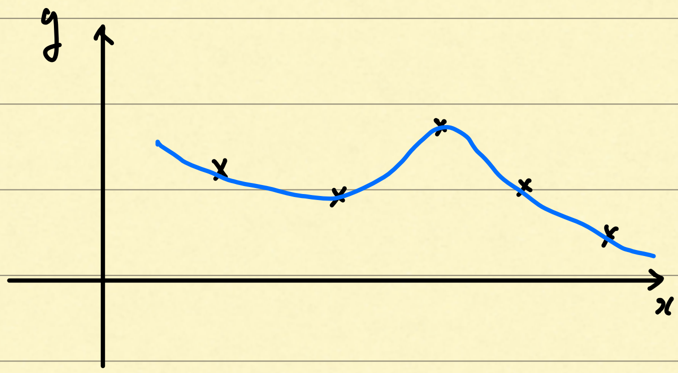
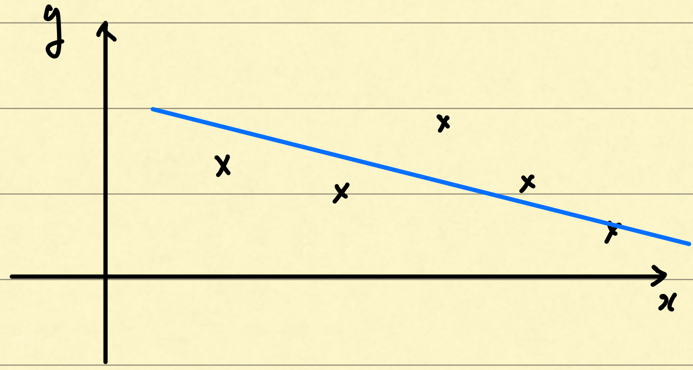
Chap 4 Interpolation and Numerical Differentiation

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n



interpolation



least square

4.1. Polynomial Interpolation

i) $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$p(x_i) = y_i$

$$\Rightarrow \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & & \cdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

↳ Vandermonde Matrix

ii) Lagrange form

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

cardinal polynomial

Lagrange form of the interpolation polynomial

$$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

① $P_n(x_j) = \sum_{i=0}^{n} l_i(x_j) f(x_i) = l_j(x_j) f(x_j) = f(x_j)$

② $l_i(x) = \prod_{\substack{j \neq i \\ j=0}}^{n} \left(\dfrac{x - x_j}{x_i - x_j} \right) \quad (0 \leq i \leq n)$

EXAMPLE 2 Write out the cardinal polynomials appropriate to the problem of interpolating the following table, and give the Lagrange form of the interpolating polynomial:

x	$\frac{1}{3}$	$\frac{1}{4}$	1
$f(x)$	2	−1	7

Solution Using Equation (2), we have

$$\ell_0(x) = \frac{\left(x - \frac{1}{4}\right)(x - 1)}{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{3} - 1\right)} = -18\left(x - \frac{1}{4}\right)(x - 1)$$

$$\ell_1(x) = \frac{\left(x - \frac{1}{3}\right)(x - 1)}{\left(\frac{1}{4} - \frac{1}{3}\right)\left(\frac{1}{4} - 1\right)} = 16\left(x - \frac{1}{3}\right)(x - 1)$$

$$\ell_2(x) = \frac{\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)} = 2\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

Therefore, the interpolating polynomial in Lagrange's form is

$$p_2(x) = -36\left(x - \frac{1}{4}\right)(x - 1) - 16\left(x - \frac{1}{3}\right)(x - 1) + 14\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$

<u>Thm 1</u> (Thm on Existence of Polynomial Interpolation)

If points $x_0, x_1, \cdots, x_n$ are distinct, then for arbitrary real values $y_0, y_1, \cdots, y_n$, there is a unique polynomial p of degree n s.t. $p(x_i) = y_i$ for $0 \leq i \leq n$

iii) Newton Form

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+ \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$
$$= \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1}(x-x_j)$$

How to find a_i ?

