内容集锦: 讨论班、课程讲义

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目录

Chapter 1

Witt rings and NK-groups

References:

part 1 J. P. Serre, Local fields.

part 1 Daniel Finkel, An overview of Witt vectors.

part 2 Hendrik Lenstra, Construction of the ring of Witt vectors.

part 2 Barry Dayton, Witt vectors, the Grothendieck Burnside ring, and Necklaces.

part 3 C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466–493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.

part 3 D. R. Grayson, Grothendieck rings and Witt vectors.

part 3 C. A. Weibel, The K-Book: An introduction to algebraic K-theory.

Ernst Witt Ernst Witt (June 26, 1911–July 3, 1991) was a German mathematician, one of the leading algebraists of his time. Witt completed his Ph.D. at the University of Göttingen in 1934 with Emmy Noether. Witt's work has been highly influential. His invention of the Witt vectors clarifies and generalizes the structure of the p-adic numbers. It has become fundamental to p-adic Hodge theory. For more information, see `https://en.wikipedia.org/wiki/Ernst_Witt` and `http://www-history.mcs.st-andrews.ac.uk/Biographies/Witt.html`.

1.1 p-Witt vectors

In this section we introduce p-Witt vectors. Witt vectors generalize the p-adics and we will see all p-Witt vectors over any commutative ring form a ring.

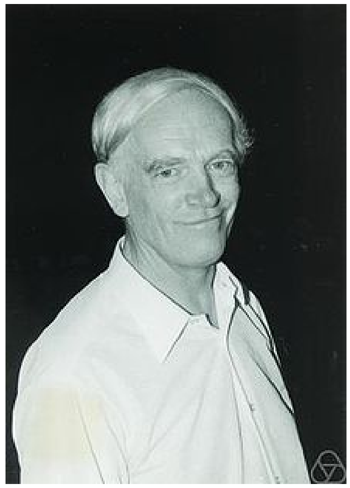

图 1.1: Ernst Witt

From now on, fix a prime number p.

Definition 1.1. A p-Witt vector over a commutative ring R is a sequence $(X_0, X_1, X_2, \cdots)$ of elements of R.

Remark 1.2. If $R = \mathbb{F}_p$, any p-Witt vector over $\mathbb{F}_p$ is just a p-adic integer $a_0 + a_1 p + a_2 p^2 + \cdots$ with $a_i \in \mathbb{F}_p$.

We introduce Witt polynomials in order to define ring structure on p-Witt vectors.

Definition 1.3. Fix a prime number p, let $(X_0, X_1, X_2, \cdots)$ be an infinite sequence of indeterminates. For every $n \geq 0$, define the n-th Witt polynomial

$$W_n(X_0, X_1, \cdots) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

For example, $W_0 = X_0,\ W_1 = X_0^p + pX_1,\ W_2 = X_0^{p^2} + pX_1^p + p^2 X_2$.

Question: how can we add and multiple Witt vectors?

Theorem 1.4. *Let $(X_0, X_1, X_2, \cdots)$, $(Y_0, Y_1, Y_2, \cdots)$ be two sequences of indeterminates. For every polynomial function $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \cdots, \varphi_n, \cdots)$ of elements of $\mathbb{Z}[X_0, \cdots, X_n, \cdots; Y_0, \cdots, Y_n, \cdots]$ such that*

$$W_n(\varphi_0, \cdots, \varphi_n, \cdots) = \Phi\big(W_n(X_0, \cdots), W_n(Y_0, \cdots)\big),\ n = 0, 1, \cdots.$$

If $\Phi = X + Y$(resp. XY), then there exist $(S_1, \cdots, S_n, \cdots)$(“$S$” stands for sum) and $(P_1, \cdots, P_n, \cdots)$ (“P” stands for product) such that

$$W_n(X_0, \cdots, X_n, \cdots) + W_n(Y_0, \cdots, Y_n, \cdots) = W_n(S_1, \cdots, S_n, \cdots),$$

$$W_n(X_0,\cdots,X_n,\cdots)W_n(Y_0,\cdots,Y_n,\cdots)=W_n(P_1,\cdots,P_n,\cdots).$$

Let R be a commutative ring, if $A=(a_0,a_1,\cdots)\in R^{\mathbb{N}}$ and $B=(b_0,b_1,\cdots)\in R^{\mathbb{N}}$ are p-Witt vectors over R, we define

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots).$$

Theorem 1.5. *The p-Witt vectors over any commutative ring R form a commutative ring under the compositions defined above (called the ring of p-Witt vectors with coefficients in R, denoted by $W(R)$).*

Example 1.6. We have

$$\begin{aligned} S_0(A,B)&=a_0+b_0 & P_0(A,B)&=a_0b_0\\ S_1(A,B)&=a_1+b_1+\tfrac{a_0^p+b_0^p-(a_0+b_0)^p}{p} & P_1(A,B)&=b_0^pa_1+a_0^pb_1+pa_1b_1 \end{aligned}$$

