# Chap 8 More on Linear Systems

## 8.1 Matrix Factorization

$n \times n$ system $Ax = b$

objective : naive Gauss algorithm applied to $A$

$\Rightarrow$ a factorization of $A$, $A = LU$

LU-factorization
$$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \qquad Ax = b$$

$$\Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix} \qquad M_1 A x = M_1 b$$

where
$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

$M_2 M_1 A x$
$= M_2 M_1 b$

where $M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

$M_3 M_2 M_1 A x$
$= M_3 M_2 M_1 b$

where $M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

$M = M_3 M_2 M_1$

$MA = U$

$A = M^{-1} U = M_1^{-1} M_2^{-1} M_3^{-1} U = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix}$$

## Formal Derivation

$M_{qp}$ : elementary matrix to subtract $\lambda$ times row $p$ from row $q$. assume $p<q$

$M_{qp} = (m_{ij})$ $\quad m_{ij} = \begin{cases} 1 & \text{if } i=j \\ -\lambda & \text{if } i=q \text{ and } j=p \\ 0 & \text{otherwise} \end{cases}$

$$(M_{qp}A)_{ij} = \sum_{s=1}^{n} m_{is} a_{sj} = \begin{cases} a_{ij} & \text{if } i \neq q \\ a_{qj} - \lambda a_{pj} & \text{if } i = q \end{cases}$$

$K^{th}$ step of Gaussian elimination corresponds to the matrix $M_K$,

$M_K = M_{nK} M_{n-1,K} \cdots M_{K+1,K}$ Lower triangular matrix

$$M_{n-1} \cdots M_2 M_1 A = U \Rightarrow A = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1} U = LU$$

This construction depends upon not encountering any 0 divisors in the algorithm.

Example of no LU factorization $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

### Thm 1 LU factorization thm

$A = (a_{ij})$ $n \times n$ matrix

Assume that the forward elimination phase of the naive Gaussian algorithm is applied to $A$ without encounting any 0 divisors. Let the resulting matrix be denoted by $\tilde{A} = (\tilde{a}_{ij})$. If

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \tilde{a}_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{n,n-1} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ & & \ddots & \vdots \\ & & & \tilde{a}_{nn} \end{bmatrix}$$

then $A = LU$

Solving Linear systems using LU factorization

$$LUx = b$$

$$Lz = b \quad , \quad Ux = z$$

## $LDL^T$ factorization

$A$ : symmetric

$$LU = A = A^T = (LU)^T = U^T L^T$$

$$\therefore \quad \underbrace{U(L^T)^{-1}}_{\text{upper}} = \underbrace{L^{-1}U^T}_{\text{lower}} \quad \Rightarrow \quad L^{-1}U^T = D$$

$$\therefore \quad A = LU = L(L^{-1}U^T L^T) = LDL^T$$

## Cholesky Factorization

If $A$ is a real, symmetric, and positive definite matrix, then it has a unique factorization, $A = LL^T$, in which $L$ is lower triangular with a positive diagonal symmetric $A = A^T$

positive definite $x^T A x > 0$ for all $x \neq 0$

$$A = LDL^T = LD^{\frac{1}{2}} D^{\frac{1}{2}} L^T = \tilde{L} \tilde{L}^T$$

## Computing $A^{-1}$

$$AX = I$$

$$A [x^{(1)}\ x^{(2)} \cdots x^{(n)}] = [I^{(1)}\ I^{(2)} \cdots I^{(n)}]$$

$$Ax^{(j)} = I^{(j)} \qquad (1 \leq j \leq n)$$

## permutation matrix

$$PA = LU$$

$$PAx = Pb \Rightarrow Ly = Pb \Rightarrow Ux = y \quad //$$

## 8.2. Eigenvalues and Eigenvectors

$$p(\lambda) = \det(A - \lambda I)$$

Thm The following statements are true for any square matrix $A$

1. $\lambda$ : eigenvalue of $A$, then $p(\lambda)$ is an eigenvalue of $P(A)$ for any polynomial $p$. In particular, $\lambda^k$ is an eigenvalue of $A^k$
2. $A$ : nonsingular, $\lambda$: eigenvalue of $A$ $\Rightarrow$ $p(1/\lambda)$ is an eigenvalue of $P(A^{-1})$
3. $A$ : real and symmetric $\Rightarrow$ its eigenvalues are real
4. $A$ : complex and Hermitian $\Rightarrow$ its eigenvalues are real

