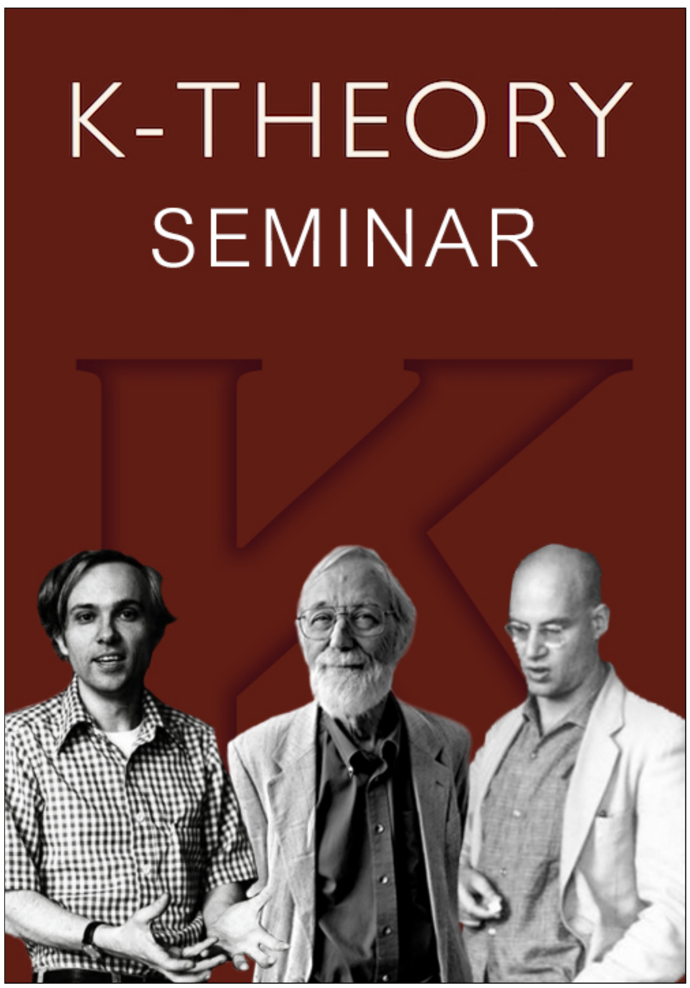
Notes on Algebraic K-theory

Hao Zhang



Quillen Milnor Grothendieck

Contents

Chapter 1

Notes on Higher K-theory of group-rings of virtually infinite cyclic groups

1.1 Introduction

Authors: Aderemi O. Kuku and Guoping Tang

1.1.1 Preliminaries

Definition 1.1 (Virtually cyclic groups)**.** A discrete group V is called virtually cyclic if it contains a cyclic subgroup of finite index, i.e., if V is finite or virtually infinite cyclic.

Virtually infinite cyclic groups are of two types:

1 $V = G \rtimes_\alpha T$ is a semi-direct product where G is a finite group, $T = \langle t \rangle$ an infinite cyclic group generated by t, $\alpha \in Aut(G)$, and the action of T is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

2 V is a non-trivial amalgam of finite groups and has the form $V = G_0 *_H G_1$ where $[G_0 : H] = 2 = [G_1 : H]$.

We denote by $\mathcal{VCYC}$ the family of virtually cyclic subgroups of G.

$$\text{virtually cyclic groups} \begin{cases} \text{finite groups} \\ \text{virtually infinite cyclic groups} \begin{cases} \text{I.} V = G \rtimes_\alpha T, G \text{ is finite}, T = \langle t \rangle \cong \mathbb{Z} \\ \text{II.} V = G_0 *_H G_1, H \text{is finite}, [G_i : H] = 2 \end{cases} \end{cases}$$

Let G be a finite group, V be a group such that $1 \to G \to V \to T \to 1$ is exact，then V is TYPE.I, i.e., $V = G \rtimes_\alpha T$, $\alpha : T \to Aut(G), \alpha(t)(g) = tgt^{-1}$. Multiplication in V [1]:

$$(g_1, t_1)(g_2, t_2) = (g_1\alpha(t_1)(g_2), t_1t_2) = (g_1t_1g_2t_1^{-1}, t_1t_2).$$

若 G 是有限群，V 满足 $V \to D_\infty \to 1$，$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, 则 V 是类型 II。

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & G_0 \arrow[d] \\
G_1 \arrow[r] & G_0 *_H G_1
\end{tikzcd}$$

is a push-out square.

Definition 1.2 (Orders)**.** Let R be a Dedekind domain with quotient field F. An R-order in a F-algebra Σ is a subring Λ of Σ, having the same unity as Σ and s.t. R is contained in the center of Λ, Λ is finitely generated R-module and $F \otimes_R \Lambda = \Sigma$.

A Λ-lattice in Σ is a Λ-bisubmodule of Σ which generates Σ as a F-space.

A maximal R-order Γ in Σ is an order that is not contained in any other R-order in Σ.

Example 1.3. We give some examples:

1. G is a finite group, then RG is an R-order in FG when $ch(F) \nmid |G|$.
2. R is a maximal R-order in F.
3. $M_n(R)$ is a maximal R-order in $M_n(F)$.

Remark 1.4. Any R-order Λ is contained in at least one maximal R-order in Σ. Any semisimple F-algebra Σ contains at least one maximal R-order. However, if Σ is commutative, then Σ contains a unique maximal order, namely, the integral closure of R in Σ.

Theorem 1.5. *R, F, Λ, Σ as above, Then $K_0(\Lambda), G_0(\Lambda)$ are finitely generated abelian groups.*

1.1.2 The Farrell-Jones conjecture

Let G be a discrete group and $\mathcal{F}$ a family of subgroups of G closed under conjugation and taking subgroups, e.g., $\mathcal{VCYC}$.

Let $Or_{\mathcal{F}}(G) := \{G/H | H \in F\}$, R any ring with identity.

There exists a “Davis -Lück”functor

$$\mathbb{K}R : Or_{\mathcal{F}}(G) \longrightarrow Spectra$$

$$G/H \mapsto \mathbb{K}R(G/H) = K(RH)$$

[1] there is a little difference between this note and the original paper about α

where $K(RH)$ is the K-theory spectrum such that $\pi_n(K(RH)) = K_n(RH)$.

There exists a homology theory

$$H_n(-, \mathbb{K}R) : G - CWcomplexes \longrightarrow \mathbb{Z} - Mod$$

$$X \mapsto H_n(X, \mathbb{K}R)$$

Let $E_{\mathcal{F}}(G)$ be a G-CW-complex which is a model for the classifying space of $\mathcal{F}$. Note that $E_{\mathcal{F}}(G)^H$ is homotopic to the one point space (i.e., contractible) if $H \in F$ and $E_{\mathcal{F}}(G)^H = \emptyset$ if $H \notin F$ and $E_{\mathcal{F}}(G)$ is unique up to homotopy.

There exists an assembly map

$$A_{R,\mathcal{F}} : H_n(E_{\mathcal{F}}(G), \mathbb{K}R) \longrightarrow K_n(RG).$$

The Farrell-Jones isomorphism conjecture says that $A_{R,\mathcal{VCYC}} : H_n(E_{\mathcal{VCYC}}(G), \mathbb{K}R) \cong K_n(RG)$ is an isomorphism for all $n \in Z$. Note that KR is the non-connective K-theory spectrum such that $\pi_n(KR)$ is Quillen's $K_n(R), n \geq 0$, and $\pi_n(KR)$ is Bass's negetive $K_n(R)$, for $n \leq 0$.

1.1.3 Notations

- F: number field, i.e, $\mathbb{Q} \subset F$ is a finite field extension.
- R: the ring of integers in F.
- Σ: a semisimple F-algebra.
- Λ: an R-order in Σ, $\alpha : \Lambda \to \Lambda$: an R-automorphism.
- $\Gamma \in \{\alpha$-invariant R-orders in Σ containing $\Lambda\}$ is a maximal element.
- $\max(\Gamma) = \{$two-sided maximal ideals in $\Gamma\}$.
- $\max_\alpha(\Gamma) = \{$two-sided maximal α-invariant ideals in $\Gamma\}$.
- $\mathcal{C}$: exact category, $K_n(\mathcal{C}) = \pi_{n+1}(BQ\mathcal{C}), n \geq 0$. If A is a unital ring, $K_n(A) = K_n(\mathcal{P}(A)), n \geq 0$. When A is noetherian, $G_n(A) = K_n(\mathcal{M}(A))$.
- $T = \langle t \rangle$: infinite cyclic group $\cong \mathbb{Z}$, T^r: free abelian group of rank r.
- $A_\alpha[T] = A_\alpha[t, t^{-1}]$: α-twisted Laurent series ring, $A_\alpha[T] = A[T] = A[t, t^{-1}]$ additively and multiplication given by $(rt^i)(st^j) = r\alpha^i(s)t^{i+j}$.（注：这里和文章有些区别）
- $A_\alpha[t]$: the subgroup of $A_\alpha[T]$ generated by A and t, that is, $A_\alpha[t]$ is the twisted polynomial ring.
- $NK_n(A, \alpha) := \ker(K_n(A_\alpha[t]) \to K_n(A)), n \in \mathbb{Z}$ where the homomorphism is induced by the augmentation $\epsilon : A_\alpha[t] \to A$. If $\alpha = \mathrm{id}$, $NK_n(A, \mathrm{id}) = NK_n(A) = \ker(K_n(A[t]) \to K_n(A))$.

1.1.4 Known results

Next, we focus on higher K-theory of virtually cyclic groups

Theorem 1.6 (A. Kuku). *For all $n \geq 1$,$K_n(\Lambda)$ and $G_n(\Lambda)$ are finitely generated Abelian groups and hence that for any finite group G, $K_n(RG)$ and $G_n(RG)$ are finitely generated.*

见 Kuku, A.O.:K_n, SK_n of integral group-ring and orders. Contemporary Mathematics Part I, 55, 333-338 (1986) 和 Kuku, A.O.:K-theory of group-rings of finite groups over maximal orders in division algebras. J. Algebra 91, 18-31 (1984).

Using the fundamental theorem for G-theory,

$$G_n(\Lambda[t]) = G_n(\Lambda)$$

$$G_n(\Lambda[t, t^{-1}]) = G_n(\Lambda) \oplus G_{n-1}(\Lambda)$$

one gets that:

Corollary 1.7. *For all $n \geq 1$, if C is a finitely generated free Abelian group or monoid, then $G_n(\Lambda[C])$ are also finitely generated.*

Remark 1.8. However we can not draw the same conclusion for $K_n(\Lambda[C])$ since for a ring A, it is known that all the $NK_n(A)$ are not finitely generated unless they are zero. 见 Weibel, C.A.: Mayer Vietoris sequences and module structures on NK_*, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 466-493,Lecture Notes in Math., 854, Springer, Berlin, 1981 的 Proposition 4.1

1.1.5 Main results

Part 1

Theorem 1.9 (1.1). *The set of all two-sided, α-invariant, Γ-lattices in Σ is a free Abelian group under multiplication and has $\max_\alpha(\Gamma)$ as a basis.*

Theorem 1.10 (1.6). *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\alpha : \Lambda \to \Lambda$ is an R-automorphism, then there exists an R-order $\Gamma \subset \Sigma$ such that*
(1) $\Lambda \subset \Gamma$,
(2) Γ is α-invariant, and
(3) Γ is a (right) regular ring. In fact, Γ is a (right) hereditary ring.

后面证明中反复用了这里的 Γ 是一个正则环。这两个定理推广了 Farrell 和 Jones 在文章 The Lower Algebraic K-Theory of Virtually Infinite Cyclic Groups.K-Theory 9, 13-30 (1995) 中的定理 1.5 和定理 1.2

Theorem 1.11 (Farrell-Jones 文章中的定理 1.5)**.** *The set of all two-sided, α-invariant, A-lattices in $\mathbb{Q}G$ is a free Abelian group under multiplication and has* $\max_\alpha(A)$ *as a basis.*

Theorem 1.12 (Farrell-Jones 文章中的定理 1.2)**.** *Given a finite group G and an automorphism $\alpha : G \to G$, then there exists a $\mathbb{Z}$-order $A \subset \mathbb{Q}G$ such that*
(1) $\mathbb{Z}G \subset A$,
(2) A is α-invariant, and
(3) A is a (right) regular ring, in fact, A is a (right) hereditary ring.

第一节的结论来源于 Farrell 和 Jones 在其文章中的结论，将 $\mathbb{Z}$ 和 $\mathbb{Q}$ 的陈述推广到数域 F 和代数整数环 R 上，并且把之前的群环 $\mathbb{Q}G$ 推广为任何半单 F 代数 Σ。

Part 2 定理 2.1 中的方法是讲过的，关键一步是证两个范畴是自然等价。（文中有笔误:718 页第一行 mt^n 应为 xt^n, 后面所谓 m_i 应为 x_i，另有一处 Hom 所在的范畴不在 $\mathcal{B}$, 应在 $\mathcal{M}(A_\alpha[T])$）

Theorem 1.13 (2.2)**.** *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, α an automorphism of Λ. Then*

(a) For all $n \geq 0$

(i) $NK_n(\Lambda, \alpha)$ is s-torsion for some positive integer s. Hence the torsion free rank of $K_n(\Lambda_\alpha[t])$ is the torsion free rank of $K_n(\Lambda)$ and is finite.

If $n \geq 2$, then the torsion free rank of $K_n(\Lambda_\alpha[t])$ is equal to the torsion free rank of $K_n(\Sigma)$.

(ii) If G is a finite group of order r, then $NK_n(RG, \alpha)$ is r-torsion, where α is the automorphism of RG induced by that of G.

对第一类 virtually infinite cyclic groups 的结论:

(b) Let $V = G \rtimes_\alpha T$ be the semi-direct product of a finite group G of order r with an infinite cyclic group $T = \langle t \rangle$ with respect to the automorphism $\alpha : G \longrightarrow G, g \mapsto tgt^{-1}$. Then
(i) $K_n(RV) = 0$ for all $n < -1$.
(ii) The inclusion $RG \hookrightarrow RV$ induces an epimorphism $K_{-1}(RG) \twoheadrightarrow K_{-1}(RV)$. Hence $K_{-1}(RV)$ is finitely generated Abelian group.
(iii) For all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group.
(iv) $NK_n(RV)$ is r-torsion for all $n \geq 0$.

第 3 节 对第二类 virtually infinite cyclic groups 的结论:

Theorem 1.14 (3.2)**.** *If R is regular, then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$. If R is quasi-regular then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \leq 0$.*

Theorem 1.15 (3.3)**.** *Let V be a virtually infinite cylic group in the second class having the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. Then the Nil-groups*

$NK_n(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_1-H])$ defined by the triple $(\mathbb{Z}H;\mathbb{Z}[G_0-H],\mathbb{Z}[G_1-H])$ are $|H|$-torsion when $n \geq 0$ and 0 when $n \leq -1$.

1.2 K-theory for the first type of virtually infinite cyclic groups

我们首先回顾 Farrell 和 Jones 在文章中的做法：

原型 G: finite group, $|G| = q$, $\mathbb{Z}G$ is a $\mathbb{Z}$-order in $\mathbb{Q}G$, then there exists a regular ring $A \subset \mathbb{Q}G$ which is a $\mathbb{Z}$-order, and we have[2] $qA \subset \mathbb{Z}G$.

Hence, we have the following Cartesian square

$$\begin{tikzcd}
\mathbb{Z}G \arrow[r] \arrow[d] & A \arrow[d] \\
\mathbb{Z}G/qA \arrow[r] & A/qA
\end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd}
\mathbb{Z}(G \rtimes_\alpha T) \arrow[r] \arrow[d] & A_\alpha[T] \arrow[d] \\
(\mathbb{Z}G/qA)_\alpha[T] \arrow[r] & (A/qA)_\alpha[T]
\end{tikzcd}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

Definition 1.16. A ring R is quasi-regular if it contains a two-sided nilpotent ideal N such that R/N is right regular.

重要的结论是

Prop1.1 If R is a (right) regular, $\alpha : R \longrightarrow R$ an automorphism, then $R_\alpha[t], R_\alpha[T] = R_\alpha[t,t^{-1}]$ are also (right) regular.

Prop1.4 $\mathbb{Z}G/qA, A/qA, (\mathbb{Z}G/qA)_\alpha[T], (A/qA)_\alpha[T]$ are all quasi-regular[3].

即得到的方块右上角是 regular ring，下方是 quasi-regular ring。于是得到 $K_n(\mathbb{Z}(G \rtimes_\alpha T)) = 0, n \leq 2$ 且有 $K_{-1}(\mathbb{Z}G) \twoheadrightarrow K_{-1}(\mathbb{Z}(G \rtimes_\alpha T))$ 是满射。

[2] 参考 Reiner, I.: Maximal Orders 中定理 41.1: $n = |G|$, Γ is an R-order in FG containing RG, then $RG \subset \Gamma \subset n^{-1}RG$ when $\mathrm{ch}(F) \nmid n$.

[3] If S is a quasi-regular ring, then $K_{-n}(S) = 0$.(正确不？)

推广到数域 F 和代数整数环 R G: finite group, $|G| = s$, $\Lambda = RG$ is a R-order in $\Sigma = FG$, then there exists a regular ring $\Gamma \subset \Sigma = FG$ which is a R-order, and we have $s\Gamma \subset RG$.

