EN.520.637: Foundations of Reinforcement Learning Homework 1

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1 Problem 1

(Anderson) The Monty Hall problem is a famous math problem loosely based on a television quiz show hosted by Monty Hall. You face three closed doors. There is a big prize behind one door, but nothing behind the other two. You are asked to pick one door out of the three, but you cannot yet see what is behind it. Monty opens one of the two other doors to reveal that there is nothing behind it. Then he offers you one chance to switch your choice. Should you switch?

I will switch. Because: Let A be the event that we achieve a big prize at the first choice, and B be the event that Monty opens one of the two other doors and nothing behind this.

(a) Suppose you will not switch. What is the probability that you win?

Since we first have 3 doors and 1 prize in one of these 3, we have $p(A) = \frac{1}{3}$. If we do not switch, then A and B are independent, i.e., $p(A|B) = p(A) = \frac{1}{3}$. So, the probability that we win is $\frac{1}{3}$.

(b) Suppose you will switch. Monty shows a door without a prize. What is the probability that you win?

If we switch, then A and B are not independent, i.e.,

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}.$$

We have the probability if nothing is behind the door Monty open is $p(B) = 1$. If we win at the first choice, then $p(B|A) = 1$, so we get the probability if we win at the first choice is

$$p(A|B) = \frac{1 * \frac{1}{3}}{1} = \frac{1}{3}.$$

Because we switch, so the probability we win, i.e., the probability the first choice we lose is

$$1 - p(A|B) = \frac{2}{3}.$$

2 Problem 2

(Anderson) 10 men and 5 women are meeting in a conference room. Four people are chosen at random from the 15 to form a committee.

(a) What is the probability that the committee consists of 2 men and 2 women?

There are 2 combinations of the set which has 10 men, i.e.,

$$C(10, 2) = \frac{10!}{2!(10-2)!} = 45.$$

Similarly, there are 2 combinations of the set which has 5 women, i.e.,

$$C(5, 2) = \frac{5!}{2!(5-2)!} = 10.$$

On the other hand, since we select random 4 over 15 total people, the total number of combinations is

$$C(15,4) = \frac{15!}{4!(15-5)!} = 1365.$$

As a result, the probability that the committee consists of 2 men and women is

$$\frac{\text{2 men and 2 women combinations}}{\text{The total of combinations}} = \frac{45 * 10}{1365} = \frac{30}{91}.$$

(b) Among the 15 is a couple, Bob and Jane. What is the probability that Bob and Jane both end up on the committee?
Since Bob and Jance both end up on the committee, there are only 2 positions left for the committee from the set of 13 remaining people, i.e.,

$$C(13,2) = \frac{13!}{2!(13-2)!} = 78.$$

On the other hand, since we still select random 4 over 15 total people, the total number of combinations is

$$C(15,4) = \frac{15!}{4!(15-5)!} = 1365.$$

As a result, the probability that Bob and Jane both end up on the committee is

$$\frac{\text{Bob and Jane in committee}}{\text{The total of combinations}} = \frac{78}{1365} = \frac{2}{35}.$$

3 Problem 3

(Anderson) Pick a uniformly chosen random point inside the triangle with vertices $(0,0)$, $(3,0)$, $0,3$.

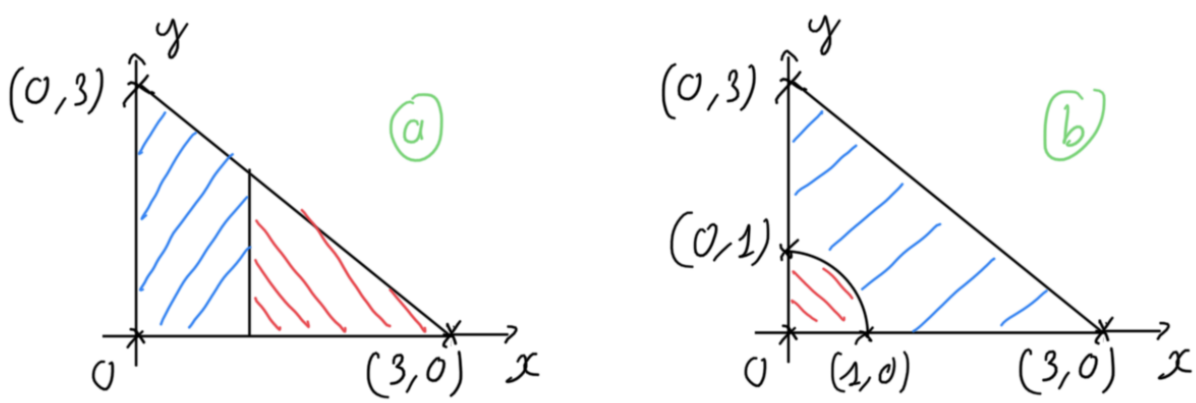

Figure 1: The visualization of two cases.

