

# Ex

## Computer Exercise 2.1

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## Computer Exercise 2.2

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## Computer Exercise 2.3

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# Com Ex

## Computer Exercise 2.2

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## Computer Exercise 2.3

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```
In [ ]: import numpy as np
import scipy as sp
import matplotlib.pyplot as plt
```

```
In [1]: # 2.2 - 4

def hilbert_matrix(n):
    return np.array([[1/(i+j+1) for j in range(n)] for i in range(n)])

def calc_b(n):
    return np.array([sum([1/(i+j+1) for j in range(n)]) for i in range(n)])

def gauss_solver(A, b):
    n = len(b)
    A = [row[:] for row in A]
    b = b[:]
```

```

s = [max(abs(A[i][j]) for j in range(n)) for i in range(n)]

for i in range(n):
    pivot_index = i
    max_ratio = abs(A[i][i]) / s[i] if s[i] != 0 else 0
    for k in range(i + 1, n):
        ratio = abs(A[k][i]) / s[k] if s[k] != 0 else 0
        if ratio > max_ratio:
            max_ratio = ratio
            pivot_index = k

    if pivot_index != i:
        A[i], A[pivot_index] = A[pivot_index], A[i]
        b[i], b[pivot_index] = b[pivot_index], b[i]
        s[i], s[pivot_index] = s[pivot_index], s[i]

    if A[i][i] == 0:
        raise ValueError("Unsolvable singular matrix")

    pivot = A[i][i]
    for j in range(i, n):
        A[i][j] /= pivot
    b[i] /= pivot

    for k in range(i + 1, n):
        factor = A[k][i]
        for j in range(i, n):
            A[k][j] -= factor * A[i][j]
        b[k] -= factor * b[i]

x = [0 for _ in range(n)]
for i in range(n - 1, -1, -1):
    x[i] = b[i]
    for j in range(i + 1, n):
        x[i] -= A[i][j] * x[j]

return x

def verify_solution(n):
    A = hilbert_matrix(n)
    b = calc_b(n)
    x_gauss = gauss_solver(A.copy(), b.copy())
    x_solve = np.linalg.solve(A.copy(), b.copy())
    print(f"Gauss Solution for {n} = {x_gauss}")
    print("Gauss Residual: ", np.linalg.norm(np.dot(A, x_gauss) - b))
    print(f"Solver Solution for {n} = {x_solve}")
    print("Solver Residual: ", np.linalg.norm(np.dot(A, x_solve) - b))
    print("Condition number: ", np.linalg.cond(A))
    print()

for n in range(2, 15):
    verify_solution(n)

```

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-
NameError                                Traceback (most recent call las
t)
Cell In[1], line 66
      62     print()
      65     for n in range(2, 15):
----> 66         verify_solution(n)

Cell In[1], line 53, in verify_solution(n)
      52     def verify_solution(n):
----> 53         A = hilbert_matrix(n)
      54         b = calc_b(n)
      55         x_gauss = gauss_solver(A.copy(), b.copy())

Cell In[1], line 4, in hilbert_matrix(n)
      3     def hilbert_matrix(n):
----> 4         return np.array([[1/(i+j+1) for j in range(n)] for i in range
(n)])

NameError: name 'np' is not defined

```

In [ ]: # 2.2 - 9

```

A = np.array(
    [[0.0001, -5.0300, 5.8090, 7.8320],
     [2.2660, 1.9950, 1.2120, 8.0080],
     [8.8500, 5.6810, 4.5520, 1.3020],
     [6.7750, -2.2530, 2.9080, 3.9700]]
).astype(np.float32)

b = np.array([9.5740, 7.2190, 5.7300, 6.2910]).astype(np.float32)

def naive_gauss_elimination(A, b):
    n = len(b)
    A = [row[:] for row in A]
    b = b[:]

    for i in range(n):
        if A[i][i] == 0:
            raise ValueError("Unsolvable singular matrix")

        pivot = A[i][i]
        for j in range(i, n):
            A[i][j] /= pivot
        b[i] /= pivot

        for k in range(i + 1, n):
            factor = A[k][i]
            for j in range(i, n):
                A[k][j] -= factor * A[i][j]
            b[k] -= factor * b[i]

    x = [0 for _ in range(n)]
    for i in range(n - 1, -1, -1):
        x[i] = b[i]
        for j in range(i + 1, n):
            x[i] -= A[i][j] * x[j]

    return x

```

```

def scaled_pivot_gauss_elimination(A, b):
    n = len(b)
    A = [row[:] for row in A]
    b = b[:]

    s = [max(abs(A[i][j]) for j in range(n)) for i in range(n)]

    for i in range(n):
        pivot_index = i
        max_ratio = abs(A[i][i]) / s[i] if s[i] != 0 else 0
        for k in range(i + 1, n):
            ratio = abs(A[k][i]) / s[k] if s[k] != 0 else 0
            if ratio > max_ratio:
                max_ratio = ratio
                pivot_index = k

        if pivot_index != i:
            A[i], A[pivot_index] = A[pivot_index], A[i]
            b[i], b[pivot_index] = b[pivot_index], b[i]
            s[i], s[pivot_index] = s[pivot_index], s[i]

        if A[i][i] == 0:
            raise ValueError("Unsolvable singular matrix")

        pivot = A[i][i]
        for j in range(i, n):
            A[i][j] /= pivot
            b[i] /= pivot

        for k in range(i + 1, n):
            factor = A[k][i]
            for j in range(i, n):
                A[k][j] -= factor * A[i][j]
            b[k] -= factor * b[i]

    x = [0 for _ in range(n)]
    for i in range(n - 1, -1, -1):
        x[i] = b[i]
        for j in range(i + 1, n):
            x[i] -= A[i][j] * x[j]

    return x

def verifyty_solution(A, b, solver=naive_gauss_elimination):
    x = solver(A, b)
    print("Solver: ", solver.__name__)
    print("Solution: ", x)
    print("Residual: ", np.linalg.norm(np.dot(A, x) - b))
    print()

verifyty_solution(A, b)
verifyty_solution(A, b, scaled_pivot_gauss_elimination)

```

Solver: naive\_gauss\_elimination  
 Solution: [np.float32(0.22265625), np.float32(-0.015712023), np.float32(0.6273985), np.float32(0.7469854)]  
 Residual: 0.0

Solver: scaled\_pivot\_gauss\_elimination  
 Solution: [np.float32(0.22265625), np.float32(-0.015712023), np.float32(0.6273985), np.float32(0.7469854)]  
 Residual: 0.0

In [ ]: # 2.3 - 6

```
def build_A(n):
    return np.diag(np.ones(n) * 5) + np.diag(np.ones(n - 1) * -1, k=1) +

def build_b(n, k):
    return np.append(np.arange(k, n + 1), np.arange(1, k))

def tridiagonal_solver(A, b):
    n = len(b)
    A = A.copy()
    b = b.copy()

    for i in range(n - 1):
        factor = A[i + 1][i] / A[i][i]
        A[i + 1][i] -= factor * A[i][i]
        A[i + 1][i + 1] -= factor * A[i][i + 1]
        b[i + 1] -= factor * b[i]

    x = [0 for _ in range(n)]
    x[n - 1] = b[n - 1] / A[n - 1][n - 1]
    for i in range(n - 2, -1, -1):
        x[i] = (b[i] - A[i][i + 1] * x[i + 1]) / A[i][i]

    return x

def verify_tridiagonal_solver(n, k):
    A = build_A(n)
    b = build_b(n, k)
    x = [x.item() for x in tridiagonal_solver(A, b)]
    x_exact = np.linalg.solve(A, b)
    print(f"Solution for n={n}, k={k}: ", x)
    print("Exact Solution: ", x_exact)
    print("Residual: ", np.linalg.norm(np.dot(A, x) - b))
    print()

n = 50
for i in range(1, n+1):
    verify_tridiagonal_solver(50, i)
```

Solution for n=50, k=1: [0.31929967043547763, 0.5964983521773881, 0.8631920904514627, 1.136128766746593, 1.4435386998032413, 1.9164104672968378, 2.1820742427415536, 2.4549458045182884, 2.7623519995512327, 3.235223565913301, 3.500887355234946, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.56552634573672, 9.831190135058222, 10.104061820349274, 10.411468610005116, 10.88434302551535, 11.150020465932752, 11.422957557553692, 11.730677727763071, 12.205053639711114, 12.477925181764018, 12.785331285124652, 13.258202412397013, 13.523864097920256, 13.796725703308264, 14.104084197247147, 14.57672721407563, 14.84129595680124, 15.108920958644976, 15.39118937766013, 15.743618623414234, 15.432208585691678, 12.940153456368959]

Exact Solution: [ 0.33333333 0.66666667 1.667 2.1.33333333 1.6666667 2.

2.33333333	2.66666667	3.	3.33333333	3.66666667	4.
4.33333333	4.66666667	5.	5.33333333	5.66666667	6.
6.33333333	6.66666667	7.	7.33333333	7.66666667	8.
8.33333333	8.66666667	9.	9.33333333	9.66666667	10.
10.33333333	10.66666667	11.	11.33333333	11.66666667	12.
12.33333333	12.66666664	12.99999988	13.33333278	13.666664	13.99998722
14.33327212	14.66637339	14.99859481	15.32660066	15.63440849	15.84544177
15.59280037	13.11856007]				

Residual: 4.267574441889994

Solution for n=50, k=2: [0.572335077279053, 0.8616753863952653, 1.1360418546972737, 1.4435338870911036, 1.9164101894538965, 2.1820742289624087, 2.4549458038429948, 2.762351999518119, 3.2352235659116135, 3.500887355234865, 3.7737590405263237, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349342, 10.411468610005437, 10.884343025516882, 11.150020465940095, 11.42295755758887, 11.730677727931614, 12.205053640518644, 12.477925185633133, 12.785331303662698, 13.258202501218122, 13.523864523487763, 13.796727742324686, 14.104093966761752, 14.576774022632234, 14.841520230069657, 15.109995516430462, 15.396337893319146, 15.768286643923828, 15.550400172580636, 13.50644337030416, 2.71325798278704]

Exact Solution: [ 0.59709595 0.98547975 1.33030278 1.66603415 1.99986799 2.33330578

2.66666092	2.9999988	3.33333308	3.66666661	3.99999999	4.33333333
4.66666667	5.	5.33333333	5.66666667	6.	6.33333333
6.66666667	7.	7.33333333	7.66666667	8.	8.33333333
8.66666667	9.	9.33333333	9.66666667	10.	10.33333333
10.66666667	11.	11.33333333	11.66666667	12.	12.33333333
12.66666665	12.9999999	13.33333287	13.66666446	13.99998944	14.33328273
14.66642422	14.99883838	15.32776768	15.64	15.8722323	15.72116152
13.73357532	2.94671506]				

Residual: 4.363599803516546

Solution for n=50, k=3: [0.8268093085029149, 1.1340465425145743, 1.4434234040699563, 1.9164038111685406, 2.182073912642312, 2.45494578834066, 2.762351998757955, 3.235223565872863, 3.5008873552330395, 3.7737590405262385, 4.081165830183727, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.411468

610005457, 10.88434302551698, 11.150020465940564, 11.422957557591127, 11.730677727942437, 12.205053640570503, 12.477925185881602, 12.785331304853196, 13.258202506922137, 13.523864550817335, 13.796727873268534, 14.10409459415142, 14.576777028636728, 14.841534632702462, 15.110064523589976, 15.3966652648391, 15.76987080258814, 15.557990332737436, 13.542810012423839, 2.8875010332286397, 0.8348486100883199]

Exact Solution: [ 0.86085856 1.30429282 1.66060556 1.99873497 2.33306931 2.66661156