Hence, we have the following Cartesian square

$$\begin{tikzcd} RG \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ RG/s\Gamma \arrow[r] & \Gamma/s\Gamma \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd} R(G \rtimes_\alpha T) \arrow[r] \arrow[d] & \Gamma_\alpha[T] \arrow[d] \\ (RG/s\Gamma)_\alpha[T] \arrow[r] & (\Gamma/s\Gamma)_\alpha[T] \end{tikzcd}$$

于是可以分别得到 Mayer-Vietoris 正合序列。

对应到这里来$\Gamma, \Gamma_\alpha[T]$ 是正则环，$RG/s\Gamma, \Gamma/s\Gamma, (RG/s\Gamma)_\alpha[T], (\Gamma/s\Gamma)_\alpha[T]$ 是 quasi-regular rings.

群环推广到半单代数 考虑 $\Lambda \subset \Gamma \subset \Sigma$ 分别是 R-order, 正则环，半单 F-代数，则存在正整数 s 使得 $\Lambda \subset \Gamma \subset \Lambda(1/s)$, 令 $q = s\Gamma$

Hence, we have the following Cartesian square

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \Lambda/q \arrow[r] & \Gamma/q \end{tikzcd}$$

Since α induces automorphisms of all 4 rings in this square, we have another cartesian square

$$\begin{tikzcd} \Lambda_\alpha[t] \arrow[r] \arrow[d] & \Gamma_\alpha[t] \arrow[d] \\ (\Lambda/q)_\alpha[t] \arrow[r] & (\Gamma/q)_\alpha[t] \end{tikzcd}$$

写出 MV 序列后每项均 $\otimes\mathbb{Z}[1/s]$ 仍然正合[4]，再分别取核得到 Nil 群的长正合列。

$\Gamma, \Gamma_\alpha[t]$ regular $\Longrightarrow NK_n(\Gamma, \alpha) = 0$.

$\Lambda/q, \Gamma/q, (\Lambda/q)_\alpha[t], (\Gamma/q)_\alpha[t]$ are all quasi-regular.

Remark 1.17. Farrell，Jones 文章中四个环是 quasi-regular 的结论证明中用到了 Artinian 性质，从而可以推广到这篇文章所讨论的情形。

[4] 文献 [16] 中是对素数 p 的陈述，对于一般的整数是否成立？

一些注记：1.A: finite, $J(A)$: its Jacobson radical, why is $A/J(A)$ regular？因为是有限环
2.720 页第四行的文献应为 [16], 引用的结论为 “I is a nilpotent ideal in a $\mathbb{Z}/p^m\mathbb{Z}$-algebra Λ with unit, then $K_*(\Lambda, I)$ is a p-group”, 这个结论对一般的正整数 s 成立。同样地在 719 页得到序列 (III) 时同样参考 [16] 里的结论以及在 721 页倒数第 8 行所引用的 [16]Cor 3.3(d) 中的 p 对任何正整数成立。

原文中 “By [9] the torsion free rank of $K_n(\Lambda)$ is finite and if $n \geq 2$ the torsion free rank of $K_n(\Sigma)$ is the torsion free rank of $K_n(\Lambda)$ (see [12])” 引用的参考文献为

[9] van der Kallen, W.: Generators and relations in algebraic K-theory, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305-310, Acad. Sci.Fennica, Helsinki, 1980

[12] Kuku, A.O.: Ranks of K_n and G_n of orders and group rings of finite groups over integers in number fields. J. Pure Appl. Algebra 138, 39-44 (1999)
但在 [9] 中并未找到相应结论。

另外文献 [10] 在网上未找到电子文档。

Open problem: What is the rank of $K_{-1}(RV)$?

1.3 Nil-groups for the second type of virtually infinite cyclic groups

范畴 $\mathcal{T}$: 对象为 $\mathbf{R} = (R; B, C)$，其中 R 是环，B, C 是 R-双模，态射为 $(\phi, f, g) : (R, B, C) \to (S, D, E)$，其中 $\phi : R \to S$ 是环同态，$f : B \otimes_R S \to D$ 与 $g : C \otimes_R S \to E$ 是 R-S 双模同态。

$$\rho : \mathcal{T} \longrightarrow Rings$$

$$\rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(C \otimes_R B) & C \otimes_R T_R(B \otimes_R C) \\ B \otimes_R T_R(C \otimes_R B) & T_R(B \otimes_R C) \end{pmatrix}$$

If M is an R-module, then its tensor algebra $T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots$.

$$\varepsilon : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

$$NK_n(\mathbf{R}) := \ker(K_n(R_\rho) \to K_n \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix})$$

Let V be a group in the second class of the form $V = G_0 *_H G_0$ where the groups $G_i, i = 0, 1$,and H are finite and $[G_i : H] = 2$. Considering $G_i - H$ as the right coset of H in G_i which is different from H, the free $\mathbb{Z}$-module $\mathbb{Z}[Gi - H]$ with basis $G_i - H$nis a $\mathbb{Z}H$-bimodule which is isomorphic to $\mathbb{Z}H$ as a left $\mathbb{Z}H$-module, but the right action is twisted by an automorphism of $\mathbb{Z}H$ induced by an automorphism of H. Then the Waldhausen's Nil-groups are defined to be $NK_n(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_0 - H])$ using the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_0 - H])$. This inspires us to consider the following general case. Let R be a ring with identity and $\alpha : R \longrightarrow R$ a ring auto-morphism. We denote by R^α the R-bimodule which is R as a left R-module but with right multiplication given by $a \cdot r = a\alpha(r)$. For any automorphisms α and β of R , we consider the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$. We will prove that $\rho(\mathbf{R})$ is in fact a twisted polynomial ring and this is important for later use.

Theorem 1.18 (3.1)**.** *Suppose that α and β are automorphisms of R. For the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$, let R_ρ be the ring $\rho(\mathbf{R})$, and let γ be a ring automorphism of $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ defined by*

$$\gamma : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} \beta(b) & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Then there is a ring isomorphism

$$\mu : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x].$$

加法群同构是显然的，只需验证乘法同态。

利用这个结论将这种形式的 Nil 群转化为 Farrell Nil 群，利用已知的命题来证明结论。如正则环的 NK_n 为 0, 拟正则环的 NK_n 当 $n \le 0$ 时为 0.

当我们接下来研究 $R = \mathbb{Z}H, h = |H|$ 时，取一个 regular order Γ，我们有相应的 4 triples，于是得到 4 个 twisted polynomial rings $R_\rho, \Gamma_\rho; (R/h\Gamma)_\rho, (\Gamma/h\Gamma)_\rho$.

之前第二节的方块

$$\begin{tikzcd} RG \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ RG/s\Gamma \arrow[r] & \Gamma/s\Gamma \end{tikzcd}$$

在这里 (之前的 R, G, s 换成 $\mathbb{Z}, H, h$) 变成了 (注意这里 $R = \mathbb{Z}H$)

$$\begin{tikzcd} R = \mathbb{Z}H \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \mathbb{Z}H/h\Gamma \arrow[r] & \Gamma/h\Gamma \end{tikzcd}$$

从而有

$$\begin{tikzcd} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \arrow[r] \arrow[d] & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \arrow[d] \\ \begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix} \arrow[r] & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix} \end{tikzcd}$$

接着有

$$\begin{array}{ccc}
\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x] & \longrightarrow & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}_\gamma [x] \\
\downarrow & & \downarrow \\
\begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix}_\gamma [x] & \longrightarrow & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix}_\gamma [x]
\end{array}$$

而这个方块恰好是

$$\begin{array}{ccc}
R_\rho & \longrightarrow & \Gamma_\rho \\
\downarrow & & \downarrow \\
(R/h\Gamma)_\rho & \longrightarrow & (\Gamma/h\Gamma)_\rho
\end{array}$$

证明中使用了 $n \leq -1$ 时 quasi-regular ring 的 NK_n 为 0.

Remark 1.19. 722 页中间参考文献 [3] 未找到 augmentation map。另外这里把 f, g 是双模同态在原文基础上进行了修改。

726 页第 8 行“(2) and (3)”应为“(3) and (4)”。

Chapter 2

Witt vectors and NK-groups

References:

part 1 J. P. Serre, Local fields.

part 1 Daniel Finkel, An overview of Witt vectors.

part 2 Hendrik Lenstra, Construction of the ring of Witt vectors.

part 2 Barry Dayton, Witt vectors, the Grothendieck Burnside ring, and Necklaces.

part 3 C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, pp. 466493 in Lecture Notes in Math. 854, Springer-Verlag, 1981.

part 3 D. R. Grayson, Grothendieck rings and Witt vectors.

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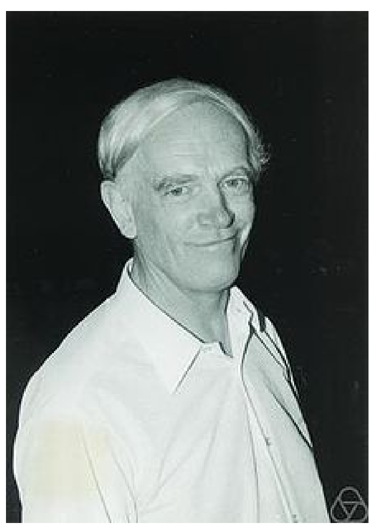

图 2.1: Ernst Witt

Ernst Witt Ernst Witt (June 26, 1911–July 3, 1991) was a German mathematician, one of the leading algebraists of his time. Witt completed his Ph.D. at the University of Göttingen in 1934 with Emmy Noether. Witt's work has been highly influential. His invention of the Witt vectors clarifies and generalizes the structure of the p-adic numbers. It has become fundamental to p-adic Hodge theory. For more information, see `https://en.wikipedia.org/wiki/Ernst_Witt` and `http://www-history.mcs.st-andrews.ac.uk/Biographies/Witt.html`.

2.1 p-Witt vectors

In this section we introduce p-Witt vectors. Witt vectors generalize the p-adics and we will see all p-Witt vectors over any commutative ring form a ring.

From now on, fix a prime number p.

Definition 2.1. A p-Witt vector over a commutative ring R is a sequence $(X_0, X_1, X_2, \cdots)$ of elements of R.

Remark 2.2. If $R = \mathbb{F}_p$, any p-Witt vector over $\mathbb{F}_p$ is just a p-adic integer $a_0 + a_1 p + a_2 p^2 + \cdots$ with $a_i \in \mathbb{F}_p$.

We introduce Witt polynomials in order to define ring structure on p-Witt vectors.

Definition 2.3. Fix a prime number p, let $(X_0, X_1, X_2, \cdots)$ be an infinite sequence of indeterminates. For every $n \geq 0$, define the n-th Witt polynomial

$$W_n(X_0, X_1, \cdots) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

For example, $W_0 = X_0,\ W_1 = X_0^p + pX_1,\ W_2 = X_0^{p^2} + pX_1^p + p^2 X_2.$

Question: how can we add and multiple Witt vectors?

Theorem 2.4. *Let $(X_0, X_1, X_2, \cdots)$, $(Y_0, Y_1, Y_2, \cdots)$ be two sequences of indeterminates. For every polynomial function $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \cdots, \varphi_n, \cdots)$ of elements of $\mathbb{Z}[X_0, \cdots, X_n, \cdots; Y_0, \cdots, Y_n, \cdots]$ such that*

$$W_n(\varphi_0, \cdots, \varphi_n, \cdots) = \Phi\big(W_n(X_0, \cdots), W_n(Y_0, \cdots)\big),\ n = 0, 1, \cdots.$$

If $\Phi = X + Y$(resp. XY), then there exist $(S_1, \cdots, S_n, \cdots)$("$S$" stands for sum) and $(P_1, \cdots, P_n, \cdots)$ ("P" stands for product) such that

$$W_n(X_0, \cdots, X_n, \cdots) + W_n(Y_0, \cdots, Y_n, \cdots) = W_n(S_1, \cdots, S_n, \cdots),$$

$$W_n(X_0, \cdots, X_n, \cdots)W_n(Y_0, \cdots, Y_n, \cdots) = W_n(P_1, \cdots, P_n, \cdots).$$

Let R be a commutative ring, if $A = (a_0, a_1, \cdots) \in R^{\mathbb{N}}$ and $B = (b_0, b_1, \cdots) \in R^{\mathbb{N}}$ are p-Witt vectors over R, we define

$$A + B = (S_0(A, B), S_1(A, B), \cdots), \quad AB = (P_0(A, B), P_1(A, B), \cdots).$$

Theorem 2.5. *The p-Witt vectors over any commutative ring R form a commutative ring under the compositions defined above (called the ring of p-Witt vectors with coefficients in R, denoted by $W(R)$).*

Example 2.6. We have

$$\begin{array}{cc} S_0(A, B) = a_0 + b_0 & P_0(A, B) = a_0 b_0 \\ S_1(A, B) = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p} & P_1(A, B) = b_0^p a_1 + a_0^p b_1 + p a_1 b_1 \end{array}$$

For more computations, see MO 92750

Theorem 2.7. *There is a ring homomorphism*

$$W_* \colon W(R) \longrightarrow R^{\mathbb{N}}$$
$$(X_0, X_1, \cdots, X_n, \cdots) \mapsto (W_0, W_1, \cdots, W_n, \cdots)$$

Proof. Only need to check this map is a ring homomorphism: since

$$A + B = (S_0(A, B), S_1(A, B), \cdots), \quad AB = (P_0(A, B), P_1(A, B), \cdots),$$

by definition we have

$$\begin{aligned} W(A) + W(B) &= (W_0(A) + W_0(B), W_1(A) + W_1(B), \cdots) \\ &= \big(W_0(S_0(A, B), S_1(A, B), \cdots), W_1(S_0(A, B), S_1(A, B), \cdots), \cdots\big) \\ &= W(S_0(A, B), S_1(A, B), \cdots) = W(A + B). \end{aligned}$$

And similarly,

$$\begin{aligned} W(A)W(B) &= (W_0(A)W_0(B), W_1(A)W_1(B), \cdots) \\ &= \big(W_0(P_0(A, B), P_1(A, B), \cdots), W_1(P_0(A, B), P_1(A, B), \cdots), \cdots\big) \\ &= W(P_0(A, B), P_1(A, B), \cdots) = W(AB). \end{aligned}$$

Indeed, we only need to show $W_n(A) + W_n(B) = W_n(A + B)$ and $W_n(A)W_n(B) = W_n(AB)$ which are obviously true. (实际上就是为了使得这个是同态而定义出了 $A + B$ 和 AB。) $\square$

Example 2.8. 1. If p is invertible in R, then $W(R) = R^{\mathbb{N}}$ — the product of countable number of R.(if p is invertible the homomorphism W_* is an isomorphism.)

2. $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$ — the ring of p-adic integers.
3. $W(\mathbb{F}_{p^n})$ is an unramified extension of the ring of p-adic integers.

Note that the functions P_k and S_k are actually only involve the variables of index $\leqslant k$ of A and B. In particular if we truncate all the vectors at the k-th entry, we can still add and multiply them.

Definition 2.9. Truncated p-Witt ring $W_k(R) = \{(a_0, a_1, \cdots, a_{k-1}) | a_i \in R\}$ (also called the ring of Witt vectors of length k.)

Example 2.10. $W_1(R) = R$, $W(R) = \varprojlim W_k(R)$. Since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, $W(\mathbb{F}_p) = \mathbb{Z}_{(p)}$.

Definition 2.11. We define two special maps as follows

- The "shift" map $V \colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$, this map is *additive*.
- When $char(R) = p$, the "Frobenius" map $F \colon W(R) \longrightarrow W(R)$, $(a_0, a_1, \cdots) \mapsto (a_0^p, a_1^p, \cdots)$, this is indeed a ring homomorphism.

