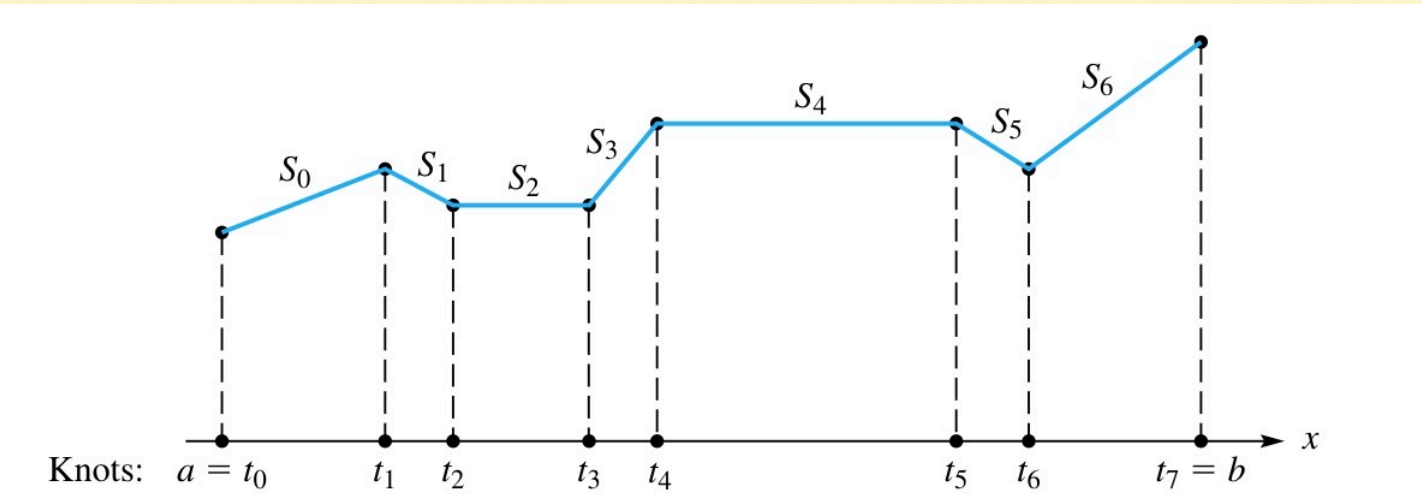
6. Spline functions

spline function: a function that consists of polynomial pieces joined together with certain smoothness conditions.

First degree Splines



$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1] \\ S_1(x), & x \in [t_1, t_2] \\ \quad \cdots & \\ S_{n-1}(x), & x \in [t_{n-1}, t_n] \end{cases}$$

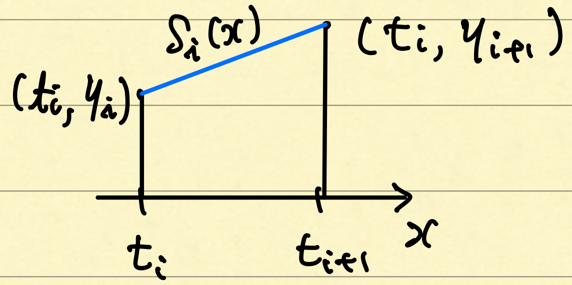
where

$S_i(x) = a_i x + b_i$

unknown

A function S is called a spline of degree 1 if

1. The domain of S is an interval $[a, b]$
2. S is continuous on $[a, b]$
3. There is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ s.t. S is a linear polynomial on each subinterval $[t_i, t_{i+1}]$.



$$S_i(x) = y_i + m_i(x - t_i)$$

$$m_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

if $x < t_0 \Rightarrow S(x) = y_0 + m_0(x - t_0)$

if $x > t_n \Rightarrow S(x) = y_{n-1} + m_{n-1}(x - t_{n-1})$

Modulus of continuity of f

$$\omega(f;h) = \sup\{ |f(u) - f(v)| : a \le u \le v \le b,\ |u - v| \le h \}$$

Thm 1 <u>First degree polynomial accuracy thm</u>

If p is a first-degree polynomial that interpolates a function f at the endpoints of an interval $[a,b]$, then with $h = b - a$, we have

$$|f(x) - p(x)| \le \omega(f;h) \qquad (a \le x \le b)$$

pf)

$$p(x) = \left(\frac{x-a}{b-a}\right) f(b) + \left(\frac{b-x}{b-a}\right) f(a)$$

$$f(x) - p(x) = \left(\frac{x-a}{b-a}\right)[f(x) - f(b)] + \left(\frac{b-x}{b-a}\right)[f(x) - f(a)]$$

$$|f(x) - p(x)| \le \left(\frac{x-a}{b-a}\right)|f(x) - f(b)| + \left(\frac{b-x}{b-a}\right)|f(x) - f(a)|$$

$$\le \left(\frac{x-a}{b-a}\right)\omega(f;h) + \left(\frac{b-x}{b-a}\right)\omega(f;h)$$

$$= \omega(f;h) \quad //$$

Thm2 First degree Spline Accuracy thm

p : first degree spline having knots $a = x_0 < x_1 < \cdots < x_n = b$
interpolates f at these knots.

