Chap2 Linear Systems

Solve $Ax = b$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$A : n \times n$, $x : n \times 1$, $b : n \times 1$

nonsingular (invertible)

$\Rightarrow \quad x = A^{-1}b$

Question:

1) Memory problem
2) answer is correct?
3) precision
4) algorithm fail? or success?
5) How long it will take?
6) operation counting
7) stable or unstable
8) pivoting
9) ill-conditioned

Cramer's rule. $\Rightarrow$ $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$

operation counting to calculate $\det A \approx O(n!)$

Naive Gauss Elimination

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{cases}$$

$$\Rightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 & \leftarrow \text{pivot eqn} \\ -4x_2 + 2x_3 + 2x_4 = -6 \\ -12x_2 + 8x_3 + x_4 = -27 \\ 2x_2 + 3x_3 - 14x_4 = -18 \end{cases}$$

$$\Rightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ -4x_2 + 2x_3 + 2x_4 = -6 \\ 2x_3 - 5x_4 = -9 \\ 4x_3 - 13x_4 = -21 \end{cases}$$

$$\Rightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ -4x_2 + 2x_3 + 2x_4 = -6 \\ 2x_3 - 5x_4 = -9 \\ -3x_4 = -3 \end{cases}$$

$\downarrow$ forward elimination

$$x_4 = \frac{-3}{-3} = 1$$

$$2x_3 - 5 = -9 \quad \Rightarrow \quad x_3 = -2$$

$$x_2 = 1$$

$$x_1 = 3$$

$\uparrow$ backward substitution

Algorithm

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

first eqn $\rightarrow$ first pivot eqn

$a_{11} \rightarrow$ pivot element

$$\begin{cases} a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{i1}}{a_{11}}\right) a_{1j} & (1 \le j \le n) \\ b_i \leftarrow b_i - \left(\dfrac{a_{i1}}{a_{11}}\right) b_1 & (i > 1) \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{cases} a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{i2}}{a_{22}}\right) a_{ij} & (2 \le j \le n) \\ b_i \leftarrow b_i - \left(\dfrac{a_{i2}}{a_{22}}\right) b_2 & (i > 2) \end{cases}$$

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$$\begin{cases} a_{ij} \leftarrow a_{ij} - \left(\dfrac{a_{ik}}{a_{kk}}\right) a_{kj} & (k \le j \le n) \\ b_i \leftarrow b_i - \left(\dfrac{a_{ik}}{a_{kk}}\right) b_k & (i > k) \end{cases}$$

$$x_n = b_n / a_n$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right)$$

$$(i = n-1, n-2, \cdots, 1)$$

Residual and Error Vectors

$Ax = b$

$e = \tilde{x} - x$ error vector $\qquad$ $\tilde{x}$: computed sol

$r = A\tilde{x} - b$ residual vector $\qquad$ x: exact sol

$$Ae = A(\tilde{x} - x) = A\tilde{x} - Ax = A\tilde{x} - b = r$$

Gauss Elimination with Scaled Partial Pivoting

Naive Gaussian Elimination Can Fail

$$\begin{cases} 0x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$$

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Rightarrow \begin{cases} \varepsilon x_1 + x_2 = 1 \\ (1-\varepsilon^{-1}) x_2 = 2 - \varepsilon^{-1} \end{cases} \Rightarrow x_2 = \frac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \approx 1$$

$$x_1 = \varepsilon^{-1}(1 - x_2) \approx 0$$

Correct sol. $\quad x_1 = \dfrac{1}{1-\varepsilon} \simeq 1, \quad x_2 = \dfrac{1-2\varepsilon}{1-\varepsilon} \approx 1$

$$\begin{cases} x_1 + x_2 = 2 \\ 0x_1 + x_2 = 1 \end{cases}$$

$$\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 2 \\ (1-\varepsilon) x_2 = 1 - 2\varepsilon \end{cases}$$

$$x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \approx 1, \quad x_1 = 2 - x_1 \approx 1$$

Partial Pivoting and Complete Partial Pivoting

Gaussian elimination with partial pivoting selects the pivot row to be the one with the maximum pivot entry in absolute value from those in the leading column of the reduced matrix.