5. $A$ : Hermitian and positive definite
$\Rightarrow$ its eigenvalues are positive

6. $P$ : nonsingular $\Rightarrow$ $A$ and $PAP^{-1}$ have the same characteristic polynomial

**Thm** Similar matrices have the same eigenvalues

$A$ is Hermitian if $A = A^*$ where $A^* = \bar{A}^T = (\bar{a}_{ji})$

Def) $A$ and $B$ are unitarily similar if $B = U^*AU$ for some unitary matrix ($UU^* = I$)

**Thm** Schur's thm

$$\boxed{\text{Every square matrix is unitarily similar to a triangular matrix}}$$

**Corollary**

Hermitian matrix is unitarily similar to a diagonal matrix.

example)

$$A = \begin{bmatrix} 3 & -2 \\ 8 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 25 = 0$$

$$\lambda = 3 \pm 4i, \quad v_1 = [i \ \ 2]^T, \quad v_2 = [-i \ \ 2]^T$$

using the G-S orthogonalization process,

$$u_1 = v_1, \qquad u_2 = v_2 - [v_2^* v_1 / u_1^* u_1] u_1$$

$$= [-2 \quad -i]^T$$

normalization $\Rightarrow$ $U = \frac{1}{\sqrt{5}} \begin{bmatrix} i & -2 \\ 2 & -i \end{bmatrix}$ , $UU^* = I$

$$\therefore \quad UAU^* = \begin{bmatrix} 3+4i & -6 \\ 0 & 3-4i \end{bmatrix}$$

**Thm** <u>Gershgorin's thm</u>

$A$ : $n \times n$ matrix

$R_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \le \sum_{j=1, j \ne i}^{n} |a_{ij}| \}$

$\Rightarrow$ The eigenvalues of $A$ are contained within $R = \bigcup_{i=1}^{n} R_i$

pf) $Ax = \lambda x$ , $\lambda$ : eigenvalue , $\|x\|_\infty = 1$

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \quad \text{for each } i = 1, 2, \cdots, n$$

If $k$ is an integer with $|x_k| = \|x\|_\infty = 1$ ,

$$\Rightarrow \sum_{\substack{j=1 \\ j \ne k}}^{n} a_{kj} x_j = \lambda x_k - a_{kk} x_k = (\lambda - a_{kk}) x_k$$

$$|\lambda - a_{kk}| |x_k| = \left| \sum_{\substack{j=1 \\ j \ne k}}^{n} a_{kj} x_j \right| \le \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}| |x_j| \le \sum_{\substack{j=1 \\ j \ne k}}^{n} |a_{kj}|$$

//

example)

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 9 \end{bmatrix}$$

$R_1 = \{ z \in \mathbb{C} \mid |z - 4| \le 2 \}$

$R_2 = \{ z \in C \mid |z - 2| \le 1 \}$

$R_3 = \{ z \in C \mid |z - 9| \le 2 \}$

# Singular Value Decomposition

singular values: nonnegative square roots of the eigenvalues of $A^TA$

## Thm5 Matrix Spectral Theorem

$A: m\times n$, $A^TA: n\times n$ symmetric

$A^TA = QDQ^{-1}$ : $Q$: orthogonal

$\quad\hookrightarrow$ diagonal $\quad (Q^TQ = QQ^T = I)$

## Thm6 Orthogonal Basis thm

If the rank of $A$ is $r$, then an orthogonal basis for the column space of $A$ is $\{Av_j : 1\le j\le r\}$

$A^TAv_j = \lambda_j v_j$

pf) i) orthogonality

$$(Av_i)^T(Av_j) = v_i^T A^TA v_j = v_i^T \lambda_j v_j = \lambda_j \delta_{ij}$$

ii) If $w$ is any vector in the column space of $A$, then $w = Ax$ for some $x \in \mathbb{R}^n$, $x = \sum_{j=1}^{n} c_j v_j$.

$$w = Ax = A\left(\sum_{j=1}^{n} c_j v_j\right) = \sum_{j=1}^{n} c_j Av_j$$

$\therefore$ $w$ is in the span of $\{Av_1, \cdots, Av_r\}$. //

Thm (SVD)