Firstly, we note that $W_k(R) = W(R)/V^k W(R)$, and if we consider $V \colon W_n(R) \hookrightarrow W_{n+1}(R)$ there are exact sequences

$$0 \longrightarrow W_k(R) \xrightarrow{V^r} W_{k+r}(R) \longrightarrow W_r(R) \longrightarrow 0, \quad \forall k,\, r.$$

The map $V \colon W(R) \longrightarrow W(R)$ is additive: for it suffices to verify this when p is invertible in R, and in that case the homomorphism $W_* \colon W(R) \longrightarrow R^{\mathbb{N}}$ transforms V into the map which sends $(w_0, w_1, \cdots)$ to $(0, pw_0, pw_1, \cdots)$.

$$\begin{tikzcd}
W(R) \arrow[r, "V"] \arrow[d, "W_*"] & W(R) \arrow[d, "W_*"] \\
R^{\mathbb{N}} \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
(a_0, a_1, \cdots) \arrow[r, mapsto, "V"] \arrow[d, mapsto, "W_*"] & (0, a_0, a_1, \cdots) \arrow[d, mapsto, "W_*"] \\
(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \cdots) \arrow[r, mapsto] \arrow[d, equal] & (0, pa_0, pa_0^p + p^2 a_1, \cdots) \arrow[d, equal] \\
(w_0, w_1, w_2, \cdots) \arrow[r, mapsto] & (0, pw_0, pw_1, \cdots)
\end{tikzcd}$$

If $x \in R$, define a map

$$\begin{aligned} r \colon R &\longrightarrow W(R) \\ x &\mapsto (x, 0, \cdots, 0, \cdots) \end{aligned}$$

When p is invertible in R, W_* transforms r into the mapping that $x \mapsto (x, x^p, \cdots, x^{p^n}, \cdots)$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\mathrm{id}"] & W(R) \arrow[d, "W_*"] \\
R \arrow[r] & R^{\mathbb{N}}
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, equal] & (x, 0, \cdots, 0, \cdots) \arrow[d, mapsto, "W_*"] \\
x \arrow[r, mapsto] & (x, x^p, \cdots, x^{p^n}, \cdots)
\end{tikzcd}$$

One deduces by the same reasoning as above the formulas:

Proposition 2.12.

$$\begin{aligned}
r(xy) &= r(x)r(y),\ x, y \in R \\
(a_0, a_1, \cdots) &= \sum_{n=0}^{\infty} V^n(r(a_n)),\ a_i \in R \\
r(x)(a_0, \cdots) &= (xa_0, x^p a_1, \cdots, x^{p^n} a_n, \cdots),\ x_i, a_i \in R.
\end{aligned}$$

Proof. The first formula: put $r(x)r(y),\ r(xy)$ to $R^{\mathbb{N}}$, we get $(x, x^p, \cdots, x^{p^n}, \cdots)(y, y^p, \cdots, y^{p^n}, \cdots)$ and $(xy, (xy)^p, \cdots, (xy)^{p^n}, \cdots)$.
The second formula: put $(a_0, a_1, \cdots)$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \cdots)$
consider $V^i(r(a_i))$: put $r(a_i)$ to $R^{\mathbb{N}}$, we get $(a_i, a_i^p, \cdots, a_i^{p^n}, \cdots) \in R^{\mathbb{N}}$, and W_* transforms V to the mapping $(w_0, w_1, \cdots, w_n, \cdots) \mapsto (0, pw_0, \cdots, pw_{n-1}, \cdots)$,
now we put $(r(a_0))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p, \cdots, a_0^{p^n}, \cdots)$
put $V^1(r(a_1))$ to $R^{\mathbb{N}}$, we get $(0, pa_1, \cdots, pa_1^{p^{n-1}}, \cdots)$
put $V^2(r(a_2))$ to $R^{\mathbb{N}}$, we get $(0, 0, p^2 a_2, \cdots, p^2 a_2^{p^{n-2}}, \cdots)$
put $V^i(r(a_i))$ to $R^{\mathbb{N}}$, we get $(\underbrace{0, 0, \cdots, 0}_{i \text{ terms}}, p^i a_i, \cdots, p^i a_i^{p^{n-i}}, \cdots)$
so put $\sum_n V^n(r(a_n))$ to $R^{\mathbb{N}}$, we get $(a_0, a_0^p + pa_1, \cdots)$.
We leave the proof of the last formula to readers. □

Proposition 2.13.

$$VF = p = FV.$$

Proof. It suffices to check this when R is perfect. Note that a ring R of characteristic p is called perfect if $x \mapsto x^p$ is an automorphism. For more details, we refer to page 44 on Serre's book *Local Fields*. □

2.2 Big Witt vectors

Now we turn to the `big`(universal) Witt vectors. J. P, May once said "This(the theory of Witt vectors) is a strange and beautiful piece of mathematics that every well-educated mathematician should see at least once".

Take the ring of all big vectors of a commutative ring is a functor

$$\mathbf{CRing} \longrightarrow \mathbf{CRing}$$
$$R \mapsto W(R).$$

In this section, R is a commutative ring with unit.

Definition 2.14. The ring of all big Witt vectors in R which also denoted by $W(R)$ is defined as follows,
as a set: $W(R) = \{a(T) \in R[\![T]\!] | a(T) = 1 + a_1 T + a_2 T^2 + \cdots\} = 1 + TR[\![T]\!]$; (we note that as a set $W(R)$ is the kernel of the map $A[\![T]\!]^* \xrightarrow{T \mapsto 0} A^*$)
addition in $W(R)$: usual multiplication of formal power series, sum $a(T)b(T)$, difference $\frac{a(T)}{b(T)}$; $(W(R), +) \cong (1 + TR[\![T]\!], \times)$ which is a subgroup of the group of units $R[\![T]\!]^\times$ of the ring $R[\![T]\!]$
multiplication in $W(R)$: denoted by $*$, this is a little mysterious, we will talk the details later. For the present purposes we only define $*$ as the unique continuous functorial operation for which $(1 - aT) * (1 - bT) = (1 - abT)$.
'zero'(additive identity) of $W(R)$: 1.
'one'(multiplicative identity) of $W(R)$: $[1] = 1 - T$. Note that $[1]$ is the image of $1 \in R$ under the multiplicative (Teichmuller) map

$$R \longrightarrow W(R)$$
$$a \mapsto [a] = 1 - aT$$

functoriality: any homomorphism $f\colon R \longrightarrow S$ induces a ring homomorphism

$$W(f)\colon W(R) \longrightarrow W(S).$$

A quick way to check multiplicative formulas in $W(R)$ is to use the `ghost` map (indeed a ring homomorphism)

$$gh\colon W(R) \longrightarrow R^{\mathbb{N}} = \prod_{i}^{\infty} R.$$

It is obtained from the abelian group homomorphism

$$-T\frac{d}{dT}\log\colon (1 + TR[\![T]\!])^\times \longrightarrow (TR[\![T]\!])^+$$
$$a(T) \mapsto -T\frac{a'(T)}{a(T)}$$

the right side of gh is $R^{\mathbb{N}}$ via $\sum a_n t^n \longleftrightarrow (a_1, a_2, \cdots)$.

A basic principle of the theory of Witt vectors is that to demonstrate certain equations it suffices to check them on vectors of the form $1 - aT$.

2.3 Module structure on NK_*

Notations Λ: a ring with 1

R: commutative ring

$W(R)$: the ring of big Witt vectors of R

End(Λ): the exact category of endomorphisms of finitely generated projective right Λ-modules.

Nil(Λ): the full exact subcategory of nilpotent endomorphisms.

P(Λ): the exact category of finitely generated projective right Λ-modules.

The fundamental theorem in algebraic K-theory states that

$$K_i(\Lambda[t]) \cong K_i(\Lambda) \oplus NK_i(\Lambda) \cong K_i(\Lambda) \oplus \mathrm{Nil}_{i-1}(\Lambda),$$

and hence $\mathrm{Nil}(\Lambda)$ is the obstruction to K-theory being homotopy invariant. By a theorem of Serre, a ring Λ is regular, if and only if every (right) Λ-module has a finite projective resolution. So the resolution theorem and the fact that G-theory is homotopy invariant show that for a regular ring, $NK_*(\Lambda) = \mathrm{Nil}_{*-1}(\Lambda) = 0$. In general, one knows that the groups $\mathrm{Nil}_*(\Lambda)$, if non-zero, are infinitely generated. It is also known that the groups $\mathrm{Nil}_*(\Lambda)$ are modules over the big Witt ring $W(R)$ (just this notes want to show you).

Goals:

- Define the $\mathrm{End}_0(R)$-module structure on $NK_*(\Lambda)$
- (Stienstra's observation) this can extend to a $W(R)$-module structure.
- Computations in $W(R)$ with Grothendieck rings.

2.3.1 $\mathrm{End}_0(\Lambda)$

Let **End**(Λ) denote the exact category of endomorphisms of finitely generated projective right Λ-modules.

Objects: pairs (M, f) with M finitely generated projective and $f \in \mathrm{End}(M)$.

Morphisms: $(M_1, f_1) \xrightarrow{\alpha} (M_2, f_2)$ with $f_2 \circ \alpha = \alpha \circ f_1$, i.e. such α make the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ M_2 & \xrightarrow{f_2} & M_2 \end{array}$$

There are two interesting subcategories of $\mathbf{End}(\Lambda)$ —
$\mathbf{Nil}(\Lambda)$: the full exact subcategory of nilpotent endomorphisms.
$\mathbf{P}(\Lambda)$: the exact category of finitely generated projective right Λ-modules. (Remark: the reflective subcategory of zero endomorphisms is natually equivalent to $\mathbf{P}(\Lambda)$. Note that a full subcategory $i\colon \mathcal{C} \longrightarrow \mathcal{D}$ is called reflective if the inclusion functor i has a left adjoint T, $(T \dashv i)\colon \mathcal{C} \rightleftarrows \mathcal{D}$.)

Since inclusions are split and all the functors below are exact, they induce homomorphisms between K-groups

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{Nil}(\Lambda)$$

$$\mathbf{P}(\Lambda) \rightleftarrows \mathbf{End}(\Lambda)$$

$$M \mapsto (M, 0)$$

$$M \mapsfrom (M, f)$$

Definition 2.15. $K_n(\mathbf{End}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{End}_n(\Lambda)$, $K_n(\mathbf{Nil}(\Lambda)) = K_n(\Lambda) \oplus \mathrm{Nil}_n(\Lambda)$

Now suppose Λ is an R-algebra for some commutative ring R, then there are exact pairings (i.e. bifunctors):

$$\otimes\colon \mathbf{End}(R) \times \mathbf{End}(\Lambda) \longrightarrow \mathbf{End}(\Lambda)$$
$$\otimes\colon \mathbf{End}(R) \times \mathbf{Nil}(\Lambda) \longrightarrow \mathbf{Nil}(\Lambda)$$
$$(M, f) \otimes (N, g) = (M \otimes_R N, f \otimes g)$$

These induce (use "generators-and-relations" tricks on K_0)

$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{End}(\Lambda)) \longrightarrow K_*(\mathbf{End}(\Lambda))$$
$$K_0(\mathbf{End}(R)) \otimes K_*(\mathbf{Nil}(\Lambda)) \longrightarrow K_*(\mathbf{Nil}(\Lambda))$$

$[(0,0)], [(R,1)] \in K_0(\mathbf{End}(R))$ act as the zero and identity maps.

I think we can fix an element $(M, f) \in \mathbf{End}(R)$, then $(M, f)\otimes$ induces an endofunctor of $\mathbf{End}(\Lambda)$. We can get endomorphisms of K-groups, then we check that this does not depent on the isomorphism classes and the bilinear property. (Can also see Weibel The K-book chapter2, chapter3 Cor 1.6.1, Ex 5.4, chapter4 Ex 1.14.)

If we take $R = \Lambda$, we see that $K_0(\mathbf{End}(R))$ is a commutative ring with unit $[(R, 1)]$. $K_0(R)$ is an ideal, generated by the idempotent $[(R, 0)]$, and the quotient ring is $\mathrm{End}_0(R)$. Since $(R, 0)\otimes$ reflects $\mathbf{End}(\Lambda)$ into $\mathbf{P}(\Lambda)$,

$$i\colon \mathbf{P}(\Lambda) \longrightarrow \mathbf{End}(\Lambda); \quad (R, 0) \otimes -\colon \mathbf{End}(\Lambda) \longrightarrow \mathbf{P}(\Lambda)$$

$K_0(R)$ acts as zero on $\mathrm{End}_*(\Lambda)$ and $\mathrm{Nil}_*(\Lambda)$. (Consider $P \in \mathbf{P}(R)$ acts on $\mathbf{End}(\Lambda)$, $(P,0) \otimes (N,g) = (P \otimes_R N, 0) \in \mathbf{P}(\Lambda)$.)

The following is immediate (and well-known):

Proposition 2.16. *If Λ is an R-algebra with* 1, $\mathrm{End}_*(\Lambda)$ *and* $\mathrm{Nil}_*(\Lambda)$ *are graded modules over the ring* $\mathrm{End}_0(R)$.

Now we focus on $* = 0$ and $\Lambda = R$:

The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $f = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \longrightarrow K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$.

Theorem 2.17 (Almkvist)**.** *The homomorphism (in fact it is a ring homomorphism)*

$$\begin{aligned} \chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = (1 + TR[\![T]\!])^{\times} \\ (M,f) &\mapsto \det(1 - fT) \end{aligned}$$

is injective and $\mathrm{End}_0(R) \cong \mathrm{Im}\chi = \left\{ \frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1 + TR[T] \right\}$

The map χ (taking characteristic polynomial) is well-difined, and we have

$$\chi([(R,0)]) = 1, \quad \chi([(R,1)]) = 1 - T$$

χ is a ring homomorphism, and $\mathrm{Im}\chi =$ the set of all rational functions in $W(R)$. Note that

$$\det(1 - fT)\det(1 - gT) = \det(1 - (f \oplus g)T), \quad \det(1 - fT) * \det(1 - gT) = \det(1 - (f \otimes g)T),$$

for more details we refer the reader to S.Lang *Algebra*, Chapter 14, Exercise 15.

Remark 2.18. when R is a algebraically closed field (for instance $\mathbb{C}$), we can use Jordan canonical forms to prove the last identity (i.e. it can be reduced to check that $\prod_i (1 - \lambda_i T) * \prod_j (1 - \mu_j T) = \prod_{i,j}(1 - \lambda_i \mu_j T)$).

Definition 2.19 (NK_*)**.** As above, we define $NK_n(\Lambda) = \ker(K_n(\Lambda[y]) \longrightarrow K_n(\Lambda))$. Grayson proved that $NK_n(\Lambda) \cong \mathrm{Nil}_{n-1}(\Lambda)$ in "Higher algebraic K-theory II". The map is given by

$$NK_n(\Lambda) \subset K_n(\Lambda[y]) \subset K_n(\Lambda[x,y]/xy = 1) \xrightarrow{\partial} K_{n-1}(\mathbf{Nil}(\Lambda)).$$

Thus $NK_n(\Lambda)$ are $\mathrm{End}_0(R)$-modules. For $n \geq 1$, this is just 2.16; for $n = 0$ (and $n < 0$) this follows from the functoriality of the module structure and the fact that $NK_0(\Lambda)$ is the "contracted functor" of $NK_1(\Lambda)$.

Note that $NK_1(\Lambda) \cong K_1(\Lambda[y], (y-\lambda)), \forall \lambda \in \Lambda$, since

$$\Lambda[y] \rightleftarrows \Lambda$$
$$y \mapsto \lambda.$$

Since $[(P,\nu)] = [(P \oplus Q, \nu \oplus 0)] - [(Q,0)] \in K_0(\mathbf{Nil}(\Lambda))$, we see $\mathrm{Nil}_0(\Lambda)$ is generated by elements of the form $[(\Lambda^n,\nu)] - n[(\Lambda,0)]$ for some n and some nilpotent matrix ν Sign convention:

$$NK_1(\Lambda) \cong \mathrm{Nil}_0(\Lambda)$$
$$[1-\nu y] \leftrightarrow [(\Lambda^n,\nu)] - n[(\Lambda,0)]$$

Example 2.20. Let k be a field, $\mathbf{End}(k)$ consists pairs (V,A) with V a finite-dimensional vector space over k and A a k-endomorphism. Two pairs are isomorphic if and only if their minimal polynomials are equal. When we consider $\mathbf{Nil}(k)$, then $K_0(\mathbf{Nil}(k)) \cong \mathbb{Z}$, we conclude that $\mathrm{Nil}_0(k) = 0$. Recall that since k is a regular ring, $NK_*(k) = 0$, we have another proof of $NK_1(k) \cong \mathrm{Nil}_0(k) = 0$.

2.3.2 Grothendieck rings and Witt vectors

We refer the reader to Grayson's paper *Grothendieck rings and Witt vectors*.

Definition 2.21. A λ-ring R is a commutative ring with 1, together with an operation λ_t which assigns to each element x of R a power series

$$\lambda_t(x) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots, \quad x \in R$$

This operation must obey $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

Let R is a commutative ring with unit, $K_0(R) = K_0(\mathbf{P}(R))$ becomes a λ-ring if we define

$$[M][N] = [M \otimes_R N], \quad \lambda_t^n([M]) = [\wedge_R^n M].$$

Recall $M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$, $\wedge^n(M \oplus M') \cong \bigoplus_{i+j=n} \wedge^i M \otimes \wedge^j M'$, and $\wedge^n(M) := M^{\otimes n}/\langle x \otimes x \mid x \in M\rangle$, rank $\wedge^n(M) = \binom{\text{rank } M}{n}$.

For instance, if R is a field, $K_0(R) = \mathbb{Z}$ and $\lambda_t(n) = (1+t)^n = 1 + \binom{n}{1}t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n$, since $\dim(\wedge^i R^n) = \binom{n}{i}$.

We make $K_0(\mathbf{End}(R))$ into a λ-ring by defining

$$\lambda^n([M,f]) = ([\wedge^n M, \wedge^n f]).$$

The ideal generated by the idempotent $[(R,0)]$ is isomorphic to $K_0(R)$, the quotient $\mathrm{End}_0(R)$ is a λ-ring. It is convenient to think of End_0 as a convariant functor on the category of rings, and the functor End_0 satisfies:

1. If $R \longrightarrow S$ is surjective ring homomorphism, then $\mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(S)$ is surjective.
2. If R is an algebraically close field, then the group $\mathrm{End}_0(R)$ is generated by the elements of the form $[(R, r)]$. (This holds because any matrix over R is triagonalizable.)

   Recall

$$\begin{aligned}\chi\colon \mathrm{End}_0(R) &\longrightarrow W(R) = 1 + TR[\![T]\!]\\ (M, f) &\mapsto \det(1 - fT)\end{aligned}$$

$W(R)$ is the underlying (additive) group of the ring of Witt vectors. The λ-ring operations on $W(R)$ are the unique operations which are continuous, functorial in R, and satisfy:

$$\begin{aligned}(1 - aT) * (1 - bT) &= 1 - abT\\ \lambda_t(1 - aT) &= 1 + (1 - aT)t\end{aligned}$$

By 2.17, χ is an injective ring homomorphism whose image consists of all Witt vectors which are quotients of polynomials. In fact χ is a λ-ring homomorphism, so we have

Theorem 2.22. $\mathrm{End}_0(R)$ *is dense sub-λ-ring of* $W(R)$.

The hard part of the theorem is the injectivity. When R is a field the injectivity follows immediately from the existence of the rational canonical form (we can see it below) for a matrix. The result is surprising when R is not a field.

