Lower Bounds for Sorting, Linear Time Sorting, Counting Sort

(Class 15)

From Book's Chapter 8

In-place, Stable Sorting

- In merge sort we needed a temporary array when we perform merging the two subarrays.
- In Quick sort, the algorithm updates the original array without the need of temporary array.
- Another fact Stable sort or unstable sort.

- For example, if the array has duplicate numbers, then which position they will appear.
- An *in-place* sorting algorithm is one that uses no additional array for storage.
- A sorting algorithm is stable if duplicate elements remain in the same relative position after sorting.

9	3	$3'$	5	6	$5'$	2	1	$3''$	unsorted
1	2	3	$3'$	$3''$	5	$5'$	6	9	stable sort
1	2	$3'$	3	$3''$	$5'$	5	6	9	unstable

- Bubble sort, insertion sort and selection sort are in-place sorting algorithms.
- Bubble sort and insertion sort can be implemented as stable algorithms, but selection sort cannot (without significant modifications).
- Merge sort is a stable algorithm but not an in-place algorithm.
- It requires extra array storage.

- Quicksort is not stable but is an in-place algorithm.
- Heapsort is an in-place algorithm but is not stable.
- Stable sorting is important because if our key has satellite data (the whole record) has to be moved after sorting.
- It can also leads to disturb the data because if two keys are same then after sorting which key points to which record?
- It can affect the performance.

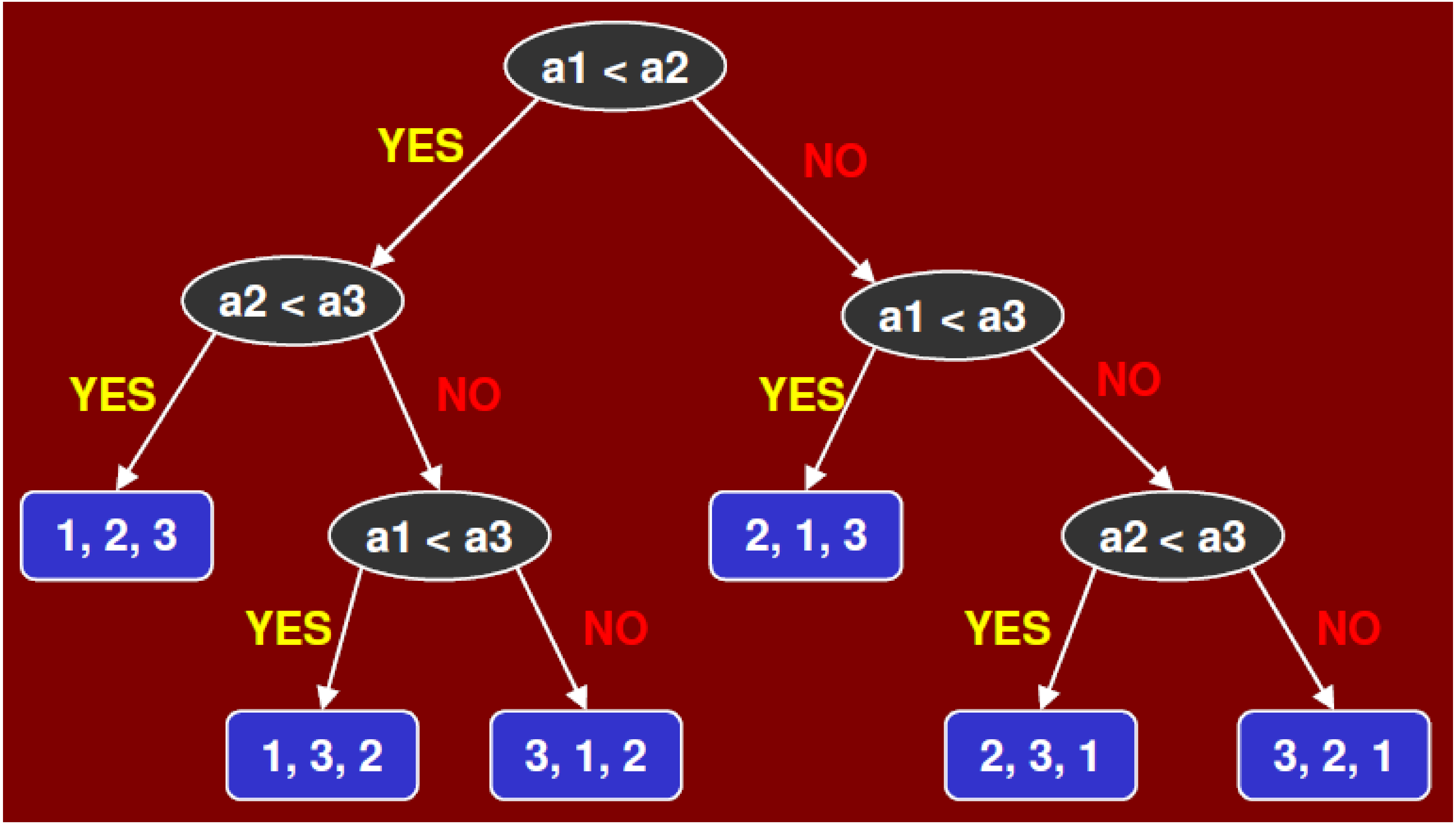
Lower Bounds for Sorting

- The best we have seen so far is $O(n \log n)$ algorithms for sorting.
- Is it possible to do better than $O(n \log n)$?
- If a sorting algorithm is solely based on comparison of keys in the array, then it is impossible to sort more efficiently than $\Omega(n \log n)$ time.
- All algorithms we have seen so far are comparison-based sorting algorithms.

