \frac{x^{mp^n}}{mp^n}) \\
&= \prod_{m \geq 1, gcd(m,p)=1} \exp(\frac{1}{m} \sum_{n \geq 0} \frac{x^{mp^n}}{p^n}) \\
&= \prod_{m \geq 1, gcd(m,p)=1} \left(\exp(\sum_{n \geq 0} \frac{(x^m)^{p^n}}{p^n}) \right)^{\frac{1}{m}} \\
&= \prod_{m \geq 1, gcd(m,p)=1} \left(AH(x^m) \right)^{\frac{1}{m}}
\end{aligned}$$

$$1 - x = \prod_{m \geq 1, gcd(m,p)=1} \left(AH(x^m)\right)^{\frac{1}{m}}$$

$$1 - r_n x^n = \prod_{m \geq 1, gcd(m,p)=1} \left(AH((r_n x^n)^m)\right)^{\frac{1}{m}}$$

$$\prod_{n=1}^{\infty}(1 - r_n x^n) = \prod_{n \geq 1} \prod_{m \geq 1, gcd(m,p)=1} \left(AH((r_n x^n)^m)\right)^{\frac{1}{m}}$$

p-Witt vector $(a_0, a_1, \cdots, a_n)$ correspond to big Witt vector $(x_1, x_2, \cdots)$ with $x_1 = a_0$, $x_p = a_1 p$, $x_{p^i} = a_i p^i$ and for other k, $x_k = 0$.

$(1, 1, 0, 0, \cdots) \in BigWitt(R)$ correspond to $(1-x)(1-x^2) \in (1 + xR[[x]])^{\times}$,$(r_1, r_2, \cdots) \in BigWitt(R)$ correspond to $\prod_{n \geq 1}(1 - r_n x^n) \in (1 + xR[[x]])^{\times}$

8.3 The computation of NK_2 groups

Define $NK_i(R) = \ker(K_i(R[x]) \xrightarrow{x \mapsto 0} K_i(R))$ for $i \in \mathbb{Z}$. We summand some properties of Bass NK-groups here.

Proposition 8.9. *Let R be a ring,*
(1) For $i \in \mathbb{Z}$, if $NK_i(R) \neq 0$, then $NK_i(R)$ is not finitely generated.
(2) ([27] Theorem B) If $H \leq NK_i(R)$ is a finite subgroup, then $\bigoplus_{\infty} H$ also appears as a subgroup of $NK_i(R)$. Moreover, if H is a direct summand in $NK_i(R)$, then so is $\bigoplus_{\infty} H$.
(3) ([27] Theorem C) If $NK_i(R)$ has finite exponent, then there exists a finite abelian group H, so that $NK_i(R) \cong \bigoplus_{\infty} H$.

Lemma 8.10. 无限群的结构，待补充

Proposition 8.11 (Vorst [47], Corollary 2.1)**.** *For all $n \geq 1$ we have*
(1) $NK_n(R[x]) = 0$ implies $NK_{n-1}(R) = 0$.
(2) K_n-regularity implies K_{n-1}-regularity.

Lemma 8.12. *Let R be a commutative ring with identity.*
(1)If $N^2K_n(R) = 0$, then $NK_n(R) = 0$ and $NK_{n-1}(R) = 0$.
(2)Moreover, let $j \geq 2$ be an integer, if $N^jK_n(R) = 0$, then $N^iK_n(R) = 0$ for $1 \leq i \leq j$ and $N^{j-1}K_{n-1}(R) = 0$. Equivalently, $N^{j-1}K_{n-1}(R) \neq 0$ implies $N^jK_n(R) \neq 0$, and $N^iK_n(R) \neq 0$ implies $N^kK_n(R) \neq 0$ for $k \geq i \geq 1$.

Proof. For any functor F, $NF(R)$ is a summand of $N^pF(R)$. If $N^sK_i(R) = 0$, it follows that $N^jK_i(R) = 0$ for $j = 1, 2, \cdots, s-1$ ([52] chapter III, 3.4.2]). So $N^2K_n(R) = 0$ implies $NK_n(R) = 0$, hence $NK_n(R[x]) \cong NK_n(R) \oplus N^2K_n(R) = 0$. By the above proposition, $NK_{n-1}(R) = 0$.

Next, replace R by $R[x]$, one has $N^2K_n(R[x]) = 0$ implies $NK_{n-1}(R[x]) = 0$. $N^2K_n(R[x]) \cong N^2K_n(R) \oplus N^3K_n(R) = 0$ and $NK_{n-1}(R[x]) = NK_{n-1}(R) \oplus N^2K_{n-1}(R) = 0$, so $N^3K_n(R) = 0$ implies $N^2K_{n-1}(R) = 0$. Hence the second part is obtained by induction. □

Corollary 8.13. *If $NK_2(R) \neq 0$, then $N^jK_i(R) \neq 0$ for all $i \geq 2$ and $j \geq i-1$.*

Lemma 8.14. *Let k be a finite field of characteristic $p > 0$, then for any integer $n > 1$, $NK_2(k[t]/(t^n)) \cong K_2(k[t,s]/(t^n), (t))$.*

Proof. First recall that $K_2(k[t,s]/(t^n)) \cong K_2(k[t,s]/(t^n), (t)) \cong K_2(k[t,s]/(t^n), (t,s))$ by proposition 8.1. For a finite field k, $K_2(k[t]/(t^n)) \cong K_2(k) = 0$ ([30] Theorem 3.2). Hence by definition, $NK_2(k[t]/(t^n)) = \ker(K_2(k[t,s]/(t^n)) \longrightarrow K_2(k[t]/(t^n))) \cong K_2(k[t,s]/(t^n))$. □

When $k = \mathbb{F}_{p^f}$, there is a group ring isomorphism

$$\mathbb{F}_{p^f}[t]/(t^{p^n}) = \mathbb{F}_{p^f}[t]/(1-t)^{p^n} \cong \mathbb{F}_{p^f}[C_{p^n}],$$
$$1 - t_1 \mapsto \sigma,$$

where C_{p^n} is the cyclic group of order p^n and $\mathbb{F}_{p^f}[C_{p^n}][x] = \mathbb{F}_{p^f}[t,s]/(t^{p^n}) = \mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})$. So

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n})) \cong K_2(\mathbb{F}_{p^f}[t_1,t_2]/(t_1^{p^n}), (t_1)).$$

8.4 The computation of $NK_2(\mathbb{F}_2[C_2])$

In this section, we compute the case of $k = \mathbb{F}_2$, $p = 2$, $n = 2$, i.e. $K_2(\mathbb{F}_2[t_1,t_2]/(t_1^2), (t_1)) \cong NK_2(\mathbb{F}_2[C_2])$.

Theorem 8.15. *(1)$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$, where the expression $\bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$ refers to a countable infinite sum of $\mathbb{Z}/2\mathbb{Z}$*
(2)$NK_2(\mathbb{F}_2[C_2]) \cong K_2(\mathbb{F}_2[t,x]/(t^2),(t))$ is generated by these Dennis-Stein symbols of order 2: $\{\langle tx^i, x\rangle \mid i \geq 0\}$, $\{\langle tx^i, t\rangle \mid i \geq 1$ is odd$\}$.

Proof. (1)Put $A=\mathbb{F}_2[t_1,t_2]/(t_1^2)=\mathbb{F}_2[C_2][x]$, $I=(t_1^2)$, $M=(t_1)$, then $A/M=\mathbb{F}_2[x]$.

$$\begin{aligned}
\Delta &= \{(\alpha_1,\alpha_2)\in\mathbb{N}^2 \mid t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1^2)\}\\
&= \{(\alpha_1,\alpha_2)\mid \alpha_1\geq 2,\alpha_2\geq 0\}\\
\Lambda &= \{((\alpha_1,\alpha_2),i)\in\mathbb{N}^2\times\{1,2\}\mid \alpha_i\geq 1, \text{ and } t_1^{\alpha_1}t_2^{\alpha_2}\in(t_1)\}\\
&= \{((\alpha_1,\alpha_2),i)\in\mathbb{N}^2\times\{1,2\}\mid \alpha_i\geq 1,\alpha_1\geq 1,\alpha_2\geq 0\}\\
&= \{((\alpha_1,\alpha_2),1)\mid \alpha_1\geq 1,\alpha_2\geq 0\}\cup\{((\alpha_1,\alpha_2),2)\mid \alpha_1\geq 1,\alpha_2\geq 1\}.
\end{aligned}$$

If $(\alpha,i)\in\Lambda$, define $[\alpha,i]=\min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^i\in\Delta\}$, one has

$$\begin{aligned}
[\alpha,1] &= \min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^1\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid (m\alpha_1-1,m\alpha_2)\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid m\alpha_1\geq 3\}.\\
[\alpha,2] &= \min\{m\in\mathbb{Z}\mid m\alpha-\varepsilon^2\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid (m\alpha_1,m\alpha_2-1)\in\Delta\}\\
&= \min\{m\in\mathbb{Z}\mid m\alpha_1\geq 2\}.
\end{aligned}$$

And

$$\begin{aligned}
[(1,\alpha_2),1] &= 3,\ \alpha_2\geq 0,\\
[(2,\alpha_2),1] &= 2,\ \alpha_2\geq 0,\\
[(\alpha_1,\alpha_2),1] &= 1,\ \alpha_1\geq 3,\alpha_2\geq 0,\\
[(1,\alpha_2),2] &= 2,\ \alpha_2\geq 1,\\
[(\alpha_1,\alpha_2),2] &= 1,\ \alpha_1\geq 2,\alpha_2\geq 1.
\end{aligned}$$

If $gcd(2,\alpha_1,\alpha_2)=1$, i.e., at least one of α_1 and α_2 is odd, set $[\alpha]=\max\{[\alpha,i]\mid \alpha_i\not\equiv 0 \bmod 2\}$, $\alpha=(\alpha_1,\alpha_2)$,if only α_1 is odd, $[\alpha]=[\alpha,1]$,if only α_2 is odd, $[\alpha]=[\alpha,2]$, if both α_1 and α_2 are odd, then $[\alpha]=\max\{[\alpha,1],[\alpha,2]\}$. Then we get the following results and put all of them into a chart for conviniece

$$
\begin{aligned}
[(1,\alpha_2)] &= \max\{[(1,\alpha_2),1],[(1,\alpha_2),2]\} = 3, \alpha_2 \geq 1 \text{ is odd} \\
[(1,\alpha_2)] &= [(1,\alpha_2),1] = 3, \alpha_2 \geq 0 \text{ is even} \\
[(3,\alpha_2)] &= \max\{[(3,\alpha_2),1],[(3,\alpha_2),2]\} = 1, \alpha_2 \geq 1 \text{ is odd} \\
[(3,\alpha_2)] &= [(3,\alpha_2),1] = 1, \alpha_2 \geq 0 \text{ is even} \\
[(2,1)] &= [(2,1),2] = 1, \\
[\alpha] &= 1, \text{ other } \alpha.
\end{aligned}
$$

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$3,\alpha_2 \geq 0$	$2,\alpha_2 \geq 1$	3
$(2,\alpha_2)$	$2,\alpha_2 \geq 0$	$1,\alpha_2 \geq 1$	1, if α_2 is odd
$(3,\alpha_2)$	$1,\alpha_2 \geq 0$	$1,\alpha_2 \geq 1$	1
$(\alpha_1,0),\alpha_1 \geq 3$	1	-	1, if α_1 is odd
$(\alpha_1,\alpha_2),\alpha_1 \geq 3,\alpha_2 \geq 1$	1	1	1, if $(\alpha_1,\alpha_2)=1$

下面我们计算 $\Lambda^{00} = \{(\alpha,i) \in \Lambda \mid gcd(\alpha_1,\alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\}\}$,

分情况来讨论

1. 对于任何的 $\alpha_2 \geq 0$, $((1,\alpha_2),1) \notin \Lambda^{00}$, 这是因为 $1 \not\equiv 0 \bmod 2$and $[(1,\alpha_2),1] = 3 = [(1,\alpha_2)]$, 从而 $\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(1,\alpha_2),j] = [(1,\alpha_2)]\} = 1$；
2. 对于任何的奇数 $\alpha_2 \geq 0$, $((2,\alpha_2),1) \in \Lambda^{00}$, 偶数 $\alpha_2 \geq 0$, $((2,\alpha_2),1) \notin \Lambda^{00}$, 因为 $\alpha_2 \not\equiv 0 \bmod 2$ 并 and $[(2,\alpha_2),2] = 1 = [(2,\alpha_2)]$, 故 $\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(2,\alpha_2),j] = [(2,\alpha_2)]\} = 2 \neq 1$, 此时 $[(2,\alpha_2),1] = 2$；
3. 对于偶数 $\alpha_1 \geq 3$ 和奇数 $\alpha_2 \geq 1$, $((\alpha_1,\alpha_2),1) \in \Lambda^{00}$, 其余情况当 $\alpha_1 \geq 3$ is odd 或 α_1,α_2 均 is even 时 $((\alpha_1,\alpha_2),1) \notin \Lambda^{00}$. 由于要求 $1 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$, 当 $\alpha_1 \geq 3$ is odd 时上式不成立, $2 = \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$ 当 and 仅当 $\alpha_1 \geq 3$ is evenand $\alpha_2 \geq 1$ is odd, 此时 $[(\alpha_1,\alpha_2),1] = 1$；
4. 对于任何的 $\alpha_2 \geq 1$, $((1,\alpha_2),2) \in \Lambda^{00}$, 由于此时 $[(1,\alpha_2),1] = 3 = [(1,\alpha_2)]$, $\min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\} = 1$, 此时 $[(1,\alpha_2),2] = 2$；
5. 对于任何的奇数 $\alpha_2 \geq 1$, $((2,\alpha_2),2) \notin \Lambda^{00}$, 由于 $[(2,\alpha_2),2] = 1 = [(2,\alpha_2)]$, 与 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha,j] = [\alpha]\}$ 矛盾；
6. 对于奇数 $\alpha_1 \geq 3$ 和任意 $\alpha_2 \geq 1$, $((\alpha_1,\alpha_2),2) \in \Lambda^{00}$, 其余情况只要当 $\alpha_1 \geq 3$ is even

时 $((\alpha_1,\alpha_2),2) \notin \Lambda^{00}$. 要求 $2 \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [(\alpha_1,\alpha_2),j] = [(\alpha_1,\alpha_2)]\}$, 当 α_1 is even 时上式不成立, 而当 α_1 is odd 时, 任意 $\alpha_2 \geq 1$, $[(\alpha_1,\alpha_2),1] = 1 = [(\alpha_1,\alpha_2)]$, 此时 $[(\alpha_1,\alpha_2),2] = 1$.

从而

$$\begin{aligned}\Lambda^{00} =&\{((2,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\}\\ &\cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}\\ &\cup \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3 \text{ is even}, \alpha_2 \geq 1 \text{ is odd}\}\\ &\cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3 \text{ is odd}, \alpha_2 \geq 1\}.\end{aligned}$$

记 $\Lambda_1^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 1\}$, $\Lambda_2^{00} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = 2\}$, 我们有

$$\Lambda_1^{00} = \{((\alpha_1,\alpha_2),1) | \alpha_1 \geq 3 \text{ is even}, \alpha_2 \geq 1 \text{ is odd}\} \cup \{((\alpha_1,\alpha_2),2) | \alpha_1 \geq 3 \text{ is odd}, \alpha_2 \geq 1\}$$

$$\Lambda_2^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\} \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$$

$$\Lambda^{00} = \Lambda_1^{00} \sqcup \Lambda_2^{00}$$

if $[\alpha,i] = 1$ 时, $(1 + x\mathbb{F}_2[x]/(x))^\times$ 是平凡的, $[\alpha,i] = 2$ 时, $(1 + x\mathbb{F}_2[x]/(x^2))^\times \cong \mathbb{Z}/2\mathbb{Z}$, 从而由定理9.1得

$$\begin{aligned}NK_2(\mathbb{F}_2[C_2]) \cong K_2(A,M) \cong& \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times\\ =& \bigoplus_{(\alpha,i)\in\Lambda_2^{00}} (1 + x\mathbb{F}_2[x]/(x^2))^\times\\ =& \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1 + x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1 \text{ is odd}}} (1 + x\mathbb{F}_2[x]/(x^2))^\times\\ =& \bigoplus_{\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\alpha_2\geq 1 \text{ is odd}} \mathbb{Z}/2\mathbb{Z},\end{aligned}$$

作为 abelian group,

$$NK_2(\mathbb{F}_2[C_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z}.$$

(2) 由9.1, 对于任意 $(\alpha,i) \in \Lambda^{00}$, $\Gamma_{\alpha,i}$ 诱导了同态

$$\begin{aligned}\Gamma_{\alpha,i} \colon (1 + xk[x]/(x^{[\alpha,i]}))^\times &\longrightarrow K_2(A,M)\\ 1 - xf(x) &\mapsto \langle f(t^\alpha)t^{\alpha-\varepsilon^i}, t_i\rangle.\end{aligned}$$

此时只需考虑 $\Lambda_2^{00} = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\} \cup \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$, 对于任意 $(\alpha,i) \in \Lambda_2^{00}$, $\Gamma_{\alpha,i}$ 均诱导了单射, 对任意 $\alpha_2 \geq 1$,

$$\begin{aligned}\Gamma_{(1,\alpha_2),2} \colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1t_2^{\alpha_2-1}, t_2\rangle.\end{aligned}$$

对任意 $\alpha_2 \geq 1$ is odd,

$$\begin{aligned}\Gamma_{(2,\alpha_2),1} \colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\ 1+x &\mapsto \langle t_1t_2^{\alpha_2}, t_1\rangle,\end{aligned}$$

我们作简单的替换令 $t=t_1, x=t_2$, 于是 $\langle t_1t_2^{\alpha_2-1}, t_2\rangle = \langle tx^{\alpha_2-1}, x\rangle$, $\langle t_1t_2^{\alpha_2}, t_1\rangle = \langle tx^{\alpha_2}, t\rangle$. 由同构9.1可知 $NK_2(\mathbb{F}_2[C_2])$ 是由 Dennis-Stein 符号 $\{\langle tx^i, x\rangle \mid i \geq 0\}$ 与 $\{\langle tx^i, t\rangle \mid i \geq 1 \text{ is odd}\}$ 生成的, 由于 $t^2=0$ 故 $\langle tx^i,x\rangle + \langle tx^i,x\rangle = \langle tx^i+tx^i-t^2x^{2i+1}, x\rangle = 0$, $\langle tx^i,t\rangle + \langle tx^i,t\rangle = \langle tx^i+tx^i-t^3x^{2i}, t\rangle = 0$. □

Remark 8.16. 对于 $i \geq 1$ is even, $\langle tx^i,t\rangle = \langle x^{i/2},t\rangle + \langle x^{i/2},t\rangle = \langle x^{i/2}+x^{i/2}+tx^i, t\rangle = 0$.

Weibel 在文献 [51] 中给出了以下可裂正合列

$$0 \longrightarrow V/\Phi(V) \xrightarrow{F} NK_2(\mathbb{F}_2[C_2]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0,$$

其中 $V = x\mathbb{F}_2[x]$, $\Phi(V) = x^2\mathbb{F}_2[x^2]$ 是 V 的子群, $\Omega_{\mathbb{F}_2[x]} \cong \mathbb{F}_2[x]\,dx$ 是绝对 Kähler 微分模, $F(x^n) = \langle tx^n, t\rangle$, $D(\langle ft, g+g't\rangle) = f\,dg$. 显然 $D(\langle tx^i,t\rangle) = 0$, $D(\langle tx^i,x\rangle) = x^i\,dx$, 可以看出 $NK_2(\mathbb{F}_2[C_2])$ 的直和项 $\bigoplus_{((2,\alpha_2),1),\alpha_2\geq 1 \text{ is odd}} \mathbb{Z}/2\mathbb{Z} \cong V/\Phi(V)$, 直和项 $\bigoplus_{((1,\alpha_2),2),\alpha_2\geq 1} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[x]\,dx$.

V 和 $\Omega_{\mathbb{F}_2[x]}$ 作为 abelian group 是同构的, 但作为 $W(\mathbb{F}_2)$-模是不同的. $V = x\mathbb{F}_2[x]$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [7]) 为

$$\begin{aligned}V_m(x^n) &= x^{mn},\\ F_d(x^n) &= \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{其它}\end{cases},\\ [a]x^n &= a^nx^n.\end{aligned}$$

$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx$ 上的 $W(\mathbb{F}_2)$-模结构 (见 [7]) 为

$$V_m(x^{n-1}\,dx) = mx^{mn-1}\,dx,$$

$$F_d(x^{n-1}\,dx) = \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{其它} \end{cases},$$

$$[a]x^{n-1}\,dx = a^n x^{n-1}\,dx.$$

结合两者我们可以得到 $NK_2(\mathbb{F}_2[C_2])$ 的 $W(\mathbb{F}_2)$-模结构为

$$V_m(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{mn}, t\rangle, & \text{if } m \text{ is odd} \\ 0, & \text{if } m \text{ is even} \end{cases}, \quad n \geq 1 \text{ is odd}$$

$$V_m(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{mn-1}, x\rangle, & \text{if } m \text{ is odd} \\ 0, & \text{if } m \text{ is even} \end{cases}, \quad n \geq 1$$

$$F_d(\langle tx^n, t\rangle) = \begin{cases} \langle tx^{n/d}, t\rangle, & \text{if } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1 \text{ is odd}$$

$$F_d(\langle tx^{n-1}, x\rangle) = \begin{cases} \langle tx^{n/d-1}, x\rangle, & \text{if } d|n \\ 0, & \text{其它} \end{cases}, \quad n \geq 1$$

$$[1]\langle tx^n, t\rangle = \langle tx^n, t\rangle, \quad n \geq 1 \text{ is odd}$$

$$[1]\langle tx^{n-1}, x\rangle = \langle tx^{n-1}, x\rangle, \quad n \geq 1.$$

Corollary 8.17. *Let $R = \mathbb{F}_2[C_2]$, then $K_2(R[x]) = NK_2(R) = \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$. For any integer $r \geq 1$, $K_2(\mathbb{F}_2[C_2 \times \mathbb{Z}^r])$ is not finitely generated.*

TODO: find a basis of $NK_2(\mathbb{F}_p[C_p])$, first consider $p = 3$, then $NK_2(\mathbb{F}_3[C_9])$

8.5 $NK_2(\mathbb{F}_2[C_4])$

这一节首先用同样的方法计算 $NK_2(\mathbb{F}_2[C_{2^2}])$, 继而对于任意 n 可以得到类似的结果.