Computation in $W(R)$ Computation in $W(R)$ which is tedious unless we perform it in $\mathrm{End}_0(R)$:

$$(1 - aT^2) * (1 - bT^2) = ?$$

Note that $\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) = \det\begin{pmatrix} 1 & -aT \\ -T & 1 \end{pmatrix} = 1 - aT^2$, $\chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \det\begin{pmatrix} 1 & -bT \\ -T & 1 \end{pmatrix} = 1 - bT^2$,

$$\chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\Big) * \chi\Big(\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Big(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\Big) = \chi\Bigg(\begin{pmatrix} 0 & 0 & 0 & ab \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\Bigg)$$

$$= \det\begin{pmatrix} 1 & 0 & 0 & -aTb \\ 0 & 1 & -bT & 0 \\ 0 & -aT & 1 & 0 \\ -T & 0 & 0 & 1 \end{pmatrix} = 1 - 2abT^2 + a^2b^2T^4.$$

If we use the previous formula

$$(1-rT^m)*(1-sT^n)=(1-r^{n/d}s^{m/d}T^{mn/d})^d,\ d=\gcd(m,n),$$

we obtain the same answer. Indeed we have the formula

$$\det(1-fT)*\det(1-gT)=\det(1-(f\otimes g)T),$$

if a polynomial is $1+a_1T+\cdots+a_nT^n\in W(R)$, we can write

$$f=\begin{pmatrix}0 & & & & -a_n\\ 1 & 0 & & & -a_{n-1}\\ & \ddots & \ddots & & \vdots\\ & & 1 & 0 & -a_2\\ & & & 1 & -a_1\end{pmatrix}\in M_n(R).$$

Operations on $W(R)$ **and** $\mathrm{End}_0(R)$ We have already known that W and End_0 can be regarded as functors from the category of commutative rings to that of (λ-)rings. The following operations $F_n,\ V_n\colon W\implies W$(resp. $\mathrm{End}_0\implies\mathrm{End}_0$) are indeed natural transformation. These auxiliary operations defined on $W(R)$ can also be computed in $\mathrm{End}_0(R)$.

1. the ghost map

$$gh\colon W(R)\xrightarrow{-T\frac{d}{dT}\log}TR[\![T]\!]\cong R^{\mathbb{N}},\quad \alpha(T)\mapsto\frac{-T}{\alpha(T)}\frac{d\alpha}{dT}.$$

and the n-th ghost coordinate

$$gh_n\colon W(R)\longrightarrow R$$

it is the unique continuous natual additive map which sends $1-aT$ to a^n.
Remark. $gh(1-aT)=\frac{aT}{1-aT}=\sum_{i=1}a^iT^i$.The exponential map is defined by

$$R^{\mathbb{N}}\longrightarrow W(R),\quad (r_1,\cdots)\mapsto\prod_{i=1}^{\infty}\exp(\frac{-r_iT^i}{i}).$$

2. the Frobenius endomorphism

$$F_n\colon W(R)\longrightarrow W(R),\quad \alpha(T)\mapsto\sum_{\zeta^n=1}\alpha(\zeta T^{\frac{1}{n}}).$$

it is the unique continuous natual additive map which sends $1-aT$ to $1-a^nT$.
Remark. $F_n(1-aT)=\sum_{\zeta^n=1}(1-a\zeta T^{\frac{1}{n}})=1-a^nT$, since "+" in $W(R)$ is the normal product.

3. the Verschiebung endomorphism

$$V_n \colon W(R) \longrightarrow W(R), \quad \alpha(T) \mapsto \alpha(T^n).$$

it is the unique continuous natual additive map which sends $1 - aT$ to $1 - aT^n$.

ghost map $gh_n \colon W(R) \longrightarrow R$	$1 - aT \mapsto a^n$	
Frobenius endomorphism $F_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - a^n T$	$\alpha(T) \mapsto \sum_{\zeta^n=1} \alpha(\zeta T^{\frac{1}{n}})$
Verschiebung endomorphism $V_n \colon W(R) \longrightarrow W(R)$	$1 - aT \mapsto 1 - aT^n$	$\alpha(T) \mapsto \alpha(T^n)$

We define similar operations on $\mathrm{End}_0(R)$ as follows:

$gh_n \colon \mathrm{End}_0(R) \longrightarrow R$	$[(M, f)] \mapsto \mathrm{tr}(f^n)$
$F_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M, f^n)]$
$V_n \colon \mathrm{End}_0(R) \longrightarrow \mathrm{End}_0(R)$	$[(M, f)] \mapsto [(M^{\oplus n}, v_n f)]$

where $v_n f$ is represented by $\begin{pmatrix} 0 & & & f \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$. The matrix $v_n f$ is close to an n-th root of f.
Another equivalent description is

$$V_n \colon [(M, f)] \mapsto [(M[y]/y^n - f, y)].$$

One easily checks that the operations just defined are additive with respect to short exact sequences in $\mathbf{End}(R)$, and thus are well-defined on $\mathrm{End}_0(R)$.

Since $\mathrm{End}_0(R) \subset W(R)$ is dense and $gh_n,\ F_n,\ V_n$ are continuous, identities among them may be verified on $W(R)$ by checking them on $\mathrm{End}_0(R)$.

$W(R)$	$\longleftrightarrow$	$\mathrm{End}_0(R)$
$gh_n(v * w) = gh_n v * gh_n w$		$\mathrm{tr}((f \otimes g)^n) = \mathrm{tr}(f^n)\mathrm{tr}(g^n)$
$F_n(v * w) = F_n v * F_n w$		$(f \otimes g)^n = f^n \otimes g^n$
$F_n V_n = n$		$(v_n f)^n = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$
$gh_n V_d(v) = \begin{cases} d\, gh_{n/d}(v), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$		$\mathrm{tr}((v_d f)^n) = \begin{cases} d\, \mathrm{tr}(f^{n/d}), \text{ if } d \mid n \\ 0, \text{ if } d \nmid n \end{cases}$

From the last row we may recover the usual expression of the ghost coordinates in terms of the Witt coordinates . The Witt coordinates of a vector v are the coefficients in the expression

$$v = \prod_{i=1} (1 - a_i T^i) = \prod_{i=1} V_i(1 - a_i T).$$

We obtain

$$gh_n(v) = \sum_{d|n} da_d^{n/d}.$$

"Many mordern treatments of the subject of Witt vectors take this latter expression as the starting point of the theory."

The logarithmic derivative of $1 - a_d T^d$ is $\frac{d}{dT}\log(1 - a_d T^d) = -\sum_{m=1}^{\infty} da_d^m T^{dm-1}$, and $-T\frac{d}{dT}\log(1 - a_d T^d) = \sum_{n=1}^{\infty} gh_n(1 - a_d T^d)T^n$. So we obtain the formula:

$$-Tv^{-1}\frac{dv}{dT} = \sum_{n=1}^{\infty} gh_n(v)T^n$$

which yields the exponential trace formula:

$$-T\chi([M,f])^{-1}\frac{d\chi([M,f])}{dT} = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n.$$

For example, when rank $M = 2$, we have $\mathrm{tr}(f^2) = (\mathrm{tr}(f))^2 - 2\det(f)$, note that $\det(1 - fT) = 1 - \mathrm{tr}(f)T + \det(f)T^2$.

Remark 2.23. When R is a field, the exponential trace formula

$$-T\frac{d}{dT}\log\det(1 - fT) = \sum_{n=1}^{\infty} \mathrm{tr}(f^n)T^n$$

can be checked by $\det(1 - fT) = \prod(1 - \lambda_i T)$ where λ_i are eigenvalues. And we also have

$$\det(1 - fT) = \exp(\sum_{n=1}^{\infty} -\mathrm{tr}(f^n)\frac{T^n}{n}),$$

since $\prod(1 - \lambda_i T) = \exp\left(\ln(\prod(1 - \lambda_i T))\right) = \exp\left(\sum \ln(1 - \lambda_i T)\right)$ and recall that formally $\ln(1 - x) = -\sum \frac{x^n}{n}$.

2.3.3 $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Recall Λ is an R-algebra, where R is a commutative ring with unit. We define a map

$$\mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$$
$$(R^n, f) * [(P, \nu)] = [(P^n, f\nu)]$$

Let $\alpha_n = \alpha_n(a_1, \cdots, a_n)$ denote the $n \times n$ matrix (looks like the rational canonical form) over R:

$$\alpha_n(a_1, \cdots, a_n) = \begin{pmatrix} 0 & & & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2 \\ & & & 1 & -a_1 \end{pmatrix}$$

Recall

$$\begin{aligned} \chi\colon \mathrm{End}_0(R) &\rightarrowtail W(R) \\ (R^n, \alpha_n) &\mapsto \det(1-\alpha_n T) \end{aligned}$$

we obtain

$$\det\begin{pmatrix} 1 & & & & a_nT \\ -T & 1 & & & a_{n-1}T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2T \\ & & & -T & 1+a_1T \end{pmatrix} = 1+a_1T+\cdots+a_nT^n$$

(Computation methods: 1. the traditional computation. 2. Note that if A is invertible,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D-CA^{-1}B).$$

In this case $A^{-1} = \begin{pmatrix} 1 & & & & \\ T & 1 & & & \\ \vdots & \ddots & \ddots & & \\ T^{n-3} & \cdots & T & 1 & \\ T^{n-2} & T^{n-3} & \cdots & T & 1 \end{pmatrix}$)

Then we can also conclude that $\mathrm{Im}\chi = \left\{\frac{g(T)}{h(T)} \in W(R) \mid g(T), h(T) \in 1+TR[T]\right\}$.

Remark 2.24. Why is a general elment of the form (R^n, α_n)? Namely how to reduce an endomorphism to a rational canonical form ?

Now we want to check some identities

$$\begin{gathered} \mathrm{End}_0(R) \times \mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda) \\ (R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n\nu)] \quad \text{by definition} \\ (R^{n+1}, \alpha_{n+1}(a_1, \cdots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)] \quad \text{compute under } \chi \\ (R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)] \quad \text{where } \beta = \alpha_n(a_1\nu, \cdots, a_n\nu^n) \end{gathered}$$

In fact, the last identity always holds when $R = \mathbb{Z}[a_1, \cdots, a_n]$. β is nilpotent because $\beta = \alpha_n\nu$.

We only show how to check the last equation: only need to show that

$$\alpha_n\nu = \alpha_n(a_1\nu, \cdots, a_n\nu^n)$$

$$LHS = \begin{pmatrix} 0 & & & & -a_n\nu \\ \nu & 0 & & & -a_{n-1}\nu \\ & \ddots & \ddots & & \vdots \\ & & \nu & 0 & -a_2\nu \\ & & & \nu & -a_1\nu \end{pmatrix}, \quad RHS = \begin{pmatrix} 0 & & & & -a_n\nu^n \\ 1 & 0 & & & -a_{n-1}\nu^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_2\nu^2 \\ & & & 1 & -a_1\nu \end{pmatrix}$$

we can check this using the characteristic polynomial since χ is injective: check

$$\det(1-\alpha_n\nu T) = \det(1-\alpha_n(a_1\nu,\cdots,a_n\nu^n)T)$$

$$LHS = \det\begin{pmatrix} 1 & & & & a_n\nu T \\ -\nu T & 1 & & & a_{n-1}\nu T \\ & \ddots & \ddots & & \vdots \\ & & -\nu T & 1 & a_2\nu T \\ & & & -\nu T & 1+a_1\nu T \end{pmatrix} = \det(1+a_1\nu T+\cdots+a_n\nu^nT^n)$$

$$RHS = \det\begin{pmatrix} 1 & & & & a_n\nu^n T \\ -T & 1 & & & a_{n-1}\nu^{n-1} T \\ & \ddots & \ddots & & \vdots \\ & & -T & 1 & a_2\nu^2 T \\ & & & -T & 1+a_1\nu T \end{pmatrix} = \det(1+a_1\nu T+\cdots+a_n\nu^nT^n).$$

Note that if $\exists N$ such that $\nu^N = 0$, β is independent of the a_i for $i \geq N$. If $\nu^N = 0$ then $\alpha_n \otimes \nu$ represents 0 in $\mathrm{Nil}_0(\Lambda)$ whenever $\chi(\alpha_n) \equiv 1 \bmod t^N$.

More operations Let $F_n\mathbf{Nil}(\Lambda)$ denote the full exact subcategory of $\mathbf{Nil}(\Lambda)$ on the (P,ν) with $\nu^n = 0$. If Λ is an algebra over a commutative ring R, the kernel $F_n\mathrm{Nil}_0(\Lambda)$ of $K_0(F_n\mathbf{Nil}(\Lambda)) \longrightarrow K_0(\mathbf{P}(\Lambda))$ is an $\mathrm{End}_0(R)$-module and $F_n\mathrm{Nil}_0(\Lambda) \longrightarrow \mathrm{Nil}_0(\Lambda)$ is a module map.

The exact endofunctor $F_m\colon (P,\nu) \mapsto (P,\nu^m)$ on $\mathbf{Nil}(\Lambda)$ is zero on $F_m\mathbf{Ni1}(\Lambda)$. For $\alpha \in \mathrm{End}_0(R)$ and $(P,\nu) \in \mathrm{Nil}_0(\Lambda)$, nota that $(V_m\alpha) * (P,\nu) = V_m(\alpha * F_m(P,\nu))$, and we can conclude that $V_m\mathrm{End}_0(R)$ acts trivially on the image of $F_m\mathrm{Nil}_0(\lambda)$ in $\mathrm{Nil}_0(\lambda)$. For more details, see Weibel, *K*-book chapter 2, pp 155 Exercise II.7.17.

2.3.4 $W(R)$-module structure on $\mathrm{Nil}_0(\Lambda)$

Once we have a nilpotent endomorphism, we can pass to infinity.

Theorem 2.25. $\mathrm{End}_0(R)$*-module structure on* $\mathrm{Nil}_0(\Lambda)$ *extends to a* $W(R)$*-module structure by the formula*

$$(1+\sum a_iT^i) * [(P,\nu)] = (R^n, \alpha_n(a_1,\cdots,a_n)) * [(P,\nu)],\ n \gg 0.$$

2.3.5 $W(R)$-module structure on $\mathrm{Nil}_*(\Lambda)$

The induced t-adic topology on $\mathrm{End}_0(R)$ is defined by the ideals

$$I_N = \{f \in \mathrm{End}_0(R) \mid \chi(f) \equiv 1 \bmod t^N\},\ I_N \supset I_{N+1},$$

and $\mathrm{End}_0(R)$ is separated (i.e. $\cap I_N = 0$) in this topology. The key fact is:

Theorem 2.26 (Almkvist). *The map $\chi\colon \mathrm{End}_0(R) \longrightarrow W(R)$ is a ring injection, and $W(R)$ is the t-adic completion of $\mathrm{End}_0(R)$, i.e. $W(R) = \varprojlim \mathrm{End}_0(R)/I_N$.*

Theorem 2.27 (Stienstra). *For every $\gamma \in \mathrm{Nil}_*(\Lambda)$ there is an N so that γ is annihilated by the ideal*

$$I_N = \{f \mid \chi(f) \equiv 1 \bmod t^N\} \subset \mathrm{End}_0(R).$$

Consequently, $NK_*(\Lambda)$ is a module over the t-adic completion $W(R)$ of $\mathrm{End}_0(R)$.

Recall the sign convention:

$$\begin{aligned} NK_1(\Lambda) &\cong \mathrm{Nil}_0(\Lambda) \\ [1-\nu y] &\leftrightarrow [(\Lambda^n, \nu)] - n[(\Lambda, 0)] \end{aligned}$$

The $W(R)$-module structure on $NK_1(\Lambda)$ is completely determined by the formula

$$\alpha(t) * [1-\nu y] = [\alpha(\nu y)].$$

And the $W(R)$-module structure on $NK_n(\Lambda)$

$$\alpha(t) * \{\gamma, 1-\nu y\} = \{\gamma, \alpha(\nu y)\} \in NK_n(\Lambda), \quad \gamma \in K_{n-1}(R).$$

2.3.6 Modern version

Reference: Weibel, K-book, chapter 4, pp. 58.

2.4 Some results

Proposition 2.28. *If R is $S^{-1}\mathbb{Z}$, $\hat{\mathbb{Z}}_p$ or $\mathbb{Q}$-algebra, then*

$$\begin{aligned} \lambda_t\colon R &\longrightarrow W(R) \\ r &\mapsto (1-t)^r \end{aligned}$$

is a ring injection.

Corollary 2.29. *Fix an integer p and a ring Λ with 1.*
(a) If Λ is an $S^{-1}\mathbb{Z}$-algebra, $NK_*(\Lambda)$ is an $S^{-1}\mathbb{Z}$-module.
(b) If Λ is a $\mathbb{Q}$-algebra, $NK_*(\Lambda)$ is a center(Λ)-module.
(c) If Λ is a $\hat{\mathbb{Z}}_p$-algebra, $NK_*(\Lambda)$ is a $\hat{\mathbb{Z}}_p$-module.
(d) If $p^m = 0$ in Λ, $NK_*(\Lambda)$ is a p-group.

Theorem 2.30 (Stienstra). *If $0 \neq n \in \mathbb{Z}$, $NK_1(R)[\frac{1}{n}] \cong NK_1(R[\frac{1}{n}])$.*

Corollary 2.31. [1] *If G is a finite group of order n, then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of n. In fact, $NK_*(\mathbb{Z}[G])$ is an n-torsion group, and $Z_{(p)} \otimes NK_*(\mathbb{Z}[G]) = NK_*(\mathbb{Z}_{(p)}[G])$, where $p \mid n$.*

[1] Weibel, *K*-book chapter3, page 27.

Chapter 3

Notes on NK_0 and NK_1 of the groups C_4 and D_4

This note is based on the paper [14].

