Chap 5 Numerical Integration

5.1 Trapezoidal Method

P : partition of $[a, b]$

$= \{ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \}$

$m_i = \inf \{ f(x) : x_i \leq x \leq x_{i+1} \}$ greatest lower bound

$M_i = \sup \{ f(x) : x_i \leq x \leq x_{i+1} \}$ least upper bound

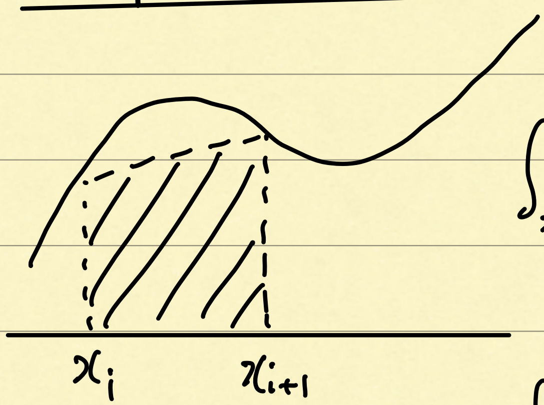
lower sums $L(f; P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$

upper sums $U(f; P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$

$$L(f; P) \leq \int_a^b f(x)\,dx \leq U(f; P)$$

Riemann integrable if $\inf_P U(f; P) = \sup_P L(f; P)$

Trapezoidal Rule



basic TR

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx A_i = \frac{1}{2}(x_{i+1} - x_i)[f(x_i) + f(x_{i+1})]$$

composite TR

$$\int_a^b f(x)\,dx \approx T(f, P) = \sum_{i=1}^{n-1} A_i$$

$$= \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)[f(x_i) + f(x_{i+1})]$$

uniform spacing: $T(f;p) = h\left\{ \sum_{i=0}^{n-1} f(x_i) + \frac{1}{2}[f(x_0) + f(x_n)] \right\}$

Thm 1 <u>Theorem on Precision of Composite Trapezoidal Rule</u>

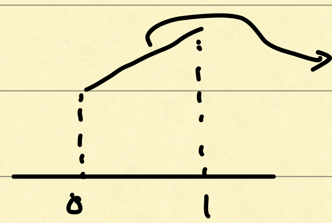
f'' exist and conti on $[a, b]$

$T(f;p)$: uniform spacing h, for some ζ in (a, b)

$I = \int_a^b f(x)\,dx \Rightarrow I - T = -\frac{1}{12}(b-a)h^2 f''(\zeta) = O(h^2)$

pf) i) $a=0$, $b=1$, $h=1$

Want to show $\int_0^1 f(x)\,dx - \frac{1}{2}[f(0) + f(1)] = -\frac{1}{12} f''(\zeta)$



$p(x) = f(0) + [f(1) - f(0)]\,x$

$\int_0^1 p(x)\,dx = \frac{1}{2}[f(0) + f(1)]$

By error formula, $f(x) - p(x) = \frac{1}{2} f''[\xi(x)]\,x(x-1)$

where $\xi(x)$ depends on x in $(0, 1)$

$$\int_0^1 f(x)\,dx - \int_0^1 p(x)\,dx = \frac{1}{2}\int_0^1 f''[\xi(x)]\,x(x-1)\,dx$$

Mean value Theorem

$$= \frac{1}{2} f''[\xi(s)] \int_0^1 x(x-1)\,dx$$

$$= -\frac{1}{12} f''(\zeta)$$

let $x = a + (b-a)t$, $g(t) = f(a + t(b-a))$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{2}[f(a) + f(b)] - \frac{1}{12}(b-a)^3 f''(\zeta)$$

$$\because \int_a^b f(x)dx = (b-a)\int_0^1 f[a+t(b-a)]dt$$

$$= (b-a)\int_0^1 g(t)\,dt$$

$$= (b-a)\left\{\frac{1}{2}[g(0)+g(1)] - \frac{1}{12}g''(\zeta)\right\}$$

$$= \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12}f''(\zeta)$$

for $[x_i, x_{i+1}]$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}[f(x_i)+f(x_{i+1})] - \frac{1}{12}h^3 f''(\xi_i)$$

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})] - \frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i)$$

$$-\frac{b-a}{12}h^2\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)\right] = -\frac{b-a}{12}h^2 f''(\zeta) = O(h^2)$$