$$\begin{cases} f(x_0) = a_0 \\ f(x_1) = a_0 + a_1(x_1-x_0) \\ f(x_2) = a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) \\ \cdots \end{cases}$$

$$a_0 = f(x_0)$$

$$a_1 = \frac{f(x_1)-a_0}{x_1-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$a_2 = \frac{f(x_2)-a_0-a_1(x_2-x_0)}{(x_2-x_0)(x_2-x_1)} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

let $f[x_0] = a_0$

$f[x_0, x_1] = a_1$

$\cdots$

$$P(x) = \sum_{i=0}^{n} \left\{ f[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1}(x-x_j) \right\}$$

Newton form of the interpolation polynomial

The first three divided differences are thus

$$f[x_i] = f(x_i)$$
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Using Formula (13), we can construct a divided-difference table for a function f. It is customary to arrange it as follows (here $n = 3$):

$$\begin{array}{c|c|c|c|c}
x & f[\,] & f[\,,\,] & f[\,,\,,\,] & f[\,,\,,\,,\,] \\
\hline
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & \\
 & & f[x_2, x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

In the table, the coefficients along the top diagonal are the ones needed to form the Newton form of the interpolating polynomial (3).

EXAMPLE 7 Construct a divided-difference diagram for the function f given in the following table, and write out the Newton form of the interpolating polynomial.

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

Solution The first entry is $f[x_0, x_1] = \left(\frac{13}{4} - 3\right)/\left(\frac{3}{2} - 1\right) = \frac{1}{2}$. After completion of column 3, the first entry in column 4 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3}$$

The complete diagram is

$$\begin{array}{c|c|c|c|c}
x & f[\,] & f[\,,\,] & f[\,,\,,\,] & f[\,,\,,\,,\,] \\
\hline
1 & 3 & & & \\
 & & \frac{1}{2} & & \\
\frac{3}{2} & \frac{13}{4} & & \frac{1}{3} & \\
 & & \frac{1}{6} & & -2 \\
0 & 3 & & -\frac{5}{3} & \\
 & & -\frac{2}{3} & & \\
2 & \frac{5}{3} & & &
\end{array}$$

Thus, we obtain

$$p_3(x) = 3 + \tfrac{1}{2}(x - 1) + \tfrac{1}{3}(x - 1)\left(x - \tfrac{3}{2}\right) - 2(x - 1)\left(x - \tfrac{3}{2}\right)x$$ ■

Thm 2 Recursive Property of Divided Difference

$$f[x_0, x_1, \cdots, x_i] = \frac{f[x_1, x_2, \ldots, x_i] - f[x_0, x_1, \cdots, x_{i-1}]}{x_i - x_0}$$

Thm 3 Invariance Thm

The divided difference $f[x_0, x_1, \cdots, x_i]$ is invariant under all permutations of the arguments $x_0, x_1, \cdots, x_k$

pf of thm 2

$P_k(x)$

$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{k-1} \quad x_k$

$P_{k-1}(x)$ $\qquad$ $g(x)$

$$P_k(x) = g(x) + \frac{x - x_k}{x_k - x_0}\left[g(x) - P_{k-1}(x)\right] \quad \cdots\cdots (*)$$

from $P_k(x) = a(x - x_k)P_{k-1}(x) + b(x - x_0)g(x)$

Take the coefficient of x^k on both side of $(*)$

$$f[x_0, x_1, \cdots x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots x_{k-1}]}{x_k - x_0}$$

//

Interpolation of Bivariate Function

$(x, y) \to f(x, y)$: tensor product interpolation

$$l_i(x) = \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{x - x_j}{x_i - x_j} \quad (1 \le i \le n), \qquad \bar{l}_j(y) = \prod_{\substack{j \neq i \\ j=1}}^{m} \frac{y - y_i}{y_i - y_j} \quad (1 \le i \le m)$$

$$P(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, l_i(x)\, \bar{l}_j(y)$$