3.1 Outline

Definition 3.1 (Bass Nil-groups). $NK_n(\mathbb{Z}G) = \ker(K_n(\mathbb{Z}G[x]) \overset{x\mapsto 0}{\longrightarrow} K_n(\mathbb{Z}G))$

G	$NK_0(\mathbb{Z}G)$	$NK_1(\mathbb{Z}G)$	$NK_2(\mathbb{Z}G)$
C_2	0	0	V
$D_2 = C_2 \times C_2$	V	$\Omega_{\mathbb{F}_2[x]}$	
C_4	V	$\Omega_{\mathbb{F}_2[x]}$	
$D_4 = C_4 \rtimes C_2$			

Note that $D_4 = \langle \sigma, \tau | \sigma^4 = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.
$V = x\mathbb{F}_2[x] = \oplus_{i=1}^{\infty} \mathbb{F}_2 x^i = \oplus_{i=1}^{\infty} \mathbb{Z}/2x^i$: continuous $W(\mathbb{F}_2)$-module. As an abelian group, it is countable direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2$ on generators $x^i, i > 0$.
$\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty} \mathbb{F}_2 e^i$, often write e^i stands for $x^{i-1}\,dx$. As an abelian group, $\Omega_{\mathbb{F}_2[x]} \cong V$. But it has a different $W(\mathbb{F}_2)$-module structure.

3.2 Preliminaries

3.2.1 Regular rings

We list some useful notations here:

R: ring with unit (usually commutative in this chapter)
R-mod: the category of R-modules,
$\mathbf{M}(R)$: the subcategory of finitely generated R-modules,
$\mathbf{P}(R)$: the subcategory of finitely generated projective R-modules.

Let $\mathbf{H}(R) \subset R$-mod be the full subcategory contains all M which has finte $\mathbf{P}(R)$-resolutions. R is called *regular* if $\mathbf{M}(R) = \mathbf{P}(R)$.

Proposition 3.2. *Let R be a commutative ring with unit, A an R-algebra and $S \subset R$ a multiplicative set, if A is regular, then $S^{-1}A$ is also regular.*

3.2.2 The ring of Witt vectors

As additive group $W(\mathbb{Z}) = (1 + x\mathbb{Z}[\![x]\!])^{\times}$, it is a module over the Cartier algebra consisting of row-and-column finite sums $\sum V_m[a_{mn}]F_n$, where $[a]$ are homothety operators for $a \in \mathbb{Z}$.

additional structure Verschiebung operators V_m, Frobenius operators F_m (ring endomorphism), homothety operators $[a]$.

$$\begin{aligned}
[a] &: \alpha(x) \mapsto \alpha(ax) \\
V_m &: \alpha(x) \mapsto \alpha(x^m) \\
F_m &: \alpha(x) \mapsto \sum_{\zeta^m=1} \alpha(\zeta x^{\frac{1}{m}}) \\
F_m &: 1 - rx \mapsto 1 - r^m x
\end{aligned}$$

Remark 3.3. $W(R) \subset Cart(R)$, $\prod_{m=1}^{\infty}(1 - r_m x^m) = \sum_{m=1}^{\infty} V_m[a_m]F_m$. See [4].

Proposition 3.4. $[1] = V_1 = F_1$*: multiplicative identity. There are some identities:*

$$\begin{aligned}
V_m V_n &= V_{mn} \\
F_m F_n &= F_{mn} \\
F_m V_n &= m \\
[a] V_m &= V_m [a^m] \\
F_m [a] &= [a^m] F_m \\
[a][b] &= [ab] \\
V_m F_k &= F_k V_m, \textit{ if } (k,m) = 1
\end{aligned}$$

We call a $W(R)$-module M continuous if $\forall v \in M$, $\mathrm{ann}_{W(R)}(v)$ is an open ideal in $W(R)$, that is $\exists k$ s.t. $(1-rx)^m * v = 0$ for all $r \in R$ and $m \geqslant k$. Note that if A is an R-module, $xA[x]$ is a continuous $W(R)$-module but that $xA[\![x]\!]$ is not.

3.2.3 Dennis-Stein symbol

Steinberg symbol Let R be a commutative ring, $u, v \in R^*$. First we construct Steinberg symbol $\{u, v\} \in K_2(R)$ as follows:

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$$

where $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.s

These symbols satisfy

(a) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ for $u_1, u_2, v \in R^*$. [Bilinear]

(b) $\{u, v\}\{v, u\} = 1$ for $u, v \in R^*$. [Skew-symmetric]

(c) $\{u, 1-u\} = 1$ for $u, 1-u \in R^*$.

Theorem 3.5. *If R is a field, division ring, local ring or even a commutative semilocal ring, $K_2(R)$ is generated by Steinberg symbols $\{r, s\}$.*

Dennis-Stein symbol version 1 If $a, b \in R$ with $1 + ab \in R^*$, Dennis-Stein symbol $\langle a, b\rangle \in K_2(R)$ is defined by

$$\langle a, b\rangle = x_{21}(-\frac{b}{1+ab})x_{12}(a)x_{21}(b)x_{12}(-\frac{a}{1+ab})h_{12}(1+ab)^{-1}.$$

Note that

$$\langle a, b\rangle = \begin{cases} \{-a, 1+ab\}, & \text{if } a \in R^* \\ \{1+ab, b\}, & \text{if } b \in R^* \end{cases}$$

and if $u, v \in R^* - \{1\}$, $\{u, v\} = \langle -u, \frac{1-v}{u}\rangle = \langle \frac{u-1}{v}, v\rangle$, thus Steinberg symbol is also a Dennis-Stein symbol. See Dennis, Stein *The functor K_2: a survey of computational problem*.

Maazen and Stienstra define the group $D(R)$ as follows:
take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1 + ab \in R^*$,
defining relations:
(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

If $I \subset R$ is an ideal, $a \in I$ or $b \in I$, we can consider $\langle a,b\rangle \in K_2(R,I)$ satisfy following relations

(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$,

(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a, b+c+abc\rangle$,

(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$ if any of a,b,c are in I.

Theorem 3.6. 1. *If R is a commutative local ring, then $D(R) \overset{\cong}{\to} K_2(R)$ is isomorphic. (Maazen-Stienstra, Dennis-Stein, van der Kallen)*

2. *Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $D(R,I) \overset{\cong}{\to} K_2(R,I)$.*

Dennis-Stein symbol version 2 In 1980s, things have changed. Dennis-Stein symbol is defined as follows (R is not necessarily commutative)

$r,s \in R$ commute and $1-rs$ is a unit, that is $rs = sr$ and $1-rs \in R^*$,

$$\langle r,s\rangle = x_{ji}(-s(1-rs)^{-1})x_{ij}(-r)x_{ji}(s)x_{ij}((1-rs)^{-1}r)h_{ij}(1-rs)^{-1}.$$

Note that if $r \in R^*$, $\langle r,s\rangle = \{r, 1-rs\}$. If $I \subset R$ is an ideal, $r \in I$ or $\in I$, we can even consider $\langle r,s\rangle \in K_2(R,I)$

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$,

(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$,

(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of r,s,t are in I).

Note that $\langle r,1\rangle = 0$ for any $r \in R$ and $\langle r,s\rangle_{version2} = \langle -r,s\rangle_{version1}$.

Theorem 3.7. 1. *If R is a commutative local ring or a field, then $K_2(R)$ is generated by $\langle r,s\rangle$ satisfying D1, D2, D3, or by all Steinberg symbols $\{r,s\}$.*

2. *Let R be a commutative ring. If $I \subset \mathrm{Rad}(R)$ (ideal I is contained in the Jacobson radical), $K_2(R,I)$ is generated by $\langle r,s\rangle$ (either $r \in R$ and $s \in I$ or $r \in I$ and $s \in R$) satisfying D1, D2, D3, or by all $\{u, 1+q\}$, $u \in R^*, q \in I$ when R is additively generated by its units.*

3. *Moreover, if R is semi-local, $K_2(R)$ is generated by either all $\langle r,s\rangle$, $r,s \in R$, $1-rs \in R^*$ or by all $\{u,v\}$, $u,v \in R^*$.*

3.2.4 Relative group and double relative group

You can skip this subsection for first reading. We will use the results in 3.4.

excision 失效就是说 if $A \longrightarrow B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A,I) \longrightarrow K_n(B,I)$ need not be an isomorphism. 由于这个不是同构，没法有 Mayer-Vietoris 序列

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1}(A/I) \arrow[r] \arrow[d] & K_i(A,I) \arrow[r] \arrow[d, shift right] & K_i(A) \arrow[r] \arrow[d] & K_i(A/I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{i+1}(B/I) \arrow[r] & K_i(B,I) \arrow[r] \arrow[u, dashed, shift right] & K_i(B) \arrow[r] & K_i(B/I) \arrow[r] & \cdots
\end{tikzcd}$$

要连接 $K_n(A,I) \longrightarrow K_n(B,I)$ 就要考虑 birelative K-groups（也称 double relative K-groups）,$K(A,B,I)$ 定义为 homotpy fiber of the map $K(A,I) \longrightarrow K(B,I)$。以下是详细的定义和性质。

Relative groups Let R be a ring (not necessarily commutative), $I \subset R$ a two-sided ideal, by definition $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$, there exists a map

$$BGL(R)^+ \longrightarrow BGL(R/I)^+$$

Definition 3.8. $K(R,I)$ is the homotopy fibre of the map $BGL(R)^+ \longrightarrow BGL(R/I)^+$. $K_i(R,I) := \pi_i(K(R,I)), i \geq 1$.

By long exact sequences of homotopy groups of a homotopy fibre, there is an exact sequence

$$\cdots \longrightarrow K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R,I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots.$$

In particular,

$$\begin{aligned} &K_3(R,I) \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R,I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow \\ \longrightarrow &K_1(R,I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \end{aligned}$$

Let R be any ring (not necessarily commutative), if $I,J \subset R$ are two-sided ideals, there is a map

$$K(R,I) \longrightarrow K(R/J, I+J/J).$$

If $I \cap J = 0$, the following diagram is a Cartesian (pullback) square with all maps surjective,

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"] & R/I \arrow[d, "g"] \\ R/J \arrow[r, "f"] & R/I+J \end{tikzcd}$$

Associated to the horizontal arrows of above diagram, we have, for $i \geq 0$, the long exact sequences of algebraic K-theory

(3.8)

$$\begin{tikzcd} \cdots \arrow[r] & K_{i+1}(R) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & K_{i+1}(R/I) \arrow[r, "\partial"] \arrow[d, "g_*"] & K_i(R,I) \arrow[r, "j"] \arrow[d, "\epsilon_i"] & K_i(R) \arrow[r, "\alpha_*"] \arrow[d] & K_i(R/I) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & K_{i+1}(R/J) \arrow[r, "f_*"] & K_{i+1}(R/I+J) \arrow[r, "\partial"] & K_i(R/J, I+J/J) \arrow[r, "j'"] & K_i(R/J) \arrow[r, "f_*"] & K_i(R/I+J) \arrow[r] & \cdots \end{tikzcd}$$

where the induced homomorphism

$$\epsilon_i \colon K_i(R,I) \longrightarrow K_i(R/J, I+J/J)$$

is called the i-th excision homomorphism for the square; its kernel is called the i-th excision kernel.

Firstly we have the MayerVietoris sequence

$$\begin{aligned} &K_2(R) \longrightarrow K_2(R/I) \oplus K_2(R/J) \longrightarrow K_2(R/I+J) \longrightarrow \\ \longrightarrow &K_1(R) \longrightarrow K_1(R/I) \oplus K_1(R/J) \longrightarrow K_1(R/I+J) \longrightarrow \cdots . \end{aligned}$$

Secondly, there is a generalized theorem

Theorem 3.9. *1. Suppose that the excision map ϵ_i in 3.8 is an isomorphism. Then there is a homomorphism $\delta_i \colon K_{i+1}(R/I+J) \longrightarrow K_i(R)$ making the sequence*

$$\begin{aligned} &K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow\ &K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{aligned}$$

exact, where $\phi(x,y) = f_*(x) - g_*(y)$ and $\psi(z) = (\beta_*(z), \alpha_*(z))$.

2. *If ϵ_i is an isomorphism, and in additionϵ_{i+1} is surjective, the sequence in (1) remains exact with $K_{i+1}(R) \longrightarrow$ appended at the left, that is*

$$\begin{gathered} K_{i+1}(R) \longrightarrow K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \xrightarrow{\delta} \\ \longrightarrow K_i(R) \xrightarrow{\psi} K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{gathered}$$

3. *Suppose instead that ϵ_i is surjective, and let $L = ker(\epsilon_i)$. If $K_{i+1}(R) \xrightarrow{\alpha_*} K_{i+1}(R/I)$ is onto (e.g. if $R \longrightarrow R/I$ is a split surjection), L is mapped injectively to $K_i(R)$, and the sequence*

$$\begin{gathered} K_{i+1}(R/I) \oplus K_{i+1}(R/J) \xrightarrow{\phi} K_{i+1}(R/I+J) \longrightarrow \\ \longrightarrow K_i(R)/L \longrightarrow K_i(R/I) \oplus K_i(R/J) \longrightarrow K_i(R/I+J) \end{gathered}$$

   is exact.

Proof. Define $\delta_i = j\epsilon_i^{-1}\partial'$. The proof is then an easy diagram chase. □

Remark 3.10. It is known that ϵ_0 and ϵ_1 are isomorphism regardless of the specific rings. Moreover Swan [11] has shown that ϵ_2 cannot be an isomorphism in general. For more discussion, see [10].

Double relative groups

Definition 3.11. Let R be any ring (not necessarily commutative), $I, J \subset R$ two-sided ideals, $K(R; I, J)$ is the homotopy fibre of the map $K(R, I) \longrightarrow K(R/J, I+J/J)$. $K_i(R, I, J) := \pi_i(K(R; I, J)), i \geq 1$.

$$\begin{tikzcd} K(R;I,J) \arrow[d, dashed] & & \\ K(R,I) \arrow[r] \arrow[d] & BGL(R)^+ \arrow[r] \arrow[d] & BGL(R/I)^+ \arrow[d] \\ K(R/J, I+J/J) \arrow[r] & BGL(R/J)^+ \arrow[r] & BGL(R/I+J)^+ \end{tikzcd}$$

Remark 3.12. $K_i(R; I, J) \cong K_i(R; J, I)$, $K_i(R; I, I) = K_i(R, I)$.

We have a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R, I) \longrightarrow K_{i+1}(R/J, I+J/J) \longrightarrow K_i(R; I, J) \longrightarrow K_i(R, I) \longrightarrow K_i(R/J, I+J) \longrightarrow \cdots.$$

Let R be any ring (not necessarily commutative), if $I, J \subset R$ are two-sided ideals such that $I \cap J = 0$, then there is an exact sequence

$$K_3(R, I) \longrightarrow K_3(R/I, I+J/J) \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

where $R^e = R \otimes_{\mathbb{Z}} R^{op}$, $\psi([a] \otimes [b]) = \langle a, b \rangle$, see [15] 3.5.10, [10], [7] or [5] p. 195.

In the case $I \cap J = 0$, $K_2(R; I, J) \cong St_2(R; I, J) \cong I/I^2 \otimes_{R^e} J/J^2$, see [6] theorem 2.

Remark 3.13. $I/I^2 \otimes_{R^e} J/J^2 = I \otimes_{R^e} J$ and if R is commutative, $K_2(R; I, J) = I \otimes_R J$. See [6].

Theorem 3.14. *Let R be a commutative ring, I, J ideals such that $I \cap J$ radical, then $K_2(R; I, J)$ is generated by Dennis-Stein symbols $\langle a, b \rangle$, where $a, b \in R$ such that a or $b \in I$, a or $b \in J$, $1 - ab \in R^*$ (if $I \cap J$ radical, the last condition $1 - ab \in R^*$ is obviously holds), and moreover in D3 a or b or $c \in I$ and a or b or $c \in J$.*

Proof. See [6] theorem 3. □

Lemma 3.15. *Let $(R; I, J)$ satisfy the following Cartesian square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/I+J \end{array}$$

suppose $f\colon (R, I) \longrightarrow (R/J, I+J/J)$ has a section g, then

$$0 \longrightarrow I/I^2 \otimes_{R^e} J/J^2 \longrightarrow K_2(R, I) \longrightarrow K_2(R/I, I+J/J) \longrightarrow 0$$

is split exact.

3.3 $W(R)$-module structure

$W(\mathbb{F}_2)$**-module structure on** $V = x\mathbb{F}_2[x]$ See Dayton& Weibel [4] example 2.6, 2.9.

$$\begin{aligned} V_m(x^n) &= x^{mn} \\ F_d(x^n) &= \begin{cases} dx^{n/d}, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases} \\ [a]x^n &= a^n x^n \end{aligned}$$

$W(\mathbb{F}_2)$**-module structure on** $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]\,dx = \oplus_{i=1}^{\infty}\mathbb{F}_2 e^i$ Dayton& Weibel [4]example 2.10

$$
\begin{aligned}
V_m(x^{n-1}\,dx) &= mx^{mn-1}\,dx \\
F_d(x^{n-1}\,dx) &= \begin{cases} x^{n/d-1}\,dx, & \text{if } d|n \\ 0, & \text{otherwise} \end{cases} \\
[a]x^{n-1}\,dx &= a^n x^{n-1}\,dx
\end{aligned}
$$

Remark 3.16. $\Omega_{\mathbb{F}_2[x]}$ is not finitely generated as a module over the $\mathbb{F}_2$-Cartier algebra or over the subalgebra $W(\mathbb{F}_2)$.

In general, for any map $R \longrightarrow S$ of communicative rings, the S-module $\Omega^1_{S/R}$(relative Kähler differential module $\Omega_{S/R}$) is defined by
generators: ds, $s \in S$,
relations: $d(s+s') = ds + ds'$, $d(ss') = sds' + s'ds$, and if $r \in R$, $dr = 0$.