Theorem 1.7. *There is a ring homomorphism*

$$W_*\colon W(R)\longrightarrow R^{\mathbb{N}}$$
$$(X_0,X_1,\cdots,X_n,\cdots)\mapsto(W_0,W_1,\cdots,W_n,\cdots)$$

Proof. Only need to check this map is a ring homomorphism: since

$$A+B=(S_0(A,B),S_1(A,B),\cdots),\quad AB=(P_0(A,B),P_1(A,B),\cdots),$$

by definition we have

$$\begin{aligned} W(A)+W(B)&=(W_0(A)+W_0(B),W_1(A)+W_1(B),\cdots)\\ &=\big(W_0(S_0(A,B),S_1(A,B),\cdots),W_1(S_0(A,B),S_1(A,B),\cdots),\cdots\big)\\ &=W(S_0(A,B),S_1(A,B),\cdots)=W(A+B). \end{aligned}$$

And similarly,

$$\begin{aligned} W(A)W(B)&=(W_0(A)W_0(B),W_1(A)W_1(B),\cdots)\\ &=\big(W_0(P_0(A,B),P_1(A,B),\cdots),W_1(P_0(A,B),P_1(A,B),\cdots),\cdots\big)\\ &=W(P_0(A,B),P_1(A,B),\cdots)=W(AB). \end{aligned}$$

Indeed, we only need to show $W_n(A)+W_n(B)=W_n(A+B)$ and $W_n(A)W_n(B)=W_n(AB)$ which are obviously true. (实际上就是为了使得这个是同态而定义出了 $A+B$ 和 AB。) □

Example 1.8. 1. If p is invertible in R, then $W(R)=R^{\mathbb{N}}$ — the product of countable number of R.(if p is invertible the homomorphism W_* is an isomorphism.)

2. $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$ — the ring of p-adic integers.

3. $W(\mathbb{F}_{p^n})$ is an unramified extension of the ring of p-adic integers.

Note that the functions P_k and S_k are actually only involve the variables of index $\leqslant k$ of A and B. In particular if we truncate all the vectors at the k-th entry, we can still add and multiply them.

Definition 1.9. Truncated p-Witt ring $W_k(R) = \{(a_0, a_1, \cdots, a_{k-1}) | a_i \in R\}$ (also called the ring of Witt vectors of length k.)

Example 1.10. $W_1(R) = R$, $W(R) = \varprojlim W_k(R)$. Since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$.

Definition 1.11. We define two special maps as follows

- The "shift" map $V\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$, this map is *additive*.
- When $char(R) = p$, the "Frobenius" map $F\colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots)$, this is indeed a ring homomorphism.

Firstly, we note that $W_k(R) = W(R)/V^k W(R)$, and if we consider $V\colon W_n(R) \hookrightarrow W_{n+1}(R)$ there are exact sequences

$$0 \longrightarrow W_k(R) \xrightarrow{V^r} W_{k+r}(R) \longrightarrow W_r(R) \longrightarrow 0, \quad \forall k,\ r.$$

The map $V\colon W(R) \longrightarrow W(R)$ is additive: for it suffices to verify this when p is invertible in R, and in that case the homomorphism $W_*\colon W(R) \longrightarrow R^{\mathbb{N}}$ transforms V into the map which sends $(w_0, w_1, \cdots)$ to $(0, pw_0, pw_1, \cdots)$.

$$\begin{tikzcd} W(R) \arrow[r, "V"] \arrow[d, "W_*"] & W(R) \arrow[d, "W_*"] \\ R^{\mathbb{N}} \arrow[r] & R^{\mathbb{N}} \end{tikzcd}$$

$$\begin{tikzcd} (a_0, a_1, \cdots) \arrow[r, mapsto, "V"] \arrow[d, mapsto, "W_*"] & (0, a_0, a_1, \cdots) \arrow[d, mapsto, "W_*"] \\ (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \cdots) \arrow[r, mapsto] \arrow[d, equal] & (0, pa_0, pa_0^p + p^2 a_1, \cdots) \arrow[d, equal] \\ (w_0, w_1, w_2, \cdots) \arrow[r, mapsto] & (0, pw_0, pw_1, \cdots) \end{tikzcd}$$

If $x \in R$, define a map

$$\begin{aligned} r\colon R &\longrightarrow W(R) \\ x &\mapsto (x, 0, \cdots, 0, \cdots) \end{aligned}$$

When p is invertible in R, W_* transforms r into the mapping that $x \mapsto (x, x^p, \cdots, x^{p^n}, \cdots)$.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\mathrm{id}"] & W(R) \arrow[d, "W_*"] \\ R \arrow[r] & R^{\mathbb{N}} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, equal] & (x, 0, \cdots, 0, \cdots) \arrow[d, mapsto, "W_*"] \\ x \arrow[r, mapsto] & (x, x^p, \cdots, x^{p^n}, \cdots) \end{tikzcd}$$

One deduces by the same reasoning as above the formulas:

Proposition 1.12.

