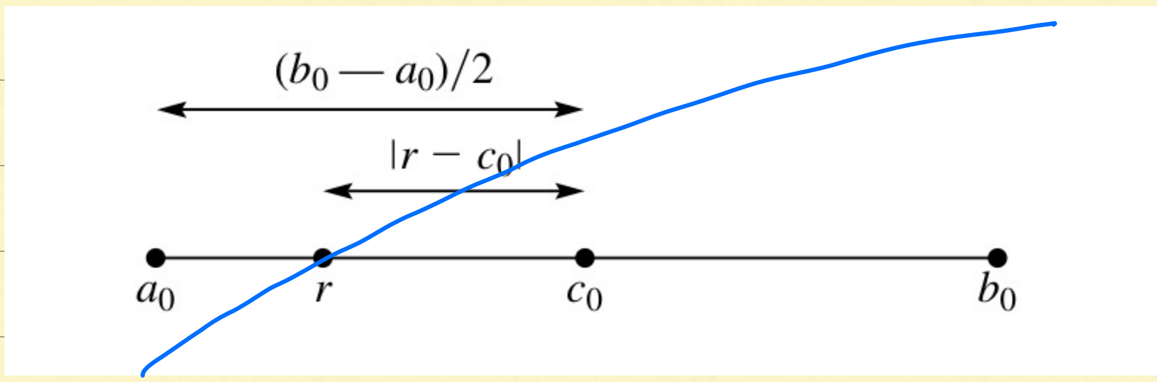
Chap3 Nonlinear equations

- Bisection Method
- Newton Method
- Secant Method
- Fixed point iteration

Bisection Method Theorm

If the bisection algorithm is applied to a continuous function f on an interval $[a,b]$, where $f(a)f(b) < 0$, then, after n steps, an approximate root will have been computed with error at most $(b-a)/2^{n+1}$



Thm If $[a_0, b_0], [a_1, b_1], \cdots, [a_n, b_n]$ denote the intervals in the bisection method, then the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, are equal, and represent a zero of f. If $r = \lim_{n\to\infty} c_n$ and $c_n = \frac{1}{2}(a_n + b_n)$, then $|r - c_n| \leq (b_0 - a_0)/2^{n+1}$

pf) $a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_0$

$b_0 \geq b_1 \geq b_2 \geq \cdots \geq a_0$

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \quad (n \geq 0) \quad \cdots (*)$$

$\{a_n\}$ increasing, bounded above $\Rightarrow$ converges

$\{b_n\}$ decreasing, bounded below $\Rightarrow$ converges

From $(*)$, we can get $b_n - a_n = (b_0 - a_0)/2^n$

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}(b_0 - a_0)/2^n = 0$$

let $r = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$

From the relation $f(a_n)f(b_n) \leq 0$,

taking a limit, we can get $[f(r)]^2 \leq 0$

$\therefore f(r) = 0$

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) \leq \frac{1}{2^{n+1}}(b_0 - a_0)$$ //