//

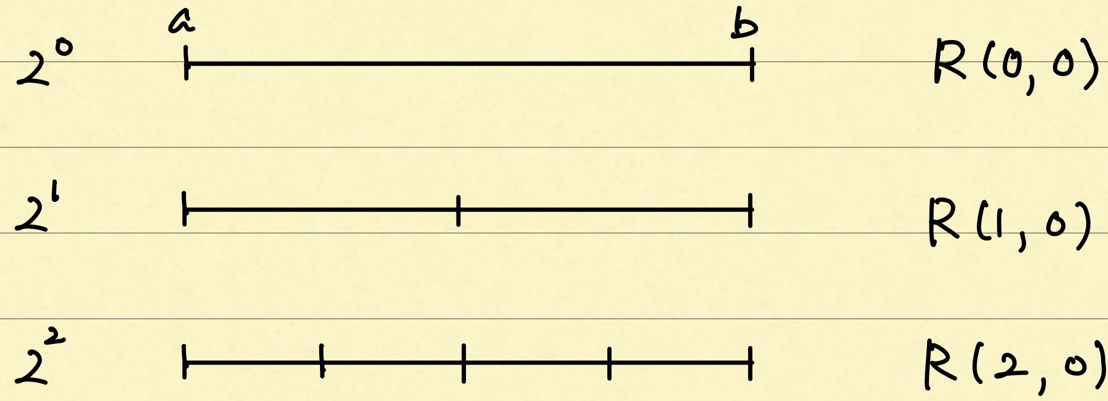
Recursive Trapezoidal Formula

$$T(f;P) = h\sum_{i=1}^{n-1} f(x_i) + \frac{h}{2}[f(x_0)+f(x_n)]$$

$$= h\sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2}[f(a)+f(b)]$$

$n \to 2^n$

$$R(n,0) = h\sum_{i=1}^{2^n-1} f(a+ih) + \frac{h}{2}[f(a)+f(b)]$$



$$R(n,0) = \frac{1}{2}R(n-1,0) + \left[R(n,0) - \frac{1}{2}R(n-1,0)\right]$$

let $C = \frac{h}{2}[f(a) + f(b)]$

$$R(n,0) = h\sum_{i=1}^{2^n-1} f(a+ih) + C$$

$$R(n-1,0) = 2h\sum_{j=1}^{2^{n-1}-1} f(a+2jh) + 2C$$

$$R(n,0) - \frac{1}{2}R(n-1,0) = h\sum_{k=1}^{2^{n-1}} f[a+(2k-1)h] \quad //$$

Thm Recursive Trapezoidal Formula

If $R(n-1,0)$ is available, then $R(n,0)$ can be computed by the formula

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f[a+(2k-1)h] \quad (n\geq 1)$$

using $h = (b-a)/2^n$. Here $R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)]$.

Multidimensional Integration

$$\int_0^1 f(x)dx \approx \frac{1}{2h}\left[f(0) + 2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right) + f(1)\right] \approx \sum_{i=0}^{n} A_i f\left(\frac{i}{n}\right)$$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy \approx \int_0^1 \sum_{i=0}^{n} A_i f\left(\frac{i}{n}, y\right) dy$$

$$= \sum_{i=0}^{n} A_i \int_0^1 f\left(\frac{i}{n}, y\right) dy$$

$$\approx \sum_{i=0}^{n} A_i \sum_{j=0}^{n} A_j f\left(\frac{i}{n}, \frac{j}{n}\right)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n} A_i A_j f\left(\frac{i}{n}, \frac{j}{n}\right)$$

with error $O(h^2)$

<u>effort</u>

one variable $O(n)$

two variable $O(n^2)$

- - -

k- variable $O(n^k)$

<u>error</u>

$N = n^k$: number of node

$$O(h^2) = O(n^{-2}) = O((n^k)^{-2/k}) = O(N^{-2/k})$$

the quality of the numerical approximation of the integral declines very quickly as the number of variable, k, increases.