2.9999885	3.33333093	3.66666617	3.9999999	4.33333331	4.66666666
5.	5.33333333	5.66666667	6.	6.33333333	6.66666667
7.	7.33333333	7.66666667	8.	8.33333333	8.66666667
9.	9.33333333	9.66666667	10.	10.33333333	10.66666667
11.	11.33333333	11.66666667	12.	12.33333333	12.66666665
12.99999991	13.33333288	13.6666645	13.9999896	14.33328351	14.66642794
14.99885619	15.32785301	15.64040888	15.8741914	15.73054811	13.77854917
3.16219774	1.03243955]				

Residual: 4.399623391818616

Solution for n=50, k=4: [1.0881773911686234, 1.4408869558431172, 1.9162573880469624, 2.182066651058362, 2.4549454324622406, 2.7623519813072885, 3.2352235649832912, 3.5008873551911086, 3.7737590405242694, 4.081165830183633, 4.55404024570144, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025516996, 11.150020465940639, 11.422957557591483, 11.730677727944132, 12.20505364057862, 12.477925185920498, 12.78533130503955, 13.25820250781501, 13.523864555095342, 13.796727893765711, 14.104094692359304, 14.57677749178979, 14.84153688720583, 15.110075325564578, 15.396720281853547, 15.770118777461715, 15.55917845173567, 13.548502632541437, 2.914776014818399, 0.9655308979195198, 0.6261364575662399]

Exact Solution: [ 1.12462118 1.6231059 1.99090834 2.33143579 2.66627063 2.99991734

3.33331608	3.66666307	3.99999925	4.33333318	4.66666663	4.99999999
5.33333333	5.66666667	6.	6.33333333	6.66666667	7.
7.33333333	7.66666667	8.	8.33333333	8.66666667	9.
9.33333333	9.66666667	10.	10.33333333	10.66666667	11.
11.33333333	11.66666667	12.	12.33333333	12.66666665	12.99999991
13.33333288	13.6666645	13.99998963	14.33328364	14.66642857	14.99885922
15.32786753	15.64047845	15.87452471	15.73214512	13.78620089	3.19885932
1.20809569	0.84161914]				

Residual: 4.431585474604479

Solution for n=50, k=5: [1.3825791646037098, 1.9128958230185493, 2.181899950489037, 2.4549372627599704, 2.762351580702121, 3.2352235445618867, 3.5008873542285253, 3.773759040479069, 4.081165830181504, 4.554040245701336, 4.819717686154689, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517, 11.15002046594066, 11.42295755759158, 11.730677727944602, 12.20505364058088, 12.477925185931321, 12.785331305091407, 13.258202508063482, 13.523864556285842, 13.796727899469726, 14.104094719688877, 14.576777630122827, 14.8415375145955, 15.110078331569072, 15.396734684486347, 15.770187784621228, 15.559509084900435, 13.55008679120575, 2.9223661749751977, 1.0018975400391996, 0.8003795080078399, 0.8348486100883199]

Exact Solution: [ 1.3883838 1.94191898 2.32121112 2.66413661 2.99947195 3.33322312

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3.66664366 3.9999952 4.33333233 4.66666646 4.99999996 5.33333332
5.66666666 6. 6.33333333 6.66666667 7. 7.33333333
7.66666667 8. 8.33333333 8.66666667 9. 9.33333333
9.66666667 10. 10.33333333 10.66666667 11. 11.33333333
11.66666667 12. 12.33333333 12.66666665 12.99999991 13.33333288
13.6666645 13.99998964 14.33328367 14.66642873 14.99885998 15.32787117
15.64049588 15.87460824 15.73254532 13.78811836 3.20804648 1.25211405
1.05252376 1.01050475]
Residual: 4.40303863667178

```

Solution for n=50, k=6: [1.635614571457097, 2.1780728572854864, 2.454749714970335, 2.7623423842328565, 3.235223075759165, 3.5008873321310268, 3.7737590394414213, 4.081165830132606, 4.554040245698961, 4.819717686154579, 5.092654777947381, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517, 11.150020465940663, 11.422957557591605, 11.730677727944727, 12.20505364058147, 12.477925185934145, 12.785331305104938, 13.258202508128305, 13.523864556596429, 13.79672790095785, 14.104094726818895, 14.57677766428479, 14.841537678275309, 15.11007911580616, 15.396738441991966, 15.770205787912229, 15.559595343849825, 13.550500082661705, 2.924346373305589, 1.0113852402351977, 0.8458378106574393, 1.0526524231403198, 1.0435607626104]

Exact Solution: [ 1.65214641 2.26073206 2.6515139 2.99683743 3.33267327 3.6665289

```

3.99997125 4.33332733 4.66666541 4.99999974 5.33333328 5.66666666
6. 6.33333333 6.66666667 7. 7.33333333 7.66666667
8. 8.33333333 8.66666667 9. 9.33333333 9.66666667
10. 10.33333333 10.66666667 11. 11.33333333 11.66666667
12. 12.33333333 12.66666665 12.99999991 13.33333288 13.6666645
13.99998964 14.33328368 14.66642877 14.99886018 15.32787212 15.6405004
15.87462988 15.73264901 13.78861518 3.21042688 1.26351922 1.10716923
1.27232692 1.25446538]

```

Residual: 4.462009386470942

Solution for n=50, k=7: [1.890088802727971, 2.450444013639855, 2.7621312654713037, 3.2352123137166635, 3.5008868248511456, 3.773759015620732, 4.081165829010088, 4.554040245644427, 4.819717686152066, 5.092654777947264, 5.40037494897967, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591614, 11.730677727944762, 12.205053640581642, 12.47792518593497, 12.78533130510889, 13.258202508147246, 13.523864556687183, 13.796727901392673, 14.104094728902266, 14.576777674266816, 14.841537726102061, 15.110079344957892, 15.396739539923885, 15.770211048420094, 15.559620548457227, 13.550620845190856, 2.924924981343928, 1.0141575178977449, 0.8591205909318369, 1.1162940468497593, 1.3484861008831999, 1.46098506765456]

Exact Solution: [ 1.91590903 2.57954514 2.98181668 3.32953825 3.66587459 3.99983468