Theorem 8.18. *$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}/4\mathbb{Z}$.*

Proof. $\mathbb{F}_2[t_1, t_2]/(t_1^4) = \mathbb{F}_2[C_4][t_2]$, 此时 $I = (t_1^4)$, $M = (t_1)$ 不变, 我们直接写出以下集合

$$\Delta = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 4, \alpha_2 \geq 0\},$$

$$\Lambda = \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\},$$

用 $\lceil x\rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 x 的最小整数,

$$[\alpha, 1] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 5\} = \lceil 5/\alpha_1 \rceil\,,$$
$$[\alpha, 2] = \min\{m \in \mathbb{Z} \mid m\alpha_1 \geq 4\} = \lceil 4/\alpha_1 \rceil\,.$$

例如

$$\begin{aligned}
[(1,\alpha_2),1] &= 5,\ \alpha_2 \geq 0,\\
[(2,\alpha_2),1] &= 3,\ \alpha_2 \geq 0,\\
[(3,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(4,\alpha_2),1] &= 2,\ \alpha_2 \geq 0,\\
[(\alpha_1,\alpha_2),1] &= 1,\ \alpha_1 \geq 5, \alpha_2 \geq 0,\\
[(1,\alpha_2),2] &= 4,\ \alpha_2 \geq 1,\\
[(2,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(3,\alpha_2),2] &= 2,\ \alpha_2 \geq 1,\\
[(\alpha_1,\alpha_2),2] &= 1,\ \alpha_1 \geq 4, \alpha_2 \geq 1.
\end{aligned}$$

(α_1,α_2)	$[(\alpha_1,\alpha_2),1]$	$[(\alpha_1,\alpha_2),2]$	$[(\alpha_1,\alpha_2)]$
$(1,\alpha_2)$	$5, \alpha_2 \geq 0$	$4, \alpha_2 \geq 1$	5
$(2,\alpha_2)$	$3, \alpha_2 \geq 0$	$2, \alpha_2 \geq 1$	2, 当 α_2 is odd 时
$(3,\alpha_2)$	$2, \alpha_2 \geq 0$	$2, \alpha_2 \geq 1$	2
$(4,\alpha_2)$	$2, \alpha_2 \geq 0$	$1, \alpha_2 \geq 1$	1当 α_2 is odd 时
$(\alpha_1,0), \alpha_1 \geq 5$	1	-	1, 当 α_1 is odd 时
$(\alpha_1,\alpha_2), \alpha_1 \geq 5, \alpha_2 \geq 1$	1	1	1, 当 $(\alpha_1,\alpha_2)=1$ 时

记 $\Lambda^{00}_d = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] = d\}$, $\Lambda^{00}_{>1} = \{(\alpha,i) \in \Lambda^{00} | [(\alpha,i)] > 1\}$

由于 $(\alpha,i) \in \Lambda^{00}_1$ 均有 $[(\alpha,i)] = 1$, 实际上要计算 $(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$ 只需确定 $\Lambda^{00}_{>1}$. 由同样的方法可得 $\Lambda^{00}_4 = \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\}$, $\Lambda^{00}_3 = \{((2,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\}$, $\Lambda^{00}_2 = \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\}$,

$$\begin{aligned}
\Lambda^{00}_{>1} =& \{((1,\alpha_2),2) \mid \alpha_2 \geq 1\} \cup \{((3,\alpha_2),2) \mid gcd(3,\alpha_2) = 1, \alpha_2 \geq 1\}\\
&\cup \{((2,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\} \cup \{((4,\alpha_2),1) \mid \alpha_2 \geq 1 \text{ is odd}\}.
\end{aligned}$$

由定理9.1, 从而

$$
\begin{aligned}
NK_2(\mathbb{F}_2[C_4]) \cong K_2(A,M) \cong & \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
= & \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1+x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times \\
= & \bigoplus_{\substack{((3,\alpha_2),2)\\ gcd(3,\alpha_2)=1\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^2))^\times \oplus \bigoplus_{\substack{((4,\alpha_2),1)\\ \alpha_2\geq 1 \text{ is odd}}} (1+x\mathbb{F}_2[x]/(x^2))^\times \\
\oplus & \bigoplus_{\substack{((2,\alpha_2),1)\\ \alpha_2\geq 1 \text{ is odd}}} (1+x\mathbb{F}_2[x]/(x^3))^\times \oplus \bigoplus_{\substack{((1,\alpha_2),2)\\ \alpha_2\geq 1}} (1+x\mathbb{F}_2[x]/(x^4))^\times
\end{aligned}
$$

By8.5, $(1+x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $(1+x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, therefore as an abelian group,

$$NK_2(\mathbb{F}_2[C_4]) \cong \bigoplus_{\infty} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}/4\mathbb{Z}.$$

For every $(\alpha,i)\in\Lambda^{00}_{>1}$, the induced homomorphism $\Gamma_{\alpha,i}$ is injective. For any $\alpha_2 \geq 1$ with $gcd(3,\alpha_2)=1$,

$$
\begin{aligned}
\Gamma_{(3,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\
1+x &\mapsto \langle t_1^3 t_2^{\alpha_2-1}, t_2\rangle,
\end{aligned}
$$

For any $\alpha_2 \geq 1$,

$$
\begin{aligned}
\Gamma_{(1,\alpha_2),2}\colon (1+x\mathbb{F}_2[x]/(x^4))^\times &\rightarrowtail K_2(A,M)\\
1+x(\text{四阶元}) &\mapsto \langle t_1 t_2^{\alpha_2-1}, t_2\rangle,\\
1+x^3(\text{二阶元}) &\mapsto \langle t_1^3 t_2^{3\alpha_2-1}, t_2\rangle,
\end{aligned}
$$

For any $\alpha_2 \geq 1$ is odd,

$$
\begin{aligned}
\Gamma_{(4,\alpha_2),1}\colon (1+x\mathbb{F}_2[x]/(x^2))^\times &\rightarrowtail K_2(A,M)\\
1+x &\mapsto \langle t_1^3 t_2^{\alpha_2}, t_1\rangle,\\
\Gamma_{(2,\alpha_1),1}\colon (1+x\mathbb{F}_2[x]/(x^3))^\times &\rightarrowtail K_2(A,M)\\
1+x(\text{四阶元}) &\mapsto \langle t_1 t_2^{\alpha_2}, t_1\rangle.
\end{aligned}
$$

Simply replacing t_1 by t and t_2 by x, it follows from the isomorphism 9.1 that $NK_2(\mathbb{F}_2[C_4])$ is generated by the following Dennis-Stein symbols
elements of order 4:

- $\{\langle tx^i, t\rangle \mid i \geq 1 \text{ is odd}\}$,
- $\{\langle tx^{i-1}, x\rangle \mid i \geq 1\}$,

elements of order 2:

- $\{\langle t^3x^{i-1}, x\rangle \mid i \geq 1\}$,
- $\{\langle t^3x^i, t\rangle \mid i \geq 1 \text{ is odd}\}$. $\square$

Remark 8.19. For any odd number $i \geq 1$, $2\langle tx^i, t\rangle = \langle t^3x^{2i}, t\rangle$, $4\langle tx^i, t\rangle = 2\langle t^3x^{2i}, t\rangle = \langle 2t^3x^{2i} - t^7x^{4i}, t\rangle = \langle 0, t\rangle = 0$. However, when $2i$ is even, then $\langle tx^{2i}, t\rangle = 2\langle x^i, t\rangle = -2\langle t, x^i\rangle = -2i\langle tx^{i-1}, x\rangle = 0$.

$2\langle tx^{i-1}, x\rangle = \langle t^2x^{2i-1}, x\rangle$, and $4\langle tx^{i-1}, x\rangle = 2\langle t^2x^{2i-1}, x\rangle = \langle 0, x\rangle = 0$.

$2\langle t^3x^{i-1}, x\rangle = \langle 0, x\rangle = 0$.

$2\langle t^3x^i, t\rangle = \langle 0, t\rangle = 0$.

More comments about $\langle t^2x^i, t\rangle$ and $\langle t^2x^{i-1}, x\rangle$: $\langle t^2x^i, t\rangle = -\langle t, t^2x^i\rangle = -(\langle t^3, x^i\rangle + \langle tx^i, t^2\rangle) = -\langle t^3, x^i\rangle - 2\langle t^2x^i, t\rangle = -\langle t^3, x^i\rangle = -i\langle t^3x^{i-1}, x\rangle$. $\langle t^2x^i, x\rangle = -\langle x, t^2x^i\rangle = -(\langle x^{i+1}, t^2\rangle + \langle xt^2, x^i\rangle) = -2\langle tx^{i+1}, t\rangle - i\langle t^2x^{i-1}, x\rangle$. If $i = 2k-1$ is odd, then $\langle t^2x^{2k-1}, x\rangle = 2\langle tx^{k-1}, x\rangle$. Else if $i = 2k$ is even, $\langle t^2x^{2k}, x\rangle = -2\langle tx^{2k+1}, t\rangle$. It follows that these elements can be generated by the above Dennis-Stein symbols.

Note that $\langle f(x,t), x^i\rangle = i\langle f(x,t)x^{i-1}, x\rangle$, hence $\langle t, x^i\rangle = i\langle tx^{i-1}, x\rangle$. $2\langle t, x^i\rangle = \langle t + t - t^2x^i, x^i\rangle = \langle -t^2x^i, x^i\rangle = \langle t^2x^i, x^i\rangle = i\langle t^2x^{2i-1}, x\rangle = 2i\langle tx^{i-1}, x\rangle$ since $-1 = 1, 2 = 0 \in \mathbb{F}_2$. 根据 [36], 存在同态

$$\rho_1\colon \mathbb{F}_2[x]dx \longrightarrow NK_2(\mathbb{F}_2[C_4])$$
$$x^i dx \mapsto \langle t^3x^i, x\rangle$$
$$\rho_2\colon x\mathbb{F}_2[x]/x^4\mathbb{F}_2[x^4] \longrightarrow NK_2(\mathbb{F}_2[C_4])$$
$$x^i \mapsto \langle t^3x^i, t\rangle$$

Note that $\langle t^3x^{4i}, t\rangle = 4\langle x^i, t\rangle = -4i\langle tx^{i-1}, x\rangle = 0$, $\langle t^3x^{4i+2}, t\rangle = 2\langle tx^{2i+1}, t\rangle$ are elements of order 2. ρ_1, ρ_2 are injective. For if $\langle \sum_{i=0}^n a_ix^it^3, t\rangle = \sum_{i=0}^n \langle a_ix^it^3, t\rangle = 0$, then $\sum_{i=0}^n a_ix^i \in (x^4)$.

$\{\langle t^3x^{i-1}, x\rangle \mid i \geq 1\} = \{\langle t^3x^{3i-1}, x\rangle \mid i \geq 1\} \cup \{\langle t^3x^{i-1}, x\rangle \mid i \geq 1, gcd(i,3) = 1\}$, 从而 $\Omega_{\mathbb{F}_2[x]} \oplus x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2]$ 是 $NK_2(\mathbb{F}_2[C_4])$ 的直和项.

8.6 $NK_2(\mathbb{F}_2[C_2 \times C_2])$

Theorem 8.20. $NK_2(\mathbb{F}_2[C_2 \times C_2]) \cong \bigoplus_\infty \mathbb{Z}/2\mathbb{Z}$.

$K_2(\mathbb{F}_2[C_2 \times C_2][x]) = K_2(\mathbb{F}_2[t_1, t_2, t_3]/(t_1^2, t_2^2), (t_1, t_2))$.

$I = (t_1^2, t_2^2), M = (t_1, t_2), \Delta = \{(\alpha_1, \alpha_2, \alpha_3) \mid t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \in (t_1^2, t_2^2)\} = \{\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \mid \alpha_1 \geq 2\} \cup \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \mid \alpha_2 \geq 2\}$. $\Lambda = \{(\alpha, i) \in \mathbb{N}^3 \times \{1, 2, 3\} \mid \alpha_i \geq 1, t^\alpha \in M\} = \{(\alpha, 1) \mid \alpha_1 \geq 1\} \cup \{(\alpha, 2) \mid \alpha_2 \geq 1\} \cup \{(\alpha, 3) \mid \alpha_1 \geq 1, \alpha_3 \geq 1\} \cup \{(\alpha, 3) \mid \alpha_2 \geq 1, \alpha_3 \geq 1\}$.

For $(\alpha, i) \in \Lambda$, $[\alpha, i] = \min\{n \in \mathbb{Z} \mid m\alpha - \varepsilon^i \in \Delta\}$,

$$[\alpha, 1] = \begin{cases} \lceil 3/\alpha_1 \rceil, \text{ if } \alpha_1 \geq 1, \alpha_2 = 0 \\ \min\{\lceil 3/\alpha_1 \rceil, \lceil 2/\alpha_2 \rceil\}, \text{if } \alpha_1, \alpha_2 \neq 0 \end{cases}$$

$$[\alpha, 2] = \begin{cases} \lceil 3/\alpha_2 \rceil, \text{ if } \alpha_1 = 0, \alpha_2 \geq 1 \\ \min\{\lceil 2/\alpha_1 \rceil, \lceil 3/\alpha_2 \rceil\}, \text{if } \alpha_1, \alpha_2 \neq 0 \end{cases}$$

$$[\alpha, 3] = \begin{cases} \lceil 2/\alpha_2 \rceil, \text{ if } \alpha_1 = 0, \alpha_2 \geq 1, \alpha_3 \geq 1 \\ \lceil 2/\alpha_1 \rceil, \text{ if } \alpha_1 \geq 1, \alpha_2 = 0, \alpha_3 \geq 1 \\ \min\{\lceil 2/\alpha_1 \rceil, \lceil 2/\alpha_2 \rceil\}, \text{if } \alpha_1, \alpha_2 \neq 0, \alpha_3 \geq 1 \end{cases}$$

If $gcd(2, \alpha_1, \alpha_2, \alpha_3) = 1$, set $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod 2\}$, $\Lambda^{00} = \{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2, \alpha_3) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod 2, [\alpha, j] = [\alpha]\}\}$

α	$[\alpha,1]$	$[\alpha,2]$	$[\alpha,3]$	$[\alpha]$	min	Λ^{00}
$(0,1,\alpha_3), \alpha_3 \geq 1$	–	3	2	3	2	$(\alpha,3)$
$(1,0,\alpha_3), \alpha_3 \geq 1$	3	–	2	3	1	$(\alpha,3)$
$(1,1,0)$	2	2	–	3	1	$(\alpha,2)$
$(1,1,\alpha_3), \alpha_3 \geq 1$	2	2	2	2	1	$(\alpha,2), (\alpha,3)$
$(0,2,\alpha_3), \alpha_3 \geq 1$	–	2	1	$1, \alpha_3 \geq 1$ odd	$3, \alpha_3 \geq 1$ odd	$(\alpha,2)$, if $\alpha_3 \geq 1$ is odd
$(2,0,\alpha_3), \alpha_3 \geq 1$	2	–	1	$1, \alpha_3 \geq 1$ odd	$3, \alpha_3 \geq 1$ odd	$(\alpha,1)$, if $\alpha_3 \geq 1$ is odd
$(1,2,0)$	1	2	–	1	1	$(\alpha,2)$
$(1,2,\alpha_3), \alpha_3 \geq 1$	1	2	1	1	1	$(\alpha,2), (\alpha,3)$
$(2,1,0)$	2	1	–	1	2	$(\alpha,1)$
$(2,1,\alpha_3), \alpha_3 \geq 1$	2	1	1	1	2	$(\alpha,1), (\alpha,3)$

In fact $\Lambda^{00}_{>1} = \Lambda^{00}_2$:

- $((1,0,\alpha_3),3)$, if $\alpha_3 \geq 1$
- $((0,1,\alpha_3),3)$, if $\alpha_3 \geq 1$

- $((1,1,\alpha_3),3)$, if $\alpha_3 \geq 1$
- $((1,1,\alpha_3),2)$, if $\alpha_3 \geq 0$
- $((2,0,\alpha_3),1)$, if $\alpha_3 \geq 1$ is odd
- $((0,2,\alpha_3),2)$, if $\alpha_3 \geq 1$ is odd
- $((2,1,\alpha_3),1)$, if $\alpha_3 \geq 0$
- $((1,2,\alpha_3),2)$, if $\alpha_3 \geq 0$

the corresponding Dennis-Stein symbols are

$$\{\langle t_1x^i, x\rangle \mid i \geq 0\}, \{\langle t_2x^i, x\rangle \mid i \geq 0\}, \{\langle t_1t_2x^i, x\rangle \mid i \geq 0\}, \{\langle t_1x^i, t_2\rangle \mid i \geq 0\},$$
$$\{\langle t_1x^i, t_1\rangle \mid i \geq 1 \text{ is odd}\}, \{\langle t_2x^i, t_2\rangle \mid i \geq 1 \text{ is odd}\}, \{\langle t_1t_2x^i, t_1\rangle \mid i \geq 0\}, \{\langle t_1t_2x^i, t_2\rangle \mid i \geq 0\}.$$

Note that $\langle t_2x^i, t_1\rangle = -\langle t_1, t_2x^i\rangle = -(\langle t_1t_2, x^i\rangle + \langle t_1x^i, t_2\rangle) = -i\langle t_1t_2x^{i-1}, x\rangle - \langle t_1x^i, t_2\rangle$.

Since $K_2(\mathbb{F}_2[C_2 \times C_2]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^2, t_2^2))$ is generated by $\langle t_1t_2, t_1\rangle$, $\langle t_1t_2, t_2\rangle$, $\langle t_1, t_2\rangle$. Therefore as the kernel of the map $K_2(\mathbb{F}_2[C_2 \times C_2][x]) \longrightarrow \mathbb{F}_2[C_2 \times C_2]$, $NK_2(\mathbb{F}_2[C_2 \times C_2])$ is generated by the following Dennis-Stein symbols

$$\{\langle t_1x^i, x\rangle \mid i \geq 0\}, \{\langle t_2x^i, x\rangle \mid i \geq 0\}, \{\langle t_1t_2x^i, x\rangle \mid i \geq 0\}, \{\langle t_1x^i, t_2\rangle \mid i \geq 1\},$$
$$\{\langle t_1x^i, t_1\rangle \mid i \geq 1 \text{ is odd}\}, \{\langle t_2x^i, t_2\rangle \mid i \geq 1 \text{ is odd}\}, \{\langle t_1t_2x^i, t_1\rangle \mid i \geq 1\}, \{\langle t_1t_2x^i, t_2\rangle \mid i \geq 1\}.$$

Remark 8.21. We know that $K_2(\mathbb{F}_2[C_2 \times C_2]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^2, t_2^2))$ is generated by $\langle t_1t_2, t_1\rangle$, $\langle t_1t_2, t_2\rangle$, $\langle t_1, t_2\rangle$, which are corresponding to $((2,1,0),1)$, $((1,2,0),2)$ and $((1,1,0),2)$. In fact, if $((\alpha_1, \alpha_2, 0), i) \in \Lambda^{00}_{>1}$, it is easy to see that $((\alpha_1, \alpha_2, \alpha_3), i) \in \Lambda^{00}_{>1}$ for $\alpha_3 \geq 0$. And for $\alpha_3 \geq 1$, the corresponding Dennis-Stein symbols $\langle t^{\alpha - \varepsilon_i}, t_i\rangle$ are generators of $NK_2(\mathbb{F}_2[C_2 \times C_2])$.

Some non-trivial elements in $NK_2(\mathbb{Z}[C_2 \times C_2])$ The following map is an isomorphism

$$\mathbb{F}_2[C_2 \times C_2] \longrightarrow \mathbb{F}_2[t_1, t_2]/(t_1^2, t_2^2)$$
$$\sigma_i \mapsto 1 - t_i$$

where σ_i are the generators of $C_2 \times C_2$. Consider the canonical surjection $\mathbb{Z}[C_2 \times C_2] \twoheadrightarrow \mathbb{F}_2[C_2 \times C_2] \cong \mathbb{F}_2[t_1, t_2]/(t_1^2, t_2^2)$, elements of the form $\langle x^n(1+\sigma_1)(1+\sigma_2), 1-\sigma_1\rangle$ in $NK_2(\mathbb{Z}[C_2 \times C_2])$ map to $\langle t_1t_2x^n, t_1\rangle \neq 0$ in $NK_2(\mathbb{F}_2[C_2 \times C_2])$. Hence for any $n \geq 1$, $\langle x^n(1+\sigma_1)(1+\sigma_2), 1-\sigma_1\rangle$ is a non-trivial element of $NK_2(\mathbb{Z}[C_2 \times C_2])$. Similarly for any $n \geq 1$, $\langle x^n(1+\sigma_1)(1+\sigma_2), 1-\sigma_2\rangle$ is a non-trivial element. If $n \geq 1$ is odd, then $\langle x^n(1+\sigma_1), 1-\sigma_1\rangle$, $\langle x^n(1+\sigma_2), 1-\sigma_2\rangle$ are non-trivial.

8.7 $NK_2(\mathbb{F}_2[C_4 \times C_4])$

8.8 $NK_2(\mathbb{F}_q[C_{2^n}])$

设 $\mathbb{F}_q$ 是特征为 2 的有限域, $q = 2^f$, C_{2^n} 是 2^n 阶循环群, 这一节计算 $NK_2(\mathbb{F}_q[C_{2^n}])$. 假设 $A = \mathbb{F}_q[t_1, t_2]/(t_1^{2^n}) = \mathbb{F}_q[C_{2^n}][x]$, 此时 $I = (t_1^{2^n})$, $M = (t_1)$, $A/M = \mathbb{F}_q[x]$.

Lemma 8.22. $\Delta = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 2^n, \alpha_2 \geq 0\}$, $\Lambda = \{((\alpha_1, \alpha_2), 1) \mid \alpha_1 \geq 1, \alpha_2 \geq 0\} \cup \{((\alpha_1, \alpha_2), 2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\}$, 对任意 $(\alpha, i) \in \Lambda$, $[\alpha, 1] = \lceil (2^n+1)/\alpha_1 \rceil$, $[\alpha, 2] = \lceil 2^n/\alpha_1 \rceil$, 其中 $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ 表示不小于 x 的最小整数.

Lemma 8.23. *Let $I_1 = \{((\alpha_1, \alpha_2), 1) \mid gcd(\alpha_1, \alpha_2) = 1, 1 < \alpha_1 \leq 2^n$ is even, $\alpha_2 \geq 1$ is odd$\}$, and $I_2 = \{((\alpha_1, \alpha_2), 2) \mid gcd(\alpha_1, \alpha_2) = 1, 1 \leq \alpha_1 < 2^n$ is odd, $\alpha_2 \geq 1\}$, then $\Lambda^{00}_{>1} = I_1 \sqcup I_2$.*

By virtue of Theorem 9.1,

$$\begin{aligned} NK_2(\mathbb{F}_q[C_{2^n}]) \cong K_2(A, M) \cong & \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ = & \bigoplus_{(\alpha,i) \in \Lambda^{00}_{>1}} (1 + x\mathbb{F}_q[x]/(x^{[\alpha,i]}))^\times \\ = & \bigoplus_{(\alpha,1) \in I_1} (1 + x\mathbb{F}_q[x]/(x^{\lceil (2^n+1)/\alpha_1 \rceil}))^\times \\ & \oplus \bigoplus_{(\alpha,2) \in I_2} (1 + x\mathbb{F}_q[x]/(x^{\lceil 2^n/\alpha_1 \rceil}))^\times. \end{aligned}$$

Recall that $BigWitt_k(R) = (1 + xR[\![x]\!])^\times/(1 + x^{k+1}R[\![x]\!])^\times \cong (1 + xR[x]/(x^{k+1}))^\times$. Under the isomorphism 8.7, one has

$$\begin{aligned} NK_2(\mathbb{F}_q[C_{2^n}]) \cong & \bigoplus_{(\alpha,1) \in I_1} \bigoplus_{\substack{1 \leq m \leq \lceil (2^n+1)/\alpha_1 \rceil - 1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1 + \left\lfloor \log_2 \frac{\lceil (2^n+1)/\alpha_1 \rceil - 1}{m} \right\rfloor}\mathbb{Z})^f \\ & \oplus \bigoplus_{(\alpha,2) \in I_2} \bigoplus_{\substack{1 \leq m \leq \lceil 2^n/\alpha_1 \rceil - 1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1 + \left\lfloor \log_2 \frac{\lceil 2^n/\alpha_1 \rceil - 1}{m} \right\rfloor}\mathbb{Z})^f. \end{aligned}$$

接下来我们证明对于任意 $1 \leq k \leq n$, $\mathbb{Z}/2^k\mathbb{Z}$ 都在 $NK_2(\mathbb{F}_q[C_{p^n}])$ 出现无限多次

Lemma 8.24. 对于任意的 $1 \leq k < n$, $1 + \left\lfloor \log_2(\frac{2^n-1}{2^k+1}) \right\rfloor = n - k$.

Proof. 当 $1 \leq k < n$ 时, $2^k - 1 \geq 1 \geq \frac{1}{2^{n-k-1}}$, 即

$$2^{n-1} - 2^{n-k-1} \geq 1$$

上式等价于 $2^n - 1 \geq 2^{n-k-1}(2^k+1)$, and $2^n - 1 < 2^{n-k}(2^k+1)$, 于是

$$2^{n-k} > \frac{2^n - 1}{2^k + 1} \geq 2^{n-k-1}$$

取对数得 $\left\lfloor \log_2(\frac{2^n-1}{2^k+1}) \right\rfloor = n - k - 1$. $\square$

考虑 $((1, \alpha_2), 2) \in I_2$,

$$\bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\leq m\leq 2^n-1 \\ gcd(m,2)=1}} (\mathbb{Z}/2^{1+\left\lfloor \log_2 \frac{2^n-1}{m} \right\rfloor}\mathbb{Z})^f$$

是 $NK_2(\mathbb{F}_{2^f}[C_{2^n}])$ 的直和项, 当 $m = 1$ 时 $1 + \lfloor \log_2(2^n - 1) \rfloor = n$, 当 $m = 2^k + 1 (1 \leq k < n)$ is odd 时, 由8.24, $1 + \left\lfloor \log_2 \frac{2^n-1}{m} \right\rfloor = n - k$, 于是对于任何的 $1 \leq k \leq n$, $\mathbb{Z}/2^k\mathbb{Z}$ 均出现在直和项中, and 对于任意 $\alpha_2 \geq 1$, 这样的项总会出现, 于是

$$NK_2(\mathbb{F}_q[C_{2^n}]) \cong \bigoplus_{k=1}^{n} (\bigoplus_{\infty} \mathbb{Z}/2^k\mathbb{Z}).$$

接下来给出一些 $NK_2(\mathbb{F}_q[C_{2^n}])$ 中的 $2^k (1 \leq k \leq n)$ 阶元素.

对任意 $\alpha_2 \geq 1, a \in \mathbb{F}_q$,

$$\begin{aligned}
\Gamma_{(1,\alpha_2),2} \colon (1 + x\mathbb{F}_q[x]/(x^{2^n}))^\times &\rightarrowtail K_2(A, M) \\
1 + ax(2^n \text{ 阶元}) &\mapsto \langle atx^{\alpha_2 - 1}, x \rangle, \\
1 + ax^3(2^{n-1} \text{ 阶元}) &\mapsto \langle at^3x^{3\alpha_2 - 1}, x \rangle, \\
1 + ax^{2^k+1}(2^{n-k} \text{ 阶元}) &\mapsto \langle at^{2^k+1}x^{(2^k+1)\alpha_2 - 1}, x \rangle.
\end{aligned}$$

8.9 NK_2 of finite abelian groups

8.9.1 NK_2 of finite cyclic p-groups

Let p be a prime number, $\mathbb{F}_{p^f}$ the finite field with p^f elements, C_{p^n} the cyclic group of order p^n. First note that $NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong K_2(\mathbb{F}_{p^f}[t_1, t_2]/(t_1^{p^n}), (t_1))$. Put $I = (t_1^{p^n})$, let M be the radical ideal (t_1), then one easily get

Lemma 8.25. $\Delta = \{(\alpha_1,\alpha_2) \mid \alpha_1 \geq p^n, \alpha_2 \geq 0\}$, $\Lambda = \{((\alpha_1,\alpha_2),1) \mid \alpha_1 \geq 1, \alpha_2 \geq 0\} \cup \{((\alpha_1,\alpha_2),2) \mid \alpha_1 \geq 1, \alpha_2 \geq 1\}$. *For any* $(\alpha,i) \in \Lambda$, $[\alpha,1] = \lceil (p^n+1)/\alpha_1 \rceil$, $[\alpha,2] = \lceil p^n/\alpha_1 \rceil$, *where* $\lceil x \rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ *denotes the smallest integer no less than* x.

