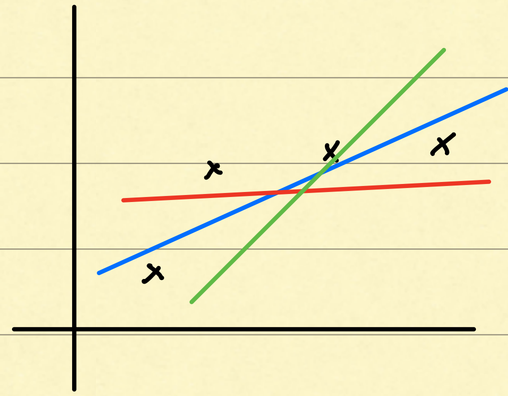
Chap 9 Least square Methods and Fourier Series

9.1 Method of Least Squares

x	x_0	x_1	$\cdots$	x_m
y	y_0	y_1	$\cdots$	y_m



$$y = ax + b$$

$$ax_k + b - y_k = 0$$

$$\psi(a,b) = \sum_{k=0}^{m} |ax_k + b - y_k| \qquad \text{minimize } \psi(a,b) : \ell_1\text{-appoximation}$$

$$\varphi(a,b) = \sum_{k=0}^{m} (ax_k + b - y_k)^2 \qquad \text{minimize } \varphi(a,b) : \ell_2\text{-approx}$$

$$\frac{\partial \varphi}{\partial a} = \sum_{k=0}^{m} 2(ax_k + b - y_k)x_k = 0$$

$$\frac{\partial \varphi}{\partial b} = \sum_{k=0}^{m} 2(ax_k + b - y_k) = 0$$

$$\begin{cases} \overset{s}{\left(\sum_{k=0}^{m} x_k^2\right)} a + \overset{p}{\left(\sum_{k=0}^{m} x_k\right)} b = \overset{r}{\sum_{k=0}^{m} y_k x_k} \\ \left(\sum_{k=0}^{m} x_k\right) a + (m+1) b = \underset{q}{\sum_{k=0}^{m} y_k} \end{cases}$$

normal eqn.

$$\begin{bmatrix} s & p \\ p & m+1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ q \end{bmatrix}$$

$(x_i, y_i) \quad i=1,2,3,4$

$$y_i = ax_i + b$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$A : m \times n \quad , \ m > n$

$A^T A : n \times n$

$$Ax = b \Leftrightarrow A^T A x = A^T b \Leftrightarrow \text{minimizing } \|Ax - b\|_2^2$$

Nonpolynomial Example

$$y = a \ln x + b \cos x + c e^x$$

$$\varphi(a, b, c) = \sum_{k=0}^{m} (a \ln x_k + b \cos x_k + c e^{x_k} - y_k)^2$$

$$\frac{\partial \varphi}{\partial a} = \frac{\partial \varphi}{\partial b} = \frac{\partial \varphi}{\partial c} = 0 \Rightarrow a, b, c$$

Basis Function $\{g_0, g_1, \cdots g_n\}$

$$y = \sum_{j=0}^{n} c_j g_j(x)$$

$$\varphi(c_0, c_1, \cdots, c_n) = \sum_{k=0}^{m} \left[\sum_{j=0}^{n} c_j g_j(x_k) - y_k \right]^2$$

$$\frac{\partial \varphi}{\partial c_i} = 0 \quad , \quad 0 \le i \le n$$

$$\frac{\partial \varphi}{\partial c_i} = \sum_{k=0}^{m} 2 \left[\sum_{j=0}^{n} c_j g_j(x_k) - y_k \right] g_i(x_k) \qquad (0 \le i \le n)$$

$$\sum_{j=0}^{n} \left[\sum_{k=0}^{m} g_i(x_k) g_j(x_k) \right] c_j = \sum_{k=0}^{m} y_k g_i(x_k) \quad (0 \le i \le n)$$

9.2 Orthogonal Systems and Chebyshev Polynomials

Orthogonal Basis Function $\{g_0, g_1, \dots g_n\}$

$$\mathcal{G} = \{ g : \exists c_i \text{ s.t. } g(x) = \sum_{j=0}^{n} c_j g_j(x) \}$$

What basis for $\mathcal{G}$ should be chosen for numerical work?