```

4.33329883 4.66665947 4.9999985 5.33333302 5.6666666 5.99999999
6.33333333 6.66666667 7. 7.33333333 7.66666667 8.
8.33333333 8.66666667 9. 9.33333333 9.66666667 10.
10.33333333 10.66666667 11. 11.33333333 11.66666667 12.
12.33333333 12.66666665 12.99999991 13.33333288 13.6666645 13.99998964
14.33328368 14.66642878 14.99886023 15.32787235 15.64050154 15.87463533
15.73267513 13.78874033 3.21102652 1.26639228 1.12093487 1.33828206
1.57047543 1.51409509]

```



Residual: 4.425068058315489

Solution for n=50, k=8: [2.151456885618927, 2.757284428094636, 3.2349652548542527, 3.5008751795099613, 3.7737584687825096, 4.0811658032410625, 4.5540402443925, 4.819717686094351, 5.092654777944609, 5.400374948979547, 5.874750864870506, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944771, 12.205053640581681, 12.477925185935167, 12.785331305109834, 13.258202508151763, 13.523864556708832, 13.796727901496391, 14.104094729399208, 14.576777676647811, 14.84153773751009, 15.110079399617037, 15.39673980181158, 15.77021230319943, 15.559626560466219, 13.550649650456458, 2.9250629956629544, 1.0148187842272753, 0.8622889082604623, 1.1314743671633563, 1.4212193851225592, 1.8094711685377598, 1.6696972201766398]

Exact Solution: [ 2.17967164 2.89835822 3.31211946 3.66223907 3.99907591 4.33314046

4.66662641	4.9999916	5.33333158	5.6666663	5.9999992	6.33333332
6.66666666	7.	7.33333333	7.66666667	8.	8.33333333
8.66666667	9.	9.33333333	9.66666667	10.	10.33333333
10.66666667	11.	11.33333333	11.66666667	12.	12.33333333
12.66666665	12.99999991	13.33333288	13.6666645	13.99998964	14.33328368
14.66642878	14.99886024	15.32787241	15.64050182	15.87463667	15.73268153
13.78877099	3.21117341	1.26709604	1.12430679	1.3544379	1.64788269
1.88497556	1.77699511]				

Residual: 4.384893153953152

Solution for n=50, k=9: [2.404191993466573, 3.0209599673328658, 3.500607843197755, 3.773745915322577, 4.081165211676003, 4.554040215652718, 4.819717684769408, 5.092654777883623, 5.400374948976729, 5.8747508648703715, 6.147622425814475, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581692, 12.477925185935213, 12.785331305110054, 13.258202508152824, 13.523864556713912, 13.796727901520745, 14.104094729515891, 14.576777677206872, 14.841537740188707, 15.110079412451066, 15.396739863303118, 15.770212597823088, 15.55962797209297, 13.55065641396657, 2.925095401586758, 1.0149740503361804, 0.8630328328811839, 1.13503872415806, 1.4382972454753558, 1.891296113307039, 2.0617440836702396, 1.8784093726987199]

Exact Solution: [ 2.44343426 3.2171713 3.64242224 3.99493989 4.3322723 4.66644624

4.999954	5.33332373	5.66666466	5.99999958	6.33333325	6.66666665
7.	7.33333333	7.66666667	8.	8.33333333	8.66666667
9.	9.33333333	9.66666667	10.	10.33333333	10.66666667
11.	11.33333333	11.66666667	12.	12.33333333	12.66666665
12.99999991	13.33333288	13.6666645	13.99998964	14.33328368	14.66642879
14.99886024	15.32787242	15.64050188	15.87463699	15.73268306	13.78877833
3.21120861	1.26726471	1.12511494	1.35830997	1.66643493	1.97386468
2.20288848	2.0405777 ]				

Residual: 4.447384635904777

Solution for n=50, k=10: [2.6988940721575103, 3.4944703607875516, 3.7734577317802493, 4.081151631447028, 4.554039555889677, 4.819717654353443, 5.092654776483599, 5.400374948912043, 5.874750864867308, 6.147622425814336, 6.455028619687651, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.10406182034934

6, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581694, 12.477925185935224, 12.785331305110107, 13.258202508153072, 13.523864556715093, 13.796727901526392, 14.10409472954295, 14.576777677336517, 14.841537740809883, 15.11007941542731, 15.396739877563153, 15.77021266614702, 15.55962829945259, 13.55065798244074, 2.925102916597993, 1.0150100569181841, 0.8632053507799673, 1.135865307069973, 1.4422576421361375, 1.9102715136990351, 2.152660688969439, 2.3140169988027197, 2.0871215252208]

Exact Solution: [ 2.70719688 3.53598438 3.97272502 4.32764071 4.66547855 4.99975203

5.33328158 5.66665586 5.99999775 6.33333286 6.66666657 6.99999998  
7.33333333 7.66666667 8. 8.33333333 8.66666667 9.  
9.33333333 9.66666667 10. 10.33333333 10.66666667 11.  
11.33333333 11.66666667 12. 12.33333333 12.66666665 12.99999991  
13.33333288 13.6666645 13.99998964 14.33328368 14.66642879 14.99886024  
15.32787243 15.6405019 15.87463706 15.73268343 13.78878007 3.21121691  
1.26730446 1.12530541 1.35922257 1.67080744 1.99481462 2.30326566  
2.52151368 2.30430274]

Residual: 4.451815152146066

Solution for n=50, k=11: [2.953368328203574, 3.7668416410178684, 4.08083987688577, 4.554024410077654, 4.819716956111193, 5.092654744344014, 5.400374947427058, 5.874750864796965, 6.14762242581113, 6.4550286196875035, 6.92790018063164, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.78533130511012, 13.258202508153133, 13.523864556715386, 13.796727901527806, 14.104094729549727, 14.57677767736899, 14.841537740965462, 15.110079416172725, 15.396739881134645, 15.770212683259063, 15.55962838144131, 13.550658375272285, 2.925104798766998, 1.0150190749316659, 0.8632485586783718, 1.1360723285485128, 1.4432495416304332, 1.9150239896919734, 2.1754311694398343, 2.4231169251617586, 2.6098506765455993, 2.5045458302649597]

Exact Solution: [ 2.97095949 3.85479746 4.3030278 4.66034153 4.99867987 5.33305781

5.66660916 5.999988 6.33333083 6.66666614 6.99999989 7.33333331  
7.66666666 8. 8.33333333 8.66666667 9. 9.33333333  
9.66666667 10. 10.33333333 10.66666667 11. 11.33333333  
11.66666667 12. 12.33333333 12.66666665 12.99999991 13.33333288  
13.6666645 13.99998964 14.33328368 14.66642879 14.99886024 15.32787243  
15.6405019 15.87463708 15.73268351 13.78878047 3.21121884 1.26731371  
1.12534971 1.35943484 1.67182449 1.99968761 2.32661357 2.63338022  
2.84028755 2.56805751]

Residual: 4.399444449681981

Solution for n=50, k=12: [3.2147365297995956, 4.073682648997978, 4.55367671519029, 4.819700926953477, 5.092654006533612, 5.400374913337091, 5.874750863182144, 6.1476224257375165, 6.455028619684137, 6.9279001806314815, 7.193563944000489, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.258202508153147, 13.523864556715454, 13.796727901528127, 14.104094729551258, 14.576777677376331, 14.841537741000637, 15.110079416341268, 15.396739881942176, 15.77021268712818, 15.559628399979355, 13.550658464093395, 2.9251052243345046, 1.0150211139480882, 0.8632583281929771, 1.1361191371051176, 1.4434738148988515, 1.9160985474774603, 2.1805796850988504, 2.4477849456713536, 2.7280422634345585, 3.0708357442001595, 2.71325798278704]

Exact Solution: [ 3.23472211 4.17361054 4.63333058 4.99304235 5.33188  
119 5.66636359  
5.99993674 6.33332013 6.66666391 6.99999942 7.33333321 7.66666664  
7.99999999 8.33333333 8.66666667 9. 9.33333333 9.66666667  
10. 10.33333333 10.66666667 11. 11.33333333 11.66666667  
12. 12.33333333 12.66666665 12.99999991 13.33333288 13.6666645  
13.99998964 14.33328368 14.66642879 14.99886024 15.32787243 15.6405019  
15.87463709 15.73268353 13.78878056 3.21121928 1.26731584 1.12535991  
1.35948369 1.67205856 2.00080912 2.33198706 2.65912616 2.96364372  
3.15909244 2.83181849]  
Residual: 4.34599793948434

Solution for n=50, k=13: [3.467472206397379, 4.3373610319868945, 4.819332  
953537091, 5.092637069031898, 5.400374130752839, 5.87475082611161, 6.14762  
2424047633, 6.455028619606828, 6.927900180627869, 7.193563944000324, 7.466  
435504944905, 7.77384169882023, 8.246713259774477, 8.51237702319248, 8.785  
248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.  
104061820349346, 10.41146861000546, 10.884343025517001, 11.15002046594066  
7, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.47792518  
5935227, 12.785331305110121, 13.258202508153149, 13.52386455671547, 13.796  
727901528199, 14.104094729551601, 14.576777677377983, 14.841537741008546,  
15.110079416379149, 15.396739882123683, 15.770212687997827, 15.55962840414  
609, 13.550658484057443, 2.9251053199880084, 1.015021572251557, 0.86326052  
40568167, 1.1361296581208467, 1.4435242241136566, 1.916340072535757, 2.181  
7369011755288, 2.453329500996448, 2.754607823983353, 3.1981189916190385,  
3.3231086593326395, 2.92197013530912]  
Exact Solution: [ 3.49848472 4.49242362 4.96363336 5.32574317 5.66508  
251 5.99966937  
6.33326433 6.6665226 6.99999699 7.33333271 7.66666654 7.99999997  
8.33333333 8.66666667 9. 9.33333333 9.66666667 10.  
10.33333333 10.66666667 11. 11.33333333 11.66666667 12.  
12.33333333 12.66666665 12.99999991 13.33333288 13.6666645 13.99998964  
14.33328368 14.66642879 14.99886024 15.32787243 15.6405019 15.87463709  
15.73268353 13.78878058 3.21121938 1.26731632 1.12536223 1.35949484  
1.67211197 2.001065 2.33321303 2.66500014 2.99178769 3.2939383  
3.47790381 3.09558076]  
Residual: 4.403561328454318

Solution for n=50, k=14: [3.7205103434672715, 4.602551717336357, 5.092248  
2432145095, 5.400356165402858, 5.874749975104129, 6.147622385253908, 6.455  
028617832073, 6.927900180544922, 7.1935639439965735, 7.466435504944735, 7.  
773841698820223, 8.246713259774477, 8.51237702319248, 8.785248584372402,  
9.092654779376096, 9.565526345736721, 9.831190135058236, 10.10406182034934  
6, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557  
591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785  
331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528213, 1  
4.10409472955168, 14.57677767737835, 14.841537741010313, 15.1100794163876  
2, 15.396739882164272, 15.770212688192297, 15.55962840507786, 13.550658488  
521806, 2.9251053413780603, 1.0150216747374534, 0.8632610150962473, 1.1361  
320108321031, 1.4435354966305092, 1.9163940824087624, 2.181995678023704,  
2.454569375364318, 2.7605484189745257, 3.2265820922070327, 3.4594835672814  
384, 3.575381574465119, 3.1306822878311995]  
Exact Solution: [ 3.76224734 4.8112367 5.29393614 5.65844399 5.99828  
383 6.33297515  
6.66659191 6.9999844 7.33333008 7.66666599 7.99999986 8.3333333  
8.66666666 9. 9.33333333 9.66666667 10. 10.33333333  
10.66666667 11. 11.33333333 11.66666667 12. 12.33333333  
12.66666665 12.99999991 13.33333288 13.6666645 13.99998964 14.33328368  
14.66642879 14.99886024 15.32787243 15.6405019 15.87463709 15.73268354  
13.78878059 3.2112194 1.26731643 1.12536276 1.35949736 1.67212406

2.00112295 2.33349071 2.66633058 2.99816218 3.32448031 3.62423937  
 3.79671654 3.35934331]  
 Residual: 4.486731915208644

Solution for n=50, k=15: [4.016664322657973, 5.083321613289866, 5.399943743791353, 5.874730439000239, 6.147621494688106, 6.455028577090016, 6.92790017864076, 7.193563943910446, 7.466435504940842, 7.7738416988200445, 8.24671325977447, 8.51237702319248, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528216, 14.104094729551697, 14.576777677378436, 14.84153774101073, 15.110079416389624, 15.396739882173872, 15.770212688238296, 15.55962840529826, 13.55065848957781, 2.9251053464376726, 1.0150216989795124, 0.8632611312469299, 1.136132567343458, 1.4435381630365998, 1.9164068579278617, 2.18205688921311, 2.4548626557922493, 2.761953609924778, 3.2333147665303614, 3.491741747947832, 3.729939803473758, 3.8712152522079992, 3.5481065928753597]

Exact Solution: [ 4.02600995 5.13004977 5.62423892 5.99114481 6.33148515 6.66628093

6.99991949 7.33331653 7.66666316 7.99999927 8.33333318 8.66666663  
 8.99999999 9.33333333 9.66666667 10. 10.33333333 10.66666667  
 11. 11.33333333 11.66666667 12. 12.33333333 12.66666665  
 12.99999991 13.33333288 13.66666645 13.99998964 14.33328368 14.66642879  
 14.99886024 15.32787243 15.6405019 15.87463709 15.73268354 13.78878059  
 3.21121941 1.26731646 1.12536288 1.35949793 1.67212679 2.001136  
 2.33355321 2.66663006 2.99959707 3.3313553 3.65717942 3.95454179  
 4.11552954 3.62310591]

Residual: 4.381902285685808

Solution for n=50, k=16: [4.278095081598254, 5.390475407991271, 5.874281958358099, 6.147601050465895, 6.455027641797465, 6.927900134927966, 7.193563941933273, 7.466435504851446, 7.773841698815991, 8.246713259774282, 8.512377023192473, 8.785248584372402, 9.092654779376096, 9.565526345736721, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378457, 14.841537741010823, 15.110079416390066, 15.396739882175993, 15.770212688248463, 15.559628405346968, 13.550658489811177, 2.925105347555793, 1.015021704336749, 0.8632611569149925, 1.1361326903265339, 1.4435387522839167, 1.91640968118137, 2.182070416233333, 2.454927467639856, 2.762264142142588, 3.2348026157718053, 3.498870461937239, 3.7640955241793512, 4.0348651417465575, 4.332200319862559, 3.7568187453974398]

Exact Solution: [ 4.28977257 5.44886285 5.9545417 6.32384563 6.66468647 6.99958671

7.33324707 7.66664866 7.99999624 8.33333255 8.6666665 8.99999997  
 9.33333333 9.66666667 10. 10.33333333 10.66666667 11.  
 11.33333333 11.66666667 12. 12.33333333 12.66666665 12.99999991  
 13.33333288 13.66666645 13.99998964 14.33328368 14.66642879 14.99886024  
 15.32787243 15.6405019 15.87463709 15.73268354 13.78878059 3.21121941  
 1.26731646 1.1253629 1.35949806 1.6721274 2.00113892 2.33356721  
 2.66669711 2.99991835 3.33289465 3.66455491 3.98987988 4.2848445  
 4.4343426 3.88686852]

Residual: 4.315110851582846

Solution for n=50, k=17: [4.531130488441811, 5.655652442209055, 6.147131722603463, 6.455006170808264, 6.9278991314378615, 7.193563896544384, 7.466435502799207, 7.773841698722897, 8.246713259769983, 8.512377023192277, 8.785248584372395, 9.092654779376096, 9.565526345736721, 9.831190135058236, 1

0.104061820349346, 10.41146861000546, 10.884343025517001, 11.1500204659406  
 67, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.4779251  
 85935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.79  
 6727901528218, 14.104094729551703, 14.576777677378459, 14.841537741010844,  
 15.110079416390164, 15.396739882176462, 15.770212688250703, 15.55962840535  
 7696, 13.550658489862586, 2.925105347802123, 1.0150217055169872, 0.8632611  
 625698533, 1.1361327174205997, 1.4435388820993855, 1.9164103031646484, 2.1  
 82073396334258, 2.4549417461612024, 2.762332554648395, 3.235130399779494,  
 3.5004409694698744, 3.771620277834837, 4.07091840249135, 4.50494186993103  
 7, 4.5844732349950394, 3.96553089791952]

Exact Solution: [ 4.55353519 5.76767593 6.28484448 6.65654645 6.99788  
 779 7.33289249

7.66657466 7.9999808 8.33332933 8.66666583 8.99999983 9.3333333  
 9.66666666 10. 10.33333333 10.66666667 11. 11.33333333  
 11.66666667 12. 12.33333333 12.66666665 12.99999991 13.33333288  
 13.6666645 13.99998964 14.33328368 14.66642879 14.99886024 15.32787243  
 15.6405019 15.87463709 15.73268354 13.78878059 3.21121941 1.26731646  
 1.12536291 1.35949809 1.67212753 2.00113957 2.33357033 2.66671206  
 2.99998996 3.33323774 3.66619872 3.99775587 4.32258063 4.61514726  
 4.75315568 4.15063114]

Residual: 4.367560046522021

Solution for n=50, k=18: [4.785604719665583, 5.9280235983279175, 6.454513  
 2719740055, 6.927876094875443, 7.193562854577124, 7.466435455687129, 7.773  
 841696585788, 8.246713259671251, 8.51237702318784, 8.785248584372194, 9.09  
 2654779376089, 9.565526345736721, 9.831190135058236, 10.104061820349346, 1  
 0.41146861000546, 10.884343025517001, 11.150020465940667, 11.4229575575916  
 16, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.7853313  
 05110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.10  
 4094729551703, 14.576777677378463, 14.84153774101085, 15.110079416390185,  
 15.396739882176563, 15.770212688251195, 15.559628405360055, 13.55065848987  
 3884, 2.925105347856241, 1.015021705776282, 0.8632611638122093, 1.13613272  
 33730847, 1.443538910619455, 1.9164104398125108, 2.182074051053499, 2.4549  
 448831095444, 2.762347584670865, 3.235202412943501, 3.5007860052674404, 3.  
 773273443658663, 4.078839195812915, 4.5428926707150294, 4.766306445593437,  
 4.836746150127519, 4.1742430504416]

Exact Solution: [ 4.8172978 6.08648901 6.61514726 6.98924727 7.33108  
 911 7.66619827

7.99990224 8.33331293 8.66666241 8.99999911 9.33333315 9.66666663  
 9.99999999 10.33333333 10.66666667 11. 11.33333333 11.66666667  
 12. 12.33333333 12.66666665 12.99999991 13.33333288 13.6666645  
 13.99998964 14.33328368 14.66642879 14.99886024 15.32787243 15.6405019  
 15.87463709 15.73268354 13.78878059 3.21121941 1.26731646 1.12536291  
 1.35949809 1.67212756 2.00113972 2.33357102 2.66671537 3.00000585  
 3.33331389 3.66656361 3.99950414 4.33095711 4.65528143 4.94545004  
 5.07196876 4.41439375]

Residual: 4.446061221610798

Solution for n=50, k=19: [5.038277150156952, 6.191385750784758, 6.7186516  
 0376684, 7.193538934716111, 7.466434374161547, 7.773841647525378, 8.246713  
 257404735, 8.512377023085929, 8.785248584367613, 9.09265477937588, 9.56552  
 6345736712, 9.831190135058236, 10.104061820349346, 10.41146861000546, 10.8  
 84343025517001, 11.150020465940667, 11.422957557591616, 11.73067772794477  
 3, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250  
 815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576  
 777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176586, 1  
 5.770212688251307, 15.559628405360595, 13.550658489876476, 2.9251053478686  
 657, 1.015021705835812, 0.8632611640974336, 1.1361327247396764, 1.44353891  
 71671886, 1.916410471184587, 2.182074201366147, 2.4549456033007093, 2.7623  
 510353140412, 3.235218945968218, 3.500865219747849, 3.7736529830359857, 4.

080657678219123, 4.55160554336875, 4.8080523264558295, 5.036762681785758, 5.1325798278704, 4.591667355485759]

Exact Solution: [ 5.08106042 6.40530209 6.94545004 7.32194809 7.66429043 7.99950405

8.33322982 8.66664506 8.99999549 9.33333239 9.66666647 9.99999996  
10.33333332 10.66666666 11. 11.33333333 11.66666667 12.  
12.33333333 12.66666665 12.99999991 13.33333288 13.6666645 13.99998964  
14.33328368 14.66642879 14.99886024 15.32787243 15.6405019 15.87463709  
15.73268354 13.78878059 3.21121941 1.26731646 1.12536291 1.3594981  
1.67212757 2.00113975 2.33357117 2.66671611 3.00000937 3.33333074  
3.66664431 3.99989083 4.33280985 4.66415842 4.98798225 5.27575282  
5.39078184 4.67815637]

Residual: 4.497725847057182

Solution for n=50, k=20: [5.341374575763903, 6.706872878819516, 7.192989818333679, 7.466409546182214, 7.773840521273048, 8.246713205373586, 8.5123770207464, 8.785248584262431, 9.09265477937114, 9.565526345736496, 9.831190135058227, 10.104061820349346, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251332, 15.559628405360712, 13.550658489877042, 2.9251053478713764, 1.0150217058488011, 0.8632611641596692, 1.1361327250378654, 1.443538918595898, 1.916410478029945, 2.182074234164227, 2.454945760445751, 2.762351788241168, 3.2352225534588115, 3.5008825042736893, 3.7737357981745943, 4.081054469386324, 4.55350668406615, 4.817161238775627, 5.080406102687348, 5.341688020058557, 5.593564895524959, 4.80037950800784]

Exact Solution: [ 5.34482303 6.72411517 7.27575282 7.65464891 7.99749175 8.33280983

8.66655741 8.9999772 9.33332857 9.66666567 9.99999979 10.33333329  
10.66666666 11. 11.33333333 11.66666667 12. 12.33333333  
12.66666665 12.99999991 13.33333288 13.6666645 13.99998964 14.33328368  
14.66642879 14.99886024 15.32787243 15.6405019 15.87463709 15.73268354  
13.78878059 3.21121941 1.26731646 1.12536291 1.3594981 1.67212757  
2.00113976 2.3335712 2.66671627 3.00001014 3.33333445 3.66666211  
3.99997609 4.33321834 4.66611562 4.99735974 5.32068307 5.6060556  
5.70959492 4.94191898]

Residual: 4.394137227516732

Solution for n=50, k=21: [5.594409982607479, 6.972049913037394, 7.46583958257949, 7.773814666526725, 8.246712010923703, 8.512376967039158, 8.78524858184784, 9.09265477926226, 9.565526345731508, 9.831190135058003, 10.104061820349337, 10.41146861000546, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.55962840536074, 13.550658489877163, 2.925105347871967, 1.01502170585163, 0.8632611641732234, 1.136132725102807, 1.4435389189070518, 1.916410479520772, 2.182074241307209, 2.4549457946698343, 2.762351952218603, 3.235223391219003, 3.500886268611699, 3.7737538342015586, 4.081140885183134, 4.553920727023231, 4.819145037764218, 5.089911054673225, 5.387228980999348, 5.8117647482430375, 5.8458378106574385, 5.009091660529919]

Exact Solution: [ 5.60858565 7.04292825 7.6060556 7.98734973 8.33069307 8.66611561

8.99988499 9.33330933 9.66666166 9.99999895 10.33333312 10.66666662  
10.99999999 11.33333333 11.66666667 12. 12.33333333 12.66666665  
12.99999991 13.33333288 13.6666645 13.99998964 14.33328368 14.66642879  
14.99886024 15.32787243 15.6405019 15.87463709 15.73268354 13.78878059

