

Thomas Spencer (1946-)

The absence of ferromagnetic order in the two-dimensional XY model

part 4 McBryan-Spencer's proof of the absence of order

Advanced Topics in Statistical Physics
by Hal Tasaki



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§ bound on the correlation function for the model in $d=2$ with $h=0$

$$\Lambda_L = \{1, \dots, L\}^2, \mathcal{B}_L = \{(u, v) \mid u \text{ and } v \text{ are nearest neighbors (periodic b.c.)}\} \quad (1)$$

$$\Theta = (\theta_u)_{u \in \Lambda_L}, \quad \theta_u \in [0, 2\pi), \quad \int d\Theta = \prod_{u \in \Lambda_L} \int_0^{2\pi} d\theta_u \quad (2)$$

$$H_L(\Theta) = - \sum_{(u, v) \in \mathcal{B}_L} \cos(\theta_u - \theta_v) \quad (3)$$

$$Z_L(\beta) = \int d\Theta e^{-\beta H_L(\Theta)} \quad (4)$$

$$\langle \dots \rangle_{L, \beta} = \frac{1}{Z_L(\beta)} \int d\Theta (\dots) e^{-\beta H_L(\Theta)} \quad (5)$$

Proved
in parts

Theorem we have for any $0 < \beta < \infty$ that

$$0 \leq \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L, \beta} = \left\langle e^{i(\theta_u - \theta_v)} \right\rangle_{L, \beta} \leq |u - v|^{-\eta(\beta)} \quad (6)$$

for any $u, v \in \Lambda_L$ s.t. $|u - v| \leq \frac{L}{2}$ with $\eta(\beta) > 0$

we also have $\eta(\beta) \simeq (2\beta C)^{-1}$ if $\beta \gg 1$ (C is a constant)

On the Decay of Correlations in $SO(n)$ -symmetric Ferromagnets*

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Abstract. We prove that for low temperatures T the spin-spin correlation function of the two-dimensional classical $SO(n)$ -symmetric Ising ferromagnet decays faster than $|x|^{-\text{const } T}$ provided $n \geq 2$. We also discuss a nearest neighbor continuous spin model, with spins restricted to a finite interval, where we show that the spin-spin correlation function decays exponentially in any number of dimensions.

1. Introduction and Results

The Mermin-Wagner theorem [1] states that at non-zero temperatures the two dimensional Heisenberg model has no spontaneous magnetization. Consequently the spin-spin correlation function decays to zero at large distances, although the Mermin-Wagner theorem gives no indication of the rate of decay. Similar results apply for the classical $SO(n)$ -symmetric ($n \geq 2$) nearest neighbor Ising ferromagnets which we study here, see for example the paper of Mermin [2]. We establish a polynomial upper bound for the decay rate of the spin-spin correlation function for these models at very low temperatures. Fisher and Jasnow [3] have previously obtained a $\log^{-1}|x|$ decay.

To describe the $SO(n)$ -symmetric ferromagnet, we consider the infinite lattice of unit spacing with sites labelled by indices $i \in \mathbb{Z}^2$. To each site i we associate an n -component classical spin s_i of unit length, $\|s_i\|=1$. The spin-spin correlation function at inverse temperature $\beta = T^{-1}$ is

$$\langle s_0 \cdot s_x \rangle(\beta) = Z^{-1} \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j} s_0 \cdot s_x, \quad (1)$$

$$Z = \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j},$$

where $\sum_{\langle i,j \rangle}$ denotes a sum over nearest neighbor pairs, $\Omega_i^{(n)}$ is the invariant measure

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ons and (1) is to be interpreted as the thermodynamic volume quantities $\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (1) but $(1+1) \times (2N+1)$ periodic lattice. Let $C(x)$ denote the ice's equation on the lattice: $-\Delta C(x) = \delta_{0,x}, C(0) = 0$.

