Asymptotic Notations

(Class 6)

From Book’s Page No 50 (Chapter 3)

- Many people sloppily use O when they should use θ.
- The book we are following also uses θ to express worst-case running time.
- For example, an algorithm analyst may end up with a time function $T(n) = n^2 + n + 2$ and immediately conclude that $T(n) = O(n^2)$ which is technically right, but a sharper assertion would be $T(n) = \theta(n^2)$.

- We can attribute this oblivious behavior to two reasons. First, many see O to be more popular and acceptable, possibly because of its long history. Recall that it was introduced more than a century ago, whereas θ and Ω were introduced only in 1976 (by Donald Knuth).
- Second, it could be because O is readily available on the keyboard, whereas θ is not!

- From a technical point of view, however, the main reason careful analysts prefer to use O over θ is that the O covers "greater territory" than the θ.
- If we take an example of some binary search and want to use θ we will have to make two assertions:
  - One for the best case, namely $\theta(1)$
  - Another for the worst case, namely $\theta(\log n)$
- With O we make only one assertion, namely $O(\log n)$.

The need of Asymptotic Notations

- If we are given running times of some algorithm:

$$T(n) = 4n^2 + 2n + 5$$

$$T(n) = n^2 + 10n + 9$$

$$T(n) = 95n^2 + 40n$$

- Which one is better?

- We may get confused by simply watching these polynomial equations.
- But if we only take the dominating term from each equation:

$$T(n) \approx n^2$$

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- Now we can easily say that all these algorithms have equal running time.

Growth of Function (Book's Page Number 32)

- We use Asymptotic Notations to describe the growth rate of the function.
- We know that for the growth of a function, the highest order term matters the most.
- e.g., the term c_1n^2 in the function $c_1n^2 + c_2n + c_3$ and thus we can neglect the other terms.

Commonly Used Functions and Their Comparison

- **Constant Functions**
- Whatever is the input size n these functions take a constant amount of time.

$$f(n) = 1$$

- **Linear Functions**
- These functions grow linearly with the input size n.

$$f(n) = n$$

- **Quadratic Functions**
- These functions grow faster than the super-linear functions i.e., $n \log(n)$.

$$f(n) = n^2$$

- **Cubic Functions**
- Faster growing than quadratic but slower than exponential.

$$f(n) = n^3$$

- **Logarithmic Functions**
- These are slower growing than even linear functions.

$$f(n) = \log(n)$$

- **Super-linear Functions**
- Faster growing than linear but slower than quadratic.