Remark 3.17. If $R = \mathbb{Z}$, we often omit it. In the previous section, $\Omega_{\mathbb{F}_2[x]} = \Omega^1_{\mathbb{F}_2[x]/\mathbb{Z}}$.

As abelian groups, $x\mathbb{F}_2[x] \xrightarrow{\sim} \Omega_{\mathbb{F}_2[x]}$, $x^i \mapsto x^{i-1}dx$. However, as $W(\mathbb{F}_2)$-modules,

$$
\begin{aligned}
V_m(x^i) &= x^{im}, \\
V_m(x^{i-1}dx) &= mx^{im-1}dx
\end{aligned}
$$

x^{im} is corresponding to $x^{im-1}dx$ but not to $mx^{im-1}dx$. So they have different $W(\mathbb{F}_2)$-module structure.

Remark 3.18. 一个不知道有没有用的结论, see [4]

There is a $W(\mathbb{F}_2)$-module homomorphism called de Rham differential

$$
\begin{aligned}
D\colon x\mathbb{F}_2[x] &\longrightarrow \Omega_{\mathbb{F}_2[x]} \\
x^i &\mapsto ix^{i-1}dx
\end{aligned}
$$

Then $\ker D = H^0_{dR}(\mathbb{F}_2[x]/\mathbb{F}_2)$ is the de Rham cohomology group and $\mathrm{coker} D = HC_1^{\mathbb{F}_2}(\mathbb{F}_2[x])$ is the cyclic homology group. Note that $HC_1(\mathbb{F}_2[x]) = \sum_{l=1}^{\infty}\mathbb{F}_2 e_{2l}$ where $e_{2l} = x^{2l-1}dx$, and $H^0_{dR}(\mathbb{F}_2[x]) = x^2\mathbb{F}_2[x^2]$.

3.4 NK_i of the groups C_2 and C_p

First, consider the simplest example $G = C_2 = \langle\sigma\rangle = \{1, \sigma\}$. There is a Rim square

$$
\begin{tikzcd}
\mathbb{Z}[C_2] \arrow[r, "\sigma\mapsto 1"] \arrow[d, "\sigma\mapsto -1"'] & \mathbb{Z} \arrow[d, "q"] \\
\mathbb{Z} \arrow[r, "q"] & \mathbb{F}_2
\end{tikzcd}
\tag{3.18}
$$

Since $\mathbb{F}_2$ (field) and $\mathbb{Z}$ (PID) are regular rings, $NK_i(\mathbb{F}_2) = 0 = NK_i(\mathbb{Z})$ for all i.

By MayerVietoris sequence, one can get $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$. Note that the similar results are true for any cyclic group of prime order.

$$
\begin{tikzcd}
NK_2\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_1\mathbb{Z}[C_2] \arrow[r] & NK_1\mathbb{Z}\oplus NK_1\mathbb{Z} \arrow[r] \arrow[d, equal] & NK_1\mathbb{F}_2 \arrow[r] \arrow[d, equal] & NK_0\mathbb{Z}[C_2] \arrow[r] & NK_0\mathbb{Z}\oplus NK_0\mathbb{Z} \arrow[d, equal] \\
0 & & 0 & 0 & & 0
\end{tikzcd}
$$

$$\ker(\mathbb{Z}[C_2] \overset{\sigma\mapsto -1}{\longrightarrow} \mathbb{Z}) = (\sigma+1)$$

By relative exact sequence,

$$0 = NK_3(\mathbb{Z}) \longrightarrow NK_2(\mathbb{Z}[C_2], (\sigma+1)) \overset{\cong}{\longrightarrow} NK_2(\mathbb{Z}[C_2]) \longrightarrow NK_2(\mathbb{Z}) = 0.$$

And from $(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow (\mathbb{Z}[C_2]/(\sigma-1), (\sigma+1)+(\sigma-1)/(\sigma-1)) = (\mathbb{Z}, (2))$ one has double relative exact sequence

$$0 = NK_3(\mathbb{Z}, (2)) \longrightarrow NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \overset{\cong}{\longrightarrow} NK_2(\mathbb{Z}[C_2], (\sigma+1)) \longrightarrow NK_2(\mathbb{Z}, (2)) = 0.$$

Note that $0 = NK_{i+1}(\mathbb{Z}/2) \longrightarrow NK_i(\mathbb{Z}, (2)) \longrightarrow NK_i(\mathbb{Z}) = 0$.

$$
\begin{tikzcd}
 & NK_3(\mathbb{Z}, (2)) = 0 \arrow[d] & & \\
 & NK_2(\mathbb{Z}[C_2]; (\sigma+1), (\sigma-1)) \arrow[d, "\cong"] & & \\
0 = NK_3(\mathbb{Z}) \arrow[r] & NK_2(\mathbb{Z}[C_2], (\sigma+1)) \arrow[r, "\cong"] \arrow[d] & NK_2(\mathbb{Z}[C_2]) \arrow[r] & NK_2(\mathbb{Z}) = 0 \\
 & NK_2(\mathbb{Z}, (2)) = 0 & &
\end{tikzcd}
$$

We obtain $NK_2(\mathbb{Z}[C_2]) \cong NK_2(\mathbb{Z}[C_2],(\sigma+1),(\sigma-1))$, from Guin-Loday-Keune [6], $NK_2(\mathbb{Z}[C_2];(\sigma+1),(\sigma-1))$ is isomorphic to $V = x\mathbb{F}_2[x]$, with the Dennis-Stein symbol $\langle x^n(\sigma-1),\sigma+1\rangle$ corresponding to $x^n \in V$. Note that $1-x^n(\sigma-1)(\sigma+1) = 1$ is invertible in $\mathbb{Z}[C_2][x]$ and $\sigma+1 \in (\sigma+1), x^n(\sigma-1) \in (\sigma-1)$.

Theorem 3.19. *$NK_2(\mathbb{Z}[C_2]) \cong V$, $NK_1(\mathbb{Z}[C_2]) = 0$, $NK_0(\mathbb{Z}[C_2]) = 0$.*

In fact, when p is a prime number, we have $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{F}_p[x]$, $NK_1(\mathbb{Z}[C_p]) = 0$, $NK_0(\mathbb{Z}[C_p]) = 0$.

Example 3.20 ($\mathbb{Z}[C_p]$)**.** $R = \mathbb{Z}[C_p]$, $I = (\sigma-1)$, $J = (1+\sigma+\cdots+\sigma^{p-1})$ such that $I \cap J = 0$. There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r, "\sigma\mapsto\zeta"] \arrow[d, "\sigma\mapsto 1"', "f"] & \mathbb{Z}[\zeta] \arrow[d, "g"] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

$I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 \cong \mathbb{Z}_p$ is cyclic of order p and generated by $(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1})$. Note that $p(\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) = 0$ since $(1+\sigma+\cdots+\sigma^{p-1})^2 = p(1+\sigma+\cdots+\sigma^{p-1})$.

And the map

$$\begin{aligned} I/I^2 \otimes_{\mathbb{Z}[C_p]^{op}} J/J^2 &\longrightarrow K_2(R,I) \\ (\sigma-1)\otimes(1+\sigma+\cdots+\sigma^{p-1}) &\mapsto \langle \sigma-1, 1+\sigma+\cdots+\sigma^{p-1}\rangle = \langle \sigma-1,1\rangle^p = 1 \end{aligned}$$

Also see [10].

Example 3.21 ($\mathbb{Z}[C_p][x]$)**.** There is a Rim square

$$\begin{tikzcd} \mathbb{Z}[C_p][x] \arrow[r] \arrow[d] & \mathbb{Z}[\zeta][x] \arrow[d] \\ \mathbb{Z}[x] \arrow[r] & \mathbb{F}_p[x] \end{tikzcd}$$

$K_2(\mathbb{Z}[C_p][x]; I[x], J[x]) \cong I[x] \otimes_{\mathbb{Z}[C_p][x]} J[x] = I \otimes_{\mathbb{Z}[C_p]} J[x] \cong \mathbb{Z}_p[x]$.

Since $\Lambda = \mathbb{Z}, \mathbb{F}_p, \mathbb{Z}[\zeta]$ are regular, $K_i(\Lambda[x]) = K_i(\Lambda)$, i.e. $NK_i(\Lambda) = 0$. Hence

$$K_2(\mathbb{Z}[C_p][x], I[x], J[x])/K_2(\mathbb{Z}[C_p], I, J) \cong K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]),$$

finally $NK_2(\mathbb{Z}[C_p]) = K_2(\mathbb{Z}[C_p][x])/K_2(\mathbb{Z}[C_p]) \cong \mathbb{Z}/p[x]/\mathbb{Z}/p = x\mathbb{Z}/p[x] = x\mathbb{F}_p[x]$.

3.5 NK_i of the group D_2

Now let us consider $G = D_2 = C_2 \times C_2$. Let $\Phi(V)$ be the subgroup (also a Cartier submodule) $x^2\mathbb{F}_2[x^2]$ of $V = x\mathbb{F}_2[x]$. Recall Ω_R is the Kähler differentials of R, $\Omega_{\mathbb{F}_2[x]} = \mathbb{F}_2[x]dx$. And we simply write $\mathbf{F}_2[\epsilon]$ stands for the 2-dimensional $\mathbb{F}_2$-algebra $\mathbb{F}_2[x]/(x^2)$.

Note that

$$\begin{array}{cccccccccc} \mathbb{F}_2[C_2] = & \mathbb{F}_2[x]/(x^2-1) \cong & \mathbb{F}_2[x]/(x-1)^2 \cong & \mathbb{F}_2[x-1]/(x-1)^2 \cong & \mathbb{F}_2[x]/(x^2) = & \mathbb{F}_2[\epsilon] \\ \sigma \quad \mapsto & x \quad \mapsto & x \quad \mapsto & x \quad \mapsto & 1+x \quad \mapsto & 1+\epsilon \end{array}$$

Lemma 3.22. *The map $q\colon \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$ in 3.18 induces an exact sequence*

$$0 \longrightarrow \Phi(V) \longrightarrow NK_2(\mathbb{Z}[C_2]) \xrightarrow{q} NK_2(\mathbb{F}_2[\epsilon]) \xrightarrow{D} \Omega_{\mathbb{F}_2[x]} \longrightarrow 0.$$

Proof. See [14] Lemma 1.2. □

Theorem 3.23.

$$NK_1(\mathbb{Z}[D_2]) \cong \Omega_{\mathbb{F}_2[x]},$$

$$NK_0(\mathbb{Z}[D_2]) \cong V,$$

$$NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2(\mathbb{Z}[C_2]) \cong V^2$$

the image of the above map is $\Phi(V) \times V$.

觉得最后一个论断有些问题。

Proof. We tensor 3.18 with $\mathbb{Z}[C_2]$. Note that $R[G_1 \times G_2] = R[G_1][G_2]$, for commutative R, $R[G_1 \times G_2] = R[G_1] \otimes R[G_2]$, $\sum_{g,h} c_g c_h g \otimes h \leftarrow\!\shortmid \sum_{g,h} c_g g \otimes c_h h$. As for infinite product, see MO 46950.

$$\begin{tikzcd} \mathbb{Z}[D_2] = \mathbb{Z}[C_2] \otimes \mathbb{Z}[C_2] \arrow[r] \arrow[d] & \mathbb{Z}[C_2] \arrow[d, two heads, "q"] \\ \mathbb{Z}[C_2] \arrow[r, two heads, "q"] & \mathbb{F}_2[C_2] \end{tikzcd} \tag{3.23}$$

Recall that $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\varepsilon]/(\epsilon^2)$. By [15] chapter 2 Ex 7.4.5,

$$NK_1(\mathbb{F}_2[C_2]) = NK_1(\mathbb{F}_2[\varepsilon]/(\epsilon^2)) = (1+\varepsilon x\mathbb{F}_2[\varepsilon]/(\epsilon^2)[x])^\times = (1+\varepsilon x\mathbb{F}_2[x])^\times \cong V = x\mathbb{F}_2[x]$$
$$[(P,\nu)] \mapsto \det(1-\nu x)$$

Remark 3.24. $(1+\varepsilon x\mathbb{F}_2[x])^\times \cong x\mathbb{F}_2[x]$, $1+\varepsilon x\sum a_i x^i \mapsto x\sum a_i x^i$ 的原因，左边是乘法群，右边的乘法是普通的多项式相加，左边 $(1+\varepsilon x\sum a_i x^i)(1+\varepsilon x\sum b_j x^j) = 1+\varepsilon x\sum a_i x^i + \varepsilon x\sum b_j x^j + (\varepsilon x\sum a_i x^i)(\varepsilon x\sum b_j x^j) = 1+\varepsilon x(\sum a_i x^i + \sum b_j x^j)$, 右边 $x\sum a_i x^i + x\sum b_j x^j = x(\sum a_i x^i + \sum b_j x^j)$.

As $W(\mathbb{F}_2)$-modules,

$$1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n) \mapsto a_0x+a_1x^2+\cdots+a_nx^{n+1}$$

and we can easily check that

$$V_m(1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon x^m(a_0+a_1x^m+\cdots+a_nx^{mn})$$

$$[a](1+\varepsilon x(a_0+a_1x+\cdots+a_nx^n)) = 1+\varepsilon ax(a_0+a_1ax+\cdots+a_na^nx^n)$$

hence the module structure of $(1+\varepsilon x\mathbb{F}_2[x])^\times$ are the same as V.

By MayerVietoris sequence for the NK-functor, one has

$$\begin{array}{l}
NK_2(\mathbb{Z}[D_2]) \longrightarrow NK_2\mathbb{Z}[C_2]\oplus NK_2\mathbb{Z}[C_2] \xrightarrow{q\times q} NK_2(\mathbb{F}_2[C_2]) \longrightarrow \\
\longrightarrow NK_1(\mathbb{Z}[D_2]) \longrightarrow \underset{\cong}{NK_1\mathbb{Z}[C_2]\oplus NK_1\mathbb{Z}[C_2]} = 0 \longrightarrow NK_1(\mathbb{F}_2[C_2]) \longrightarrow \\
\longrightarrow NK_0(\mathbb{Z}[D_2]) \longrightarrow NK_0\mathbb{Z}[C_2]\oplus NK_0\mathbb{Z}[C_2] = 0
\end{array}$$

Hence $NK_0(\mathbb{Z}[D_2]) \cong V$, $NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}(q\times q) \cong NK_2(\mathbb{F}_2[C_2])/\mathrm{Im}q \cong \Omega_{\mathbb{F}_2[x]}$ since $\mathrm{Im}(q\times q) = \mathrm{Im}q$. $\square$

最后一个论断，若对则有 $(q\times q)(\Phi(V)\times V)=0$，然而这个等式是不成立的。

3.5.1 A result from the K-book

For the convenience of the reader we copy [15] chapter 2 Ex 7.4.5 as follows.

Let R ba a commutative regular ring, $A = R[x]/(x^N)$, we claim that

$$\mathrm{Nil}_0(A) \rightarrowtail \mathrm{End}_0(A)$$

is an injection, and

$$\begin{aligned}
\mathrm{Nil}_0(A) &\cong (1+xtA[t])^\times \\
[(A,x)] &\mapsto 1-xt \\
[(P,\nu)] &\mapsto \det(1-\nu t)
\end{aligned}$$

the isomorphism $NK_1(A) \cong \mathrm{Nil}_0(A) \cong (1 + xtA[t])^\times$ is universal in the following sense:

Let B be a R-algebra, $(P, \nu) \in \mathbf{Nil}(B)$ with $\nu^N = 0$, regard P as an A-B-bimodule

$$\begin{aligned} \mathrm{Nil}_0(A) &\longrightarrow \mathrm{Nil}_0(B) \\ (A, x) &\mapsto (P, \nu) \end{aligned}$$

there is an $\mathrm{End}_0(R)$-module homomorphism

$$\begin{aligned} (1 + xtA[t])^\times &\longrightarrow \mathrm{Nil}_0(B) \\ 1 - xt &\mapsto [(P, \nu)]. \end{aligned}$$

3.5.2 About the lemma

In this subsection, we concentrate on the lemma 3.22.

For a complete proof, see [13].

3.6 NK_i of the group C_4

3.7 NK_i of the group D_4

Chapter 4

Lower Bounds for the Order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$

2016.3.19 阅读这篇文章 [9] 1976 年发表在 *Math. Ann.*。

基本假设：p: rational prime, G: elementary abelian p-group.

用的方法：Bloch; van der Kallen K_2 of truncated polynomial rings

结论：the p-rank of $K_2(\mathbb{Z}G)$[1] grows expotentially with the rank of G.

$Wh_2(G)$: "pseudo-isotopy" group is nontrivial if G has rank at least 2.

这篇文章之前已知的结论 (exact computations) Dunwoody, G cyclic of order 2 or 3, $K_2(\mathbb{Z}G)$ is an elementary abelian 2-group of rank 2 if G has order 2 and of rank 1 if G has order 3. 两者都有 $Wh_2(G)$ 平凡。

一些记号和基本结论 R commutative ring, A a subring of R. $\Omega^1_{R/A}$ the module of Kähler differentials of R considerd as an algebra over A and R^* will denote the group of units of R.

the p-rank of an abelian group G is $\dim_{\mathbb{F}_p}(G \otimes_{\mathbb{Z}} \mathbb{F}_p)$.

elementary abelian p-groups An elementary abelian p-group is an abelian group in which every nontrivial element has order p. The number p must be prime, and the elementary abelian groups are a particular kind of p-group. The case where $p = 2$, i.e., an elementary abelian 2-group, is sometimes called a Boolean group.

结构：Every elementary abelian p-group is a vector space over the prime field $\mathbb{F}_p$ with p elements, and conversely every such vector space is an elementary abelian group.