$A \in \mathbb{R}^{m \times n}$ : $r$ positive singular values, $m \geq n$

$\Rightarrow U$ : $m \times m$ orthogonal, $V$ : $n \times n$ orthogonal

$\tilde{\Sigma}$ : $m \times n$ diagonal

s.t. $A = U \tilde{\Sigma} V^T$, $\tilde{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Rmk) 1) if $m < n$, then find the SVD of $A^T$

$$A^T = U \tilde{\Sigma} V^T \Rightarrow A = V \tilde{\Sigma} U^T$$

2) $A = U \tilde{\Sigma} V^T$, $A v_j = \sigma_j u_j$

the columns of $U$ and $V$ are called the left and right singular vectors, respectively.

pf of thm

$A^T A$ symmetric, $\Rightarrow \sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 \geq \sigma_{r+1}^2 \geq \cdots \geq \sigma_n^2 \geq 0$

Assume $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$

Let $\tilde{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$ where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$

The eigenvectors of $A^T A$ are an orthonormal basis for $\mathbb{R}^n$.

Let $V = [v_1 \; v_2 \; \cdots \; v_{n-1} \; v_n]$

We need to find $U$

let $u_i = \dfrac{Av_i}{\sigma_i} \quad 1 \leq i \leq r$

$$\langle u_i, u_j \rangle = \frac{(Av_i)^T(Av_j)}{\sigma_i\sigma_j} = \frac{v_i^T A^T A v_j}{\sigma_i \sigma_j} = \frac{\sigma_j v_i^T v_j}{\sigma_i} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

If $r < m$, we still need $m-r$ additional vectors $\{u_{r+1}, u_{r+2}, \cdots, u_{m-1}, u_m\} \Leftarrow$ using a G-S process

We need to show that $A = U\tilde{\Sigma}V^T$ or $U^TAV = \tilde{\Sigma}$

$$\begin{aligned} U^TAV &= U^TA[v_1 \ v_2 \ \cdots \ v_m] \\ &= U^T[\sigma_1 u_1 \ \sigma_2 u_2 \ \cdots \ \sigma_m u_m] \\ &= [\sigma_1 U^T u_1 \ \sigma_2 U^T u_2 \ \cdots \ \sigma_m U^T u_m] \\ &= [\sigma_1 e_1 \ \sigma_2 e_2 \ \cdots \ \sigma_m e_m] \\ &= \tilde{\Sigma} \end{aligned}$$

//

condition number $\kappa(A) = \sqrt{\dfrac{\sigma_{max}}{\sigma_{min}}}$

$\|A\|_2^2 = \rho(A^TA) = \sigma_{max}(A)$, $\quad \|A^{-1}\|_2^2 = \rho(A^{-T}A^{-1}) = \sigma_{min}(A)$

Example 6) Calculate the SVD

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

sol)

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = 3 \qquad [1 \quad 1]^T \to [\tfrac{1}{2}\sqrt{2} \quad \tfrac{1}{2}\sqrt{2}]^T$$

$$\lambda_2 = 1 \qquad [1 \quad -1]^T \to [\tfrac{1}{2}\sqrt{2} \quad -\tfrac{1}{2}\sqrt{2}]^T$$

$$D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \sigma_1^{-1} A v_1 = [\tfrac{1}{3}\sqrt{6} \quad \tfrac{1}{6}\sqrt{6} \quad \tfrac{1}{6}\sqrt{6}]^T$$

$$u_2 = \sigma_2^{-1} A v_2 = [0 \quad -\tfrac{1}{2}\sqrt{2} \quad \tfrac{1}{2}\sqrt{2}]^T$$

$$\text{G-S} \Rightarrow \quad u_3 = \sigma_1 - (u_1^T e_1) u_1 - (u_2^T e_2) u_2 = [\tfrac{1}{3} \quad -\tfrac{1}{3} \quad -\tfrac{1}{3}]^T$$

$$\therefore \quad \tilde{u}_3 = [\tfrac{1}{3}\sqrt{3} \quad -\tfrac{1}{3}\sqrt{3} \quad -\tfrac{1}{3}\sqrt{3}]^T$$

$$\therefore \quad U = [u_1 \quad u_2 \quad \tilde{u}_3]^T$$

$$A = UDV^T$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sqrt{6} & 0 & \frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \quad //$$