$$\begin{aligned} r(xy) &= r(x)r(y),\ \ x, y \in R \\ (a_0, a_1, \cdots) &= \sum_{n=0}^{\infty} V^n(r(a_n)),\ \ a_i \in R \\ r(x)(a_0, \cdots) &= (xa_0, x^p a_1, \cdots, x^{p^n} a_n, \cdots),\ \ x_i, a_i \in R. \end{aligned}$$

Proof. The first formula: put $r(x)r(y),\ r(xy)$ to $R^{\mathbb{N}}$, we get $(x, x^p, \cdots, x^{p^n}, \cdots)(y, y^p, \cdots, y^{p^n}, \cdots)$ and $(xy, (xy)^p, \cdots, (xy)^{p^n}, \cdots)$.
The second formula: put $(a_0, a_1, \cdots)$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \cdots)$
consider $V^i(r(a_i))$: put $r(a_i)$ to $R^{\mathbb{N}}$, we get $(a_i, a_i^p, \cdots, a_i^{p^n}, \cdots) \in R^{\mathbb{N}}$, and W_* transforms V to the mapping $(w_0, w_1, \cdots, w_n, \cdots) \mapsto (0, pw_0, \cdots, pw_{n-1}, \cdots)$,
now we put $(r(a_0))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p, \cdots, a_0^{p^n}, \cdots)$
put $V^1(r(a_1))$ to $R^{\mathbb{N}}$, we get $(0, pa_1, \cdots, pa_1^{p^{n-1}}, \cdots)$
put $V^2(r(a_2))$ to $R^{\mathbb{N}}$, we get $(0, 0, p^2 a_2, \cdots, p^2 a_2^{p^{n-2}}, \cdots)$
put $V^i(r(a_i))$ to $R^{\mathbb{N}}$, we get $(\underbrace{0, 0, \cdots, 0}_{i \text{ terms}}, p^i a_i, \cdots, p^i a_i^{p^{n-i}}, \cdots)$
so put $\sum_n V^n(r(a_n))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, \cdots)$.
We leave the proof of the last formula to readers. $\square$

Proposition 1.13.

$$VF = p = FV.$$

Proof. It suffices to check this when R is perfect. Note that a ring R of characteristic p is called perfect if $x \mapsto x^p$ is an automorphism. For more details, we refer to page 44 on Serre's book *Local Fields*. $\square$

1.2 Big Witt vectors

Now we turn to the `big` Witt vectors. J. P, May once said "This(the theory of Witt vectors) is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once".

Take the ring of all big vectors of a commutative ring is a functor

$$\mathbf{CRing} \longrightarrow \mathbf{CRing}$$
$$R \mapsto W(R).$$

In this section, R is a commutative ring with unit.

Definition 1.14. The ring of all big Witt vectors in R which also denoted by $W(R)$ is defined as follows,
as a set: $W(R) = \{a(T) \in R[[T]] | a(T) = 1 + a_1 T + a_2 T^2 + \cdots\} = 1 + TR[[T]]$; (we note that as a set $W(R)$ is the kernel of the map $A[[T]]^* \xrightarrow{T \mapsto 0} A^*$)
addition in $W(R)$: usual multiplication of formal power series, sum $a(T)b(T)$, difference $\frac{a(T)}{b(T)}$;($(W(R), +) \cong (1 + TR[[T]], \times)$ which is a subgroup of the group of units $R[[T]]^\times$ of the ring $R[[T]]$)
multiplication in $W(R)$: denoted by $*$, this is a little mysterious, we will talk the details later. For the present purposes we only define $*$ as the unique continuous functorial operation for which $(1 - aT) * (1 - bT) = (1 - abT)$.
'zero'(additive identity) of $W(R)$: 1.
'one'(multiplicative identity) of $W(R)$: $[1] = 1 - T$. Note that $[1]$ is the image of $1 \in R$ under the multiplicative (Teichmuller) map

$$R \longrightarrow W(R)$$
$$a \mapsto [a] = 1 - aT$$

functoriality: any homomorphism $f \colon R \longrightarrow S$ induces a ring homomorphism $W(f) \colon W(R) \longrightarrow W(S)$.

A quick way to check multiplicative formulas in $W(R)$ is to use the `ghost` map (indeed a ring homomorphism)

$$gh \colon W(R) \longrightarrow R^{\mathbb{N}} = \prod_i^{\infty} R.$$

It is obtained from the abelian group homomorphism

$$-T\frac{d}{dT}\log\colon (1+TR[[T]])^\times \longrightarrow (TR[[T]])^+$$
$$a(T) \mapsto -T\frac{a'(T)}{a(T)}$$

the right side of gh is $R^{\mathbb{N}}$ via $\sum a_n t^n \longleftrightarrow (a_1, a_2, \cdots)$.

A basic principle of the theory of Witt vectors is that to demonstrate certain equations it suffices to check them on vectors of the form $1-aT$.