- Consider sorting three distinct numbers a_1, a_2, a_3.
- There are $3! = 6$ possible combinations:

$$(a_1, a_2, a_3), (a_1, a_3, a_2), (a_3, a_2, a_1)$$
$$(a_3, a_1, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1)$$

- One of these permutations leads to the numbers in sorted order.
- To make these combinations we have to perform the comparisons.
- The comparison-based algorithm defines a decision tree.
- Here is the tree for the three numbers.


a1 < a2
YES
NO
a2 < a3
a1 < a3
YES
NO
YES
NO
1, 2, 3
a1 < a3
2, 1, 3
a2 < a3
YES
NO
YES
NO
1, 3, 2
3, 1, 2
2, 3, 1
3, 2, 1

- For n elements, there will be $n!$ possible permutations.
- The height of the tree is exactly equal to $T(n)$, the running time of the algorithm.
- The height is $T(n)$ because any path from the root to a leaf corresponds to a sequence of comparisons made by the algorithm.
- Any binary tree of height $T(n)$ has at most $2^{T(n)}$ leaves. ($n = 2^h$)

- Thus, a comparison-based sorting algorithm can distinguish between at most $2^{T(n)}$ different final outcomes.
- So, we have:

$$2^{T(n)} = n!$$

- And therefore (taking log on both sides):

$$T(n) = \log n!$$

- We can use *Stirling's approximation* for $n!$

$$n! \approx \sqrt{2\pi n}\,(\frac{n}{e})^n$$

- Where e is the base of natural log.

- Therefore

$$T(n) = \log(\sqrt{2\pi n}\,(\frac{n}{e})^n)$$

$$= \log(\sqrt{2\pi n}) + \log(\frac{n}{e})^n$$

$$= \log(\sqrt{2\pi n}) + n\log n + n\log e$$

- The dominating term is $n\log n$.

$$\boldsymbol{T(n) \in \Omega(n\log n)}$$

- We thus have the following theorem.
- **Theorem 1:** Any comparison-based sorting algorithm has worst-case running time $(n\log n)$.

Linear Time Sorting

- The lower bound implies that if we hope to sort numbers faster than O(n log n), we cannot do it by making comparisons alone.
- Is it possible to sort without making comparisons?
- The answer is yes, but only under very restrictive circumstances.

- Many applications involve sorting small integers (e.g., sorting characters, exam scores, etc.).
- We present three algorithms based on the theme of speeding up sorting in special cases, by not making comparisons.

Counting Sort

- We will consider three algorithms that are faster and work by not making comparisons:
  - Counting Sort
  - Bucket or Bin Sort
  - Radix Sort

- Counting sort assumes that the numbers to be sorted are in the range 1 to k where k is small.
- The basic idea is to determine the rank of each number in final sorted array.
- Recall that the rank of an item is the number of elements that are less than or equal to it.
- Once we know the ranks, we simply place all the numbers to their final rank position in an output array.

- The question is how to find the rank of an element without comparing it to the other elements of the array?.
- The algorithm uses three arrays.:
  - $A[1 \ldots n]$: holds the initial input.
  - $B[1 \ldots n]$: holds the sorted output.
  - $C[1 \ldots k]$: is an array of integers.

- $C[x]$ is the rank of x in A, where $x \in [1..k]$.

- The algorithm is remarkably simple, but deceptively clever.
- The algorithm operates by first constructing C.
- This is done in two steps.
- First, we set $C[x]$ to be the number of elements of $A[j]$ that are equal to x.

- We can do this initializing C to zero, and then for each j, from 1 to n, we increment $C[\ A[j]\ ]$ by 1.
- Thus, if $A[j] = 5$, then the 5th element of C is incremented, indicating that we have seen one more 5.
- To determine the number of elements that are less than or equal to x, we replace $C[x]$ with the sum of elements in the sub array $R[1{:}\,x]$.
- This is done by just keeping a running total of the elements of C.

- $C[x]$ now contains the rank of x.
- This means that if $x = A[j]$ then the final position of $A[j]$ should be at position $C[x]$ in the final sorted array.
- Thus, we set $B[\ C[x]\ ] = A[j]$.
- Notice we need to be careful if there are duplicates, since we do not want them to overwrite the same location of B.
- To do this, we decrement $C[i]$ after copying.

Counting Sort Algorithm

```
COUNTING-SORT (array A, array B, int k)
  for i ← 1 to k
    C[i] ← 0                                k times
  for j ← 1 to length[A]
    C[A[j]] ← C[A[j]] + 1                   n times

 // C[i] now contains the number of elements = i

  for i ← 2 to k
    C[i] ← C[i] + C[i-1]                    k-1 times

 // C[i] now contains the number of elements ≤ i

  for j ← length[A] downto 1
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1                   n times
```

