7. Initial Values Problem

$$\begin{cases} \dfrac{dx(t)}{dt} = f(t, x(t)) \\ x(a) \text{ is given} \end{cases}$$

IVP

Solving Differential Equations and Integration

$$\begin{cases} \dfrac{dx}{dr} = f(r, x) \\ x(a) = s \end{cases}$$

$$\int_t^{t+h} dx = \int_t^{t+h} f(r, x(r))\,dr$$

$$\Rightarrow \quad x(t+h) = x(t) + \int_t^{t+h} f(r, x(r))\,dr$$

$$\int_t^{t+h} f(r, x(r))\,dr \approx h f(t, x(t))$$

or

$$\approx \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right]$$

$$\Rightarrow \quad x(t+h) = x(t) + h f(t, x(t))$$

Euler's method.

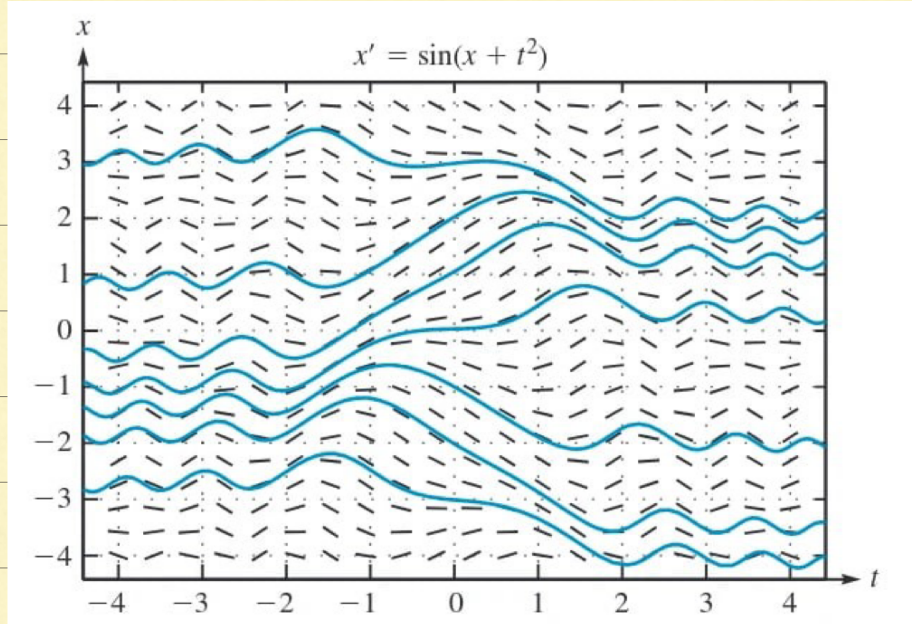
$$\Rightarrow \quad x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right]$$

implicit method

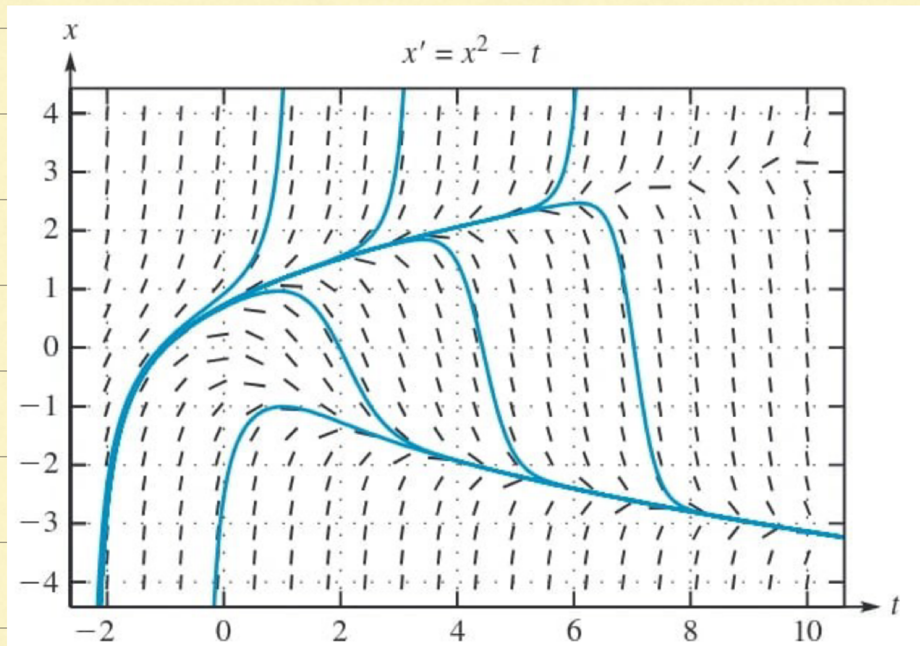
Vector fields

$x' = \sin(x + t^2)$

$x(0) = 0$



$x' = x^2 - t$



* initial condition at $t = -2$:

exceedingly sensitive to the initial condition.

