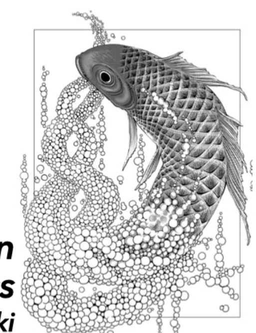
# The absence of ferromagnetic order in the two-dimensional XY model

## part 5 exponential decay of correlations at high temperatures (appendix)



Advanced Topics in Statistical Physics
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## § Setting and the goal

$$\Lambda_L = \{1, \dots, L\}^d \quad (1) \qquad d = 1, 2, \cdots$$

$$\mathcal{B}_L = \{\{u, v\} \mid u \text{ and } v \text{ are nearest neighbors (periodic b.c.)}\}$$

$$\Theta = (\theta_u)_{u \in \Lambda_L}, \quad \theta_u \in [0, 2\pi), \quad \int d\Theta = \prod_{u \in \Lambda_L} \int_0^{2\pi} d\theta_u \qquad (2)$$

$$H_L(\Theta) = -\sum_{\{u,v\} \in \mathcal{B}_L}{}' \cos(\theta_u - \theta_v) \qquad (3)$$

$$Z_L(\beta) = \int d\Theta \, e^{-\beta H_L(\Theta)} \qquad (4)$$

$$\langle \cdots \rangle_{L,\beta} = \frac{1}{Z_L(\beta)} \int d\Theta \, (\cdots) \, e^{-\beta H_L(\Theta)} \qquad (5)$$

theorem for any $0 < \beta < \frac{2}{2d-1}$

$$0 \leq \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq \left(1 - \frac{2d-1}{2}\beta\right)^{-1} \left(\frac{2d-1}{2}\beta\right)^{|u-v|} \qquad (6)$$

for any $u, v \in \Lambda_L$ s.t. $|u - v| \leq \frac{L}{2}$

## § random current representation

$\{w, w'\} \in \mathcal{B}_L$

$$e^{\beta\cos(\theta_w - \theta_{w'})} = e^{\frac{\beta}{2}\{e^{i(\theta_w-\theta_{w'})} + e^{-i(\theta_w-\theta_{w'})}\}}$$

$$= \sum_{n,n'=0}^{\infty} \left(\frac{\beta}{2}\right)^{n+n'} \frac{1}{n!\,n'!} e^{in(\theta_{w'}-\theta_w)} e^{in'(\theta_w-\theta_{w'})} \quad (1)$$

$$\boxed{\begin{array}{ll} n \to n_{ww'} & \text{current from } w \text{ to } w' \\ n' \to n_{w'w} & \text{current from } w' \text{ to } w \end{array}}$$

$(w, w') \neq (w', w)$

the set of ordered bonds $\quad \overline{\mathcal{B}}_L = \{(w, w') \mid \{w, w'\} \in \mathcal{B}_L\} \quad (2)$

current configuration $\quad \mathbf{n} = (n_{ww'})_{(w,w')\in\overline{\mathcal{B}}_L} \quad (3)$

$n_{ww'} = 0, 1, 2, \cdots$

$$|\mathbf{n}| = \sum_{(w,w')\in\overline{\mathcal{B}}_L} n_{ww'} \quad (4) \qquad \mathbf{n}! = \prod_{(w,w')\in\overline{\mathcal{B}}_L} n_{ww'}! \quad (5)$$

(with $0! = 1$)

$$e^{-\beta H_L(\Theta)} = \prod_{\{w,w'\}\in \mathcal{B}_L} e^{\beta \cos(\theta_w - \theta_{w'})}$$

$$= \sum_{\mathbf{n}} \left(\frac{\beta}{2}\right)^{\mathbf{n}} \frac{1}{\mathbf{n}!} \prod_{(w,w')\in \vec{\mathcal{B}}_L} e^{i n_{ww'}(\theta_{w'} - \theta_w)}$$

$$= \sum_{\mathbf{n}} \left(\frac{\beta}{2}\right)^{\mathbf{n}} \frac{1}{\mathbf{n}!} \prod_{w\in \Lambda_L} e^{-i(\operatorname{div}\mathbf{n})_w \theta_w} \qquad (1)$$

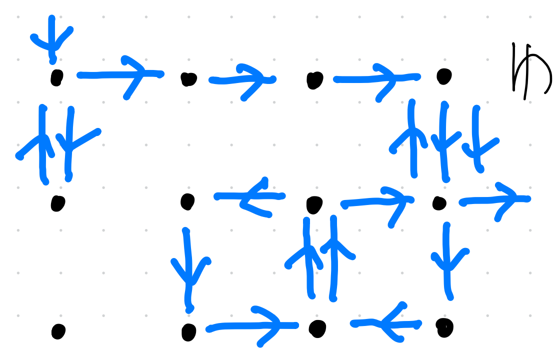
with $(\operatorname{div}\mathbf{n})_w = \sum_{w'\in n(w)} (n_{ww'} - n_{w'w})$ (2)

note $\int_0^{2\pi} e^{im\theta} d\theta = \begin{cases} 2\pi & m=0 \\ 0 & m\in \mathbb{Z}\setminus\{0\} \end{cases}$ (3)

random current representation of $Z(\beta)$ (a stochastic geometric representation)

$$Z(\beta) = \int d\Theta\, e^{-\beta H_L(\Theta)}$$

$$= (2\pi)^{|\Lambda_L|} \sum_{\substack{\mathbf{n} \\ (\operatorname{div}\mathbf{n} = 0)}} \left(\frac{\beta}{2}\right)^{|\mathbf{n}|} \frac{1}{\mathbf{n}!} \qquad (4)$$