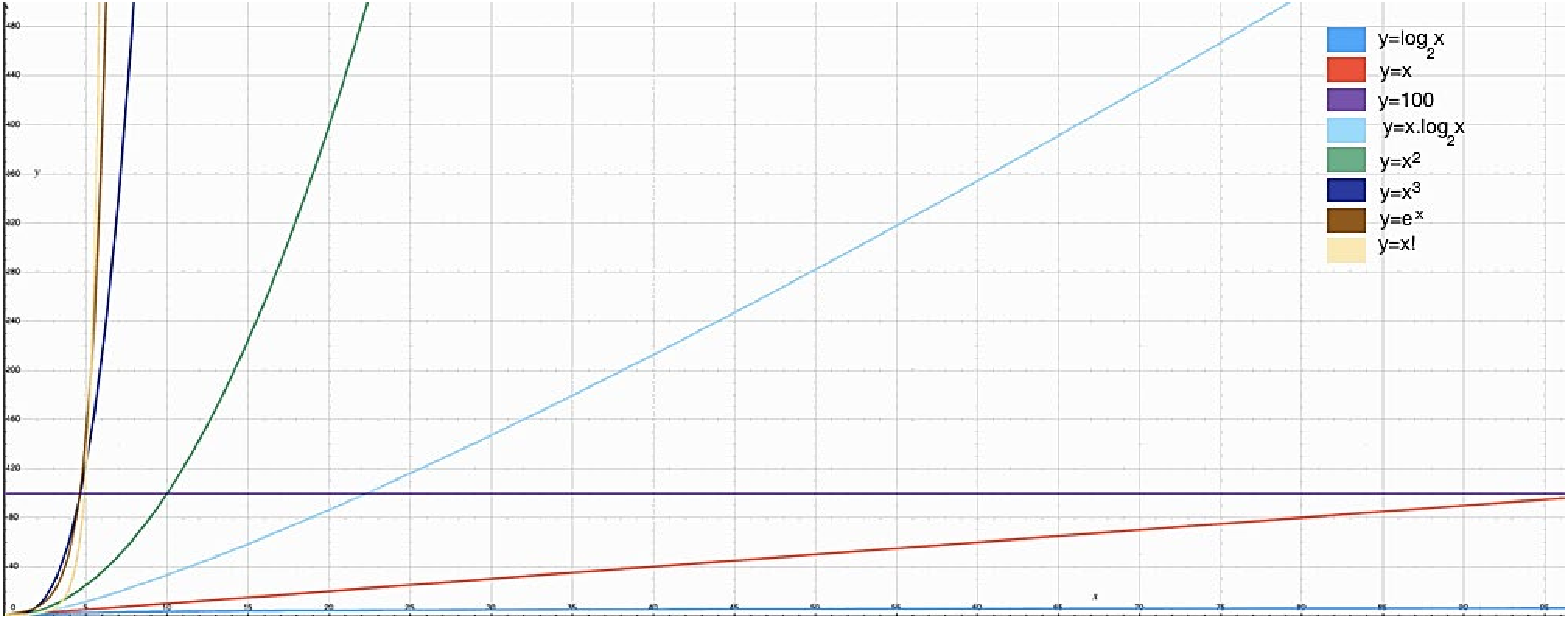
$$f(n) = n\ log(n)$$

- **Exponential Functions**
- Faster than all of the functions mentioned here except the factorial functions.

$$f(n) = c^n$$

- **Factorial Functions**
- Fastest growing than all these functions mentioned here.

$$f(n) = n!$$



y=log2x
y=x
y=100
y=x.log2x
y=x2
y=x3
y=ex
y=x!