If $\alpha_1 \not\equiv 0 \bmod p$, then $[\alpha] = [\alpha,1] = \lceil (p^n+1)/\alpha_1 \rceil$. If $p|\alpha_1$ and $\alpha_2 \not\equiv 0 \bmod p$, then $[\alpha] = [\alpha,2] = \lceil p^n/\alpha_1 \rceil$. In addition, if $\alpha_1 \not\equiv 0 \bmod p$, then for any $\alpha_2 \geq 1$ with $gcd(\alpha_1,\alpha_2) = 1$, we have $(\alpha,2) \in \Lambda^{00}$ and $[\alpha,2] = \lceil p^n/\alpha_1 \rceil$. Similarly, if $p|\alpha_1$, then for any $\alpha_2 \not\equiv 0 \bmod p$ with $gcd(\alpha_1,\alpha_2) = 1$, we have $(\alpha,1) \in \Lambda^{00}$ and $[\alpha,1] = \lceil (p^n+1)/\alpha_1 \rceil$. In fact,

Lemma 8.26. *Let* $I_1 = \{(\alpha,1) \mid gcd(\alpha_1,\alpha_2) = 1, \alpha_1 \equiv 0 \bmod p$ *and* $\alpha_2 \not\equiv 0 \bmod p\}$, $I_2 = \{(\alpha,2) \mid gcd(\alpha_1,\alpha_2) = 1, \alpha_1 \not\equiv 0 \bmod p$ *and* $\alpha_2 \geq 1\}$, *then* $\Lambda^{00}_{>1} = I_1 \sqcup I_2$.

By theorem 9.1,

$$\begin{aligned}
NK_2(\mathbb{F}_{p^f}[C_{p^n}]) &\cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1+x\mathbb{F}_{p^f}[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1+x\mathbb{F}_{p^f}[x]/(x^{[\alpha,i]}))^\times \\
&= \bigoplus_{(\alpha,1)\in I_1} (1+x\mathbb{F}_{p^f}[x]/(x^{\lceil (p^n+1)/\alpha_1\rceil}))^\times \\
&\oplus \bigoplus_{(\alpha,2)\in I_2} (1+x\mathbb{F}_{p^f}[x]/(x^{\lceil p^n/\alpha_1\rceil}))^\times.
\end{aligned}$$

Recall that $BigWitt_k(R) = (1+xR[\![x]\!])^\times/(1+x^{k+1}R[\![x]\!])^\times \cong (1+xR[x]/(x^{k+1}))^\times$, by 8.7,

$$\begin{aligned}
NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong & \bigoplus_{(\alpha,1)\in I_1} \bigoplus_{\substack{1\leq m\leq \lceil (p^n+1)/\alpha_1\rceil-1 \\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\left\lfloor \log_p \frac{\lceil (p^n+1)/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f \\
&\oplus \bigoplus_{(\alpha,2)\in I_2} \bigoplus_{\substack{1\leq m\leq \lceil p^n/\alpha_1\rceil-1 \\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\left\lfloor \log_p \frac{\lceil p^n/\alpha_1\rceil -1}{m}\right\rfloor}\mathbb{Z})^f.
\end{aligned}$$

We claim that for any $1 \leq k \leq n$, $\mathbb{Z}/p^k\mathbb{Z}$ appears infinite times in $NK_2(\mathbb{F}_{p^f}[C_{p^n}])$.

Lemma 8.27. *For any* $1 \leq k < n$, $1 + \left\lfloor \log_p(\frac{p^n-1}{p^k+1}) \right\rfloor = n-k$.

Proof. On the one hand, $p^n - 1 < p^n + p^{n-k} = p^{n-k}(p^k + 1)$. On the other hand, if $1 \le k < n$ and $p \ge 2$, $p^{n-1} - p^{n-k-1} = p^{n-k-1}(p^k - 1) \ge 1$, hence

$$p^{n-1} \ge 1 + p^{n-k-1} \ge \frac{1 + p^{n-k-1}}{p - 1}.$$

Then $p^n - p^{n-1} = p^{n-1}(p-1) \ge 1 + p^{n-k-1}$, that is $p^{n-1} + p^{n-k-1} \le p^n - 1$, the $LFS = p^{n-k-1}(p^k + 1)$. So one has

$$p^{n-k-1}(p^k + 1) \le p^n - 1 < p^{n-k}(p^k + 1),$$
$$p^{n-k-1} \le \frac{p^n - 1}{p^k + 1} < p^{n-k}.$$

The lemma is obvious by taking the logarithm of both sides. □

Consider $((1, \alpha_2), 2) \in I_2$ with $\alpha_2 \ge 1$, then

$$\bigoplus_{((1,\alpha_2),2)\in I_2} \bigoplus_{\substack{1\le m\le p^n-1 \\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\left\lfloor \log_p \frac{p^n-1}{m} \right\rfloor}\mathbb{Z})^f$$

is a direct summand of $NK_2(\mathbb{F}_{p^f}[C_{p^n}])$. When $m = 1$, $1 + \left\lfloor \log_p(p^n - 1) \right\rfloor = n$, while if $m = p^k + 1 (1 \le k < n)$, $1 + \left\lfloor \log_p \frac{p^n-1}{m} \right\rfloor = n - k$ by 8.27. Therefore for any $1 \le k \le n$, $\mathbb{Z}/p^k\mathbb{Z}$ is a direct summand, and for every $\alpha_2 \ge 1$, it appears at least once, so we concluded that

$$NK_2(\mathbb{F}_{p^f}[C_{p^n}]) \cong \bigoplus_{k=1}^{n} (\bigoplus_{\infty} \mathbb{Z}/p^k\mathbb{Z}).$$

8.9.2 NK_2 of finite abelian p-groups

要将前面的 A, M 改成多元的

Let $\mathbb{F}$ be the finite field $\mathbb{F}_{p^f}$, G the finite abelian p-group $C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_r}}$, then $\mathbb{F}[t_1, t_2, \cdots, t_r]/(t_1^{p^{n_1}}, t_2^{p^{n_2}}, \cdots, t_r^{p^{n_r}}) \cong \mathbb{F}[G]$.

Proposition 8.28 ([11] Proposition 3.4)**.** *Let $\mathbb{F}$ be a finite field with p^f elements and G a finite abelian p-group of exponent p^e. Let r_i denote the p^i-rank of G. Then*

$$K_2(\mathbb{F}[G]) = \bigoplus_{i=1}^{e} C_{p^i}^{f\rho_i(G)},$$

where $\rho_e(G) = (r_e - 1)(|G^{p^{e-1}}| - |G^{p^e}|)$, and $\rho_i(G) = (r_i - 1)(|G^{p^{i-1}}| - |G^{p^i}|) - (r_{i+1} - 1)(|G^{p^i}| - |G^{p^{i+1}}|)$, $1 \le i < e$.

Theorem 8.29. *Given $\mathbb{F}$ and G as above, then $NK_2(\mathbb{F}[G]) \cong \bigoplus_\infty(\bigoplus_{k=1}^{n_1} \mathbb{Z}/p^k\mathbb{Z})$.*

Proof. Since $NK_i(R) = \ker(K_i(R[x]) \xrightarrow{x\mapsto 0} K_i(R))$, one get a split exact sequence

$$0 \longrightarrow NK_2(\mathbb{F}[G]) \longrightarrow K_2(\mathbb{F}[G][x]) \longrightarrow K_2(\mathbb{F}[G]) \longrightarrow 0.$$

Put $A = \mathbb{F}[t_1, t_2, \cdots, t_r, t_{r+1}]/(t_1^{p^{n_1}}, t_2^{p^{n_2}}, \cdots, t_r^{p^{n_r}})$ and $M = (t_1, \cdots, t_r)$ in this section. Obviously, $K_2(\mathbb{F}[G][x]) \cong K_2(A)$, and one can compute $K_2(\mathbb{F}[G])$ by proposition 8.28, therefore

$$NK_2(\mathbb{F}[G]) \cong K_2(A)/K_2(\mathbb{F}[G]) \cong K_2(A, M)/K_2(\mathbb{F}[G]).$$

Without loss of generality, assume $n_1 \geq n_2 \geq \cdots \geq n_r$. Then

$$\begin{aligned}
\Delta =& \bigcup_{i=1}^{r} \{\alpha \in \mathbb{N}^{r+1} \mid \alpha_i \geq p^{n_i}\}\\
\Lambda =& \{(\alpha, i) \in \mathbb{N}^{r+1} \times \{1, 2, \cdots, r+1\} \mid \alpha_i \geq 1, \text{ and } t^\alpha \in M\}\\
=& \bigcup_{i=1}^{r} \{(\alpha, i) \in \mathbb{N}^{r+1} \times \{1, 2, \cdots, r\} \mid \alpha_i \geq 1\}\\
& \cup \bigcup_{i=1}^{r} \{(\alpha, r+1) \in \mathbb{N}^{r+1} \times \{r+1\} \mid \alpha_{r+1} \geq 1, \text{ and } \alpha_i \geq 1\}
\end{aligned}$$

For $(\alpha, i) \in \Lambda$,

$$\begin{aligned}
[\alpha, 1] &= \min\{\left\lceil \frac{p^{n_1}+1}{\alpha_1} \right\rceil, \left\lceil \frac{p^{n_2}}{\alpha_2} \right\rceil, \cdots, \left\lceil \frac{p^{n_r}}{\alpha_r} \right\rceil\},\\
[\alpha, 2] &= \min\{\left\lceil \frac{p^{n_1}}{\alpha_1} \right\rceil, \left\lceil \frac{p^{n_2}+1}{\alpha_2} \right\rceil, \cdots, \left\lceil \frac{p^{n_r}}{\alpha_r} \right\rceil\},\\
&\vdots\\
[\alpha, r] &= \min\{\left\lceil \frac{p^{n_1}}{\alpha_1} \right\rceil, \left\lceil \frac{p^{n_2}}{\alpha_2} \right\rceil, \cdots, \left\lceil \frac{p^{n_r}+1}{\alpha_r} \right\rceil\},\\
[\alpha, r+1] &= \min\{\left\lceil \frac{p^{n_1}}{\alpha_1} \right\rceil, \left\lceil \frac{p^{n_2}}{\alpha_2} \right\rceil, \cdots, \left\lceil \frac{p^{n_r}}{\alpha_r} \right\rceil\},
\end{aligned}$$

if $\alpha_j = 0$ for some j, then one can delete the item $\left\lceil \frac{p^{n_j}}{\alpha_j} \right\rceil$ from above formulas or simply regard it as ∞.

Now consider the pair $((1, 0, \cdots, 0, \alpha_{r+1}), r+1)$ with $\alpha_{r+1} \geq 1$, set $\hat{\alpha} = (1, 0, \cdots, 0, \alpha_{r+1})$, then we have $[\hat{\alpha}, 1] = p^{n_1} + 1$ and $[\hat{\alpha}, r+1] = p^{n_1}$. One has $[\hat{\alpha}] = [\hat{\alpha}, 1]$, and $r + 1 \neq 1 = \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\hat{\alpha}, j] = [\hat{\alpha}]\}$. Therefore this pair $(\hat{\alpha}, r+1)$ is in the set Λ^{00}.

The map $\Gamma_{(\hat{\alpha},r+1)}$ from $(1+x\mathbb{F}[x]/(x^{p^{n_1}}))^\times$ to $K_2(A,M)$ is injective. Firstly,

$$(1+x\mathbb{F}[x]/(x^{p^{n_1}}))^\times \cong \bigoplus_{\substack{1\le m\le p^{n_1}-1\\ gcd(m,p)=1}} (\mathbb{Z}/p^{1+\left\lfloor \log_p \frac{p^{n_1}-1}{m}\right\rfloor}\mathbb{Z})^f$$

by corollary 8.7. Then by lemme 8.27, $\mathbb{Z}/p^k\mathbb{Z}$ is a direct summand for any $1\le k\le n_1$. Under the assumption that p^{n_1} is the exponent of G, the biggest number of $[\alpha,i]$ is p^{n_1}. And for any other $(\alpha,i)\in\Lambda^{00}$, the factors in the decomposition of $(1+x\mathbb{F}[x]/(x^{[\alpha,i]}))^\times$ are included in the set $T=\{\mathbb{Z}/p^k\mathbb{Z}\mid 1\le k\le n_1\}$. Therefore each groups in T will appear infinite times in $K_2(A,M)$. And since everyone in T appear finite times in $K_2(\mathbb{F}[G])$ by proposition 8.28, then for any $1\le k\le n_1$, $\mathbb{Z}/p^k\mathbb{Z}$ will appear infinite times in $NK_2(\mathbb{F}[G])$. Moreover there are no other direct summands.

By proposition 8.9 (2) and (3), we can choose H to be the finite abelian group $\bigoplus_{k=1}^{n_1}\mathbb{Z}/p^k\mathbb{Z}$ so that

$$NK_2(\mathbb{F}[G]) \cong \bigoplus_{\infty}(\bigoplus_{k=1}^{n_1}\mathbb{Z}/p^k\mathbb{Z}).$$ □

From the above disscusion, the Dennis-Stein symbol $\langle t_1x^{i-1},x\rangle$ with $i\ge 1$ is an element of order p^{n_1} and for any $1\le k\le n_1$, the symbol $\langle t_1^{p^k+1}x^{(p^k+1)i-1},x\rangle$ is of order p^{n_1-k}. But it is not easy to write a set of generators in general.

8.9.3 Computation of $NK_2(\mathbb{F}[G])$

Theorem 8.30. *Let $\mathbb{F}$ be a finite field with p^f elements and G a finite abelian group. Let $G(p)$ be the p-Sylow subgroup of G. If p^e is the exponent of $G(p)$, then*

$$NK_2(\mathbb{F}[G]) = \bigoplus_{\infty}(\bigoplus_{k=1}^{e}\mathbb{Z}/p^k\mathbb{Z}).$$

Proof. Firstly, note that $G=G(p)\times H$ where $G(p)$ is the p-Sylow subgroup of G and $gcd(p,|H|)=1$. There is a ring isomorphism

$$\mathbb{F}[G]\cong\mathbb{F}[H\times G(p)]\cong(\mathbb{F}[H])[G(p)].$$

Since $gcd(p,|H|)=1$, it follows from Maschke's theorem that $\mathbb{F}[H]$ is semisimple, then by Wedderburn-Artin theorem,

$$\mathbb{F}[H]\cong\prod_{i=1}^{m}\mathbb{F}_{p^{f_i}},$$

where m is a positive integer and $\mathbb{F}_{p^{f_i}}$ is the finite field with p^{f_i} elements such that $\sum_{i=1}^{m} f_i = f|H|$. Hence one gets

$$\mathbb{F}[G] \cong (\mathbb{F}[H])[G(p)] \cong \prod_{i=1}^{m} \mathbb{F}_{p^{f_i}}[G(p)],$$

$$\mathbb{F}[G][x] \cong \prod_{i=1}^{m} \mathbb{F}_{p^{f_i}}[G(p)][x],$$

and $K_2(\mathbb{F}[G]) \cong \bigoplus_{i=1}^{m} K_2(\mathbb{F}_{p^{f_i}}[G(p)])$, $K_2(\mathbb{F}[G][x]) \cong \bigoplus_{i=1}^{m} K_2(\mathbb{F}_{p^{f_i}}[G(p)][x])$, so $NK_2(\mathbb{F}[G]) \cong \bigoplus_{i=1}^{m} NK_2(\mathbb{F}_{p^{f_i}}[G(p)])$.

Now the exponent of $G(p)$ is p^e, it follows from theorem 8.29 that

$$NK_2(\mathbb{F}_{p^{f_i}}[G(p)]) \cong \bigoplus_{\infty}(\bigoplus_{k=1}^{e} \mathbb{Z}/p^k\mathbb{Z}),$$

then we have

$$NK_2(\mathbb{F}[G]) \cong \bigoplus_{\infty}(\bigoplus_{k=1}^{e} \mathbb{Z}/p^k\mathbb{Z}). \qquad \square$$

Corollary 8.31. *If $K_2(\mathbb{F}G) \neq 0$, then $NK_2(\mathbb{F}G)$ is not finitely generated.*

Proof. If $\{\langle r, s\rangle\}$ is a generating set of $K_2(\mathbb{F}G)$, then $\{\langle rx^i, s\rangle \mid i \geq 1\}$ generate a subgroup of $NK_2(\mathbb{F}G)$ which is infinitely generated. $\square$

Corollary 8.32. *$K_2(\mathbb{F}_p[C_p \times T])$ where T is the infinite cyclic group generated by t.*
$K_2(\mathbb{F}_{p^f}[G \times T])$

By corollary 8.13, if $NK_2(\mathbb{F}G) \neq 0$, then $N^jK_2(\mathbb{F}G) \neq 0$ for all $j \geq 1$ and $N^jK_i(\mathbb{F}G) \neq 0$ for all $i \geq 2$ and $j \geq i-1$.

Chapter 9

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

9.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

9.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [53] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

9.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1, t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:
generators: $\langle a, b\rangle$, $(a, b) \in A \times M \cup M \times A$;
relations: $\langle a, b\rangle = -\langle b, a\rangle$,
$\langle a, b\rangle + \langle c, b\rangle = \langle a + c - abc, b\rangle$,
$\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [46]

- $\mathbb{N}$: the monoid of natural numbers,
- $\epsilon^1 = (1,0) \in \mathbb{N}^2$, $\epsilon^2 = (0,1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,
- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1,2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha) t^{\alpha - \epsilon^i}, t_i \rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 9.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [46]。 □

Lemma 9.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x \rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3 \rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

9.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 9.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. $\square$

Theorem 9.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$, then $I = (t_1^4, t_2^4)$, $M = (t_1, t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha, i) \in \Lambda^{00} \mid [(\alpha, i)] > 1\}$, and then

$$K_2(A, M) \cong \bigoplus_{(\alpha,i) \in \Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i) \in \Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha, 1) \in \Lambda^{00}_{>1}$ and $[\alpha, 1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ and $[\alpha, 2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1, \alpha_2 \leq 4$ are both odd and $gcd(\alpha_1, \alpha_2) = 1$, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha, 1]$.

By the computation 9.2, we can get the following table

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^{\times}$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Consider $((1,1),2) \in \Lambda^{00}$,

$$\begin{aligned}
\Gamma_{(1,1),2} \colon (1 + x\mathbb{F}_2[x]/(x^4))^{\times} &\rightarrowtail K_2(A, M) \\
1 + x(\text{order } 4) &\mapsto \langle t_1, t_2 \rangle, \\
1 + x^3(\text{order } 2) &\mapsto \langle t_1^3 t_2^2, t_2 \rangle.
\end{aligned}$$

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below,
the generators of order 4:

$$\langle t_1 t_2, t_1 \rangle, \langle t_1 t_2, t_2 \rangle, \langle t_1, t_2 \rangle,$$

the generators of order 2:

$$\langle t_1 t_2^3, t_1 \rangle, \langle t_1^3 t_2, t_1 \rangle, \langle t_1^3 t_2^3, t_1 \rangle, \langle t_1 t_2^3, t_2 \rangle, \langle t_1^3 t_2^2, t_2 \rangle, \langle t_1 t_2^2, t_2 \rangle, \langle t_1^3 t_2, t_2 \rangle, \langle t_1^3 t_2^3, t_2 \rangle, \langle t_1^3, t_2 \rangle. \quad \square$$

Remark 9.5. Compared with [53], note that $\langle t_1^3, t_2\rangle = \langle t_1^2 t_2, t_1\rangle$, because

$$\begin{aligned}\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1 t_2\rangle - \langle t_1^2 t_2, t_1\rangle \\ &= \langle t_1, t_1^2 t_2\rangle - \langle t_1^2 t_2, t_1\rangle - \langle t_1^2 t_2, t_1\rangle \\ &= -3\langle t_1^2 t_2, t_1\rangle \\ &= -\langle t_1^2 t_2, t_1\rangle \\ &= \langle t_1^2 t_2, t_1\rangle,\end{aligned}$$

since $\langle t_1^2 t_2, t_1\rangle + \langle t_1^2 t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

$K_2(\mathbb{F}_2[C_2 \times C_2])$ 推广至 $K_2(\mathbb{F}_p[C_p \times C_p])$ 接着算 $K_2(\mathbb{F}_3[C_9 \times C_9])$ 推广至 $K_2(\mathbb{F}_p[C_{p^2} \times C_{p^2}])$. 另外 $K_2(\mathbb{F}_2[C_2 \times C_4])$ 推广至 $K_2(\mathbb{F}_2[C_{2^i} \times C_{2^j}])$

9.5 $K_2(\mathbb{F}_p[C_p \times C_p])$

$K_2(\mathbb{F}_p[C_p \times C_p]) = K_2(\mathbb{F}_p[t_1, t_2]/(t_1^p, t_2^p), (t_1, t_2))$

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq p \text{ or } \alpha_2 \geq p\},$$
$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{p+1}{\alpha_1} \right\rceil, \left\lceil \frac{p}{\alpha_2} \right\rceil\},$$
$$[\alpha, 2] = \min\{\left\lceil \frac{p}{\alpha_1} \right\rceil, \left\lceil \frac{p+1}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

$gcd(p, \alpha_1, \alpha_2) = 1$, $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$

1. If $p \mid \alpha_1$, $p \nmid \alpha_2$, $[\alpha] = [\alpha, 2]$.
2. If $p \nmid \alpha_1$, $p \mid \alpha_2$, $[\alpha] = [\alpha, 1]$.
3. If $p \nmid \alpha_1$, $p \nmid \alpha_2$, $[\alpha] = \max\{[\alpha, 1], [\alpha, 2]\}$.

$\Lambda^{00}_{>1}$:

- $gcd(\alpha_1, \alpha_2) = 1, p \mid \alpha_1, 1 \leq \alpha_1 \leq p, p \nmid \alpha_2, (\alpha, 1) \in \Lambda^{00}_{>1}$, $[\alpha, 1] = \min\{\left\lceil \frac{p+1}{\alpha_1} \right\rceil, \left\lceil \frac{p}{\alpha_2} \right\rceil\}$.
- $gcd(\alpha_1, \alpha_2) = 1$, $p \nmid \alpha_1$, $1 \leq \alpha_1 \leq p$, if $p \mid \alpha_2$, then $[\alpha] = [\alpha, 1]$, $(\alpha, 2) \in \Lambda^{00}_{>1}$. Else if $p \nmid \alpha_2$, then $(\alpha, 2) \in \Lambda^{00}_{>1}$ only if $[\alpha] = [\alpha, 1]$. In both case, $[\alpha, 2] = \min\{\left\lceil \frac{p}{\alpha_1} \right\rceil, \left\lceil \frac{p+1}{\alpha_2} \right\rceil\}$.

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$
$((p, \alpha_2), 1), 1 \leq \alpha_2 \leq p-1$	2
$((\alpha_1, p), 2), 1 \leq \alpha_1 \leq p-1$	2
$((1, \alpha_2), 2), 1 \leq \alpha_2 \leq p-1$	$\left\lceil \frac{p+1}{\alpha_2} \right\rceil$
$((\alpha_1, 1), 2), 1 \leq \alpha_1 \leq p-1$	$\left\lceil \frac{p}{\alpha_1} \right\rceil$

上面最后一行有些问题，需要订正 when p is a small prime, such as 2 or 3, when $p = 2$:

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_p[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

When $p = 3$:

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_p[x]/(x^{[\alpha,i]}))^\times$
$((3,1),1)$	2	$\mathbb{Z}/3\mathbb{Z}$
$((3,2),1)$	2	$\mathbb{Z}/3\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/3\mathbb{Z}$
$((2,3),2)$	2	$\mathbb{Z}/3\mathbb{Z}$
$((1,1),2)$	3	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$((1,2),2)$	2	$\mathbb{Z}/3\mathbb{Z}$
$((2,1),2)$	2	$\mathbb{Z}/3\mathbb{Z}$

Note that $(1 + x\mathbb{F}_3[x]/(x^3))^\times = BigWitt_2(\mathbb{F}_3) = (\mathbb{Z}/3\mathbb{Z})^2$. Hence $K_2(\mathbb{F}_2[C_2 \times C_2]) \cong (\mathbb{Z}/2\mathbb{Z})^3$, $K_2(\mathbb{F}_3[C_3 \times C_3]) \cong (\mathbb{Z}/3\mathbb{Z})^8$.

9.6 $K_2(\mathbb{F}_p[C_{p^2} \times C_{p^2}])$

$K_2(\mathbb{F}_p[C_{p^2} \times C_{p^2}]) \cong K_2(\mathbb{F}_p[t_1, t_2]/(t_1^{p^2}, t_2^{p^2}), (t_1, t_2))$, let $A = \mathbb{F}_p[t_1, t_2]/(t_1^{p^2}, t_2^{p^2})$, $I = (t_1^{p^2}, t_2^{p^2})$, M is the nilradical (t_1, t_2).

$$\Delta = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq p^2 \text{ or } \alpha_2 \geq p^2\},$$

$$\Lambda = \{(\alpha, i) \mid \alpha_i \geq 1\}.$$

For $(\alpha, i) \in \Lambda$,

$$[\alpha, 1] = \min\{\left\lceil \frac{p^2+1}{\alpha_1} \right\rceil, \left\lceil \frac{p^2}{\alpha_2} \right\rceil\},$$

$$[\alpha, 2] = \min\{\left\lceil \frac{p^2}{\alpha_1} \right\rceil, \left\lceil \frac{p^2+1}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

If $gcd(p, \alpha_1, \alpha_2) = 1$, $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$

1. If $p \mid \alpha_1$, $p \nmid \alpha_2$, $[\alpha] = [\alpha, 2]$.
2. If $p \nmid \alpha_1$, $p \mid \alpha_2$, $[\alpha] = [\alpha, 1]$.
3. If $p \nmid \alpha_1$, $p \nmid \alpha_2$, $[\alpha] = \max\{[\alpha, 1], [\alpha, 2]\}$.