## random current representation of (unnormalized) correlation

$$Z_L(\beta)\langle \vec{S}_u\cdot\vec{S}_v\rangle_{L,\beta} = \int d\Theta\, e^{i(\theta_u-\theta_v)}\, e^{-\beta H_L(\Theta)}$$

$$= \int d\Theta \sum_{\mathbf{n}} \left(\frac{\beta}{2}\right)^{|\mathbf{n}|} \frac{1}{\mathbf{n}!}\, e^{i(\theta_u-\theta_v)} \prod_{w\in\Lambda_L} e^{-i(\mathrm{div}\,\mathbf{n})_w \theta_w}$$

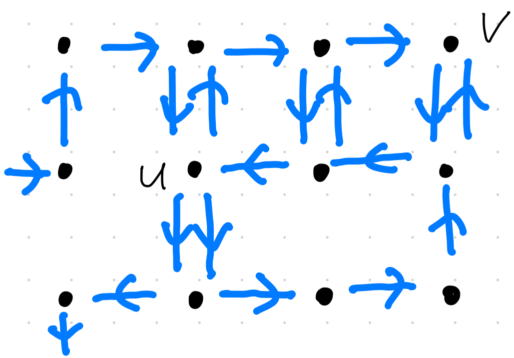
$$= (2\pi)^{|\Lambda_L|} \sum_{\mathbf{n}\in\mathcal{C}_{u\to v}} \left(\frac{\beta}{2}\right)^{|\mathbf{n}|} \frac{1}{\mathbf{n}!} \geq 0 \qquad (1)$$

this proves $\langle \vec{S}_u\cdot\vec{S}_v\rangle_{L,\beta} \geq 0$

$$\mathcal{C}_{u\to v} = \{\mathbf{n} \mid (\mathrm{div}\,\mathbf{n})_u = 1,\ (\mathrm{div}\,\mathbf{n})_v = -1,\ (\mathrm{div}\,\mathbf{n})_w = 0 \text{ for } w\neq u,v\} \qquad (2)$$

the set of current configurations with a current from u to v

self-avoiding walk with $|W| = m$ steps




**lemma** let $\mathbf{n}\in\mathcal{C}_{u\to v}$

There exists a sequence $W = (w_0, w_1, \ldots, w_m)$

s.t. $w_0 = u$, $w_m = v$, $w_j \neq w_{j'}$ if $j \neq j'$,

$(w_{j-1}, w_j) \in \overline{\mathcal{B}}_L$, and $n_{w_{j-1}w_j} \geq 1$ for all $j = 1, \ldots, m$

## upper bound on the correlation function

for $\mathfrak{n} \in \mathcal{C}_{u\to v}$ and a corresponding self-avoiding walk $W$

let $$n'_{ww'} = \begin{cases} n_{ww'} - 1 & \text{if } (w,w') = (w_{j-1}, w_j) \text{ for some } j \\ n_{ww'} & \text{otherwise} \end{cases} \quad (1)$$

clearly $\operatorname{div} \mathfrak{n}' = 0$ (2) $\quad |\mathfrak{n}| = |\mathfrak{n}'| + |W|$ (3) $\quad \dfrac{1}{\mathfrak{n}!} \leq \dfrac{1}{\mathfrak{n}'!}$ (4)

$$Z_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq \sum_{W: u\to v}{}' (2\pi)^{|\Lambda_L|} \sum_{\substack{\mathfrak{n} \in \mathcal{C}_{u\to v} \\ (W \text{ is obtained from } \mathfrak{n})}}{}' \left(\frac{\beta}{2}\right)^{|\mathfrak{n}|} \frac{1}{\mathfrak{n}!}$$

all self-avoiding walks from u to v

$$\leq \sum_{W: u\to v}{}' \left(\frac{\beta}{2}\right)^{|W|} \underbrace{(2\pi)^{|\Lambda_L|} \sum_{\substack{\mathfrak{n}' \\ (\operatorname{div} \mathfrak{n}' = 0)}}{}' \left(\frac{\beta}{2}\right)^{|\mathfrak{n}'|} \frac{1}{\mathfrak{n}'!}}_{= Z_L(\beta)} \quad (5)$$

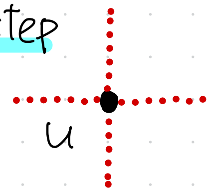
P3-(4)

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq \sum_{W: u \to v} \left(\frac{\beta}{2}\right)^{|W|} = \sum_{m=|u-v|}^{\infty} N_m \left(\frac{\beta}{2}\right)^m \quad (1)$$

when $|u-v| \leq \frac{L}{2}$

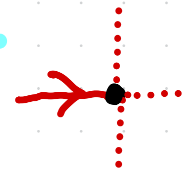
$N_m$ : the number of $m$-step self-avoiding walks from $u$ to $v$

1st step



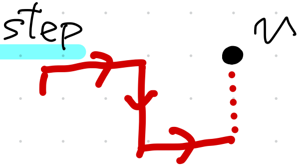

2d choices

2nd ~ (m-1)-th steps



at most $(2d-1)$ choices

m-th step




at most 1 choice

$$N_m \leq 2d\,(2d-1)^{m-2} \leq (2d-1)^m \quad (2)$$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq \sum_{m=|u-v|}^{\infty} (2d-1)^m \left(\frac{\beta}{2}\right)^m = \left(1 - \frac{2d-1}{2}\beta\right)^{-1} \left(\frac{2d-1}{2}\beta\right)^{|u-v|} \quad (3)$$

if $\frac{2d-1}{2}\beta < 1 \quad (4)$