- From the graph, you can see that for any sufficiently larger n:

$$n! \geq \; c^n \geq \; n^3 \geq \; n^2 \geq \; n\, log(n) \geq n \geq log(n) \geq 1$$

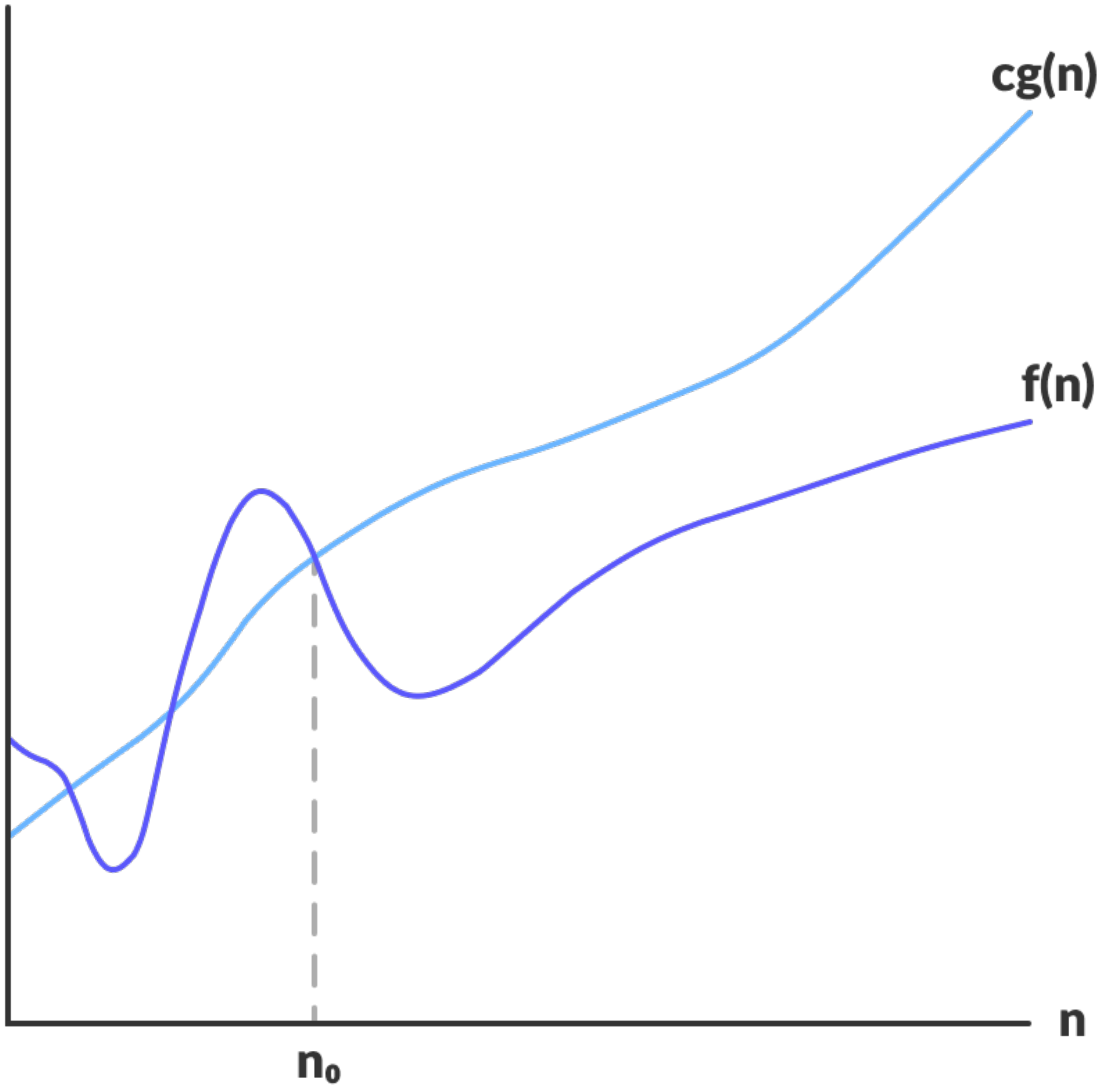
- We always want to keep the rate of the growth as low as possible.
- We try to make an algorithm to follow the function with least growth rate to accomplish a task.