Gaussian elimination with complete pivoting selects the pivot entry as the maximum pivot entry from all entries in the submatrix.

Partial Pivoting and Scaled Partial Pivoting

Consider the Augmented matrix

$$\left[\begin{array}{cc|c} 2 & 2c & 2c \\ 1 & 1 & 2 \end{array}\right] \qquad c: \text{large number}$$

i) first row as a pivot row (pp)

$$\left[\begin{array}{cc|c} 2 & 2c & 2c \\ 0 & 1-c & 2-c \end{array}\right]$$

$$1-c \approx -c \ , \quad 2-c \approx -c$$

$$\therefore \ y=1, \ x=0$$

ii) second row as a pivot row (spp)

$$\{ 2/2c \ , \ 1 \}$$

$$\left[\begin{array}{cc|c} 0 & 2c-2 & 2c-4 \\ 1 & 1 & 2 \end{array}\right]$$

$$2c-2 \approx 2c \ , \quad 2c-4 \approx 2c$$

$\therefore \ y=1, \quad x=1$ $\Leftarrow$ correct sol

Gauss Elimination with Scaled Partial Pivoting

$\ell = [\ell_1, \ell_2, \cdots, \ell_n]$ index vector (row vector)

$= [1, 2, \cdots, n]$ initial

scale factor $\displaystyle s_i = \max_{1 \le j \le n} |a_{ij}| \quad (1 \le i \le n)$

scale vector $s = [s_1, s_2, \cdots, s_n]$

First step

i) select j to be the first index associated with the largest ratio in the set $\left\{ \dfrac{|a_{\ell_i 1}|}{s_{\ell_i}} : 1 \le i \le n \right\}$

ii) interchange l_j with l_1 in the index vector l.

iii) Use multipliers $a_{l_i 1} / a_{l_1 1}$ times row l_1, and subtract from eqns l_i for $2 \le i \le n$

k Step

i) $\left\{ \dfrac{|a_{l_i, k}|}{s_{l_i}} : k \le i \le n \right\}$

ii) interchange l_j with l_k

iii) $a_{l_i, k} / a_{l_k, k}$

$\boxed{\begin{array}{l}\text{Thm on Long Operations} \\ \text{The forward elimination phase of the Gaussian elimination algorithm with scaled partial pivoting,} \\ \text{if applied only to the } n\times n \text{ coefficient arrarys, involves approximately } n^3/3 \text{ long operations} \\ \text{(multiplications or divisions). Solving for } x \text{ requires an additional } n^2 \text{ long operations}\end{array}}$

Tridiagonal and Banded System

tridiagonal, superdiagonal, subdiagonal

banded structure if $\exists k$ s.t. $a_{ij} = 0$ whenever $|i-j| \ge k$

$$
\begin{bmatrix}
d_1 & c_1 & & & & & & \\
a_1 & d_2 & c_2 & & & & & \\
 & a_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & a_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & a_{n-2} & d_{n-1} & c_{n-1} \\
 & & & & & & a_{n-1} & d_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}
$$

$$
\begin{cases}
d_i \leftarrow d_i - \left(\dfrac{a_{i-1}}{d_{i-1}} \right) c_{i-1} \\
b_i \leftarrow b_i - \left(\dfrac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \qquad (2 \leq i \leq n)
\end{cases}
$$

$$
\begin{bmatrix}
d_1 & c_1 & & & & & \\
 & d_2 & c_2 & & & & \\
 & & d_3 & c_3 & & & \\
 & & & \ddots & \ddots & & \\
 & & & & d_i & c_i & \\
 & & & & & \ddots & \ddots \\
 & & & & & d_{n-1} & c_{n-1} \\
 & & & & & & d_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}
$$

$$
x_n \leftarrow b_n / d_n
$$

$$
x_i \leftarrow (b_i - c_i x_{i+1}) / b_i \quad , \quad i = n-1, n-2, \cdots, 1
$$

Strictly diagonal dominance

$A = (a_{ij})_{n\times n}$ is strictly diagonal dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1 \leq i \leq n)$$

pentadiagonal systems

Block Pentadiagonal systems