CS 477/677 Causal Inference: Homework 3
Mediation analysis, Instrumental Variables, and d-separation

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1 Analytical (32 Points)

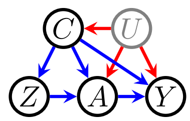


Figure 1: The IV setting in Card's paper.

1 (6 Points) Consider the setting in the graph in Fig 1 where we are interested in the ACE (of A on Y). Compare the performance of the ordinary least squares (OLS) estimator (fitting a linear regression model $E[Y \mid A, \mathbf{C}]$, and using the regression coefficient for A as an estimate of the ACE) vs the two stage least squares (2SLS) estimator you implemented.

If the U is present in the graph, how would you expect the OLS and the 2SLS to compare in terms of bias and variance? If the U is absent in the graph, how would you expect the OLS and the 2SLS to compare in terms of bias and variance?

Solution:

From the graph, we have linear regressions from baseline confounders $\mathbf{C}$ for Z, A, and Y:

$$\begin{cases} Z = t_0 + t_1\mathbf{C} + \epsilon_1 \\ A = u_0 + u_1\mathbf{C} + u_2 Z + \epsilon_2 \\ Y = v_0 + v_1\mathbf{C} + v_2 A + \epsilon_3 \end{cases} \tag{1}$$

If fitting a linear regression model $E[Y \mid A, \mathbf{C}]$, then:

$$\begin{cases} bias_{OLS} = \mathbb{E}[Y \mid A, \mathbf{C}] - Y \\ variance_{OLS} = \mathbb{E}[((Y \mid A, \mathbf{C}) - (\mathbb{E}[Y \mid A, \mathbf{C})])^2] \end{cases}$$

we have biased estimator $\mathbb{E}[Y \mid A, \mathbf{C}] \neq Y$ due to omitted instrumental variable Z in 1.

If fitting with 2SLS, then:

$$\begin{cases} \mathbb{E}[A \mid Z, \mathbf{C}; \alpha] = \alpha_0 + \alpha_1\mathbf{C} + \alpha_2 Z \\ \mathbb{E}[Y \mid Z, \mathbf{C}; \beta] = \beta_0 + \beta_1\mathbf{C} + \beta_2\mathbb{E}[A \mid Z, \mathbf{C}; \alpha] \end{cases}$$

, and:

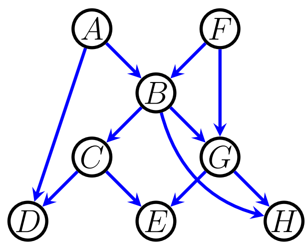
$$\begin{cases} bias_{2SLS} = \mathbb{E}[Y \mid Z, \mathbf{C}; \beta] - Y \\ variance_{2SLS} = \mathbb{E}[((Y \mid Z, \mathbf{C}; \beta) - (\mathbb{E}[Y \mid Z, \mathbf{C}; \beta)])^2] \end{cases}$$

we have an unbiased estimator $\mathbb{E}[Y \mid Z, \mathbf{C}; \beta] = Y$ due to considered instrumental variable Z in 1. So the performance of OLS is worse than 2SLS by $bias_{OLS} > bias_{2SLS}$.

If U is present in the graph, then linear regressions in 1 don't hold by Z will no longer be a strong predictor of A, and $A \not\perp\!\!\!\perp U$. Therefore, if fitting OLS and 2SLS like above, we should expect both low-bias and high-variance estimators because the instrumental variable Z here doesn't have any meaning.

If U is absent in the graph, then Z will be a strong predictor of A. If fitting OLS, we should expect a high-bias and low-variance estimator due to omitted instrumental variable Z. If fitting 2SLS, we should expect a low-bias and high-variance estimator due to considering instrumental variable Z.

2 d-separation (9 points)



In a statistical DAG model for the graph shown, let $\mathbf{V} = \{A, B, C, D, E, F, G, H\}$.

1 Do the following independence relations hold? Explain why or why not.

- $A \perp\!\!\!\perp H \mid B, E$
- $A \perp\!\!\!\perp H \mid C, D$

Solution:

$A \perp\!\!\!\perp H \mid B, E$? No: because if block B and E, there is still a way from A to H $A \to B \leftarrow F \to G \to H$ (because A, B, F is a collider triplet).

$A \perp\!\!\!\perp H \mid C, D$? No: because if block C and D, there is still a way from A to H $A \to B \to H$.

2 Give the smallest subset $\mathbf{S}$ of nodes in $\mathbf{V}$ such that $C \perp\!\!\!\perp \mathbf{V} \setminus (\mathbf{S} \cup \{C\}) \mid \mathbf{S}$.