Asymptotic Notations

- We use some mathematical tools to describe the behavior of the running time.
- There are mainly three asymptotic notations:
  - O-Notation (Big O)
  - Ω-Notation (Omega)
  - θ-Notation (Theta)

O-Notation

- Big-O notation represents the upper bound of the running time of an algorithm.
- Thus, it gives the worst-case complexity of an algorithm.
- O-Notation characterizes an *upper bound* on the asymptotic behavior of a function.



$f(n) = O(g(n))$

- In other words, it says that a function grows no faster than a certain rate, based on the highest-order term.
- Consider, for example, the function $7n^3 + 100n^2 + 20n + 6$.
- Its highest-order term is $7n^3$, and so we say that this function's rate of growth is n^3.
- Because this function grows no faster than n^3 , we can write that it is $O(n^3)$.
- Also, the restriction is not applied to the region of $n \leq n_\circ$.

Formal Definition of O-Notation

- For a given function $g(n)$, we denote by $O(g(n))$ the set of functions.

$$O\big(g(n)\big) = \{\, f(n) :$$

$$\textit{there exist positive constants } c \textit{ and } n_{\circ} \textit{ such that}$$

$$0 \le f(n) \le cg(n)$$

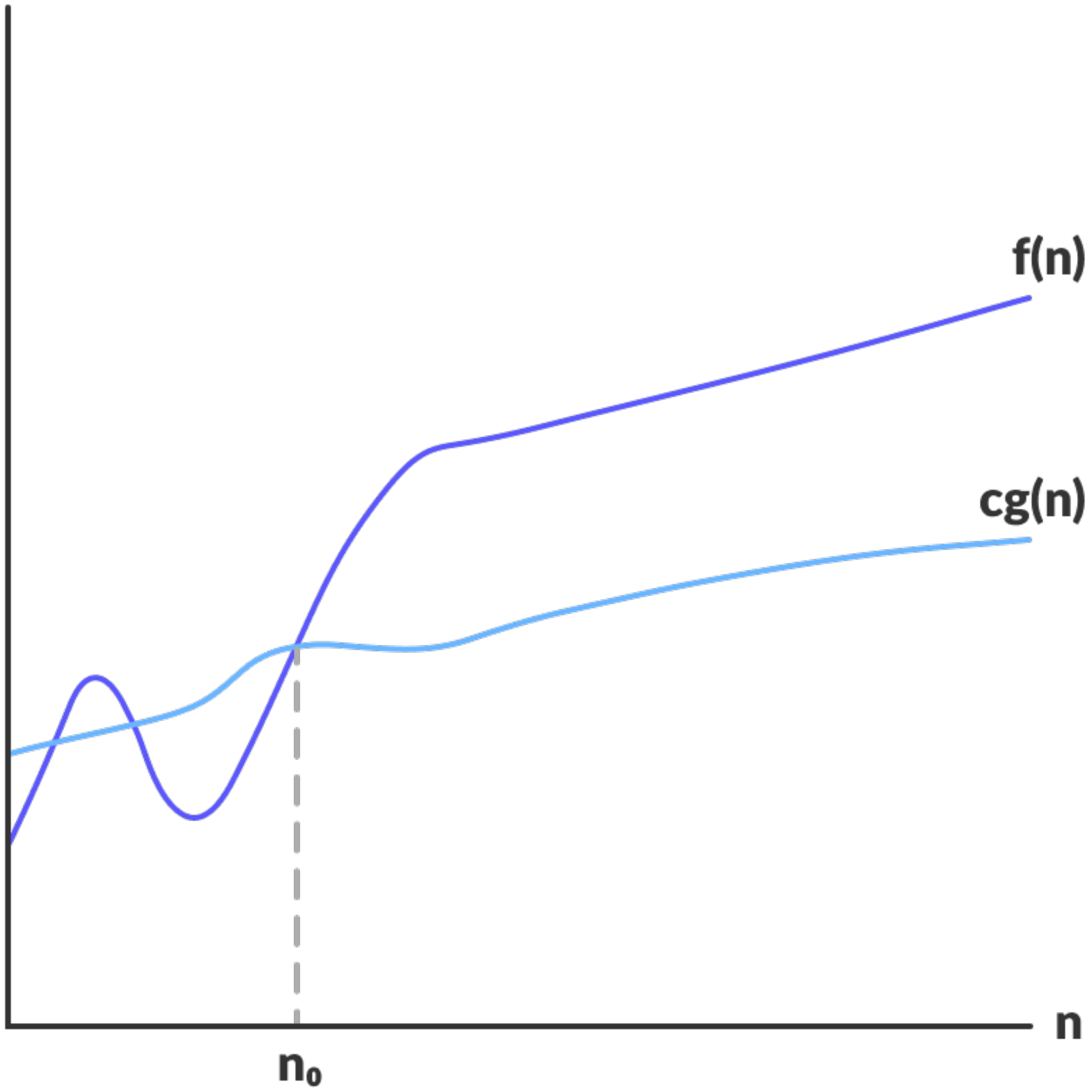
$$\textit{for all } n \ \ge n_{\circ} \,\}$$

$$f(n) \in O\big(g(n)\big)$$

- A function $f(n)$ belongs to the set $O\big(g(n)\big)$ if there exists a positive constant c such that $f(n) \leq cg(n)$ for sufficiently large n.

Ω-Notation

- Ω-Notation characterizes a *lower bound* on the asymptotic behavior of a function.
- Omega notation represents the lower bound of the running time of an algorithm.
- Thus, it provides the best-case complexity of an algorithm.



$f(n) = \Omega(g(n))$

- In other words, it says that a function grows *at least as fast* as a certain rate.
- Because the highest-order term in the function $7n^3 + 100n^2 + 20n + 6$ grows at least as fast as n^3, this function is $\Omega(n^3)$.
- This function is also $\Omega(n^2)$ and $\Omega(n)$.
- More generally, it is $\Omega(n^c)$ for any constant c ≤ 3).

Formal Definition of Ω-Notation

- For a given function $g(n)$, we denote by $\Omega(g(n))$ the set of functions.