1.3 Module structure on NK_*

Notations Λ: a ring with 1

R: commutative ring

$W(R)$: the ring of big Witt vectors of R

$\mathbf{End}(\Lambda)$: the exact category of endomorphisms of finitely generated projective right Λ-modules.

$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.

$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules.

The fundamental theorem in algebraic K-theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus NK_i(\Lambda) \cong K_i(\Lambda) \oplus \mathrm{Nil}_{i-1}(\Lambda),$$

and hence $\mathrm{Nil}(\Lambda)$ is the obstruction to K-theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (right) Λ-module has a finite projective resolution. So the resolution theorem and the fact that G-theory is homotopy invariant show that for a regular ring, $NK_*(\Lambda) = \mathrm{Nil}_{*-1}(\Lambda) = 0$. In general, one knows that the groups $\mathrm{Nil}_*(\Lambda)$, if non-zero, are infinitely generated. It is also known that the groups $\mathrm{Nil}_*(\Lambda)$ are modules over the big Witt ring $W(R)$ (just this notes want to show you).

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.
- Computations in $W(R)$ with Grothendieck rings.

1.3.1 $\mathrm{End}_0(\Lambda)$

Let $\mathbf{End}(\Lambda)$ denote the exact category of endomorphisms of finitely generated projective right Λ-modules.

Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.
Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & M_1 \arrow[d, "\alpha"] \\ M_2 \arrow[r, "f_2"] & M_2 \end{tikzcd}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathscr{C} \longrightarrow \mathscr{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathscr{C} \rightleftarrows \mathscr{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \leftarrow\!\shortmid (M, f)$$

Definition 1.15. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) \longrightarrow \mathbf{End}(\Lambda)$$
$$\otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) \longrightarrow \mathbf{Nil}(\Lambda)$$
$$(M, f) \otimes (N, g) = (M \otimes_R N, f \otimes g)$$

These induce (use "generators-and-relations" tricks on K_0)

$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) \longrightarrow K_*(\mathbf{End}(\Lambda))$$
$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) \longrightarrow K_*(\mathbf{Nil}(\Lambda))$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M, f) \in \mathbf{End}(R)$, then $(M, f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R,1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R,0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R,0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R,0)\otimes\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. (Consider $P \in \mathbf{P}(R)$ acts on $\mathbf{End}(\Lambda)$, $(P,0)\otimes(N,g) = (P\otimes_R N, 0) \in \mathbf{P}(\Lambda)$.)

The following is immediate (and well-known):

Proposition 1.16. *If Λ is an R-algebra with 1, $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$ are graded modules over the ring $\mathrm{End}_0(R)$.*

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 1.17 (Almkvist)**.** *The homomorphism (in fact it is a ring homomorphism)*

$$\begin{aligned}\chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = (1+TR[[T]])^\times\\ (M,f) &\mapsto \det(1-fT)\end{aligned}$$

is injective and $\mathrm{End}_0(R) \cong \mathrm{Im}\chi = \left\{\frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1+TR[T]\right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1-T$$

χ is a ring homomorphism, and $\mathrm{Im}\chi =$ the set of all rational functions in $W(R)$. Note that

$$\det(1-fT)\det(1-gT) = \det(1-(f\oplus g)T), \quad \det(1-fT)*\det(1-gT) = \det(1-(f\otimes g)T),$$

for more details we refer the reader to S.Lang *Algebra*, Chapter 14, Exercise 15.

Remark 1.18. when R is a algebraically closed field (for instance $\mathbb{C}$), we can use Jordan canonical forms to prove the last identity (i.e. it can be reduced to check that $\prod_i(1-\lambda_i T) * \prod_j(1-\mu_j T) = \prod_{i,j}(1-\lambda_i\mu_j T)$).

Definition 1.19 (NK_*). As above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in "Higher algebraic K-theory II". The map is given by

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/xy=1) \xrightarrow{\partial} K_{n-1}(\mathbf{Nil}(\Lambda)).$$

Thus $NK_n(\Lambda)$ are $\mathrm{End}_0(R)$-modules. For $n \geq 1$, this is just 1.16; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the "contracted functor" of $NK_1(\Lambda)$.

Note that $NK_1(\Lambda) \cong K_1(\Lambda[y], (y-\lambda))$, $\forall \lambda \in \Lambda$, since

$$\Lambda[y] \rightleftarrows \Lambda$$
$$y \mapsto \lambda.$$

Since $[(P,\nu)] = [(P \oplus Q, \nu \oplus 0)] - [(Q,0)] \in K_0(\mathbf{Nil}(\Lambda))$, we see $\mathrm{Nil}_0(\Lambda)$ is generated by elements of the form $[(\Lambda^n,\nu)] - n[(\Lambda,0)]$ for some n and some nilpotent matrix ν Sign convention:

$$NK_1(\Lambda) \cong \mathrm{Nil}_0(\Lambda)$$
$$[1-\nu y] \leftrightarrow [(\Lambda^n,\nu)] - n[(\Lambda,0)]$$

Example 1.20. Let k be a field, $\mathbf{End}(k)$ consists pairs (V, A) with V a finite-dimensional vector space over k and A a k-endomorphism. Two pairs are isomorphic if and only if their minimal polynomials are equal. When we consider $\mathbf{Nil}(k)$, then $K_0(\mathbf{Nil}(k)) \cong \mathbb{Z}$, we conclude that $\mathrm{Nil}_0(k) = 0$. Recall that since k is a regular ring, $NK_*(k) = 0$, we have another proof of $NK_1(k) \cong \mathrm{Nil}_0(k) = 0$.