By the classification of finitely generated abelian groups, or by the fact that every vector space has a basis, every finite elementary abelian group must be of the form $(\mathbb{Z}/p\mathbb{Z})^n$ for n a

[1]this is a finite group

non-negative integer (sometimes called the group's rank). Here, $C_p = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p.

In general, a (possibly infinite) elementary abelian p-group is a direct sum of cyclic groups of order p. (Note that in the finite case the direct product and direct sum coincide, but this is not so in the infinite case.)

4.1 Part 1

环是 $\mathbb{F}_q$ 有限域的情况。

先说结论

首先是一个奇素数的结论

Proposition 4.1. *Let $q = p^f$ be odd and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary p-group of rank $f(n-1)(p^n-1)$.*

接着是素数 2 的结论

Proposition 4.2. *Let $q = 2^f$ be odd and let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{F}_q G)$ is an elementary 2-group of rank $f(n-1)(2^n-1)$.*

结论实际上是可以统一的，但是方法有些区别，因此原文中分开表述。

我们引进方法时借鉴了 van der Kallen 的方法和记号

Let R be a commutative ring. The abelian group $TD(R)$ is the universal R-module having generators $Da, Fa, a \in R$, subject to the relations

$$\begin{aligned} D(ab) &= aDb + bDa, \\ D(a+b) &= Da + Db + F(ab), \\ F(a+b) &= Fa + Fb, \\ Fa &= D(1+a) - Da. \end{aligned}$$

There is a natural surjective homomorphism of R-modules

$$TD(R) \twoheadrightarrow \Omega^1_{R/\mathbb{Z}} \longrightarrow 1$$

whose kernel is the submodule of $TD(R)$ generated by the $Fa, a \in R$. Relations imply

$$F(c^2 a) = cFa$$

$(F(c^2a) = F(ca \cdot c) = D(ca+c) - D(ac) - D(c) = D(c(a+1)) - D(ac) - D(c) = cD(a+1) - (a+1)D(c) - aD(c) - cD(a) - D(c) = cF(a)$, $0 = F(0) = F(a-a) = F(a) + F(-a)$, $\Rightarrow F(a) = -F(a) = F(-a)$, $\Rightarrow F(2a) = 0)$

for all $a, c \in R$ see [12]p. 1204.
Hence $F(2a) = 2F(a) = 0$, if 2 is a unit of R, $F(a) = 0$, then the kernel is trivial and $\Omega^1_{R/\mathbb{Z}} \cong TD(R)$,

$$1 \longrightarrow TD(R) \xrightarrow{\cong} \Omega^1_{R/\mathbb{Z}} \longrightarrow 1.$$

Example 4.3. $R = \mathbb{Z}$, then the kernel of the above surjection is $\mathbb{Z}/2\mathbb{Z}$.
If R is a field of characteristic $\neq 2$, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.
If R is a perfect field, then $TD(R) \cong \Omega^1_{R/\mathbb{Z}}$.

Definition 4.4. We define groups $\Phi_i(R)$, $i \geq 2$, by the exact sequence

$$1 \longrightarrow \Phi_i(R) \longrightarrow K_2(R[x]/(x^i)) \longrightarrow K_2(R[x]/(x^{i-1})) \longrightarrow 1. \tag{4.4}$$

This sequence is exact at the right as $SK_1(R[x]/(x^i), (x^{i-1})/(x^i)) = 1$ (cf. [8] Theorem 6.2 and [2]9.2, p. 267).

4.1.1 Remarks

我们把 Bass 书 [2] 中相关的结论 (p. 267) 放在这里以备今后查询和使用

A semi-local ring is a ring for which $R/rad(R)$ is a semisimple ring, where $rad(R)$ is the Jacobson radical of R. In commutative algebra, semi-local means "finitely many maximal ideals", for instance, all rational numbers r/s with s prime to 30 form a semi-local ring, with maximal ideals generated respectively by $2, 3$, and 5. This is a PID, as a matter of fact, any semi-local Dedekind domain is a PID. And if R is a commutative noetherian ring, the set of zero-divisors is a union of finitely many prime ideals (namely, the "associated primes" of (0)), thus its classical ring of quotients (obtained from R by inverting all of its non zero-divisors) is a semi-local ring. See [1] p.174. and [2] p. 86.

In studying the stable structure of general linear groups in algebraic K-theory, Bass proved the following basic result (ca. 1964) on the unit structure of semilocal rings.

Theorem 4.5. *If R is a semi-local ring, then R has stable range* 1*, in the sense that, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb \in R^*$.*

Example 4.6. Some classes of semi-local rings: left(right) artinian rings, finite direct products of local rings, matrix rings over local rings, module-finite algebras over commutative semi-local rings.
The quotient $\mathbb{Z}/m\mathbb{Z}$ is a semi-local ring. In particular, if m is a prime power, then $\mathbb{Z}/m\mathbb{Z}$ is a

local ring.

A finite direct sum of fields $\bigoplus_{i=1}^{n} F_i$ is a semi-local ring. In the case of commutative rings with unit, this example is prototypical in the following sense: the Chinese remainder theorem shows that for a semi-local commutative ring R with unit and maximal ideals $m_1, \cdots, m_n$

$$R/\bigcap_{i=1}^{n} m_i \cong \bigoplus_{i=1}^{n} R/m_i$$

(The map is the natural projection). The right hand side is a direct sum of fields. Here we note that $\cap_i m_i = rad(R)$, and we see that $R/rad(R)$ is indeed a semisimple ring.

The classical ring of quotients for any commutative Noetherian ring is a semilocal ring.

The endomorphism ring of an Artinian module is a semilocal ring.

Semi-local rings occur for example in commutative algebra when a (commutative) ring R is localized with respect to the multiplicatively closed subset $S = \cap(R - p_i)$, where the p_i are finitely many prime ideals.

Theorem 4.7. *Let I be a two-sided ideal in a ring R. Assume either that R is semi-local or that $I \subset rad(R)$. Then*

$$GL_1(R, I) \longrightarrow K_1(R, I)$$

is surjective, and, for all $m \geq 2$,

$$GL_m(R, I)/E_m(R, I) \longrightarrow K_1(R, I)$$

is an isomorphism. Moreover $[GL_m(R), GL_m(R, I)] \subset E_m(R, I)$, with equality for $m \geq 3$.

Corollary 4.8. *Suppose that R above is commutative, then $E_n(R, I) \stackrel{\cong}{\to} SL_n(R, I)$ is an isomorphism for all $n \geq 1$, and $SK_1(R, I) = 0$.*

Proof. The determinant induces the inverse,

$$\det\colon K_1(R, I) \longrightarrow GL_1(R, I).$$

In particular, if $\alpha \in GL_n(R, I)$ and $\det(\alpha) = 1$ then $\alpha \in E_n(R, I)$, i.e. $SL_n(R, I) \subset E_n(R, I)$. The opposite inclusion is trivial. Finally $SK_1(R, I) = SL(R, I)/E(R, I) = 0$. $\square$

还有一个小插曲，当 k 是域时，$k[x]/(x^m)$ 是局部环的证明

Proposition 4.9. *Let I be an ideal in the ring R.*

a) If $rad(I)$ is maximal, then R/I is a local ring.

b) In particular, if m is a maximal ideal and $n \in \mathbb{Z}^+$ then R/m^n is a local ring.

Proof. a) We know that $rad(I) = \bigcap_{P \supset I} P$, so if $rad(I) = m$ is maximal it must be the only prime ideal containing I. Therefore, by correspondence R/I is a local ring. (In fact it is a ring with a unique prime ideal.)
b)$rad(m^n) = rad(m) = m$, so part a) applies. □

Example 4.10. For instance, for any prime number p, $\mathbb{Z}/(p^k)$ is a local ring, whose maximal ideal is generated by p. It is easy to see (using the Chinese Remainder Theorem) that conversely, if $\mathbb{Z}/(n)$ is a local ring then n is a prime power.

The ring $\mathbb{Z}_p$ of p-adic integers is a local ring. For any field k, the ring $k[\![t]\!]$ of formal power series with coefficients in k is a local ring. Both of these rings are also PIDs. A ring which is a local PID is called a discrete valuation ring. Note that a local ring is connected, i.e., $e^2 = e \Rightarrow e \in \{0, 1\}$.

令 R 是 $k[x]$, I 是 (x^m), 有 $rad(x^m) = (x)$ 是极大理想 (由于 $0 \to (x) \to k[x] \to k \to 0$ 正合)，从而 $k[x]/(x^i)$ 是局部环。

Remarks 到此结束

4.1.2 Theorem

The first part of the following theorem is due to van der Kallen [12] and the second to Bloch [3].

Theorem 4.11. *Let R be a commutative ring. Then*
(1) $\Phi_2(R) \cong TD(R)$;
(2) If R is a local $\mathbb{F}_p$*-algebra and p is odd prime, then*

$$\Phi_i(R) \cong \begin{cases} \Omega^1_{R/\mathbb{Z}}, i \not\equiv 0, 1 \bmod p \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r}, i = mp^r, (p, m) = 1. \end{cases}$$

当 p 是 odd prime 时，这一定理 (2) 可应用于 $\mathbb{F}_p[C_p]$, 因为 $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[t]/(t^p)$

Lemma 4.12. *Let $q = p^f$ and let H be a finite abelian p-group. Then $\Omega^1_{\mathbb{F}_q H/\mathbb{Z}}$is a free $\mathbb{F}_q H$-module of rank equal to the p-rank of H.*

Proof. In terms of polynomials, we have

$$\mathbb{F}_q H \cong \mathbb{F}_q[x_1, \cdots, x_n]/I$$

where n is the p-rank of H and I is the ideal of $\mathbb{F}_q[x_1, \cdots, x_n]$ generated by polynomials of the form $F_i = x_i^{q_i} - 1$ where q_i is a power of p. By [BoreI,A.: Linear algebraic groups. New York:

W. A. Benjamin 1969, p. 61], $\Omega^1_{\mathbb{F}_qH/\mathbb{Z}}$ is the $\mathbb{F}_qH$-module with generators $dx_1, \cdots, dx_n$ subject to the relations

$$\sum_i \frac{dF_i}{dx_i} dx_i = 0.$$

Since the ring has characteristic p, the relations are trivial and the module is free. As $\mathbb{F}_q$ is perfect, its module of differentials is trivial. Hence $\Omega^1_{\mathbb{F}_qH/\mathbb{F}_q} = \Omega^1_{\mathbb{F}_qH/\mathbb{Z}}$, yielding the result. $\square$

由这个引理得到了4.1.

下面是节选一些可能用到的陈述。

- $\mathbb{F}_q G$ is a local ring, where G is an elementary abelian p-group, for example $G = (\mathbb{Z}/p\mathbb{Z})^n$.

对 odd prime 的证明如下

Proof. We begin by showing that $K_2(\mathbb{F}_qG)$ is an elementary abelian p-group even in case $p = 2$. As $\mathbb{F}_qG$ is a local ring, it follows that $K_2(\mathbb{F}_qG)$ is generated by the Steinberg symbols $\{u, v\}$, $u, v \in \mathbb{F}_qG^*$. Now $u^p, v^P \in \mathbb{F}^*$ as G is an elementary abelian p-group (p 次后 G 中的元就变成单位元了). Choose $w \in \mathbb{F}_q^*$ so that $w^p = u^p$.(这里注意之前的 u 是群环里的，这里的 w 取在域里) Then

$$\begin{aligned}\{u, v\}^p &= \{u^p, v\} \\ &= \{w^p, v\} \\ &= \{w, v^p\}.\end{aligned}$$

Thus $\{w, v^p\}$ is trivial as it lies in the image of $K_2(\mathbb{F}_q) = 1$(有限域的 K_2 是平凡的，并且这个符号是在 K_2 中). Hence $K_2(\mathbb{F}_qG)$ has exponent p.

Let H be generated by $x_1, \cdots, x_{n-1}$ where $x_1, \cdots, x_n$ are independent generators of G. Then (由于特征是 p 才有下面的最后一步，对于 $\mathbb{Z}$ 是不对的)

$$\mathbb{F}_qG = \mathbb{F}_qH[x_n]/(x_n^p - 1) \cong \mathbb{F}_qH[x]/(x^p).$$

Exact sequence 4.4 together with Theorem yield

$$\begin{aligned}\text{rank } K_2(\mathbb{F}_qG) &= \text{rank } K_2(\mathbb{F}_qH) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_qH/\mathbb{Z}} + \text{rank } \mathbb{F}_qH/\mathbb{F}_q \\ &= \text{rank } K_2(\mathbb{F}_qH) + f(p-1)(n-1)p^{n-1} + f(p^{n-1} - 1)\end{aligned}$$

and the result follows by induction.

上面的结论我们详细写出来是

$$1 \longrightarrow \Phi_p(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q G) = K_2(\mathbb{F}_q H[x]/(x^p)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow 1,$$
$$1 \longrightarrow \Phi_{p-1}(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-1})) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^{p-2})) \longrightarrow 1,$$
$$\cdots$$
$$1 \longrightarrow \Phi_2(\mathbb{F}_q H) \longrightarrow K_2(\mathbb{F}_q H[x]/(x^2)) \longrightarrow K_2(\mathbb{F}_q H[x]/(x)) \longrightarrow 1.$$

Note that $\mathbb{F}_q H[x]/(x) = \mathbb{F}_q H$, $G = (\mathbb{Z}/p\mathbb{Z})^n$, $H = (\mathbb{Z}/p\mathbb{Z})^{n-1}$ then

$$\begin{aligned}
\text{rank } K_2(\mathbb{F}_q G) &= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H[x]/(x^{p-1})) \\
&= \text{rank } \Phi_p(\mathbb{F}_q H) + \text{rank } \Phi_{p-1}(\mathbb{F}_q H) + \cdots + \text{rank } \Phi_2(\mathbb{F}_q H) + \text{rank } K_2(\mathbb{F}_q H) \\
&= \text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + (p-2)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } K_2(\mathbb{F}_q H) \\
&= (p-1)\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} + \text{rank } \mathbb{F}_q H/\mathbb{F}_q + \text{rank } K_2(\mathbb{F}_q H) \\
&= f(p-1)(n-1)p^{n-1} + f(p^{n-1}-1) + \text{rank } K_2(\mathbb{F}_q H)
\end{aligned}$$

since

$$\Phi_p(\mathbb{F}_q H) = \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} \oplus \mathbb{F}_q H/\mathbb{F}_q H^p,$$
$$\Phi_i(\mathbb{F}_q H) = \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} = (\mathbb{F}_q H)^{n-1}, 2 \le i \le p-1,$$
$$\mathbb{F}_q H/\mathbb{F}_q H^p = \mathbb{F}_q H/\mathbb{F}_q$$

$\mathbb{F}H$ 是以 H 中元素为基的自由 F 模并且

$$\text{rank } \Omega^1_{\mathbb{F}_q H/\mathbb{Z}} = \text{rank } (\mathbb{F}_{p^f} H)^{n-1} = (n-1)f|H| = (n-1)fp^{n-1}$$
$$\text{rank } \mathbb{F}_q H/\mathbb{F}_q = \text{rank } \mathbb{F}_q H - \text{rank } \mathbb{F}_q = f(p^{n-1}-1).$$

接下来是归纳计算,首先我们看它截至到哪一步:最后一步应该是 $\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]$,因为 $K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]) = 0$, 这时有

$$\begin{aligned}
\text{rank } K_2(\mathbb{F}_q[(\mathbb{Z}/p\mathbb{Z})^2]) &= \text{rank } K_2(\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]) + (p-1)\text{rank } \Omega^1_{\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{Z}} + \text{rank } \mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]/\mathbb{F}_q \\
&= 0 + f(p-1)(2-1)p^{2-1} + f(p^{2-1}-1)
\end{aligned}$$

从而我们知道

$$\begin{aligned}
\text{rank } K_2(\mathbb{F}_q G) &= f(p-1)(n-1)p^{n-1} + f(p^{n-1}-1) + \cdots + f(p-1)p^1 + f(p^1-1) \\
&= \sum_{i=1}^{n-1}(f(p-1)ip^i + f(p^i-1)) \\
&= -f\frac{p-p^n}{1-p} + f(n-1)p^n + f\frac{p-p^n}{1-p} - (n-1)f \\
&= f(n-1)(p^n-1)
\end{aligned}$$

这里的计算用到等比数列求和，记 $S = \sum_{i=1}^{n-1} ip^i$

$$pS = \sum_{i=1}^{n-1} ip^{i+1} = \sum_{i=2}^{n}(i-1)p^i$$

$$S - pS = \sum_{i=1}^{n-1} p^i - (n-1)p^n$$

因此

$$S = \frac{p-p^n}{(1-p)^2} - \frac{(n-1)p^n}{(1-p)}$$

$$\begin{aligned}\sum_{i=1}^{n-1}(f(p-1)ip^i + f(p^i-1)) &= f(p-1)S + f\frac{p-p^n}{1-p} - (n-1)f \\ &= f(p-1)(\frac{p-p^n}{(1-p)^2} - \frac{(n-1)p^n}{(1-p)}) + f\frac{p-p^n}{1-p} - (n-1)f \\ &= -f\frac{p-p^n}{1-p} + f(n-1)p^n + f\frac{p-p^n}{1-p} - (n-1)f \\ &= f(n-1)(p^n-1)\end{aligned}$$

$\square$

In case $p = 2$ the details become more complicated.(暂且略过这个情形)

4.2 Part 2

第二部分是考了系数环是 $\mathbb{Z}$ 的情形，如何将上面的有限域和这里的整数环联系起来，就是用了一个相对 K 群的正合列。

We now exploit these computations of $K_2(\mathbb{F}_qG)$ to obtain lower bounds for $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. There is an exact sequence

(4.12) $$K_2(\mathbb{Z}G) \longrightarrow K_2(\mathbb{F}_pG) \longrightarrow SK_1(\mathbb{Z}G, p\mathbb{Z}G) \longrightarrow SK_1(\mathbb{Z}G) \longrightarrow 1$$

This sequence is exact on the right because $\mathbb{F}_pG$ is a local ring, which implies $SK_1(\mathbb{F}_pG) = 1$ [2], p. 267.