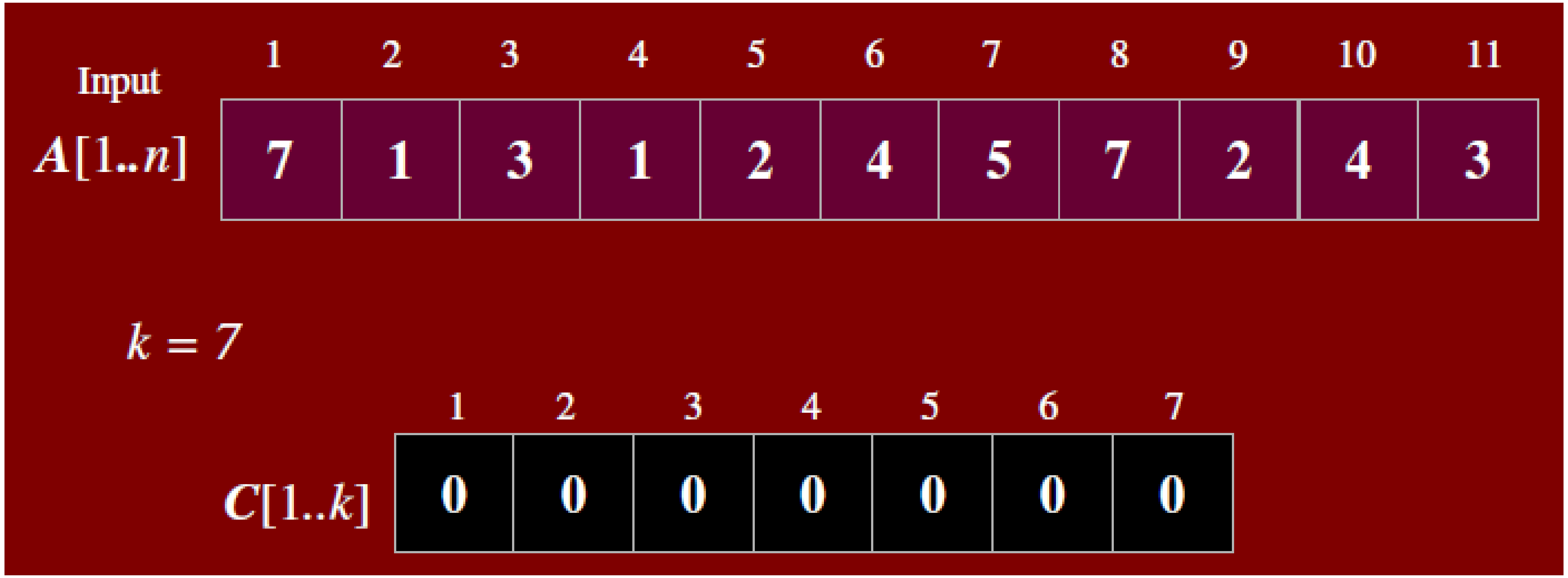
- There are four (unnested) loops, executed k times, n times, $k - 1$ times, and n times, respectively.
- So, the total running time is:

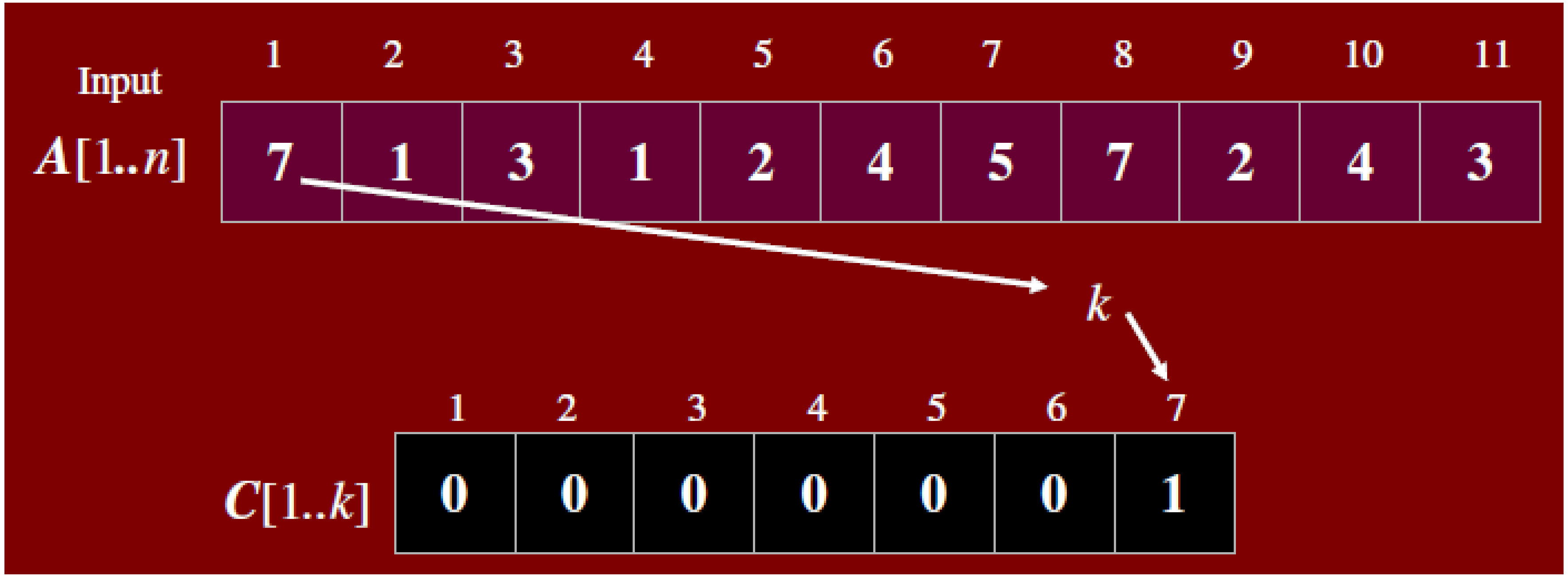
$$T(n) = n + k + k - 1 + n$$

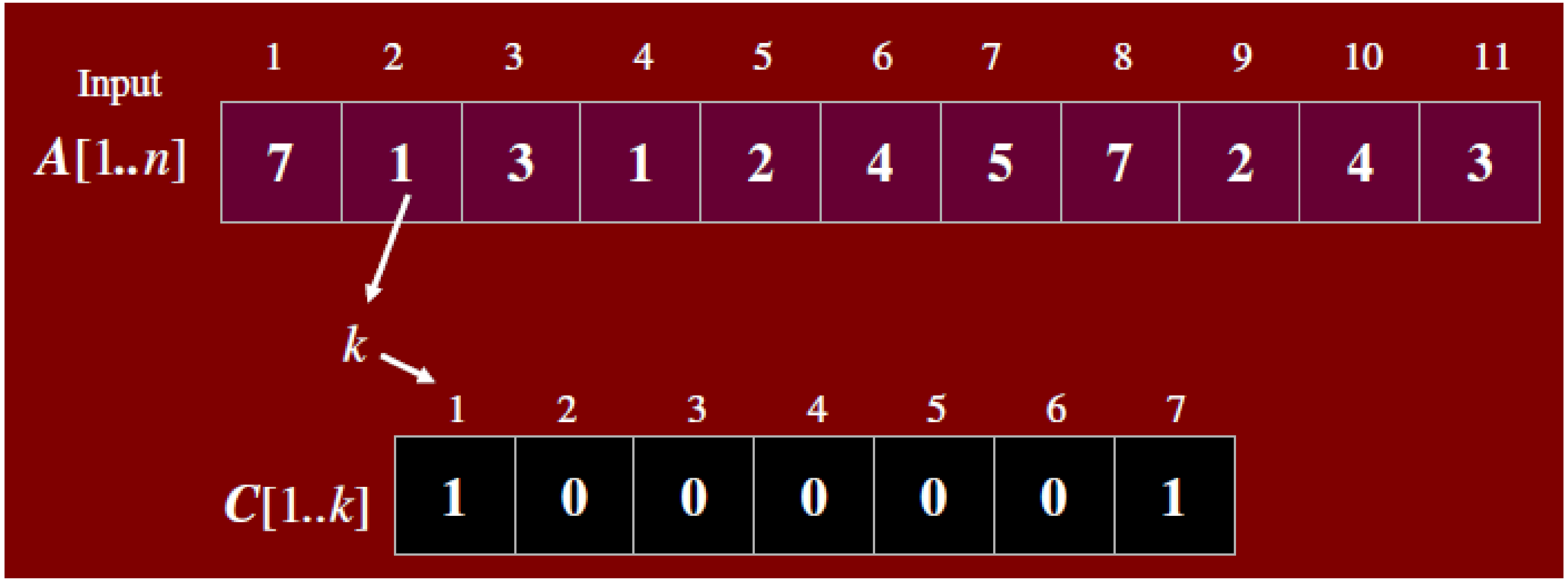