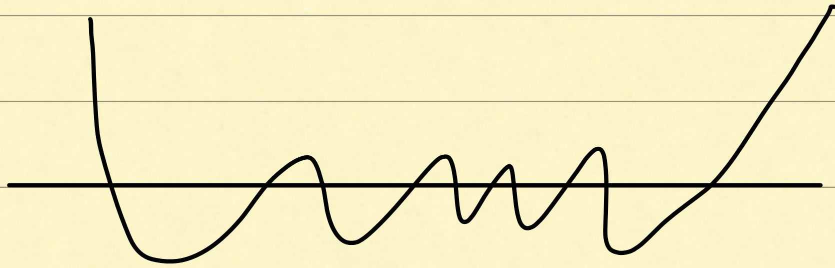
Example

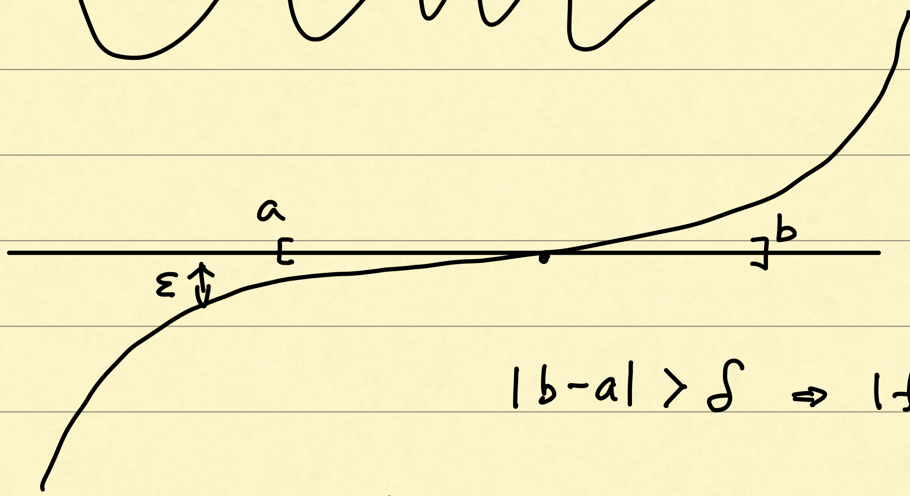
Suppose that the bisection method is stored with the interval $[50, 63]$. How many steps should be taken to compute a root with relative accuracy 10^{-12}?

sol) $$\frac{|r - c_n|}{|r|} \leq 10^{-12} \Rightarrow \frac{|r - c_n|}{50} \leq 10^{-12}$$

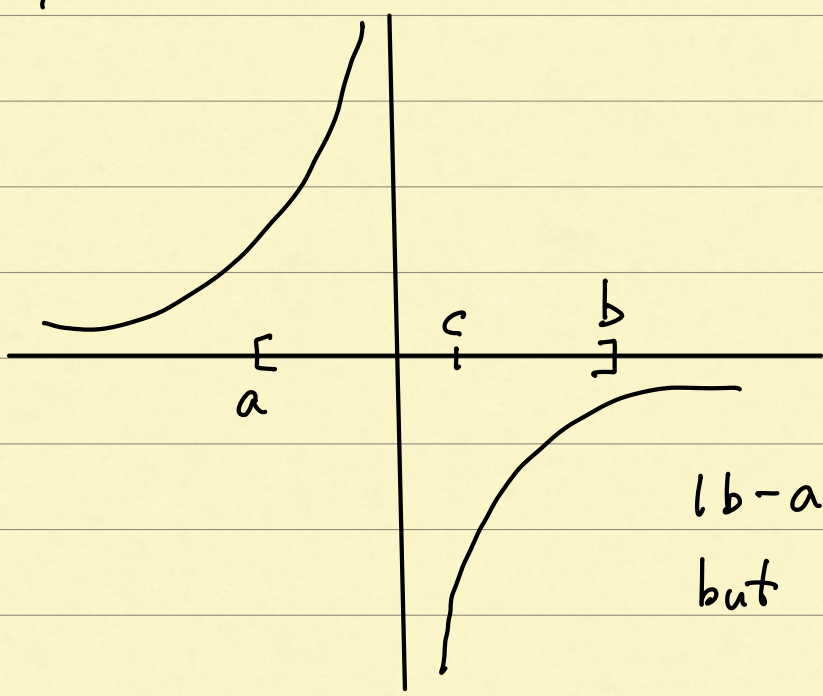
By thm $|r - c_n| \leq 2^{-(n+1)} \cdot 13 \leq 50 \times 10^{-12}$

