CS 477/677 Causal Inference: Homework 5
Causal Decision Theory and Structure Learning

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1 Analytical (30 Points)

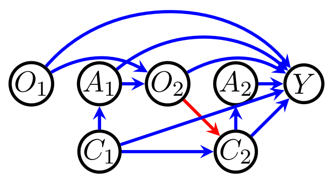


Figure 1: A causal model corresponding to a variation of the simulation study in *Statistical Methods for Dynamic Treatment Regimes*. The red edge is absent in the original simulation study and is added to create *time-dependent confounding* by C_2 and O_2.

1 Optimal Trajectories (5 Points) Consider Section 1.2 and Figure 1, as well as the models you fit in that section. Assume we are given the following trajectory: $O_1 = 1, C_1 = 0.5$. What is the optimal treatment assignment for A_1?

Assume further that after assigning A_1, we obtained the following second part of the trajectory, $O_2 = 1, C_2 = -1.0$. What is the optimal treatment assignment for A_2 given the full trajectory and the optimal assignment of A_1?

You may need to write additional code to interface with the previously fitted models to answer this question.

Solution.

Let A_1^* be the optimal treatment assignment for A_1. As we know f_{A_2} from Section 1.2, apply Learning Multi-Stage Policies with the Recursive Stage, we have:

$$\begin{aligned} A_1^* &= \underset{f_{A_1}}{\operatorname{argmax}} \mathbb{E}\left[Y(A_1 = f_{A_1}(O_1, C_1)) \mid O_1, C_1\right] \\ &= \underset{f_{A_1}}{\operatorname{argmax}} \mathbb{E}\left[Y \mid A_1 = f_{A_1}(O_1, C_1)), O_1, C_1\right]. \end{aligned} \tag{1}$$

From Section 1.2, we have Q-functions – the fitted expected reward after a full trajectory and two treatments:

$$\mathbb{E}[Y \mid o_1, c_1, a_1, o_2, c_2, a_2; \widehat{\boldsymbol{\alpha}}] \tag{2}$$

and the fitted expected reward:

$$\mathbb{E}\left[\max_{a_2 \in \{-1,1\}} \mathbb{E}[Y \mid o_1, c_1, a_1, O_2, C_2, a_2; \widehat{\boldsymbol{\alpha}}] \middle| o_1, c_1, a_1; \widehat{\boldsymbol{\beta}}\right] \tag{3}$$

after a partial trajectory where the first treatment A_1 is applied.
Fit $O_1 = 1, C_1 = 0.5$ into the fitted model 3 in the code, then find A_1^* according to equation 1, we obtain:

$$
\begin{aligned}
A_1^* &= \underset{a_1 \in \{-1,1\}}{\operatorname{argmax}} \left(\mathbb{E}\left[Y \mid a_1, o_1 = 1, c_1 = 0.5; \widehat{\boldsymbol{\beta}} \right] \right) \\
&= -1.
\end{aligned}
$$

Let A_2^* be the optimal treatment assignment for A_2, then apply Learning Multi-Stage Policies with the Last Stage, we have:

$$
\begin{aligned}
A_2^* &= \underset{f_{A_2}}{\operatorname{argmax}} \mathbb{E}\left[Y(A_2 = f_{A_2}(O_1, C_1, A_1^*, O_2, C_2)) \mid O_1, C_1, A_1^*, O_2, C_2) \right] \\
&= \underset{f_{A_2}}{\operatorname{argmax}} \mathbb{E}\left[Y \mid A_2 = f_{A_2}(O_1, C_1, A_1^*, O_2, C_2), O_1, C_1, A_1^*, O_2, C_2) \right].
\end{aligned} \tag{4}
$$

Fit $O_1 = 1, C_1 = 0.5, A_1^* = -1, O_2 = 1, C_2 = -1.0$ into the fitted model 2 in the code, then find A_2^* according to equation 4, we obtain:

$$
\begin{aligned}
A_2^* &= \underset{a_2 \in \{-1,1\}}{\operatorname{argmax}} \left(\mathbb{E}\left[Y \mid a_2, o_1 = 1, c_1 = 0.5, A_1^* = -1, o_2 = 1, c_2 = -1.0); \widehat{\boldsymbol{\alpha}} \right] \right) \\
&= -1.
\end{aligned}
$$

2 Observational Equivalence (7 points)

- List all DAGs observationally equivalent to the DAG in Fig. 2.