5.2. Romberg Algorithm

$$
\begin{array}{lllll}
R(0,0) & & & & \\
R(1,0) & R(1,1) & & & \\
R(2,0) & R(2,1) & R(2,2) & & \\
\vdots & \vdots & \vdots & \ddots & \\
R(n,0) & R(n,1) & R(n,2) & \cdots & R(n,n)
\end{array}
$$

$$R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]$$

$$R(1,0) = \frac{1}{4}(b-a)\left[f(a) + f\left(\frac{a+b}{2}\right)\right] + \frac{1}{4}(b-a)\left[f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{1}{2}(b-a) f\left(\frac{a+b}{2}\right) + \frac{1}{4}(b-a)\left[f(a) + f(b)\right]$$

$$= \frac{1}{2} R(0,0) + \frac{1}{2}(b-a) f\left(\frac{a+b}{2}\right)$$

...

$$R(n,0) = \frac{1}{2} R(n-1,0) + h \sum_{k=1}^{2^{n-1}} f\left[a + (2k-1)h\right]$$

where $h = (b-a)/2^n$, $n \geq 1$

$$R(n,m) = R(n,m-1) + \frac{1}{4^n - 1}\left[R(n,m-1) + R(n-1,m-1)\right]$$

$n \geq 1$, $m \geq 1$ $\therefore$ (*)

↑ using a Richardson extrapolation.

Euler-Maclaurin Formula

$$\int_a^b f(x)\,dx = R(n-1,0) + a_2h^2 + a_4h^4 + \cdots \qquad \cdots (3)$$

$n \to n+1, \qquad h \to \frac{h}{2}$

$$\int_a^b f(x)\,dx = R(n,0) + \frac{1}{4}a_2h^2 + \frac{1}{16}a_4h^4 + \cdots \qquad (4)$$

$4\times(4) - (3)$

$$\int_a^b f(x)\,dx = R(n,1) - \frac{1}{4}a_4h^4 - \frac{5}{16}a_6h^6 - \cdots$$

where $R(n,1) = R(n,0) + \frac{1}{3}[R(n,0) - R(n-1,0)]$

$n \geq 1$

with similar procedure,

$$\int_a^b f(x)\,dx = R(n,2) + \frac{1}{4^3}a_6h^6 + \frac{21}{4^5}a_8h^8 + \cdots$$

where $R(n,2) = R(n,1) + \frac{1}{15}[R(n,1) - R(n-1,1)]$

$n \geq 2$

Euler-Maclaurin Formula and Error Term

If $f^{(2m)}$ exists and is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \frac{h}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})] + E$$

where $h = (b-a)/n$, $x_i = a + ih$ for $0 \leqq i \leqq n$, and

$$E = \sum_{k=1}^{m-1} A_{2k}h^{2k}[f^{(2k-1)}(a) - f^{(2k-1)}(b)] - A_{2m}(b-a)h^{2m}f^{(2m)}(\xi)$$

for some ξ in the interval (a, b).

$$\lim_{n\to\infty} R(n,m) = \int_a^b f(x)\,dx \quad (m \geq 0)$$

$$\frac{R(n,m) - R(n-1,m)}{R(n+1,m) - R(n,m)} \approx 4^{m+1} \quad \text{as } h \to 0$$

In case $m=0$,

$$R(n,0) - R(n-1,0) = \frac{3}{4} a_2 h^2 + \frac{15}{16} a_4 h^4 + \cdots$$

$$R(n+1,0) - R(n,0) = \frac{3}{4^2} a_2 h^2 + \frac{15}{16^2} a_4 h^4 + \cdots$$

$$\Rightarrow \frac{R(n,0) - R(n-1,0)}{R(n+1,0) - R(n,0)} = 4\left[1 + \frac{15}{4^2}\left(\frac{a_4}{a_2}\right) h^2 + \cdots\right] \approx 4 \quad \text{as } h \to 0$$

General Extrapolation

$$L = \varphi(h) + a h^\alpha + b h^\beta + c h^\gamma + \cdots$$

$$= \varphi\left(\frac{h}{2}\right) + a\left(\frac{h}{2}\right)^\alpha + b\left(\frac{h}{2}\right)^\beta + \cdots$$

$$(2^\alpha - 1)L = 2^\alpha \varphi(h/2) - \varphi(h) + (2^{\alpha-\beta} - 1) b h^\beta + \cdots$$

$$\Rightarrow L = \underline{\frac{2^\alpha}{2^\alpha - 1}\varphi\left(\frac{h}{2}\right) - \frac{1}{2^\alpha - 1}\varphi(h)} + \tilde{b} h^\beta + \tilde{c} h^\gamma + \cdots$$

$$\hookrightarrow \varphi\left(\frac{h}{2}\right) + \frac{1}{2^\alpha - 1}\left[\varphi\left(\frac{h}{2}\right) - \varphi(h)\right]$$