1.3.2 Grothendieck rings and Witt vectors

We refer the reader to Grayson's paper *Grothendieck rings and Witt vectors.*

Definition 1.21. A λ-ring R is a commutative ring with 1, together with an operation λ_t which assigns to each element x of R a power series

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots, \quad x \in R$$

This operation must obey $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

Let R is a commutative ring with unit, $K_0(R) = K_0(\mathbf{P}(R))$ becomes a λ-ring if we define

$$[M][N] = [M \otimes_R N], \quad \lambda_t^n([M]) = [\wedge_R^n M].$$

Recall $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$, $\wedge^n(M \oplus M') \cong \bigoplus_{i+j=n} \wedge^i M \otimes \wedge^j M'$, and $\wedge^n(M) := M^{\otimes n}/\langle x \otimes x \mid x \in M\rangle$, rank $\wedge^n(M) = \binom{\mathrm{rank} M}{n}$.

For instance, if R is a field, $K_0(R) = \mathbb{Z}$ and $\lambda_t(n) = (1+t)^n = 1 + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$, since $\dim(\wedge^i R^n) = \binom{n}{i}$.

We make $K_0(\mathbf{End}(R))$ into a λ-ring by defining

$$\lambda^n([M, f]) = ([\wedge^n M, \wedge^n f]).$$

The ideal generated by the idempotent $[(R, 0)]$ is isomorphic to $K_0(R)$, the quotient $\mathrm{End}_0(R)$ is a λ-ring. It is convenient to think of End_0 as a convariant functor on the category of rings, and the functor End_0 satisfies:

1. If $R \longrightarrow S$ is surjective ring homomorphism, then $\mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(S)$ is surjective.

2. If R is an algebraically close field, then the group $\mathrm{End}_0(R)$ is generated by the elements of the form $[(R, r)]$. (This holds because any matrix over R is triagonalizable.)

Recall

$$\begin{aligned} \chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = 1 + TR[[T]] \\ (M, f) &\mapsto \det(1 - fT) \end{aligned}$$

$W(R)$ is the underlying (additive) group of the ring of Witt vectors. The λ-ring operations on $W(R)$ are the unique operations which are continuous, functorial in R, and satisfy:

$$\begin{aligned} (1 - aT) * (1 - bT) &= 1 - abT \\ \lambda_t(1 - aT) &= 1 + (1 - aT)t \end{aligned}$$

By 1.17, χ is an injective ring homomorphism whose image consists of all Witt vectors which are quotients of polynomials. In fact χ is a λ-ring homomorphism, so we have

Theorem 1.22. $\mathrm{End}_0(R)$ *is dense sub-λ-ring of* $W(R)$*.*

The hard part of the theorem is the injectivity. When R is a field the injectivity follows immediately from the existence of the rational canonical form (we can see it below) for a matrix. The result is surprising when R is not a field.

Computation in $W(R)$ Computation in $W(R)$ which is tedious unless we perform it in $\mathrm{End}_0(R)$:

$$(1-aT^2)*(1-bT^2)=?$$

Note that $\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\Big)=\det\begin{pmatrix}1 & -aT\\ -T & 1\end{pmatrix}=1-aT^2$, $\chi\Big(\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\det\begin{pmatrix}1 & -bT\\ -T & 1\end{pmatrix}=1-bT^2$,

$$\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\Big)*\chi\Big(\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\chi\Big(\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & b\\ 1 & 0\end{pmatrix}\Big)=\chi\Big(\begin{pmatrix}0 & 0 & 0 & ab\\ 0 & 0 & b & 0\\ 0 & a & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\Big)$$

$$=\det\begin{pmatrix}1 & 0 & 0 & -aTb\\ 0 & 1 & -bT & 0\\ 0 & -aT & 1 & 0\\ -T & 0 & 0 & 1\end{pmatrix}=1-2abT^2+a^2b^2T^4.$$

If we use the previous formula

$$(1-rT^m)*(1-sT^n)=(1-r^{n/d}s^{m/d}T^{mn/d})^d,\ d=\gcd(m,n),$$

we obtain the same answer. Indeed we have the formula

$$\det(1-fT)*\det(1-gT)=\det(1-(f\otimes g)T),$$

if a polynomial is $1+a_1T+\cdots+a_nT^n\in W(R)$, we can write $f=\begin{pmatrix}0 & & & & -a_n\\ 1 & 0 & & & -a_{n-1}\\ & \ddots & \ddots & & \vdots\\ & & 1 & 0 & -a_2\\ & & & 1 & -a_1\end{pmatrix}\in M_n(R)$.