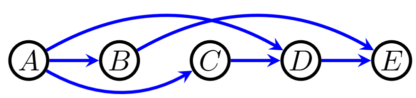
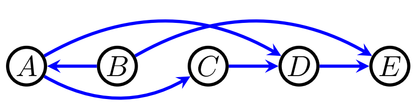
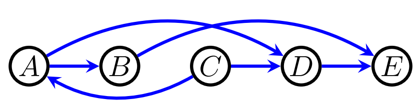


Figure 2: A DAG model for question 2.1.

Solution.

From the DAG in Fig.!2, we have two collider triplets: $B \to D \leftarrow C$ and $B \to E \leftarrow D$, then the DAGs observationally equivalent must also share the same these colliders and same skeleton.
Therefore, we have two following DAGs give the same model with DAG 2:





- List an undirected graph observationally equivalent to the DAG in Fig. 2, if it exists. If it does not exist, explain why.

Solution.

There is no undirected graph observationally equivalent to the DAG in Fig. 2. Because in order to satisfy the observationally equivalent, the undirected graph or

Markov Random Fields (MRF) must share skeletons and unshielded colliders with DAG in Fig. 2. MRF is a totally undirected graph while the DAG in Fig. 2 includes two collider triplets: $B \to D \leftarrow C$ and $B \to E \leftarrow D$, therefore, there is no MRF that can satisfy the unshielded colliders sharing condition.

3 The PC Algorithm (13 points) Assume a conditional independence oracle gave you the following list of conditional independences on (A, B, C, D, E) variables:

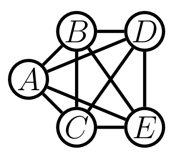
- $A \perp\!\!\!\perp C \mid B$;
- $A \perp\!\!\!\perp D \mid C$;
- $A \perp\!\!\!\perp D \mid B$;
- $A \perp\!\!\!\perp E \mid B$;
- $B \perp\!\!\!\perp D \mid C$.

- Use the PC algorithm to reconstruct the pattern corresponding to the equivalence class of models represented by this list of constraints.

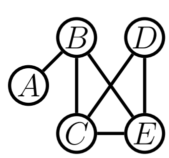
  Solution.

  Apply PC algorithm, step 1: Construct the skeleton:

  - Start with a complete undirected graph G with 5 vertices (A, B, C, D, E);

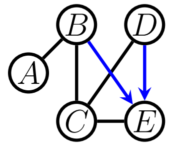


  - Due to $A \perp\!\!\!\perp C \mid B$, remove path A-C, add B to sepset(A,C) = sepset(C,A) = {B};
  - Due to $A \perp\!\!\!\perp D \mid C$, remove path A-D, add C to sepset(A,D) = sepset(D,A) = {C};
  - Due to $A \perp\!\!\!\perp D \mid B$, add B to sepset(A,D) = sepset(D,A) = {C,B}.
  - Due to $A \perp\!\!\!\perp E \mid B$, remove path A-E, add B to sepset(A,E) = sepset(E,A) = {B};
  - Due to $B \perp\!\!\!\perp D \mid C$, remove path B-D, add C to sepset(B,D) = sepset(D,B) = {C};
  - We obtain the following skeleton:



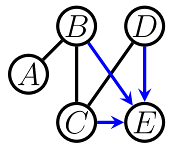
  Step 2: Identify colliders:

  - Consider non-adjacent B-D and B-E-D, due to $E \notin$ sep(B,D) and sep(D,B), we have a collider $B \to E \leftarrow D$.
  - We obtain the following pattern:



Step 3: Orient more edges based on step 2:

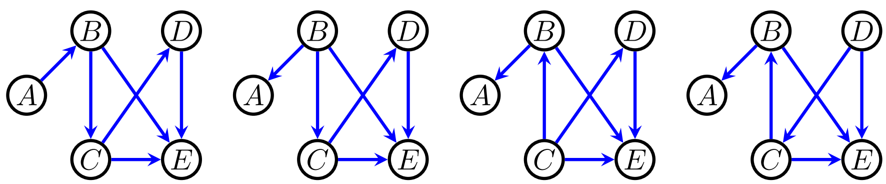
- Consider C-B$\rightarrow E$, C-D$\rightarrow E$, C-E, due to B not adjacent to D, we can orient $C \rightarrow E$.
- We obtain the final pattern:



- How many DAGs are there in this class?

  Solution.

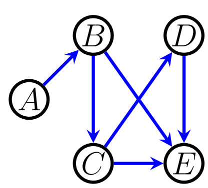
  From the above pattern, we have 4 following DAGs in this class:



- Assume, in addition to conditional independence tests on (A, B, C, D, E) listed above, you also learn that interventions on A affect B (and possibly other variables, but this was not measured in the experiment). What can we conclude about the true graph?

