Implementation of Network Effects

IEOR E8100 Networks Final Project Report

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Summary

This project proposes a Python program which is based on the set-up of network effect model we learnt in class. The program allows the user to specify their own assumptions on type distribution of the agents and the network effect function, as well as other relevant parameters. With the specified inputs, the program automatically calculates the equilibria states and provides 2 functions (Finite Agent Analysis and Analytical Form Analysis) for the user to test any initial state and find out the equilibrium that the given initial state will converge to.

Motivation

This project is inspired by the following observations:

- *Network effect is a key characteristic of networks:* Network effects captured one widely observed phenomenon of the behavior of agents in a network, especially in social network settings.
- *It is also an ideal model for implementation:* The majority of the analysis is based on the generic form of the utility function: $u(s_{i,t}, x_{i,t}, S_t, S_{t-1}) = x_i[\lambda \mathrm{h}(S_{t-1}) + (1-\lambda)\mathrm{h}(S_t)]s_{i,t} - cs_{i,t}$ and the distribution $G(x)$. It allows an implementation of generic framework with the component functions being editable and adjustable for a variety of assumptions and applications.
- *It will be an intuitive study tool:* A graphical representation of the result we got in class will be a very intuitive study tool for students to better understand the theories on network effect

Theoretical Setup

The setup of the model is a simple extension of the model we learnt from Lecture 19 and Lecture 20. There are 5 main components in the model:

- *Infinite agents:* This model considers a society with a large number of agents. For instance, a continuum $I \coloneqq [0,1]$ of agents.
- *Homogeneous utility function*: Each agent has the utility function: $u(s_{i,t}, x_i, S_t, S_{t-1}) = [x_i[\lambda \mathrm{h}(S_{t-1}) + (1-\lambda)\mathrm{h}(S_t)] - c]s_{i,t}$, where $s_{i,t} \in \{0,1\}$ denotes the strategy of agent i at time t, and S_t denotes the fraction of population that choose $s = 1$
- *Type distribution of agents:* Each agent has a type x_i, which describes his/her utility from taking action $s = 1$. x_i comes from the distribution described by the cumulative density function $G(x)$.
- *Network effect function*: $h(S)$ is an increasing function that imposes positive network effect on agents.
- *Other constant parameters*: c denotes the cost of taking action $s = 1$; λ denotes the weight of the network effect from the previous period.

With this setup, the most important result we have is the following equation[1]:

[1] While the type distribution and network effect function is changeable, the utility function is a necessary condition for deriving this equation, which means the program is not able to accommodate other kind of utility functions.

$$S_t = 1 - G\left(\frac{c}{\lambda h(S_{t-1}) + (1-\lambda)h(S_t)}\right) = k(S_t, S_{t-1})$$

This equation gives the condition satisfied by the equilibrium states. It follows from the fact that only the agents with the following type will take action $s = 1$:

$$x_i > \frac{c}{\lambda h(S_{t-1}) + (1-\lambda)h(S_t)}$$

Program Structure

The program has the following structure:

