FYS 4110/9110 Modern Quantum Mechanics
Midterm Exam, Fall Semester 2018

Return of solutions:
The problem set is available from Monday morning, 15 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 22 October, at 12:00.
Use candidate numbers rather than full names.

Language:

Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems:

Please ask Joakim Bergli (room V405, or on the Piazza page).

The problem set consists of 1 problem written on 6 pages.

Problem 1: Sqeezed states for enhancing the sensitivity of gravitaional wave detectors

We have in the lectures studied coherent states of the harmonic oscillator as examples of minimal uncertainty states. Here we will consider a related class of minimal uncertainty states called squeezed states. We will first study their general properties, and then see how they can be used to enhance the sensitivity of interferometers used in gravitational wave detectors.

We define the squeeze operator

$$S(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$$

where ζ is a complex number and $\hat{a}$ and $\hat{a}^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator. The squeezed vacuum state is defined as

$$|sq_\zeta\rangle = S(\zeta)|0\rangle$$

a) Show that the action of the squeeze operator on $\hat{a}$ and $\hat{a}^\dagger$ is given by

$$S^\dagger(\zeta)\hat{a}S(\zeta) = \cosh r\hat{a} - e^{i\theta}\sinh r\hat{a}^\dagger$$
$$S^\dagger(\zeta)\hat{a}^\dagger S(\zeta) = \cosh r\hat{a}^\dagger - e^{-i\theta}\sinh r\hat{a}$$

where $\zeta = re^{i\theta}$.

b) In the state $|sq_\zeta\rangle$, find the variance of the position and momentum operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right).$$

That is, calculate

$$\Delta x^2 = \langle sq_\zeta|\hat{x}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{x}|sq_\zeta\rangle^2$$
$$\Delta p^2 = \langle sq_\zeta|\hat{p}^2|sq_\zeta\rangle - \langle sq_\zeta|\hat{p}|sq_\zeta\rangle^2$$

c) The Heisenberg uncertainty relation tells us that $\Delta x \Delta p \geq \frac{\hbar}{2}$ with equality only for minimal uncertainty states. Calculate the product $\Delta x \Delta p$ for the states $|sq_\zeta\rangle$ and show that for certain θ they are minimal uncertainty states. For those θ which gives minimal uncertainty, compare Δx and Δp with the corresponding values in vacuum and describe what happens to the uncertainties.

d) Find the expectation value of the number operator $\hat{a}^\dagger \hat{a}$ in the state $|sq_\zeta\rangle$. Later we will apply the theory of squeezed states to a mode of the electromagnetic field, which we know is equivalent to a harmonic oscillator. This expectation value is then interpreted as the mean number of photons in the mode.

The squeezed vacuum state can be displaced to create the squeezed coherent states

$$|\alpha, sq_\zeta\rangle = D(\alpha)S(\zeta)|0\rangle.$$

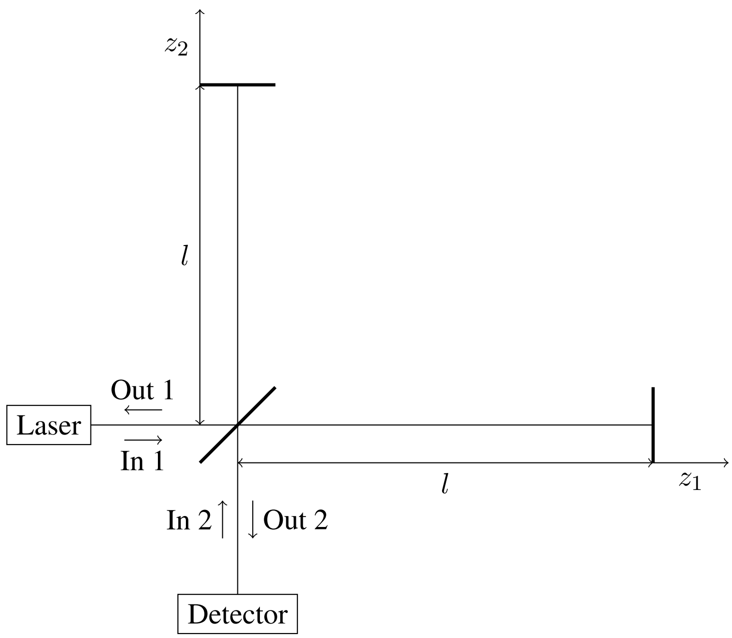
We will now study some properties of these states.

e) Show that these states are still minimal uncertainty states, and that their uncertainties are the same as for the squeezed state $|sq_\zeta\rangle$. Find the expectation values of position and momentum in terms of α and ζ.

f) We have defined the sqeezed coherent states as $|\alpha, sq_\zeta\rangle = D(\alpha)S(\zeta)|0\rangle$. That is, we first sqeeze the vacuum, and then displace. The operators $= D(\alpha)$ and $S(\zeta)$ do not commute. Investigate the states $|sq_\zeta, \alpha\rangle = S(\zeta)D(\alpha)|0\rangle$. That is, we first displace and then squeeze. You may find information on this in the literature, and you should include references to all sources that you use.