Solution:

If block $\mathbf{S}$ then there is no path from C to $\mathbf{V} \setminus (\mathbf{S} \cup \{C\})$. So, $\mathbf{S} = \{B, D, E, A, F, G\}$ (because there is no path from C to $\mathbf{V} \setminus (\mathbf{S} \cup \{C\}) = \{H\}$ if we block {B, D, E, A, F, G})

3 Identify a minimal (not necessarily unique) set of edges to add to this graph such that the new graph obtains the property that for any $\mathbf{V}_4$ such that $\mathbf{V}_3 \subset \mathbf{V}_4$,

$$\mathbf{V}_1 \perp\!\!\!\perp \mathbf{V}_2 \mid \mathbf{V}_3 \Rightarrow \mathbf{V}_1 \perp\!\!\!\perp \mathbf{V}_2 \mid \mathbf{V}_4.$$

Solution:

Consider $\mathbf{V}_1 = \{D\}$, $\mathbf{V}_2 = \{E\}$, $\mathbf{V}_3 = \{A, C\}$,

From the graph, we have: $D \perp\!\!\!\perp E \mid A, C$, i.e., $\mathbf{V}_1 \perp\!\!\!\perp \mathbf{V}_2 \mid \mathbf{V}_3$.

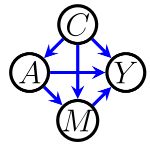
We also have, if $D \perp\!\!\!\perp E \mid A, C$, then $D \perp\!\!\!\perp E \mid A, C, x$ with $x \in \mathbf{V}$, i.e., we have $\mathbf{V}_3 \subset \mathbf{V}_4$,

$$\mathbf{V}_1 \perp\!\!\!\perp \mathbf{V}_2 \mid \mathbf{V}_3 \Rightarrow \mathbf{V}_1 \perp\!\!\!\perp \mathbf{V}_2 \mid \mathbf{V}_4$$

, with any $\mathbf{V}_4$.

So, the minimal set of new edges $= \{\varnothing\}$.

3 Structural likelihood reparameterization (5 Points)



Given a treatment A, mediator M, outcome Y, and a set of baseline variables C, the natural direct effect can be defined as the following counterfactual mean contrast (with the corresponding identifying functionals):

$$E[Y(1, M(0))] - E[Y(0)] = \left(\sum_{C,M} E[Y \mid A=1, M, C]p(M \mid A=0, C)p(C)\right) - \left(\sum_{C} E[Y \mid A=0, C]p(C)\right).$$

These identifying functionals are obtained using the assumptions $Y(a,m) \perp\!\!\!\perp M(a') \mid C$, $Y(a,m) \perp\!\!\!\perp A, M \mid C$ (which is the same as $Y(a,m) \perp\!\!\!\perp A, M(a) \mid C$), and $M(a) \perp\!\!\!\perp A \mid C$, for $A = a$. Now assume further that:

$$E[Y|A, M, C] = f(A, M, C) - \sum_{M,C} f(A, M, C)p(M|A=0, C)p(C) + w_0 + w_a A.$$

Show that if we take $f(A, M, C) = E[Y|A, M, C] - E[Y|A, M=0, C=0]$, then w_a is equal to the natural direct effect.

Solution:

If we take $f(A, M, C) = \mathbb{E}[Y|A, M, C] - \mathbb{E}[Y|A, M=0, C=0]$, then:

$$\begin{aligned}
&\mathbb{E}[Y|A, M, C] = f(A, M, C) - \sum_{M,C} f(A, M, C)p(M|A=0, C)p(C) + w_0 + w_a A \\
&\Leftrightarrow \mathbb{E}[Y|A, M=0, C=0] = -\sum_{M,C} f(A, M, C)p(M|A=0, C)p(C) + w_0 + w_a A \\
&\Rightarrow f(A, M, C) = 0 \text{ (since linear dependent from the above graph),}
\end{aligned}$$

so, we obtain:

$$\mathbb{E}[Y|A, M, C] = \mathbb{E}[Y|A, M=0, C=0] = w_0 + w_a A \tag{2}$$

We also have:

$$\begin{aligned}
\sum_{C} \mathbb{E}[Y \mid A=0, C]p(C) &= \sum_{C}\sum_{y} yp(y|A=0, C)p(C) \\
&= \sum_{C}\sum_{y} y \sum_{M} p(y, M|A=0, C)p(C) \\
&= \sum_{C}\sum_{y} y \sum_{M} p(y|A=0, C, M)p(M|A=0, C)p(C) \\
&= \sum_{C,M}\sum_{y} yp(y|A=0, M, C)p(M|A=0, C)p(C) \\
&= \sum_{C,M}\sum_{y} yp(y|A=0, M, C)p(M|A=0, C)p(C) \\
&= \sum_{C.M} \mathbb{E}[Y \mid A=0, M, C]p(M|A=0, C)p(C)
\end{aligned}$$

so, we obtain:

$$
\begin{aligned}
E[Y(1, M(0))] - E[Y(0)] &= \left(\sum_{C,M} E[Y \mid A = 1, M, C] p(M \mid A = 0, C) p(C)\right) - \left(\sum_{C} E[Y \mid A = 0, C] p(C)\right) \\
&= \sum_{C,M} \left(E[Y \mid A = 1, M, C] - E[Y \mid A = 0, M, C]\right) p(M \mid A = 0, C) p(C) \\
&= \sum_{C,M} \left(w_0 + w_a - w_0\right) p(M \mid A = 0, C) p(C) \text{ (since 2)} \\
&= \sum_{C,M} \left(w_a\right) p(M \mid A = 0, C) p(C) \\
&= w_a.
\end{aligned}
$$