## Power Method

$A$: $n \times n$ matrix

has the following two properties

1. $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$
2. There is a linearly independent set of $n$ eigenvectors

$$\{u^{(1)}, u^{(2)}, \ldots, u^{(n)}\} : \text{basis for } \mathbb{C}^n$$

$$Au^{(j)} = \lambda_j u^{(j)}, \quad j=1,\cdots,n, \quad u^{(j)} \neq 0$$

$$x^{(0)} \in \mathbb{C}^n \Rightarrow \quad x^{(0)} = c_1 u^{(1)} + c_2 u^{(2)} + \cdots + c_n u^{(n)}$$

assume $c_1 \neq 0$

WLOG $\quad x^{(0)} = u^{(1)} + u^{(2)} + \cdots + u^{(n)}$

$$\Rightarrow \quad \begin{aligned} x^{(1)} &= Ax^{(0)} \\ x^{(2)} &= Ax^{(1)} = A^2 x^{(0)} \\ &\cdots \\ x^{(k)} &= Ax^{(k-1)} = A^k x^{(0)} \\ &\cdots \end{aligned}$$

$$\begin{aligned} x^{(k)} = A^k x^{(0)} &= A^k (u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \\ &= \lambda_1^k u^{(1)} + \lambda_2^k u^{(2)} + \cdots + \lambda_n^k u^{(n)} \\ &= \lambda_1^k \left[ u^{(1)} + \left(\frac{\lambda_2}{\lambda_1}\right)^k u^{(2)} + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k u^{(n)} \right] \end{aligned}$$

$$\left|\frac{\lambda_j}{\lambda_1}\right| < 1 \text{ for } j > 1 \Rightarrow \left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0 \quad \text{as } k \to \infty$$

$$\therefore \quad x^{(k)} = \lambda_1^k \left[ u^{(1)} + \varepsilon^{(k)} \right] \quad \text{where } \varepsilon^{(k)} \to 0 \text{ as } k \to \infty$$

$\cdots (*)$

let $\varphi$ be any complex-valued linear function on $\mathbb{C}^n$

s.t. $\varphi(u^{(1)}) \neq 0$, $\quad \varphi(a\vec{x} + b\vec{y}) = a\varphi(\vec{x}) + b\varphi(\vec{y})$

Apply $\varphi$ to $(*)$

$$\varphi(x^{(k)}) = \lambda_1^k \left[ \varphi(u^{(1)}) + \varphi(\varepsilon^{(k)}) \right]$$

$$r_k \equiv \frac{\varphi(x^{(k+1)})}{\varphi(x^{(k)})} = \lambda_1 \left[ \frac{\varphi(u^{(1)}) + \varphi(\varepsilon^{(k+1)})}{\varphi(u^{(1)}) + \varphi(\varepsilon^{(k)})} \right] \to \lambda_1 \quad \text{as } k \to \infty$$

## Inverse Power Method

$$Ax = \lambda x \quad \Rightarrow \quad x = A^{-1}(\lambda x)$$

$$\Rightarrow \quad A^{-1}x = \frac{1}{\lambda} x$$

So the smallest eigenvalue of $A$ in magnitude is the reciprocal of the largest eigenvalue of $A^{-1}$.

Suppose $A$ has eigenvalues such as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$$

The eigenvalues of $A^{-1}$ are $\lambda_j^{-1}$ for $j=1, \cdots, n$

$$|\lambda_n^{-1}| > |\lambda_{n-1}^{-1}| \geq \cdots \geq |\lambda_1^{-1}| > 0$$

We can use the Power method on $A^{-1}$

$$x^{(k+1)} = A^{-1}x^{(k)} \quad \Leftrightarrow \quad Ax^{(k+1)} = x^{(k)}$$

$$LUx^{(k+1)} = x^{(k)}$$

$$Ux^{(k+1)} = L^{-1}x^{(k)}$$

## Shifted (Inverse) Power method

$$Ax = \lambda x \quad \Leftrightarrow \quad (A - \mu I)x = (\lambda - \mu)x$$

$$\Leftrightarrow \quad (A - \mu I)^{-1}x = \frac{1}{\lambda - \mu} x$$

* Suppose for some eigenvalue $\lambda_j$ of matrix $A$, we have $|\lambda_j - \mu| > \varepsilon$ and $0 < |\lambda_i - \mu| < \varepsilon$ for all $i \neq j$

Applying the power method to the shifted matrix $A - \mu I$, we compute ratios $r_k$ that converge to $\lambda_j - \mu$

$\Leftarrow$ shifted power method.