$\Rightarrow |f(x) - p(x)| \leq \omega(f; h)$

where $h = \max_i (x_i - x_{i-1})$, $a \leq x \leq b$

Second degree Splines

Q : spline of degree 2

1. domain Q is $[a, b]$
2. Q, Q' continuous on $[a, b]$
3. Q is a polynomial of degree at most 2 on $[t_i, t_{i+1}]$.

$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

$Q_i(x) = a_i x^2 + b_i x + c_i$

unknown : $3n$

conditions :

$$\left.\begin{array}{ll} Q_i(t_i) = y_i & \Rightarrow n+1 \\ Q \text{ conti} \Rightarrow Q_i(t_{i+1}) = Q_{i+1}(t_{i+1}) & \Rightarrow n-1 \\ Q' \text{ conti} \Rightarrow Q_i'(t_{i+1}) = Q_{i+1}'(t_{i+1}) & \Rightarrow n-1 \end{array}\right\} 3n-1$$

need one more condition :

impose $Q'(t_0) = 0$ or $Q''(t_0) = 0$

Finding $Q(x)$

$Q_i(t_i) = y_i$, $Q_i'(t_i) = z_i$ for $0 \leq i \leq n$

$Q_i(x) = a(x - t_i)^2 + z_i(x - t_i) + y_i$

$$Q_{i+1}(x) = \tilde{a}(x - t_{i+1})^2 + z_{i+1}(x - t_{i+1}) + y_{i+1}$$

<u>Q' conti</u>

$$Q_i'(t_{i+1}) = 2a(t_{i+1} - t_i) + z_i$$

$$Q_{i+1}'(t_{i+1}) = z_{i+1}$$

$$\Rightarrow \quad a = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}$$

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i$$

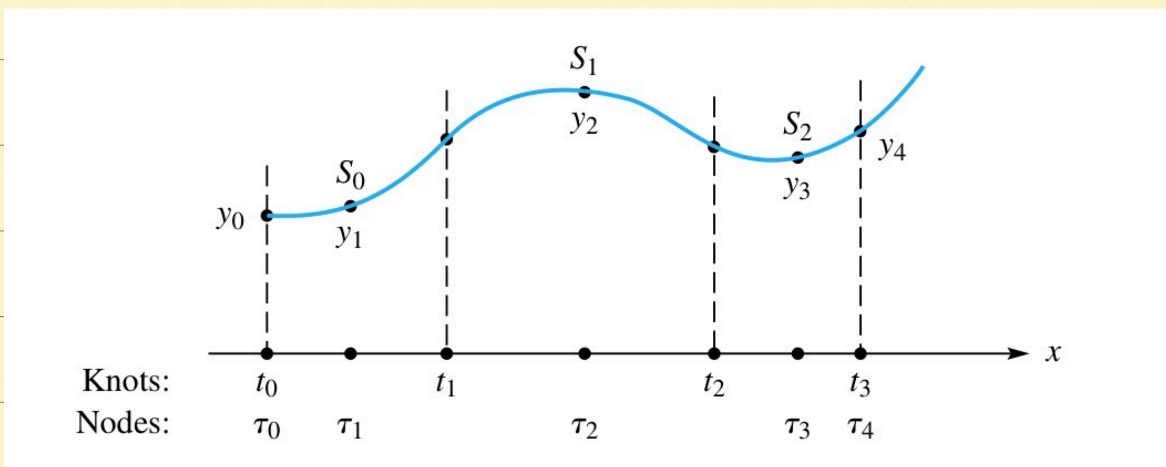
<u>Q conti</u>

$$Q_i(t_{i+1}) = y_{i+1}$$

$$\Rightarrow z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) \quad (0 \le i \le n)$$

z_0: arbitrary $\Rightarrow$ get z_i $(1 \le i \le n)$

Subbotin Quadratic Spline



knots : the points where the spline ftn is permitted to change in form from one poly to another.

nodes : the points where the value of the spline are specified.

$$\begin{cases} \tau_0 = t_0, & \tau_{n+1} = t_n \\ \tau_i = \frac{1}{2}(t_i + t_{i-1}) & (1 \leq i \leq n) \end{cases}$$

$Q(\tau_i) = y_i \quad (0 \leq i \leq n+1) \Rightarrow n+2$ conditions.