(a) What is the probability that the distance of this point to the y-axis is less than 1?
Following Figure 1 a, we have the probability that the distance of this point to the y-axis is less than 1 is the area of the blue trapezium over the area of the original triangle, i.e.,

$$\frac{\frac{1}{2} * (\frac{2}{3} * 3 + 3) * 1}{\frac{1}{2} * 3 * 3} = \frac{5}{9}.$$

(b) What is the probability that the distance of this point to the origin is more than 1?
Following Figure 1 b, we have the area of the blue shape equal to the area of the original triangle minus the area of the red shape, i.e.,

$$\frac{1}{2} * 3 * 3 - \frac{1}{4} * \pi * 1^2 = \frac{9}{2} - \frac{\pi}{4}.$$

On the other hand, we have the probability that the distance of this point to the origin is more than 1 is the area of the blue shape over the area of the original triangle, i.e.,

$$\frac{\frac{9}{2} - \frac{\pi}{4}}{\frac{1}{2} * 3 * 3} = 1 - \frac{\pi}{18}.$$

4 Problem 4

(Anderson) An urn contains one 6-sided die, two 8-sided dice, three 10-sided dice, and four 20-sided dice. One die is chosen at random and then rolled.

(a) What is the probability that the roll gave a five?
Let A be the outcome of the rolled dice and B_i be the event that we chose i-sided dice. Since A and B_i are not independent, we have the probability that the roll gave a five is

$$\begin{aligned} p(A = 5) &= \sum_{i=\{6,8,10,20\}} p(A = 5, B_i) = \sum_{i=\{6,8,10,20\}} p(A = 5 \mid B_i)p(B_i) \\ &= \frac{1}{6} * \frac{1}{10} + \frac{1}{8} * \frac{2}{10} + \frac{1}{10} * \frac{3}{10} + \frac{1}{20} * \frac{4}{10} = \frac{11}{120}. \end{aligned}$$

(b) What is the probability that the die rolled was the 20-sided die, given that the outcome of the roll was seven?
The probability that the die rolled was the 20-sided die, given that the outcome of the roll was seven is

$$\begin{aligned} p(B_{20} \mid A = 7) &= \frac{p(A = 7 \mid B_{20}) * p(B_{20})}{p(A = 7)} = \frac{p(A = 7 \mid B_{20}) * p(B_{20})}{\sum_{i=\{8,10,20\}} p(A = 7|B_i)p(B_i)} \\ &= \frac{(1/20) * (4/10)}{(1/8) * (2/10) + (1/10) * (3/10) + (1/20) * (4/10)} = \frac{4}{15}. \end{aligned}$$

5 Problem 5

(Anderson) Two factories I and II produce phone for brand ABC. Factory I produces 60% of all ABC phones, and factory II produces 40%. 10% of phone from factory I are defective, and 20% produced from factory II are defective. If you purchase a brand ABC phone, and assume that it's not defective. What's the probability that it comes from factory II.
Let A be the event we buy a phone that is not defective and B be the event we buy a phone from factory II. We have the probability that we can buy a phone that it's not defective is

$$p(A) = 1 - (0.6 * 0.1 + 0.2 * 0.4) = 0.86.$$

And since factory II produces 40%, we have $p(B) = 0.4$, and 20% of them are defective, so $p(A|B) = 1 - 0.2 = 0.8$. As a consequence, the probability that our non-defective phone comes from factory II is

$$p(B|A) = \frac{p(A|B)p(B)}{p(A)} = \frac{0.8 * 0.4}{0.86} = \frac{16}{43}.$$

6 Problem 6

(Anderson) Let X have possible values $\{1,2,3,4,5\}$ and probability mass function

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

Table 1: The PMF table.

(a) Calculate $P(X \leq 3)$
From the Table 1, we have

$$P(X \leq 3) = \sum_{i=1}^{3} p_X(x=i) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{3}{7}.$$

(b) Calculate $P(X < 3)$
From the Table 1, we have

$$P(X < 3) = p_X(x=1) + p_X(x=2) = \frac{1}{7} + \frac{1}{14} = \frac{3}{14}.$$

7 Problem 7

Suppose that the random variable X has expected value $\mathbb{E}[X] = 3$ and variance $Var(X) = 4$. Compute $\mathbb{E}[(2X+3)^2]$.
We have

$$\begin{aligned}
\mathbb{E}[(2X+3)^2] &= \mathbb{E}[4X^2 + 12X + 9] \\
&= 4\mathbb{E}[X^2] + 12\mathbb{E}[X] + 9 \\
&= 4\left(Var(X) + \mathbb{E}[X]^2\right) + 12\mathbb{E}[X] + 9 \\
&= 4\left(4 + 3^2\right) + 12 * 3 + 9 = 97.
\end{aligned}$$

8 Problem 8

Suppose that a random variable X has the normal distribution with mean 95 and standard deviation 11. Find the probability $P(X \geq 100)$.
Let the mean and the standard deviation of random variable X is μ and σ respectively, and let the raw score of X be x, then apply the standard z-score, i.e.,

$$z = \frac{x - \mu}{\sigma},$$

we obtain

$$P(X \geq 100) = P(Z \geq \frac{100-95}{11}) = P(Z \geq \frac{5}{11}) = 1 - P(Z < \frac{5}{11}).$$

Using the z-score table, we know $P(Z < \frac{5}{11}) = P(Z < 0.45) \approx 0.67$, as a consequence, we obtain

$$P(X \geq 100) = 1 - 0.67 = 0.33.$$