Neville's Algorithm

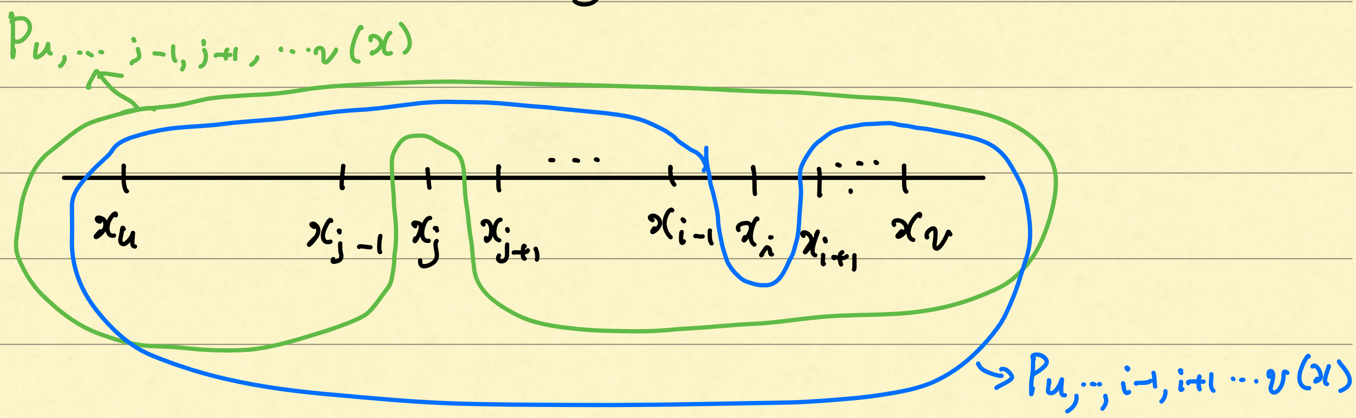
x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

$P_{a,b,\dots,s}(x)$: interpolation poly. at $x_a, x_b, \dots, x_s$

$$P_i(x) = y_i = f(x_i)$$

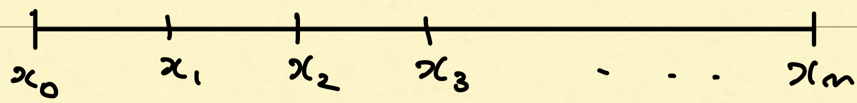
Define $\quad (j < i)$

$$P_{u,\dots v}(x) = \left(\frac{x - x_j}{x_i - x_j}\right) P_{u,\dots,j-1,j+1,\dots v}(x) + \left(\frac{x_i - x}{x_i - x_j}\right) P_{u,\dots,i-1,i+1,\dots v}(x)$$



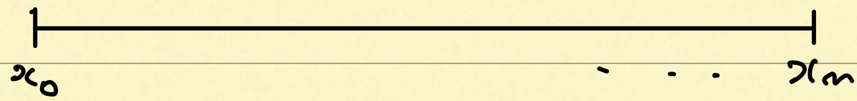
Using this formula repeately,

x_0	$P_0(x)$			
x_1	$P_1(x)$	$P_{0,1}(x)$		
x_2	$P_2(x)$	$P_{1,2}(x)$	$P_{0,1,2}(x)$	
x_3	$P_3(x)$	$P_{2,3}(x)$	$P_{1,2,3}(x)$	$P_{0,1,2,3}(x)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\ddots$



$P_0(x) = f(x_0)$: constat

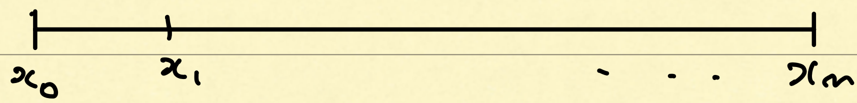
x_0 :



$P_{0,1}(x)$

$P_0(x)$ $P_1(x)$

x_1 :

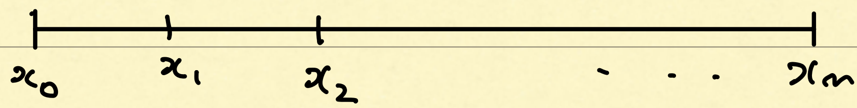


$P_{0,1,2}(x)$

$P_{0,1}(x)$ $P_{1,2}(x)$

$P_0(x)$ $P_1(x)$ $P_2(x)$

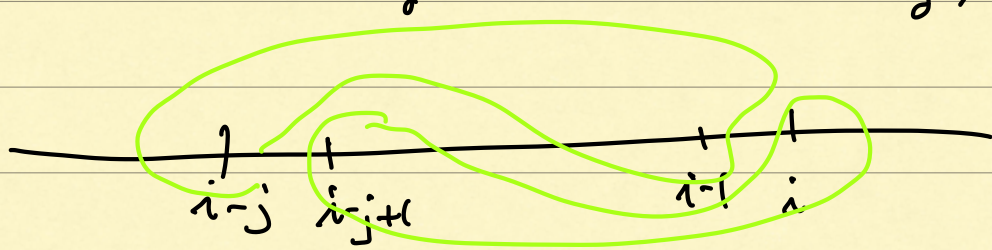
x_2 :