Q, Q' conti $\Rightarrow 2(n-1)$ conditions

$\Rightarrow 3n$ conditions

$z_i \equiv Q'(t_i)$

$$\Rightarrow Q_i(x) = y_{i+1} + \frac{1}{2}(z_{i+1} + z_i)(x - \tau_{i+1}) + \frac{1}{2h_i}(z_{i+1} - z_i)(x - \tau_{i+1})^2$$

where $h_i = t_{i+1} - t_i$

$Q_i(\tau_{i+1}) = y_{i+1}, \quad Q_i'(t_i) = z_i, \quad Q'(t_{i+1}) = z_{i+1}$

conti : $\lim\limits_{x \to t_i^-} Q_{i-1}'(x) = \lim\limits_{x \to t_i^+} Q_i'(x) \quad (1 \leq i \leq n-1)$

$h_{i-1} z_{i-1} + 3(h_{i-1} + h_i) z_i + h_i z_{i+1} = 8(y_{i+1} - y_i) \quad (1 \leq i \leq n-1)$

$Q(\tau_0) = y_0 \Rightarrow 3h_0 z_0 + h_0 z_1 = 8(y_1 - y_0)$

$Q(\tau_{n+1}) = y_{n+1} \Rightarrow h_{n-1} z_{n-1} + 3h_{n-1} z_n = 8(y_{n+1} - y_n)$

$$\begin{bmatrix} 3h_0 & h_0 & & & & \\ h_0 & 3(h_0+h_1) & h_1 & & & \\ & h_1 & 3(h_1+h_2) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 3(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & h_{n-1} & 3h_{n-1} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = 8 \begin{bmatrix} y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ \vdots \\ y_n - y_{n-1} \\ y_{n+1} - y_n \end{bmatrix}$$

General definition of spline functions of arbitrary degree k

A function S is called a spline of degree k if

1. The domain S is an interval $[a, b]$
2. $S, S', S'', \cdots, S^{(k-1)}$ are all continuous ftns on $[a, b]$
3. There are points t_i (the knots of S) such that $a = t_0 < t_1 < \cdots < t_n = b$ and such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$

Natural Cubic Splines

x	t_0	t_1	$\cdots$	t_n
y	y_0	y_1	$\cdots$	y_n

cubic polynomial

$$S(x) = \begin{cases} S_0(x) & t_0 \leq x \leq t_1 \\ S_1(x) & t_1 \leq x \leq t_2 \\ \quad \cdots & \\ S_{n-1}(x) & t_{n-1} \leq x \leq t_n \end{cases}$$

unknowns

$$\left.\begin{array}{lll} S(t_i) = y_i & (0 \leq i \leq n) & \rightarrow n+1 \\ S, S', S'' \text{ conti} & & \rightarrow 3(n-1) \end{array}\right\} 4n-2$$

need two more conditions

$$S''(t_0) = S''(t_n) = 0$$

$\Rightarrow$ natural cubic spline

Thm Cubic Spline Smoothness theorem

S: natural cubic spline function of f at $a = t_0 < t_1 < \cdots < t_n = b$,

$$\Rightarrow \int_a^b [S''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx$$

pf) let $g(x) = f(x) - S(x)$. $g(t_i) = 0$ for $0 \leq i \leq n$

$f'' = g'' + S''$

$$\int_a^b (f'')^2 dx = \int_a^b (S'')^2 dx + \int_a^b (g'')^2 dx + 2\int_a^b S''g'' dx$$

$$\int_a^b S''g'' dx = S''g' \Big|_a^b - \int_a^b S'''g' dx = -\int_a^b S'''g' dx$$

$S''(a) = S''(b) = 0$

$$\int_a^b S'''g' dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} S'''g' dx = \sum_{i=0}^{n-1} C_i \int_{t_i}^{t_{i+1}} g' dx$$

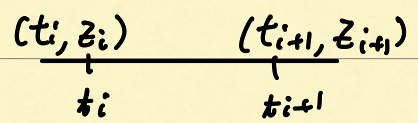
$$= \sum_{i=0}^{n-1} C_i [g(t_{i+1}) - g(t_i)] = 0$$

$$\therefore \int_a^b (f'')^2 dx \geq \int_a^b (S'')^2 dx \quad //$$

Algorithm for Natural Cubic Spline

① let $z_i \equiv S''(t_i)$, $0 \leq i \leq n$

$z_0 = z_n = 0$



, $h_i = t_{i+1} - t_i$ for $0 \leq i \leq n-1$

$$S_i''(x) = \frac{z_{i+1}}{h_i}(x-t_i) + \frac{z_i}{h_i}(t_{i+1}-x)$$

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x-t_i)^3 + \frac{z_i}{6h_i}(t_{i+1}-x)^3 + cx + d$$

$$= \frac{z_{i+1}}{6h_i}(x-t_i)^3 + \frac{z_i}{6h_i}(t_{i+1}-x)^3 + C_i(x-t_i) + D_i(t_{i+1}-x)$$