* Want to compute an eigenvalues of $A$ that is closest to a given number $\mu$.

Suppose $0 < |\lambda_j - \mu| < \varepsilon$ and $|\lambda_i - \mu| > \varepsilon$, $i \neq j$

$$(A - \mu I) x^{(k+1)} = x^{(k)}$$

$\Rightarrow$ $r_k$ converge to $(\lambda_j - \mu)^{-1}$

$$\lambda_j = \mu + \lim_{k \to \infty} \frac{1}{r_k}$$

$\Leftarrow$ shifted Inverse Power Method

## 8.4. Iterative Solutions of Linear Systems

· Basic Concepts

$$Ax = b$$

$$A = Q - (Q - A)$$

$$Qx = (Q - A)x + b$$

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b \qquad (k \geq 1)$$

1. The sequence $[x^{(k)}]$ is easily computed
2. The sequence $[x^{(k)}]$ converges rapidly to a sol.

$$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b \quad \cdots (*)$$

$$x = (I - Q^{-1}A)x + Q^{-1}b$$

$$\|x^{(k)} - x\| \leq \|I - Q^{-1}A\| \|x^{(k-1)} - x\|$$

if $\|I - Q^{-1}A\| < 1$, then $\lim_{k \to \infty} \|x^{(k)} - x\| = 0$

$\boxed{\text{Thm 1} \quad \text{If } \|I - Q^{-1}A\| < 1 \text{ for some subordinate matrix norm, the sequenco produced by } (*) \text{ converges to the solution of } Ax = b \text{ for any initial vector } x^{(0)}.}$

- Richardson Method

$Q = I$

$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$

- Jacobi Method

$Q = \text{diag}(A)$ , $Q^{-1}A = [a_{ij}/a_{ii}]$

$$\|I - Q^{-1}A\|_\infty = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}/a_{ii}|$$

Thm 2 Convergence of Jacobi Method

$A$ : diagonally dominant

$\Rightarrow$ the sequence produced by the Jacobi iteration converges to the solution of $Ax=b$ for any starting vector

pf) diagonally dominant $\quad |a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

$$\Rightarrow \|I - Q^{-1}A\|_\infty < 1$$

By thm 1, the Jacobi iteration converges.

$$A = D + L + U$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

Jacobi : $\quad Q = D$

$$Dx^{(k)} = -(L+U)x^{(k-1)} + b$$

Gauss-Seidel: $\quad Q = D + L$

$$(D+L)x^{(k)} = -Ux^{(k-1)} + b$$

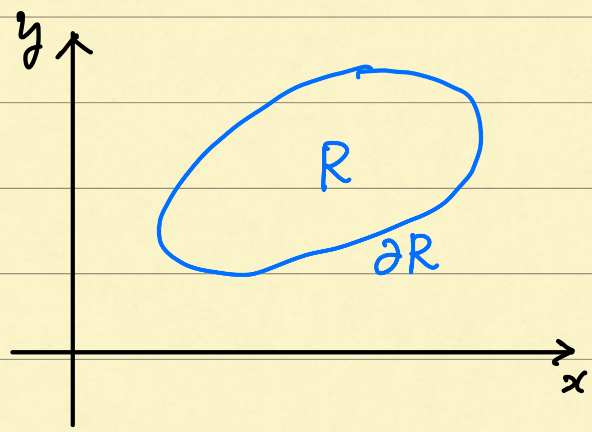
SOR (successive over relaxation) : $\quad Q = \frac{1}{\omega}D + L$

$$(D+\omega L)x^{(k)} = ((1-\omega)D - \omega U)x^{(k-1)} + \omega b, \quad 0 < \omega < 2$$

Thm 3

A: spd $\Rightarrow$ G-S and SOR ( $0<\omega<2$ ) converges for any starting vector.