$\beta \geq \beta_0(e)$ sufficiently large

(2)

spin correlation functions are known to decay

case $n=2$ in Section II; other values of n are normalization. For $n=2$ we use the representation of the form

$$\sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) \cos(\phi_0 - \phi_x). \quad (3)$$

vated by the approximation [4]

$$(\phi_i - \phi_j)^2 = \frac{1}{2} (\partial A \phi). \quad (4)$$

ing the limit of integration in (3) to extend to correlation

$$d\phi_i e^{\frac{1}{2}\beta\phi_i\phi_j} \cos(\phi_0 - \phi_x) / \left(\prod_i \int_{-\pi}^{\pi} d\phi_i e^{\frac{1}{2}\beta\phi_i\phi_0} \right) \quad (5)$$

Theorem 1. It is difficult to justify the two use (3) is the integral of a periodic function of the extended to infinity without changing (3), on (4) is then unreasonable, since it makes sense below we show that there is a marked difference depending on whether the integration range is correlations on a n -dimensional lattice by

$$\langle \phi_0 \phi_x \rangle / \left(\prod_i \int_{-\pi}^{\pi} d\phi_i e^{\frac{1}{2}\beta\phi_i\phi_0} \right) \quad (6)$$

η -symmetric Ferromagnets

te μ, β , y there is an $\tilde{m} > 0$ such that

$t e^{-\tilde{m}|x|}, |x| \rightarrow \infty$.

y be chosen at least as large as $\cosh^{-1}(1+m^2/4)$, m contrast to the exponential decay for finite μ , (5) always for $\mu = \infty$ in two dimensions.

and for the Plane Rotator

se the representation (3) for $n=2$, replacing $\cos(\phi_0 - \phi_x)$ by $\langle \phi_0 \phi_x \rangle(\beta) = 0$. Using the periodicity of the integrand, we

$$\langle C(j) - C(j-x) \rangle,$$

means that we deform the path of integration and use $x^2 = 1, z$ real, yields

$$\begin{aligned} & \prod_i \int d\phi_i e^{\frac{1}{2}\beta\sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) \cosh(a_i - a_j)} \\ & \sum_{\langle i,j \rangle} (\cosh(a_i - a_j) - 1) \end{aligned}$$

amental solution $C(x)$ we prove below that

$= 1$, uniformly in x, i, j .

nd $\beta_0(e)$ such that for $\beta \geq \beta_0$

$$+ e \sum_{\langle i,j \rangle} (a_i - a_j)^2 = \frac{1}{2} (1 + \varepsilon) (a - \Delta a)$$

$$= (1 + \varepsilon) (2\beta)^{-1} (a_0 - a_x). \quad (9)$$

(x) we obtain the bound of Theorem 1 from (7) and

been formal in that (3) should be interpreted as a $\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (3) but with sites in a $(2N$ All of the steps above are valid for $\langle s_0 \cdot s_x \rangle(\beta, N)$ where by the corresponding fundamental solution

$$x-1)/(4-2\cos k_1-2\cos k_2), \quad (10)$$

$+1)^{-1} r_p r_i$ integers, $|r_i| \leq N$. To prove (8) we use

$$\sum_{\substack{i,j \\ L \\ x}} |k_1|/(k_1^2 + k_2^2) < 2.$$

y, uniformly in x, i, j, N .

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$n > 2$ we parameterize the n -sphere by angles ϕ . We then treat $\langle s_0^{(1)} s_x^{(1)} + s_0^{(2)} s_x^{(2)} \rangle = (2/n) \langle s_0 \cdot s_x \rangle$ as only the variables ϕ_i . Alternatively one may apply compare $N=2$ with $N \geq 3$. See [5, 6].

or the Square Well Model

ge of variable $\phi \rightarrow \phi/\mu$ reduces the problem to the $\langle \phi_{-A,I} \rangle(\beta)$ by $\langle \phi_0 \phi_x \rangle$ and note that it is the limit as expectations $\langle \phi_0 \phi_x \rangle_p$ defined by replacing $\int_{-1}^1 d\phi_i$ at using integration by parts we have for any $m > 0$

$$\langle p-I \rangle \langle \phi_0 \phi_x^{p-1} \rangle_p - \beta m \langle \phi_0 \phi_x \rangle_p \}.$$