Now we want to know when $\min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}$ is 1 or 2.

Firstly, we can easily get if $p \nmid \alpha_1$ and $p \mid \alpha_2$ or $p \nmid \alpha_1$ and $p \nmid \alpha_2$ with $[\alpha, 2] \leq [\alpha, 1]$, then $p \nmid \alpha_1$ and $[\alpha] = [\alpha, 1]$, hence $\min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\} = 1$. In both cases, $(\alpha, 2) \in \Lambda^{00}$.

Similarly, if $p \mid \alpha_1$ and $p \nmid \alpha_2$ or $p \nmid \alpha_1$ and $p \nmid \alpha_2$ with $[\alpha, 1] < [\alpha, 2]$, then $(\alpha, 1) \in \Lambda^{00}$.

When $p \mid \alpha_2$,i.e. $\alpha_2 = p, 2p, \cdots, (p-1)p$ $((1, \alpha_2), 2) \in \Lambda^{00}$, $[(1, ip), 2] = \left\lceil \frac{p^2+1}{ip} \right\rceil$.

Chapter 10

$NK_i(FG)$

10.1 Preliminaries

Definition 10.1. If F is any functor from the category of rings to the category of abelian groups,we write $NF(R)$ for the cokernel of the natural map $F(R) \xrightarrow{x \mapsto 1} F(R[x])$; NF is also a functor on rings. Moreover, the ring map $R[x] \longrightarrow R$ provides a splitting $F(R[x]) \longrightarrow F(R)$ of the natural map, so we have a decomposition $F(R[x]) \cong F(R) \oplus NF(R)$.

In particular, when F is K_i we have functors NK_i and a decomposition $K_i(R[x]) = K_i(R) \oplus NK_i(R)$. Since the ring maps $R[x] \xrightarrow{x \mapsto r} R$ are split surjections for every $r \in R$, hence for every r we also have $NK_0(R) \cong K_0(R[x], (x-r))$ and $NK_1(R) \cong K_1(R[x], (x-r))$.

Let U be a functor from the category of commutative rings to the category of abelian groups,

$$\begin{aligned} U\colon \mathbf{CRing} &\longrightarrow \mathbf{Ab} \\ R &\mapsto U(R) = R^\times. \end{aligned}$$

We can also define the functor NU.

Lemma 10.2. *Let R be a commutative ring with nilradical $\mathfrak{N}$. If $r_0 + r_1x + \cdots + r_nx^n$ is a unit of $R[x]$ then $r_0 \in R^\times$ and $r_1, \cdots, r_n$ are nilpotent. We have*

1. *$NU(R)$ is the subgroup $1 + x\mathfrak{N}[x]$ of $R[x]^\times$;*
2. *$R[x]^\times = R^\times \oplus NU(R)$;*

3. *R is reduced if and only if $R^\times = R[x]^\times$;*
4. *Suppose that R is an algebra over a field k. If $char(k) = p$, $NU(R)$ is a p-group. If $char(k) = 0$, $NU(R)$ is a uniquely divisible abelian group (= a $\mathbb{Q}$-module).*

Definition 10.3. Let R be a commutative ring, the determinant of a matrix provides a group homomorphism from $GL(R)$ onto the group $R^\times$ of units of R. Define $SK_1(R)$ to be the kernel of the induced surjection $\det\colon K_1(R) \longrightarrow R^\times$. Moreover, there is a direct sum decomposition $K_1(R) = R^\times \oplus SK_1(R)$.

Example 10.4. If F is a field then $SK_1(F) = 0$. Similarly, if R is a Euclidean domain such as $\mathbb{Z}$ or $F[x]$ then $SK_1(R) = 0$ and hence $K_1(R) = R^\times$. In particular, $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ and $K_1(F[x]) = F^\times$. If F is a finite field extension of $\mathbb{Q}$ (a number field) and R is an integrally closed subring of F , then Bass, Milnor and Serre proved that $SK_1(R) = 0$.

Lemma 10.5. *If R is a commutative semilocal ring, then $SK_1(R) = 0$ and $K_1(R) = R^\times$. In particular, if $(R, \mathfrak{m})$ is a commutative local ring, then $SK_1(R) = 0$.*

Proof. See [52] lemma 3.1.4. □

Lemma 10.6. *Let R ba a commutative regular ring, $A = R[t]/(t^N)$, then*

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$\begin{aligned} NK_1(A) \cong \mathrm{Nil}_0(A) &\cong (1 + txA[x])^\times = (1 + xA[x])^\times \\ [(A,t)] &\mapsto 1 - tx \\ [(P,\nu)] &\mapsto \det(1 - \nu x) \end{aligned}$$

Proof. See [52] chapter 2 example 7.4.5 and chapter 3 example 3.8.1. □

10.2 Main Theorem

Theorem 10.7. *(1) Let R be a commutative semilocal ring, if $NK_1(R) = 0$, then $SK_1(R[x]) = 0$ and $R[x]^\times = R^\times$ (which means R is reduced);*
(2) If R is a reduced commutative semilocal ring, then $NK_1(R) = SK_1(R[x])$;
(3) If R is a commutative semilocal ring but not reduced, then $NK_1(R) \neq 0$, this is equivalent to say that $NK_1(R)$ is not finitely generated.

Proof. Let R be a commutative semilocal ring. There are two split exact sequences

$$0 \longrightarrow SK_1(R[x]) \longrightarrow K_1(R[x]) \longrightarrow R[x]^\times \longrightarrow 0,$$

$$0 \longrightarrow NK_1(R) \longrightarrow K_1(R[x]) \longrightarrow K_1(R) \longrightarrow 0.$$

Recall that $SK_1(R) = 0$ for any commutative semilocal ring R, hence $K_1(R) \cong R^\times$. We have

$$\begin{aligned} K_1(R[x]) &= NK_1(R) \oplus R^\times \\ &= SK_1(R[x]) \oplus R[x]^\times \\ &= SK_1(R[x]) \oplus R^\times \oplus NU(R) \\ &= SK_1(R[x]) \oplus R^\times \oplus (1 + x\mathfrak{N}[x]), \end{aligned}$$

hence $NK_1(R) \cong SK_1(R[x]) \oplus (1 + x\mathfrak{N}[x])$.

(1) If $NK_1(R) = 0$, one can obtain $SK_1(R[x]) = 0$ and $NU(R) = 0$;

(2) Since $NU(R) = 0$;

(3) In this case $NK_1(R) \supset 1 + x\mathfrak{N}[x] \neq 0$, and it is known that NK_i is either 0 or is not finitely generated. □

Theorem 10.8. *If F is a field of characteristic $p > 0$ and G is a finite abelian p-group, then $NK_1(FG) \neq 0$. In particular $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \neq 0$.*

Proof. First we claim that $R = FG$ is a commutative local ring. In fact suppose F is a field of characteristic p and G is a finite group, then FG is a local ring if and only if G is a finite p-group, in this case the maximal ideal is the augmentation ideal $\mathfrak{m} = IG$ which is also nilpotent, see [20]. And it is easy to see that R is not reduced: take an element a of the maximal order $o(a)$, $o(a) = p^m \leq |G|$ for some m, then $a^{o(a)/p} - 1$ is a nilpotent element since $(a^{o(a)/p} - 1)^p = 0$.

Since local rings are semilocal, we conclude that $NK_1(R) \neq 0$ by Theorem 10.7(3). □

Theorem 10.9. *Let $\mathbb{F}$ be a finite field of characteristic $p > 0$ and H is a finite group such that $(p, |H|) = 1$, then $NK_i(\mathbb{F}H) = 0$ for all i.*

Proof. By Maschke's theorem, $\mathbb{F}H$ is semisimple. Since it is finite, by Wedderburn-Artin theorem $\mathbb{F}H$ is a direct product of matrix rings over finite fields F of characteristic

p, $\mathbb{F}H \cong \prod_m M_m(F)$ and $\mathbb{F}H[x] \cong \prod_m M_m(F[x])$. Hence $K_i(\mathbb{F}H)$ is isomorphic to a direct product of groups $K_i(F)$, and $NK_i(\mathbb{F}H)$ is isomorphic to a direct product of groups $NK_i(F)$for all i. Since finite fields are regular, we obtain $NK_i(\mathbb{F}H) = 0$. □

Since $\mathbb{F}_{p^m}[C_{p^n}] = \mathbb{F}_{p^m}[t]/(t^{p^n})$ and $\mathbb{F}_{p^m}$ is regular, we can use Lemma 10.6 to describe $NK_1(\mathbb{F}_{p^m}[C_{p^n}])$ as follows.

Theorem 10.10. $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \cong$. 还没算完

这个证明还有问题

Proof. By lemma 10.6, $NK_1(\mathbb{F}_{p^m}[C_{p^n}]) \cong (1+tx\mathbb{F}_{p^m}[t,x]/(t^{p^n}))^\times \cong (1+x\mathbb{F}_{p^m}[t,x]/(t^{p^n}))^\times$. As an abelian group, we have

$$(1+tx\mathbb{F}_{p^m}[t]/(t^{p^n})[x])^\times = (1+tx\mathbb{F}_{p^m}[x][\![t]\!])^\times/(1+t^{p^n}x\mathbb{F}_{p^m}[x][\![t]\!])^\times = W_{p^n-1}(x\mathbb{F}_{p^m}[x]).$$

□

由同态

$$(1+t\mathbb{F}_3[\![t]\!])^\times \longrightarrow (1+t\mathbb{F}_3[t]/(t^3))^\times$$

它的核是 $(1+t^3\mathbb{F}_3[\![t]\!])^\times$. 从而 $(1+t\mathbb{F}_3[t]/(t^3))^\times \cong (1+t\mathbb{F}_3[\![t]\!])^\times/(1+t^3\mathbb{F}_3[\![t]\!])^\times$, 又有 $(1+t\mathbb{F}_3[\![t]\!])^\times/(1+t^3\mathbb{F}_3[\![t]\!])^\times = W_2(\mathbb{F}_3)$. 注意这里的 $W(R)$ 是 big Witt 向量, $W_2(\mathbb{F}_3) \neq \mathbb{Z}/\mathbb{Z}^9$

算 $(1+t\mathbb{F}_3[t]/(t^3))^\times$ 中的元素, 只有 3 阶元:

$(1+t(a_0+a_1t))(1+t(b_0+b_1t)) = 1+t(a_0+b_0+(a_1+b_1+a_0b_0)t)$, 考虑 pair$(a_0,a_1)$, 想找到它和 $\mathbb{Z}/9\mathbb{Z}$ 之间的对应,但是经过计算这样的 pair 没有 9 阶元,比如 $(2,1)(2,1)(2,1) = (1,0)(2,1) = (0,0)$.

最终算出来 $(1+t\mathbb{F}_3[t]/(t^3))^\times = \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

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致谢

Chapter 11

Note on Some Formulas Pertaining to the K-theory of Commutative Groupschemes

Author: Spencer BLOCH

Journal of algebra 1978

k: ground ring

$k - alg$: the category of k-algebras (all rings and algebras commutative with 1)

$\widehat{k - alg}$: the category of functors $F : k - alg \longrightarrow Ab$, i.e. $\widehat{k - alg} = Ab^{k-alg}$

Some examples of objects in $\widehat{k - alg}$: (a) Any commutative group scheme G over k ($G(A) = Mor_k(Spec(A), G)$).
我们主要关注的是 (b) Any of the K-functors of Grothendieck-Bass-Milnor-Quillen.
(c) Let $F, G \in Obj(\widehat{k - alg})$. There is an internal hom, $\mathrm{Hom}(G, F) \in Obj(\widehat{k - alg})$, defined by

$$\mathrm{Hom}(G, F)(A) = \underset{\widehat{A-alg}}{\mathrm{Hom}}(G|_{A-alg}, F|_{A-alg}),$$

character group $\mathrm{Hom}(G, G_m)$ associated to a groupscheme G ($G_m =$ "multiplicative group", $G_m(A) = A^* =$ units in A).

Under certain reasonable hypotheses $\mathrm{Hom}(G, G_m)$ is representable and one has a duality (Cartier duality)

$$\mathrm{Hom}(\mathrm{Hom}(G, G_m), G_m) = G.$$

研究 $\mathrm{Hom}(G, G_m)$, 但是替换成变种 $\mathrm{Hom}(G, K_n)$

首先定理

Theorem 11.1. $n \geq 1$, $\mathrm{Hom}(G_m, K_n) = K_{n-1}$.

Theorem 11.2. *Let G be representable and assume G satisfies Cartier duality,* $\mathrm{Hom}(\mathrm{Hom}(G, G_m), G_m) = G$*. Then the natural map*

$$G \longrightarrow \mathrm{Hom}(\mathrm{Hom}(G, K_n), K_n)$$

is a (canonically) split injection for any $n \geq 1$.

Definition 11.3. Define a covariant functor "translation by n" from the category of groupschemes over k satisfying Cartier duality to $\widehat{k - alg}$:

$$G\{n\} := \mathrm{Hom}(\mathrm{Hom}(G, G_m), K_{n+1}), n \geq 0.$$

Example 11.4. $G_m\{n\} = K_{n+1}$, $\mathbb{Z}\{n\} = K_n$.

Concern the functors
$C_nK_q =$ curves of length n on K_q,
$CK_q =$ curves on K_q ,
$\widehat{CK}_q =$ formal curves on K_q
defined by

$$C_nK_q(A) = \ker(K_q(A[T]/(T^{n+1})) \xrightarrow{T \mapsto 0} K_q(A))$$

$$CK_q(A) = \varprojlim_n C_nK_q(A)$$

$$\widehat{CK}_q(A) = \ker(K_q(A[T]) \xrightarrow{T \mapsto 0} K_q(A))$$

用 NK 的语言看最熟悉的是 $\widehat{CK}_q = NK_q$. 我们关心的 $\mathbb{F}_p$-algebra $\mathbb{F}_p[C_p]$,

$$C_{p-1}K_q(\mathbb{F}_p) = \ker(K_q(\mathbb{F}_p[T]/(T^p)) \longrightarrow K_q(\mathbb{F}_p))$$

这个对于 K_2 来讲是平凡的，因为 $K_2(\mathbb{F}_q) = 0$, $K_2(\mathbb{F}_p[T]/(T^n)) = 0$，实际上对于 perfect field of char $p > 0$, $K_2(k[T]/(T^n)) = K_2(k)$ 参考"Derivarions of witt vectors with application to K_2 of truncated polynomial rings and laurent series"

考虑 smooth $\mathbb{F}_p$-algebra $\mathbb{F}_p[x]$,

$$C_{p-1}K_2(\mathbb{F}_p[x]) = \ker(K_2(\mathbb{F}_p[C_p][x]) \longrightarrow K_2(\mathbb{F}_p)[x]) = 0$$

实际上有 $C_{p-1}K_2(\mathbb{F}_p[x]) = \widehat{CK}_2(\mathbb{F}_p[C_p]) = NK_2(\mathbb{F}_p[C_p])$, 并且对于一般的 n 有

$$C_nK_2(\mathbb{F}_p[x]) = K_2(\mathbb{F}_p[x,T]/(T^{n+1}))$$

RHS is the K_2 of truncated polynomial rings.

Back to paper

$$CK_1(A) = \varprojlim_n C_nK_1(A) = BigWitt(A) = (1 + TA[[T]])^*$$

the big Witt vectors over A.

$\widehat{CK}_1(A) = NK_1(A)$ contains as a direct factor the group $\widehat{BigWitt}(A)$ = invertible elements in $(1 + TA[T])$ (big formal Witt vectors on A) 参考 P. CARTIER, Groupes formels associes aux anneaux de Witt generalises。

When k is a $\mathbb{Z}_{(p)}$-algebra (like $\mathbb{F}_p$), $\mathbb{Z}_{(p)} = \mathbb{Z}[1/l \mid (l,p) = 1]$. The functors $CK_q, \widehat{CK}_q = NK_q, BigWitt, \widehat{BigWitt}$ split into product of p-typical pieces denoted

$$TCK_q, T\widehat{CK}_q = TNK_q, W, \widehat{W}$$

respectively. (Jan Stienstra 中 On K_2 and K_3 of truncated polynomial rings 中还有一些更多的内容)

Theorem 11.5. *Assume some prime number p is nilpotent in k, then there are isomorphisms*

$$TCK_q \cong \mathrm{Hom}(\widehat{W}, K_q),$$
$$TNK_q = T\widehat{CK}_q \cong \mathrm{Hom}(W, K_q)$$

As an application, we prove a more precise form of Cartier duality for $\widehat{W}$ and K_2:

Theorem 11.6. *Assume some prime number p is nilpotent in k. Then*

$$\mathrm{Hom}(\mathrm{Hom}(\widehat{W}, K_2), K_2) = T\widehat{CK}_1 = TNK_1.$$

11.1 DUALITY AND TRANSLATION

k: ground commutative ring

$k-alg$: the category of k-algebras (all rings and algebras commutative with 1)

Definition 11.7. We define $\mathrm{Hom}(F, G) \in Obj(\widehat{k-alg})$ by

$$\mathrm{Hom}(F, G)(A) = \underset{\widehat{A-alg}}{\mathrm{Hom}}(F|_{A-alg}, G|_{A-alg}).$$

Viewing F, G as functors $k-alg \longrightarrow Sets$, we can define $Mor(F, G)$ by considering morphisms of functors which are not necessarily compatible with the Abelian group structure. Clearly, $\mathrm{Hom}(F, G) \subset Mor(F, G)$.

Definition 11.8. $F \in Obj(\widehat{k-alg})$ is said to be representable if there exists an $A \in Obj(k-alg)$ and an isomorphism of functors

$$F(-) = \underset{k-alg}{\mathrm{Hom}}(A, -).$$

When this is the case, there is an isomorphism (Yoneda)

$$Mor(F, G) = G(A).$$

Suppose for example $F = G_m$ is the functor $G_m(A) = A^* =$ units in A. G_m is represented by the algebra $k[T, T^{-1}]$.

We obtain for any $G \in \widehat{k-alg}$, an inclusion $\mathrm{Hom}(G_m, G) = \mathrm{Hom}(\mathrm{Hom}(k[T, T^{-1}], -), G(-)) \subset G(k[T, T^{-1}])$.

Particularly interesting for us will be the case $G = K_i$, one of the K-functors of Grothendieck-Bass-MiInor-Quillen.

Theorem 11.9. $\mathrm{Hom}(G_m, K_n) = K_{n-1}$

A basic theorem (proved independently by Quillen and Waldhausen) gives a split-exact sequence

$$0 \longrightarrow K_n(R[T]) \oplus K_n(R[T^{-1}]) \longrightarrow K_n(R[T, T^{-1}]) \longrightarrow K_{n-1}(R) \longrightarrow 0.$$

Elements of $K_n(R)$ correspond to constant morphisms $G_m \longrightarrow K_n$.

A group scheme G over $Spec(k)$ will be said to satisfy Cartier duality if the natural map

$$G \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Hom}(G, G_m), G_m).$$

is an isomorphism. When k is Noetherian, finite flat group schemes and formal group schemes satisfy Cartier duality.

Theorem 11.10. *Let G be a groupscheme satisfying Cartier duality, and let $n \geq 1$ be an integer. Then the natural map*

$$G \rightarrowtail \mathrm{Hom}(\mathrm{Hom}(G, K_n), K_n)$$

is a (canonically) split injection.

一些 $\mathrm{Hom}(G, K_n)$ 的计算, 某些条件下 $\mathrm{Hom}(\widehat{W}, K_q) \cong TCK_q, \mathrm{Hom}(W, K_q) \cong T\widehat{CK}_q$

Definition 11.11. Let $n \geq 0$ be an integer and let $Cartier/k$ denote the category of group schemes over k satisfying Cartier duality. The functor translation by n: $Cartier/k \longrightarrow \widehat{k-alg}$, $G \mapsto G\{n\}$ is given by

$$G\{n\} = \mathrm{Hom}(\mathrm{Hom}(G, G_m), K_{n+1}).$$

For example, $\mathbb{Z}\{n\} = K_n$, $G_m\{n\} = K_{n+1}$.

Proposition 11.12. *(i) G is a direct summand of $G\{0\}$. When $\mathrm{Hom}(G, G_m)$ is infinitesimal, we have $G \cong G\{0\}$.*
(ii) $\mathrm{Hom}(G_m, G\{n+1\}) \cong G\{n\}$, $n \geq 0$.
(iii) Translation by n is a faithful functor.

Given an extension of rings $0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$ with I nilpotent, we get $\ker(K_1(A) \longrightarrow K_1(B)) \cong (1+I)^*$.

Definition 11.13. Let $F \in Obj(\widehat{k-alg})$. The tangent space of F, $t_F \in Obj(\widehat{k-alg})$, is defined by

$$t_F(A) = \ker(F(A[\varepsilon]/(\varepsilon^2)) \xrightarrow{\varepsilon \mapsto 0} F(A)).$$

The bitangent space $bi-t_F$ is given by

$$bi-t_F(A) = \ker(F(A[\varepsilon, \delta]/(\varepsilon^2, \delta^2, \varepsilon\delta)) \longrightarrow F(A[\varepsilon]) \oplus F(A[\delta])).$$

当我们考虑 $F = K_2$ 时,

$$t_{K_2}(\mathbb{F}_2) = \ker(K_2(\mathbb{F}_2[C_2]) \xrightarrow{\varepsilon \mapsto 0} K_2(\mathbb{F}_2)) = 0$$

$$t_{K_2}(\mathbb{F}_2[x]) = NK_2(\mathbb{F}_2[C_2]) = K_2(\mathbb{F}_2[C_2][x])$$

$$t_{K_2}(\mathbb{F}_{2^f}[x]) = NK_2(\mathbb{F}_{2^f}[C_2])$$

看看一般的能不能推出来

Proposition 11.14. *(i) Assume F is representable, i.e., there exists a $\Lambda \in Obj(k-alg)$ and a $\lambda \in F(\Lambda)$ inducing an isomorphism $F(-) \cong \mathrm{Hom}_{k-alg}(\Lambda, -)$. The identity element $0 \in F(k)$ gives a k-homomorphism $\rho\colon \Lambda \longrightarrow k$. Let $I = \ker \rho$. Then*

$$t_F(A) = \underset{k-mod}{\mathrm{Hom}}(I/I^2, A),$$
$$bi - t_F = 0.$$

Assume now that $1/2 \in k$. Then
(ii) $t_{K_2}(A) = \Omega_A =$ module of absolute Kähler differentials of A. $bi - t_{K_2}(A) = A$.
(iii) Let G/k be a group scheme satisfying Cartier duality and such that t_G is locally free, $G^* = \mathrm{Hom}(G, G_m)$. Then

$$t_{G\{1\}}(A) \cong (t_G(A) \otimes_A \Omega^1_A) \oplus t^*_{G^*}(A)$$
$$bi - t_{G\{1\}}(A) \cong t_G(A).$$

where $t^*(A) = \mathrm{Hom}_A(t(A), A)$.

By definition, an element in $t_F(A)$ is a k-algebra homomorphism $f : \Lambda \longrightarrow A[\varepsilon]$ such that the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "f"] \arrow[d, "\rho"'] & A[\varepsilon] \arrow[d, "\varepsilon\mapsto 0"] \\ k \arrow[r] & A \end{tikzcd}$$

commutes.

$$\begin{tikzcd} \mathrm{Hom}(\Lambda,\Lambda) = F(\Lambda) \arrow[r, "F(f)"] \arrow[d, "\rho"'] & F(A[\varepsilon]) = \mathrm{Hom}(\Lambda, A[\varepsilon]) \arrow[d] \\ \mathrm{Hom}(\Lambda, k) = F(k) \arrow[r] & F(A) = \mathrm{Hom}(\Lambda, A) \end{tikzcd}$$

$$\rho \mapsfrom 0$$

后面还没看

Chapter 12

有限域总结

基本的结果：

- Every finite field has prime power order. Every finite field must have characteristic p for some prime p.
- For every prime power $q = p^n$, there is a finite field of that order. Any finite field with $q = p^n$ elements is isomorphic to the splitting field of $x^q - x$ over $\mathbb{F}_p$.
- Any two finite fields of the same size are isomorphic (usually not in just one way).
- A subfield of $\mathbb{F}_{p^n}$ has order p^d where $d|n$, and there is one such subfield for each d.
- Let F be a finite field containing a subfield K with q elements. Then F has q^m elements, where $m = [F : K]$.
- Let F be a finite field. Then F has p^n elements, where the prime p is the characteristic of F and n is the degree of F over its prime subfield.
- For a prime p and positive integer n, there is an irreducible $g(x)$ of degree n in $\mathbb{F}_p[x]$, and $\mathbb{F}_p[x]/(g(x))$ is a field of order p^n. 这样构造的域也称为 Galois 域，the term Galois field lives on today among coding theorists in computer science and electrical engineering as a synonym for finite field and Moore's notation $GF(q)$ is often used in place of $\mathbb{F}_q$.
- If $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = d$, the $\mathbb{F}_p$-conjugates of α are $\alpha, \alpha^p, \alpha^{p^2}, \cdots, \alpha^{p^{d-1}}$.
- Every finite extension of $\mathbb{F}_p$ is a Galois extension whose Galois group over $\mathbb{F}_p$ is generated by the pth power map. The Galois group $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic and a generator is the p-th power map $\phi_p \colon t \mapsto t^p$ on $\mathbb{F}_{p^n}$.
- If F is a finite field with q elements, then every $a \in F$ satisfies $a^q = a$.