Application : Poisson's equation




$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{on } R \quad \cdots (*)$$

$$u(x,y) = g(x,y) \quad \text{on } \partial R$$

approximation

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{1}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

$$(*) \Rightarrow \quad -u_{i-1,j} - u_{i+1,j} + 4u_{i,j} - u_{i,j-1} - u_{i,j+1} = h^2 f(x_i, y_j)$$

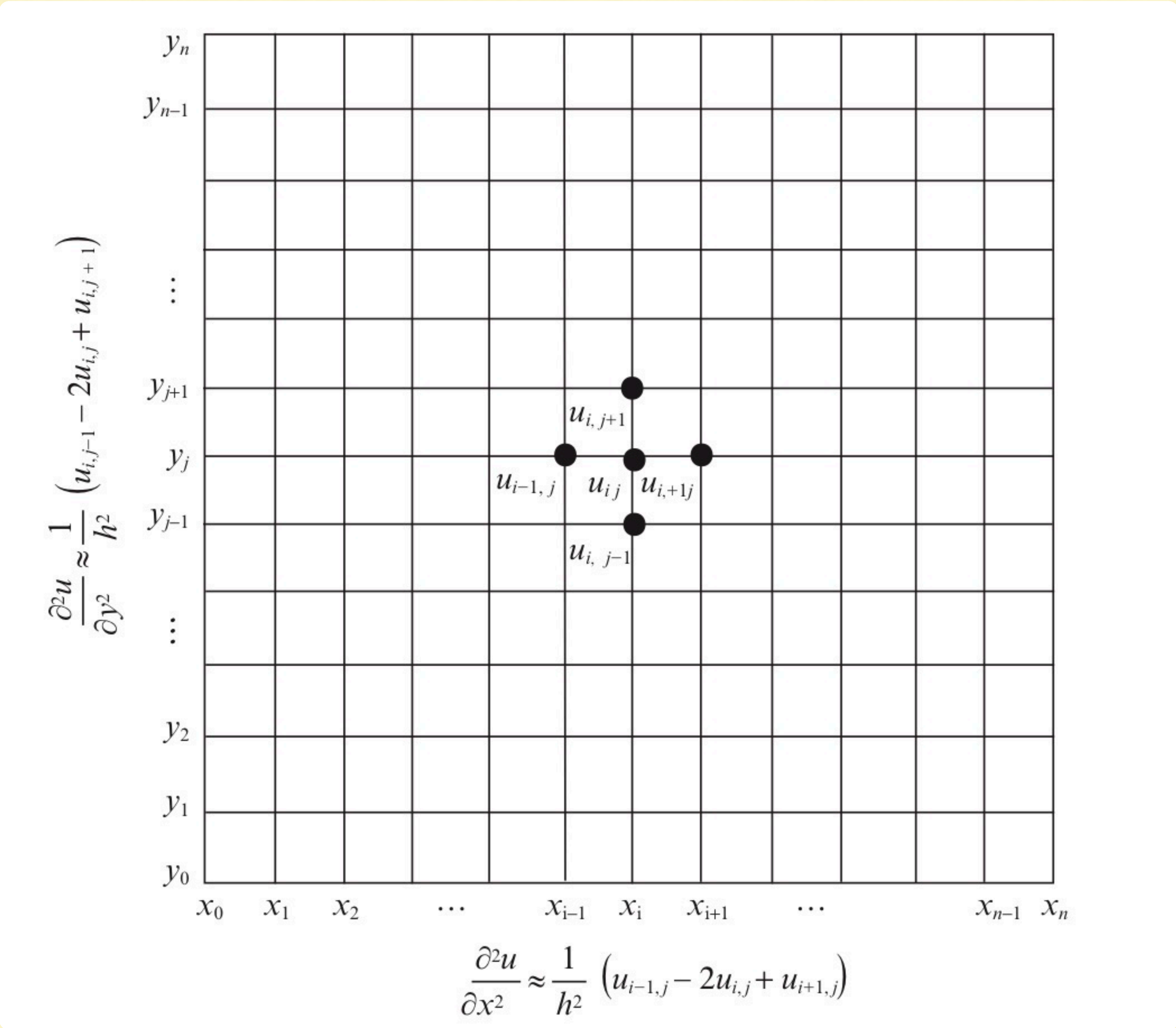
$$1 \le i, j \le n-1$$

$$Au = b + \text{boundary} \quad , \quad b_{i,j} = h^2 f(x_i, y_j)$$

where

$$u = [\, u_{1,1} \; u_{2,1} \cdots u_{n-1,1} \,\vdots\, u_{1,2} \; u_{2,2} \cdots u_{n-1,2} \,\vdots\, \cdots \,\vdots\, u_{1,n-1}, u_{2,n-1}, \cdots u_{n-1,n-1} \,]^T$$

$$b = [\, b_{1,1} \; b_{2,1} \cdots b_{n-1,1} \,\vdots\, b_{1,2} \; b_{2,2} \cdots b_{n-1,2} \,\vdots\, \cdots \,\vdots\, b_{1,n-1}, b_{2,n-1}, \cdots b_{n-1,n-1} \,]^T$$