$|x-x'|, \tilde{m} \equiv \cosh^{-1}(1+m^2/4)$,

ind $m > 0$ such that

$\varepsilon \geq 0$, all p .

t side of (11) is positive if

(11)

o eliminate the ferromagnetic couplings:

$$\begin{aligned} & \phi^2 e^{-\phi \rho} \phi^{\rho-2} / \int d\phi e^{-\nu \phi} \phi^2 e^{-\phi \rho} \\ & d\phi \phi^{\rho-2} / \int_{-\infty}^{\infty} d\phi e^{-\nu \phi} \phi^2 \end{aligned}$$

stisfies (12) for all p .
shown using inequalities for log concave
g range order, see [8].

317, 1133 (1960)

1971

67)

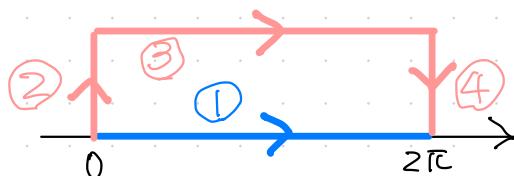
1976

1976

Lemma let $f(z)$ be analytic in $z \in \mathbb{C}$ and satisfies $f(z+2\pi) = f(z)$

then $\int_0^{2\pi} f(\theta) d\theta = \int_0^{2\pi} f(\theta + i\varphi) d\theta \quad (1) \quad \text{for any } \varphi \in \mathbb{R}$

proof $\int_0^{2\pi} f(\theta) d\theta = i \int_0^{\varphi} f(iy) dy + \int_0^{2\pi} f(\theta + iy) d\theta - i \int_0^{\varphi} f(iy + 2\pi) dy \quad (2)$



$i \int_0^{\varphi} f(iy) dy$

thus

$$\mathcal{Z}_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} = \mathcal{Z}_L(\beta) \langle e^{i(\theta_u - \theta_v)} \rangle_{L,\beta}$$

$$= \int d\Theta e^{i(\theta_u - \theta_v) + \beta \sum_{(w,w') \in B_c} \cos(\theta_w - \theta_{w'})}$$

$\theta_w \rightarrow \theta_w + i\varphi_w$

$$= \int d\Theta e^{i(\theta_u + i\varphi_u - \theta_v - i\varphi_v) + \beta \sum_{(w,w') \in B_c} \cos(\theta_w + i\varphi_w - \theta_{w'} - i\varphi_{w'})} \quad (3)$$

for any $\varphi_w \in \mathbb{R} \quad (w \in \Lambda_L)$

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$$\text{since } \cos(\theta + i\varphi) = \cos\theta \cosh\varphi - i \sin\theta \sinh\varphi$$

$$\mathbb{X}_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta}$$

$$= \int d\Theta e^{i(\theta_u + i\varphi_u - \theta_v - i\varphi_v) + \beta \sum_{\{w, w'\} \in B_L} \cos(\theta_w + i\varphi_w - \theta_{w'} - i\varphi_{w'})}$$

$$= e^{-\varphi_u + \varphi_v} \int d\Theta \left[e^{i\{\theta_u - \theta_v - \beta \sum_{\{w, w'\}} \sin(\theta_w - \theta_{w'}) \sinh(\varphi_w - \varphi_{w'})\}} \right.$$

$\times e^{\beta \sum_{\{w, w'\}} \cos(\theta_w - \theta_{w'}) \cosh(\varphi_w - \varphi_{w'})} \right]$

nonnegative

real (2)

$$\mathbb{X}_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq e^{-\varphi_u + \varphi_v} \int d\Theta e^{\beta \sum_{\{w, w'\}} \cos(\theta_w - \theta_{w'}) \cosh(\varphi_w - \varphi_{w'})}$$

$$\cos\theta \cosh\varphi = \cos\theta (\cosh\varphi - 1) + \cos\theta \leq (\cosh\varphi - 1) + \cos\theta$$

$$\leq e^{-\varphi_u + \varphi_v + \beta \sum_{\{w, w'\} \in B_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}} \int d\Theta e^{\beta \sum_{\{w, w'\} \in B_L} \cos(\theta_w - \theta_{w'})}$$