② $S_i(t_i) = y_i$, $S_i(t_{i+1}) = y_{i+1}$

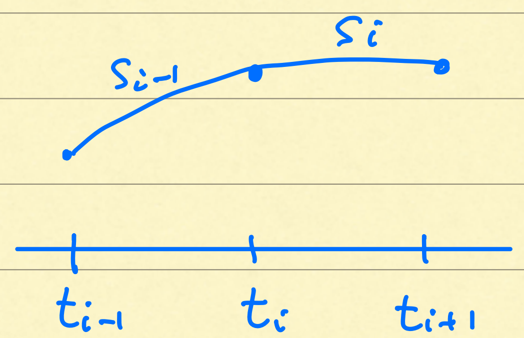
$$\Rightarrow S_i(t_i) = \frac{z_i}{6h_i}h_i^3 + D_i h_i$$

$$= \frac{z_i}{6}h_i^2 + D_i h_i = y_i \Rightarrow D_i = \left(\frac{y_i}{h_i} - \frac{h_i}{6}z_i\right)$$

$$S_i(t_{i+1}) = \frac{z_{i+1}}{6}h_i^2 + C_i h_i = y_{i+1} \Rightarrow C_i = \left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}\right)$$

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x-t_i)^3 + \frac{z_i}{6h_i}(t_{i+1}-x)^3$$

$$+ \left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}\right)(x-t_i) + \left(\frac{y_i}{h_i} - \frac{h_i}{6}z_i\right)(t_{i+1}-x)$$



$$S_{i-1}'(t_i) = S_i'(t_i)$$

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x-t_i)^2 - \frac{z_i}{2h_i}(t_{i+1}-x)^2 + \frac{y_{i+1}}{h_i} - \frac{h_i}{6}z_{i+1}$$

$$- \frac{y_i}{h_i} + \frac{h_i}{6}z_i$$

$$\begin{cases} S_i'(t_i) = -\frac{h_i}{6} z_{i+1} - \frac{h_i}{3} z_i + b_i , \quad b_i = \frac{1}{h_i}(y_{i+1} - y_i) \\ S_{i-1}'(t_i) = \frac{h_{i-1}}{6} z_{i-1} + \frac{h_{i-1}}{3} z_i + b_{i-1} \end{cases}$$

$$\Rightarrow \quad h_{i-1} z_{i-1} + u_i z_i + h_i z_{i+1} = v_i$$

$$u_i = 2(h_{i-1} + h_i), \quad v_i = 6(b_i - b_{i-1})$$

$$\Rightarrow \begin{cases} z_i = 0 & \leftarrow S''(t_0) = 0 \\ h_{i-1} z_{i-1} + u_i z_i + h_i z_{i+1} = v_i & (1 \le i \le n-1) \\ z_n = 0 & \leftarrow S''(t_n) = 0 \end{cases}$$

Algorithm

Given the interpolation pts (t_i, y_i) for $i = 0, 1, \cdots, n$

1. Compute for $i = 0, 1, \cdots, n-1$

$$\begin{cases} h_i = t_{i+1} - t_i \\ b_i = \frac{1}{h_i}(y_{i+1} - y_i) \end{cases}$$

2. Set $u_1 = 2(h_0 + h_1)$, $v_1 = 6(b_1 - b_0)$

   and compute inductively for $i = 2, 3, \cdots, n-1$:

$$u_i = 2(h_i + h_{i-1}) - \frac{h_{i-1}^2}{u_{i-1}}, \quad v_i = 6(b_i - b_{i-1}) - \frac{h_{i-1} v_{i-1}}{u_{i-1}}$$

3. Set $z_n = 0$, $z_0 = 0$

   and compute inductively for $i = n-1, n-2, \cdots, 1$

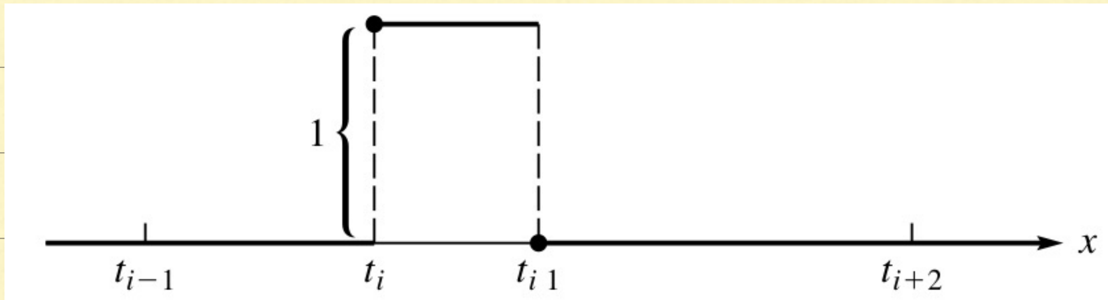
$$z_i = \frac{v_i - h_i z_{i+1}}{u_i}$$

B Splines: Interpolation and Approximation

basis, bell

$$\begin{cases} \cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots \\ \lim_{i \to \infty} t_i = \infty = -\lim_{i \to \infty} t_{-i} \end{cases}$$