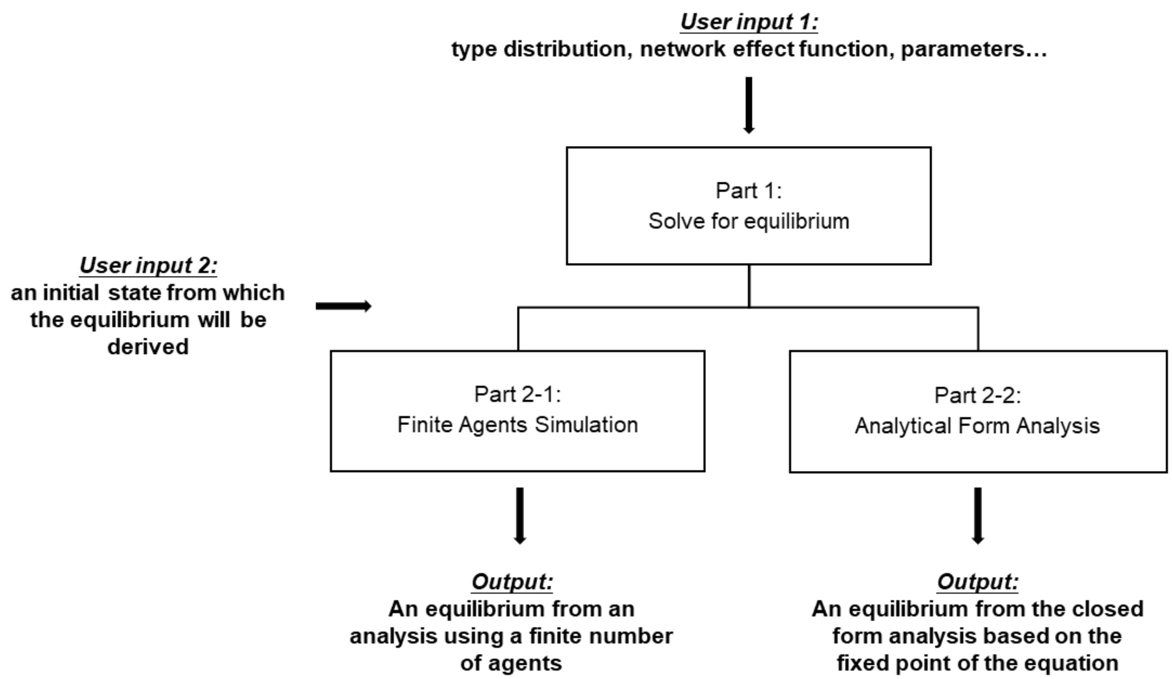


Figure 1: Structure of the Program

After the user has specified the forms of the type distribution and the network effect function as well as the relevant parameters, the program will compute the theoretical equilibria and present them to the user. An example of such presentation is as follows:

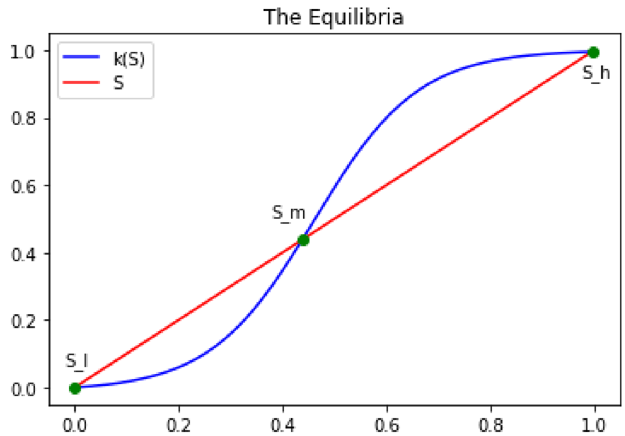


Figure 2: An Example of Equilibria Presentation

Then, the user has the chance to choose whether to use finite agent simulation, or analytical form analysis, or both tools to conduct further analysis.

In the finite agent analysis, the program uses a finite number of agents, which will be a user input, to approximate the dynamics of the game. After the user has given the number of agents, the initial state and the small perturbation to the state, the program will use the inverse function of $G(x)$ to sample the type of each agent. Then, the program will iterate through each agent's utility function to decide each agent's action. Only the agents whose type satisfies the following condition will take action $s_i = 1$: $x_i > \frac{c}{\lambda h(S_{t-1})+(1-\lambda)h(S_t)}$. After each agent has made their decision, the iteration for this period is done. The system will calculate this period's fraction of agents that have taken $s = 1$, and will record the result. This iteration will continue until the program obtains the same fraction S of agents taking strategy $s = 1$ for 3 consecutive periods. This fraction S will be regarded as equilibrium and the result will be reported to the user.

In the analytical form analysis, the program computes the best response of the agents to until equilibrium is reached. For a given initial state S_0 and a perturbation ε, the best response of the agents is given by $k(S_0 + \varepsilon, S_0)$. The program will keep iterating the process with the formula $S_i = k(S_{i-1}, S_{i-2})$ until it obtains the same S for 2 consecutive periods. This fraction S will be regarded as equilibrium and the result will be reported to the user.

For more detailed user instructions, please refer to the *Instruction Manual*.

Some Experimental Results

Example 1: Self-defined distribution and trivial h(S)

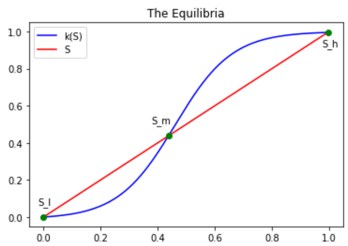
With this tool in hand, we are able to do a relatively comprehensive test of the model we learnt in class. The following is a demonstration of an experiment on the speed of convergence that we can conduct by utilizing this program.