  Solution.

  If we learn that interventions on A affect B, then we can conclude that there is only 1 DAG satisfy the causal association from A to B, that is:



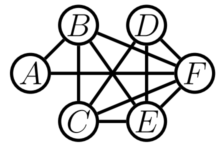
- Assume, in addition to the above variables and constraints, there is an extra variable F, and an extra constraint $(F \perp\!\!\!\perp A, C, D, E)$. What can we conclude from applying the PC algorithm to the new list of constraints?

  Solution.

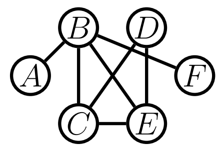
  Due to $(F \perp\!\!\!\perp A, C, D, E)$, apply decomposition, we obtain the additional flowing constraints: $F \perp\!\!\!\perp A$, $F \perp\!\!\!\perp C$, $F \perp\!\!\!\perp D$, and $F \perp\!\!\!\perp E$.

  Combine with the PC algorithm from step 1 in the first solution, we have:

- Continue with the skeleton with 6 vertices (A, B, C, D, E, F);

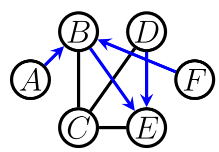


- Due to $F \perp\!\!\!\perp A \mid \{\}$, remove path A-A, sepset(F,A) = sepset(A,F) = $\{\}$;
- Due to $F \perp\!\!\!\perp C \mid \{\}$, remove path F-C, sepset(F,C) = sepset(C,F) = $\{\}$;
- Due to $F \perp\!\!\!\perp D \mid \{\}$, remove path F-D, sepset(F,D) = sepset(D,F) = $\{\}$;
- Due to $F \perp\!\!\!\perp E \mid \{\}$, remove path F-E, sepset(F,E) = sepset(E,F) = $\{\}$;
- We obtain the following skeleton:



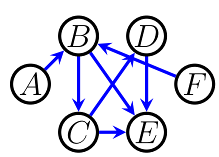
Step 2: Identify colliders:

- Consider non-adjacent A-F and A-B-F, due to $B \notin$ sep(A,F) and sep(F,A), we have a collider $A \to B \leftarrow F$.
- Consider non-adjacent B-D and B-E-D, due to $E \notin$ sep(B,D) and sep(D,B), we have a collider $B \to E \leftarrow D$.
- We obtain the following pattern:



Step 3: Orient more edges based on step 2:

- Consider A$\to$ B-C, due to A not adjacent to C, we can orient $B \to C$.
- Continue to consider B$\to$ C-D, due to B not adjacent to D, we can orient $C \to D$.
- Continue to consider $C \to D \to E$, due to C-E, we can orient $C \to E$.
- So, we obtain a complete DAG:



- How would you modify the PC algorithm to learn the structure of undirected graphs from data (under the Markov random field model that obeys a version of faithfulness that implies path separation holds if and only if conditional independence holds)?

  Solution.

  To learn the structure of undirected graphs from data, we can modify PC algorithm to only contain the step that construct the skeleton, that is:

- Start with a complete undirected graph G with n vertices:
  * For $i = 0 \rightarrow n-2$,
    · For every adjacent pair A, B in G,
    · if $(A \perp\!\!\!\perp B \mid \vec{S})$, $(|\vec{S}| = i, \vec{S} \subset nb_G(A) \cup nb_G(B) \setminus \{A, B\})$, remove edge A-B from G, add $\vec{S}$ to sepset(A,B), sepset(B,A).

4 Faithfulness And Observational Equivalence In Hidden Variable Models (5 points)
Assume $p^{(a)}(A,B,C,D)$ is faithful with respect to the DAG in Fig. 3 (a), and $p^{(b)}(A,B,C,D) \equiv \sum_{H_1} p^{(b)}(A,B,C,D,H_1)$, where $p^{(b)}(A,B,C,D,H_1)$ is faithful with respect to the DAG in Fig. 3 (b).

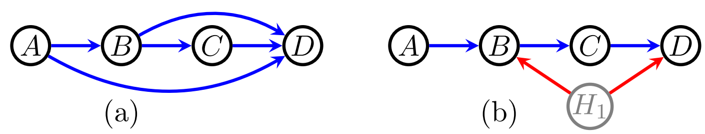


Figure 3: A DAG and a hidden variable DAG.

- Prove that $p^{(a)}(A,B,C,D)$ and $p^{(b)}(A,B,C,D)$ agree on all conditional independence constraints.