B splines of degree 0 : $B_i^0(x) = \begin{cases} 1 & t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$



Continuous from the right at all points even where the jumps occur. Thus $\lim_{x \to t_i^+} B_i^0(x) = 1 = B_i^0(t_i)$ and

$\lim_{x \to t_{i+1}^+} B_i^0(x) = 0 = B_i^0(t_{i+1})$.

$$\begin{cases} B_i^0(x) \geq 0 \quad \text{for all } x \text{ and for all } i. \\ \sum_{i=-\infty}^{\infty} B_i^0(x) = 1 \quad \text{for all } x \end{cases}$$

B spline of degree 0 :

If $S(x) = b_i$ for $t_i \le x < t_{i+1}$,

$$S = \sum_{i=-\infty}^{\infty} b_i B_i^0$$

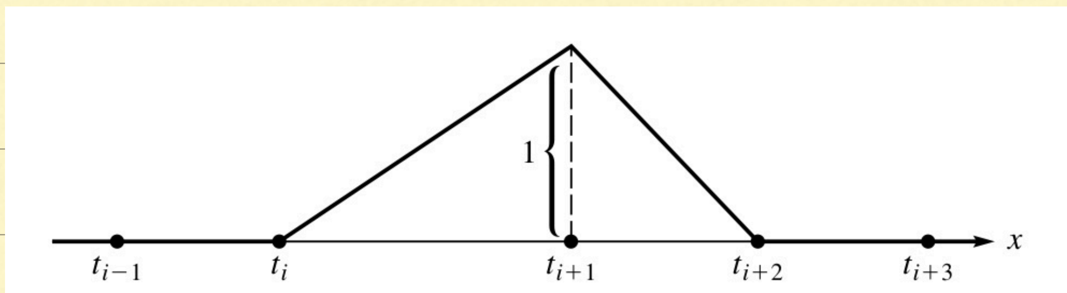
Higher degree B splines

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+1} - t_i}\right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x), \quad (k \geq 1)$$

$k = 1, 2, \cdots \quad , \quad i = 0, \pm 1, \pm 2, \cdots$

$$B_i^1(x) = \left(\frac{x - t_i}{t_{i+1} - t_i}\right) B_i^0(x) + \left(\frac{t_{i+2} - x}{t_{i+2} - t_{i+1}}\right) B_{i+1}^0(x)$$

$$= \begin{cases} 0 & , \quad x \geq t_{i+2} \text{ or } x \leq t_i \\ \dfrac{x - t_i}{t_{i+1} - t_i} & , \quad t_i \leq x < t_{i+1} \\ \dfrac{t_{i+2} - x}{t_{i+2} - t_{i+1}} & , \quad t_{i+1} \leq x < t_{i+2} \end{cases}$$



$$\sum_{i=-\infty}^{\infty} B_i^1(x) = 1 \quad \text{for all } x$$

$$\Rightarrow \quad B_i^k(x) = 0 \quad , \quad x \notin [t_i, t_{i+k+1}) \qquad (k \geq 0)$$

$$B_i^k(x) > 0 \quad , \quad x \in (t_i, t_{i+k+1}) \qquad (k \geq 0)$$

To evaluate a function

$$f(x) = \sum_{i=-\infty}^{\infty} C_i^k B_i^k(x)$$

$$= \sum_{i=-\infty}^{\infty} C_i^k \left[\left(\frac{x - t_i}{t_{i+k} - t_i} \right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} \right) B_{i+1}^{k-1}(x) \right]$$

$$= \sum_{i=-\infty}^{\infty} \left[C_i^k \left(\frac{x - t_i}{t_{i+k} - t_i} \right) + C_{i-1}^k \left(\frac{t_{i+1} - x}{t_{i+k} - t_i} \right) \right] B_i^{k-1}(x)$$

$$= \sum_{i=-\infty}^{\infty} C_i^{k-1} B_i^{k-1}(x)$$

repeating $k-1$ times

$$f(x) = \sum_{i=-\infty}^{\infty} C_i^0 B_i^0(x) .$$

If $t_m \le x < t_{m+1}$, then $f(x) = C_m^0$

$$C_i^{j-1} = C_i^j \left(\frac{x - t_i}{t_{i+j} - t_i} \right) + C_{i-1}^j \left(\frac{t_{i+j} - x}{t_{i+j} - t_i} \right)$$

$C_m^k, C_{m-1}^k, \cdots C_{m-k}^k$ are needed to compute $f(x)$ if $t_m \le x < t_{m+1}$.