Example

$$\varphi(x) = L + a_1 x^{-1} + a_2 x^{-2} + a_3 x^{-3} + \cdots$$

How can be estimated using Richardson extrapolation?

sol)

$$\varphi(2x) = L + 2^{-1} a_1 x^{-1} + 2^{-2} a_2 x^{-2} + \cdots$$

$$\psi(x) = 2\varphi(2x) - \varphi(x) = L - 2^{-1} a_2 x^{-2} + 3 \cdot 2^{-2} a_3 x^{-3} + \cdots$$

Here is a concrete illustration of the preceding example. We want to estimate $\lim_{x\to\infty} \varphi(x)$ from the following table of numerical values:

x	1	2	4	8	16	32	64	128
$\varphi(x)$	21.1100	16.4425	14.3394	13.3455	12.8629	12.6253	12.5073	12.4486

A tentative hypothesis is that φ has the form in the preceding example. When we compute the values of the function $\psi(x) = 2\varphi(2x) - \varphi(x)$, we get a new table of values:

x	1	2	4	8	16	32	64
$\psi(x)$	11.7750	12.2363	12.3516	12.3803	12.3877	12.3893	12.3899

It seems reasonable to believe that the value of $\lim_{x\to\infty} \varphi(x)$ is approximately 12.3899. If we do another extrapolation, we should compute $\theta(x) = [4\psi(2x) - \psi(x)]/3$; values for this table are

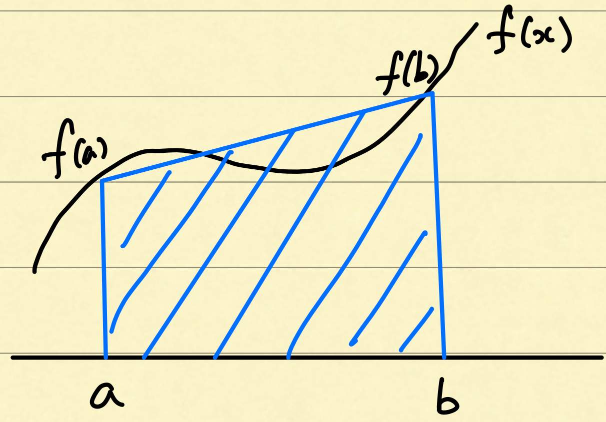
x	1	2	4	8	16	32
$\theta(x)$	12.3901	12.3900	12.3899	12.3902	12.3898	12.3901

For the precision of the given data, we conclude that $\lim_{x\to\infty} \varphi(x) = 12.3900$ to within roundoff error.

5.3 Simpson's Rule and Newton–Cotes Rule

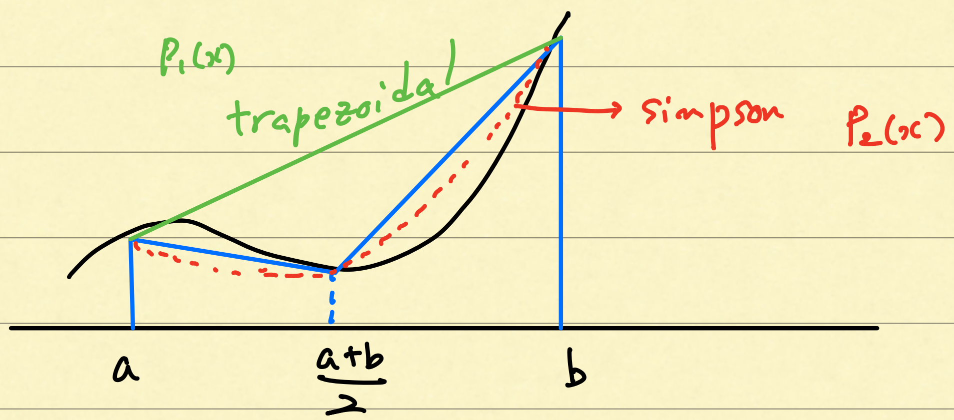
Basic trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$