Our work is to solve Normal eq.

$$\sum_{j=0}^{n} \left[\sum_{k=0}^{m} g_i(x_k)\, g_j(x_k) \right] c_j = \sum_{k=0}^{m} y_i\, g_i(x_k) \qquad (0 \le i \le n) \quad \cdots (*)$$

Orthogonality

$$\sum_{k=0}^{m} g_i(x_k) g_j(x_k) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

$$(*) \Rightarrow \quad c_j = \sum_{k=0}^{m} y_k\, g_j(x_k) \qquad (0 \le j \le n)$$

Using a Gram-Schmidt Process, we can obtain such a basis.

One simple choice

$$g_0(x) = 1, \quad g_1(x) = x, \quad g_2(x) = x^2, \quad \cdots, \quad g_n(x) = x^n$$

$$g(x) = \sum_{j=0}^{n} c_j g_j(x)$$

Using a Chebyshev Polynomials

$x \in [-1, 1]$, $\quad T_0(x) = 1, \quad T_1(x) = x$

$$T_j(x) = 2x T_{j-1}(x) - T_{j-2}(x) \qquad (j \ge 2)$$

$$g(x) = \sum_{j=0}^{n} c_j T_j(x)$$

An Algorithm to compute $g(x)$ for given x

$$\begin{cases} w_{n+2} = w_{n+1} = 0 \\ w_j = c_j + 2x w_{j+1} - w_{j+2} & (j = n, n-1, \cdots, 0) \\ g(x) = w_0 - x w_1 \end{cases}$$

$$z \qquad\qquad x$$
$$[-1, 1] \rightarrow [a, b]$$
$$x = \frac{1}{2}(b-a)z + \frac{1}{2}(a+b)$$

Algorithm

1. Find the smallest interval $[a, b] \ni x_k$. let $a = \min\{x_k\}$, $b = \max\{x_k\}$
2. Change the interval to $[-1, 1]$ from $[a, b]$
$$z_k = \frac{2x_k - a - b}{b - a} \qquad (0 \leq k \leq m)$$
3. Decide on the value of n to be used
4. Using chebyshev poly. as a basis. Generate $(n+1)\times(n+1)$ normal eqns.
$$\sum_{j=0}^{n}\left[\sum_{k=0}^{m} T_i(z_k)\, T_j(z_k)\right] c_j = \sum_{k=0}^{m} y_k T_i(z_k) \quad (0 \leq i \leq n)$$
5. Find c_j and let $g(x) = \sum_{j=0}^{n} c_j T_j(x)$
6. The polynomial that is being sought is
$$g\left(\frac{2x - a - b}{b - a}\right)$$

Smoothing Data : Polynomial Regression

x	x_0	x_1	$\cdots$	x_m
y	y_0	y_1	$\cdots$	y_m

Goal: obtain a suitable polynomial of modest degree, with the experimental errors of the table somehow suppresed. We don't know what degree of polynomial should be used.

$$P_N(x) = \sum_{i=0}^{N} a_i x^i$$

$$y_i = P_N(x_i) + \varepsilon_i \qquad (0 \le i \le m)$$

ε_i : represents an observational error that is present in y_i independent random variables that are normally distributed.

1) Find a fixed value n, we have already discussed a method of determining P_n by the method of least squares. Once determine the coefficients of P_n.

