**Problem Set 8**

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**solutions**

**Problem 1**

Consider the production function Y = √N √K, where Y is output, N is labor, and K is capital.

1. Derive the relation between the growth rate of output, gY, the growth rate of capital, gK, and the growth rate of labor, gN.

The growth rate of a product is the sum of the growth rates of single variables. Hence,

Moreover, the growth rate of a power function equals the exponent times the growth rate of the base variable. Hence

B) Assume that N is constant, and suppose we want to achieve output growth equal to 2% per year. What is the required rate of growth of capital?

From the equation (1), we see that if , the equation boils down to . Hence, if we want , we need .

C) In the situation described in B), what happens to the ratio of capital to output, K/Y, over time?

Since the growth rate of a ratio is a difference between the growth rates of the numerator and that of the denominator, and we see that is growing over time at 2% per year. This means that over time the capital stock would become infinitely larger than output.

**Problem 2**

Consider a Solow model with production function Y = N1–a × Ka, where Y is output, N is labor, K is capital, and 0 < a < 1 is a parameter. The number of workers, N, is fixed. The saving rate is s > 0 and the depreciation rate is δ > 0.

1. Use the production function to compute output per worker y as a function of capital per worker k.

Dividing the production function by , we get that . Hence, since , and , we get that

1. Express investment per worker as a function of the saving rate s and capital per worker k. What are the two assumptions required to reach this result?

First, we assume that the economy is closed and private (i.e. that there is no foreign and government savings in the economy; or, alternatively, that the net savings of these sectors are always zero). Second, we assume that people save a fixed fraction of their incomes.

In that case, we know that investment equals private savings: . Finally, we divide this equation by in order to express everything in per-capita terms:

where , and .

C) Find an equation relating capital in period t+1, K(t+1), to capital in period t, K(t), and investment in period t, I(t).

Capital stock is growing due to investment, ,while it is decreasing because of depreciation, . Therefore, the next period capital stock is given by

D) Using the results from questions B) and C), derive the law of motion for capital per worker. That is, find an equation relating capital per worker in period t+1, k(t+1), to capital per worker in period t, k(t), and parameters of the model.

Since the population is constant in this version of the model, we divide equation (3) by , to get

And then, using (2), we finally get

E) Derive the steady-state level of capital per worker in terms of the saving rate, s, and the depreciation rate, δ, and the production-function parameter, a.

By definition, the steady-state capital per worker is a level of capital per worker, such that

Hence, applying this condition to equation (4), we get . Simplifying yields , meaning that depreciation and investment per capita are the same in the steady-state. Solving this equation for , we get

F) Derive the steady-state levels of output per worker and consumption per worker in terms of the saving rate, s, and the depreciation rate, δ, and the production-function parameter, a.

From (1), we get that . Hence, using (5), the steady-state output per worker is

Next, the consumption per worker equals output per worker net of savings per worker. Hence, . Thus, using (6), the steady-state consumption per worker is

G) Set a = 0.5 and δ = 5%. Compute steady-state output per worker and steady-state consumption per worker for s = 0%; s = 5%; s = 10%; s = 20%; s = 30%; s=50%; s=70%; and s=100%. Explain the intuition behind your results.

From (6), we see that . Therefore, steady-state output per capita for these levels of s would be: 0; 1; 2; 4; 6; 10; 14; 20.

From (7), we see that . Therefore, steady-state consumption per worker for these levels of s would be: 0; 0.95; 1.8; 3.2; 4.2; 5; 4.2; 0.

So, we see that the steady-state output per worker is monotonically increasing in savings rate: capital is accumulating faster. At the same time, while higher savings increase consumption per worker via higher output per worker, higher savings also mean that the part of output left for consumption is lower. Therefore, there is a trade-off. At very low and very high levels of saving rate, consumption per worker goes to zero.

H) Given a production-function parameter a and depreciation rate δ, find the value of the saving rate s that maximizes consumption per worker. What is the name of this value?

From (7), we can find the so-called “Golden Rule” level of saving rate that solves the problem of

The easiest way could be to first take the logs of this expression:

Where we can ignore the last term with . Using the first order condition, , we get that

I) Compute consumption per worker when the saving rate takes the value obtained in question H).

When , from (7), we get that

J) How does the value of the saving rate found in H) and the value of consumption found in I) depend on the production-function parameter a and depreciation rate δ? Discuss.

From (8), we see that is increasing one-to-one with , and does not depend on . From (9), we also see that is decreasing in , and increasing in .

**Problem 3**

Consider a Solow model with production function Y = K2/3 × N1/3. The saving rate is 10% and the depreciation rate is 6%. The number of workers in the economy is N = 100.

A) In steady state, at what rates do output, output per worker, consumption, and consumption per worker grow? Why?

In a steady-state, capital per worker is constant, and since the number of workers is also constant, the total stock of capital is constant in a steady-state. Therefore, output, output per worker are constant in a steady-state. Moreover, since consumption and consumption per worker are just a fraction of, respectively, output and output per worker, they are also constant.

B) Draw the equilibrium diagram of this Solow model. Show where the steady state is. Give the equations for all the curves you have placed on the diagram.



Depreciation per worker is given by , which is is our case. Output per worker is given by , which is in our case. Investment per worker equals saving per worker and is thus equal to , which in our case is .

C) Solve for capital per worker and output per worker in steady state.

As usual, the steady-state capital per worker is defined as , which also means that depreciation per worker equals investment per worker. Thus, should be equal to in the steady state. Hence, . From this we get that . Therefore, output per worker in a steady state would be equal to

D) Solve for consumption per worker and investment per worker in steady state.

Consumption per worker in a steady-state is , which, using (1), equals to

Investment per worker is the remaining part of output per worker, , which equals to

E) Does the economy have more or less capital per worker than at the golden-rule steady state? To achieve the golden-rule steady state, does the saving rate need to increase or decrease?

We have shown in the previous problem (that shares the same structure) that the maximum consumption per worker in a steady-state is achieved when the savings rate is at its Golden Rule level: . In our case, therefore, . However, savings are set at the level . Thus, savings are lower compared to the level that gives maximum consumption. Moreover, since at the golden-rule steady state is higher, capital per worker would also be higher (see equation (5) in the previous problem). To achieve the golden-rule steady-state, saving rate needs to be increased.

F) Suppose the change in saving rate described in question E) occurs. Draw the evolution of consumption per worker over time: in the old steady state, after the change in saving rate, and in the golden-rule steady state. Explain.



G) Suppose the change in saving rate described in question E) occurs. Draw the evolution of output per worker over time: in the old steady state, after the change in saving rate, and in the golden-rule steady state. Explain.

On the graph above, the dynamics of output per worker is also depicted. Before the change in the savings rate, output per worker is constant (see part (A)). At the moment of the change, output per worker does not change sharply because the output per worker at time t is determined by current capital per worker, which is unaffected by current change in s (it is predetermined by the previous levels of capital and the parameters of the model). However, at the subsequent periods, output per worker starts to grow due to faster accumulation of capital (higher investment due to higher savings). Eventually, output per capita approaches a new (higher) steady state level.