```

3.21121941 1.26731646 1.12536291 1.3594981 1.67212757 2.00113976
2.33357121 2.66671631 3.00001031 3.33333527 3.66666602 3.99999483
4.33330815 4.66654591 4.99942139 5.33056106 5.65338389 5.93635838
6.028408 5.2056816 ]
Residual: 4.440220719455894

```

Solution for n=50, k=22: [5.84888421383134, 7.244421069156703, 7.773221131952172, 8.246684590604163, 8.512375734112117, 8.785248526417408, 9.092654776762798, 9.565526345616993, 9.831190135052879, 10.104061820349107, 10.41146861000545, 10.884343025517001, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877193, 2.9251053478721003, 1.0150217058522697, 0.863261164176288, 1.1361327251174904, 1.443538918977405, 1.9164104798578554, 2.182074242922272, 2.454945802408065, 2.762351989294693, 3.235223516764121, 3.500887119746712, 3.7737579122344034, 4.081160424212345, 4.55401434413644, 4.819593584301056, 5.092060170244199, 5.397526012317381, 5.861100789262227, 6.082220984435357, 6.141671488400319, 5.42651596557408]

Exact Solution: [ 5.87234827 7.36174133 7.93635838 8.32005055 8.66389439 8.99942139

```

9.33321257 9.66664146 9.99999474 10.33333224 10.66666644 10.99999995
11.33333332 11.66666666 12. 12.33333333 12.66666665 12.99999991
13.33333288 13.66666645 13.99998964 14.33328368 14.66642879 14.99886024
15.32787243 15.6405019 15.87463709 15.73268354 13.78878059 3.21121941
1.26731646 1.12536291 1.3594981 1.67212757 2.00113976 2.33357121
2.66671631 3.00001035 3.33333545 3.66666688 3.99999894 4.33332784
4.66664027 4.99987349 5.33272717 5.66376238 5.98608471 6.26666116
6.34722107 5.46944421]

```

Residual: 4.397855344275549

Solution for n=50, k=23: [6.101556644323137, 7.50778322161568, 8.037359463755266, 8.512347430493984, 8.785247253932047, 9.092654719384035, 9.565526342988123, 9.831190134935184, 10.10406182034384, 10.411468610005214, 10.884343025516992, 11.150020465940667, 11.422957557591616, 11.730677727944773, 12.205053640581696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721296, 1.0150217058524085, 0.8632611641769522, 1.1361327251206732, 1.4435389189926535, 1.9164104799309145, 2.1820742432723197, 2.4549458040852454, 2.7623519973305477, 3.2352235552662147, 3.500887304221327, 3.7737587961053785, 4.0811646590926065, 4.554034634666774, 4.819690802072466, 5.092525968570914, 5.3997577861795465, 5.871793860246338, 6.1334545654937465, 6.387146322708157, 6.602656556054878, 5.635228118096159]

Exact Solution: [ 6.13611088 7.68055441 8.26666116 8.65275137 8.99709571 9.33272717