The use of squeezed states to reduce the noise in gravitational wave interferometers was first proposed by C. Caves, Phys. Rev. D **23**, 1693 (1981). A recent overview is provided by R. Schnabel *et al.*, Nat. Commun. **1**, 121 (2010) and demonstration of the practical use is shown in J. Asai *et al.*, Nat. Photonics **7**, 613 (2013).

To detect gravitational waves, one can use a Michelson interferometer as shown:



Light is aimed at a semitransparent mirror (beam splitter), which splits it into two perpendicular beams. These are reflected back from distant mirrors, and recombined at the beam splitter. Interference beween the two beams will give rise to interference fringes with alternating constructive and destructive interference depending on the exact path length difference. The interferometer is normally operating with the detector at a dark point in the interference pattern, so that in the absence of a signal, there are (ideally) no photons reaching the detector. The end mirrors, where the light is reflected back to the beamspliter, are mounted on large suspended masses (with mass m), which ideally do not move. When a gravitational wave passes through the interferometer, the lengths of the arms change, the fringes move, and the light intensity (photon counting rate) oscillates.

g) In the LIGO-detector (which was the first to detect a real gravitational wave), the distance from the beam splitter to the mirrors is $l = 4$ km. The strain amplitude (ratio of length change to initial length) of a realistic gravitational wave of cosmic origin (inspiraling of two black holes) is 10^{-21}. How small displacement differences $\Delta z = z_2 - z_1$ of the interferometer mirrors do we have to detect to see the gravitational wave signal? Compare your answer to some relevant physical dimension.

There are several sources of noise that will reduce the sensitivity of the interferometer. In this problem we will focus on two fundamental quantum mechanical noise limits, and ignore any practical problems (which are not trivial in practice). The first effect is called photon-counting error (or shot noise) and is a consequence of the fact that the laser light used is not in a number eigenstate, but rather close to a coherent state. This means that the photon number is not a sharply defined quantity, and it will fluctuate in time as a result of quantum uncertainty. The second effect is called radiation-pressure error, and is a result of the fluctuating motion of the mirrors because of the fluctuating radiation pressure in the laser beams. This is again because the photon number is not well-defined, and is therefore also a fundamental quantum restriction.

Normally, one would input coherent light (from a powerful laser) in the input port 1, and arrange the interferometer so that in the absense of any gravitational wave signal all the light would exit back in the same direction, while there will be complete destructive interference in the output port 2. Port

2 would be used only for output, with no (that is, the vacuum state) input. Surprisingly, the noise can be modified by the input of a squeezed vacuum state in port 2, instead of the normal vacuum. To investigate this effect, we need to understand how to find the combined state of the field from the two sources. One has to add the electric fields from each source, and it can be shown that this leads to relations between the creation and anihilation operators for the modes.

h) For the radiation pressure noise we need to consider the relation between the field before and after passing the beamsplitter. Let $\hat{a}_1^\dagger$ and $\hat{a}_1$ be the creation and annihilation operators for photons in input mode 1 (moving horizontally in the figure), while $\hat{a}_2^\dagger$ and $\hat{a}_2$ are the corresponding operators for mode 2 (moving vertically). The operators for the horizontal mode after the beamsplitter is $\hat{b}_1^\dagger$ and $\hat{b}_1$, and those for the vertical mode are $\hat{b}_2^\dagger$ and $\hat{b}_2$. The relation between the operators are similar to those we have used to relate states passing beamsplitters:

$$\begin{aligned}\hat{b}_1 &= \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2)\\ \hat{b}_2 &= \frac{1}{\sqrt{2}}(\hat{a}_2 + i\hat{a}_1)\end{aligned}$$

The momentum of a photon is $p = E/c = \hbar\omega/c$. The momentum transfer to the mirror is twice the momentum of a single photon times the number of photons. The change in the interferometer output depends only on the difference in the change in path length, and therefore only on the difference in the transferred momenta to the two end mirrors. The difference in the transferred momentum is then

$$P = \frac{2\hbar\omega}{c}(\hat{b}_2^\dagger\hat{b}_2 - \hat{b}_1^\dagger\hat{b}_1)$$

Find the expectation values of P and P^2 if the input state is

$$|\psi\rangle = S_2(\zeta)D_1(\alpha)|0\rangle$$

where $S_2(\zeta) = e^{\frac{1}{2}(\zeta^*\hat{a}_2^2 - \zeta\hat{a}_2^{\dagger 2})}$ is the squeezing operator in incoming mode 2 and $D_1(\alpha) = e^{\alpha\hat{a}_1^\dagger - \alpha^*\hat{a}_1}$ is the displacement operator in incoming mode 1. That is, we have a coherent state (with typically large intensity) in mode 1 and a squeezed vacuum state in mode 2. You need only consider the case where both α and $\zeta = r$ are real.

i) The effect of the radiation pressure fluctutaions builds up over time as the momentum transferred to the end mirrors leads to displacement. If we define the variance of P as $(\Delta P)^2 = \langle\psi|P^2|\psi\rangle - \langle\psi|P|\psi\rangle^2$, argue that the variance in path difference after a time τ is $\Delta z_{rp} = \frac{\tau}{2m}\Delta P$ and show that it is given by

$$\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\sqrt{\alpha^2 e^{2r} + \sinh^2 r}.$$

In what way does Δz_{rp} depend on the power of the laser beam in input 1? On the mass of the and mirrors? How can we reduce Δz_{rp}?

j) For the photon counting error we need to consider the output modes, after the light has passed through the beamsplitter, reflected from the mirrors and passed the beamsplitter the second time.