Simplify the notation

$$S_{ij}(x) = P_{i-j,\, i-j+1,\, \dots,\, i-1,\, i}(x) \quad \text{for } i \geq j$$

$$S_{ij}(x) = \left(\frac{x - x_{i-j}}{x_i - x_{i-j}}\right) S_{i,j-1}(x) + \left(\frac{x_i - x}{x_i - x_{i-j}}\right) S_{i-1,j-1}(x)$$



x_0	$S_{00}(x)$			
x_1	$S_{10}(x)$	$S_{11}(x)$		
x_2	$S_{20}(x)$	$S_{21}(x)$	$S_{22}(x)$	
x_3	$S_{30}(x)$	$S_{31}(x)$	$S_{32}(x)$	$S_{33}(x)$
$\vdots$	.	—	—	—

Change the notation : $P_i^0(x) = y_i$ for $0 \le i \le n$

$P_i^j(x) = S_{ij}(x)$ for $1 \le j \le i \le n$

Thm : $P_i^j(x_k) = y_k \quad (0 \le i-j \le k \le i \le n)$

pf) We use induction on j

$j=0 \qquad P_i^0(x_k) = y_k \quad (0 \le i \le k \le i \le n)$

(i.e. $P_i^0(x_i) = y_i$)

Assume $P_i^{j-1}(x_k) = y_k$ for some $j \ge 1$

$(0 \le i-j+1 \le k \le i \le n)$

For $k = i-j$, $P_i^j(x_{i-j}) = \left(\dfrac{x_i - x_{i-j}}{x_i - x_{i-j}}\right) P_{i-1}^{j-1}(x_{i-j}) = y_{i-j}$

For $k = i$, $P_i^j(x_i) = \left(\dfrac{x_i - x_{i-j}}{x_i - x_{i-j}}\right) P_i^{j-1}(x_i) = y_i$

Now let $i-j < k < i$,

$$P_i^j(x_k) = \left(\frac{x_k - x_{i-j}}{x_i - x_{i-j}}\right) P_i^{j-1}(x_k) + \left(\frac{x_i - x_k}{x_i - x_{i-j}}\right) P_{i-1}^{j-1}(x_k)$$