```

9.66654015 9.9999736 10.33332782 10.66666552 10.99999976 11.33333328
11.66666666 12. 12.33333333 12.66666665 12.99999991 13.33333288
13.66666645 13.99998964 14.33328368 14.66642879 14.99886024 15.32787243
15.6405019 15.87463709 15.73268354 13.78878059 3.21121941 1.26731646
1.12536291 1.3594981 1.67212757 2.00113976 2.33357121 2.66671631
3.00001036 3.33333549 3.66666707 3.99999984 4.33333215 4.66666091
4.9999724 5.33320107 5.66603295 5.9969637 6.31878553 6.59696394
6.66603415 5.73320683]

```

Residual: 4.457714093973955

Solution for n=50, k=24: [6.362987403265469, 7.814937016327345, 8.511697678371254, 8.785218042195591, 9.092653402171925, 9.565526282638622, 9.83119

0132233301, 10.104061820222874, 10.411468609999785, 10.884343025516742, 1  
 1.150020465940656, 11.422957557591616, 11.730677727944773, 12.205053640581  
 696, 12.477925185935227, 12.785331305110121, 13.25820250815315, 13.5238645  
 56715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.8  
 4153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338,  
 15.559628405360746, 13.550658489877199, 2.925105347872136, 1.0150217058524  
 384, 0.863261164177096, 1.1361327251213618, 1.4435389189959538, 1.91641047  
 99467276, 2.182074243348085, 2.454945804448258, 2.762351999069846, 3.23522  
 3563599694, 3.5008873441494237, 3.773758987412385, 4.0811655756995435, 4.5  
 54039026394453, 4.8197118441039235, 5.092626787000524, 5.40024083629614,  
 5.874108292399695, 6.144543676143936, 6.440277443805746, 6.85722305089263  
 7, 6.854929471187359, 5.84394027061824]

Exact Solution: [ 6.3998735 7.99936749 8.59696394 8.98545219 9.33029  
 703 9.66603295

9.99986774 10.33330573 10.66666091 10.9999988 11.33333308 11.66666661  
 11.99999999 12.33333333 12.66666665 12.99999991 13.33333288 13.66666645  
 13.99998964 14.33328368 14.66642879 14.99886024 15.32787243 15.6405019  
 15.87463709 15.73268354 13.78878059 3.21121941 1.26731646 1.12536291  
 1.3594981 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036  
 3.33333549 3.66666711 4.00000004 4.33333309 4.66666542 4.99999399  
 5.33330453 5.66652865 5.99933873 6.33016502 6.65148635 6.92726672  
 6.98484723 5.99696945]

Residual: 4.448959722757915

Solution for n=50, k=25: [6.6576894767855235, 8.288447383927615, 8.784547  
 442852551, 9.092623163668474, 9.565524897228956, 9.831190070207704, 10.104  
 061817445928, 10.411468609875124, 10.884343025511068, 11.1500204659404, 1  
 1.422957557591605, 11.730677727944773, 12.205053640581696, 12.477925185935  
 227, 12.785331305110121, 13.25820250815315, 13.523864556715472, 13.7967279  
 01528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11  
 007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746,  
 13.550658489877199, 2.9251053478721367, 1.0150217058524449, 0.863261164177  
 1272, 1.1361327251215108, 1.4435389189966674, 1.916410479950146, 2.1820742  
 43364463, 2.454945804526729, 2.7623519994458237, 3.23522356540111, 3.50088  
 73527805267, 3.7737590287664857, 4.0811657738389435, 4.554039975737353, 4.  
 819716392679019, 5.092648580533106, 5.40034525538395, 5.874608594306166,  
 6.146940766588483, 6.451762594122013, 6.912251712029427, 7.11858762655503  
 7, 7.107202386319838, 6.052652423140319]

Exact Solution: [ 6.66363611 8.31818057 8.92726672 9.31815301 9.66349  
 835 9.99933873

10.33319532 10.66663786 10.99999399 11.33333208 11.6666664 11.99999994  
 12.33333332 12.66666664 12.99999991 13.33333288 13.6666645 13.99998964  
 14.33328368 14.66642879 14.99886024 15.32787243 15.6405019 15.87463709  
 15.73268354 13.78878059 3.21121941 1.26731646 1.12536291 1.3594981  
 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036 3.3333355  
 3.66666712 4.00000008 4.3333333 4.6666664 4.99999869 5.33332707  
 5.66663666 5.99985624 6.33264451 6.66336634 6.98418717 7.2575695  
 7.30366031 6.26073206]

Residual: 4.428639015720036

Solution for n=50, k=26: [6.912163708056397, 8.560818540281984, 9.0919289  
 93353516, 9.56549309315227, 9.83118864632088, 10.10406175369714, 10.411468  
 607013306, 10.884343025380769, 11.15002046593459, 11.422957557591344, 11.7  
 30677727944762, 12.205053640581696, 12.477925185935227, 12.78533130511012  
 1, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729  
 551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.39673  
 9882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.  
 925105347872137, 1.0150217058524462, 0.863261164177134, 1.136132725121544,  
 1.4435389189968264, 1.9164104799509085, 2.182074243368117, 2.4549458045442  
 365, 2.7623519995297063, 3.235223565803017, 3.500887354706179, 3.773759037



992838, 4.081165818045051, 4.554040187541538, 4.819717407493842, 5.0926534  
428030354, 5.4003685519187785, 5.874720214710377, 6.147475572074712, 6.454  
325001148944, 6.92452894167785, 7.177411367770225, 7.389043862747356, 7.40  
3036064062719, 6.4700767281844795]

Exact Solution: [ 6.92739873 8.63699364 9.2575695 9.65085383 9.99669  
967 10.33264451

10.6665229 10.99996999 11.33332707 11.66666536 11.99999973 12.33333327  
12.66666664 12.99999999 13.33333288 13.6666645 13.99998964 14.33328368  
14.66642879 14.99886024 15.32787243 15.6405019 15.87463709 15.73268354  
13.78878059 3.21121941 1.26731646 1.12536291 1.3594981 1.67212757  
2.00113976 2.33357121 2.66671632 3.00001036 3.3333355 3.66666712  
4.00000009 4.33333334 4.66666661 4.99999972 5.33333197 5.66666015  
5.99996879 6.33318382 6.66595029 6.99656766 7.31688799 7.58787228  
7.62247339 6.52449468]

Residual: 4.370717204811605

Solution for n=50, k=27: [7.16483613877344, 8.824180693867197, 9.35606733  
0562542, 9.831155958945512, 10.104060290251976, 10.411468541316184, 10.884  
343022389542, 11.15002046580111, 11.42295755758539, 11.730677727944494, 1  
2.205053640581685, 12.477925185935227, 12.785331305110121, 13.258202508153  
15, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.5767776  
77378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770  
212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376,  
1.0150217058524467, 0.8632611641771355, 1.1361327251215512, 1.443538918996  
8606, 1.9164104799510728, 2.182074243368904, 2.4549458045480086, 2.7623519  
995477785, 3.235223565889606, 3.5008873551210504, 3.7737590399806074, 4.08  
1165827569028, 4.55404023317365, 4.819717626130422, 5.092654490353821, 5.4  
00373571036126, 5.874744262746329, 6.147590793137124, 6.45487705842505, 6.  
92717400699597, 7.190084637084727, 7.449765144001743, 7.693969201020156,  
7.864021131717279, 6.678788880706559]

Exact Solution: [ 7.19116134 8.95580672 9.58787228 9.98355465 10.32990  
099 10.66595029

10.99985048 11.33330213 11.66666015 11.99999864 12.33333305 12.66666659  
12.99999989 13.33333288 13.6666645 13.99998964 14.33328368 14.66642879  
14.99886024 15.32787243 15.6405019 15.87463709 15.73268354 13.78878059  
3.21121941 1.26731646 1.12536291 1.3594981 1.67212757 2.00113976  
2.33357121 2.66671632 3.00001036 3.3333355 3.66666712 4.00000009  
4.33333335 4.66666666 4.99999994 5.33333304 5.66666526 5.99999324  
6.33330093 6.6665114 6.99925608 7.32976898 7.64958881 7.91817506  
7.94128647 6.78825729]

Residual: 4.425490261823382

Solution for n=50, k=28: [7.426266898794995, 9.131334493974979, 9.8304055  
710799, 10.104026694757858, 10.41146703314418, 10.884342953721665, 11.1500  
20462736869, 11.422957557448639, 11.730677727938378, 12.205053640581404, 1  
2.477925185935215, 12.785331305110121, 13.25820250815315, 13.5238645567154  
72, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.8415377  
4101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559  
628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467,  
0.8632611641771358, 1.1361327251215527, 1.4435389189968681, 1.916410479951  
1084, 2.182074243369074, 2.4549458045488204, 2.7623519995516683, 3.2352235  
659082424, 3.5008873552103448, 3.773759040408444, 4.081165829618914, 4.554  
040242995249, 4.819717673188536, 5.0926547158227935, 5.400374651322873, 5.  
874749438711095, 6.1476155926741995, 6.454995880145662, 6.927743316061957,  
7.192812360694039, 7.462834452982324, 7.756588022313743, 8.16404592920463  
7, 8.116294046849758, 6.88750103322864]

Exact Solution: [ 7.45492396 9.2746198 9.91817506 10.31625547 10.66310  
231 10.99925608

11.33317807 11.66663426 11.99999324 12.33333192 12.66666635 12.99999984  
13.33333287 13.6666645 13.99998964 14.33328368 14.66642879 14.99886024

```

15.32787243 15.6405019 15.87463709 15.73268354 13.78878059 3.21121941
1.26731646 1.12536291 1.3594981 1.67212757 2.00113976 2.33357121
2.66671632 3.00001036 3.3333355 3.66666712 4.00000009 4.33333335
4.66666667 4.99999999 5.33333327 5.66666636 5.99999854 6.33332632
6.66663306 6.99983898 7.33256186 7.6629703 7.98228963 8.24847784
8.26009955 7.05201991]
Residual: 4.403807216321526

```

Solution for n=50, k=29: [7.6793023108192555, 9.396511554096277, 10.103255459662126, 10.411432410881021, 10.88434137735168, 11.150020392392806, 11.422957554309315, 11.730677727797927, 12.205053640575043, 12.47792518593493, 12.78533130511011, 13.25820250815315, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968697, 1.9164104799511157, 2.18207424336911, 2.454945804548995, 2.762351999552505, 3.2352235659122504, 3.5008873552295467, 3.7737590405004444, 4.081165830059716, 4.554040245107255, 4.819717683307761, 5.0926547643069116, 5.400374883624239, 5.874750551733803, 6.147620925486381, 6.455021431183861, 6.927865738440768, 7.193398921549903, 7.465644834882829, 7.770053370960403, 8.228562290537424, 8.425410504867036, 8.36856696198224, 7.096213185750719]

Exact Solution: [ 7.71868658 9.59343288 10.24847784 10.64895629 10.99630363 11.33256186

```

11.66650565 11.99996639 12.33332632 12.66666518 12.9999996 13.33333282
13.66666449 13.99998963 14.33328368 14.66642879 14.99886024 15.32787243
15.6405019 15.87463709 15.73268354 13.78878059 3.21121941 1.26731646
1.12536291 1.3594981 1.67212757 2.00113976 2.33357121 2.66671632
3.00001036 3.3333355 3.66666712 4.00000009 4.33333335 4.66666667
5. 5.33333332 5.66666666 5.99999968 6.33333182 6.6666594
6.99996519 7.33316657 7.66586764 7.99617162 8.31499045 8.57878061
8.57891263 7.31578253]

```

Residual: 4.474066208230061

Solution for n=50, k=30: [7.975443233531931, 9.877216167659657, 10.410637604766357, 10.884305189505472, 11.150018777543615, 11.422957482241634, 11.730677724573644, 12.205053640428936, 12.477925185928425, 12.78533130510982, 13.258202508153138, 13.523864556715472, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.9164104799511177, 2.1820742433691183, 2.4549458045490336, 2.7623519995526897, 3.235223565913136, 3.50088735523379, 3.7737590405207757, 4.081165830157128, 4.554040245573986, 4.819717685544003, 5.092654775021385, 5.400374934960364, 5.874750797699956, 6.147622103981014, 6.455027077690877, 6.927892792481214, 7.193528545245116, 7.466265899318449, 7.773029069443289, 8.242819718516238, 8.493721946278223, 8.695866741059355, 8.664400639725118, 7.5136374907948795]

Exact Solution: [ 7.98244919 9.91224596 10.57878061 10.98165711 11.32950495 11.66586764