- If F is a finite field with q elements and K is a subfield of F, then the polynomial $x^q - x$ in $K[x]$ factors in $F[x]$ as $x^q - x = \prod_{a \in F}(x-a)$ and F is a splitting field of $x^q - x$ over K.
- If F is a finite field of order q, the group $F^\times$ is cyclic of order $q-1$.

例子：

- $\mathbb{F}_4$ as $\mathbb{F}_2(\theta) = \{0, 1, \theta, \theta+1\}$, where $\theta^2+\theta+1=0$, we find that both θ and $\theta+1$ are primitive elements.
- Two fields of order 8 are $\mathbb{F}_2[x]/(x^3+x+1)$ and $\mathbb{F}_2[x]/(x^3+x^2+1)$.
- Two fields of order 9 are $\mathbb{F}_3[x]/(x^2+1)$ and $\mathbb{F}_3[x]/(x^2+x+2)$.
- The polynomial x^3-2 is irreducible in $\mathbb{F}_7[x]$, so $\mathbb{F}_7[x]/(x^3-2)$ is a field of order $7^3 = 343$.

警告：The ring $\mathbb{Z}/(m)$ is a field only when m is a prime number. In order to create fields of non-prime size we must do something other than look at $\mathbb{Z}/(m)$. Every finite field is isomorphic to a field of the form $\mathbb{F}_p[x]/(f(x))$

In the field $\mathbb{F}_3[x]/(x^2+1)$, the nonzero numbers are a group of order 8. The powers of x only take on 4 values, so x is not a generator. The element $x+1$ is a generator: its successive powers are exhaust all the nonzero elements of $\mathbb{F}_3[x]/(x^2+1)$.

k	1	2	3	4	5	6	7	8
x^k	$x+1$	$2x$	$2x+1$	2	$2x+2$	x	$x+2$	1

计算 For every $f(x) \in \mathbb{F}_p[x]$, $f(x)^{p^m} = f(x^{p^m})$ for $m \geq 0$.

Combinatorics. An important theme in combinatorics is q-analogues, which are algebraic expressions in a variable q that become classical objects when $q=1$, or when $q \mapsto 1$. For example, the q-binomial coefficient is

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})},$$

which for $n \geq k$ is a polynomial in q with integer coefficients. When $q \mapsto 1$ this has the value $\binom{n}{k}$. While $\binom{n}{k}$ counts the number of k-element subsets of a finite set, when q is a prime power the number $\binom{n}{k}_q$ counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$. Identities involving q-binomial coefficients can be proved by checking them when q runs through prime powers, using linear algebra over the fields $\mathbb{F}_q$.

Chapter 13

问题

$K_2(\mathbb{F}_2C_2) = K_2(\mathbb{F}_2[x]/(x^2)) =?$

$K_2(\mathbb{F}_pC_p) = K_2(\mathbb{F}_p[x]/(x^p)) =?$

$\mathbb{Z}[Q_8]$, $Q_8 = \langle a, b | a^2 = b^2 = -1, (ab)^2 = -1 \rangle$

slogan: "you can do for vector spaces you can do for finitely generated projective modules"

拆成两篇：一篇简单的涉及 $F_2[C_2]$ 和 $F_2[C_4]$ 的，这样里面可以不涉及 witt 向量的分解之后另一篇再算上 witt 向量的分解，对于一般的 p 都写出来

regular ring 与 regular local ring 关系

Theorem 13.1 (Auslander-Buchsbaum Theorem, 1959)**.** *regular local rings are unique factorization domains.*

13.1 其他可以考虑的问题

$NK_2(\mathbb{F}_{p^m}[C_{p^n}]) =?$

$\mathbb{F}_2[C_2 \times C_2] \cong \mathbb{F}_2[C_2] \otimes \mathbb{F}_2[C_2] \cong \mathbb{F}_2[x, y]/(x^2, y^2)$, 看看能否用同样的方法得到一些结果.

$$0 \longrightarrow K_2(k[t_1, t_2, t_3]/(t_1^n, t_2^n), (t_1, t_2)) \longrightarrow K_2(k[t_1, t_2, t_3]/(t_1^n, t_2^n)) \longrightarrow K_2(k[t_3]) \longrightarrow 0$$

对于有限域 k 来讲 $K_2(k[t_3]) = 0$,

$$0 \longrightarrow NK_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2][x]) \longrightarrow K_2(\mathbb{F}_2[C_2 \times C_2]) \longrightarrow 0,$$

中间那项就是可以用这篇文章里的方法确定, 又 $K_2(\mathbb{F}_2[C_2 \times C_2])$ 可以通过 Gao Yubin 等文章得到, 应该是 C_2^3, 于是可以得到 $NK_2(\mathbb{F}_2[C_2 \times C_2])$, 猜测也是 $\oplus_\infty \mathbb{Z}/2\mathbb{Z}$.

另外可以考虑直接用本章里的方式重新计算高玉彬师兄文章里的结果, 看是否更简洁, 或者是否更繁复, 复杂在哪里, 哪里可以进行简化, 简化后是否可以用到算 NK 的内容中.

一个关于模结构的问题, 在 Weibel 的文章 [49] 中 5.5 和 5.7 给出的模结构和本文上面的模结构并不一致, 用 V_m 作用差一个 t^m.

$C_{p^{n-1}} \rtimes C_p$

13.2 TODO

NK

- 好好写一下 NK 的 introduction，中文的可以按照毕业论文的要求写出来,NIl 要送尽可能介绍的详细
- 把算 NK 的程序变成有界面的。python 改成面向对象的，设计实现流程。比如把打印出表格一行那一部分单独写成函数，def printrow(arg1, arg2, arg3)。请输入一个素数 p, 请输入阶 $ord = n =?$, 输出 α_1 变化范围到 ALPHAMAX, 输出 α 和对应的 $[\alpha], [\alpha, i] =?$. 写程序时遇到的情况，要求 $gcd(p, \alpha_1, \alpha_2) = 1$, 分解成几种情况，最简单的是 $(\alpha_1, \alpha_2) = 1$, since p is a prime number, if $gcd(p, \alpha_1, \alpha_2) \neq 1$, then $gcd(p, \alpha_1, \alpha_2) = p$. PF: Assume $gcd(p, \alpha_1, \alpha_2) = d$, because $d|p$, then $d = 1$ or $d = p$. So if $gcd(p, \alpha_1, \alpha_2) \neq 1$, then $p|\alpha_1, p|\alpha_2$.
- $\mathbb{F}_q[G]$ 的 NK 出来后做 $\mathbb{Z}[G]$ 的 NK，然后考虑 $(\mathbb{Z}/n\mathbb{Z})G$ 的 NK（看下张亚坤的）
- NK 与 localization，考虑有没有例子
- NK 与半直积
- NK 的"有限性"，参考 Grunewald, The finiteness of $NK_1(\mathbb{Z}[G])$. 看 $\mathbb{Z}[Q_8], \mathbb{Z}[C_4]$ 是不是 f.g. 那个 xx 模。$NK_2(\mathbb{Z}[C_p])$ 是 f.g.。Ve，Frobenius 模，$NK_1(\mathbb{Z}[C_2 \times C_2])$ 不是。看看 $NK_2(\mathbb{Z}[C_2 \times C_2])$ 这些是不是 f.g. witt module。
- NK 与 Grayson 用代数的方法定义能不能联系起来。比如用代数的方法定义 NK_i
- NK 的乘积公式，把 weibel 的那个 $NK_1 \longrightarrow NK_2$ 的写清楚看看
- NK 的模结构介绍，把几篇文章综合一下，看能不能应用于 $NK(\mathbb{F}_q[G])$ 中
- induction,restriction 的作用有没有互反律之类的，如何给分次

NK_1

- NK_1 检验 Weible 的 witt vectors module structure 和 Madsen 他们的方法正合列

得到的一样吗。有篇文章说 NK_i 都同构，写出 NK_1, NK_2 看看。

- 看 $SK1$ 怎么做出来的，还有 NK_1 做么做,$NK_1^{\alpha}(\mathbb{Z}[G])$ 是如何证的？带扭和不带扭计算时的区别
- 看看 wang kun,还有之前 NIL 非零就无穷多的那里面 α, 尽管之前是 $NK_1(\mathbb{F}_p[C_p]_{\alpha}[t])$ 是什么样子

NK_2

- 检查计算是否正确
- check the case of $\mathbb{F}_{2^f}[C_{2^n}]$, 用 $\mathbb{F}_2[C_{2^2}], \mathbb{F}_2[C_{2^3}]$ 对一下结果，看看还有哪些元素是落下的，看看用什么办法可以确定这些元素。
- 算一下 $\mathbb{F}_3[C_3], \mathbb{F}_3[C_{3^2}]$ 的 NK，看一些问题主要出在哪里。进而 $\mathbb{F}_{p^f}C_{p^n}$
- 把 $NK_2(\mathbb{F}_2[C_2 \times C_2))$ 和 $NK_2(\mathbb{F}_2[C_2])$ 做比较, 看看一般的 p 是否也有类似的规律。之前有一篇文章中出现了 $NK_2(\mathbb{F}_2[C_2 \times C_2])$，看一下现在的结果和那篇文章中的过程对一下是否一致。
- write the case of $NK_2(F[G \times H])$, for example $NK_2(\mathbb{F}_2[C_2 \times C_2])$. The general case $NK_2(\mathbb{F}_p[C_p \times C_p]), NK_2(\mathbb{F}_{p^f}[C_{p^m} \times C_{p^n}])$。推广到两个群的直积，首先是阶互素的两个群。推广到一般的 G 上一般这里的 G 是一个 p 群。
- 之前写的看能否推广到 $NK_2(\mathbb{F}_p[C_p \times C_q])$, 或者 C_q 换成更一般的阶与 p 互素的群。参考 magurn 之前的文章，还有唐高华的关于群环分解的一些 prop，一般交换群环的结构
- 论文中，$\mathbb{Z}/p^k\mathbb{Z}$ 都出现无限多次，要加上，当 $k > n$ 时 $\mathbb{Z}/p^k\mathbb{Z}$ 不会出现。
- $NK_2^{\alpha}(\mathbb{F}_2[C_2])$ 有没有结果，搞清 α 怎么来的及什么时候有 α
- 对于 $NK_2(\mathbb{F}[C_p \times C_p])$，检查前面 $NK \cong K_2(A, M)$ 的过程是否也对，前面的转化至关重要，一定要确保 NK_2 可以转化成相对 K_2
- 看看这样的问题对吗，若 $K_2(\mathbb{Z}/n\mathbb{Z}G) \neq 0$, 则 $NK_2(\mathbb{Z}/n\mathbb{Z}G)$ 是无限生成的。若 n 是素数 p, 看看成立不。推广到一般的环肯定是不对的，因为 $K_2(\mathbb{Z}) \neq 0$ 但是 $NK_2(\mathbb{Z}) = 0$. 何时 $K_2(\mathbb{F}_q[G]) = 0$?, 写出条件来，和 NK 对比

  Proposition 13.2. *当 $K_2(\mathbb{F}_q[G])$ 中有 p^n 阶元，则 $NK_2(\mathbb{F}_q[G])$ 中有无限多个 p^n 阶元。(看下对不对，最好写出元素来，证明确实是 p^n 阶元)*

  从 $K_2(\mathbb{F}_2[C_4])$ 中先看是否成立，$K_2(\mathbb{F}_2[C_4]) = C_2^? \oplus C_4^?$, $NK_2(\mathbb{F}_2[C_4]) = \oplus_\infty C_2 \oplus \oplus_\infty C_4$.
- $0 \longrightarrow NK_2(\mathbb{F}[G]) \longrightarrow K_2(\mathbb{F}[G][x]) \longrightarrow K_2(\mathbb{F}[G]) \longrightarrow 0$, 右边满射，每一个 D-S 符号 $< a, b >$ 有提升 $< a, b >$, 若满足条件 $1 - ab$ 可逆，$< ax, b >$ 映为 $< 0, b >$ 为 0 所以在核中，应该 $< a, b > \in K_2(\mathbb{F}[G])$, 应该有 $< ax, b > \in NK_2(\mathbb{F}[G])$, 考虑对

称性 $< ax, b >=< a, bx >$ 看看对不对或者差一点东东。这样写出 $K_2(\mathbb{F}[G])$ 中的一组基，就可以写成无限多 NK_2 中的线性无关元，写出 (α, i) 需要判断在 Λ^{00} 中并判断是多少阶元，另外找其他规律写出 $< ax, x >$ 这样的符号

- $\mathbb{F}_2[C_{2^n}]$ 与 $HH_n(\mathbb{F}_2[x])$ 联系，看看有没有映射。

D-S symbols $\langle t^3x^2, x\rangle = -\langle x, t^3x^2\rangle = -(\langle xt^3, x^2\rangle + \langle x^3, t^3\rangle) = -(\langle x^2t^3, x\rangle + \langle x^2t^3, x\rangle + 3\langle x^3t^2, t\rangle)$, $3\langle t^3x^2, x\rangle = -3\langle x^3t^2, t\rangle$, if 3 is invertible, then $\langle t^3x^2, x\rangle = -\langle x^3t^2, t\rangle$.

  - $\langle f(x,t), t^n\rangle = n\langle f(x,t)t^{n-1}, t\rangle$
  - $\langle f(x,t), x^m\rangle = m\langle f(x,t)x^{m-1}, x\rangle$

  $\rho_2\colon x\mathbb{F}_2[x]/x^2\mathbb{F}_2[x^2] \longrightarrow NK_2(\mathbb{F}_2[C_4]), a \mapsto \langle at^3, t\rangle$, 加法群同态, $\langle at^3, t\rangle + \langle at^3, t\rangle = \langle at^3 + bt^3 - abt^7, t\rangle = \langle (a+b)t^3, t\rangle$.

  If $t^n = 0$, then $-\langle at^{n-1}, t\rangle = \langle t, at^{n-1}\rangle = \langle at, t^{n-1}\rangle = (n-1)\langle at^{n-1}, t\rangle$, hence $n\langle at^{n-1}, t\rangle = 0$.

Witt vectors 和模作用

- Module structures on the K-theory of graded rings weibel 这篇文章最后面又模结构的讨论
- witt vectors decompostion。Artin-Hasse 公式写出来。COMPLEX ORIENTED COHOMOLOGY THEORIES AND THE LANGUAGE OF STACKS https://ncatlab.org/nlab/files/HopkinsLecture.pdf 这篇笔记第 52 页有一个比较详细的证明。
- 研究 $(1 + xk[x])/(x^m)$ 中 $1 - ax$ 的阶多少，还有其他生成元是什么样子。google 关键词“truncated witt vectors generators”
- $NK_2(R)$ 上 Witt vector 模结构的问题,$NK_2(\mathbb{F}_2[C_2])$,$NK_2(\mathbb{F}_2[C_4])$ 上 $W(\mathbb{F}_2)$-module. 参考文章把 D-S 符号的模结构（witt vectors 作用在 symbols 上）搞清楚，公式整理出来 (注意如果是老式的 D-S 符号要转化成现在常用的符号)，套在 NK 上试一下是否和 Weibel 之前的一致。weibel 有篇文章说两种不同的分次会有两种不同的结构，考虑一下分别的公式。
- $F_m(1 + ax) = 1 + ax^m$, $V_m(1 + ax) = 1 + a^m x$.
- $D = -x\frac{d}{dx}\log\colon 1 + xR[[x]] \longrightarrow xR[[x]]$, $D(1 - xf(x)) = \frac{1+x^2f'(x)}{1-xf(x)} - 1$. $D(1-x) = \frac{1}{1-x} - 1 = x + x^2 + x^3 + \cdots$. 得到是一个无穷级数，如果是有限的就好了，简单的例子看看何时是有限的。

$$
\begin{aligned}
W(R) \times NK_1(\Lambda) &\longrightarrow NK_1(\Lambda) \\
\alpha(t) * [1 - vx] &= [\alpha(vx)] \\
W(R) \times NK_2(\Lambda) &\longrightarrow NK_2(\Lambda) \\
\alpha(t) * \{r, 1 - vx\} &= \{r, \alpha(vx)\}, r \in K_1(\Lambda)
\end{aligned}
$$

$$
\begin{aligned}
V_n([1 - vx]) &= [1 - vx^n] \\
F_n([1 - vx]) &= [1 - v^n x] \\
[a]([1 - vx]) &= [1 - avx]
\end{aligned}
$$

$$
\begin{aligned}
d\log\colon K_2(R[t]/(t^2), (t)) &\longrightarrow \Omega^2_{R[t]/(t^2)} \\
\langle a, b \rangle &\mapsto \frac{da \wedge db}{1 - ab}
\end{aligned}
$$

[49]$A = \mathbb{F}[t_1, t_2]/I$, $I = (t_1^n)$. We grade A by putting $A_0 = \mathbb{F}[t_2]$ and letting t_1 belong to A_1.

$$
\begin{aligned}
\Gamma_{\alpha,i}\colon (1 + xk[[x]])^\times &\longrightarrow K_2(A, M) \\
1 - xf(x) &\mapsto \langle f(t^\alpha) t^{\alpha - \varepsilon^i}, t_i \rangle
\end{aligned}
$$

Lemma 13.3. *Given α, let $e = \deg(t^\alpha) = \alpha_1$, and identify $(1 + xk[[x]])^\times$ with the ideal $V_e W(k)$ of $W(k) = (1 + tk[[T]])^\times$ via $x = T^e$. Then for all i, the map $\Gamma_{\alpha,i}$ is a $W(k)$-module homomorphism.*

$(1 - rT^m) * \Gamma_{\alpha,i}(1 + x) = \Gamma_{\alpha,i}((1 - rT^m) * (1 + x))$.

Formula from [49]:

- $V_m(1 - rT) * \langle a, s \rangle = (1 - rT^m) * \langle a, s \rangle = d\langle a^{m/d} r^{i/d} s^{m/d - 1}, s \rangle$, where $r, s \in A_0$, $a \in A_i$ and $d = gcd(m, i)$.
- $(l - rT^m) * \langle a, b \rangle = (um + iv)\langle a^k b^{k-1} r^n, b \rangle - jv\langle a^{k-1} b^k r^n, a \rangle + ju\langle (ab)^k r^{n-1}, r \rangle + j(d-1)\langle -(ab)^k r^n, -1 \rangle$, where $a \in A_i$, $b \in A_j$, $r \in R$, $d = gcd(i + j, m)$, $k = m/d$, $n = (i + j)/d$ and u and v are integers such that $d = um + v(i + j)$.

Since $tx^{n-1} \in A_1$ and $x \in A_0$, one has $d = 1$, $V_m(\langle tx^{n-1}, x \rangle) = V_m(1 - T) * (\langle tx^{n-1}, x \rangle) = \langle (tx^{n-1})^m x^{m-1}, x \rangle = \langle t^m x^{mn-1}, x \rangle$.

MAYBE WRONG! If m is even, $t^m = 0$, hence $V_m(\langle tx^{n-1}, x\rangle) = 0$; if m is odd, then $V_m(\langle tx^{n-1}, x\rangle) = \langle t^m x^{mn-1}, x\rangle$.

Consider $V_m(\langle tx^n, t\rangle)$, if m is odd, $d = 1$, $i = j = 1$, let $u = 1$ and $v = (1-m)/2$ such that $1 = 2v + mu$, then

$$\begin{aligned}
V_m(\langle tx^n, t\rangle) &= (\frac{m+1}{2})\langle t^{2m-1}x^{mn}, t\rangle - (\frac{1-m}{2})\langle t^{2m-1}x^{mn-n}, tx^n\rangle \\
&= (\frac{m+1}{2})\langle t^{2m-1}x^{mn}, t\rangle - (\frac{1-m}{2})(n\langle t^{2m}x^{mn-1}, x\rangle + \langle t^{2m-1}x^{mn}, t\rangle) \\
&= m\langle t^{2m-1}x^{mn}, t\rangle - \frac{(1-m)n}{2}\langle t^{2m}x^{mn-1}, x\rangle \\
&= \langle t^{2m-1}x^{mn}, t\rangle
\end{aligned}$$

Frobenius Jan Stienstra, On K_2 and K_3 of truncated polynomial rings [41] 下面这个应该有问题，要对着文章写出来，这里只是把草稿上的打出来

$$\begin{aligned}
F_2\colon NK_2(\mathbb{F}_2[C_4]) &\longrightarrow NK_2(\mathbb{F}_2[C_2]) \\
\text{order 4 } \langle tx^{i-1}, x\rangle &\mapsto \langle tx^{2i-1}, x\rangle \\
\text{order 2 } \langle t^3x^{3i-1}, x\rangle &\mapsto \langle t^3x^{6i-1}, x\rangle \\
\text{order 2 } \langle t^3x^{i-1}, x\rangle &\mapsto \langle t^3x^{2i-1}, x\rangle \\
\text{order 4, } i \geq 1 \text{ odd } \langle tx^i, t\rangle &\mapsto \langle tx^{i-1}, x\rangle \\
\text{order 2, } i \geq 1 \text{ odd } \langle t^3x^i, t\rangle &\mapsto \langle tx^i, t\rangle
\end{aligned}$$

前面四个在 $\Omega_{\mathbb{F}_2[x]}$, 最后一个 $V/\Phi(V)$.

整群环

- $NK_2(\mathbb{Z}[C_p]) \neq 0$, $NK_2(\mathbb{Z}[G]) \cong NK_2(\hat{\mathbb{Z}}_{|G|}[G])$（查一下出处）
- 将有限群环中的 p^n 阶元提升到整群环中。$NK_2(\mathbb{Z}G)$，把 Pineda 文章找出来还有当时的笔记，试着用最近算出来的结果去修改证明，整群环和有限域上群环的联系，争取证明一般的 NK_2 都非零。
- $NK_2(\mathbb{Z}[C_4])$ 中有没有 4 阶元 (naive idea: $x\mathbb{F}_{p^2}[x]$)，有没有 $NK_2(\mathbb{Z}[C_4]) \longrightarrow NK_2(\mathbb{F}_2[C_4])$ 这样的映射，还有类似 $\mathbb{F}_2[C_{2^m}] \longrightarrow \mathbb{F}_2[C_{2^n}]$ 诱导的 NK 的映射
- prove $NK_2(\mathbb{Z}[C_{p^2}]) \neq 0$ (find nontrivial elements), then generalize to $NK_2(\mathbb{Z}[C_{p^n}])$. prove $NK_2(\mathbb{Z}[C_n]) \neq 0$ (find nontrivial elements)
- $NK_2(\mathbb{Z}[C_2 \times C_2])$ 中的元素，从 $NK_2(\mathbb{F}_2[C_2 \times C_2])$ 中提升上去，之后再类似看 $NK_2(\mathbb{Z}[C_4 \times C_4])$

- $2\mathbb{Z}[C_4] \longrightarrow \mathbb{Z}[C_4] = \mathbb{Z}[x]/(x^4-1) \longrightarrow \mathbb{F}_2[C_4], \sigma \mapsto \sigma, \sum_{i=0}^{3} a_i\sigma^i \mapsto \sum_{i=0}^{3} \overline{a_i}\sigma^i$,the kernel is $2\mathbb{Z}[C_4]$ ($\sum_{i=0}^{3} \overline{a_i}\sigma^i = 0, \overline{a_i} = 0$) relative exact sequence

$$\begin{aligned} NK_3(\mathbb{F}_2[C_4]) \longrightarrow NK_2(\mathbb{Z}[C_4], 2\mathbb{Z}[C_4]) \longrightarrow NK_2(\mathbb{Z}[C_4]) \\ \longrightarrow NK_2(\mathbb{F}_2[C_4]) \longrightarrow NK_1(\mathbb{Z}[C_4], 2\mathbb{Z}[C_4]) \end{aligned}$$

  最后一项用别的方法算，看看 NK_1 的文章，比如哥伦比亚大学那篇毕业论文
- $NK_1(\mathbb{Z}[C_2 \times C_2]), NK_1(\mathbb{Z}[C_4]), NK_1(\mathbb{Z}[C_p \times C_p]), NK_1(\mathbb{Z}[C_{p^2}])$ 中的非平凡元
- $(p,q) = 1, NK_2[\mathbb{Z}[C_p \times C_q] \longrightarrow NK_2[\mathbb{F}_p[C_p \times C_q]$ 或者 $NK_2[\mathbb{F}_q[C_p \times C_q]$. 想证 $NK_1(\mathbb{Z}[C_p \times C_q], p\mathbb{Z}[C_p \times C_q]) = 1$, $NK_1(\mathbb{Z}[C_p \times C_q], q\mathbb{Z}[C_p \times C_q]) = 1$ 成立的话，若 $NK_2(\mathbb{F}_p[C_p \times C_q])$ 有 p 阶元的话，提升到 $NK_2(\mathbb{Z}[C_p \times C_q])$ 有大于等于 p 阶的元，同理把 p 换成 q，由于两个都是素数 $(p,q) = 1$, 在整群环的 NK_2 中有大于等于 pq 阶的元。注意这都要建立在 NK_1 那个成立的基础上，看看有关 NK_1 的相对群。
- $\mathbb{Z}[C_p \times C_q] \twoheadrightarrow \mathbb{Z}[C_p]$ 是不是可裂满射？用 on the higher nils 里面的证明 $NK_2(\mathbb{Z}[G])$ 不为 0. $NK_2(\mathbb{Z}[C_p \times C_q])$ 中非零元，比如 $\langle x^n(1-\sigma_p), (1+\sigma_p+\cdots+\sigma_p^{p-1})\rangle + \langle x^n(1-\tau_q), (1+\tau_q+\cdots+\tau_q^{q-1})\rangle$.
- 极大 order $\mathbb{Z}[G] \longrightarrow \Gamma$
- Dennis-Stein symbols 化成 Steinberg symbols, lift to $\mathbb{Z}[G]$
- 非交换群:S_3 的用半直积写, 对于一般的有限交换群的整群环映到 $\mathbb{F}_p$
- 群环中，找个整群环的 DS 符号，证明映射过去不平凡，就可以说明对于一般的有限交换群整群环都 NK 不为 0
- 有限交换群的可裂的扩张（半直积）也如此，再证任意有限交换群的 NK 都非 0, 具有多少阶元. 现证 square-free 的，参考 G-Theory of Group Rings for Groups of Square-Free Order
- 关键是找符号，可能并不容易
- 对于 S_3 还有其他例子是否有天然的映射，不用找符号。prove $NK_2(\mathbb{Z}[S_3]) \neq 0$
- 对一般有限交换群要找符号比如 $\mathbb{Z}[C_p \times C_p]$ 试着找符号

$\mathbb{Z}[C_{p^r}] \longrightarrow \mathbb{Z}[C_p]$, $\sigma^{p^{r-1}} \mapsto \tau$. $(\sigma^{p^{r-1}})^p = \sigma^{p^r} = 1 \in \mathbb{Z}[C_{p^r}]$. Set $\tau = \sigma^{p^{r-1}}$, $(1-\tau^p) = (1-\tau)(1+\tau+\cdots+\tau^{p-1})$, $\langle (1+\tau+\cdots+\tau^{p-1})x^i, (1-\tau)\rangle$.