(3)

$$\mathbb{X}_L(\beta)$$

Lemma (McByran-Spencer) for any $\varphi_w \in \mathbb{R}$ ($w \in \Lambda_L$), we have

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq e^{-\varphi_u + \varphi_v + \beta \sum_{\{w, w' \in \mathcal{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}} \quad (1)$$

proved only by the change of variables!!

we shall choose proper φ_w

$$(\text{RHS of (1)}) \simeq e^{-A(\varphi)} \quad \varphi = (\varphi_w)_{w \in \Lambda_L}$$

$$\text{with } A(\varphi) = \varphi_u - \varphi_v - \frac{\beta}{2} \sum_{\{w, w' \in \mathcal{B}_L\}} (\varphi_w - \varphi_{w'})^2 \quad (2)$$

We can try maximizing $A(\varphi)$

$$\begin{aligned} \frac{\partial}{\partial \varphi_w} A(\varphi) &= S_{uw} - S_{vw} - \beta \sum_{w' \in N(w)} (\varphi_w - \varphi_{w'}) \\ &= S_{uw} - S_{vw} + \beta \sum_{w' \in \Lambda_L} \Delta_{ww'} \varphi_{w'} \end{aligned} \quad \begin{matrix} \text{part 3-p4-(5)} \\ (3) \end{matrix}$$

$$\frac{\partial}{\partial \varphi_w} A(\varphi) = 0 \quad (1) \Rightarrow -\sum_{w' \in \mathcal{N}_L} \Delta_{ww'} \varphi_{w'} = \frac{1}{\beta} (S_{uw} - S_{rw}) \quad (2)$$

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Poisson equation! part 3 - p11-(1)

if we choose φ_w as the solution of (2)

$$A(\varphi) = \varphi_u - \varphi_v - \frac{\beta}{2} \sum_{\{w, w' \in \mathcal{B}_L\}} (\varphi_w - \varphi_{w'})^2 = \varphi_u - \varphi_v + \frac{\beta}{2} \sum_{\{w, w' \in \mathcal{B}_L\}} \varphi_w \Delta_{ww'} \varphi_{w'}$$

$$= \frac{1}{2} \{ \varphi_u - \varphi_v \} \quad (3)$$

p5-(1) then implies

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{\beta, L} \lesssim e^{-\frac{1}{2} \{ \varphi_u - \varphi_v \}} \quad (4)$$

recovers part 3 - p10-(5) from the harmonic approximation!

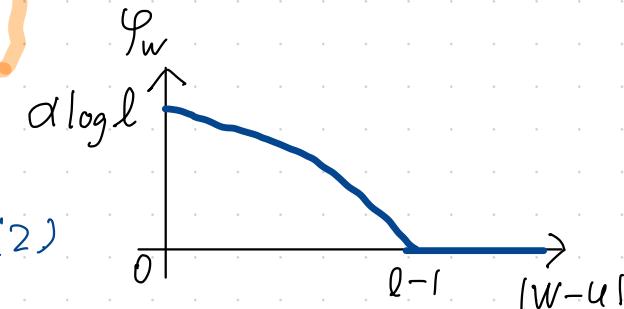
We use simpler and explicit choice of φ_w