Operations on $W(R)$ and $\mathrm{End}_0(R)$ We have already known that W and End_0 can be regarded as functors from the category of commutative rings to that of (λ-)rings. The following operations $F_n,\ V_n\colon W\Longrightarrow W$(resp. $\mathrm{End}_0\Longrightarrow\mathrm{End}_0$) are indeed natural transformation. These auxiliary operations defined on $W(R)$ can also be computed in $\mathrm{End}_0(R)$.

1. the ghost map

$$gh\colon W(R)\xrightarrow{-T\frac{d}{dT}\log}TR[[T]]\cong R^{\mathbb{N}},\quad \alpha(T)\mapsto\frac{-T}{\alpha(T)}\frac{d\alpha}{dT}.$$

and the n-th ghost coordinate

$$gh_n \colon W(R) \longrightarrow R$$

it is the unique continuous natual additive map which sends $1-aT$ to a^n.

Remark. $gh(1-aT) = \frac{aT}{1-aT} = \sum_{i=1} a^i T^i$.The exponential map is defined by

$$R^{\mathbb{N}} \longrightarrow W(R), \quad (r_1, \cdots) \mapsto \prod_{i=1}^{\infty} \exp(\frac{-r_i T^i}{i}).$$

2. the Frobenius endomorphism

$$F_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}}).$$

it is the unique continuous natual additive map which sends $1-aT$ to $1-a^nT$.

Remark. $F_n(1-aT) = \sum_{\zeta^n=1}(1-a\zeta T^{\frac{1}{n}}) = 1-a^nT$, since "+" in $W(R)$ is the normal product.

3. the Verschiebung endomorphism

$$V_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \alpha(T^n).$$

it is the unique continuous natual additive map which sends $1-aT$ to $1-aT^n$.

ghost map $gh_n \colon W(R) \longrightarrow R$	$1-aT \mapsto a^n$	
Frobenius endomorphism $F_n \colon W(R) \longrightarrow W(R)$	$1-aT \mapsto 1-a^nT$	$\alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}})$
Verschiebung endomorphism $V_n \colon W(R) \longrightarrow W(R)$	$1-aT \mapsto 1-aT^n$	$\alpha(T) \mapsto \alpha(T^n)$

We define similar operations on $\mathrm{End}_0(R)$ as follows:

$gh_n \colon \mathrm{End}_0(R) \longrightarrow R$	$[(M,f)] \mapsto \mathrm{tr}(f^n)$
$F_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M,f)] \mapsto [(M,f^n)]$
$V_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M,f)] \mapsto [(M^{\oplus n}, v_n f)]$

where $v_n f$ is represented by $\begin{pmatrix} 0 & & & f \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$. The matrix $v_n f$ is close to an n-th root of f. Another equivalent description is

$$V_n \colon [(M,f)] \mapsto [(M[y]/y^n - f, y)].$$

One easily checks that the operations just defined are additive with respect to short exact sequences in $\mathbf{End}(R)$, and thus are well-defined on $\mathrm{End}_0(R)$.

Since $\mathrm{End}_0(R) \subset W(R)$ is dense and $gh_n,\ F_n,\ V_n$ are continuous, identities among them may be verified on $W(R)$ by checking them on $\mathrm{End}_0(R)$.

$W(R)$	$\longleftrightarrow$	$\mathrm{End}_0(R)$
$gh_n(v * w) = gh_n v * gh_n w$		$\mathrm{tr}((f \otimes g)^n) = \mathrm{tr}(f^n)\mathrm{tr}(g^n)$
$F_n(v * w) = F_n v * F_n w$		$(f \otimes g)^n = f^n \otimes g^n$
$F_n V_n = n$		$(v_n f)^n = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$
$gh_n V_d(v) = \begin{cases} d\, gh_{n/d}(v), & \text{if } d \mid n \\ 0, & \text{if } d \nmid n \end{cases}$		$\mathrm{tr}((v_d f)^n) = \begin{cases} d\,\mathrm{tr}(f^{n/d}), & \text{if } d \mid n \\ 0, & \text{if } d \nmid n \end{cases}$

From the last row we may recover the usual expression of the ghost coordinates in terms of the Witt coordinates . The Witt coordinates of a vector v are the coefficients in the expression

$$v = \prod_{i=1}(1 - a_i T^i) = \prod_{i=1} V_i(1 - a_i T).$$

We obtain

$$gh_n(v) = \sum_{d \mid n} d a_d^{n/d}.$$

"Many mordern treatments of the subject of Witt vectors take this latter expression as the starting point of the theory."