If C_{p^r} is a subgroup of C_n, then $\mathbb{Z}[C_n] \twoheadrightarrow \mathbb{Z}[C_{p^r}]$, it induces split surjection $K_i(\mathbb{Z}[C_n]) \longrightarrow K_i(\mathbb{Z}[C_{p^r}])$?

其他

- 有限群环的结构，有限群环的商环。

- 特殊有限环的 K_2 群
- 有限交换群的正群环的 K_2
- 截断多项式环的 K 理论
- 翻译成英文，中文的保留做毕业论文用。
- 毕业论文: 介绍 farrell-jones conj 列举哪些类型的群是被证明了的
- 群论: 把所有常见群论定理如三百题中总结出来，p 群的定理有限 p 群。
- 下一步：拓扑循环同调与 NK: -定义写清楚 - 计算过程大致算一下 - 主要运用的定理 - 和 NK 有关的结论群环的研究 => Madsen trace formula => TC, THH => SPECTRA F-J conj

T. Gersten, Some exact sequence in the higher K-theory of rings.

13.3 Nil-groups

下面是从 SOME REMARKS ON NIL GROUPS IN ALGEBRAIC K-THEORY 借鉴来的，需要修改 The fundamental theorem of algebraic K-theory states that

$$K_nR[t,t^{-1}] \cong K_nR \oplus K_{n-1}R \oplus NK_nR \oplus NK_nR.$$

For any functor $F\colon \text{Rings} \longrightarrow \text{Ab}$, Bass [3] defines two functors

$$\begin{aligned} NF(R) &= \ker(F(R[t]) \longrightarrow F(R)), \\ LF(R) &= \operatorname{coker}\left(F(R[t]) \oplus F(R[t^{-1}]) \longrightarrow F(R[t,t^{-1}])\right). \end{aligned}$$

The functor F is called a contracted functor if the sequence

$$0 \longrightarrow F(R) \longrightarrow F(R[t]) \oplus F(R[t^{-1}]) \longrightarrow F(R[t,t^{-1}]) \longrightarrow LF(R) \xrightarrow{\partial} 0$$

is exact and there is a splitting $h_{t,R}$ of the surjection ∂ which is natural in both t and R. Note that $NLF(R) \cong LNF(R)$ if F is contracted, then so are NL and LN. The fundamental theorem of K-theory may be stated as the assertions that K_n ($n \in \mathbb{Z}$) are contracted functors and there are natural identification $K_{n-1} = LK_n$.

This leads to the calculation of the K-theory of polynomial rings and Laurent polynimial rings.

$$K_q(R[t_1, ...t_n]) \cong (I+N)^n K_q(R)$$
$$K_q(R[t_1,t_1^{-1},\cdots,t_n,t_n^{-1}]) \cong (I+2N+L)^n K_q(R)$$

放在 intro 里摘自 wangkun ON PASSAGE TO OVER-GROUPS OF FINITE INDICES OF THE FARRELL-JONES CONJECTURE

It is known that if the Bass Nil-groups of a ring R vanish, i.e. $NK_n(R) = 0$, $n \in \mathbb{Z}$, then $NK_n(R[H])$ is rationally trivial for any finite group H. This was proved by Weibel [48] for some special rings and by Hambleton and Lück [18] for general rings.

13.4 Topological cyclic homology

Cyclotomic trace

$$K(\mathbb{F}_2[t,x]/(t^2),(t)) \xrightarrow{\sim} TC(\mathbb{F}_2[t,x]/(t^2),(t))$$

$\mathrm{Nil}_q(A[x]/(x^m))$ is non-zero for all integer $q \geq 0$ and $m > 1$. If $q < 0$, it is trivial.

$\mathrm{Nil}_k = NK_{k+1}$. If A is a regular noetherian ring and $\mathbb{F}_p$-algebra,

$$\cdots \longrightarrow \bigoplus_{i\geq 0} W_{i+1}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{V_m} \bigoplus_{i\geq 0} W_{m(i+1)}\Omega^{q-2i}_{(A[t],(t))} \xrightarrow{\varepsilon} \mathrm{Nil}_q(A[x]/(x^m)) \longrightarrow \cdots$$

where $W_r\Omega^q_{A[t]} = W_r\Omega^q_A \oplus W_r\Omega^q_{(A[t],(t))}$.

If $p \nmid m$, then V_m is injective.

13.5 Milnor square

$C_n = \langle\sigma\rangle$, $I = (\sigma - 1)$ and $J = 1 + \sigma + \cdots + \sigma^{n-1}$ are two disjoint ideals of $\mathbb{Z}[C_n]$, there is a Cartesian square

$$\begin{array}{ccc} \mathbb{Z}[C_n] & \longrightarrow & \mathbb{Z}[C_n]/J \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \end{array}$$

since $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are regular, hence we get a exact sequence

$$NK_2(\mathbb{Z}[C_n]) \longrightarrow NK_2(\mathbb{Z}[C_n]/J) \longrightarrow 0.$$

For $n = 4$, $J = (1 + \sigma + \sigma^2 + \sigma^3)$, $\mathbb{Z}[C_4]/J = \mathbb{Z}[C_4]/(1 + \sigma + \sigma^2 + \sigma^3) \cong \mathbb{Z}[x]/(1 + x)(1 + x^2)$. If we can prove $\mathbb{Z}[C_4]/J \neq 0$, then $NK_2(\mathbb{Z}[C_4]) \neq 0$. In fact $NK_2(\mathbb{Z}[C_4]) \neq 0$ is proved in [51], from the surjection $NK_2(\mathbb{Z}[C_4]) \twoheadrightarrow x^2\mathbb{F}_2[x^2]$.

from $(\mathbb{Z}[C_4], J) \longrightarrow (\mathbb{Z}[C_4]/(\sigma-1), (4)) = (\mathbb{Z}, (4))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (4)) \longrightarrow NK_2(\mathbb{Z}[C_4]; J, (\sigma-1)) \longrightarrow NK_2(\mathbb{Z}[C_4], J) \longrightarrow NK_2(\mathbb{Z}, (4)) = 0.$$

$$NK_3(\mathbb{Z}[C_4]/J) \longrightarrow NK_2(\mathbb{Z}[C_4], J) \longrightarrow NK_2(\mathbb{Z}[C_4]) \longrightarrow NK_2(\mathbb{Z}[C_4]/J).$$

In [51], Weibel stated $NK_2(\mathbb{Z}[C_4], (\sigma^2+1), (\sigma^2-1)) \cong \mathbb{F}_2[C_2] \otimes x\mathbb{F}_2[x]$ on symbols $\langle \sigma^2+1, x^n(\sigma-1)\rangle$. We can get $K_2(\mathbb{Z}[C_4], (\sigma^2+1), (\sigma^2-1)) \cong (\sigma^2+1) \otimes (\sigma^2-1)$, $NK_2(\mathbb{Z}[C_4], (\sigma^2+1), (\sigma^2-1)) \cong (\sigma^2+1) \otimes (\sigma^2-1)[x]/(\sigma^2+1) \otimes (\sigma^2-1)$. $K_2(\mathbb{Z}[C_4], (\sigma-1), (1+\sigma+\sigma^2+\sigma^3)) \cong (\sigma-1) \otimes (1+\sigma+\sigma^2+\sigma^3)$?

考虑 C_6

$$\begin{array}{ccc} \mathbb{Z}[C_6] & \longrightarrow & \mathbb{Z}[x]/(1+x)(1+x^2+x^4) = R_1 \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/6\mathbb{Z} \end{array}$$

By Mayer-Vietoris sequence, $NK_i(R_1) = 0$ for $i \leq 1$. And $NK_2(\mathbb{Z}[C_6]) \twoheadrightarrow NK_2(R_1) \longrightarrow 0$.

Recall that $R_1 = \mathbb{Z}[C_6]/(1+\sigma+\cdots+\sigma^5)$,

$$\begin{array}{ccc} R_1 = \mathbb{Z}[C_6]/(1+\sigma+\cdots+\sigma^5) & \longrightarrow & \mathbb{Z}[C_6]/(1+\sigma) = \mathbb{Z} \\ \downarrow & & \downarrow \\ R_2 = \mathbb{Z}[C_6]/(1+\sigma^2+\sigma^4) & \longrightarrow & \mathbb{Z}[C_6]/3\mathbb{Z}[C_6] = \mathbb{F}_3[C_6] \end{array}$$

$NK_2(R_1) \longrightarrow NK_2(R_2) \longrightarrow NK_2(\mathbb{F}_3[C_6]) \longrightarrow 0.$

$\mathbb{Z}[C_6]/3\mathbb{Z}[C_6] \cong \mathbb{F}_3[C_6]$? right or not. We have $NK_2(\mathbb{F}_3[C_6]) \cong \bigoplus_\infty \mathbb{Z}/3\mathbb{Z}$. Hence $NK_2(R_2) \neq 0$.

$I = (1+\sigma+\sigma^2)/(1+\sigma^2+\sigma^4)$, $J = (1-\sigma+\sigma^2)/(1+\sigma^2+\sigma^4)$ are ideals of R_2

$$\begin{array}{ccc} R_2 = \mathbb{Z}[C_6]/(1+\sigma^2+\sigma^4) & \longrightarrow & R_2/I = \mathbb{Z}[\zeta_3] \\ \downarrow & & \downarrow \\ R_2/J = \mathbb{Z}[\zeta_3] & \longrightarrow & \mathbb{F}_2[\zeta_3] \cong \mathbb{F}_4 \end{array}$$

this square is obtained by $\mathbb{Z}[C_2]$ tensor with $\mathbb{Z}[\zeta_3] \cong \mathbb{Z}[C_3]/(1+\tau+\tau^2)$. 这儿会导出一个矛盾，看看出在哪里 $NK_2(R_2) \longrightarrow 0 \longrightarrow 0 \longrightarrow NK_1(R_2) \longrightarrow 0$, 但是 $NK_1(R_2) \cong NK_1(\mathbb{F}_3[C_6]) \neq 0$.

$K_2(R_2, I, J) \cong I \otimes J$ 对于 double relative 可以直接用 NK 函子吧？

$$NK_3(R_2/J, I+J/J) \longrightarrow NK_2(R_2, I, J) \longrightarrow NK_2(R_2, I) \longrightarrow NK_2(R_2/J, I+J/J)$$

第一项和最后一项应该是 0 吧

$$NK_3(R_2/I) \longrightarrow NK_2(R_2, I) \longrightarrow NK_2(R_2) \longrightarrow NK_2(R/I)$$

第一项和最后一项应该也是 0。

所以应该有 $NK_2(R_2, I, J) = NK_2(R_2) \cong I \otimes J[x]/I \otimes J$. $NK_2(R_2)$ is generated by $\langle 1+\overline{\sigma}+\overline{\sigma}^2, (1-\overline{\sigma}+\overline{\sigma}^2)x^n \rangle$.

$I \otimes J$ is generated by $\beta_1 = \langle 1+\overline{\sigma}+\overline{\sigma}^2, 1-\overline{\sigma}+\overline{\sigma}^2 \rangle$ and $\beta_2 = \langle 1+\overline{\sigma}+\overline{\sigma}^2, \overline{\sigma}(1-\overline{\sigma}+\overline{\sigma}^2) \rangle$, $\beta_1^2 = 1$, $\beta_2 = 1$(可以得到), so is generated by β_1.

$NK_2(R_2) \cong x\mathbb{F}_2[x]$, $NK_2(\mathbb{Z}[C_6]/(1+\sigma^2+\sigma^4)) = NK_2(\mathbb{Z}[C_2] \otimes \mathbb{Z}[\zeta_3]) \cong x\mathbb{F}_2[x]$. Witt vector module structure is same as $NK_2(\mathbb{Z}[C_2])$.

Question: if R regular, $NK_2(\mathbb{Z}[C_2] \otimes R) \cong NK_2(\mathbb{Z}[C_2])$?

$I' = (1+\sigma)$, $J' = (1+\sigma^2+\sigma^4)$ are ideals of R_1, $R_1/J' = R_2$, $R_1/I' = \mathbb{Z}$. $NK_2(R_1; J', I') \cong NK_2(R_1, J')$,

$$NK_3(R_2) \longrightarrow NK_2(R_1, J') \longrightarrow NK_2(R_1) \longrightarrow NK_2(R_2) \longrightarrow NK_1(R_1, J') \longrightarrow NK_1(R_1)$$

Given a Milnor square

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R/I \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle g} \\ R/J & \xrightarrow{f} & R/I+J \end{array}$$

if R/I, R/J, $R/I+J$ are all regular (or $NK_i(-) = 0$), then

$$\begin{array}{ccccccc} & & NK_3(R/J, I+J/J) = 0 & & & & \\ & & \downarrow & & & & \\ & & NK_2(R; I, J) & & & & \\ & & \downarrow{\scriptstyle \cong} & & & & \\ 0 = NK_3(R/I) & \longrightarrow & NK_2(R, I) & \xrightarrow{\cong} & NK_2(R) & \longrightarrow & NK_2(R/I) = 0 \\ & & \downarrow & & & & \\ & & NK_2(R/J, I+J/J) = 0 & & & & \end{array}$$

$$0 = NK_{i+1}(R/I+J) \longrightarrow NK_i(R/J, I+J/J) \longrightarrow NK_i(R/J) = 0 \Longrightarrow NK_i(R/J, I+J/J) = 0.$$

Therefore $NK_2(R) \cong NK_2(R; I, J)$.

$K_2(R; I, J) \cong I \otimes J$, $K_2(R[x]; I[x], J[x]) \cong I \otimes J[x]$, $NK_2(R) \cong NK_2(R; I, J) = I \otimes J[x]/I \otimes J$.

If R/J, $R/I+J$ are regular (or $NK_i(-) = 0$, 换成 quasi-regular 会怎么样)

$$\begin{tikzcd}
 & NK_3(R/J, I+J/J) = 0 \arrow[d] & & \\
 & NK_2(R; I, J) \arrow[d, "\cong"] & & \\
NK_3(R/I) \arrow[r] & NK_2(R, I) \arrow[r] \arrow[d] & NK_2(R) \arrow[r] & NK_2(R/I) \\
 & NK_2(R/J, I+J/J) = 0 & &
\end{tikzcd}$$

If moreover $R \longrightarrow R/I$ is a split surjection, then

$$0 \longrightarrow NK_2(R, I) \longrightarrow NK_2(R) \longrightarrow NK_2(R/I) \longrightarrow 0,$$

hence $NK_2(R) \cong NK_2(R; I, J) \oplus NK_2(R/I) = (I \otimes J) \oplus NK_2(R/I)$.

if R, S are regular, is $R \times S$ also regular? For example $\mathbb{F}_p \times \mathbb{F}_p$ regular or not? Note that hereditary rings are regular rings.

$NK_2(\mathbb{Z}[C_{pq}])$

$$\begin{tikzcd}
\mathbb{Z}[C_p] \arrow[r, "\sigma \mapsto \zeta_p"] \arrow[d, "\sigma \mapsto 1"'] & \mathbb{Z}[\zeta_p] \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{F}_p
\end{tikzcd}$$

tensor with $\mathbb{Z}[C_q]$

$$\begin{tikzcd}
\mathbb{Z}[C_{pq}] \cong \mathbb{Z}[C_p] \otimes \mathbb{Z}[C_q] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta_p][C_q] \arrow[d] \\
\mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q]
\end{tikzcd}$$

By Mayer-Vietoris sequence for the NK-functor, one has

$$\begin{array}{l} NK_2(\mathbb{Z}[C_{pq}]) \longrightarrow NK_2(\mathbb{Z}[\zeta_p][C_q]) \oplus NK_2(\mathbb{Z}[C_q]) \longrightarrow NK_2(\mathbb{F}_p[C_q]) \longrightarrow \\ \longrightarrow NK_1(\mathbb{Z}[C_{pq}]) \longrightarrow NK_1(\mathbb{Z}[\zeta_p][C_q]) \oplus NK_1(\mathbb{Z}[C_q]) \longrightarrow NK_1(\mathbb{F}_p[C_q]) \longrightarrow \\ \longrightarrow NK_0(\mathbb{Z}[C_{pq}]) \longrightarrow NK_0(\mathbb{Z}[\zeta_p][C_q]) \oplus NK_0(\mathbb{Z}[C_q]) \end{array}$$

Note that for any finite group G of aquare-free order, $NK_i(\mathbb{Z}[G]) = 0$ for any $i \leq 1$. And $NK_i(\mathbb{F}_p[C_q]) = 0$ when p and q are coprime. Since $NK_2(\mathbb{Z}[C_q]) \neq 0$, then $NK_2(\mathbb{Z}[C_{pq}]) \neq 0$. By induction, for a finite cyclic group C_n of square-free order, we have $NK_2(\mathbb{Z}[C_n]) \neq 0$.

If $G = C_{2p} = C_p \times C_2 = \{(\sigma, \iota) \mid \sigma^p = 1, \iota^2 = 1\}$ with p prime to 2.

$$\begin{array}{ccc} \mathbb{Z}[C_{2p}] & \longrightarrow & \mathbb{Z}[C_p] \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_p] & \longrightarrow & \mathbb{F}_2[C_p] \end{array}$$

gives

$$NK_2(\mathbb{Z}[C_{2p}]) \longrightarrow 2NK_2(\mathbb{Z}[C_p]) \longrightarrow NK_2(\mathbb{F}_2[C_p]) \longrightarrow NK_1(\mathbb{Z}[C_{2p}]) \longrightarrow 0,$$

$NK_2(\mathbb{F}_2[C_p]) = 0 \implies NK_1(\mathbb{Z}[C_{2p}]) = 0$. Then $\langle x^n \iota^i(\sigma - 1), (1 + \sigma + \cdots + \sigma^{p-1})\rangle$ $(i = 0, 1)$ map to $\langle x^n(\sigma - 1), (1 + \sigma + \cdots + \sigma^{p-1})\rangle$.

For cyclic group of order p^2, there are two Milnor squares

$$\begin{array}{ccc} \mathbb{Z}[C_{p^2}] & \longrightarrow & \mathbb{Z}[C_{p^2}]/J = \mathbb{Z}[x]/(1 + \sigma + \cdots + \sigma^{p-1})(1 + \sigma^p + \cdots + \sigma^{(p-1)p}) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^2\mathbb{Z} \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}[C_{p^2}] & \xrightarrow{\sigma \mapsto \zeta_{p^2}} & \mathbb{Z}[\zeta_{p^2}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_p] & \longrightarrow & \mathbb{Z}[\zeta_{p^2}]/(1 - \zeta_{p^2}^p) \end{array}$$

Note that $\mathbb{Z}[\zeta_{p^2}]/(1-\zeta_{p^2}^p) \cong \mathbb{F}_p[t]/(t^p)$?

Let $S_3 = \langle h, f \mid h^3 = 1 = f^2, fhf^{-1} = h^{-1} \rangle$

$$\begin{tikzcd}
\mathbb{Z}[S_3] \arrow[r] \arrow[d] & A = \mathbb{Z}[\zeta, f] \arrow[d] \\
\mathbb{Z}[C_2] \arrow[r] & \mathbb{F}_3[C_2]
\end{tikzcd}$$

A is hereditary.

$$NK_2(\mathbb{Z}[S_3]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \longrightarrow 0,$$

$NK_2(\mathbb{Z}[C_2]) \cong x\mathbb{F}_2[x]$, $\langle x^n(\sigma-1), (1+\sigma)\rangle$ corresponding to x^n. So we give some non-trivial elements in $NK_2(\mathbb{Z}[S_3])$: $\langle x^n(f-1), f+1\rangle$, $\langle x^n(f-h^{-1}), f+h\rangle$ (because $(f-h^{-1})(f+h) = f^2+fh-h^{-1}f-1 = 1+fh-fh-1 = 0$, $(f+h)(f-h^{-1}) = f^2+hf-fh^{-1}-1 = 1+hf-hf-1 = 0$, note that $f^{-1} = f$), $\langle x^n(f-h), f+h^{-1}\rangle$.

13.6 Explicit examples in NK_1

c.f. Scott Schmieding

Lemma 13.4 (Higman's trick). *$NK_1(R)$ is the set of elements of $K_1(R[x])$ which contain a matrix of the form $I - xN$, with N a nilpotent matrix over R.*

$$\begin{aligned}
&NK_1(R) \longrightarrow \mathrm{Nil}_0(R) \\
&I - xN \mapsto [R^n, N], \text{ where } N \text{ is a } n \times n \text{ nilpotent matrix.}
\end{aligned}$$

endomorphism

$$\begin{aligned}
V_k\colon NK_1(R) &\longrightarrow NK_1(R) \\
[1 - xN] &\mapsto [1 - x^k N] \\
F_k\colon NK_1(R) &\longrightarrow NK_1(R) \\
[1 - xN] &\mapsto [1 - xN^k] \\
V_k\colon \mathrm{Nil}_0(R) &\longrightarrow \mathrm{Nil}_0(R) \\
[N] &\mapsto \left[\begin{pmatrix} 0 & & & N \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}\right] \\
F_k\colon \mathrm{Nil}_0(R) &\longrightarrow \mathrm{Nil}_0(R) \\
[N] &\mapsto [N^k]
\end{aligned}$$

$NK_1(\mathbb{Z}[C_{p^n}]) \neq 0$, $n \geq 2$.

For general computation

$$\begin{tikzcd}
\mathbb{Z}[C_{p^n}] \arrow[r, "\sigma \mapsto \zeta_{p^n}"] \arrow[d] & \mathbb{Z}[\zeta_{p^n}] \arrow[d] \\
\mathbb{Z}[C_p] \arrow[r] & \mathbb{Z}[\zeta_{p^n}]/(1 - \zeta_{p^n}^p)
\end{tikzcd}$$

Note that $\mathbb{Z}[\zeta_{p^n}]/(1 - \zeta_{p^n}^p) \cong \mathbb{F}_p[t]/(t^p) = \mathbb{F}_p[C_p]$?

ζ_{p^n}: a primitive p^n-th root of unity. $\mathbb{Z}[\zeta_{p^n}]$: the ring of integers of $\mathbb{Q}[\zeta_{p^n}]$. By Mayer-Vietoris sequence for the NK-functor, one has

$$\begin{aligned}
& NK_2(\mathbb{Z}[C_{p^n}]) \longrightarrow NK_2(\mathbb{Z}[\zeta_{p^n}] \oplus NK_2(\mathbb{Z}[C_p]) \longrightarrow NK_2(\mathbb{F}_p[t]/(t^p)) \longrightarrow \\
\longrightarrow\ & NK_1(\mathbb{Z}[C_{p^n}]) \longrightarrow NK_1(\mathbb{Z}[\zeta_{p^n}] \oplus NK_1(\mathbb{Z}[C_p]) \longrightarrow NK_1(\mathbb{F}_p[t]/(t^p)) \longrightarrow \\
\longrightarrow\ & NK_0(\mathbb{Z}[C_{p^n}]) \longrightarrow NK_0(\mathbb{Z}[\zeta_{p^n}] \oplus NK_0(\mathbb{Z}[C_p])
\end{aligned}$$

$NK_i(\mathbb{Z}[\zeta]) = 0$ for all i. $NK_1(\mathbb{Z}[C_{p^n}]) \cong \operatorname{coker}(NK_2(\mathbb{Z}[C_p]) \longrightarrow NK_2(\mathbb{F}_p[t]/(t^p)))$. Since elements of $NK_2(\mathbb{F}_p[t]/(t^p))$ are p-torsion, hence $NK_1(\mathbb{Z}[C_{p^n}]) \cong \bigoplus_\infty \mathbb{Z}/p\mathbb{Z}$.

$$0 \longrightarrow NK_1(\mathbb{F}_p[t]/(t^p)) \longrightarrow NK_0(\mathbb{Z}[C_{p^n}]) \longrightarrow 0$$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $NK_1(\mathbb{Z}[C_4])$ with

$$\begin{aligned}
A =&1-(1-\sigma^2)(x-2x^2+2x^3-\sigma+x\sigma+x^2\sigma)\\
B =&(\sigma^2-1)(1+2x-x^2-x^3-2x^4+\sigma-x\sigma-2x^2\sigma-3x^3\sigma+2x^4\sigma)\\
C =&(\sigma^2-1)(-1+2x-5x^2+7x^3-3x^4+2x^5-\sigma+2x\sigma-2x^3\sigma+3x^4\sigma-2x^5\sigma)\\
D =&1-(1-\sigma^2)(2+x-2x^2-4x^4-2x^5+\sigma-3x\sigma-x^2\sigma-4x^3\sigma+6x^4\sigma-4x^5\sigma+4x^6\sigma)
\end{aligned}$$

is non-zero.