$\therefore \quad n \geq 37$ //





$|b-a| > \delta \Rightarrow |f(c)| < \varepsilon$



$|b-a| < \delta$

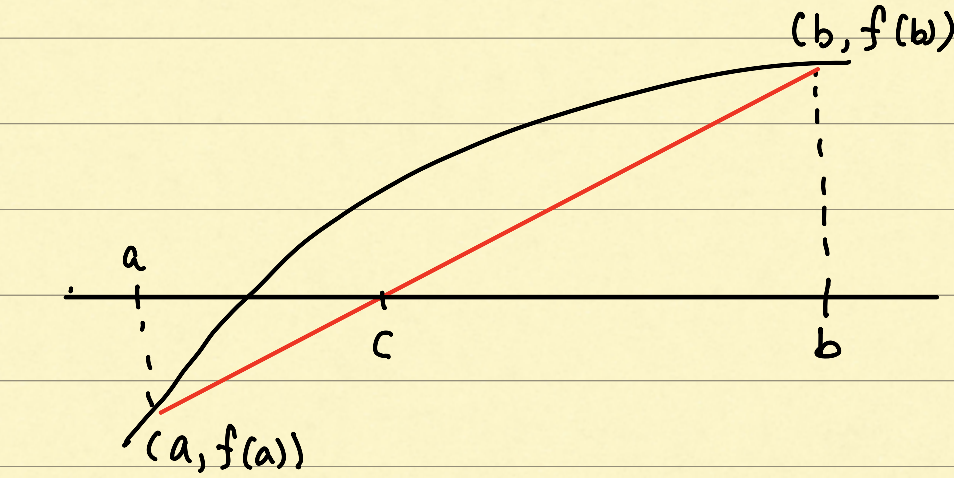
but $|f(c)| > \varepsilon$

A sequence $\{x_n\}$ is said to be have a linear convergence to a limit x if $\exists C \in [0,1)$ s.t. $|x_{n+1} - x| \leq C|x_n - x|$. $(n \geq 1)$

If this inequality is true for all n, then

$$|x_{n+1} - x| \leq C|x_n - x| \leq C^2|x_{n-1} - x| \leq \cdots \leq C^n|x_1 - x|$$

False Position (Regular Falsi) Method



$$\frac{b-c}{f(b)} = \frac{c-a}{-f(a)}$$

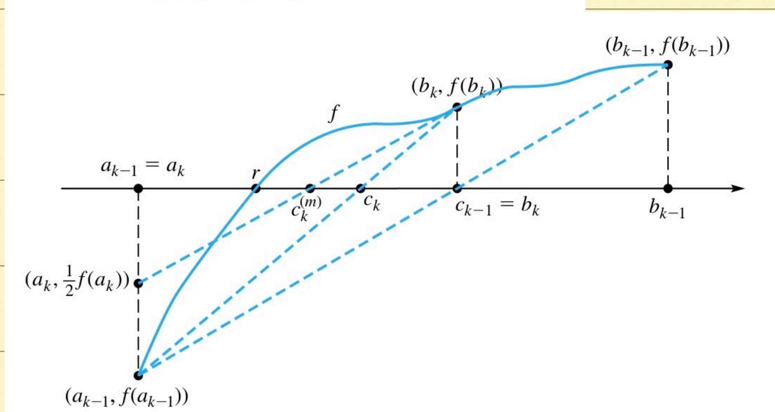
$$\Rightarrow c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$c_k = \frac{a_k f(b_k) - b_k f(a_k)}{f(b_k) - f(a_k)}$$

