# Lecture 6: Iterative methods

Michael S. Floater

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These notes are based on Sections 2.11–2.12 of Chapter 2 of the book.

## 1 Iterative schemes and splitting

Given a linear system $A\mathbf{x} = \mathbf{b}$ where $A$ is an $n \times n$ matrix and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$, solving it by factorization can be expensive for large $n$. It might be more efficient to use an iterative method. The simplest iterative schemes are based on so-called *splitting*. We choose a matrix $B$ and rewrite the system as

$$(A - B)\mathbf{x} = -B\mathbf{x} + \mathbf{b}. \tag{1}$$

We choose $B$ such that $A - B$ is non-singular and such that the system

$$(A - B)\mathbf{x} = \mathbf{y}$$

is easy to solve for any right-hand side $\mathbf{y}$. For example, we might choose $B$ to make $A - B$ diagonal or triangular.

Having chosen $B$ we use an iterative method to find $\mathbf{x}$. We make an initial guess (approximation) $\mathbf{x}^{(0)}$ for the solution $\mathbf{x}$ although $\mathbf{x}^{(0)}$ can be arbitrary. We then generate a sequence $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$, and so on, by solving

$$(A - B)\mathbf{x}^{(k+1)} = -B\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2, \ldots. \tag{2}$$

If this sequence converges to a limit

$$\mathbf{x} = \lim_{k \to \infty} \mathbf{x}^{(k)},$$

then by taking the limit of both sides of (2), we obtain equation (1) and therefore $\mathbf{x}$ solves $A\mathbf{x} = \mathbf{b}$.

What are the necessary and sufficient conditions for convergence? Suppose that $A$ is non-singular so that there is a unique solution $\mathbf{x}$. Consider the $k$-th error,

$$\mathbf{e}^{(k)} := \mathbf{x}^{(k)} - \mathbf{x}.$$

By subtracting equation (1) from equation (2), we deduce that

$$(A - B)\mathbf{e}^{(k+1)} = -B\mathbf{e}^{(k)}, \qquad k = 0, 1, 2, \ldots.$$

Under our assumption that $A - B$ is non-singular this means that

$$\mathbf{e}^{(k+1)} = H\mathbf{e}^{(k)}, \qquad k = 0, 1, 2, \ldots, \tag{3}$$

where the matrix

$$H := -(A - B)^{-1}B$$

is the *iteration matrix*. In practical applications we do not calculate $H$. We are just using it here to analyze the convergence of (2). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues (real or complex) of $H$. Recall that the *spectral radius* of $H$ is

$$\rho(H) = \max_{i=1,\ldots,n} |\lambda_i|.$$

**Theorem 1** *The error vectors* $\mathbf{e}^{(k)}$ *converge to zero as* $k \to \infty$ *if and only if* $\rho(H) < 1$.

*Proof.* Applying (3) recursively gives

$$\mathbf{e}^{(k)} = H^k\mathbf{e}^{(0)}.$$

Thus the theorem is equivalent to the fact that $H^k \to 0$ as $k \to \infty$ if and only if $\rho(H) < 1$. This can be proved using the Jordan normal form of $H$ (see R. Varga, *Matrix iterative analysis*). $\square$

# 2 Jacobi and Gauss-Seidel iterations

Both of these splitting methods can be used when $A$ has non-zero diagonal elements. We write $A$ in the form $A = L + D + U$ where $L$ is the strictly lower triangular (subdiagonal) part of $A$, $D$ is the diagonal, and $U$ is the strictly upper triangular (superdiagonal) part of $A$.