$G = C_4 = \langle\sigma\rangle, \sigma^4 = 1.$

$$\begin{tikzcd}
\mathbb{Z}[C_4] \arrow[r, "\sigma\mapsto i"] \arrow[d, "\sigma^2\mapsto 1"'] & \mathbb{Z}[i] \arrow[d, "i\mapsto 1+\varepsilon"] \\
\mathbb{Z}[C_2] \arrow[r, "q"] & \mathbb{F}_2[\varepsilon]/(\varepsilon^2)
\end{tikzcd}$$

$NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\varepsilon]/(\varepsilon^2)) \xrightarrow{\partial} NK_1(\mathbb{Z}[C_4]) \longrightarrow 0,$
$\ker\partial = \operatorname{Im} q = \ker(D\colon NK_2(\mathbb{F}_2[\varepsilon]/(\varepsilon^2)) \longrightarrow \Omega_{\mathbb{F}_2[x]}), \langle f\varepsilon, g+g'\varepsilon\rangle \mapsto f dg.$

For example, $D(\langle\varepsilon, x+\varepsilon\rangle) = dx \neq 0$, hence $\langle\varepsilon, x+\varepsilon\rangle \notin \operatorname{Im}(q)$. $\ker\partial = \operatorname{Im} q$, so $\partial(\langle\varepsilon, x+\varepsilon\rangle) \neq 0$ in $NK_1(\mathbb{Z}[C_4])$.

Compute $\partial(\langle\varepsilon, x+\varepsilon\rangle)$

$$\begin{tikzcd}
 & K_1(\mathbb{Z}[C_4][x], (1-\sigma^2)) \arrow[r, "j"] \arrow[d, "\psi", "\cong"'] & K_1(\mathbb{Z}[C_4]) \\
K_2(\mathbb{F}_2[\varepsilon, x]/(\varepsilon^2)) \arrow[r, "\partial_1"] & K_1(\mathbb{Z}[i][x], (2)) &
\end{tikzcd}$$

since $\sigma \mapsto i$ takes $(1-\sigma^2) \mapsto (2)$.

$K_2(\mathbb{F}_2[\varepsilon, x]/(\varepsilon^2))$ 中的 Dennis-Stein symbols: $\langle\varepsilon x^i, x\rangle$ $(i \geq 0)$, $\langle\varepsilon x^i, \varepsilon\rangle$ ($i \geq i$ is odd).

$NK_2(\mathbb{Z}[C_p]) = x\mathbb{F}_p[x]$, $\langle(1-\sigma), (1+\sigma+\cdots+\sigma^{p-1})x^n\rangle$ corresponding to x^n.

$G = C_4 \times C_2$.

$$\begin{tikzcd}
\mathbb{Z}[C_2 \times C_4] \arrow[r] \arrow[d] & \mathbb{Z}[C_4] \arrow[d, "\psi_1"] \\
\mathbb{Z}[C_4] \arrow[r, "\psi_2"] & \mathbb{F}_2[C_4]
\end{tikzcd}$$

ψ_i are reduction mod 2 for $i = 1, 2$.

$NK_2(\mathbb{Z}[C_4 \times C_2]) \longrightarrow 2NK_2(\mathbb{Z}[C_4]) \longrightarrow NK_2(\mathbb{F}_2[C_4]) \longrightarrow NK_1(\mathbb{Z}[C_4 \times C_2]) \longrightarrow 2NK_1(\mathbb{Z}[C_4])$,

$NK_2(\mathbb{F}_2[C_4]) = \bigoplus_\infty(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$.

13.7 Excision and group rings

[40] 的笔记：

之前的研究：groups of order 2 and 3, Dunwoody $K_2(\mathbb{Z}\pi)$ for π a group of order two or three.(方法不适用于 C_4,C_6...).
Lower bounds for elementary abelian p-groups, Dennis, Keating, Stein [39].
relative K_2:
Keune, [25], J.-L. Loday, Cohomologie et groupe de Steinberg relatifs.

In this paper, talk about the following
$K_2(\mathbb{Z}[G])$ for $G = C_p, C_4, C_6$ with p prime; non-abelian metacyclic groups $K_2(\mathbb{Z}[S_3])$

cokernels of maps of K_3's: Stein, Maps of rings which induce surjections on K_3.

这篇文章中用的 Dennis-Stein symbols 是旧的，应用时改成新的，也就是以前的 $< a, b >$ 现在应该为 $< -a, b >$.

R, $I \subset R$ two-sided ideal, define $R(I)$ by the Cartesian square

$$\begin{array}{ccc} R(I) & \xrightarrow{p_1} & R \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow \\ R & \longrightarrow & R/I \end{array}$$

$R(I) = \{(r_1, r_2) \in R \times R \mid r_1 \equiv r_2 \bmod I\}$. Let

$$K_2^s(R, I) = \ker(p_{1*} \colon K_2(R(I)) \longrightarrow K_2(R)),$$

there exists $p_{2*} \colon K_2^s(R, I) \longrightarrow K_2(R)$ induced by p_2. Keune defined $K_2(R, I) = K_2^s(R, I)/V$ where V is generated by the elements

$$\langle(0, s_2), (s_1, 0)\rangle_{\text{old}} = [x_{12}(-s_1, 0), x_{21}(0, s_2)], \quad s_1, s_2 \in I.$$

用这些可以算 $\mathbb{Z}[C_n]$

NK_i 的 M-V 序列，一般只对 $i \leq 2$ 成立，(一个观察，若 NK_3 适用，则会得出 $NK_2(\mathbb{Z}[C_p]) = 0$ 的错误结论)。但是对于 $NK_i(\mathbb{Z}[G])$ tensor 了 $\mathbb{Z}[\frac{1}{p}]$ 后的 M-V 序列是可以用的, $p \mid |G|$.

13.8 From *Higher K'-Groups of Integral Group Rings*

Let R be a (not necessary commutative) ring with 1 and let G be a (multiplicative) finite group. We will denote the group ring associated with R and G by $R[G]$ and the canonical basis elements of $R[G]$ by $[g]$, $g \in G$.

Definition 13.5. For any normal subgroup N of G, the ideal I_N is defined to be the kernel of the canonical ring epimorphism $R[G] \longrightarrow R[G/N]$.

Lemma 13.6. *The two-sided ideal I_N is generated by the elements $[g] - [1]$, $g \in N$, as left and as right ideal. In particular: If N_1, N_2 are elementwise commuting normal subgroups of G, then $I_{N_1} I_{N_2} = I_{N_2} I_{N_1}$.*

13.9 Notes on group theory

抽象代数拾遗 $\mathbb{Q}$ is far from $\mathbb{Q}/\mathbb{Z}$, since the order of any element $1 \neq x \in \mathbb{Q}$ is infinite while any element of $\mathbb{Q}/\mathbb{Z}$ has finite order. 于是 $\mathbb{Q}/\mathbb{Z}$ 是一个每个元的阶都有限但是本身不是有限群的例子.

群按照元素个数和是否交换可以分成：交换群与非交换群；有限群与无限群。对于有限交换群后面会有结构定理，是研究比较透彻的。把交换群扩进来的下一个群类是可解群

$$\text{可解群}\begin{cases}\text{abelian groups}\\ \text{素数幂阶群, 阶为}p^{\alpha}\\ S_3\text{可解}, \{1\} \lhd A_3 \lhd S_3\\ p^{\alpha}q^{\beta}\text{阶群可解, Burnside 定理}\\ \text{奇数阶群可解, Feit-Thompson 定理}\end{cases}$$

$\sigma(i_1 \cdots i_r)\sigma^{-1} = (\sigma(i_1)\cdots\sigma(i_r))$, because $\sigma(i_1 \cdots i_r)\sigma^{-1}(\sigma(i_k)) = \sigma(i_1 \cdots i_r)(i_k) = \sigma(i_{k+1})$, $(\sigma(i_1)\cdots\sigma(i_r))(\sigma(i_k)) = \sigma(i_{k+1})$.

A_4 没有 6 阶子群，说明 Lagrange 定理的逆是不对的。$|A_4| = 12 = 2^2 \cdot 3$.

13.9.1 The structure of finite abelian groups

Theorem 13.7. *Let G be a finite abelian group, then G is (in a unique way) a direct product of cyclic groups of order p^k with p prime(not necessarily distinct).*

Theorem 13.8 (Cauchy). *If G is a finite group, and $p \mid |G|$ is a prime, then G has an element of order p (or, equivalently, a subgroup of order p).*

Definition 13.9. Given a prime p, a p-group is a group in which every element has order p^k for some k. Let G be a group such that $|G| = p^k a$ where p is prime and $(p, a) = 1$. A subgroup of order p^k is called a Sylow p-subgroup of G.

Corollary 13.10. *A finite group is a p-group if and only if its order is a power of p.*

Lemma 13.11. *Let G be an abelian group such that $|G| = p^k a$ where p is prime and $(p, a) = 1$. Then there exists a unique subgroup of order p^k.*

Theorem 13.12. *Let G be a finite abelian group such that $|G| = p_1^{k_1} \cdots p_l^{k_l}$ where each of the p_i are distinct primes. Then $G \cong G(p_1) \oplus \cdots \oplus G(p_l)$ where $G(p) = \{x \in G | x^{p^k} = e\}$.*
If $G(p)$ is a finite abelian p-group of order p^k, then there exist $n_1, \cdots, n_r \in \mathbb{N}$ with $\sum_{i=1}^r n_i = k$ such that

$$G(p) \cong \bigoplus_{i=1}^{r} \mathbb{Z}/p^{n_i}\mathbb{Z}.$$

Theorem 13.13 (Structure of finite cyclic groups). *Let $G = \langle x \rangle$ be a finite cyclic group of order n. The following hold:*
(i) Every subgroup of G is cyclic and is equal to $\langle x^d \rangle$ where $d > 0$ and $d|n$.
(ii) If d and $d' \neq d$ are positive divisors of n, then $\langle x^d \rangle \neq \langle x^{d'} \rangle$.
(iii) If $k \in \mathbb{Z}$, then x^k is a generator of G iff k and n are coprime.
(iv) For any $k \in \mathbb{Z}$ we have $\langle x^k \rangle = \langle x^d \rangle$ where $d = gcd(n, k)$.
(v) For any $k \in \mathbb{Z}$ we have $o(x^k) = n/gcd(n, k)$.

13.9.2 groups of square-free order

Let G be a group of square-free order. Then G is metacyclic (see, e.g., [37], (10.1.10)), so it can be written as $G = \pi \rtimes \Gamma$, where π, Γ are cyclic of square-free order.

Theorem 13.14 (Hölder, Burnside, Zassenhaus). *If G is a finite group all of whose Sylow subgroups are cyclic, then G has a presentation*

$$G = \langle a, b \mid a^m = 1 = b^n, b^{-1}ab = a^r \rangle$$

where $r^n \equiv 1 (\bmod m)$, m is odd, $0 <\leq r < m$, and m and $n(r-1)$ are coprime.

Conversely in a group with such a presentation all Sylow subgroups are cyclic.

This means that a finite group whose Sylow subgroups are cyclic is an extension of one cyclic group by another; such groups are called metacyclic.In particular the group is supersoluble.

Prominent among the groups with cyclic Sylow subgroups are the groups with square-free order: such groups are therefore classified by the above theorem.

13.9.3 Some classes of groups

Can see https://terrytao.wordpress.com/2010/01/23/some-notes-on-group-extensions/

In the study of infinite groups, the adverb *virtually* is used to modify a property so that it need only hold for a subgroup of finite index. Let χ be a property of groups. A group G is virtually χ if it has a subgroup of finite index with the property χ. A group G is χ-by-finite if it has a normal subgroup of finite index with the property χ [38].

Definition 13.15. A group G is virtually abelian (or abelian-by-finite) if it has an abelian subgroup of finite index. Similarly, one can also define virtually nilpotent groups(nilpotent-by-finite), virtually solvable groups, virtually polycyclic groups(polycyclic-by-finite), virtually free groups.

Note that every χ-by-finite group is virtually χ, and the converse also holds if the property χ is inherited by subgroups. Note that all finite groups are virtually trivial (and trivial-by-finite).

Definition 13.16 (Virtually cyclic groups)**.** A group V is virtually cyclic if it contains a cyclic subgroup of finite index. Equivalently:
(I) V is a finite group, or
(II) V is a group extension $1 \longrightarrow F \longrightarrow V \longrightarrow C_\infty \longrightarrow 1$ for some finite group F, or
(III) V is a group extension $1 \longrightarrow F \longrightarrow V \longrightarrow D_\infty \longrightarrow 1$for some finite group F.

Example 13.17. The following groups are virtually abelian.

- Any abelian group.
- Any semidirect product $N \rtimes H$ where N is abelian and H is finite. (For example, any generalized dihedral group.)
- Any semidirect product $N \rtimes H$ where N is finite and H is abelian.
- Any finite group (since the trivial subgroup is abelian).

Virtually nilpotent groups:

- Any virtually abelian group.
- Any nilpotent group.
- Any semidirect product $N \rtimes H$ where N is nilpotent and H is finite.
- Any semidirect product $N \rtimes H$ where N is finite and H is nilpotent.

Virtually free groups:

- Any free group.
- Any virtually cyclic group.
- Any semidirect product $N \rtimes H$ where N is free and H is finite.
- Any semidirect product $N \rtimes H$ where N is finite and H is free.
- Any free product $H * K$, where H and K are both finite. (For example, the modular group $PSL(2, \mathbb{Z})$.)

It follows from Note that torsion-free virtually free group is free by Stalling's theorem https://en.wikipedia.org/wiki/Stallings_theorem_about_ends_of_groups#Applications_and_generalizations.

A polycyclic group is a solvable group that satisfies the maximal condition on subgroups (that is, every subgroup is finitely generated). Polycyclic groups are finitely presented, and this makes them interesting from a computational point of view.

Definition 13.18. A group G is polycyclic if it admits a subnormal series with cyclic factors, that is a finite set of subgroups $G_0, \cdots, G_n$ such that $G_0 = G, G_n = \{1\}$, $G_{i+1} \lhd G_i$ normal and the quotient group G_i/G_{i+1} is a cyclic group for $0 \leq i \leq n-1$.

A virtually polycyclic group is a group that has a polycyclic subgroup of finite index. Such a group necessarily has a normal polycyclic subgroup of finite index, and therefore such groups are also called polycyclic-by-finite groups. Note that polycyclic-by-finite groups need not be solvable.

A metacyclic group is a polycyclic group with $n \leq 2$, or in other words an extension of a cyclic group by a cyclic group. That is, it is a group G for which there is a short exact sequence

$$1 \to K \to G \to H \to 1,$$

where H and K are cyclic. Equivalently, a metacyclic group is a group G having a cyclic normal subgroup K, such that the quotient G/K is also cyclic. Also see Keating, Class groups of metacyclic groups of order $p^r q$, p a regular prime.

A group G is metabelian if there is an abelian normal subgroup K such that the quotient group G/K is abelian.

Examples of metacyclic groups can be found at https://en.wikipedia.org/wiki/Metacyclic_group

13.9.4 Elementary groups

Definition 13.19. A p-elementary group is a direct product of a finite cyclic group of order relatively prime to p and a p-group.

A finite group is an elementary group if it is p-elementary for some prime number p. An elementary group is nilpotent.

More generally, a finite group G is called a p-hyperelementary if it has the extension (automatically split)

$$1 \longrightarrow C \longrightarrow G \longrightarrow P \longrightarrow 1$$

where C is a cyclic group of order prime to p and P is a p-group. Note that not every hyperelementary group is elementary: for instance the non-abelian group of order 6 (S_3) is 2-hyperelementary, but not 2-elementary.

Brauer's theorem on induced characters states that a character on a finite group is a linear combination with integer coefficients of characters induced from elementary subgroups.

13.9.5 Elementary abelian groups

两个容易搞混的概念。

Definition 13.20. An elementary abelian group (or elementary abelian p-group) is an abelian group in which every nontrivial element has order p.

In other word, it is a direct product of isomorphic subgroups, each being cyclic of prime order.

The number p must be prime, and the elementary abelian groups are a particular kind of p-group.The case where $p = 2$, i.e., an elementary abelian 2-group, is sometimes called a Boolean group.

Every elementary abelian p-group is a vector space over the prime field with p elements, and conversely every such vector space is an elementary abelian group. By the classification of finitely generated abelian groups, or by the fact that every vector space has a basis, every finite elementary abelian group must be of the form $(\mathbb{Z}/p\mathbb{Z})^n$ for n a non-negative integer (sometimes called the group's rank).

In general, a (possibly infinite) elementary abelian p-group is a direct sum of cyclic groups of order p. (Note that in the finite case the direct product and direct sum coincide, but this is not so in the infinite case.)

Every finite elementary abelian group has a fairly simple finite presentation.

$$(\mathbb{Z}/p\mathbb{Z})^n \cong \langle e_1, \ldots, e_n \mid e_i^p = 1,\ e_i e_j = e_j e_i \rangle$$

13.10 A square

Rognes and Weibel

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta_p"] \arrow[d, "\sigma\mapsto 1"'] & \mathbb{Z}[\zeta_p] \arrow[d, "q"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{F}_p \end{tikzcd} \tag{13.20}$$

There is a homotopy Cartesian square

$$\begin{tikzcd} K(\mathbb{Z}[C_p];\mathbb{Z}[\frac{1}{p}]) \arrow[r] \arrow[d] & K(\mathbb{Z}[\zeta_p];\mathbb{Z}[\frac{1}{p}]) \arrow[d] \\ K(\mathbb{Z};\mathbb{Z}[\frac{1}{p}]) \arrow[r] & K(\mathbb{F}_p;\mathbb{Z}[\frac{1}{p}]) \end{tikzcd} \tag{13.20}$$

$$K_n(\mathbb{F}_p) = \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}/(p^i - 1)\mathbb{Z}, & n = 2i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let F be a field, $\mathcal{O}_F$ the ring of algebraic integers in F, G a finite group, then

$$\text{rank}\ (K_n(\mathcal{O}_F)) = \begin{cases} 1, & n = 0, \\ r_1 + r_2 - 1, & n = 1, \\ r_1 + r_2, & n \equiv 1 \bmod 4, n \neq 1, \\ r_2, & n \equiv 3 \bmod 4, \\ 0, & n > 0 \text{ even.} \end{cases}$$

For $n > 1$

$$\text{rank}\ (K_n(\mathbb{Z}G)) = \begin{cases} r, & n \equiv 1 \bmod 4, \\ c, & n \equiv 3 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

r, c are the number of irreducible real and complex representations respectively.

13.11 Negative K-theory

[52] chapter III, EX 4.7 Let G be a finite group of order n. Let $\mathcal{O}$ be a maximal order in $\mathbb{Q}G$, $\mathbb{Z}G \subset \mathcal{O} \subset \mathbb{Q}G$. It is well-known that $\mathcal{O}$ is a regular ring containing $\mathbb{Z}G$ and that $I = n\mathcal{O}$ is an ideal of $\mathbb{Z}G$. Then

- (Bass, [3] p. 560) $K_{-n}(\mathbb{Z}G) = 0$ for $n \geq 2$.
- K_{-1} has the following resolution by free abelian groups:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_0(\mathcal{O}) \oplus H_0(\mathbb{Z}/n[G]) \longrightarrow H_0(\mathcal{O}/n\mathcal{O}) \longrightarrow K_{-1}(\mathbb{Z}[G]) \longrightarrow 0.$$

- D. Carter has shown that $K_{-1}(\mathbb{Z}[G]) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^s$, where s equals the number of simple components $M_{n_i}(D_i)$ of the semisimple ring $\mathbb{Q}[G]$ such that the Schur index of D is even, but the Schur index of D_p is odd at each prime p dividing n.
- In particular, if G is abelian then $K_{-1}(\mathbb{Z}[G])$ is torsionfree (Bass, [3] p. 695).
- (Bass) If G is a finite abelian group of prime power order, then $K_{-1}(\mathbb{Z}[G]) = 0$. For example, $K_{-1}(\mathbb{Z}[C_4 \times C_2]) = 0$.

13.12 Truncated polynomial rings

13.12.1 Kernels of truncated polynomials

可能用到的定理

Theorem 13.21. *Let R be a smooth local ring which is essentially of finite type over a perfect field k of characteristic $p > 0$, and let $n \geq 1$. Then*

$$\ker\left(K_2(R[t]/(t^{n+1})) \longrightarrow K_2(R[t]/(t^n))\right)$$

is isomorphic with one of the following (unless $n = 1$, $p = 2$):

$$\begin{gathered} \Omega^1_{R/\mathbb{Z}} \text{ if } n \neq 0, -1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r} \text{ if } n = mp^r - 1, (m,p) = 1, r \geq 1, n \geq 2 \\ \Omega^1_{R/\mathbb{Z}}/D_{r,R} \text{ if } n = mp^r, (m,p) = 1, r \geq 1. \end{gathered}$$

Here $D_{r,R}$ is the subgroup of $\Omega^1_{R/\mathbb{Z}}$ generated by the forms $a^{p^j-1}\mathrm{d}a$ with $0 \leq j < r$.

If $n = 1$ and $p = 2$, then there is an exact sequence of $\mathbb{F}_2$-vector spaces

$$0 \longrightarrow R/R^2 \longrightarrow K_2(R[t]/(t^2),(t)) \longrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 0,$$

which splits, but not naturally.

Note that $\Omega^1_{R/\mathbb{Z}} = \Omega^1_{R/k}$ since k is perfect of $ch(k) = p > 0$.

$\mathbb{F}_p[x]$ is a smooth algebra over $\mathbb{F}_p$.

13.12.2 K_2 of some truncated polynomial rings

这篇笔记是关于 Roberts 的 K_2 of some truncated polynomial rings [36] 笔记，文章见http://link.springer.com/chapter/10.1007%2FBFb0103163，这篇文章收录在书《Ring Theory Waterloo 1978 Proceedings, University of Waterloo, Canada, 12-16 June, 1978》LNM734，可于http://link.springer.com/book/10.1007/BFb0103151下载.

主要集中于第 5 节 p263 的笔记.

The case $R = \mathbb{F}_p[x][t]/(t^{p^n})$. 文中的 $k = \mathbb{F}_p[x]$, 文中的 $\tilde{K}_2(R) := \ker(K_2(R) \to K_2(k))$, i.e.

$$\tilde{K}_2(\mathbb{F}_p[x][t]/(t^{p^n}) := \ker(K_2(\mathbb{F}_p[x][t]/(t^{p^n})) \to K_2(\mathbb{F}_p[x]))$$

$K_2(\mathbb{F}_p[x])=0$, so $\tilde{K}_2(\mathbb{F}_p[x][t]/(t^{p^n})\cong K_2(\mathbb{F}_p[x][t]/(t^{p^n}))$.

这里 $p^n!$ is not a unit.

下面是关于一些与 NK 有关的想法 $R=\mathbb{F}_2[x]$, $A=\mathbb{F}_2[x,t]/(t^4)$

$$\begin{aligned}\rho_1\colon \Omega_R &\longrightarrow K\\ a\,db &\mapsto \langle at^3,b\rangle\\ x^i\,dx &\mapsto \langle x^it^3,x\rangle\\ \rho_2\colon R/R^? &\longrightarrow K\\ a &\mapsto \langle at^3,t\rangle\end{aligned}$$

$$R/R^4\oplus\Omega_R\longrightarrow K_2(R[t]/(t^4),(t))=NK_2(\mathbb{F}_2[C_4])\longrightarrow K_2(R[t]/(t^3),(t))\longrightarrow 0$$

$$R/R^3\oplus\Omega_R\longrightarrow K_2(R[t]/(t^3),(t))\longrightarrow K_2(R[t]/(t^2),(t))\longrightarrow 0$$

$$0\longrightarrow R/R^2\longrightarrow K_2(R[t]/(t^2),(t))=NK_2(\mathbb{F}_2[C_2])\longrightarrow\Omega_R\longrightarrow 0$$

$k=\mathbb{F}_2[x]$, $R=k[t]/(t^4)$, 原文中的 X 这里是 t

$\tilde{K}_2(\mathbb{F}_2[x,t]/(t^4)) = \ker(K_2(\mathbb{F}_2[x,t]/(t^4)) \longrightarrow K_2(\mathbb{F}_2[x])) = K_2(\mathbb{F}_2[x,t]/(t^4)) = NK_2(\mathbb{F}_2[C_4])$.