  Solution.

  As $p^{(a)}(A,B,C,D)$ and $p^{(b)}(A,B,C,D,H_1)$ is faithful w.r.t. DAGs in Fig. 3 (a) and (b), the conditional independence in $p^{(a)}(A,B,C,D)$ and $p^{(b)}(A,B,C,D,H_1)$ $\Leftrightarrow$ d-separation in DAGs in Fig. 3 (a) and (b) respectivelly.

  Consider DAG in Fig. 3 (a), we have a directed path $A \rightarrow B \rightarrow C \rightarrow D$, three collider triplets: $A \rightarrow D \leftarrow B$, $A \rightarrow D \leftarrow C$, and $B \rightarrow D \leftarrow C$. By using d-separation, we have only 1 conditional independent constraint in this DAG, that is:

  $$A \perp\!\!\!\perp C \mid B.$$

  Apply latent projection on DAG in Fig. 3 (b), we also have a directed path $A \rightarrow B \rightarrow C \rightarrow D$ and a collider triplet $B \rightarrow D \leftarrow C$ which are similar to Fig. 3 (a). We additionally have a new collider triplet $A \rightarrow B \leftarrow D$. By using m-separation, we have:

  $$A \perp\!\!\!\perp C \mid B.$$

  Consider others in ADMG after apply latent projection on DAG in Fig. 3 (b):

  - $A \not\!\perp\!\!\!\perp C \mid B, D$ because the two collider triplets $A \rightarrow B \leftarrow D$ and $B \rightarrow D \leftarrow C$ will open a path from $A \rightarrow B \rightarrow D \leftarrow C$ if we block both B and D.
  - $A \not\!\perp\!\!\!\perp D \mid B$ because the collider triplet $A \rightarrow B \leftarrow D$.
  - $A \not\!\perp\!\!\!\perp D \mid C$ and $A \not\!\perp\!\!\!\perp D \mid B, C$ because the collider triplet $A \rightarrow B \leftarrow D$ and $B \rightarrow C$.
  - The others are marginally dependent because of their directed paths.

  So, we obtain $p^{(a)}(A,B,C,D)$ and $p^{(b)}(A,B,C,D)$ agree on all conditional independence constraints which is $A \perp\!\!\!\perp C \mid B$.

- Prove that $p^{(a)}(A, B, C, D)$ and $p^{(b)}(A, B, C, D)$ are not observationally equivalent (there is a test that can use observed data to verify whether the data came from the first or second distribution).

  Solution.

  1. As $p^{(a)}(A, B, C, D)$ and $p^{(b)}(A, B, C, D, H_1)$ is faithful w.r.t. DAGs in Fig. 3 (a) and (b), the conditional independence in $p^{(a)}(A, B, C, D)$ and $p^{(b)}(A, B, C, D, H_1) \Leftrightarrow$ d-separation in DAGs in Fig. 3 (a) and (b) respectivelly.

  So $p^{(a)}(A, B, C, D)$ and $p^{(b)}(A, B, C, D)$ are observationally equivalent $\Leftrightarrow$ DAGs in Fig. 3 (a) and (b) are observationally equivalent.

  Consider the observationally equivalent of DAGs in Fig. 3 (a) and (b), they don't share the same skeleton and colliders, therefore, they are not observationally equivalent.

  2. We also can consider linear regressions from observed data in Fig. 3 to verify whether data from $p^{(a)}(A, B, C, D)$ or $p^{(b)}(A, B, C, D, H_1)$ by:

  - From the DAG in Fig. 3 (a), we have the following structure equations from observed data A, B, C, D:

    $$\begin{cases} B = xA \\ C = yB = yxA \\ D = tB + zC + wA = txA + wA + zyxA \end{cases}$$

  - From the DAG in Fig. 3 (b), we have the following structure equations from observed data A, B, C, D:

    $$\begin{cases} B = xA \\ C = yB = yxA \\ D = zC = zyxA \end{cases}$$

  So, we have $D \perp\!\!\!\perp A$ in DAG in Fig. 3 (b) if and only if the coefficient $zyx = 0$. Consider D and A in DAG in Fig. 3 (a), assume $zyx = 0$ but $x \neq \frac{-w}{t}$, then $D \not\perp\!\!\!\perp A$. Therefore, in this case, by test whether $x \neq \frac{-w}{t}$ given $zyx = 0$, we can verify whether the data came from $p^{(a)}(A, B, C, D)$ or $p^{(b)}(A, B, C, D)$.