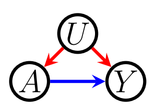
4 Mediation analysis and the null effect (2 Points) Assume the ACE of A on Y is (close to) 0. Is performing a mediation analysis with respect to a known mediator M of the influence of A on Y useful in this context? If not, explain why. If so, give an example where this might be useful to do.

Solution: Yes. Because:

If we know mediator M, then the mediator exists. If ACE of A on Y is 0, then the total effect is 0. However, the direct and indirect effects can be not equal to zero, just have a different sign. As a result, the relation between A and Y is not significant, but mediation still exists.

For example in the PhD hiring process, where A is gender, M is lazy characteristic, and Y is hiring. Males are often lazy and tend to produce less paper, but they may be smarter to make more paper than females. Therefore, the overall relation between gender and PhD hiring produced may actually be zero, yet there are two opposing mediational processes. The mediation analysis here however still useful, because it discovers whether laziness characteristic across gender actually affects the hiring process.

5 Non-Identification (5 Points)



Consider the above graph. Prove that, if $p(Y(a))$ is not identified in the above graph, then $p(Z(a))$ will not be identified in the graph constructed by taking the above graph and adding Z that is a child of Y.

Solution:

Let's consider the following: A = U, and U is drawn from a fair coin.

Construct two models that agree on $p(A, Y)$:

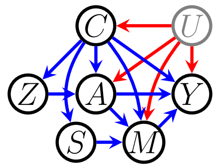
$$
\begin{aligned}
&\text{Model 1: } Y = A \text{ xor } U + \mathcal{N}(0, 1) \\
&\text{Model 2: } Y = \mathcal{N}(0, 1)
\end{aligned}
$$

But these models disagree on $p(Y(a))$. Because if $a = 1$, then:

$$
\begin{aligned}
&\text{Model 1: } Y(1) \sim \mathcal{N}(0, 1) + \text{ fair coin flip} \\
&\text{Model 2: } Y(1) \sim \mathcal{N}(0, 1)
\end{aligned}
$$

Because Z is a child of Y, consider $Z = Y$, then these two models also agree on $p(A, Z)$ and disagree on $p(Z(a))$. Therefore, if $p(Y(a))$ is not identified, then $p(Z(a))$ will not be identified.

6 Mediation and unobserved confounding (5 Points)



Consider the graph above, where we are interested in direct and indirect effects of A on Y, with M acting as a mediator. Unlike earlier cases, we have an unobserved confounder, U, for A,M, and Y. However, we have two instrumental variables Z and S. Assuming all models are linear Gaussian in this setting, is it possible to use these instruments to perform mediation analysis in this case? If not, explain why not. If so, sketch how you might go about doing this.

Solution: No. Because:

From the graph, we have linear regressions from baseline confounders $\mathbf{C}$ for Z, S, A, M, and Y:

$$\begin{cases} Z = s_0 + z_1\mathbf{C} + \epsilon_z \\ S = s_0 + s_1\mathbf{C} + \epsilon_s \\ A = a_0 + a_1\mathbf{C} + a_2 Z + \epsilon_a \\ M = m_0 + m_1\mathbf{C} + m_2 A + m_3 S + \epsilon_m \\ Y = y_0 + y_1\mathbf{C} + y_2 A + y_3 M + \epsilon_y \end{cases} \tag{3}$$

As M is a mediator, we have a mediation analysis of A on Y:

$$\mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0, M(1)] + \mathbb{E}[Y(0, M(1)] - \mathbb{E}[Y(0)]$$

From 3, we have:

$$\mathbb{E}[Y(0, M(1)] = y_0 + y_1\mathbf{C} + y_3\left(m_0 + m_1\mathbf{C} + m_2 + m_3 S + \epsilon_m\right) + \epsilon_y$$

So, $\mathbb{E}[Y(0, M(1)]$ is unidentified here.

If Z is an instrumental variable, then $Z \not\perp\!\!\!\perp A$ and Z is a strong predictor of A, so A is an dependent variable on Z.

If S is an instrumental variable, then $S \not\perp\!\!\!\perp M$ and S is a strong predictor of M. But there is no relationship between Z and S in the above graph.

What we are interested in direct and indirect effects of A on Y, with M acting as a mediator. However, if we use Z and S to do mediation analysis, the mediator M is not associated with the independent variables A anymore (by two reasons above). Therefore, we can not do mediation analysis by using Z and S here.