$n=mp^r$, $p=2$, $n=4=1\cdot 2^2$

$$\begin{aligned}\rho_2\colon \mathbb{F}_2[x]/(\mathbb{F}_2[x])^4 &\longrightarrow NK_2(\mathbb{F}_2[C_4])\\ 2\text{ 阶元}a &\mapsto \langle at^3,t\rangle\text{ 2 阶元}\end{aligned}$$

$R=\mathbb{F}_2[x]$ smooth algebra over perfect field $\mathbb{F}_2$, $ch(R)=2$.

$$0\longrightarrow\Phi_3(R)\longrightarrow K_2(R[t]/(t^4))\longrightarrow K_2(R[t]/(t^3))\longrightarrow 0$$

$$0\longrightarrow\Phi_2(R)\longrightarrow K_2(R[t]/(t^3))\longrightarrow K_2(R[t]/(t^2))\longrightarrow 0$$

$\Phi_3(R)=\Omega_R\oplus R/R^4$, $\Phi_2(R)=\Omega_R/\langle da|a\in R\rangle=\Omega_R/\langle dx^i\rangle$, $K_2(R[t]/(t^2))\cong NK_2(\mathbb{F}_2[C_2])\cong\Omega_R\oplus R/R^2$.

13.13 A lower bounds for the order of $K_2(\mathbb{Z}[C_2^k \times C_{2^n}])$

$$SK_1(\mathbb{Z}[C_{p^n} \times C_{p^2}]) \cong (\mathbb{Z}/p\mathbb{Z})^{(n-1)(p-1)}$$
$$SK_1(\mathbb{Z}[C_{p^3} \times C_{p^3}]) \cong \begin{cases} (\mathbb{Z}/p^2\mathbb{Z})^{p-1} \times (\mathbb{Z}/p\mathbb{Z})^{p^2-1}, & p \text{ odd prime} \\ (\mathbb{Z}/2\mathbb{Z})^4, & p = 2 \end{cases}$$

13.14 Notes on Witt vectors

13.14.1 Witt decomposition

Proposition 13.22. *If R is a $\mathbb{Z}_{(p)}$-algebra, then*

$$bigWitt(R) \cong \prod_{gcd(m,p)=1} W(R)$$

as rings.

Proof. Let us define the map in the universal case $A = \mathbb{Z}_{(p)}[a_1, a_2, \cdots]$. Take $\sigma_p(a_i) = a_i^p$ and $\sigma_q(a_i) = 0$ for primes q other than p. Notice that $(w_1(a), w_2(a), \cdots)$ splits to $w_m(a) = (w_{p^0m}(a), w_{p^1m}(a), w_{p^2m}(a), \cdots)$ for $gcd(m,p) = 1$. Each sequence $w_m(a)$ has the property of the Dwork lemma, and therefore has the form $w(b^m)$ for some unique $b^m \in A^{\mathbb{N}}$. This defines the map

$$\begin{aligned} bigWitt(R) &\longrightarrow \prod_{gcd(m,p)=1} W(R) \\ a &\mapsto b^m. \end{aligned}$$

It is easy to see this is an isomorphism. □

We have an explicit isomorphism

$$(1 + x\mathbb{Z}/p[\![x]\!])^\times \cong \prod_{gcd(m,p)=1} \mathbb{Z}_p.$$

We can do something similar for

$$(1 + x\mathbb{Z}/p[\![x]\!]/(x^n))^\times \cong \text{ product of cyclic groups of determined orders.}$$

13.14.2 construction of the product on $\mathbb{W}(R)$

关于 Witt vectors 还有文献 [34] 要加上.

Kaledin [23], construction of the product on $\mathbb{W}(R) \subset R[[T]]^\times$ where $A^\times = U(A)$ denotes the unit group of A.

$R[[T]]^\times \xrightarrow{T\mapsto 0} R$ induce

$$\begin{tikzcd} R[[T]]^\times \arrow[r, hook] \arrow[d, "T\mapsto 0"'] & K_1(R[[T]]) \arrow[d] \\ R^\times & K_1(R) \end{tikzcd}$$

the upper left corner is a direct summand.

$K_1(R) = R^\times \oplus SK_1(R)$, $K_1(R[[T]]) = R[[T]]^\times \oplus SK_1(R[[T]])$

$1 \longrightarrow \mathbb{W}(R) \longrightarrow R[[T]]^\times \longrightarrow R^\times \longrightarrow 1$. one in fact has a split short exact sequence $0 \longrightarrow \mathbb{W}(R) \longrightarrow K_1(R[[T]]) \longrightarrow K_1(R) \longrightarrow 0$, so that Witt vectors can be understood as elements in $K_1(R[[T]])$. Given two such elements $f, g \in K_1(R[[T]])$, we can take their external product and obtain an element in $K_2(R[[T_1, T_2]])$. Then one has to cut the number of variables down to one, and pass from K_2 back to K_1. Both these tasks are easily accomplished at the same time by taking an appropriate tame symbol. The resulting formula for the Witt vector product is

$$K_1(R[[T]]) \otimes K_1(R[[T]]) \longrightarrow K_2(R[[T_1, T_2]]) \longrightarrow K_1(R[[T]])$$

$$(f * g)(T) = \mathrm{res}_z\{f(\frac{T}{z}), g(z)\}$$

where z is an additional formal variable, and $\mathrm{res}_z\{-,-\}$ is the tame symbol extended to a map

$$K_1(R((z))[[T]]) \otimes K_1(R((z))[[T]]) \longrightarrow K_1(R[[T]])$$

by taking a truncation at T^n and then taking the limit with respect to n.

$K_1(R[[T]]) \cong \varprojlim_n K_1(R[T]/(T^{n+1}))$

下面正合列有一些问题，需要找原作者核实，需要考虑 SK_1 之间的映射 $0 \longrightarrow \mathbb{W}(R) \longrightarrow K_1(R[[T]]) \longrightarrow K_1(R) \longrightarrow 0$, if so

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{W}(R) \arrow[r] & K_1(R[[T]]) \arrow[r] & K_1(R) \arrow[r] & 0 \\ 0 \arrow[r] & NK_1(R) \arrow[r] & K_1(R[T]) \arrow[r] \arrow[u, "i_*"] & K_1(R) \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

$NK_1(R)$ 是 $W(R)$-module, 一个无穷的元素作用在 $K_1(R[T])$ 后还在其中，也说明在足够大时零化了一些东西。

$$
\begin{aligned}
R^{\mathbb{N}} &\xrightarrow{\sim} \mathbb{W}(R)\\
a_{\cdot} &\mapsto \prod(1-a_nT^n)
\end{aligned}
$$

group scheme

$$
\begin{aligned}
\mathbb{G}_{a/R}\colon R-\mathrm{Alg} &\longrightarrow \mathrm{Ab}\\
A &\mapsto (A,+)\\
\mathbb{G}_{m/R}\colon R-\mathrm{Alg} &\longrightarrow \mathrm{Ab}\\
A &\mapsto (A^{\times},\cdot)\\
\mu_{n/R} = \ker(\mathbb{G}_{m/R} &\longrightarrow \mathbb{G}_{m/R})\\
x &\mapsto x^n\\
(\underline{\mathbb{Z}/n\mathbb{Z}})_R\colon R-\mathrm{Alg} &\longrightarrow \mathrm{Ab}\\
A &\mapsto (\mathbb{Z}/n\mathbb{Z})^{\pi_0(\mathrm{Spec} A)}
\end{aligned}
$$

Witt ring scheme and Witt ring scheme of length n

$$
\begin{aligned}
W, W_n\colon \mathrm{Rings} &\longrightarrow \mathrm{Rings}\\
W(\mathbb{F}_p) &= \mathbb{Z}_p\\
W_n(\mathbb{F}_p) &= \mathbb{Z}/p^n\mathbb{Z}\\
W(\mathbb{F}_{p^n}) &= \mathbb{Z}_{p^n}(\text{待核实})
\end{aligned}
$$

13.14.3 Another definition

$\Lambda_n(A) = \ker((A[t]/(t^{n+1}))^* \longrightarrow A^*)$, 记 $\oplus$ 为其中乘法, $\mathbb{Z}$-module. $a \in A$, scaling operator

$$
\begin{aligned}
\phi_a\colon A[t]/(t^{n+1}) &\longrightarrow A[t]/(t^{n+1})\\
t &\mapsto at
\end{aligned}
$$

$\phi_a \in \mathrm{End}_{\mathbb{Z}}(\Lambda_n(A))$, $\phi_a\phi_b = \phi_{ab}$. $E = \langle \phi_a \mid a \in A\rangle \subset \mathrm{End}_{\mathbb{Z}}(\Lambda_n(A))$. $(\Lambda_n(A), \oplus)$: E-module

$$\begin{aligned} E &\longrightarrow \Lambda_n(A) \\ \phi_a &\mapsto \phi_a((1-t)^{-1}) = (1-at)^{-1} \end{aligned}$$

$L_n(A) = \langle (1-at)^{-1}\rangle$, $(1-at)^{-1}\cdot(1-bt)^{-1} = (1-abt)^{-1}$. $L_n(A) = \{u(t)\}$. $u(t) = (1-a_1t)^{-1}\cdots(1-a_kt)^{-1} \bmod t^{n+1}$, $a_i \in A$. $u(t) \otimes v(t) = \prod_{i,j}(1-a_ib_jt)^{-1} \bmod t^{n+1}$.

13.15 K_2 of fields

R: the ring of integers in an algebraic number field, $\widetilde{K_0}(R) = \mathrm{Cl}(R)$, $K_1(R) = R^\times$.
Garland proved that $K_2(R)$ is finite in [12].
Quillen's localization sequence

$$0 \longrightarrow K_2(R) \longrightarrow K_2(F) \xrightarrow{T} \bigoplus_v k(v)^\times \longrightarrow 0$$

where F is the fraction field, $k(v)$ is the residue field and T is the sum of the tame symbols. The first map is injective because K_2 of a finite field is trivial, and the last map is surjective by Matsumoto's theorem.

$K_2(R)$ is a subgroup of $K_2(F)$. [24] For a field F, $K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / \langle a \otimes (1-a) \mid a \neq 0, 1\rangle$.

If F is a global field, $K_2(F) \cong \bigoplus_\ell H^2(G_F, \mathbb{Z}_\ell^{(2)})_{tor}$, where $\ell \neq ch(F)$, see Tate [43].

13.16 Regularity

参考 K-Theory of Free Rings

Jean-Pierre Serre found a homological characterization of regular local rings: A local ring A is regular if and only if A has finite global dimension, i.e. if every A-module has a projective resolution of finite length.

13.17 Tate conjecture

- K: field, $\overline{K}$: its algebraic closure

- $G_K = \mathrm{Gal}(\overline{K}/K)$: absolute Galois group. $K = \mathbb{F}_q, \mathbb{C}$ or algebraic number field
- $\mathbb{A}^n(K)$: affine n-space
- affine variety over $\overline{K}$:

  $$V = \{(x_1, \cdots, x_n) \in \mathbb{A}^n(\overline{K}) \mid p(x_1, \cdots, x_n) = 0, \forall p \in I \subset \overline{K}[X_1, \cdots, X_n] \text{ prime ideal}\}.$$

  If $I(V)$ can be generated by polynomials with coefficients in K, we say V is defined over K, and a point $x \in V$ is called a K-rational point if the coordinates of x lie in K.
- affine coordinate ring $K[V] = K[X_1, \cdots, X_n]/I(V/K)$ where $I(V/K) := I(V) \cap K[X_1, \cdots, X_n]$.
- $K(V)$: field of fractions of $K[V]$, function field of V over K.
- $\dim V = \mathrm{tr.deg}(\overline{K}(V)/\overline{K})$. curve: a variety of dimension 1, surface: a variety of dimension 2.

nonsingular varieties: $X \subset \mathbb{A}^n$, ideal is generated by polynomials $p_1, \cdots, p_m$ is nonsingular at the point x if $m \times n$ matrix $(\frac{\partial p_i(x)}{\partial x_j})$ has rank $n - \dim(V)$.

Variety $\overset{\text{embeded}}{\hookrightarrow}$ projective space.

projective variety $\mathbb{P}^n(K)$, points $(x_0 : x_1 : \cdots : x_n)$

elliptic curves: nonsingular projective curves

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

which is the homogenization of the affine equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The points of an elliptic curve form an abelian group, identity element $(0 : 1 : 0)$, isogeny is a morphism between all curves which fixes the identity point. Any isogeny is a group homomorphism.

Given an isogeny

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, dash] & & F \arrow[dl, dash] \\ & K & \end{tikzcd}$$

we may define a map of function fields

$$\begin{aligned} \phi^* \colon K(F) &\longrightarrow K(E) \\ f &\mapsto f \circ \phi \end{aligned}$$

$K(E)/\phi^*(K(F))$ is a field extension, and $\deg(\phi) := \deg\big(K(E)/\phi^*(K(F))\big)$ is finite.

If X is a variety over $\mathbb{F}_q$ (finite field). Consider the number of points of X over each finite extension $\mathbb{F}_{q^r}$. encode these information $\Longrightarrow$ zeta function of X.

Let N_r denote the number of points of X over $\mathbb{F}_{q^r}$, define

$$Z(X,T) := \exp(\sum_{r=1}^{\infty} \frac{N_r}{r} T^r),$$

this function has many useful properties.

Example 13.23. $X = \mathbb{P}^n$. Over $\mathbb{F}_{q^r}$, the number of points in $\mathbb{P}^n$ is

$$1 + q^r + q^{2r} + \cdots + q^{nr}$$

$$\begin{aligned} Z(\mathbb{P}^n, T) &= \exp(\sum_{r=1}^{\infty} \sum_{i=0}^{n} \frac{q^{ir}}{r} T^r) \\ &= \exp(\sum_{i=0}^{n} -\log(1 - q^i T)) \\ &= \prod_{i=0}^{n} (1 - q^i T)^{-1} \end{aligned}$$

is a rational function of T. This is true for any nonsingular projective variety.

In 1949, André Weil gave some conjecture on this zeta function, and Pierre Deligne proved them in 1973.

X: nonsingular projective variety of dimension n over the finite field $\mathbb{F}_q$.

Theorem 13.24 (Rationality)**.** *$Z(X,T)$ is a rational function of T.*

Theorem 13.25 (Function Equation)**.** *There is an integer E, called the Euler characteristic of X such that*

$$Z(X, \frac{1}{q^n T}) = \pm q^{nE/2} T^E Z(X,T).$$

Theorem 13.26 (Riemann Hypothesis)**.**

$$Z(X,T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)}$$

s.t. P_i are polynomials with integer coefficients, $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^n T$, each of the P_i factors over $\mathbb{C}$ as $P_i(T) = \prod_j (1 - \alpha_{i,j} T)$ where $\alpha_{i,j}$ is an algebraic integer of absolute value $q^{i/2}$.

Theorem 13.27 (Comparison)**.** *$\widetilde{X}$: variety defined over an algebraic number field K such that X is the reduction of $\widetilde{X}$ modulo a prime ideal $P \subset \mathcal{O}_K$. Then the degree of each P_i is the i-th Betti number B_i of $\widetilde{X}$ viewed as a variety over $\mathbb{C}$.*

The above Euler characteristic of X = top Euler characteristic $\sum_{i=0}^{2n}(-1)^i B_i$.

$$\text{Betti number of } \mathbb{P}^n = \begin{cases} 1, & 0 \leq i \leq 2n \text{ even} \\ 0, & i \text{ odd} \end{cases}$$

$$P_i = \begin{cases} 1 - q^{i/2}T, & \text{for even } i \\ 1, & \text{for odd } i \end{cases}$$

$e(\mathbb{P}^n) = n + 1$.

$$Z(\mathbb{P}^n, \frac{1}{q^n T}) = \pm q^{n(n+1)/2} T^{n+1} Z(\mathbb{P}^n, T).$$

$$\begin{aligned} LHS &= \prod_{i=0}^{n}(1 - q^{i-n}T^{-1})^{-1} = \prod_{i=0}^{n} \frac{1}{1 - \frac{q^i}{q^n T}} \\ &= \prod_{i=0}^{n} \frac{q^n T}{q^n T - q^i} = \prod_{i=0}^{n} \frac{q^i(q^{n-i}T)}{-q^i(1 - q^{n-i}T)} \\ &= \prod_{i=0}^{n} \frac{(q^{n-i}T)}{-(1 - q^{n-i}T)} = \prod_{i=0}^{n} \frac{-(q^i T)}{(1 - q^i T)} \\ &= (-1)^{n+1} q^{n(n+1)/2} T^{n+1} \prod_{i=0}^{n}(1 - q^i T)^{-1}, \end{aligned}$$

$$RHS = q^{n(n+1)/2} T^{n+1} \prod_{i=0}^{n}(1 - q^i T)^{-1}.$$

13.17.1 ℓ-adic cohomology theory

X: nonsingular projective variety of dimension n over the field K. For any prime $\ell \neq ch(K)$, there exist ℓ-adic cohomology groups $H^i_\ell(X)$ for $0 \leq i \leq 2n$, which are vector spaces over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

Functoriality: cup product $y \cup x = (-1)^{ij}(x \cup y)$.
Poincaré Duality: $H^{2n}_\ell(X) \cong \mathbb{Q}_\ell$

$$H^i_\ell(X) \times H^{2n-i}_\ell(X) \longrightarrow \mathbb{Q}_\ell$$

is nondegenerate, $H^i_\ell(X)$ and $H^{2n-i}_\ell(X)$ are dual vector space.

Lefschetz Trace Formula Let $\phi\colon X \longrightarrow X$ be a morphism, define two subvarieties of $X \times X$:

- diagonal $\Delta = \{(x, x) \mid x \in X\}$
- graph $\Gamma_\phi = \{(x, \phi(x)) \mid x \in X\}$

intersection number $\Delta \cdot \Gamma_\phi$ counts the number of fixed points of ϕ,

$$\Delta \cdot \Gamma_\phi = \sum_{i=0}^{2n} (-1)^i \mathrm{tr}(\phi^*\colon H^i_\ell(X) \longrightarrow H^i_\ell(X))$$

Lefschetz formula is the key to understand Weil conjecture. Note that N_r is same as the number of fixed points of the r-th power of the Frobenius morphism $F\colon X \longrightarrow X$, $(x_0 : x_1 : \cdots : x_n) \mapsto (x_0^q : x_1^q : \cdots : x_n^q)$.

Let F_i be the matrix of F^* on $H^i_\ell(X)$, Lefschetz trace formula

$$N_r = \sum_{i=0}^{2n} (-1)^i \mathrm{tr}(F_i^r).$$

$$Z(X, T) = \exp(\sum_{i=0}^{2n} (-1)^i \sum_{r=1}^{\infty} (\mathrm{tr}(F_i^r) \frac{T^r}{r})).$$

Note that $\sum_{r=1}^{\infty} (\mathrm{tr}(F_i^r) \frac{T^r}{r}) = -\log(\det(1 - F_i T))$ as formal power series. Thus

$$Z(X, T) = \prod_{i=0}^{2n} \det(1 - F_i T)^{(-1)^{i+1}}$$

$\deg P_i = \dim H^i_\ell(X)$.

13.17.2 Tate Conjecture

K: finitely generated over its prime subfield. G_K acts on the ℓ-adic cohomology groups, giving a representation

$$G_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(H^i_\ell(X))$$

for each i.

1963, Tate conjecture, one other representation of G_K, ℓ-adic cyclotomic chracter. ℓ-adic Tate module of roots of unity in the algebraic closure $\overline{K}$, μ_{ℓ^n}: the group of ℓ^n-th roots of unity in $\overline{K}$.

$$\mu_{\ell^{n+1}} \longrightarrow \mu_{\ell^n}$$

given by raising to the ℓ-th power.

inverse system, ℓ-adic Tate module

- $T_\ell(\mu) = \varprojlim_n \mu_{\ell^n}$.
- $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$ group isomorphism
- $T_\ell(\mu) \cong \mathbb{Z}_\ell$ of ℓ-adic integers

G_K acts on $T_\ell(\mu)$ giving a representation $G_K \longrightarrow \mathrm{Aut}(\mathbb{Z}_\ell) \cong (\mathbb{Z}_\ell)^\times$, embedding $\mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$ gives a 1-dimensional representation of G_K over $\mathbb{Q}_\ell$ called the ℓ-adic cyclotomic character of G_K denoted by $\mathbb{Q}_\ell(1)$. $\mathbb{Q}_\ell(k) = \mathbb{Q}_\ell(1)^{\otimes k}$.

For any finite dimensional representation V of G_K over $\mathbb{Q}_\ell$, define the k-fold Tate twist $V(k) = V \otimes \mathbb{Q}_\ell(k)$.

Tate conjecture: the classes in $H_\ell^{2d}(X)(d)$, $0 \le d \le n$, which are fixed by a finite-index subgroup of G_K. algebraic cycle group $Z^d(X_{\overline{K}})$: the free abelian group generated by the subvarieties of X of codimension d, there is a homomorphism $c\colon Z^d(X_{\overline{K}}) \longrightarrow H_\ell^{2d}(X)(d)$ which commutes with the action of G_K.

Let $H^{2d}_{\mathrm{alg}} := c(Z^d(X_{\overline{K}})) \otimes \mathbb{Q}_\ell \subset H_\ell^{2d}(X)(d)$. By Hilbert Basis theorem, any subvariety of X over $\overline{K}$ is defined over some finite extension L/K. So any cycle class in $Z^d(X_{\overline{K}})$ is fixed by a finite-index subgroup of G_K.

Denote the set of all classes in $H_\ell^{2d}(X)(d)$ fixed by a finite-index subgroup of G_K by H^{2d}_{Tate}.

Theorem 13.28. *c commutes with the action of G_K, we have $H^{2d}_{alg} \subset H^{2d}_{Tate}$.*

Conjecture 13.29 (Tate conjecture). $H^{2d}_{alg} = H^{2d}_{Tate}$.

13.18 一些其他关系不大的笔记

（在一次李代数的报告上，$V \xrightarrow{Y(\cdot,\cdot)} \mathrm{End}(V)[[z,z^{-1}]]$, $\hat{g} = g \oplus \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}o$ affine Kac-Moody algebra. twisted 变成一个 fixed. $M((x))$ 下方截断，$\deg x^n = n$, $\mathbb{Z}$-graded algebra.）

完备化是遗忘函子的左伴随 $\mathrm{Hom}(M, res(N)) \xrightarrow{\sim} \mathrm{Hom}(\hat{M}, N)$.

Prüfer ring$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p = \varprojlim(\mathbb{Z}/1\mathbb{Z} \to \mathbb{Z}/2!\mathbb{Z} \to \mathbb{Z}/3!\mathbb{Z} \to \cdots)$(这里箭头还需要再验证), topology $\mathcal{F} = \{n\mathbb{Z} \mid n \text{ is an integer}\}$.

13.18.1 $\mathbb{F}_q^r$

https://sbseminar.wordpress.com/ $\mathbb{F}_3^2$, define an addtion $(x_0, x_1) \oplus (y_0, y_1) = (x_0 + y_0, x_1 + y_1 - x_0^2 y_0 - x_0 y_0^2)$,

$$\mathbb{F}_3^2 \xrightarrow{\sim} C_9.$$

For example $(1,0) \oplus (1,0) \oplus (1,0) = (2,1) \oplus (1,0) = (0,1)$, $(0,1) \oplus (0,1) \oplus (0,1) = (0,0)$. In fact

$$\mathbb{F}_p^r \xrightarrow{\sim} C_{p^r},$$

and the addtion $x \oplus y = (x_0, x_1, \cdots, x_{r-1}) \oplus (y_0, y_1, \cdots, y_{r-1}) = (s_0(x,y), s_1(x,y), \cdots, s_{r-1}(x,y))$ where s_i are Witt polynimials.

For any $0 \le m \le p^r - 1$,

$$\begin{aligned} C_{p^r} &\longrightarrow \mathbb{F}_p, \\ x &\mapsto \binom{x}{m}. \end{aligned}$$

Lucas's theorem

$$\binom{x-1}{p^r-1} = \begin{cases} 1, & x \equiv 0 \bmod p^r, \\ 0, & \text{otherwise.} \end{cases}$$

13.18.2 Differential modules

$\overline{B} = B/I$ for some ideal $I \subset B$, the fundamental exact sequence

$$I/I^2 \longrightarrow \Omega_{B/A} \otimes_B \overline{B} \longrightarrow \Omega_{\overline{B}/A} \longrightarrow 0$$

the fisrt map is induced by $f \mapsto df$.

For example, $A = k$, $B = k[X_1, \cdots, X_n]$, $\overline{B} = B/I$

$$I/I^2 \longrightarrow \Omega_{k[X_1,\cdots,X_n]/k} \otimes_{k[X_1,\cdots,X_n]} \overline{B} \longrightarrow \Omega_{\overline{B}/k} \longrightarrow 0$$

$\Omega_{k[X_1,\cdots,X_n]/k}$ is a free $k[X_1, \cdots, X_n]$-module on $dX_1, \cdots, dX_n$. $\Omega_{\overline{B}/k}$ is the quotient of $\bigoplus_{i=1}^n \overline{B} \cdot dX_i$ by the submodule generated by

$$df = \sum \frac{\partial f}{\partial X_i} dX_i, \text{ for } f \in I.$$

13.18.3 Homological Algebra

$\mathrm{Hom}(\varinjlim F, Z) \cong \varprojlim \mathrm{Hom}(F(-), Z)$, $\mathrm{Hom}(Z, \varprojlim F) \cong \varprojlim \mathrm{Hom}(Z, F(-))$
left exact if commutes with finite projecive limit, i.e. inverse limit (limit) $\varprojlim$
right exact if commutes with finite colimit $\varinjlim$

i.e. $\varinjlim$ right exact; $\varprojlim$ left exact

If I is a filtering category, then $\varinjlim_I F$ is exact.

Cofinality(共尾性) 是为了变 index category 把一个大的 index category 变成更小的，比如 the category of projecive module and the category of free module.

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