```

11.99983323 12.33329852 12.66665938 12.99999839 13.33333257 13.66666444
13.99998962 14.33328368 14.66642878 14.99886024 15.32787243 15.6405019
15.87463709 15.73268354 13.78878059 3.21121941 1.26731646 1.12536291
1.3594981 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036
3.3333355 3.66666712 4.00000009 4.33333335 4.66666667 5.
5.33333333 5.66666665 5.99999993 6.33333301 6.6666651 6.99999248
7.33329733 7.66649415 7.99917342 8.32937294 8.64769127 8.90908339
8.89772571 7.57954514]

```

Residual: 4.351698301267146

Solution for n=50, k=31: [8.228115782953783, 10.140578914768914, 10.674778790890791, 11.149981706351712, 11.422955827824289, 11.73067765055558, 12.205053637074817, 12.477925185779048, 12.785331305103153, 13.258202508152836, 13.52386455671546, 13.796727901528218, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.9164104799511177, 2.1820742433691196, 2.4549458045490415, 2.7623519995527293, 3.2352235659133255, 3.5008873552347, 3.773759040525136, 4.081165830178024, 4.554040245674104, 4.819717686023697, 5.092654777319743, 5.4003749459724615, 5.874750850462084, 6.147622356779559, 6.455028288921473, 6.927898595835647, 7.193556350786686, 7.466399123671862, 7.773667385668788, 8.245878075290317, 8.508375413923115, 8.766075722509742, 9.000792079332156, 9.125385707379678, 7.722349643316959]

Exact Solution: [ 8.24621181 10.23105904 10.90908339 11.31435793 11.66270627 11.99917342

12.33316081 12.66663064 12.99999239 13.33333131 13.66666418 13.99998957  
 14.33328367 14.66642878 14.99886024 15.32787243 15.6405019 15.87463709  
 15.73268354 13.78878059 3.21121941 1.26731646 1.12536291 1.3594981  
 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036 3.3333355  
 3.66666712 4.00000009 4.33333335 4.66666667 5. 5.33333333  
 5.66666666 5.99999999 6.33333326 6.66666633 6.99999838 7.33332557  
 7.66662946 7.99982173 8.3324792 8.66257426 8.98039209 9.23938617  
 9.21653879 7.84330776]

Residual: 4.401225413102068

Solution for n=50, k=32: [8.489547111724246, 10.447735558621234, 11.149130681381925, 11.42291784828839, 11.730675951364384, 12.205053560076184, 12.477925182349866, 12.785331304950079, 13.258202508145924, 13.52386455671515, 13.796727901528206, 14.104094729551703, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.18207424336912, 2.4549458045490433, 2.762351999552738, 3.235223565913367, 3.500887355234896, 3.7737590405260715, 4.081165830182503, 4.554040245695564, 4.819717686126521, 5.092654777812403, 5.400374948332938, 5.874750861771806, 6.147622410967691, 6.455028548552411, 6.927899839802205, 7.193562310988535, 7.466427680714555, 7.773804210680402, 8.246533643305693, 8.511516428988385, 8.781125229820713, 9.072898600821741, 9.470868807516636, 9.37765862251216, 7.93106179583904]

Exact Solution: [ 8.50997442 10.54987212 11.23938617 11.64705875 11.99590759 12.33247919

12.66648838 12.9999627 13.33332512 13.66666288 13.9999893 14.33328361  
 14.66642877 14.99886024 15.32787243 15.6405019 15.87463709 15.73268354  
 13.78878059 3.21121941 1.26731646 1.12536291 1.3594981 1.67212757  
 2.00113976 2.33357121 2.66671632 3.00001036 3.3333355 3.66666712  
 4.00000009 4.33333335 4.66666667 5. 5.33333333 5.66666667  
 6. 6.33333332 6.66666659 6.99999965 7.33333166 7.66665865  
 7.99996159 8.33314931 8.66578498 8.99577558 9.31309291 9.56968895  
 9.53535187 8.10707037]

Residual: 4.366447680681222

Solution for n=50, k=33: [8.742585248788235, 10.712926243941169, 11.42204597091761, 11.73063694398022, 12.205051792461761, 12.477925103628042, 12.785331301436026, 13.258202507987214, 13.523864556708102, 13.796727901527891, 14.10409472955169, 14.576777677378463, 14.84153774101085, 15.11007941639019, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.76235199955274, 3.2352235659133752, 3.50088735523

4937, 3.7737590405262718, 4.081165830183463, 4.554040245700161, 4.81971768  
6148547, 5.092654777917933, 5.400374948838563, 5.874750864194399, 6.147622  
422575036, 6.455028604166547, 6.927900106265535, 7.193563587691055, 7.4664  
33797763822, 7.773833519224218, 8.246674068975507, 8.51218924879364, 8.784  
348903177174, 9.08834414779879, 9.54487286904542, 9.732233383179034, 9.629  
931537644637, 8.13977394836112]

Exact Solution: [ 8.77373704 10.8686852 11.56968895 11.97975957 12.32910  
891 12.66578496

12.99981589 13.33329447 13.66665649 13.99998796 14.33328334 14.66642871  
14.99886023 15.32787243 15.6405019 15.87463709 15.73268354 13.78878059  
3.21121941 1.26731646 1.12536291 1.3594981 1.67212757 2.00113976  
2.33357121 2.66671632 3.00001036 3.3333355 3.66666712 4.00000009  
4.33333335 4.66666667 5. 5.33333333 5.66666667 6.  
6.33333333 6.66666665 6.99999992 7.33333297 7.66666494 7.99999173  
8.33329373 8.6664769 8.99909076 9.3289769 9.64579373 9.89999173  
9.85416494 8.37083299]

Residual: 4.431854550336578

Solution for n=50, k=34: [8.997072561283975, 10.985362806419872, 11.72974  
147081538, 12.2050112143237, 12.4779232964553, 12.78533122076587, 13.25820  
2504343751, 13.52386455654621, 13.796727901520697, 14.104094729551369, 14.  
576777677378448, 14.84153774101085, 15.11007941639019, 15.396739882176592,  
15.770212688251338, 15.559628405360746, 13.550658489877199, 2.925105347872  
1376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.443538  
9189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7  
6235199955274, 3.235223565913377, 3.500887355234946, 3.7737590405263153,  
4.081165830183673, 4.55404024570117, 4.819717686153379, 5.092654777941089,  
5.400374948949504, 5.874750864725954, 6.147622425121867, 6.45502861636914  
1, 6.927900164731678, 7.193563867819173, 7.466435139938266, 7.773839949968  
319, 8.24670488052157, 8.512336875779853, 8.785056226562185, 9.09173313773  
7634, 9.561110495354626, 9.81003252478622, 10.002689619371354, 9.925765215  
387518, 8.557198253405279]

Exact Solution: [ 9.03749966 11.18749828 11.89999173 12.31246039 12.66231  
021 12.99909066

13.33314311 13.6666249 13.99998137 14.33328196 14.66642843 14.99886017  
15.32787241 15.6405019 15.87463709 15.73268354 13.78878059 3.21121941  
1.26731646 1.12536291 1.3594981 1.67212757 2.00113976 2.33357121  
2.66671632 3.00001036 3.3333355 3.66666712 4.00000009 4.33333335  
4.66666667 5. 5.33333333 5.66666667 6. 6.33333333  
6.66666666 6.99999998 7.33333326 7.6666663 7.99999822 8.33332482  
8.66662586 8.99980448 9.33239654 9.66217822 9.97849455 10.23029451  
10.17297802 8.6345956 ]

Residual: 4.37885398238645

Solution for n=50, k=35: [9.29147433458144, 11.457371672907202, 11.995384  
029954568, 12.477881810198975, 12.785329368866392, 13.258202420702863, 13.  
523864552829737, 13.796727901355537, 14.104094729544016, 14.57677767737811  
6, 14.841537741010836, 15.11007941639019, 15.396739882176592, 15.770212688  
251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150  
217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701,  
1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527  
404, 3.2352235659133775, 3.500887355234948, 3.773759040526325, 4.081165830  
183719, 4.554040245701386, 4.819717686154412, 5.092654777946039, 5.4003749  
48973224, 5.874750864839601, 6.147622425666386, 6.455028618978089, 6.92790  
0177231897, 7.193563927711319, 7.466435426898776, 7.773841324878725, 8.246  
711468113089, 8.51236843882704, 8.7852074542066, 9.092457712912523, 9.5645  
8214358466, 9.826666190761502, 10.082386301017738, 10.307614957644155, 10.  
386750283042078, 8.76591040592736]

Exact Solution: [ 9.30126227 11.50631136 12.23029451 12.64516119 12.99551  
146 13.33239609

13.66646898 13.99994883 14.33327517 14.66642701 14.99885987 15.32787235  
 15.64050189 15.87463708 15.73268353 13.78878059 3.21121941 1.26731646  
 1.12536291 1.3594981 1.67212757 2.00113976 2.33357121 2.66671632  
 3.00001036 3.3333355 3.66666712 4.00000009 4.33333335 4.66666667  
 5. 5.33333333 5.66666667 6. 6.33333333 6.66666667  
 7. 7.33333332 7.66666659 7.99999962 8.3333315 8.6666579  
 8.99995799 9.33313206 9.66570232 9.99537954 10.31119537 10.56059729  
 10.4917911 8.89835822]  
 Residual: 4.385052013213822

Solution for n=50, k=36: [9.553205392949362, 11.766026964746812, 12.476929430784695, 12.785286855843331, 13.258200500605884, 13.523864467512768, 13.796727897564004, 14.104094729375156, 14.576777677370517, 14.841537741010498, 15.110079416390176, 15.396739882176592, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.0811658301837275, 4.554040245701432, 4.819717686154633, 5.0926547779470965, 5.400374948978292, 5.874750864863886, 6.147622425782739, 6.455028619535572, 6.927900179902961, 7.1935639405091605, 7.466435488216924, 7.773841618671628, 8.246712875759457, 8.512375183265977, 8.785239768754911, 9.09261254121514, 9.56532397054943, 9.83022049728273, 10.099416006659101, 10.38920917932974, 10.777691685828634, 10.639023198174558, 8.97462255844944]

Exact Solution: [ 9.56502489 11.82512443 12.56059727 12.97786194 13.32871242 13.66570016

13.99978837 14.33324168 14.66642002 14.99885841 15.32787205 15.64050182  
 15.87463707 15.73268353 13.78878059 3.21121941 1.26731646 1.12536291  
 1.3594981 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036  
 3.3333355 3.66666712 4.00000009 4.33333335 4.66666667 5.  
 5.33333333 5.66666667 6. 6.33333333 6.66666667 7.  
 7.33333333 7.66666665 7.99999992 8.33333294 8.66666478 8.99999098  
 9.33329013 9.66645965 9.9990081 10.32858086 10.64389619 10.89090007  
 10.81060418 9.16212084]

Residual: 4.337090383752677

Solution for n=50, k=37: [9.807679621060965, 12.03839810530482, 12.784310905463132, 13.258156422010838, 13.523862508938887, 13.796727810523896, 14.104094725498763, 14.576777677196025, 14.841537741002762, 15.110079416389832, 15.396739882176577, 15.770212688251338, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701442, 4.819717686154681, 5.092654777947323, 5.400374948979374, 5.874750864869073, 6.147622425807589, 6.455028619654632, 6.9279001804734115, 7.193563943242344, 7.466435501312393, 7.773841681415781, 8.246713176384754, 8.512376623648306, 8.785246670041264, 9.092645607264574, 9.565482399510246, 9.830979576037377, 10.103052971471518, 10.406634924637178, 10.861183447553417, 11.039056261491034, 10.891296113307039, 9.183334710971518]

Exact Solution: [ 9.8287875 12.1439375 12.89089998 13.3105624 13.66191203 13.99899774

14.33307666 14.66638558 14.99885122 15.32787055 15.64050151 15.87463701  
 15.73268352 13.78878059 3.21121941 1.26731646 1.12536291 1.3594981  
 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036 3.3333355  
 3.66666712 4.00000009 4.33333335 4.66666667 5. 5.33333333  
 5.66666667 6. 6.33333333 6.66666667 7. 7.33333333  
 7.66666666 7.99999998 8.33333325 8.66666626 8.99999807 9.33332406  
 9.66662226 9.99978723 10.33231388 10.66178218 10.97659701 11.22120285  
 11.12941726 9.42588345]

Residual: 4.397617734790607

Solution for n=50, k=38: [10.060352036641024, 12.301760183205122, 13.048448879384587, 13.523817547051156, 13.796725812392928, 14.104094636510581, 14.576777673190279, 14.841537740825098, 15.110079416381957, 15.396739882176227, 15.770212688251323, 15.559628405360746, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701444, 4.819717686154691, 5.092654777947372, 5.400374948979611, 5.8747508648702045, 6.147622425813011, 6.4550286196806095, 6.927900180597883, 7.193563943838722, 7.466435504169811, 7.773841695106497, 8.246713241980913, 8.51237693793839, 8.785248175895518, 9.092652822245759, 9.565516968561926, 9.831145206314593, 10.103846553805921, 10.410437206031979, 10.879401272193014, 11.126343103294218, 11.309512497683354, 11.187129791049918, 9.60075901601568]

Exact Solution: [10.0925501 12.4627505 13.2212024 13.64326151 13.99510515 14.33226423

14.66621601	14.99881583	15.32786316	15.64049997	15.87463668	15.73268345
13.78878057	3.21121941	1.26731646	1.12536291	1.3594981	1.67212757
2.00113976	2.33357121	2.66671632	3.00001036	3.3333355	3.66666712
4.00000009	4.33333335	4.66666667	5.	5.33333333	5.66666667
6.	6.33333333	6.66666667	7.	7.33333333	7.66666667
8.	8.33333332	8.66666658	8.99999959	9.33333135	9.66665715
9.99995439	10.33311481	10.66561966	10.9949835	11.30929783	11.55150563
11.44823034	9.68964607]				

Residual: 4.430953446777571

Solution for n=50, k=39: [10.31308707196303, 12.565435359815153, 13.31408972711273, 13.796679942415171, 14.104092593658772, 14.57677758123259, 14.841537736746597, 15.11007941620108, 15.396739882168202, 15.770212688250963, 15.559628405360732, 13.550658489877199, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154693, 5.0926547779473825, 5.400374948979661, 5.874750864870445, 6.147622425814166, 6.45502861968615, 6.927900180624425, 7.1935639439659, 7.466435504779154, 7.773841698026033, 8.246713255969253, 8.512377004960554, 8.785248497017994, 9.092654360835976, 9.56552434039053, 9.831180526867396, 10.104015784741339, 10.411248040156261, 10.883286211879007, 11.144956967599892, 11.398696879525737, 11.614437835956153, 11.648114858704478, 9.809471168537758]

Exact Solution: [10.35631264 12.78156322 13.55150347 13.97595413 14.32826718 14.66538178

14.99864172	15.32782682	15.64049238	15.8746351	15.73268312	13.7887805
3.21121939	1.26731646	1.12536291	1.3594981	1.67212757	2.00113976
2.33357121	2.66671632	3.00001036	3.3333355	3.66666712	4.00000009
4.33333335	4.66666667	5.	5.33333333	5.66666667	6.
6.33333333	6.66666667	7.	7.33333333	7.66666667	8.
8.33333333	8.66666665	8.99999991	9.33333291	9.66666463	9.99999023
10.33328652	10.66644239	10.99892544	11.32818482	11.64199865	11.88180841
11.76704342	9.95340868]				

Residual: 4.5037181066873

Solution for n=50, k=40: [10.616484455336664, 13.082422276683316, 13.795626928079912, 14.104045697049589, 14.576775470211523, 14.841537643118722, 15.110079412048744, 15.396739881983883, 15.770212688242745, 15.559628405360376, 13.550658489877186, 2.9251053478721376, 1.0150217058524467, 0.8632611641771358, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378,

3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.55404024570144  
 5, 4.819717686154694, 5.092654777947384, 5.400374948979672, 5.874750864870  
 497, 6.147622425814413, 6.455028619687331, 6.927900180630083, 7.1935639439  
 930075, 7.466435504909037, 7.77384169864834, 8.246713258950907, 8.51237701  
 9246518, 8.78524856546616, 9.092654688790844, 9.565525911716708, 9.8311880  
 55543418, 10.104051856795268, 10.411420871749879, 10.884114297793166, 11.1  
 48924565577076, 11.41770678349749, 11.705519757837736, 12.084514564140633,  
 11.900387773836957, 10.018183321059839]

Exact Solution: [10.62007492 13.10037459 13.88179805 14.30861566 14.66128  
 027 14.99778569

15.32764816	15.64045509	15.87462732	15.7326815	13.78878016	3.21121932
1.26731645	1.12536291	1.3594981	1.67212757	2.00113976	2.33357121
2.66671632	3.00001036	3.3333355	3.66666712	4.00000009	4.33333335
4.66666667	5.	5.33333333	5.66666667	6.	6.33333333
6.66666667	7.	7.33333333	7.66666667	8.	8.33333333
8.66666666	8.99999998	9.33333324	9.66666623	9.99999791	10.33332331