$$\text{boundary} = \begin{bmatrix} g_{0,1}+g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n-2,0} & g_{n-1,0}+g_{n,1} \\ g_{0,2} & 0 & 0 & \cdots & 0 & g_{n,2} \\ g_{0,3} & 0 & 0 & \cdots & 0 & g_{n,3} \\ & & & \vdots & & \\ g_{0,n-2} & 0 & 0 & \cdots & 0 & g_{n,n-2} \\ g_{0,n-1}+g_{1,n} & g_{2,n} & g_{3,n} & \cdots & g_{n-2,n} & g_{n,n-1}+g_{n-1,n} \end{bmatrix}^T$$

where $g_{i,j} = g(x_i, y_j)$

$u$, $b$, boundary : $(n-1)^2$ vectors

let $\tilde{b} = b + \text{boundary}$

$$A = \left[\begin{array}{c|c|c|c} D_1 & R_1 & & \\ \hline L_2 & D_2 & R_2 & \\ \hline & \ddots & \ddots & \ddots \\ \hline & & L_{n-1} & D_{n-1} \end{array}\right]$$

$D_i,\ R_i,\ L_i$ $\quad (n-1) \times (n-1)$ matrix

$A$ $\quad [(n-1)\times(n-1)] \times [(n-1)\times(n-1)]$ matrix

$$D_i = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{bmatrix} \qquad 1 \le i \le n-1$$

$$L_i = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} \qquad R_i = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$$

$2 \le i \le n-1$ $\qquad 1 \le i \le n-2$

Solve $Ax = \tilde{b}$

## Vector and Matrix Norms

- vector norm : $\|x\| > 0$ if $x \neq 0$

$$\|\alpha x\| = |\alpha| \, \|x\|$$

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad , \quad \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

$l_1$ norm $\qquad$ $l_2$ norm

$$\|x\|_\infty = \max_{1 \leq j \leq n} |x_i|$$ $l_\infty$ norm

- Matrix norm : $\|A\| > 0$ if $A \neq 0$

$$\|\alpha A\| = |\alpha| \, \|A\|$$

$$\|A+B\| \leq \|A\| + \|B\|$$

subordinate matrix norm (matrix norm induced by the vector norm)

$$\|A\| = \sup\{ \|Ax\| : x \in \mathbb{R}^n \text{ and } \|x\| = 1 \}$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)} = \sigma_{max}(A)$$

## Condition number

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

solve $Ax = b$

$$A(x + \delta x) = Ax + A\delta x = b + \delta b$$

- $$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

$$\|b\| = \|Ax\| \leq \|A\| \|x\| \Rightarrow \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

$$A\delta x = \delta b \Rightarrow \|\delta x\| \leq \|A^{-1}\| \|\delta b\|$$

$$\therefore \frac{\|\delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}$$

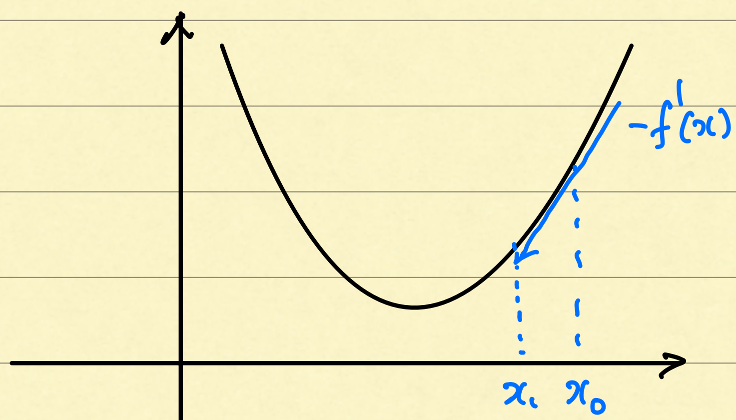
## Conjugate Gradient Method

$A$: SPD

Consider $f(x) = \frac{1}{2} x^T A x - b^T x + c$, $\quad x, b : n \times 1$, $c$ : scalar

$$f'(x) = Ax - b$$

i.e. minimize $f(x)$ is equal to solve $Ax = b$.




GD

$$x_1 = x_0 + (-f'(x))\alpha$$