* How do we know that the differential equation $x'(t) = x^2 - t$, together with an initial value, $x(t_0) = x_0$, has a unique solution?

Uniqueness of Initial-Value Problems

If f and $\partial f/\partial y$ are continuous in the rectangle defined by $|t-t_0| < \alpha$ and $|x-x_0| < \beta$, then the initial-value problem $x' = f(t,x)$, $x(t_0) = x_0$ has a unique continuous solution in some interval $|t-t_0| < \epsilon$.

Taylor Series Methods

$$x(t+h) = x(t) + h x'(t) + \frac{1}{2!} h^2 x''(t) + \cdots + \frac{1}{m!} h^m x^{(m)}(t) + \cdots$$

Euler's Method

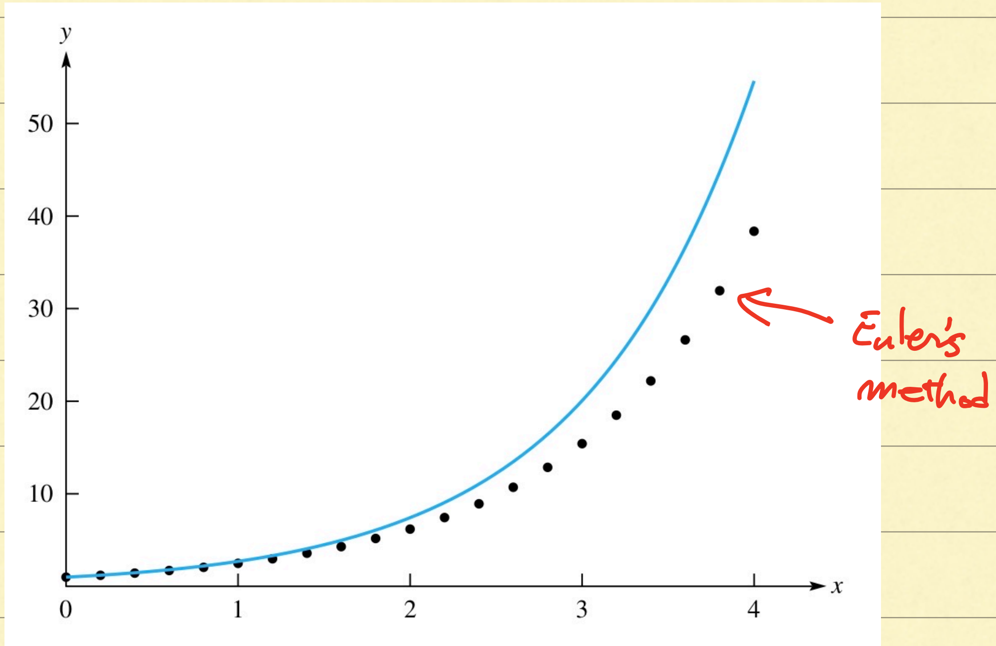
$$\begin{cases} x' = f(t, x(t)) \\ x(a) = x_a \end{cases}$$

first order taylor series method

$$x(t+h) \approx x(x) + h x'(t) \qquad x'(t) \approx \frac{x(t+h) - x(t)}{h}$$

$$\Rightarrow \quad x(t+h) = x(t) + h f(t, x(t))$$

$$\begin{cases} x'(t) = x \\ x(0) = 1 \end{cases}$$



Taylor series Method of higher order

$$\begin{cases} x'(t) = 1 + x(t)^2 + t^3 \\ x(1) = -4 \end{cases}$$

$$x' = 1 + x^2 + t^3$$

$$x'' = 2xx' + 3t^2$$

$$x''' = 2xx'' + 2x'x' + 6t$$

$$x'''' = 2xx''' + 6x'x'' + 6$$

$$x(t+h) = x(t) + hx'(t) + \frac{1}{2!}h^2x''(t) + \frac{1}{3!}h^3x'''(t) + \frac{1}{4!}h^4x''''(t)$$