2) Compute variance

$$\sigma_n^2 = \frac{1}{m-n} \sum_{i=0}^{m} [y_i - P_n(x_i)]^2 \qquad (m > n)$$

Statistical theory tells us that if the trend of the table is truly a polynomial of degree N (but infected by noise), then

$$\sigma_0^2 > \sigma_1^2 > \cdots > \sigma_N^2 = \sigma_{N+1}^2 = \cdots = \sigma_{m-1}^2 .$$

N is not known

3) Compute $\sigma_0^2, \sigma_1^2, \cdots$ until $\sigma_N^2 \approx \sigma_{N+1}^2 \approx \cdots$ stop declare P_N to be the polynomial sought

def) inner product $\langle f, g \rangle = \sum_{i=0}^{m} f(x_i) g(x_i)$

i) $\langle f, g \rangle = \langle g, f \rangle$

ii) $\langle f, f \rangle > 0$ unless $f(x_i) = 0$ for all i

iii) $\langle af, g \rangle = a \langle f, g \rangle$, $a \in \mathbb{R}$

iv) $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$

orthogonal $\langle f, g\rangle = 0$

An orthogonal set of polynomials can be generated recursively by

$$\begin{cases} g_0(x) = 1 \\ g_1(x) = x - \alpha_0 \\ g_{n+1}(x) = x g_n(x) - \alpha_n g_n(x) - \beta_n g_{n-1}(x), \quad (n \geq 1) \end{cases}$$

where

$$\begin{cases} \alpha_n = \dfrac{\langle x g_n, g_n\rangle}{\langle g_n, g_n\rangle} \\ \beta_n = \dfrac{\langle x g_n, g_{n-1}\rangle}{\langle g_{n-1}, g_{n-1}\rangle} \end{cases}$$

orthogonal ?

$$\langle g_1, g_0\rangle = \langle x-\alpha_0, g_0\rangle = \langle x g_0 - \alpha_0 g_0, g_0\rangle = \langle x g_0, g_0\rangle - \alpha_0 \langle g_0, g_0\rangle = 0$$

$$\langle g_2, g_1\rangle = \langle x g_1 - \alpha_1 g_1 - \beta_1 g_0, g_1\rangle = \langle x g_1, g_1\rangle - \alpha_1\langle g_1, g_1\rangle - \beta_1\langle g_0, g_1\rangle = 0$$

$\vdots$

p of degree n $(n \leq m-1)$ ← given

1) $P = \sum_{i=0}^{n} a_i g_i$ ← we can choose a_n

$\Rightarrow p - a_n g_n = \sum_{i=0}^{n-1} a_i g_i$ ← choose $a_{n-1}, \ldots$

2) another way to choose a_i

$$\langle p, g_j\rangle = \sum_{i=0}^{n} a_i \langle g_i, g_j\rangle \qquad (0 \leq j \leq n)$$

$$\langle p, g_j\rangle = a_j \langle g_j, g_j\rangle$$

$$\Rightarrow a_j = \frac{\langle p, g_j\rangle}{\langle g_j, g_j\rangle}$$

let F be a ftn that we wish to fit by a polynomial P_n

minimize $\sum_{i=0}^{m} [F(x_i) - P_n(x_i)]^2$

$$P_n = \sum c_i g_i \qquad c_i = \frac{\langle F, g_i\rangle}{\langle g_i, g_i\rangle} \quad \cdots (*)$$

To prove that p_n in $(*)$ solves our problem.

$\Rightarrow$ normal eqn

$$\sum_{j=0}^{n}\left[\sum_{k=0}^{m} q_i(x_k)\, q_j(x_k)\right] c_j = \sum_{k=0}^{m} y_k\, q_i(x_k) \qquad (0 \le i \le n)$$

inner product notation

$$\sum_{j=0}^{n} \langle q_i, q_j \rangle c_j = \langle F, q_i \rangle \qquad (0 \le i \le n)$$

where $F(x_k) = y_k$ for $0 \le k \le m$

The result $\quad \langle q_i, q_i \rangle c_i = \langle F, q_i \rangle \qquad (0 \le i \le n)$

Now return to the $\sigma_0^2, \sigma_1^2, \cdots$ and show how they can be easily computed.