$$\Omega\big(g(n)\big) = \{\, f(n) :$$

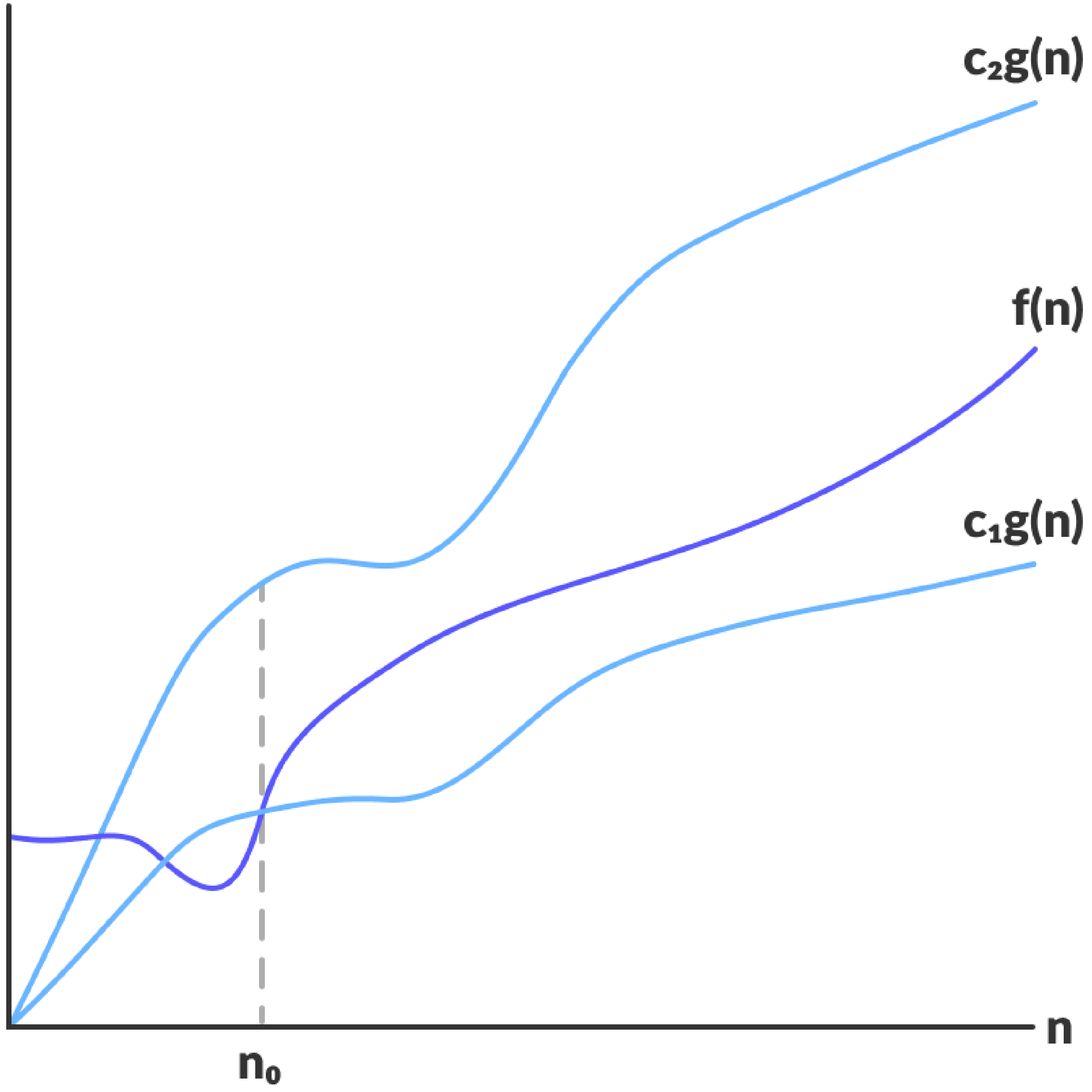
$$there\ exist\ positive\ constants\ c\ and\ n_\circ\ such\ that$$

$$0 \le cg(n) \le f(n)$$

$$for\ all\ n\ \ \ge n_\circ \,\}$$

$\boldsymbol{\theta}$-Notation

- θ-Notation characterizes a *tight bound* on the asymptotic behavior of a function.
- Theta notation encloses the function from above and below. Since it represents the *upper* and the *lower bound* of the running time of an algorithm, it is used for analyzing the average-case complexity of an algorithm.
- It says that a function grows *precisely* at a certain rate, based on the highest-order term.



$f(n) = \Theta(g(n))$

Formal Definition of $\boldsymbol{\theta}$-Notation

- For a given function $g(n)$, we denote by $\theta(g(n))$ the set of functions.

$$\theta\big(g(n)\big) = \{\, f(n) :$$

$$there\ exist\ positive\ constants\ c_1, c_2 and\ n_\circ\ such\ that$$

$$0 \leq c_1 g(n) \leq f(n) \leq c_2\ g(n)$$

$$for\ all\ n\ \geq n_\circ \}$$