Theorem 4.13. *(1) Let G be an elementary abelian 2-group of rank n. Then $K_2(\mathbb{Z}G)$ has 2-rank at least $(n-1)2^n + 2$ and $Wh_2(G)$ has 2-rank at least $(n-1)2^n - \frac{(n+2)(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n \geq 2$.*
(2) Let p be an odd prime and let G be an elementary abelian p-group of rank n. Then $K_2(\mathbb{Z}G)$ has p-rank at least $(n-1)(p^n-1) - \binom{p+n-1}{p}$ and $Wh_2(G)$ has p-rank at least $(n-1)(p^n-1) - \binom{p+n-1}{p} - \frac{n(n-1)}{2}$. In particular, $Wh_2(G)$ is non-trivial if $n \geq 2$.

Proof. (1) Since $K_1(\mathbb{Z}G, 2\mathbb{Z}G) \longrightarrow K_1(\mathbb{Z}G)$ is injective [Keating, M.E.: On the K-theory of the quaternion group. Mathematika 20, 59–62 (1973), Remark 2.4], we see that $K_2(\mathbb{Z}G) \longrightarrow K_2(\mathbb{F}_2 G)$ is surjective.

If $g_1, \cdots, g_n$, are the generators of G, then the $n+1$ symbols $\{-1,-1\}, \{-1, g_1\}, \cdots, \{-1, g_n\}$ are independent [[8], p. 65] and lie in the kernel of this map. Hence

$$\operatorname{rank} K_2(\mathbb{Z}G) \geq (n-1)(2^n - 1) + (n+1) = (n-1)2^n + 2.$$

Recall that for G abelian, $Wh_2(G)$ is the quotient of $K_2(\mathbb{Z}G)$ by the subgroup generated by all symbols of the form $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ [Hatcher, A.E.: Pseudo-isotopy and K_2, pp. 328-336. Lecture Notes in Mathematics 342. Berlin, Heidelberg, New York: Springer 1973]. It is easy to see from the bimuttiplicative and anti-symmetric properties of symbols that this subgroup has rank at most $\binom{n+1}{2} + 1$. Moreover, by using the various maps $\mathbb{Z}G \longrightarrow \mathbb{Z}$ which send elements of G to ± 1, it can be shown that the rank of this subgroup is precisely $\binom{n+1}{2} + 1$. $(n-1)2^n + 2 - \binom{n+1}{2} - 1 = (n-1)2^n - \frac{(n+2)(n-1)}{2}$.

(2) 以下这一段没有完全读懂。Let B be the integral chosure of $\mathbb{Z}G$ in $\mathbb{Q}G$. Then $SK_1(B, p^{n+1}B)$ has p-rank $\frac{p^n-1}{p-1}$ [Bass, H., Milnor, J., Serre, J. P.: Solution of the congruence subgroup problem for $SL_n(n \geq 3)$ and $Sp_{2n}(n \geq 2)$. Publ. Math. IHES 33, 59–137 (1967), Corollary 4.3, p. 95].

But $SK_1(B, p^{n+1}B) \cong SK_1(\mathbb{Z}G, p^{n+1}B)$ [[2], p. 484] since $p^n B$ lies in the conductor of B over $\mathbb{Z}G$, and $SK_1(\mathbb{Z}G, p^{n+1}B)$ maps onto $SK_1(\mathbb{Z}G, p\mathbb{Z}G)$ [[2], 9.3, p. 267]. Hence p-rank $SK_1(\mathbb{Z}G, p\mathbb{Z}G) \leq \frac{p^n-1}{p-1}$. The p-rank of $SK_1(\mathbb{Z}G)$ is $\frac{p^n-1}{p-1} - \binom{p+n-1}{p}$ [Alperin, R.C., Dennis, R. K., Stein, M. R.: The non-triviality of $SK_1(\mathbb{Z}\pi)$, pp. 1-7. Lecture Notes in Mathematics 353. Berlin, Heidelberg, New York: Springer t973, Theorem 2]. The result now follows from exact sequence 4.12.

And noting that the subgroup generated by the symbols $\{\sigma, \tau\}$, $\sigma, \tau \in \pm G$ has p-rank at most $\frac{n(n-1)}{2}$. □

Remark 4.14. The subgroup of $K_2(\mathbb{Z}G)$ generated by elements of the form $\langle a, b\rangle, 1 + ab \in (\mathbb{Z}G)^*$ maps onto $K_2(\mathbb{F}_2 G)$ for G an elementary abelian 2-group of rank ≤ 2. W. van der Kallen has shown that this subgroup maps onto in general. This follows from the rank 2 case via

Lemma (van der Kallen). Let I be a nilpotent ideal of the commutative ring R. Let $v_i \in R$ additively generate R/I and let $w_j \in I$ additively generate I. Then $K_2(I) = \ker(K_2(R) \longrightarrow K_2(R/I))$ is generated by all elements of the form $\langle v_i, w_j\rangle$ and $\langle w_j, w_i^{2^k-1} w_j\rangle$.

我的一些问题：$NK_2(\mathbb{F}_q G)$ 如何算，$NK_1(\mathbb{Z}G, p\mathbb{Z}G)$ =?，最简单的可以考虑 $NK_2(\mathbb{F}_p C_p)$, 接着是 $NK_2(\mathbb{F}_{p^2} C_p)$.

4.3 Generalizations

之前考虑的是 $\mathbb{Z}G$, G elementary. 可以推广到 G finite group, $\mathcal{O}$ be the ring of integers of an algebraic number field.

If S is a Sylow p-subgroup of G, then $\mathcal{O}G$ is a free module over $\mathcal{O}S$ and the composition

$$K_2(\mathcal{O}S) \longrightarrow K_2(\mathcal{O}G) \longrightarrow K_2(\mathcal{O}S)$$

(where the second map is the transfer) is multiplication by $(G : S)$. Hence p-rank $K_2(\mathcal{O}G) \geq$ p-rank $K_2(\mathcal{O}S)$ and estimates may be obtained by restricting to the case of a p-group.

Theorem 4.15. *Let $\mathcal{O}$ be the ring of integers in an algebraic number field which is Galois over $\mathbb{Q}$ and let G be an elementary abelian p-group of rank n. If p is unramfied in $\mathcal{O}$ with each residue field having degree f over $\mathbb{F}_p$, then $K_2(\mathcal{O}G)$ has p-rank at least*
(i) $f(n-1)(2^n-1)$ if $p=2$ and $\mathcal{O}$ has a real embedding,
(ii) $f(n-1)(2^n-l)-\binom{n+1}{2}$ if $p=2$ and $\mathcal{O}$ is totally imaginary,
(iii) $f(n-1)(p^n-l)-\binom{p+n-1}{p}$ if p is odd.

abelian p-groups which are not elementary 有以下一个结论

Proposition 4.16. *Let p be an odd prime and suppose $G = H \times C$ where C is cyclic of order p^t, $|H| = p^k$ and $s = p$-rank H. Let $\mathcal{O}$ be the ring of integers in a number field. Choose a prime $\mathfrak{p}$ of $\mathcal{O}$ lying over p and having residue degree f over $\mathbb{F}_p$. Then*

$$\begin{aligned}&\operatorname{ord}_p|K_2(\mathcal{O}G/\mathfrak{p}G)| - \operatorname{ord}_p|K_2(\mathcal{O}H/\mathfrak{p}H)| \\ \geq &f\left(p^k\left(s(p-1)p^{t-1}+1\right) - |H^{p^t}|\right) + p^k(p^{t-1}-1) - (p-1)\sum_{r=1}^{t-1}|H^{p^r}|p^{t-r-1}.\end{aligned}$$

Chapter 5

On the calucation of $K_2(\mathbb{F}_2[C_4 \times C_4])$

5.1 Abstract

We calulate $K_2(\mathbb{F}_2[C_4 \times C_4])$ by using relative K_2-group $K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

5.2 Introduction

Let C_n denote the cyclic group of order n. Chen et al. [16] calculated $K_2(\mathbb{F}_2[C_4 \times C_4])$ by the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. In this short notes, we use another method to calculate $K_2(\mathbb{F}_2[C_4 \times C_4])$ directly.

5.3 Preliminaries

Let k be a finite field of characteristic $p > 0$. Let $I = (t_1^m, t_2^n)$ be a proper ideal in the polynomial ring $k[t_1, t_2]$. Put $A = k[t_1, t_2]/I$. We will write the image of t_i in A also as t_i. Let $M = (t_1, t_2)$ be the nilradical of A. Note that $A/M = k$. One has a presentation for $K_2(A, M)$ in terms of Dennis-Stein symbols:

generators: $\langle a, b \rangle$, $(a, b) \in A \times M \cup M \times A$;

relations: $\langle a, b \rangle = -\langle b, a \rangle$,

$\langle a, b \rangle + \langle c, b \rangle = \langle a + c - abc, b \rangle$,

$\langle a, bc \rangle = \langle ab, c \rangle + \langle ac, b \rangle$ for $(a, b, c) \in A \times M \times A \cup M \times A \times M$.

Now we introduce some notations followed [13]

- $\mathbb{N}$: the monoid of non-negative integers,
- $\epsilon^1 = (1, 0) \in \mathbb{N}^2$, $\epsilon^2 = (0, 1) \in \mathbb{N}^2$,
- for $\alpha \in \mathbb{N}^2$, one writes $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2}$, so $t^{\epsilon^1} = t_1$, $t^{\epsilon^2} = t_2$,

- $\Delta = \{\alpha \in \mathbb{N}^2 \mid t^\alpha \in I\}$,
- $\Lambda = \{(\alpha, i) \in \mathbb{N}^2 \times \{1, 2\} \mid \alpha_i \geq 1, t^\alpha \in M\}$,
- for $(\alpha, i) \in \Lambda$, set $[\alpha, i] = \min\{m \in \mathbb{Z} \mid m\alpha - \epsilon^i \in \Delta\}$,
- if $gcd(p, \alpha_1, \alpha_2) = 1$, let $[\alpha] = \max\{[\alpha, i] \mid \alpha_i \not\equiv 0 \bmod p\}$
- $\Lambda^{00} = \big\{(\alpha, i) \in \Lambda \mid gcd(\alpha_1, \alpha_2) = 1, i \neq \min\{j \mid \alpha_j \not\equiv 0 \bmod p, [\alpha, j] = [\alpha]\}\big\}$,

If $(\alpha, i) \in \Lambda$, $f(x) \in k[x]$, put

$$\Gamma_{\alpha,i}(1 - xf(x)) = \langle f(t^\alpha)t^{\alpha-\epsilon^i}, t_i\rangle,$$

then $\Gamma_{\alpha,i}$ induces a homomorphism

$$(1 + xk[x]/(x^{[\alpha,i]}))^\times \longrightarrow K_2(A, M).$$

Lemma 5.1. *The $\Gamma_{\alpha,i}$ induce an isomorphism*

$$K_2(A, M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + xk[x]/(x^{[\alpha,i]}))^\times.$$

Proof. See Corollary 2.6 in [13]。 □

Lemma 5.2. $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$, $(1 + x\mathbb{F}_2[x]/(x^4))^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. It is easy to see that $(1 + x\mathbb{F}_2[x]/(x^3))^\times$ is generated by $1 + x$, and the order of $1 + x$ is 4, we conclude that $(1 + x\mathbb{F}_2[x]/(x^3))^\times \cong \mathbb{Z}/4\mathbb{Z}$.

Obeserve that the orders of the elements $1 + x, 1 + x^3 \in (1 + x\mathbb{F}_2[x]/(x^4))^\times$ are 4 and 2 respectively. The subgroups $\langle 1 + x\rangle = \{1, 1 + x, 1 + x^2, 1 + x + x^2 + x^3\}$, $\langle 1 + x^3\rangle = \{1, 1 + x^3\}$. Let σ, τ be the generators of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, then the homomorphism

$$\begin{aligned} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} &\longrightarrow (1 + x\mathbb{F}_2[x]/(x^4))^\times \\ (\sigma, \tau) &\mapsto (1 + x)(1 + x^3) = 1 + x + x^3. \end{aligned}$$

is an isomorphism. □

5.4 Main result

Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then we have $\mathbb{F}_2[C_4 \times C_4] \cong \mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4)$ since the characteristic of $\mathbb{F}_2$ is 2.

Lemma 5.3. $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2[t_1, t_2]/(t_1^4, t_2^4), (t_1, t_2))$.

Proof. The following sequence is split exact

$$0 \longrightarrow K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4),(t_1,t_2)) \xrightarrow{f} K_2(\mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)) \xrightarrow{t_i \mapsto 0} K_2(\mathbb{F}_2) \longrightarrow 0.$$

The homomorphism f is an isomorphism since K_2-group of any finite field is trivial. $\square$

Theorem 5.4. *Let $C_4 \times C_4$ be the direct product of two cyclic groups of order 4, then $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.*

Proof. Set $A = \mathbb{F}_2[t_1,t_2]/(t_1^4,t_2^4)$, then $I = (t_1^4,t_2^4)$, $M = (t_1,t_2)$, $A/M = \mathbb{F}_2$. Thus

$$\Delta = \{(\alpha_1,\alpha_2) \in \mathbb{N}^2 \mid \alpha_1 \geq 4 \text{ or } \alpha_2 \geq 4\},$$

$$\Lambda = \{(\alpha,i) \mid \alpha_i \geq 1\}.$$

For $(\alpha,i) \in \Lambda$,

$$[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\},$$

$$[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\},$$

where $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$.

Next we want to compute the set Λ^{00}. Since $(1 + x\mathbb{F}_2[x]/(x))^\times$ is trivial, it is sufficient to consider the subset $\Lambda^{00}_{>1} := \{(\alpha,i) \in \Lambda^{00} \mid [(\alpha,i)] > 1\}$, and then

$$K_2(A,M) \cong \bigoplus_{(\alpha,i)\in\Lambda^{00}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times = \bigoplus_{(\alpha,i)\in\Lambda^{00}_{>1}} (1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times.$$

(1) If $1 \leq \alpha_1 \leq 4$ is even and $1 \leq \alpha_2 \leq 4$ is odd, then $(\alpha,1) \in \Lambda^{00}_{>1}$ and $[\alpha,1] = \min\{\left\lceil \frac{5}{\alpha_1} \right\rceil, \left\lceil \frac{4}{\alpha_2} \right\rceil\}$.

(2) If $1 \leq \alpha_1 \leq 4$ is odd and $1 \leq \alpha_2 \leq 4$ is even, then $(\alpha,2) \in \Lambda^{00}_{>1}$ and $[\alpha,2] = \min\{\left\lceil \frac{4}{\alpha_1} \right\rceil, \left\lceil \frac{5}{\alpha_2} \right\rceil\}$.

(3) If $1 \leq \alpha_1,\alpha_2 \leq 4$ are both odd and $gcd(\alpha_1,\alpha_2) = 1$, then $(\alpha,2) \in \Lambda^{00}_{>1}$ only when $[\alpha] = [\alpha,1]$.

By the computation 5.2, we can get the following table

$(\alpha, i) \in \Lambda^{00}_{>1}$	$[\alpha, i]$	$(1 + x\mathbb{F}_2[x]/(x^{[\alpha,i]}))^\times$
$((2,1),1)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((2,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,1),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((4,3),1)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,2),2)$	3	$\mathbb{Z}/4\mathbb{Z}$
$((1,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((1,1),2)$	4	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
$((1,3),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,2),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,4),2)$	2	$\mathbb{Z}/2\mathbb{Z}$
$((3,1),2)$	2	$\mathbb{Z}/2\mathbb{Z}$

Hence $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong (\mathbb{Z}/4\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^9$.

Furthermore, one can use the homomorphism $\Gamma_{\alpha,i}$ to determine the generators as below, the generators of order 4:

$$\langle t_1t_2, t_1\rangle, \langle t_1t_2, t_2\rangle, \langle t_1, t_2\rangle,$$

the generators of order 2:

$$\langle t_1t_2^3, t_1\rangle, \langle t_1^3t_2, t_1\rangle, \langle t_1^3t_2^3, t_1\rangle, \langle t_1t_2^3, t_2\rangle, \langle t_1^3t_2^2, t_2\rangle, \langle t_1t_2^2, t_2\rangle, \langle t_1^3t_2, t_2\rangle, \langle t_1^3t_2^3, t_2\rangle, \langle t_1^3, t_2\rangle.$$

□

Remark 5.5. Compared with [16], note that $\langle t_1^3, t_2\rangle = \langle t_1^2t_2, t_1\rangle$, because

$$\begin{aligned}
\langle t_1^3, t_2\rangle &= \langle t_1^2, t_1t_2\rangle - \langle t_1^2t_2, t_1\rangle \\
&= \langle t_1, t_1^2t_2\rangle - \langle t_1^2t_2, t_1\rangle - \langle t_1^2t_2, t_1\rangle \\
&= -3\langle t_1^2t_2, t_1\rangle \\
&= -\langle t_1^2t_2, t_1\rangle \\
&= \langle t_1^2t_2, t_1\rangle,
\end{aligned}$$

since $\langle t_1^2t_2, t_1\rangle + \langle t_1^2t_2, t_1\rangle = \langle 0, t_1\rangle = 0$ and $\langle t_1^3, t_2\rangle = -\langle t_1^3, t_2\rangle$.

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