10.66661866	10.99976998	11.33223122	11.66138614	11.97469947	12.21211119
12.0858565	10.2171713	]			

Residual: 4.4187040623957605

Solution for n=50, k=41: [10.87095704645228, 13.35478523226139, 14.102969  
 114854679, 14.57672700867867, 14.841535493756064, 15.110079316725972, 15.3  
 96739877752584, 15.770212688054022, 15.559628405352278, 13.55065848987688,  
 2.925105347872135, 1.0150217058524467, 0.8632611641771358, 1.1361327251215  
 532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.45494580  
 45490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737  
 590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.09  
 2654777947385, 5.400374948979674, 5.874750864870508, 6.147622425814467, 6.  
 455028619687587, 6.927900180631314, 7.1935639439989005, 7.466435504937275,  
 7.773841698783636, 8.246713259599145, 8.512377022352407, 8.78524858034737  
 3, 9.092654760091019, 9.565526253336364, 9.831189692341521, 10.10405969916  
 6122, 10.411458446806053, 10.884294330703185, 11.14978715507099, 11.421839  
 698057054, 11.725321741141645, 12.179391566100614, 12.354970800332953, 12.  
 196221451579838, 10.435607626104]

Exact Solution: [10.8838359 13.41917949 14.21206155 14.64112825 14.99357  
 971 15.32677032

15.64027188	15.87458908	15.73267352	13.7887785	3.21121897	1.26731637
1.12536289	1.35949809	1.67212757	2.00113976	2.33357121	2.66671632
3.00001036	3.3333355	3.66666712	4.00000009	4.33333335	4.66666667
5.	5.33333333	5.66666667	6.	6.33333333	6.66666667
7.	7.33333333	7.66666667	8.	8.33333333	8.66666667
9.	9.33333331	9.66666657	9.99999955	10.33333119	10.6666564
10.99995079	11.33309756	11.665537	11.99458746	12.30740029	12.54241397
12.40466958	10.48093392	]			

Residual: 4.355357962306849

Solution for n=50, k=42: [11.123621618713363, 13.618108093566818, 14.3669  
 1884912072, 14.841486152036792, 15.110077128454543, 15.39673978061705, 15.  
 770212683721612, 15.559628405166281, 13.550658489869868, 2.925105347872068  
 3, 1.0150217058524456, 0.8632611641771358, 1.1361327251215532, 1.443538918  
 9968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623  
 519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.  
 08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385,  
 5.400374948979675, 5.87475086487051, 6.147622425814478, 6.455028619687642,  
 6.927900180631575, 7.1935639440001555, 7.466435504943287, 7.77384169881243  
 9, 8.246713259737145, 8.51237702301361, 8.785248583515381, 9.0926547752698  
 55, 9.56552632606254, 9.831190040793569, 10.104061368700187, 10.4114664460  
 24326, 10.884332657260483, 11.149970788639209, 11.42271953934085, 11.72953  
 73139924, 12.199589589070602, 12.451745342332135, 12.659896138605754, 12.6  
 57206519234398, 10.644319778626079]

Exact Solution: [11.14759067 13.73795336 14.54217611 14.9729272 15.32245989 15.63937224

15.87440131 15.73263433 13.78877032 3.21121727 1.26731602 1.12536282  
 1.35949808 1.67212757 2.00113976 2.33357121 2.66671632 3.00001036  
 3.3333355 3.66666712 4.00000009 4.33333335 4.66666667 5.  
 5.33333333 5.66666667 6. 6.33333333 6.66666667 7.  
 7.33333333 7.66666667 8. 8.33333333 8.66666667 9.  
 9.33333333 9.66666665 9.9999999 10.33333288 10.66666447 10.99998948  
 11.33328292 11.66642514 11.99884278 12.32778878 12.64010111 12.87271675  
 12.72348266 10.74469653]

Residual: 4.397186676903689

Solution for n=50, k=43: [11.376319074436477, 13.881595372182387, 14.631657786475461, 15.110026893528257, 15.396737550731041, 15.770212584264886, 15.559628400896413, 13.550658489708844, 2.9251053478705527, 1.0150217058524227, 0.863261164177135, 1.1361327251215532, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.4550286196876545, 6.927900180631632, 7.193563944000423, 7.466435504944565, 7.773841698818567, 8.246713259766512, 8.512377023154315, 8.785248584189546, 9.09265477849998, 9.565526341539002, 9.83119011494575, 10.104061723984628, 10.411468148294352, 10.884340813326173, 11.150009866697632, 11.42290677356727, 11.730434407066076, 12.20388782021255, 12.472339404968201, 12.758568220644134, 13.129972866790235, 12.909479434366878, 10.85303193114816]

Exact Solution: [11.41131571 14.05657856 14.87157709 15.30130689 15.63495735 15.87347987

15.73244201 13.78873018 3.21120889 1.26731427 1.12536245 1.359498  
 1.67212755 2.00113975 2.33357121 2.66671632 3.00001036 3.3333355  
 3.66666712 4.00000009 4.33333335 4.66666667 5. 5.33333333  
 5.66666667 6. 6.33333333 6.66666667 7. 7.33333333  
 7.66666667 8. 8.33333333 8.66666667 9. 9.33333333  
 9.66666666 9.99999998 10.33333324 10.6666662 10.99999775 11.33332256  
 11.66661506 11.99975272 12.33214856 12.6609901 12.97280193 13.20301953  
 13.04229574 11.00845915]

Residual: 4.465096993770099

Solution for n=50, k=44: [11.63786973646796, 14.189348682339803, 15.10887367523105, 15.396686360482116, 15.77021030109258, 15.559628302875518, 13.550658486012285, 2.925105347835751, 1.0150217058518942, 0.8632611641771154, 1.136132725121552, 1.4435389189968701, 1.916410479951118, 2.1820742433691205, 2.4549458045490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687656, 6.927900180631643, 7.193563944000481, 7.466435504944838, 7.773841698819871, 8.24671325977276, 8.512377023184248, 8.78524858433296, 9.09265477918711, 9.565526344831245, 9.831190130719834, 10.10406179956281, 10.411468510411185, 10.88434254833216, 11.15001817961074, 11.422946603126814, 11.730625241950694, 12.204802165076103, 12.47672029440134, 12.779558322946277, 13.230542488867814, 13.391337442452633, 13.16175234949936, 11.06174408367024]

Exact Solution: [11.67489829 14.37449147 15.19755908 15.61330392 15.86896054 15.73149877

13.78853331 3.2111678 1.26730569 1.12536066 1.35949763 1.67212747  
 2.00113974 2.33357121 2.66671631 3.00001036 3.3333355 3.66666712  
 4.00000009 4.33333335 4.66666667 5. 5.33333333 5.66666667  
 6. 6.33333333 6.66666667 7. 7.33333333 7.66666667  
 8. 8.33333333 8.66666667 9. 9.33333333 9.66666667  
 10. 10.33333331 10.66666657 10.99999952 11.33333103 11.66665564



11.99994719 12.33308031 12.66545435 12.99419142 13.30550275 13.53332231  
 13.36110882 11.27222176]  
 Residual: 4.474560343980878

Solution for n=50, k=45: [11.933146300471709, 14.665731502358542, 15.3955  
 11211320992, 15.77015788757976, 15.559626052664772, 13.550658401152443, 2.  
 9251053470368316, 1.015021705839762, 0.8632611641766642, 1.13613272512152  
 6, 1.4435389189968688, 1.916410479951118, 2.1820742433691205, 2.4549458045  
 490438, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.773759  
 0405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.0926  
 54777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.45  
 5028619687657, 6.927900180631646, 7.193563944000492, 7.466435504944895, 7.  
 773841698820153, 8.246713259774113, 8.51237702319073, 8.785248584364021,  
 9.092654779335938, 9.565526345544306, 9.831190134136312, 10.10406181593214  
 5, 10.411468588841377, 10.884342924113781, 11.15001998008865, 11.422955229  
 734754, 11.730666574512487, 12.205000201277121, 12.477669142844647, 12.784  
 1045289618, 13.252324670502114, 13.495702144608613, 13.661793678644955, 1  
 3.457586027242238, 11.479168388714399]  
 Exact Solution: [11.93779832 14.68899161 15.50715971 15.84680695 15.72687  
 505 13.78756829

3.21096639	1.26726366	1.12535189	1.3594958	1.67212709	2.00113966
2.33357119	2.66671631	3.00001036	3.3333355	3.66666712	4.00000009
4.33333335	4.66666667	5.	5.33333333	5.66666667	6.
6.33333333	6.66666667	7.	7.33333333	7.66666667	8.
8.33333333	8.66666667	9.	9.33333333	9.66666667	10.
10.33333333	10.66666665	10.9999999	11.33333284	11.66666431	11.99998873
12.33327932	12.66640789	12.99876013	13.32739274	13.63820357	13.86362509
13.67992189	11.53598438]				

Residual: 4.334905455830818

Solution for n=50, k=46: [12.18167745851448, 14.9083872925724, 15.5602590  
 04347518, 15.559574395831856, 13.550656453072635, 2.925105328696474, 1.015  
 02170556125, 0.863261164166302, 1.1361327251209326, 1.4435389189968357, 1.  
 9164104799511161, 2.1820742433691205, 2.4549458045490438, 2.76235199955274  
 04, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.0811658301  
 8373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948  
 979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.9279001  
 80631646, 7.193563944000495, 7.46643550494491, 7.773841698820212, 8.246713  
 259774399, 8.512377023192107, 8.785248584370622, 9.092654779367564, 9.5655  
 26345695837, 9.83119013486234, 10.104061819410742, 10.411468605508334, 10.  
 884343003969974, 11.150020362702664, 11.42295706294863, 11.73067535796784  
 5, 12.205042285340038, 12.477870779703869, 12.785070629194987, 13.25695353  
 4808828, 13.51788036590899, 13.768055920840132, 13.966719016917756, 13.918  
 571094896798, 11.68788054123648]  
 Exact Solution: [12.19742802 14.98714011 15.73827254 15.7042226 13.78284  
 045 3.20997963

1.26705771	1.12530891	1.35948682	1.67212522	2.00113927	2.33357111
2.66671629	3.00001036	3.3333355	3.66666712	4.00000009	4.33333335
4.66666667	5.	5.33333333	5.66666667	6.	6.33333333
6.66666667	7.	7.33333333	7.66666667	8.	8.33333333
8.66666667	9.	9.33333333	9.66666667	10.	10.33333333
10.66666666	10.99999998	11.33333323	11.66666616	11.9999976	12.33332181
12.66661146	12.99973547	13.33206591	13.66059406	13.97090439	14.19392787
13.99873497	11.79974699]				

Residual: 4.371409940243094

Solution for n=50, k=47: [12.414570536800785, 15.072852684003925, 15.3496  
 92883218832, 13.550611732090244, 2.9251049076671785, 1.0150216991676058,  
 0.8632611639284258, 1.1361327251073006, 1.4435389189960812, 1.916410479951  
 0726, 2.1820742433691183, 2.4549458045490438, 2.7623519995527404, 3.235223

565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.773841698820226, 8.246713259774461, 8.512377023192402, 8.785248584372026, 9.092654779374282, 9.565526345728028, 9.831190135016575, 10.104061820149733, 10.41146860904905, 10.884343020934558, 11.150020443984863, 11.422957452395035, 11.73067722391768, 12.205051225642814, 12.477913615267909, 12.785275866712407, 13.257936886831892, 13.522591888506897, 13.790630181806588, 14.074878799152131, 14.436795745102234, 14.17084401002928, 11.896592693758558]

Exact Solution: [12.44138866 15.20694328 15.59332772 13.75969534 3.20514897 1.26604949

1.12509848	1.35944291	1.67211605	2.00113735	2.33357071	2.66671621
3.00001034	3.33333549	3.66666712	4.00000009	4.33333335	4.66666667
5.	5.33333333	5.66666667	6.	6.33333333	6.66666667
7.	7.33333333	7.66666667	8.	8.33333333	8.66666667
9.	9.33333333	9.66666667	10.	10.33333333	10.66666667
11.	11.33333331	11.66666656	11.99999949	12.33333088	12.66665489
12.99994359	13.33306306	13.66537169	13.99379538	14.30360521	14.52423065
14.31754805	12.06350961]				

Residual: 4.434273064744804

Solution for n=50, k=48: [12.581232712252165, 14.906163561260826, 13.549585094051965, 2.9250952423323304, 1.015021552392298, 0.863261158467635, 1.1361327247943611, 1.4435389189787524, 1.9164104799500725, 2.182074243369068, 2.4549458045490415, 2.7623519995527404, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.773841698820228, 8.246713259774474, 8.512377023192464, 8.785248584372322, 9.092654779375708, 9.565526345734863, 9.831190135049331, 10.104061820306674, 10.411468609801004, 10.88434302453739, 11.150020461247069, 11.422957535103237, 11.730677620196483, 12.205053124328611, 12.4779227124181, 12.78531945377757, 13.258145725007513, 13.523592492319839, 13.795424362695684, 14.097849099784666, 14.54685306737581, 14.69816032076463, 14.423116925161755, 12.105304846280639]

Exact Solution: [12.61027427 15.05137134 13.64658246 3.18154093 1.26112221 1.1240701

1.35922827	1.67207125	2.001128	2.33356876	2.6667158	3.00001026
3.33333547	3.66666711	4.00000009	4.33333335	4.66666667	5.
5.33333333	5.66666667	6.	6.33333333	6.66666667	7.
7.33333333	7.66666667	8.	8.33333333	8.66666667	9.
9.33333333	9.66666667	10.	10.33333333	10.66666667	11.
11.33333333	11.66666664	11.99999989	12.33333281	12.66666416	12.99998798
13.33327572	13.66639064	13.99867747	14.3269967	14.63630603	14.85453343
14.63636113	12.32727223]				

Residual: 4.431017048553082

Solution for n=50, k=49: [12.380203056663364, 12.901015283316816, 2.924873359920716, 1.0150181829534328, 0.8632610331073177, 1.136132717610379, 1.4435389185809417, 1.9164104799271064, 2.18207424336793, 2.4549458045489856, 2.7623519995527377, 3.235223565913378, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.512377023192476, 8.785248584372386, 9.092654779376016, 9.56552634573634, 9.831190135056403, 10.10406182034056, 10.411468609963359, 10.884343025315273, 11.150020464974137, 11.422957552960693, 11.730677705756689, 12.205053534272198, 12.477924676575823, 12.785328864622597, 13.258190815074922, 13.52380853181186, 13.796459470088385, 14.1028085

97256145, 14.570615447340503, 14.812012723116611, 14.968616556956954, 14.718950602904638, 12.522729151324798]

Exact Solution: [12.41945386 13.09726929 3.0668926 1.23719371 1.11907593 1.35818593

1.6718537	2.0010826	2.33355928	2.66671383	3.00000984	3.3333539
3.6666671	4.00000009	4.33333335	4.66666667	5.	5.33333333
5.66666667	6.	6.33333333	6.66666667	7.	7.33333333
7.66666667	8.	8.33333333	8.66666667	9.	9.33333333
9.66666667	10.	10.33333333	10.66666667	11.	11.33333333
11.66666666	11.99999998	12.33333322	12.66666613	12.99999744	13.33332106
13.66660786	13.99971822	14.33198325	14.66019802	14.96900685	15.18483621
14.95517421	12.59103484]				

Residual: 4.372740889894089

Solution for n=50, k=50: [10.500622534683599, 2.5031126734179927, 1.0149408324063631, 0.8632581552804902, 1.1361325526917403, 1.4435389094486288, 1.9164104793998884, 2.1820742433417837, 2.4549458045477044, 2.7623519995526746, 3.235223565913375, 3.500887355234949, 3.7737590405263273, 4.08116583018373, 4.554040245701445, 4.819717686154694, 5.092654777947385, 5.400374948979675, 5.874750864870511, 6.147622425814481, 6.455028619687657, 6.927900180631646, 7.193563944000495, 7.466435504944912, 7.77384169882023, 8.246713259774477, 8.512377023192478, 8.7852485843724, 9.092654779376081, 9.565526345736652, 9.831190135057904, 10.10406182034775, 10.411468609997808, 10.884343025480334, 11.150020465764985, 11.422957556749878, 11.730677723911768, 12.20505362125841, 12.477925093351802, 12.785330861516282, 13.258200382767367, 13.523854373380395, 13.796679110238607, 14.103860956438721, 14.57565760310317, 14.83617114274738, 15.08436649934813, 15.273541895229753, 15.179935670559198, 12.731441303846879]

Exact Solution: [10.50517834 2.52589172 1.12428025 1.09550952 1.35326733 1.67082713

2.00086834	2.33351457	2.66670449	3.00000789	3.33333498	3.66666701
4.00000007	4.33333335	4.66666667	5.	5.33333333	5.66666667
6.	6.33333333	6.66666667	7.	7.33333333	7.66666667
8.	8.33333333	8.66666667	9.	9.33333333	9.66666667
10.	10.33333333	10.66666667	11.	11.33333333	11.66666667
12.	12.33333331	12.66666655	12.99999945	13.33333072	13.66665414
13.99993999	14.3330458	14.66528903	14.99339934	15.30170767	15.51513899
15.27398729	12.85479746]				

Residual: 4.271483287195057

In [ ]: # 2.3 - 10