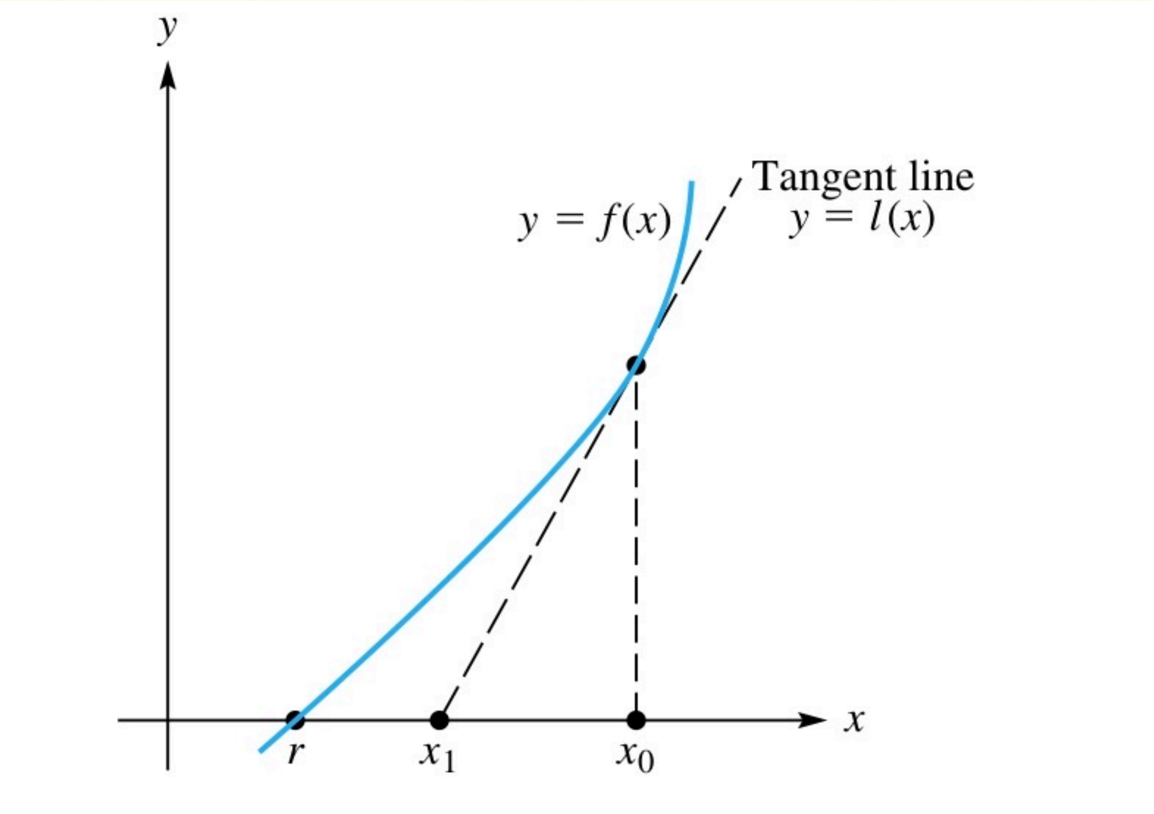
Modified false position Method

For some function, the false position method may repeatedly select the same endpoint, and the process may degrade to linear convergence. When the same endpoint is to be retained twice, the modified false position method uses

$$c_k^{(m)} = \begin{cases} \dfrac{a_k f(b_k) - 2b_k f(a_k)}{f(b_k) - 2f(a_k)}, & \text{if } f(a_k)f(b_k) < 0 \\[2ex] \dfrac{2a_k f(b_k) - b_k f(a_k)}{2f(b_k) - f(a_k)}, & \text{if } f(a_k)f(b_k) > 0 \end{cases}$$



Newton's Method



$$0 = f(r) = f(x+h) = f(x) + hf'(x) + O(h^2)$$

$$\Rightarrow \quad f(x) + hf'(x) = 0$$

$$h = -\frac{f(x)}{f'(x)}$$

$$r = x + h = x - \frac{f(x)}{f'(x)}$$

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Question $\lim_{n\to\infty} x_n = r$

Thm

If f, f', f'' conti in a nbhd of a root r of f and $f'(r) \neq 0$, then $\exists \delta > 0$ with the following property: If the initial point in Newton's method satisfies $|r - x_0| \leq \delta$, then all subsequent points x_n satisfy the same inequality, converge to r, and do so quadratically; that is,

$$|r - x_{n+1}| \leq C(\delta) |r - x_n|^2 \text{ where } C(\delta) = \frac{1}{2} \frac{\max_{|x-r| \leq \delta} |f''(x)|}{\min_{|x-r| \leq \delta} |f'(x)|}$$

pf) let $e_n = r - x_n$

$$e_{n+1} = r - x_{n+1} = r - x_n + \frac{f(x_n)}{f'(x_n)} = e_n + \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) + f(x_n)}{f'(x_n)}$$

$$0 = f(r) = f(x_n + e_n) = f(x_n) + e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n)$$

$$\Rightarrow e_{n+1} = -\frac{1}{2} \left(\frac{f''(\xi_n)}{f'(x_n)} \right) e_n^2$$

let $C(\delta) = \dfrac{1}{2} \dfrac{\max_{|x-r| \leq \delta} |f''(x)|}{\min_{|x-r| \leq \delta} |f'(x)|}$

$\therefore \ |e_{n+1}| \leq C(\delta) |e_n|^2 \quad \cdots (*)$

Choose δ so small that $\delta C(\delta) < 1$ and suppose that $|e_n| = |r - x_n| \leq \delta$

$(*) \Rightarrow |e_{n+1}| \leq \rho |e_n|$ where $0 < \rho < 1$

$\therefore \ |e_n| \leq \rho |e_{n-1}| \leq \cdots \leq \rho^n |e_0| \rightarrow 0$

$\therefore \ \lim_{n \to \infty} x_n = r$ //

Thm

If $f \in C^2(\mathbb{R})$, increasing ($f' > 0$), convex ($f'' > 0$) and has a zero, then zero is unique and the Newton iteration will converge to it from any starting point.

pf)
$$e_{n+1} = -\frac{1}{2}\left(\frac{f''}{f'}\right) e_n^2 < 0$$

$\therefore \quad r < x_{n+1}$ for all n.