The logarithmic derivative of $1 - a_d T^d$ is $\frac{d}{dT}\log(1 - a_d T^d) = -\sum_{m=1}^{\infty} d a_d^m T^{dm-1}$, and $-T\frac{d}{dT}\log(1 - a_d T^d) = \sum_{n=1}^{\infty} gh_n(1 - a_d T^d) T^n$. So we obtain the formula:

$$-T v^{-1}\frac{dv}{dT} = \sum_{n=1}^{\infty} gh_n(v) T^n$$

which yields the exponential trace formula:

$$-T\chi([M,f])^{-1}\frac{d\chi([M,f])}{dT} = \sum_{n=1}^{\infty} \mathrm{tr}(f^n) T^n.$$

For example, when $\mathrm{rank} M = 2$, we have $\mathrm{tr}(f^2) = (\mathrm{tr}(f))^2 - 2\det(f)$, note that $\det(1 - fT) = 1 - \mathrm{tr}(f)T + \det(f)T^2$.

Remark 1.23. When R is a field, the exponential trace formula

$$-T\frac{d}{dT}\log\det(1 - fT) = \sum_{n=1}^{\infty} \mathrm{tr}(f^n) T^n$$

can be checked by $\det(1-fT) = \prod(1-\lambda_i T)$ where λ_i are eigenvalues. And we also have

$$\det(1-fT) = \exp(\sum_{n=1}^{\infty} -\mathrm{tr}(f^n)\frac{T^n}{n}),$$

since $\prod(1-\lambda_i T) = \exp\big(\ln(\prod(1-\lambda_i T))\big) = \exp\big(\sum \ln(1-\lambda_i T)\big)$ and recall that formally $\ln(1-x) = -\sum \frac{x^n}{n}$.

1.3.3 $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Recall Λ is an R-algebra, where R is a commutative ring with unit. We define a map

$$\begin{aligned} \mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) &\longrightarrow \mathrm{Nil}_0(\Lambda) \\ (R^n, f) * [(P, \nu)] &= [(P^n, f\nu)] \end{aligned}$$

Let $\alpha_n = \alpha_n(a_1, \cdots, a_n)$ denote the $n \times n$ matrix (looks like the rational canonical form) over R:

$$\alpha_n(a_1, \cdots, a_n) = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

Recall

$$\begin{aligned} \chi \colon \mathrm{End}_0(R) &\rightarrowtail W(R) \\ (R^n, \alpha_n) &\mapsto \det(1-\alpha_n T) \end{aligned}$$

we obtain

$$\det \begin{pmatrix} 1 & & & & a_n T \\ -T & 1 & & & a_{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2 T \\ & & & -T & 1 + a_1 T \end{pmatrix} = 1 + a_1 T + \cdots + a_n T^n$$

(Computation methods: 1. the traditional computation. 2. Note that if A is invertible,

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B).$$

In this case $A^{-1} = \begin{pmatrix} 1 & & & & \\ T & 1 & & & \\ \vdots & \ddots & \ddots & & \\ T^{n-3} & \cdots & T & 1 & \\ T^{n-2} & T^{n-3} & \cdots & T & 1 \end{pmatrix}$)

Then we can also conclude that $\mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$.

Remark 1.24. Why is a general elment of the form (R^n, α_n)? Namely how to reduce an endomorphism to a rational canonical form ?

Now we want to check some identities

$$\mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n \nu)] \quad \text{by definition}$$
$$(R^{n+1}, \alpha_{n+1}(a_1, \cdots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)] \quad \text{compute under } \chi$$
$$(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)] \quad \text{where } \beta = \alpha_n(a_1\nu, \cdots, a_n\nu^n)$$

In fact, the last identity always holds when $R = \mathbb{Z}[a_1, \cdots, a_n]$. β is nilpotent because $\beta = \alpha_n \nu$.

We only show how to check the last equation: only need to show that

$$\alpha_n \nu = \alpha_n(a_1\nu, \cdots, a_n\nu^n)$$

$$LHS = \begin{pmatrix} 0 & & & & -a_n\nu \\ \nu & 0 & & & -a_{n-1}\nu \\ & \ddots & \ddots & & \vdots \\ & & \nu & 0 & -a_2\nu \\ & & & \nu & -a_1\nu \end{pmatrix}, \quad RHS = \begin{pmatrix} 0 & & & & -a_n\nu^n \\ 1 & 0 & & & -a_{n-1}\nu^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2\nu^2 \\ & & & 1 & -a_1\nu \end{pmatrix}$$

we can check this using the characteristic polynomial since χ is injective: check

$$\det(1 - \alpha_n \nu T) = \det(1 - \alpha_n(a_1\nu, \cdots, a_n\nu^n)T)$$

$$LHS = \det \begin{pmatrix} 1 & & & & a_n\nu T \\ -\nu T & 1 & & & a_{n-1}\nu T \\ & \ddots & \ddots & & \vdots \\ & & -\nu T & 1 & a_2\nu T \\ & & & -\nu T & 1 + a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n)$$

$$RHS = \det \begin{pmatrix} 1 & & & & a_n\nu^n T \\ -T & 1 & & & a_{n-1}\nu^{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2\nu^2 T \\ & & & -T & 1+a_1\nu T \end{pmatrix} = \det(1 + a_1\nu T + \cdots + a_n\nu^n T^n).$$

Note that if $\exists N$ such that $\nu^N = 0$, β is independent of the a_i for $i \geq N$. If $\nu^N = 0$ then $\alpha_n \otimes \nu$ represents 0 in $\mathrm{Nil}_0(\Lambda)$ whenever $\chi(\alpha_n) \equiv 1 \bmod t^N$.