$$= y_k$$

//

4.2 Errors in Polynomial Interpolation

Runge function

$$f(x) = \frac{1}{1+x^2}$$

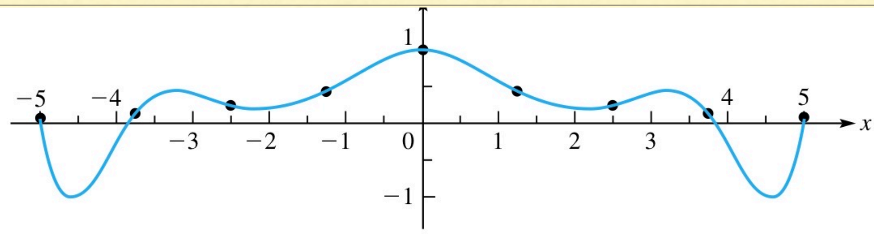


FIGURE 4.7 Polynomial interpolant with nine equally spaced nodes

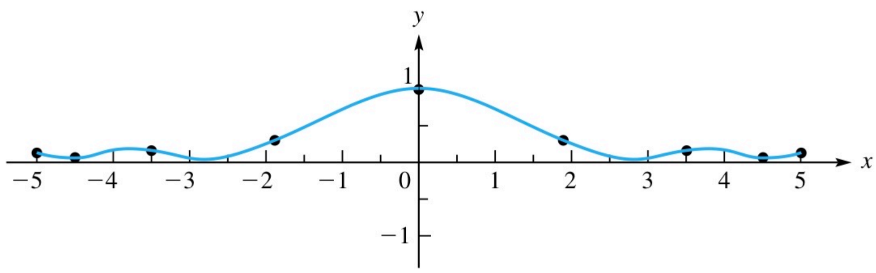


FIGURE 4.8 Polynomial interpolant with nine Chebyshev nodes

Chebyshev nodes : $x_i = \cos\left[\left(\frac{2i+1}{2n+2}\right)\pi\right]$

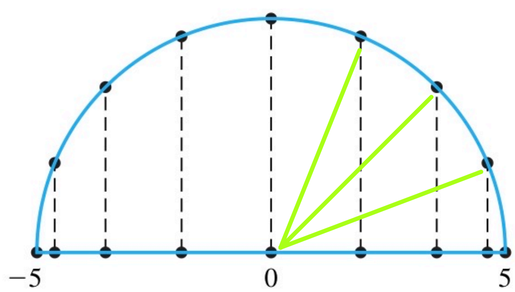


FIGURE 4.9 Interpolation with Chebyshev points

Fig 4.7 : equally spaced nodes

$$\lim_{n\to\infty} \max_{-5\le x\le 5} |f(x) - p_n(x)| = \infty$$

Thm 1 First Interpolation Error Thm

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

interpolation polynomial of degree at most n, and $x_0, x_1, \ldots, x_n$ distinct.

Lemma 1 Upper bound Lemma

$x_i = a + ih$, $h = (b-a)/n$, $i = 0, 1, \cdots, n$

For any $x \in [a, b]$, $\prod_{i=0}^{n} |x - x_i| \leq \frac{1}{4} h^{n+1} n!$

Thm 2 Second Interpolation Error Thm

$|f^{(n+1)}(x)| \leq M$, p interpolation polynomial

$n+1$ equally spaced node

degree $\leq n$

$$\Rightarrow \quad |f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1}$$

Thm 3 Third interpolation Error Thm

degree of $p = n$, x not a node

$$f(x) - p(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

Thm 4 Divided Difference and Differences

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

Corollary 1 Divided Difference Corollary

f : polynomial of degree n

$\Rightarrow f[x_0, x_1, \cdots, x_i] = 0$ for $i \geq n+1$

pf of thm 1

i) $x = x_i$ ok

ii) $x \neq x_i$

$$w(t) = \prod_{i=0}^{n} (t - x_i)$$

$$c = \frac{f(x) - p(x)}{w(x)} \quad : \text{constant}$$

$\varphi(t) = f(t) - p(t) - c\,w(t)$

$\Rightarrow \varphi(t) = 0$ at $t = x_0, x_1, \cdots, x_n, x$

φ' has at least $n+1$ roots

...

$\varphi^{(n+1)}$ has at least 1 roots

$$\Rightarrow 0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \overset{\nearrow 0}{p^{(n+1)}(\xi)} - c\,\overset{\nearrow (n+1)!}{w^{(n+1)}(\xi)}$$

$$0 = f^{(n+1)}(\xi) - c\,(n+1)!$$

$$= f^{(n+1)}(\xi) - \frac{f(x) - p(x)}{w(x)} (n+1)!$$

//

pf of Lemma 1

select j so that $x_j \le x \le x_{j+1}$,

$$|x-x_j|\,|x-x_{j+1}| \le \frac{h^2}{4}$$

$$\prod_{i=0}^{n}|x-x_i| \le \frac{h^2}{4}\prod_{i=0}^{j-1}(x-x_i)\prod_{i=j+2}^{n}(x_i-x)$$

$$\le \frac{h^2}{4}\prod_{i=0}^{j-1}(x_{j+1}-x_i)\prod_{i=j+2}^{n}(x_i-x_j)$$

$$\left(\begin{aligned} x_{j+1}-x_i &= a+(j+1)h-a-ih \\ &= (j-i+1)h \\ x_i-x_j &= (i-j)h \end{aligned}\right)$$

$$\le \frac{h^2}{4}\prod_{i=0}^{j-1}(j-i+1)h\prod_{i=j+2}^{n}(i-j)h$$

$$\le \frac{1}{4}h^{n+1}(j+1)!\,(n-j)!$$

$$\le \frac{1}{4}h^{n+1}n!$$

//

pf of Thm 2

From thm 1 and Lemma 1.