$$T(n) = 2n + 2k - 1$$

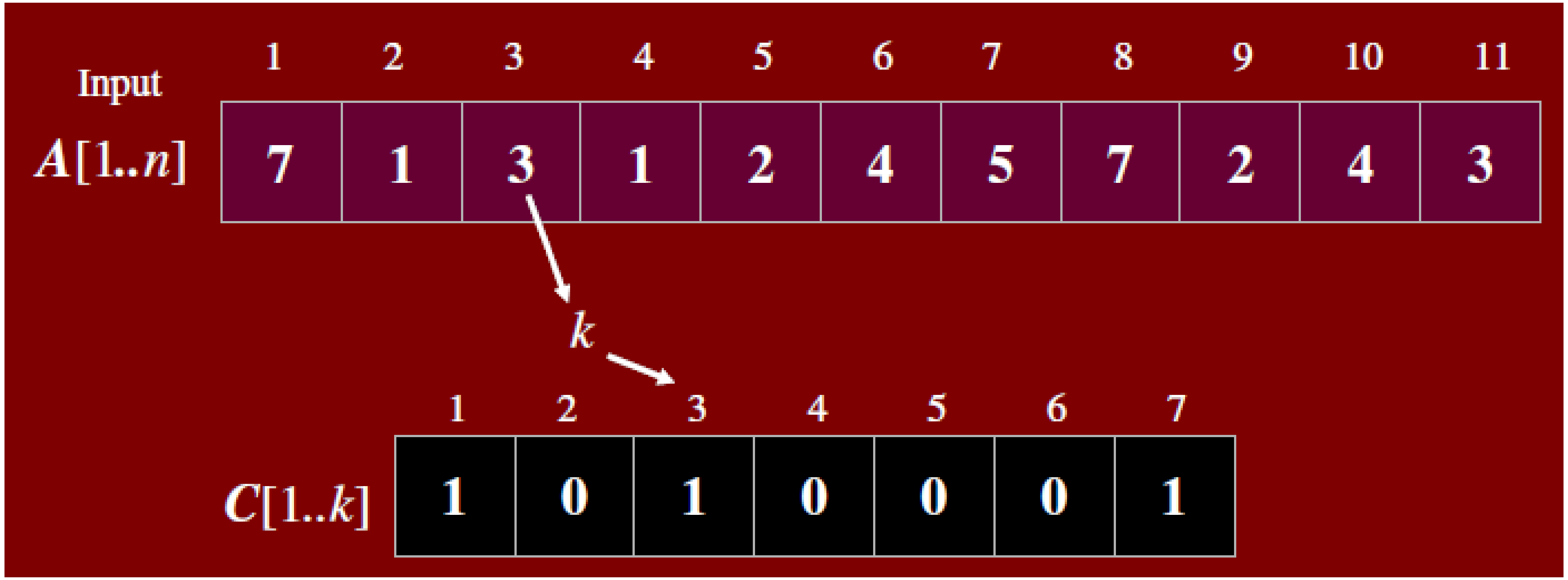
- As $k \leq n$:

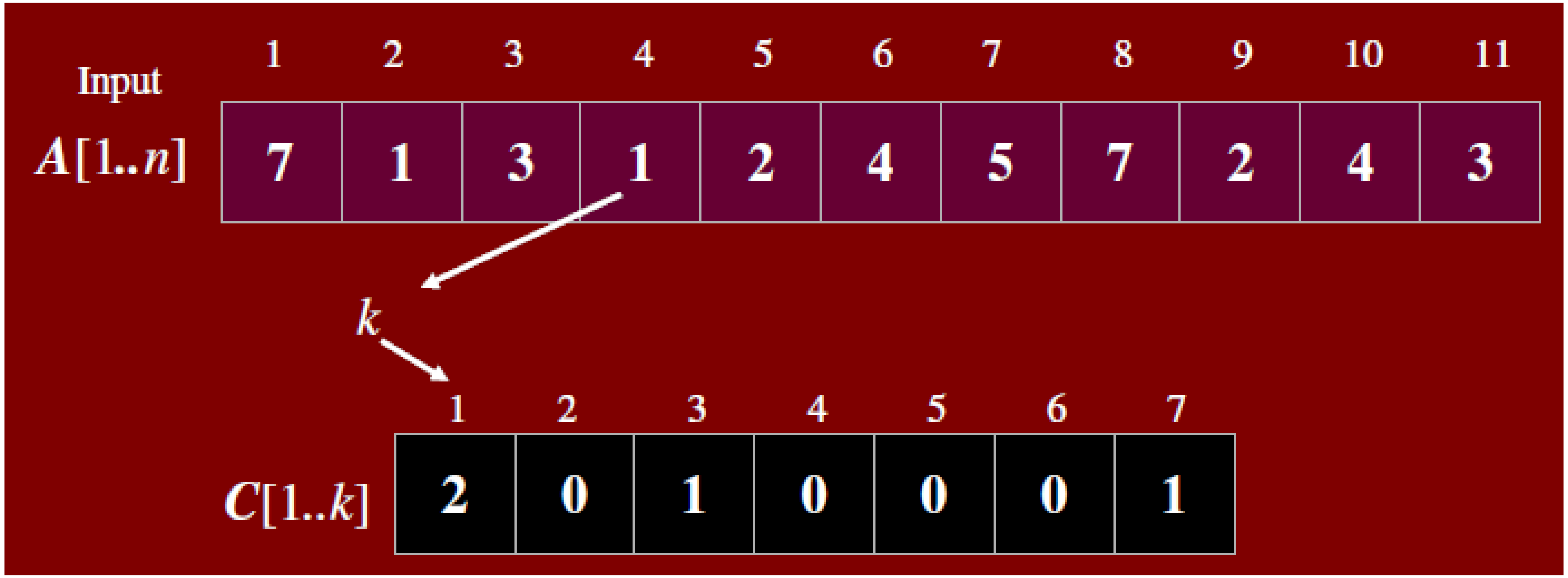
$$\boldsymbol{T(n) = O(n)}$$

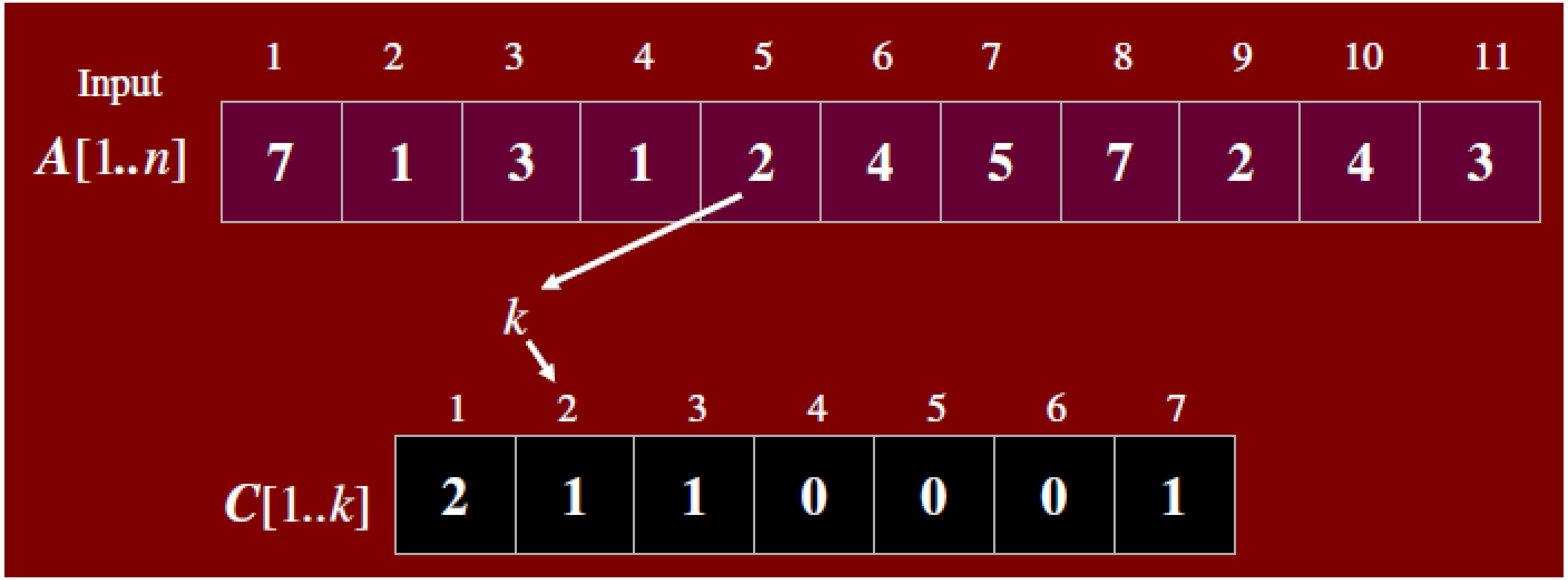

Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k = 7
1 2 3 4 5 6 7
C[1..k]
0 0 0 0 0 0 0


Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k
1 2 3 4 5 6 7
C[1..k]
0 0 0 0 0 0 1


Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k
C[1..k]
1 2 3 4 5 6 7
1 0 0 0 0 0 1


Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k
C[1..k]
1 2 3 4 5 6 7
1 0 1 0 0 0 1


Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k
C[1..k]
1 2 3 4 5 6 7
2 0 1 0 0 0 1


Input
A[1..n]
1 2 3 4 5 6 7 8 9 10 11
7 1 3 1 2 4 5 7 2 4 3
k
C[1..k]
1 2 3 4 5 6 7
2 1 1 0 0 0 1

Input

$A[1..n]$	1	2	3	4	5	6	7	8	9	10	11
	7	**1**	**3**	**1**	**2**	**4**	**5**	**7**	**2**	**4**	**3**

finally

$C[1..k]$	1	2	3	4	5	6	7
	2	**2**	**2**	**2**	**1**	**0**	**2**

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	**7**	**1**	**3**	**1**	**2**	**4**	**5**	**7**	**2**	**4**	**3**

	1	2	3	4	5	6	7
$C[1..k]$	2	2	2	2	1	0	2

for i = 2 to 7
do $C[i] = C[i] + C[i\text{-}1]$

	1	2	3	4	5	6	7
C	2	4	6	8	9	9	11

6 elements <= 3


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
3
B[6] = B[C[3]] = B[C[A[11]]] = A[11] = 3
1 2 3 4 5 6 7
C
2 4 6 8 9 9 11
C[A[11]] = C[A[11]] - 1
C
2 4 5 8 9 9 11

Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
3
4
B[8] = B[C[4]] = B[C[A[10]]] = A[10] = 4
1 2 3 4 5 6 7
C
2 4 5 8 9 9 11
C[A[10]] = C[A[10]] - 1
C
2 4 5 7 9 9 11


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
2 3 4
B[4] = B[C[2]] = B[C[A[9]]] = A[9] = 2
1 2 3 4 5 6 7
C
2 4 5 7 9 9 11
C[A[9]] = C[A[9]] - 1
C
2 3 5 7 9 9 11