$$\begin{array}{llll} C_m^k & C_m^{k-1} & \cdots & C_m^0 \\ C_{m-1}^k & C_{m-1}^{k-1} & \cdots & \\ \vdots & & \iddots & \\ C_{m-k}^k & & & \end{array}$$

$$\sum_{i=-\infty}^{\infty} B_i^k(x) = 1 \text{ for all } x \text{ and all } k \ge 0$$

i) $k=0$, clear

ii) $k>0$, $f(x) = \sum_{i=-\infty}^{\infty} C_i^k B_i^k(x)$ $\cdots$ (*1)

$f(x) = \sum_{i=-\infty}^{\infty} C_i^0 B_i^0(x)$ $\cdots$ (*2) let $C_i^k = 1$ for all i $\Rightarrow$ By the relation

$$C_i^{j-1} = C_i^j \left(\frac{x - t_i}{t_{i+j} - t_i} \right) + C_{i-1}^j \left(\frac{t_{i+j} - x}{t_{i+j} - t_i} \right)$$

$$\Rightarrow C_i^0 = 1$$

$$\therefore \quad (*2) \Rightarrow f(x) = 1$$

$$(*1) \Rightarrow \sum_{i=-\infty}^{\infty} B_i^k(x) = 1$$

The smoothness of the B splines B_i^k increases with the index k. In fact, we can show by induction that B_i^k has a continuous k-1st derivative.

Derivative of B splines

$$\frac{d}{dx} \sum_{i=-\infty}^{\infty} C_i B_i^k(x) = \sum_{i=-\infty}^{\infty} C_i \frac{d}{dx} B_i^k(x)$$

$$= \sum_{i=-\infty}^{\infty} C_i \left[\left(\frac{k}{t_{i+k} - t_i} \right) B_i^{k-1}(x) - \left(\frac{k}{t_{i+k+1} - t_{i+1}} \right) B_{i+1}^{k-1}(x) \right]$$

$$= \sum_{i=-\infty}^{\infty} \left[\left(\frac{C_i k}{t_{i+k} - t_i} \right) - \left(\frac{C_{i-1} k}{t_{i+k} - t_i} \right) \right] B_i^{k-1}(x)$$

$$= \sum_{i=-\infty}^{\infty} d_i B_i^{k-1}(x), \qquad d_i = k \left(\frac{C_i - C_{i-1}}{t_{i+k} - t_i} \right)$$

Numerical Integration of B splines

$$\int_{-\infty}^{x} B_i^k(s)\, ds = \left(\frac{t_{i+k+1} - t_i}{k+1} \right) \sum_{j=i}^{\infty} B_j^{k+1}(x)$$

$$\int_{-\infty}^{x} \sum_{i=-\infty}^{\infty} c_i B_i^k(s)\,ds = \sum_{i=-\infty}^{\infty} e_i B_i^{k+1}(x)$$

where $$e_i = \frac{1}{k+1} \sum_{j=-\infty}^{\infty} c_j \left(t_{j+k+1} - t_j\right)$$

Interpolation and Approximation by B splines

$$S(x) = \sum_{i=-\infty}^{\infty} A_i B_{i-k}^k(x) \quad \cdots \quad (*)$$

x	t_0	t_1	$\cdots$	t_n
y	y_0	y_1	$\cdots$	y_n

interpolate $S(t_i) = y_i$, $\quad 0 \leq i \leq n$

$k=0 \Rightarrow B_i^0(t_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

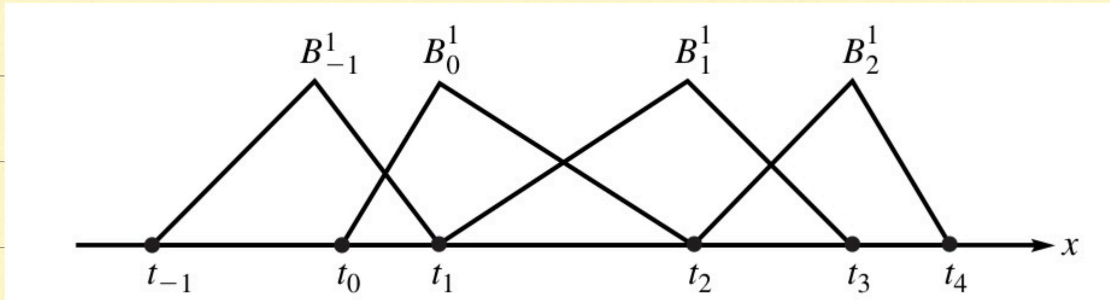
Set $A_i = y_i$ for $0 \leq i \leq n$.