Basic Simpson Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$



For uniform spacing, a, $a+h$, $a+2h$

$$\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}[f(a)+4f(a+h)+f(a+2h)]$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \cdots$$

$$f(a+2h) = f(a) + 2hf'(a) + \frac{(2h)^2}{2} f''(a) + \cdots$$

$$\frac{h}{3}\left[f(a) + 4f(a+h) + f(a+2h)\right] = 2hf + 2h^2 f' + \cdots \quad (*1)$$

$$F(a+2h) = F(a) + 2hF' + \frac{(2h)^2}{2} F'' + \cdots \quad (*2)$$

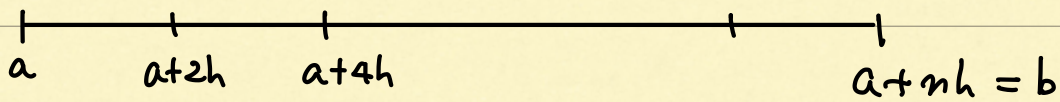
$(*2) - (*1)$

$$\int_a^{a+2h} f(x)\,dx = \frac{h}{3}\left[f(a) + 4f(a+h) + f(a+2h)\right]$$

$$- \frac{h^5}{90} f^{(4)} + \cdots$$

Composite Simpson's Rule

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n/2} \int_{a+2(i-1)h}^{a+2ih} f(x)\,dx$$

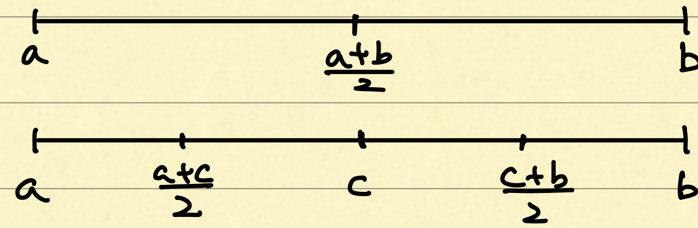


$$\approx \sum_{i=1}^{n/2} \frac{h}{3}\left[f(a+2(i-1)h) + 4f(a+(2i-1)h) + f(a+2ih)\right]$$

$$\approx \frac{h}{3}\left\{ f(a) + f(b) + 4\sum_{i=1}^{n/2} f[a+(2i-1)h] + 2\sum_{i=1}^{(n-2)/2} f(a+2ih) \right\}$$

with error term $-\dfrac{1}{180}(b-a)h^4 f^{(4)}(\xi)$, $h = \dfrac{b-a}{n}$

Adaptive Simpson's Scheme



One simpson's rule

two simpson's rule

$$I = \int_a^b f(x)\,dx = S(a,b) + E(a,b)$$

where $S(a,b) = \frac{1}{6}(b-a)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$$E(a,b) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

$$I = S^{(1)} + E^{(1)}, \quad S^{(1)} = S(a,b), \quad E^{(1)} = E(a,b)$$

$$I = S^{(2)} + E^{(2)}, \quad S^{(2)} = S(a,c) + S(c,b)$$

$$E^{(2)} = \frac{1}{16}\left[-\frac{1}{90}\left(\frac{h}{2}\right)^5 C\right]$$

$$= \frac{1}{16}E^{(1)}$$

$$S^{(2)} - S^{(1)} = E^{(1)} - E^{(2)} = 15E^{(2)}$$

$$I = S^{(2)} + E^{(2)} = S^{(2)} + \frac{1}{15}\left(S^{(2)} - S^{(1)}\right)$$

$$\left|\frac{1}{15}\left(S^{(2)} - S^{(1)}\right)\right| < \varepsilon$$

Newton - Cotes Rules

Closed when they involve function values at the ends of the interval of integration. Otherwise, they are said to be open

Some **closed Newton-Cotes rules** with error terms are as follows. Here, $a = x_0, b = x_n$, $h = (b-a)/n$, $x_i = x_0 + ih$, for $i = 0, 1, \ldots, n$, where $h = (b-a)/n$, $f_i = f(x_i)$, and $a = x_0 < \xi < x_n = b$ in the error terms.