Euler method

$x(2) \approx 4.2358541$

error $O(h^2)$

Taylor Series Method (order 4)

$x(2) \approx 4.3712096$

$O(h^5)$

exact $x(2) \approx 4.37122\,1866$

$$\begin{cases} \text{local truncation error} \\ \text{roundoff error} \end{cases}$$

7.2. Runge-Kutta Methods

$$x' = f(t, x) \quad , \quad x(a) = x_a$$

$$x_{k+1} = x_k + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4$$

$$f_1 = h f(t_k, x_k)$$

$$f_2 = h f(t_k + \alpha_2 h, \; x_k + \alpha_2 f_1)$$

$$f_3 = h f(t_k + \alpha_3 h, \; x_k + \alpha_3 f_2)$$

$$f_4 = h f(t_k + \alpha_4 h, \; x_k + \alpha_4 f_3)$$

order	α_2	α_3	α_4	β_1	β_2	β_3	β_4
1	0	0	0	1	0	0	0
2	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
3	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{4}$	0	$\frac{3}{4}$	0
4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Advantages of R-K:
- one step method
- need no starting value.

Disadvantages:
- hard to analyze the stability of the method.
- require more computing of f

Deriving RK2

$$\begin{cases} x' = f(t, x) \\ x(a) = x_a \end{cases}$$

$$f(t+h,\ x+k) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^n f(t,x)$$

$$= f + (hf_t + kf_x) + \frac{1}{2!}(h^2 f_{tt} + 2hk f_{tx} + k^2 f_{xx}) + \cdots$$

$$\begin{cases} \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^0 f(t,x) = f \\ \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^1 f(t,x) = h\frac{\partial f}{\partial t} + k\frac{\partial f}{\partial x} \\ \left(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right)^2 f(t,x) = h^2\frac{\partial^2 f}{\partial t^2} + 2hk\frac{\partial^2 f}{\partial t\partial x} + k^2\frac{\partial^2 f}{\partial x^2} \end{cases}$$

<u>RK-2</u>

$$\begin{cases} K_1 = hf(t,x) \\ K_2 = hf(t+\alpha h,\ x+\beta K_1) \end{cases}$$

$$x(t+h) = x(t) + \omega_1 K_1 + \omega_2 K_2$$

$$= x(t) + \omega_1 hf(t,x) + \omega_2 h f(t+\alpha h,\ x+\beta hf(t,x))$$

$$f(t+\alpha h,\ x+\beta hf)$$

$$= f + \alpha hf_t + \beta h f f_x + \frac{1}{2}\left(\alpha h\frac{\partial}{\partial t} + \beta hf\frac{\partial}{\partial x}\right)^2 f(\bar{t},\bar{x})$$

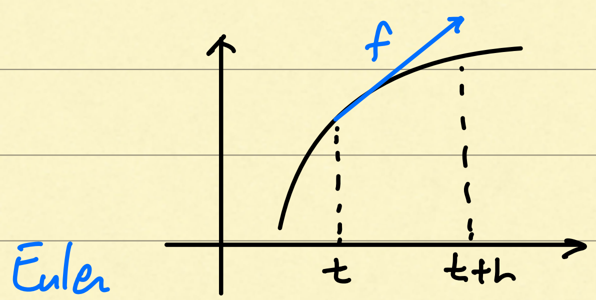
$$\therefore\ x(t+h) = x(t) + (\omega_1+\omega_2)hf$$

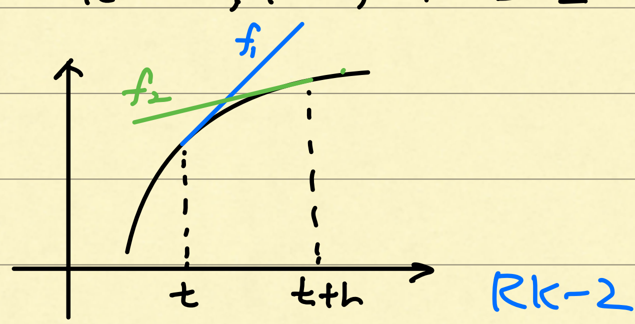
$$+ \alpha\omega_2 h^2 f_t + \beta\omega_2 h^2 f f_x + O(h^3)$$

$$x(t+h) = x(t) + h\, \underset{f}{x'(t)} + \frac{1}{2}h^2\, \underset{f_t + f_x f}{x''(t)}$$

$$\omega_1 + \omega_2 = 1,\quad \alpha\omega_2 = \frac{1}{2},\quad \beta\omega_2 = \frac{1}{2}$$

$$\Rightarrow \alpha = \beta,\quad \omega_1 = 1 - 1/2\alpha,\quad \omega_2 = 1/2\alpha \quad \Rightarrow \text{let } \alpha = 1,\ \beta = 1,\ \omega_1 = \omega_2 = \frac{1}{2}$$