We let $\hat{c}_1^\dagger$,$\hat{c}_1$ and $\hat{c}_2^\dagger$,$\hat{c}_2$ denote the creation and annihilation operators of the two output modes. Show that

$$\hat{c}_1 = ie^{i\Phi}\left[-\hat{a}_1 \sin\phi + \hat{a}_2 \cos\phi\right]$$
$$\hat{c}_2 = ie^{i\Phi}\left[\hat{a}_1 \cos\phi + \hat{a}_2 \sin\phi\right]$$

Find the expressions for Φ and ϕ and explain their physical meaning.

k) Show that the expectation value of the number operator $\hat{N}_2 = \hat{c}_2^\dagger \hat{c}_2$ in output port 2 is (for real α and ζ)

$$\langle\psi|\hat{N}_2|\psi\rangle = \alpha^2 \cos^2\phi + \sinh^2 r \sin^2\phi$$

and that the variance is

$$(\Delta N_2)^2 = \langle\psi|\hat{N}_2^2|\psi\rangle - \langle\psi|\hat{N}_2|\psi\rangle^2 = \alpha^2\cos^4\phi + 2\sinh^2 r\cosh^2 r\sin^4\phi + (\alpha^2 e^{-2r} + \sinh^2 r)\cos^2\phi\sin^2\phi.$$

l) As for the radiation pressure noise we can convert this into an uncertainty in the difference in the displacements $z = z_2 - z_1$ of the two mirrors. Show that a change of z by Δz gives a change in ϕ by $\Delta\phi = \frac{\omega}{c}\Delta z$. Show that this gives the noise in position difference due to photon counting noise

$$\Delta z_{pc} = \frac{c}{2\omega}\sqrt{\frac{\cot^2\phi}{\alpha^2} + \frac{2\tan^2\phi\sinh^2 r\cosh^2 r}{\alpha^4} + \frac{e^{-2r}}{\alpha^2} + \frac{\sinh^2 r}{\alpha^4}}.$$

We can reduce the photon counting noise by choosing the proper phase difference in the absence of a signal. Working near a dark point in the interference pattern we have $\cos\phi \approx 0$ the first term in the above expression is small. If α is sufficiently large, the second term can also be small provided we are not exactly on the dark point so that $\tan\phi$ is not too large. The last term can also be neglected compared to the third, and we are left with the approximate expression

$$\Delta z_{pc} = \frac{c}{2\omega}\frac{e^{-r}}{\alpha}.$$

Similarly we have for the radiation pressure noise approximately

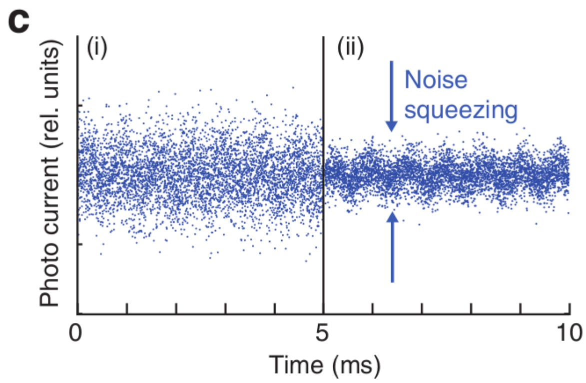
$$\Delta z_{rp} = \frac{\hbar\omega\tau}{mc}\alpha e^r.$$

If we assume that the noise sources are independent (which probably is not true), we get that the total noise is

$$\Delta z = \sqrt{\Delta z_{pc}^2 + \Delta z_{rp}^2}.$$

m) Discuss the dependence of the two noise sources on the laser power and the squezzing parameter r. The power is proportional to the number of photons, which has an average value of α^2. Minimize the total noise as a function of α^2 and determine how the optimal power and the minimal noise depend on r.

Here is a figure [Figure 6c of R. Schnabel *et al.*, Nat. Commun. **1**, 121 (2010)] showing the simulated change in the signal from the detector without (left) and with (right) input of sqeezed light.



As we see, the signal is virtually invisible without squeezing, and is clearly seen with squeezing.

n) Create a plot similar to the one shown above. You are at this point allowed to use any simplifying assumptions you need and any method that you find useful. But you should carefully describe your procedure and any assuptions made, discussing how realistic they are.