Picco (1984)

for $\alpha > 0$, we set

$$\varphi_w = \begin{cases} \alpha \log \frac{l}{|w-u|+1}, & |w-u| \leq l-1 \\ 0, & |w-u| \geq l-1 \end{cases} \quad (1)$$

with $l = |u-w|$



we can show

$$\varphi_u = \alpha \log l, \quad \varphi_v = 0 \quad (2)$$

$$\sum_{\{(w,w') \in \mathcal{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} \leq C (\cosh \alpha - 1) \log l \quad (3)$$

with constant $C > 0$

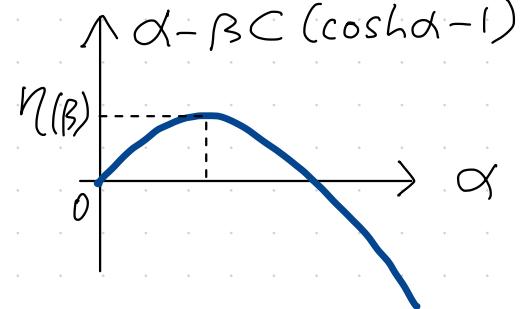
we then get

$$\begin{aligned} \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,B} &\stackrel{\text{PS-(1)}}{\leq} e^{-\varphi_u + \varphi_v + \beta \sum_{\{(w,w') \in \mathcal{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}} \\ &\leq e^{-\alpha \log l + \beta C (\cosh \alpha - 1) \log l} \\ &= l^{-(\alpha - \beta C (\cosh \alpha - 1))} \end{aligned} \quad (4)$$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq e^{-(\alpha - \beta C(\cosh \alpha - 1))} \quad (1)$$

let

$$\eta(\beta) = \max_{\alpha} \{\alpha - \beta C(\cosh \alpha - 1)\} > 0 \quad (2)$$



$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq |u-v|^{-\eta(\beta)} \quad (3)$$

the main result!

max is attained when $\sinh \alpha = \frac{1}{\beta C}$

$$\eta(\beta) = \sinh^{-1}\left(\frac{1}{\beta C}\right) - \beta C \left\{ \sqrt{1 + \frac{1}{(\beta C)^2}} - 1 \right\} \approx \frac{1}{2\beta C} \quad (5)$$



for $\beta C \gg 1$

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derivation of p7-(3)

$$\sum_{\{w, w'\} \in \mathcal{B}_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} \leq C(\cosh \alpha - 1) \log l \quad (1)$$

for $\{w, w'\} \in \mathcal{B}_L$ s.t. $|w - u| \leq |w' - u| \leq l$ (2)

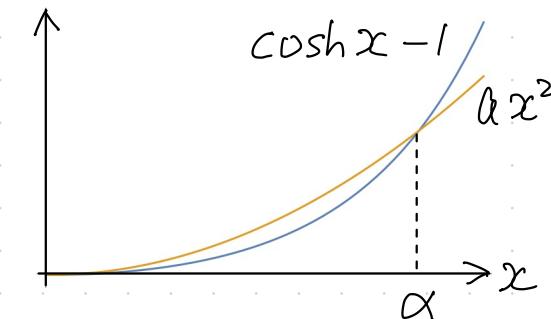
$$\begin{aligned} |\varphi_w - \varphi_{w'}| &= \alpha \log \frac{|w' - u| + 1}{|w - u| + 1} \leq \alpha \log \frac{|w - u| + l + 1}{|w - u| + 1} \\ \text{P7-(1)} \quad &= \alpha \log \left(1 + \frac{l}{|w - u| + 1} \right) \leq \frac{\alpha}{|w - u| + 1} \end{aligned} \quad (3)$$

note $\cosh x - 1 \leq \frac{\cosh \alpha - 1}{\alpha^2} x^2$ (4)

for any $x \in [0, \alpha]$

since $|\varphi_w - \varphi_{w'}| \leq \alpha$ (5)

$$\cosh(\varphi_w - \varphi_{w'}) - 1 \leq \frac{\cosh \alpha - 1}{\alpha^2} (\varphi_w - \varphi_{w'})^2 \leq \frac{\cosh \alpha - 1}{(l + |w - u|)^2} \quad (6)$$