Trapezoid Rule:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{1}{2}h[f_0 + f_1] - \frac{1}{12}h^3 f''(\xi)$$

Simpson's $\frac{1}{3}$ Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{1}{3}h[f_0 + 4f_1 + f_2] - \frac{1}{90}h^5 f^{(4)}(\xi)$$

Simpson's $\frac{3}{8}$ Rule:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3] - \frac{3}{80}h^5 f^{(4)}(\xi)$$

Boole's Rule:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2}{45}h[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4] - \frac{8}{945}h^7 f^{(6)}(\xi)$$

Six-Point Newton-Cotes Closed Rule:

$$\int_{x_0}^{x_5} f(x)\,dx = \frac{5}{288}h[19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f_5] - \frac{275}{12096}h^7 f^{(6)}(\xi)$$

Some of the **open Newton-Cotes rules** are as follows:

Midpoint Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = 2hf_1 + \frac{1}{24}h^3 f''(\xi)$$

Two-Point Newton-Cotes Open Rule:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{2}h[f_1 + f_2] + \frac{1}{4}h^3 f''(\xi)$$

Three-Point Newton-Cotes Open Rule:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{4}{3}h[2f_1 - f_2 + 2f_3] + \frac{28}{90}h^5 f^{(4)}(\xi)$$

Four-Point Newton-Cotes Open Rule:

$$\int_{x_0}^{x_5} f(x)\,dx = \frac{5}{24}h[11f_1 + f_2 + f_3 + 11f_4] + \frac{95}{144}h^5 f^{(4)}(\xi)$$

Five-Point Newton-Cotes Open Rule:

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{6}{20}h[11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5] - \frac{41}{140}h^7 f^{(6)}(\xi)$$

5.4. Gaussian Quadrature Formulas

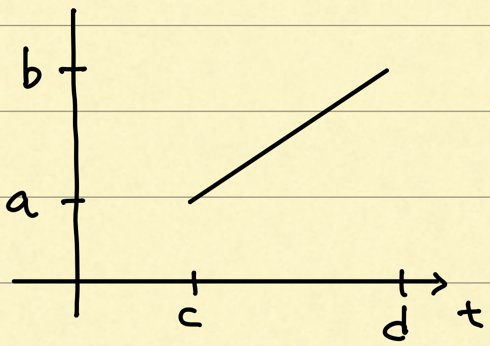
$$\int_a^b f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n)$$

lagrangian interpolation

$$p(x) = \sum_{i=0}^{n} f(x_i)\,l_i(x) \quad \text{where } l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \left(\frac{x-x_j}{x_i-x_j}\right)$$

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx = \sum_{i=0}^{n} f(x_i)\int_a^b l_i(x)\,dx = \sum_{i=0}^{n} A_i f(x_i)$$

Change of intervals



$$\int_c^d f(x)\,dx \approx \sum_{i=0}^{n} A_i f(t_i)$$

$$x = \lambda(t) = \left(\frac{b-a}{d-c}\right)t + \left(\frac{ad-bc}{d-c}\right)$$

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{d-c}\right)\int_c^d f(\lambda(t))\,dt$$

$$\approx \left(\frac{b-a}{d-c}\right)\sum_{i=0}^{n} A_i f(\lambda(t_i))$$

Gaussian Quadrature theorem

q : nontrivial poly. $\deg q = n+1$, $\int_a^b x^k q(x)\,dx = 0 \quad (0 \le k \le n)$

$q(x_i) = 0 \quad , \quad 0 \le i \le n$

$\Rightarrow \int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad$ where $A_i = \int_a^b l_i(x)\,dx \quad \cdots (*)$

will be exact for all polynomials of degree at most $2n+1$.

Furthermore, the nodes lie in the open interval (a,b)

pf) $\deg f \leq 2n+1$

$f = pq + r$, $\deg p, r \leq n$

By the hypothesis, $\int_a^b q(x)p(x)\,dx = 0$

also $f(x_i) = p(x_i)q(x_i) + r(x_i) = r(x_i)$

$$\int_a^b f(x)\,dx = \int_a^b p(x)q(x)\,dx + \int_a^b r(x)\,dx = \int_a^b r(x)\,dx$$

$$= \sum_{i=0}^{n} A_i r(x_i) = \sum_{i=0}^{n} A_i f(x_i) \quad //$$

With arbitrary nodes, (*) is exact for all poly. $\leq \deg n$.