Since f is increasing, $0 = f(r) < f(x_n)$

From $e_{n+1} = e_n + \dfrac{f(x_n)}{f'(x_n)}$, $\quad 0 > e_{n+1} > e_n$

Since e_n is increasing, bdd above by 0

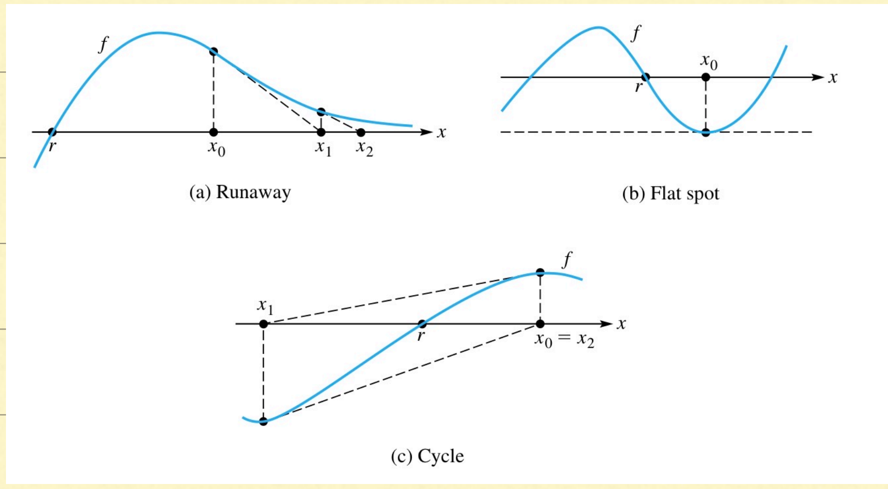
x_n is decreasing, bdd below by r,

$e^* = \lim_{n\to\infty} e_n$, $\quad x^* = \lim_{n\to\infty} x_n$ exist.

From $e_{n+1} = e_n + \dfrac{f(x_n)}{f'(x_n)}$, $\quad e^* = e^* + \dfrac{f(x^*)}{f'(x^*)}$

$\therefore \quad f(x^*) = 0$ and $r = x^*$ //

* Failure of Newton's Method



(a) Runaway (b) Flat spot

(c) Cycle

Suppose $f'(r)=0$ (i.e. r is a zero of f and f')

multiple zero of f

Newton's iteration for a multiple zero converges only linearly!

Could accelerated $\quad x^{n+1} = x_n - m\dfrac{f(x_n)}{f'(x_n)}$

$$(f^{(m)}(r) \neq 0)$$

Systems of Nonlinear Equation

$$\begin{cases} f_1(x_1, x_2, \cdots, x_N) = 0 \\ f_2(x_1, x_2, \cdots, x_N) = 0 \\ \quad \vdots \\ f_m(x_1, x_2, \cdots, x_N) = 0 \end{cases} \qquad \begin{aligned} &\vec{f}(\vec{x}) = 0 \\ & \\ &\vec{f} = [f_1 \cdots f_m]^T \\ &\vec{x} = [x_1 \cdots x_N]^T \end{aligned}$$

$$m = N$$

$$\vec{x}^{n+1} = \vec{x}^n - [\vec{f}'(\vec{x}^n)]^{-1}\vec{f}(\vec{x}^n)$$

where $\vec{f}'(\vec{x}^n)$ is the Jacobian matrix.