All other coefficients in $(*)$ are arbitrary.

zero degree B spline $\quad S(x) = \sum_{i=0}^{n} y_i B_i^0(x)$

$k=1$, using the fact $B_{i-1}^1(t_j) = \delta_{ij}$

$$S(x) = \sum_{i=0}^{n} y_i B_{i-1}^1(x) \quad \text{so } A_i = y_i$$



$n=3$, knots t_{-1}, t_4 arbitrary

$\underline{k=2}$,

$$\sum_{i=-\infty}^{\infty} A_i B_{i-2}^2(t_j) = \frac{1}{t_{j+1}-t_{j-1}}\left[A_j(t_{j-1}-t_j) + A_{j+1}(t_j - t_{j-1})\right]$$

$S(t_i) = y_i \quad (0 \leq i \leq m)$ $\cdots$ (*1)

$\Rightarrow A_j(t_{j+1}-t_j) + A_{j+1}(t_j - t_{j-1}) = y_j(t_{j+1}-t_{j-1}), \quad 0 \leq j \leq m$

$m+1$ equations, $m+2$ unknowns $A_0, A_1, \cdots, A_{m+1}$

Assign any value to A_0.

Solve (*1)

1) Assign any value to A_0

2) $A_{j+1} = \alpha_j + \beta_j A_j \qquad (0 \leq j \leq m) \qquad \cdots$ (*2)

where
$$\begin{cases} \alpha_j = y_j\left(\dfrac{t_{j+1}-t_{j-1}}{t_j - t_{j-1}}\right) \\ \beta_j = \dfrac{t_j - t_{j+1}}{t_j - t_{j-1}} \qquad (0 \leq j \leq m) \end{cases}$$

To keep the coefficients small in magnitude, selecting A_0 s.t. $\Phi = \sum_{i=0}^{m+1} A_i^2$ will be minimum.

(*2) $\Rightarrow A_{j+1} = \gamma_j + \delta_j A_0 \qquad (0 \leq j \leq m)$

where
$$\begin{cases} \gamma_0 = \alpha_0 & \delta_0 = \beta_0 \\ \gamma_j = \alpha_j + \beta_j \gamma_{j-1} & \delta_j = \beta_j \delta_{j-1} \quad (1 \leq j \leq m) \end{cases}$$

$\Rightarrow \Phi = A_0^2 + A_1^2 + \cdots + A_{m+1}^2$

$= A_0^2 + (\gamma_0 + \delta_0 A_0)^2 + (\gamma_1 + \delta_1 A_0)^2 + \cdots + (\gamma_m + \delta_m A_0)^2$

To find a minimum Φ

$$\frac{d\Phi}{dA_0} = 2A_0 + 2(\gamma_0 + \delta_0 A_0)\delta_0 + \cdots + 2(\gamma_n + \delta_n A_0)\delta_n = 0$$

$$\Longleftrightarrow \quad qA_0 + p = 0$$

where $\begin{cases} q = 1 + \delta_0^2 + \delta_1^2 + \cdots + \delta_n^2 \\ p = \gamma_0\delta_0 + \gamma_1\delta_1 + \cdots + \gamma_n\delta_n \end{cases}$

Schoenberg's Process

$$S(x) = \sum_{i=-\infty}^{\infty} f(\tau_i) B_i^2(x) \quad \text{where } \tau_i = \frac{1}{2}(t_{i+1} + t_{i+2})$$

properties

1. If $f(x) = ax + b$, then $S(x) = f(x)$
2. If $f(x) \geq 0$ everywhere, then $S(x) \geq 0$ everywhere
3. $\max_x |S(x)| \leq \max_x |f(x)|$
4. If f is conti on $[a,b]$, if $\delta = \max_i |t_{i+1} - t_i|$, and if $\delta < b - a$, then for x in $[a,b]$,

$$|S(x) - f(x)| \leq \frac{3}{2} \max_{a \leq u \leq v \leq u+\delta \leq b} |f(u) - f(v)|$$

5. The graph of S does not cross any line in the plane a greater number of times than does the graph of f.

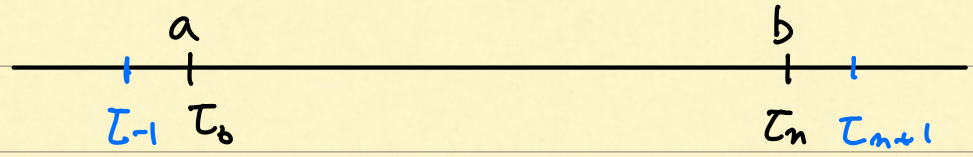
Pseudocode

nodes $\tau_i = a + ih$, $\quad h = (b-a)/n$

$\tau_i = \frac{1}{2}(t_{i+1} + t_{i+2})$, $\quad$ knots $t_i = a + (i - \frac{3}{2})h$

$B_{-1}^2, B_0^2, \cdots B_{n+1}^2$ are active on $[a,b]$

so $\quad S(x) = \sum_{i=-1}^{n+1} f(\tau_i) B_i^2(x)$



we need $f(\tau_{-1})$, $f(\tau_{n+1})$

$$f(\tau_{-1}) = 2f(\tau_0) - f(\tau_{-1})$$
$$f(\tau_{n+1}) = 2f(\tau_n) - f(\tau_{n-1})$$

$$\Rightarrow \quad S(x) = \sum_{i=1}^{n+3} D_i B_{i-2}^{2}(x), \text{ where } D_i = f(\tau_{i-2})$$