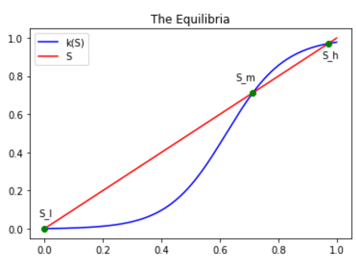
The cost of action c is fixed to be 0.5. The inputs of type distribution and network effect function were given as:

$$G(x) = \frac{\gamma + \beta}{1 + \beta} \frac{(1 + \beta)e^{-\alpha'/x}}{\gamma + \beta e^{-\alpha'/x}} \qquad h(S) = S$$

The result of stabilities of equilibria and convergence is as follows:

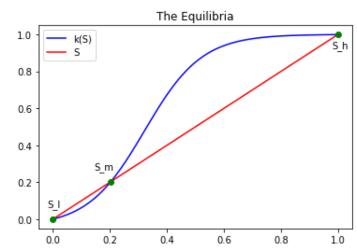
Table 1: Result of Study on Equilibrium Stability and Convergence – Example 1

<table>
<tr><th rowspan="2">Equilibria</th><th rowspan="2">λ</th><th rowspan="2">Initial state</th><th rowspan="2">Pertur-bation</th><th colspan="2">Finite agent analysis</th><th colspan="2">Analytical form analysis</th></tr>
<tr><th>Equilibrium</th><th>Time period spent</th><th>Equilibrium</th><th>Time period spent</th></tr>
<tr><td rowspan="10">

$(\alpha' = 5,\ \beta = 100,\ \gamma = 1)$
S_low: 0.0
S_medium: 0.437743092241162
S_high: 0.9952125446663069</td><td>0.0</td><td>S_medium</td><td>+0.01</td><td>0.99519</td><td>8</td><td>0.9952125</td><td>17</td></tr>
<tr><td>0.0</td><td>S_medium</td><td>-0.01</td><td>0.0</td><td>9</td><td>0.0</td><td>21</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>+0.01</td><td>0.99532</td><td>13</td><td>0.9952125</td><td>27</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>-0.01</td><td>0.0</td><td>16</td><td>0.0</td><td>33</td></tr>
<tr><td>0.0</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>3</td><td>0.0</td><td>15</td></tr>
<tr><td>0.5</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>4</td><td>0.0</td><td>23</td></tr>
<tr><td>0.0</td><td>S_high</td><td>+0.004[2]</td><td>0.99523</td><td>3</td><td>0.9952125</td><td>11</td></tr>
<tr><td>0.0</td><td>S_high</td><td>-0.004</td><td>0.99545</td><td>2</td><td>0.9952125</td><td>11</td></tr>
<tr><td>0.5</td><td>S_high</td><td>+0.004</td><td>0.99542</td><td>5</td><td>0.9952125</td><td>17</td></tr>
<tr><td>0.5</td><td>S_high</td><td>-0.004</td><td>0.99522</td><td>5</td><td>0.9952125</td><td>19</td></tr>
<tr><td rowspan="6">

$(\alpha' = 5,\ \beta = 500,\ \gamma = 1)$
S_low: 0.0
S_medium: 0.712493798659861
S_high: 0.9702949200993856</td><td>0.0</td><td>S_medium</td><td>+0.01</td><td>0.97075</td><td>12</td><td>0.9702949</td><td>34</td></tr>
<tr><td>0.0</td><td>S_medium</td><td>-0.01</td><td>0.0</td><td>9</td><td>0.0</td><td>15</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>+0.01</td><td>0.97080</td><td>19</td><td>0.9702949</td><td>52</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>-0.01</td><td>0.0</td><td>13</td><td>0.0</td><td>24</td></tr>
<tr><td>0.0</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>2</td><td>0.0</td><td>9</td></tr>
<tr><td>0.5</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>3</td><td>0.0</td><td>15</td></tr>
</table>

[2] It is different from the other experiments because it would make no sense if $S_h + \varepsilon$ exceeds 1.