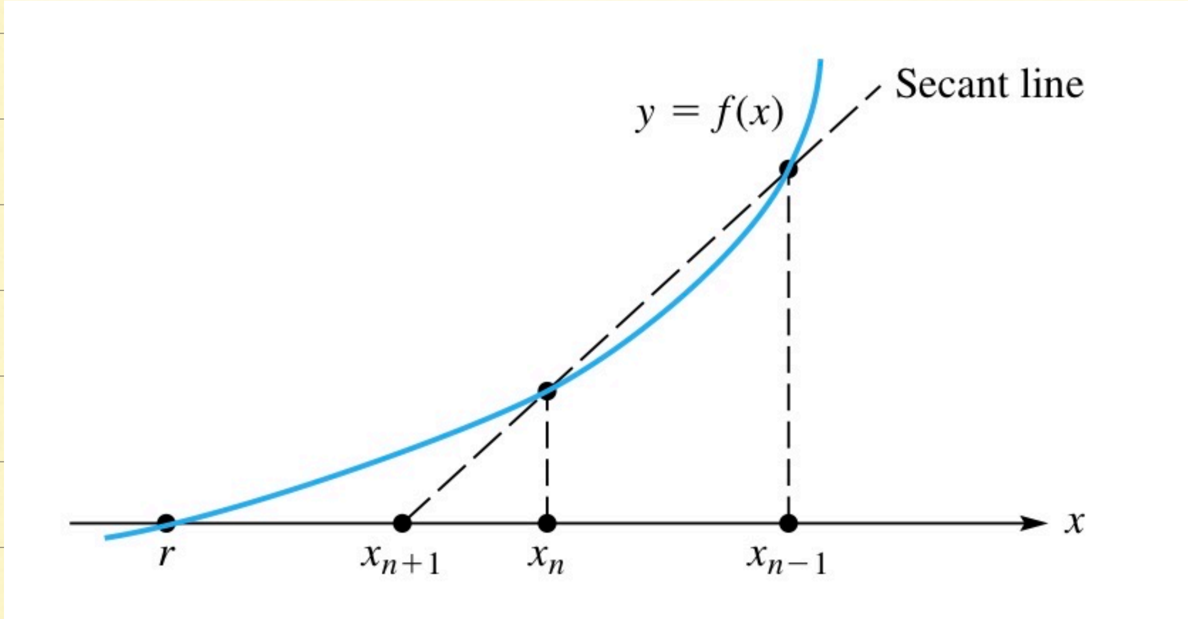
(example) 3×3

$$\begin{cases} f_1(x_1, x_2, x_3) = 0 \\ f_2(x_1, x_2, x_3) = 0 \\ f_3(x_1, x_2, x_3) = 0 \end{cases} \qquad J = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix}$$

$$\vec{x}^{n+1} = \vec{x}^n \underbrace{- J(\vec{x}^n)^{-1}\vec{f}(\vec{x}^n)}_{\vec{h}^n}$$

$$J(\vec{x}^n)\,\vec{h}^n = -\vec{f}(\vec{x}^n)$$

Secant Method



Newton's method $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n)$$

error analysis

$$e_{n+1} = r - x_{n+1} = r - x_n + \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n)$$

$$= \frac{f(x_n)\, e_{n-1} - f(x_{n-1})\, e_n}{f(x_n) - f(x_{n-1})}$$

(*)

$$= \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \; \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \, e_n e_{n-1}$$

$$f(x_n) = f(r - e_n) = f(r) - e_n f'(r) + \frac{1}{2} e_n^2 f''(r) + O(e_n^3)$$

$$\therefore \quad \frac{f(x_n)}{e_n} = -f'(r) + \frac{1}{2} e_n f''(r) + O(e_n^2)$$

$$\frac{f(x_{n-1})}{e_{n-1}} = -f'(r) + \frac{1}{2} e_{n-1} f''(r) + O(e_{n-1}^2)$$

$$(*) \quad \approx -\frac{1}{2} f''(r)$$

$$\therefore \quad e_{n+1} \approx -\frac{1}{2} \frac{f''(r)}{f'(r)} e_n e_{n-1} \sim C e_n e_{n-1}$$

To discover the order of convergence of the secant method, we can express the error relation in term of the inequality