More operations Let $F_n\mathbf{Nil}(\Lambda)$ denote the full exact subcategory of $\mathbf{Nil}(\Lambda)$ on the (P,ν) with $\nu^n = 0$. If Λ is an algebra over a commutative ring R, the kernel $F_n\mathrm{Nil}_0(\Lambda)$ of $K_0(F_n\mathbf{Nil}(\Lambda)) \longrightarrow K_0(\mathbf{P}(\Lambda))$ is an $\mathrm{End}_0(R)$-module and $F_n\mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$ is a module map.

The exact endofunctor $F_m\colon (P,\nu) \mapsto (P,\nu^m)$ on $\mathbf{Nil}(\Lambda)$ is zero on $F_m\mathbf{Ni1}(\Lambda)$. For $\alpha \in \mathrm{End}_0(R)$ and $(P,\nu) \in \mathrm{Nil}_0(\Lambda)$, nota that $(V_m\alpha) * (P,\nu) = V_m(\alpha * F_m(P,\nu))$, and we can conclude that $V_m\mathrm{End}_0(R)$ acts trivially on the image of $F_m\mathrm{Nil}_0(\lambda)$ in $\mathrm{Nil}_0(\lambda)$. For more details, see Weibel, K-book chapter 2, pp 155 Exercise II.7.17.

1.3.4 $W(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Once we have a nilpotent endomorphism, we can pass to infinity.

Theorem 1.25. *$\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$ extends to a $W(R)$-module structure by the formula*

$$(1 + \sum a_i T^i) * [(P,\nu)] = (R^n, \alpha_n(a_1, \cdots, a_n)) * [(P,\nu)],\ n \gg 0.$$

1.3.5 $W(R)$-module structure on $\mathrm{Nil}_*(\Lambda)$

The induced t-adic topology on $\mathrm{End}_0(R)$ is defined by the ideals

$$I_N = \{f \in \mathrm{End}_0(R) \mid \chi(f) \equiv 1 \bmod t^N\},\ I_N \supset I_{N+1},$$

and $\mathrm{End}_0(R)$ is separated (i.e. $\cap I_N = 0$) in this topology. The key fact is:

Theorem 1.26 (Almkvist)**.** *The map $\chi\colon \mathrm{End}_0(R) \longrightarrow W(R)$ is a ring injection, and $W(R)$ is the t-adic completion of $\mathrm{End}_0(R)$, i.e. $W(R) = \varprojlim \mathrm{End}_0(R)/I_N$.*

Theorem 1.27 (Stienstra)**.** *For every $\gamma \in \mathrm{Nil}_*(\Lambda)$ there is an N so that γ is annihilated by the ideal*

$$I_N = \{f \mid \chi(f) \equiv 1 \bmod t^N\} \subset \mathrm{End}_0(R).$$

Consequently, $NK_*(\Lambda)$ is a module over the t-adic completion $W(R)$ of $\mathrm{End}_0(R)$.

Recall the sign convention:

$$NK_1(\Lambda) \cong \mathrm{Nil}_0(\Lambda)$$
$$[1-\nu y] \leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)]$$

The $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula

$$\alpha(t) * [1-\nu y] = [\alpha(\nu y)].$$

And the $W(R)$-module structure on $NK_n(\Lambda)$

$$\alpha(t) * \{\gamma, 1-\nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda), \quad \gamma \in K_{n-1}(R).$$

1.3.6 Modern version

Reference: Weibel, K-book, chapter 4, pp. 58.

1.4 Some results

Proposition 1.28. *If R is $S^{-1}\mathbb{Z}, \hat{\mathbb{Z}}_p$ or $\mathbb{Q}$-algebra, then*

$$\lambda_t \colon R \longrightarrow W(R)$$
$$r \mapsto (1-t)^r$$

is a ring injection.

Corollary 1.29. *Fix an integer p and a ring Λ with 1.*
(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.
(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ)-module.
(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.
(d) If $p^m = 0$ in Λ, $NK_*(\Lambda)$ is a p-group.

Theorem 1.30 (Stienstra)**.** *If $0 \neq n \in \mathbb{Z}$, $NK_1(R)[\frac{1}{n}] \cong NK_1(R[\frac{1}{n}])$.*

Corollary 1.31. [1] *If G is a finite group of order n, then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n. In fact, $NK_*(\mathbb{Z}[G])$ is an n-torsion group, and $Z_{(p)} \otimes NK_*(\mathbb{Z}[G]) = NK_*(\mathbb{Z}_{(p)}[G])$, where $p \mid n$.*

[1] Weibel, K-book chapter3, page 27.

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