<table>
<tr><td rowspan="4"></td><td>0.0</td><td>S_high</td><td>+0.01</td><td>0.97208</td><td>5</td><td>0.9702949</td><td>27</td></tr>
<tr><td>0.0</td><td>S_high</td><td>-0.01</td><td>0.96967</td><td>9</td><td>0.9702949</td><td>27</td></tr>
<tr><td>0.5</td><td>S_high</td><td>+0.01</td><td>0.97020</td><td>9</td><td>0.9702949</td><td>40</td></tr>
<tr><td>0.5</td><td>S_high</td><td>-0.01</td><td>0.96950</td><td>9</td><td>0.9702949</td><td>41</td></tr>
<tr><td rowspan="10">

($\alpha' = 5,\ \beta = 100,\ \gamma = 4$)
S_low: 0.0
S_medium: 0.20323372659673733
S_high: 0.998806803525748</td><td>0.0</td><td>S_medium</td><td>+0.01</td><td>0.99881</td><td>8</td><td>0.9988068</td><td>14</td></tr>
<tr><td>0.0</td><td>S_medium</td><td>-0.01</td><td>0.00003</td><td>15</td><td>0.0</td><td>42</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>+0.01</td><td>0.99885</td><td>13</td><td>0.9988068</td><td>23</td></tr>
<tr><td>0.5</td><td>S_medium</td><td>-0.01</td><td>0.0001</td><td>21</td><td>0.0</td><td>66</td></tr>
<tr><td>0.0</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>7</td><td>0.0</td><td>34</td></tr>
<tr><td>0.5</td><td>S_low</td><td>+0.01</td><td>0.0</td><td>13</td><td>0.0</td><td>53</td></tr>
<tr><td>0.0</td><td>S_high</td><td>+0.001</td><td>0.99878</td><td>2</td><td>0.9988068</td><td>8</td></tr>
<tr><td>0.0</td><td>S_high</td><td>-0.001</td><td>0.99883</td><td>2</td><td>0.9988068</td><td>7</td></tr>
<tr><td>0.5</td><td>S_high</td><td>+0.001</td><td>0.99873</td><td>1</td><td>0.9988068</td><td>13</td></tr>
<tr><td>0.5</td><td>S_high</td><td>-0.001</td><td>0.99894</td><td>3</td><td>0.9988068</td><td>13</td></tr>
</table>

Several conclusions may be drawn from this experiment:

- We verified the conclusion we had drawn from class that S_m is indeed asymptotically unstable, while S_l and S_h are stable.
- Past-dependency may increase the time needed to converge to equilibrium.
- The higher proportion of the concave (convex) part is, the longer it may take to converge from S_m with corresponding perturbation to S_h (S_l).

Example 2: Normal distribution and different h(S)

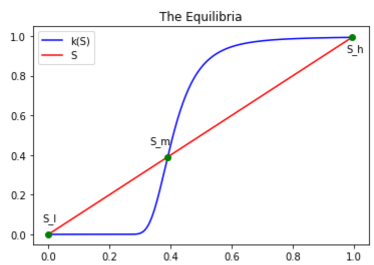
To demonstrate that this program is able to accommodate different type distribution and network effect functions, the following setup was used to conduct another experiment.

The cost of action c is fixed to be 0.5. The inputs of type distribution and network effect function were given as:

$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du \qquad\qquad h(S) = S^2$$

The result of stabilities of equilibria and convergence is as follows:

Table 2: Result of Study on Equilibrium Stability and Convergence – Example 2

Equilibria	λ	Initial state	Pertur-bation	Finite agent analysis		Analytical form analysis	
				Equilibrium	Time period spent	Equilibrium	Time period spent
 ($\mu = 3,\ \sigma = 1$) S_low: 0.0 S_medium: 0.39056218699877393 S_high: 0.9936775563950575	0.0	S_medium	+0.01	0.99388	6	0.99367755	12
	0.0	S_medium	-0.01	0.0	3	0.0	3
	0.5	S_medium	+0.01	0.99345	8	0.99367755	19
	0.5	S_medium	-0.01	0.0	4	0.0	4
	0.0	S_low	+0.01	0.0	1	0.0	1
	0.5	S_low	+0.01	0.0	1	0.0	1
	0.0	S_high	+0.004[3]	0.99350	2	0.99367755	9
	0.0	S_high	-0.004	0.99349	2	0.99367755	8
	0.5	S_high	+0.004	0.99323	3	0.99367755	14
	0.5	S_high	-0.004	0.99416	3	0.99367755	14

Conclusion Remarks

Similar to the above demonstration, the program may be utilized to conduct a variety of thorough studies of the network effect model we have learnt in class. It could be helpful in both helping the professor teach the network effect lecture as well as helping the students have a more comprehensive understanding of this topic.

[3] It is different from the other experiments because it would make no sense if $S_h + \varepsilon$ exceeds 1.