$$|e_{n+1}| \sim A |e_n|^{\alpha} \sim c\, |e_n|\, |e_{n-1}|$$

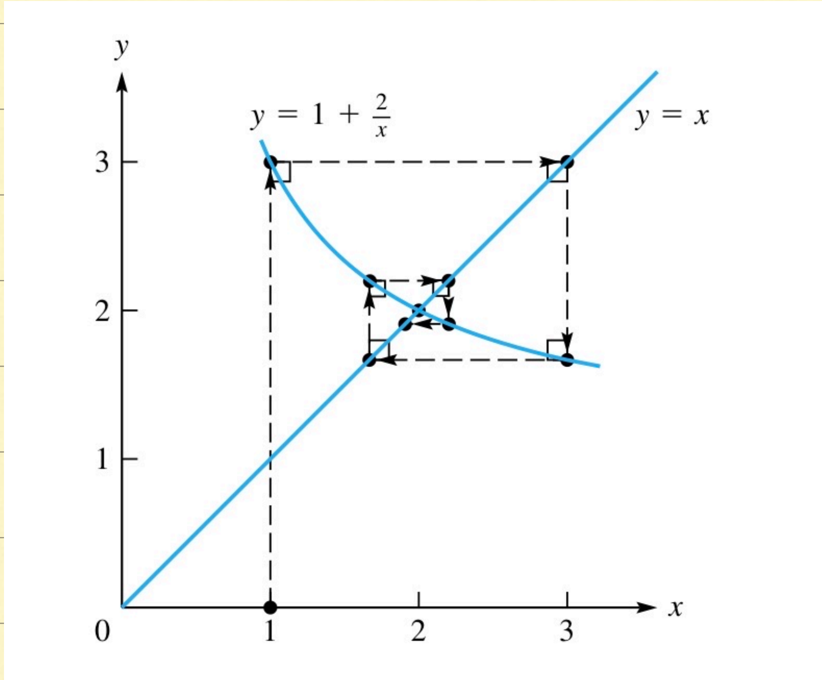
$$\Rightarrow \quad \alpha = \frac{1}{2}(1+\sqrt{5}) \qquad \text{superlinear}$$

Fixed Points and Functional Iteration

$$f(x) = 0$$

$$\Rightarrow \quad x = g(x)$$

$$x_{n+1} = g(x_n)$$



example) $f(x) = x^2 - x - 2 = 0 \Rightarrow x = 1 + \frac{2}{x} = g(x)$

$$x_{n+1} = 1 + \frac{2}{x_n}$$

$x_{n+1} = g(x_n)$ locally convergent for x^*

if $x^* = g(x^*)$ and $|g'(x^*)| < 1$

$x_{n+1} = F(x_n) \quad (n \geq 0) \quad \cdots (*)$

Newton's Method $\Rightarrow F(x) = x - \dfrac{f(x)}{f'(x)}$

Suppose $\lim_{n \to \infty} x_n = s$ and F is continuous

$$F(s) = F\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = s$$

Fixed point: $F(s) = s$

A mapping (or function) F is contractive

if $\exists \lambda < 1$ s.t. $|F(x) - F(y)| \leq \lambda |x - y|$

for all $x, y \in D$

Thm Contractive Mapping theorem

C: closed subset of the real line

$F: C \to C$ contractive

$\Rightarrow$ F has a unique fixed point.

Moreover, this fixed point is the limit of every sequence obtained from $(*)$ with a starting point $x_0 \in C$

example) $\begin{cases} x_0 = -15 \\ x_{n+1} = 3 - \frac{1}{2}|x_n| \end{cases}$ $(n \geq 0)$

$$F(x) = 3 - \frac{1}{2}|x|$$

$$|F(x) - F(y)| = \left| 3 - \frac{1}{2}|x| - 3 + \frac{1}{2}|y| \right| = \frac{1}{2}\big||y| - |x|\big| \leq \frac{1}{2}|y-x|$$

$\therefore$ the sequence must converge to the unique fixed point of F.

<u>proof of the theorem</u>

i) Convergence

$$|x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})| \leq \lambda |x_{n-1} - x_{n-2}|$$

$$\therefore \quad |x_n - x_{n-1}| \leq \lambda^{n-1} |x_1 - x_0|$$

Since x_n can be written in the form

$$x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}),$$

x_n converges $\Leftrightarrow \sum_{n=1}^{\infty} (x_n - x_{n-1})$ converges

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}| \leq \sum_{n=1}^{\infty} \lambda^{n-1} |x_1 - x_0| = \frac{1}{1-\lambda} |x_1 - x_0|$$

$\therefore$ $\{x_n\}$ converges,

let $s = \lim_{n\to\infty} x_n$, then $F(s) = s$.

ii) Uniqueness

if x, y are fixed points,

then $|x-y| = |F(x) - F(y)| \leq \lambda |x-y|$.

Since $\lambda < 1$, $|x-y| = 0 \Rightarrow x = y$. //