```
def build_A(a, d):
    n = len(a)
    ret = np.zeros((n, n))
    for i in range(n):
        ret[n - (i+1)][i] = a[i]
        ret[i][i] = d[i]

    return ret

def naive_gauss_elimination(A, b):
    n = len(b)
    A = [row[:] for row in A]
    b = b[:]

    for i in range(n):
        if A[i][i] == 0:
            raise ValueError("Unsolvable singular matrix")
```

```

    pivot = A[i][i]
    for j in range(i, n):
        A[i][j] /= pivot
    b[i] /= pivot

    for k in range(i + 1, n):
        factor = A[k][i]
        for j in range(i, n):
            A[k][j] -= factor * A[i][j]
        b[k] -= factor * b[i]

x = [0 for _ in range(n)]
for i in range(n - 1, -1, -1):
    x[i] = b[i]
    for j in range(i + 1, n):
        x[i] -= A[i][j] * x[j]

return x

def verifyty_solution(A, b, solver=naive_gauss_elimination):
    x = [x.item() for x in solver(A, b)]
    x_exact = np.linalg.solve(A, b)
    print("Solver: ", solver.__name__)
    print("Solution: ", x)
    print("Exact Solution: ", x_exact)
    print("Residual: ", np.linalg.norm(np.dot(A, x) - b))
    print()

verifyty_solution(build_A([1, 2, 3, 4, 5], [6, 7, 8, 9, 10]), np.array([11, 12, 13, 14, 15]))

```

Solver: naive\_gauss\_elimination  
 Solution: [0.16666666666666663, 0.4285714285714286, 1.0, 1.0, 1]  
 Exact Solution: [0.16666667 0.42857143 1. 1. 1]  
 Residual: 0.0