pf of Thm 3

Newton's form

$$g(x) = p(x) + f[x_0, x_1, \cdots, x_n, t]\prod_{i=0}^{n}(x-x_i)$$

$$g(t) = f(t) \Rightarrow f(t) = p(t) + f[x_0, x_1, \cdots, x_n, t]\prod_{i=0}^{n}(t-x_i)$$ //

pf of Thm 4

$p(x)$: interpolate f at $x_0, x_1, \cdots, x_{n-1}$

By thm 1, $f(x_n) - p(x_n) = \frac{1}{n!} f^{(n)}(\xi) \prod_{i=0}^{n-1} (x_n - x_i)$

By thm 3, $f(x_n) - p(x_n) = f[x_0, x_1, \cdots, x_{n-1}, x_n] \prod_{i=0}^{n-1} (x_n - x_i)$

//

4.3 Estimating Derivatives and Richardson Extrapolation

First Derivative Formula via Taylor Series

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x)$$

$$\Rightarrow f'(x) = \frac{1}{h}[f(x+h) - f(x)] - \frac{1}{2}hf''(x)$$

$$= \frac{1}{h}[f(x+h) - f(x)] + O(h)$$

$$\begin{array}{rl} & f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{3!}h^3 f'''(x) + \cdots \\ - & f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{3!}h^3 f'''(x) + \cdots \\ \hline \end{array}$$

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] + O(h^2)$$

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] + a_2 h^2 + a_4 h^4 + \cdots$$

Richardson Extrapolation

let $$\varphi(h) = \frac{1}{2h}[f(x+h) - f(x-h)]$$

$$\varphi(h) = f'(x) - a_2 h^2 - a_4 h^4 - a_6 h^6 - \cdots$$

$$\varphi\left(\frac{h}{2}\right) = f'(x) - a_2\left(\frac{h}{2}\right)^2 - a_4\left(\frac{h}{2}\right)^4 - a_6\left(\frac{h}{2}\right)^6 - \cdots$$

$$\varphi(h) - 4\varphi\left(\frac{h}{2}\right) = -3f'(x) - \frac{3}{4}a_4 h^4 - \frac{15}{16}a_6 h^6 - \cdots$$

$$\Rightarrow \varphi\left(\frac{h}{2}\right) + \frac{1}{3}\left[\varphi\left(\frac{h}{2}\right) - \varphi(h)\right] = f'(x) + \frac{1}{4}a_4 h^4 + \cdots$$

let $$\Phi(h) = \frac{4}{3}\varphi\left(\frac{h}{2}\right) - \frac{1}{3}\varphi(h)$$

$$\Phi(h) = f'(x) + b_4 h^4 + b_6 h^6 + \cdots$$

$$\Phi\left(\frac{h}{2}\right) = f'(x) + b_4\left(\frac{h}{2}\right)^4 + b_6\left(\frac{h}{2}\right)^6 + \cdots$$

$$\Rightarrow \Phi(h) - 16\,\Phi\left(\frac{h}{2}\right) = -15f' - \frac{7}{8}b_6 h^6 - \cdots$$

$$\Phi\left(\frac{h}{2}\right) + \frac{1}{15}\left[\Phi\left(\frac{h}{2}\right) - \Phi(h)\right] = f'(x) + \frac{7}{120}b_6 h^6 + \cdots$$

---

let $$\varphi(h) = L - \sum_{k=1}^{\infty} a_{2k} h^{2k}$$

$$D(n,0) = \varphi\left(\frac{h}{2^n}\right) \qquad (n \geq 0)$$

$$D(n,0) = L + \sum_{k=1}^{\infty} A(k,0)\left(\frac{h}{2^n}\right)^{2k} \quad \text{where } A(k,0) = -a_{2k}$$

extrapolation formula

$$D(n,m) = \frac{4^m}{4^m - 1} D(n, m-1) - \frac{1}{4^m - 1} D(n-1, m-1), \quad (1 \le m \le n)$$