Input $A[1..n]$

1	2	3	4	5	6	7	8	9	10	11
7	1	3	1	2	4	5	7	2	4	3

Output $B[1..n]$

1	2	3	4	5	6	7	8	9	10	11
			2		3		4			7

$$B[11] = B[C[7]] = B[C[A[8]]] = A[8] = 7$$

C

1	2	3	4	5	6	7
2	3	5	7	9	9	11

$$C[A[8]] = C[A[8]] - 1$$

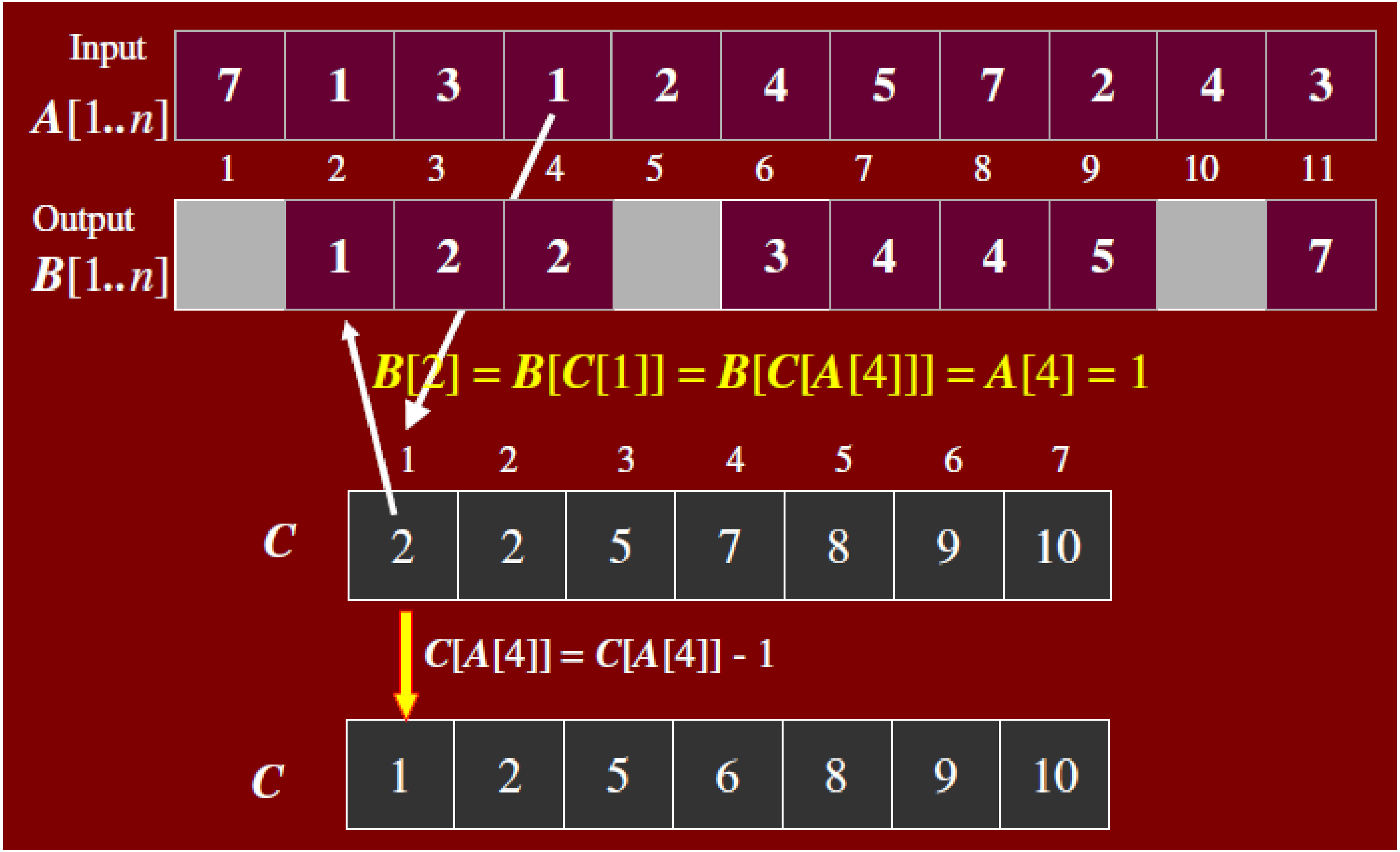
C

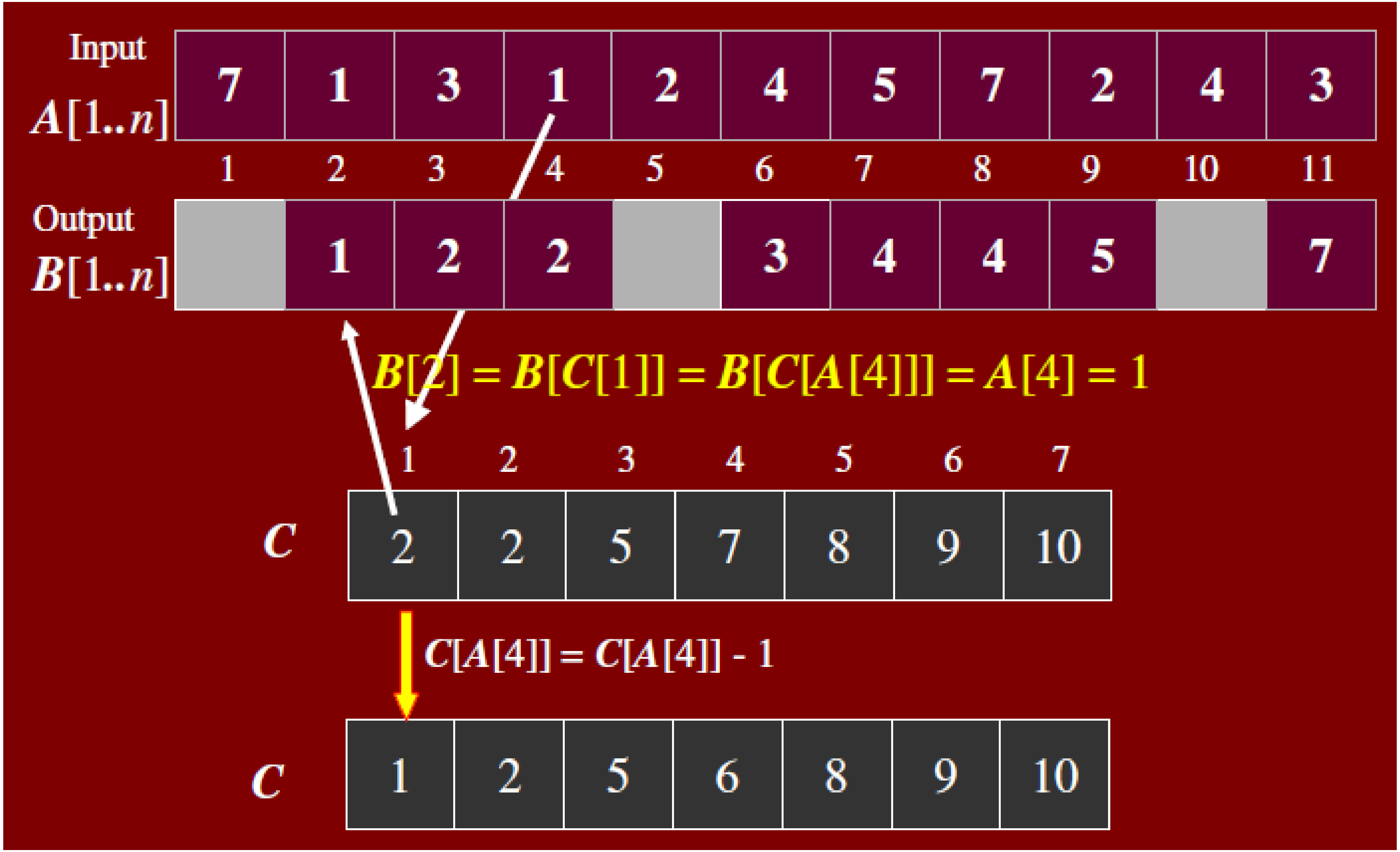
2	3	5	7	9	9	10


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
2 3 4 5 7
B[9] = B[C[5]] = B[C[A[7]]] = A[7] = 5
1 2 3 4 5 6 7
C
2 3 5 7 9 9 10
C[A[5]] = C[A[5]] - 1
C
2 3 5 7 8 9 10


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
2 3 4 4 5 7
B[7] = B[C[4]] = B[C[A[6]]] = A[6] = 4
1 2 3 4 5 6 7
C
2 3 5 7 8 9 10
C[A[6]] = C[A[6]] - 1
C
2 3 5 6 8 9 10


Input
$A[1..n]$
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
$B[1..n]$
2 2 3 4 4 5 7
$B[3] = B[C[2]] = B[C[A[5]]] = A[5] = 2$
1 2 3 4 5 6 7
C
2 3 5 7 8 9 10
$C[A[5]] = C[A[5]] - 1$
C
2 2 5 6 8 9 10


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
1 2 2 3 4 4 5 7
B[2] = B[C[1]] = B[C[A[4]]] = A[4] = 1
1 2 3 4 5 6 7
C
2 2 5 7 8 9 10
C[A[4]] = C[A[4]] - 1
C
1 2 5 6 8 9 10



Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
1 2 2 3 4 4 5 7
B[2] = B[C[1]] = B[C[A[4]]] = A[4] = 1
1 2 3 4 5 6 7
C
2 2 5 7 8 9 10
C[A[4]] = C[A[4]] - 1
C
1 2 5 6 8 9 10


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
1 1 2 2 3 3 4 4 5 7
B[1] = B[C[1]] = B[C[A[2]]] = A[2] = 1
1 2 3 4 5 6 7
C
1 2 4 7 8 9 10
C[A[3]] = C[A[3]] - 1
C
0 2 4 6 8 9 10

	1	2	3	4	5	6	7	8	9	10	11
Input $A[1..n]$	7	1	3	1	2	4	5	7	2	4	3