Bézier Curves

Bézier Curves use as a basis for the space Π_n.
(all polynomials of degree not exceeding n)
on $[0,1]$ and fix a value of n.

$$\varphi_{ni}(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (0 \le i \le n)$$

Bernstein polynomial

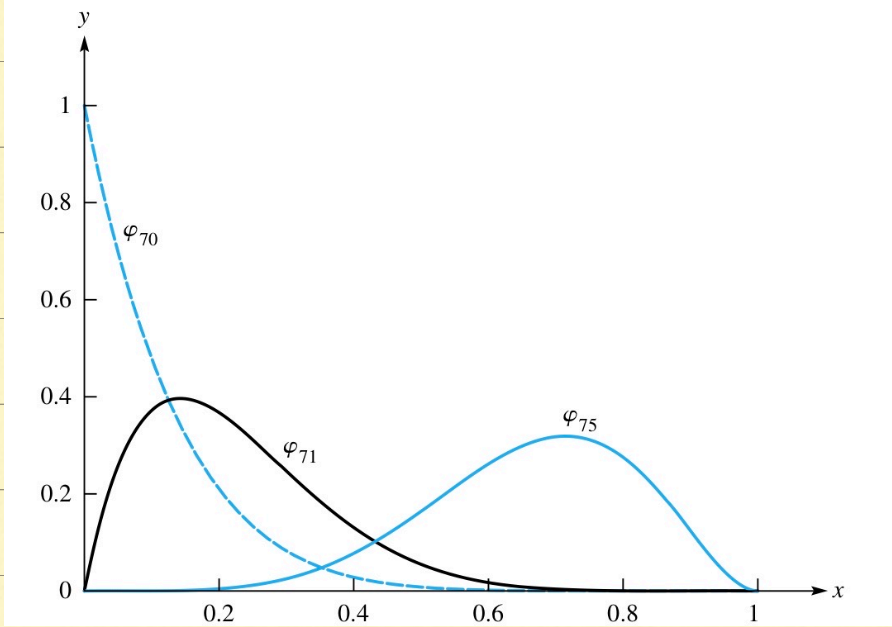
$$P_n(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \varphi_{ni}(x) \qquad (n \ge 1)$$

converges uniformly to f.

1. $\varphi_{ni}(x) \ge 0$
2. $\sum_{i=0}^{n} \varphi_{ni}(x) = 1$

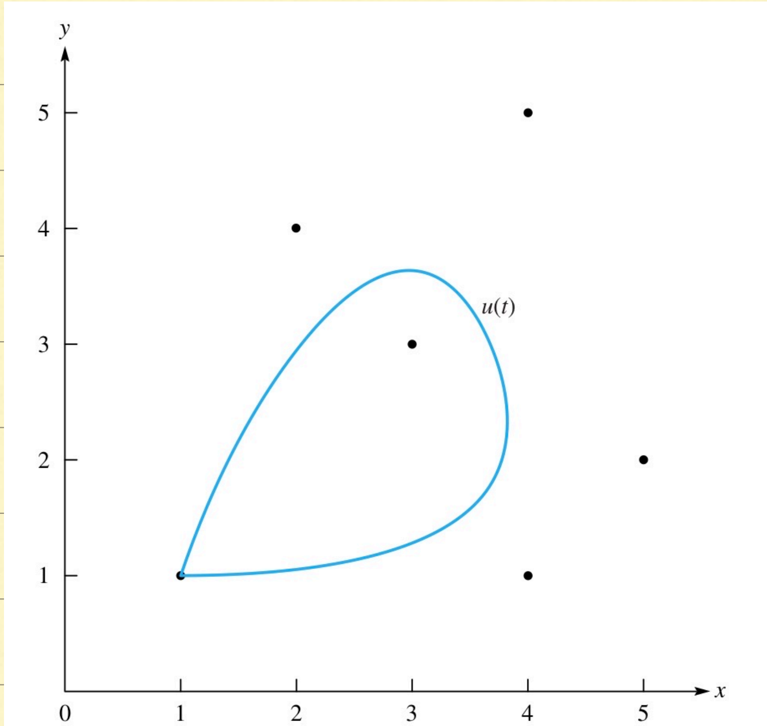
} partition of unity

$\{\varphi_{n0}, \varphi_{n1}, \cdots \varphi_{nn}\}$ is a basis for the space Π_n
every polynomials of degree at most n has
a representation $\sum_{i=0}^{n} a_i \varphi_{ni}(x)$



$$u(t) = \sum_{i=0}^{n} \varphi_{ni}(t)\, v_i \quad (0 \leq t \leq 1)$$

where v_i are the prescribed vector.



curve using control points