7.3. Adaptive Runge-Kutta and Multistep Methods

Overview of Adaptive Process

1. Given a step size h and an initial value $x(t)$, the *RK45* routine computes the value $x(t+h)$ and an error estimate ε.
2. If $\varepsilon_{\min} \leqq \varepsilon \leqq \varepsilon_{\max}$, then the step size h is not changed and the next step is taken by repeating step 1 with initial value $x(t+h)$.
3. If $\varepsilon < \varepsilon_{\min}$, then h is replaced by $2h$, provided that $|2h| \leqq h_{\max}$.
4. If $\varepsilon > \varepsilon_{\max}$, then h is replaced by $h/2$, provided that $|h/2| \geqq h_{\min}$.
5. If $h_{\min} \leqq |h| \leqq h_{\max}$, then the step is repeated by returning to step 1 with $x(t)$ and the new h value.

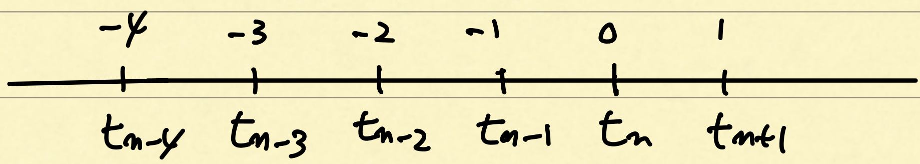
Multi-Step method

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

$$\int_{t_n}^{t_{n+1}} x'(t)\,dt = x(t_{n+1}) - x(t_n)$$

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(t, x(t))\,dt$$

Adam-Bashford formula of order 5



$$\int_{t_n}^{t_{n+1}} f(t, x)\,dt \approx h\left[A f_n + B f_{n-1} + C f_{n-2} + D f_{n-3} + E f_{n-4}\right]$$

If the integrand is a polynomial of degree ≤ 4, this will be exact.

$$\int_0^1 f(x)\,dx \approx \sum_{i=-n}^{0} A_i f(i) \Rightarrow \int_{t_0}^{t_0+h} f(x)\,dx \approx h \sum_{i=-n}^{0} A_i f(t_0+ih)$$

$$\Rightarrow \int_0^1 f(t_0+sh)\,ds \approx \sum_{i=-n}^{0} A_i f(t_0+ih)$$

$P_0(t)=1, \quad P_1(t)=t, \quad P_2(t)=t(t+1), \quad P_3(t)=t(t+1)(t+2)$

$P_4(t)=t(t+1)(t+2)(t+3)$

$$\int_0^1 P_n(t)\,dt = AP_n(0)+BP_n(-1)+CP_n(-2)+DP_n(-3)+EP_n(-4)$$

$$\Rightarrow \quad \begin{aligned} A+B+C+D+E &= 1 \\ -B-2C-3D-4E &= \tfrac{1}{2} \\ 2C+6D+12E &= 5/6 \\ -6D-24E &= 9/4 \\ 24E &= 251/30 \end{aligned}$$

$$\Rightarrow A=\frac{1901}{720},\quad B=\frac{-2774}{720},\quad C=\frac{2616}{720},\quad D=\frac{-1274}{720},\quad E=\frac{251}{720}$$

Adams - Moulton Formula

$$x_{n+1} = x_n + af_{n+1} + bf_n + cf_{n-1} + \cdots$$

$$x_{n+1} = x_n + \frac{h}{720}\left[251f_{n+1} + 646f_n - 246f_{n-1} + 106f_{n-2} - 19f_{n-3}\right]$$

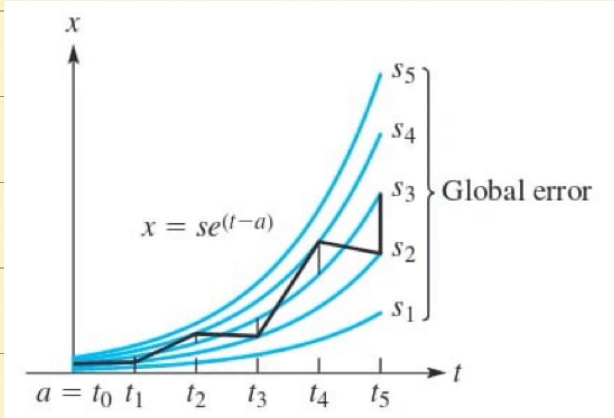
predictor - corrector method

1) use the Adam - Bashford formula to predict a tentative value for x_{n+1}, say x_{n+1}^*

2) use the Adam - Moulton formula to compute a corrected value of x_{n+1}.