With Gaussian nodes, " $\leq \deg 2n-1$

Example

Determine the Gaussian quadrature formula with three Gaussian nodes and three weights for the integral $\int_{-1}^{1} f(x)\,dx$

sol) let $q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

conditions: $\int_{-1}^{1} q(x)\,dx = \int_{-1}^{1} x\,q(x)\,dx = \int_{-1}^{1} x^2 q(x)\,dx = 0$

$$c_0 = c_2 = 0$$

$$\Rightarrow \int_{-1}^{1} x\,q(x)\,dx = \int_{-1}^{1} x(c_1 x + c_3 x^3)\,dx = 0$$

$$c_1 = -3, \quad c_3 = 5$$

$$q(x) = 5x^3 - 3x = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{3/5}$$

$$\int_{-1}^{1} f(x)\,dx \approx A_0 f\left(-\sqrt{\tfrac{3}{5}}\right) + A_1 f(0) + A_2 f\left(\sqrt{\tfrac{3}{5}}\right)$$

is exact for $1, x, x^2$

f	$\int_{-1}^{1} f(x)\,dx$	rhs
1	0	$A_0 + A_1 + A_2$
x	0	$-\sqrt{\frac{3}{5}}\,A_0 + \sqrt{\frac{3}{5}}\,A_2$
x^2	$2/3$	$3/5\,A_0 + 3/5\,A_2$

$$\Rightarrow \quad A_0 = A_2 = \frac{5}{9}, \qquad A_1 = \frac{8}{9}$$

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \quad //$$

Legendre Polynomials

$q_0(x) = 1$

$q_1(x) = x$

$q_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$\vdots$

$$q_n(x) = \left(\frac{2n-1}{n}\right) x\, q_{n-1}(x) - \left(\frac{n-1}{n}\right) q_{n-2}(x)$$

Weighted Gaussian Quadrature Theorem

Let q be a nonzero polynomial of degree $n+1$ s.t.

$$\int_a^b x^k q(x)\, w(x)\, dx = 0 \qquad (0 \le k \le n)$$

Let $x_0, x_1, \cdots, x_n$ be the roots of q. Then the formula

$$\int_a^b f(x)\, w(x)\, dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

where $l_i(x) = \prod_{j=0, j\neq i}^{n} (x - x_j)/(x_i - x_j)$ and $A_i = \int_a^b l_i(x)\, w(x)\, dx$ will be exact whenever f is a polynomial of degree at most $2n+1$

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

n	Nodes x_i	Weights A_i
1	$-\sqrt{\frac{1}{3}}$	1
	$+\sqrt{\frac{1}{3}}$	1
2	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$+\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
3	$-\sqrt{\frac{1}{7}(3-4\sqrt{0.3})}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$-\sqrt{\frac{1}{7}(3+4\sqrt{0.3})}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}(3-4\sqrt{0.3})}$	$\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$
	$+\sqrt{\frac{1}{7}(3+4\sqrt{0.3})}$	$\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$
4	$-\sqrt{\frac{1}{9}\left(5-2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{-0.7+5\sqrt{0.7}}{-2+5\sqrt{0.7}}\right)$
	$-\sqrt{\frac{1}{9}\left(5+2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{0.7+5\sqrt{0.7})}{2+5\sqrt{0.7}}\right)$
	0	$\frac{128}{225}$
	$+\sqrt{\frac{1}{9}\left(5-2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{-0.7+5\sqrt{0.7}}{-2+5\sqrt{0.7}}\right)$
	$+\sqrt{\frac{1}{9}\left(5+2\sqrt{\frac{10}{7}}\right)}$	$0.3\left(\frac{0.7+5\sqrt{0.7}}{2+5\sqrt{0.7}}\right)$

Integral with Singularities

- change variables to remove the sigularity
- then use a standard approximation techniques.

$$\int_0^1 \frac{dx}{e^x\sqrt{x}} = 2\int_0^1 \frac{dt}{e^{t^2}}$$

$$\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx = 2\int_0^{\sqrt{\pi/2}} \cos t^2\,dt \qquad \text{using } x=t^2$$

Others $x=-\log t$, $x=t/(1-t)$, $x=\tan t$, $x=\sqrt{(1+t)(1-t)}$

$\int_0^1 \frac{\sin x}{x}\,dx$, $\int_0^1 \frac{dx}{\sqrt{x}}$: Gaussian nodes are inside the domain