Output $B[1..n]$	1	1	2	2	3	3	4	4	5	7	7

$$B[10] = B[C[7]] = B[C[A[1]]] = A[1] = 7$$

	1	2	3	4	5	6	7
C	0	2	4	7	8	9	10

$$C[A[1]] = C[A[1]] - 1$$

C	0	2	4	6	8	9	9

- Counting sort is not an in-place sorting algorithm but it is stable.
- Stability is important because data are often carried with the keys being sorted. radix sort (which uses counting sort as a subroutine) relies on it to work correctly.
- Stability achieved by running the last loop down from n to 1 and not the other way around.

- The numbers 1, 2, 3, 4, and 7, each appear twice. The two 4's have been given the superscript "*".
- Numbers are placed in the output B array starting from the right.
- The two 4's maintain their relative position in the B array.
- If the sorting algorithm had caused 4^{**} to end up on the left of 4^{*}, the algorithm would be termed unstable.

Input
A[1..n]
7 1 3 1 2 4* 5 7 2 4** 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
3 4**
B[8] = B[C[4]] = B[C[A[10]]] = A[10] = 4
1 2 3 4 5 6 7
C
2 4 5 8 9 9 11
C[A[10]] = C[A[10]] - 1
C
2 4 5 7 9 9 11

Input $A[1..n]$	7	1	3	1	2	4*	5	7	2	4**	3
	1	2	3	4	5	6	7	8	9	10	11
Output $B[1..n]$				2		3	4*	4**	5		7

$$B[7] = B[C[4]] = B[C[A[6]]] = A[6] = 4$$

	1	2	3	4	5	6	7
C	2	3	5	7	8	9	10

$C[A[6]] = C[A[6]] - 1$

C	2	3	5	6	8	9	10

Input $A[1..n]$

7’	1^	$3^{\#}$	1^^	2^{+}	4^{*}	5	7’’	2^{++}	4^{**}	$3^{\#\#}$
1	2	3	4	5	6	7	8	9	10	11

Output $B[1..n]$

1^	1^^	2^{+}	2^{++}	$3^{\#}$	$3^{\#\#}$	4^{*}	4^{**}	5	7’	7’’