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$$\sum_{\{w, w' \in \mathcal{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} \leq 4 (\cosh \alpha - 1) \sum_{\substack{w \in \mathcal{N}_L \\ |w-u| \leq l}} \frac{1}{(|w-u|+1)^2} \quad (1)$$

choice of w' ↑

$$\mathbf{r} = (r_1, r_2)$$

$$\begin{aligned} \sum_{\substack{w \in \mathcal{N}_L \\ |w-u| \leq l}} \frac{1}{(|w-u|+1)^2} &\leq C' \int_{|\mathbf{r}| \leq l} \frac{1}{(|\mathbf{r}| + C'')^2} d\mathbf{r} \\ &= C' \int_0^l \frac{2\pi r}{(r + C'')^2} dr = \frac{C}{4} \log l \quad (2) \end{aligned}$$

$$\sum_{\{w, w' \in \mathcal{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} \leq C (\cosh \alpha - 1) \log l \quad (3)$$



§ proof of the Hohenberg–Mermin–Wagner theorem

based on the McBryan–Spencer argument

Tasaki (2020)

the two-dimensional XY model under magnetic field h

$$H_{L,h}(\Theta) = - \sum_{\langle u,v \rangle \in B_L} \cos(\Theta_u - \Theta_v) - h \sum_{u \in \Lambda_L} \cos \Theta_u \quad (1)$$

$$Z_L(\beta, h) = \int d\Theta e^{-\beta H_{L,h}(\Theta)} \quad (2)$$

$$\langle \dots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta,h)} \int d\Theta (\dots) e^{-\beta H_{L,h}(\Theta)} \quad (3)$$

theorem for any $0 < \beta < \infty$

$$\lim_{h \downarrow 0} \lim_{L \uparrow \infty} \left\langle \frac{1}{L^\alpha} \sum_{u \in \Lambda_L} S_u^{(x)} \right\rangle_{L,\beta,h} = \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \langle S_u^{(x)} \rangle_{L,\beta,h}$$

$$= \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \langle e^{i\Theta_u} \rangle_{L,\beta,h} = 0 \quad (4)$$

repeating the argument in p3, p4

$$|\langle e^{i\theta_u} \rangle_{L,\beta,h}| \leq e^{-\varphi_u + \beta \sum_{\{w,w' \in \mathbb{B}_L\}} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} + \beta h \sum_{w \in \Lambda_L} (\cosh \varphi_w - 1)} \quad (1)$$

corresponds to P5-(1)

as in P7-(1) $\varphi_w = \begin{cases} \alpha \log \frac{l}{|w-u|+1}, & |w-u| \leq l-1 \\ 0, & |w-u| \geq l-1 \end{cases}$ (2).

with $0 < l \leq \frac{L}{2}$

$$|\langle S_u^{(x)} \rangle_{L,\beta,h}| \leq e^{-\eta(\beta) \log l + h G(\beta, l)} \quad (3)$$

with $G(\beta, l) = \beta \sum_{w \in \Lambda_L} (\cosh \varphi_w - 1)$

$$= \beta \sum_{w \in \Lambda_L} \left\{ \cosh \left(\alpha(\beta) \log \frac{l}{|w-u|+1} \right) - 1 \right\} \quad (4)$$

($|w-u| \leq l$)

$$|\langle S_u^{(x)} \rangle_{L, \beta, h}| \leq e^{-\eta(\beta) \log l + h G(\beta, l)} \quad (1)$$

take any $l > 0$

for $\begin{cases} L \geq 2l \\ h > 0 \text{ s.t. } h G(\beta, l) \leq \frac{1}{2} \eta(\beta) \log l \end{cases}$ (2)

we have $|\langle S_u^{(x)} \rangle_{L, \beta, h}| \leq e^{-\frac{1}{2} \eta(\beta) \log l}$ (3)

thus

$$\lim_{h \downarrow 0} \lim_{L \uparrow \infty} |\langle S_u^{(x)} \rangle_{L, \beta, h}| \leq e^{-\frac{1}{2} \eta(\beta) \log l} \quad (4)$$

since l is arbitrary

$$\lim_{h \downarrow 0} \lim_{L \uparrow \infty} |\langle S_u^{(x)} \rangle_{L, \beta, h}| = 0 \quad (5)$$

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