Stability analysis

$$\begin{cases} x' = f(t, x) \\ x(a) = s \end{cases}$$

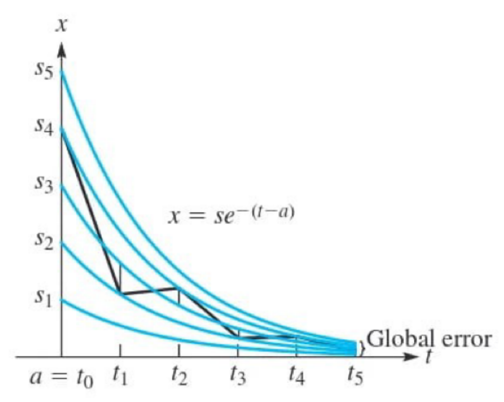


i) $\begin{cases} x' = x \\ x(a) = s \end{cases}$

$\Rightarrow x = se^{(t-a)}$

ii) $\begin{cases} x' = -x \\ x(a) = s \end{cases}$

$\Rightarrow x = se^{-(t-a)}$



If $f_x > \delta$ for some positive δ, the curves diverge.
However, if $f_x < -\delta$, they converge.
To see why, consider two nearly solution curves that correspond to initial value s and $s+h$

$$x(t, s+h) = x(t, s) + h\frac{\partial}{\partial s}x(t,s) + \frac{1}{2}h^2\frac{\partial^2}{\partial s^2}x(t,s) + \cdots$$

$$\Rightarrow x(t, s+h) - x(t,s) \approx h\frac{\partial}{\partial s}x(t,s)$$

The divergence means $\lim_{t\to\infty} |x(t,s+h) - x(t,s)| = \infty$

$$\Rightarrow \lim_{t\to\infty}\left|\frac{\partial}{\partial s}x(t,s)\right| = \infty$$

$$\frac{\partial}{\partial t}x(t,s) = f(t, x(t,s))$$

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}x(t,s) = \frac{\partial}{\partial s}f(t, x(t,s)) = \underbrace{f_x}_{g(t)}\frac{\partial}{\partial s}x(t,s) + f_t\underbrace{\frac{\partial t}{\partial s}}_{\to 0}$$

let $\frac{\partial}{\partial s}x(t,s) = u(t)$. $\quad u' = gu \Rightarrow u(t) = ce^{Q(t)}$

i.e. $\lim_{t\to\infty}|u(t)| = \infty$ if $\lim_{t\to\infty} Q(t) = \infty$

$$Q(t) = \int_a^t g(\theta)\,d\theta > \int_a^t \delta\, d\theta = \delta(t-a) \to \infty \quad \text{as } t\to\infty$$

if $f_x = g > \delta > 0$

ex) $x' = t + \tan x$

$f_x(t,x) = \sec^2 x > 1 \Rightarrow$ solution diverges as $t\to\infty$

7.4 Methods for First and High-Order system

- Coupled System

$$\begin{cases} x'(t) = x(t) - y(t) + 2t - t^2 - t^3 & , \quad x(0) = 1 \\ y'(t) = x(t) + y(t) - 4t^2 + t^3 & , \quad y(0) = 0 \end{cases}$$

- Uncoupled System

$$\begin{cases} x'(t) = x(t) + 2t - t^2 - t^3 & , \quad x(0) = 1 \\ y'(t) = y(t) - 4t^2 + t^3 & , \quad y(0) = 0 \end{cases}$$

- Systems of ODEs

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \cdots, x_n) \\ x_2' = f_2(t, x_1, x_2, \cdots, x_n) \\ \quad \cdots \\ x_n' = f_n(t, x_1, x_2, \cdots, x_n) \\ x_1(a) = s_1, \quad x_2(a) = s_2, \quad \cdots \quad x_n(a) = s_n \end{cases}$$