Thm Richardson Extrapolation Thm

$$D(n,m) = L + \sum_{k=m+1}^{\infty} A(k,m) \left(\frac{h}{2^n}\right)^{2k}, \quad (0 \le m \le n)$$

pf) $m = 0$ OK

Suppose $m-1$ OK

then for m

$$D(n,m) = \frac{4^m}{4^m - 1} D(n, m-1) - \frac{1}{4^m - 1} D(n-1, m-1), \quad (1 \le m \le n)$$

$$= \frac{4^m}{4^m - 1} \left[L + \sum_{k=m}^{\infty} A(k, m-1) \left(\frac{h}{2^n}\right)^{2k} \right]$$

$$- \frac{1}{4^m - 1} \left[L + \sum_{k=m}^{\infty} A(k, m-1) \left(\frac{h}{2^{n-1}}\right)^{2k} \right]$$

$$= L + \sum_{k=m}^{\infty} A(k, m-1) \left(\frac{4^m - 4^k}{4^m - 1}\right) \left(\frac{h}{2^n}\right)^{2k}$$

let $A(k,m) = A(k, m-1) \left(\dfrac{4^m - 4^k}{4^m - 1}\right)$

note $A(m,m) = 0$

$$\Rightarrow D(n,m) = L + \sum_{k=m+1}^{\infty} A(k,m) \left(\frac{h}{2^n}\right)^{2k} \quad //$$

$$D(n,m) = L + O\left(\frac{h^{2(m+1)}}{2^{2n(m+1)}}\right)$$

$$\begin{array}{lllll}
D(0,0) & & & & \\
D(1,0) & D(1,1) & & & \\
D(2,0) & D(2,1) & D(2,2) & & \\
\vdots & \vdots & \vdots & \ddots & \\
D(N,0) & D(N,1) & D(N,2) & \cdots & D(N,N)
\end{array}$$

First derivative formulas via interpolation polynomials

$$P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$f'(x) \approx P_1'(x) = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

if $x_0 = x$, $x_1 = x + h$, $\quad f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$

if $x_0 = x - h$, $x_1 = x + h$, $\quad f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$

$$P_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$P_2'(x) = f[x_0, x_1] + f[x_0, x_1, x_2](2x - x_0 - x_1)$$

$$f(x) - P_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) w(x)$$

$$f'(x) - P_n'(x) = \frac{1}{(n+1)!} w(x) \frac{d}{dx} f^{(n+1)}(\xi) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) w'(x)$$

at x_i node

$$f'(x_i) = P_n'(x_i) + \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, w'(x_i)$$

ex) $n=1,\ i=0$

$$f'(x_0) = f[x_0,x_1] + \frac{1}{2} f''(\xi) \frac{d}{dx}\left[(x-x_0)(x-x_1)\right]\Big|_{x=x_0}$$

$$= f[x_0,x_1] + \frac{1}{2} f''(\xi)(x_0-x_1)$$

//

$$P_3(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

$$P_3'(x) = f[x_0,x_1] + f[x_0,x_1,x_2](2x-x_0-x_1)$$
$$+ f[x_0,x_1,x_2,x_3]\big((x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)\big)$$

$x_0 = x-h,\quad x_1 = x+h,\quad x_2 = x-2h,\quad x_3 = x+2h$

$$\Rightarrow f'(x) \approx -\frac{2}{3h}\left[f(x+h)-f(x-h)\right] - \frac{1}{12h}\left[f(x+2h)-f(x-2h)\right]$$

$$= \frac{1}{2h}\left[f(x+h)-f(x-h)\right] - \frac{1}{12h}\Big[f(x+2h) - 2\left[f(x+h)-f(x-h)\right] - f(x-2h)\Big]$$
$$+ O(h^4)$$

Second-Derivative Formulas via Taylor Series

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + 2\left[\frac{1}{4!} h^4 f^{(4)}(x) + \cdots\right]$$

$$\Rightarrow f''(x) = \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] + E$$

$$E = -2\left[\frac{1}{4!} h^2 f^{(4)}(x) + \frac{1}{6!} h^4 f^{(6)}(x) + \cdots\right] = -\frac{1}{12} h^2 f^{(4)}(\xi)$$

$$\therefore\ f''(x) \approx \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right]$$

Noise in Computation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \underbrace{\frac{h^2}{6} f'''(\xi)}_{\to \text{ truncation error}}$$

Suppose $h = 10^{-2}$, $|f'''(s)| \leq 6 \Rightarrow |E| \leq 10^{-4}$

If $f(x \pm h)$ has a noise of magnitude $d = h$,

$\dfrac{f(x+h) - f(x-h)}{2h}$ has error $\dfrac{2d}{2h} = 1$. //