## 2.1 Jacobi iteration

We choose $B = L + U$, so that $A - B = D$, the diagonal part of $A$. Then

$$Dx^{(k+1)} = -(L + U)x^{(k)} + b, \qquad k = 0, 1, 2, \ldots.$$

Written out in full,

$$x_i^{(k+1)} = \frac{1}{A_{i,i}} \left( -\sum_{j \neq i} A_{i,j} x_j^{(k)} + b_i \right), \qquad i = 1, \ldots, n.$$

## 2.2 Gauss-Seidel iteration

We choose $B = U$, so that $A - B = L + D$, which is lower triangular. Then

$$(L + D)\mathbf{x}^{(k+1)} = -U\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2, \ldots.$$

There is no need to invert $L + D$: we just use forward substitution,

$$x_i^{(k+1)} = \frac{1}{A_{i,i}} \left( -\sum_{j<i} A_{i,j} x_j^{(k+1)} - \sum_{j>i} A_{i,j} x_j^{(k)} + b_i \right), \qquad i = 1, \ldots, n.$$

An advantage of Gauss-Seidel iteration compared to Jacobi iteration is that in the implementation we do not need to keep both of the vectors $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$. We just maintain a single vector $\mathbf{y}$ and update it, replacing $y_i$ by

$$\frac{1}{A_{i,i}} \left( -\sum_{j \neq i} A_{i,j} y_j + b_i \right)$$

in sequence for $i = 1, \ldots, n$.

Sufficient conditions for these two iterations to converge are as follows. A matrix $A$ is said to be *strictly diagonally dominant* if

$$|A_{i,i}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{i,j}| \qquad \text{for } i = 1, \ldots, n.$$

Strictly diagonally dominant matrices are non-singular (due to the Gerschgoring theorem).

**Theorem 2** *If $A$ is strictly diagonally dominant, then both the Jacobi and the Gauss-Seidel methods converge.*

**Theorem 3** *If $A$ is symmetric and positive definite, then the Gauss-Seidel method converges. If $A$ is symmetric and both $A$ and $2D - A$ are positive definite, then the Jacobi method converges.*

We will just go through the proof of the first one.

*Proof of Theorem 2.* For the Jacobi method we need to show that the eigenvalues of

$$H = -D^{-1}(L + U)$$

are less than one in absolute value. Suppose $\lambda$ is an eigenvalue of $H$. Then

$$\det(H - \lambda I) = 0.$$

Since $A$ is strictly diagonally dominant, $D$ is non-singular and we can multiply by $\det(D)$ so that

$$\det(L + U + \lambda D) = 0.$$

Suppose that $|\lambda| \geq 1$. Then since $A$ is strictly diagonally dominant, so is $L + U + \lambda D$ which is therefore non-singular and its determinant cannot be zero. We thus have a contradiction and we conclude that $|\lambda| < 1$ as required.

For the Gauss-Seidel method we need to show that the eigenvalues of

$$H = -(L + D)^{-1}U$$

are less than one in absolute value. We suppose that

$$\det(H - \lambda I) = 0.$$

Since $L + D$ is non-singular, we can multiply by $\det(L + D)$ so that

$$\det(U + \lambda(L + D)) = 0.$$

Suppose that $|\lambda| \geq 1$. Then since $A$ is strictly diagonally dominant, so is $U + \lambda(L + D)$ which is therefore non-singular and its determinant cannot be zero. This is a contradiction and we conclude that $|\lambda| < 1$. $\square$

# 3 Exercises

**Exercise 2.10** *Consider the iteration* $\mathbf{e}^{(k+1)} = H\mathbf{e}^{(k)}$ *for* $k = 0, 1, 2, \ldots,$ *where*

$$H = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix},$$

*with* $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $\gamma$ *large and* $|\alpha| < 1$, $|\beta| < 1$. *Calculate* $H^k$ *and show that its elements tend to zero as* $k \to \infty$. *Hence deduce that* $\mathbf{e}^{(k)} \to 0$ *as* $k \to \infty$.

**Exercise 2.11** *Starting with an arbitrary* $\mathbf{x}^{(0)}$ *the sequence* $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots,$ *is calculated by*

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}^{(k+1)} + \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{bmatrix} \mathbf{x}^{(k)} = \mathbf{b}$$

*in order to solve the linear system*

$$\begin{bmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ \gamma & \beta & 1 \end{bmatrix} \mathbf{x} = \mathbf{b},$$

*where* $\alpha, \beta, \gamma$ *are constants. Find all values for* $\alpha, \beta, \gamma$ *such that the sequence converges for every* $\mathbf{x}^{(0)}$ *and* $\mathbf{b}$. *What happens when* $\alpha = \beta = \gamma = -1$ *and when* $\alpha = \beta = 0$*?*