$X = [x_1 \ x_2 \ \cdots \ x_n]^T$

$F = [f_1 \ f_2 \ \cdots \ f_n]^T$ $\quad \Rightarrow \quad X' = F$

with $X(a) = S$

$S = [s_1 \ s_2 \ \cdots \ s_n]^T$

<u>R-K 4</u> $\quad X(t+h) = X + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

where
$$\begin{cases} K_1 = F(t, X) \\ K_2 = F(t + \frac{1}{2}h, \ x + \frac{1}{2}hK_1) \\ K_3 = F(t + \frac{1}{2}h, \ x + \frac{1}{2}hK_2) \\ K_4 = F(t + h, \ x + hK_3) \end{cases}$$

- **Autonomous ODE**

$$X' = F(t, X), \quad X(a) = S \qquad : n \text{ eqns}$$

introduce new variable $x_0 = t$

$$\Rightarrow \begin{cases} X' = F(X) \\ X(a) = S \end{cases} \qquad : n+1 \text{ eqn}$$

ex)
$$\begin{cases} x'(t) = x(t) - y(t) + 2t - t^2 - t^3, & x(0) = 1 \\ y'(t) = x(t) + y(t) - 4t^2 + t^3, & y(0) = 0 \end{cases}$$

$$x_0 = t, \quad x_1 = x, \quad x_2 = y$$

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 - x_2 + 2x_0 - x_0^2 - x_0^3 \\ x_1 + x_2 - 4x_0^2 + x_0^3 \end{bmatrix}$$

$$X(0) = [0 \quad 1 \quad 0]^T$$

7.5

- **High-Order DE**

$$\begin{cases} x^{(n)} = f(t, x, x', x'', \cdots, x^{(n-1)}) \\ x(a), x'(a), x''(a), \cdots, x^{(n-1)}(a) \text{ are given} \end{cases}$$

$$x_1 = x, \quad x_2 = x', \quad x_3 = x'', \quad \cdots, \quad x_n = x^{(n-1)}$$

$$x_1' = x' = x_2$$

$$x_2' = x'' = x_3$$

$$\vdots$$

$$x_{n-1}' = x^{(n-1)} = x_n$$

$$x_n' = x^{(n)} = f(t, x, \cdots, x^{(n-1)})$$

$$X = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$$

$$F = [x_2 \quad x_3 \quad \cdots \quad x_n \quad f]^T$$

$$\therefore \begin{cases} X = F(t, X) \\ X(a) = S \quad \text{given} \end{cases}$$

ex)
$$\begin{cases} x''' = \cos x + \sin x' - e^{x''} + t^2 \\ x(0) = 3, \quad x'(0) = 7, \quad x''(0) = 13 \end{cases}$$

old	new	IV	DE
x	x_1	3	$x_1' = x_2$
x'	x_2	7	$x_2' = x_3$
x''	x_3	13	$x_3' = \cos x_1 + \sin x_2 - e^{x_3} + t^2$

- Adaptive Scheme

- Stiff ODE

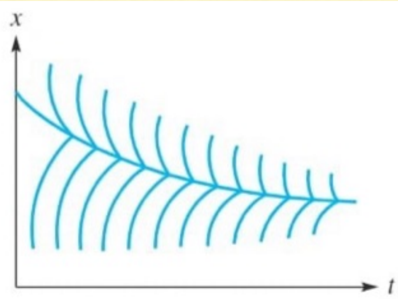


FIGURE 7.7
Solution curves for a stiff ODE

$$\begin{cases} x' = -20x - 19y, & x(0) = 2 \\ y' = -19x - 20y, & y(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x(t) = e^{-39t} + e^{-t} \\ y(t) = e^{-39t} - e^{-t} \end{cases}$$

Explict Euler Method

$$\begin{cases} x_{n+1} = x_n + h(-20x_n - 19y_n), & x_0 = 2 \\ y_{n+1} = y_n + h(-19x_n - 20y_n), & y_0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_n = (1-39h)^n + (1-h)^n \\ y_n = (1-39h)^n - (1-h)^n \end{cases}$$

numerical solution converges to 0

if $h < 2/39$

Implicit Euler Method

$$\begin{cases} x_{n+1} = x_n + h(-20x_{n+1} - 19y_{n+1}) \\ y_{n+1} = y_n + h(-19x_{n+1} - 20y_{n+1}) \end{cases}$$

$$\Rightarrow X_{n+1} = X_n + A X_{n+1}$$

$$(I - A) X_{n+1} = X_n$$

$$X_{n+1} = (I - A)^{-1} X_n$$

$$\Rightarrow X_n = (I - A)^{-n} X_0$$