First observation $\{q_0, q_1, \cdots, q_n, F - p_n\}$ is orthogonal!

$$\begin{aligned}
\langle F - p_n, q_i \rangle &= \langle F, q_i \rangle - \langle p_n, q_i \rangle \\
&= \langle F, q_i \rangle - \langle \sum_{j=0}^{n} c_j q_j, q_i \rangle \\
&= \langle F, q_i \rangle - \sum_{j=0}^{n} c_j \langle q_j, q_i \rangle \\
&= \langle F, q_i \rangle - c_i \langle q_i, q_i \rangle = 0
\end{aligned}$$

$$\sigma_n^2 = \frac{\rho_n}{m-n}, \qquad \rho_n = \sum_{i=0}^{m} \left[y_i - p_n(x_i) \right]^2$$

$$\begin{aligned}
\rho_n &= \langle F - p_n, F - p_n \rangle = \langle F - p_n, F \rangle = \langle F, F \rangle - \langle F, p_n \rangle \\
&= \langle F, F \rangle - \sum_{i=0}^{n} c_i \langle F, q_i \rangle = \langle F, F \rangle - \sum_{i=0}^{n} \frac{\langle F, q_i \rangle^2}{\langle q_i, q_i \rangle}
\end{aligned}$$

Thus, the numbers $\rho_1, \rho_2, \ldots$ can be generated recursively

$$\rho_0 = \langle F, F \rangle - \langle F, q_0 \rangle^2 / \langle q_0, q_0 \rangle$$

$$\rho_n = \rho_{n-1} - \langle F, q_n \rangle^2 / \langle q_n, q_n \rangle \qquad (n \ge 1)$$

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9.3 Examples of the Least-Squares Principles

$$\sum_{j=0}^{n} a_{kj} x_j = b_k \qquad (0 \le k \le m) \quad \cdots \quad (*)$$
$$(m > n)$$

m+1 eqns, n+1 unknown

$(x_0, x_1, \cdots, x_n) \Rightarrow$ in $(*)$

k^{th} residual $\xrightarrow{\text{(hope)}}$ all residuals should be zero.

If not, $(*)$ is inconsistent or incomplete.

minimize $\varphi(x_0, x_1, \cdots, x_n) = \sum_{k=0}^{m} \left(\sum_{j=0}^{n} a_{kj} x_j - b_k \right)^2$

$$\frac{\partial \varphi}{\partial x_i} = 0 \Rightarrow \sum_{j=0}^{n} \left(\sum_{k=0}^{m} a_{ki} a_{kj} \right) x_j = \sum_{k=0}^{m} b_k a_{ki} \quad \cdots \quad (*1)$$
$$(0 \le i \le n)$$

n+1 unknowns

The solution of $(*1)$ is a best approximate solution of eqn $(*)$ in the least-square sense.

i) $Ax = b$

factoring $A = QR$

$Q : (m+1) \times (n+1) \qquad Q^T Q = I$

$R : (n+1) \times (n+1) \qquad r_{ii} > 0$

$r_{ij} = 0$ for $j < i$

By Modified Gram-Schmidt process, we can get the least square solution.

ii) Singular value decomposition (SVD)

$A = U \Sigma V^T, \quad U^T U = I_{m+1}, \; V^T V = I_{n+1},$

$\Sigma : (m+1) \times (n+1)$, diagonal, > 0

Use of a Weight Ftn $w(x)$

Another important example of the principle of least squares occurs in fitting or approximating ftns on intervals rather than discrete sets.

For example, a given ftn f defined on an interval $[a,b]$ may have to be approximated by a ftn such as

$$g(x) = \sum_{j=0}^{n} c_j g_j(x)$$

minimize $\varphi(c_0, c_1, \cdots, c_n) = \int_a^b [g(x) - f(x)]^2 dx$

using a weighted ftn

$$\varphi(c_0, c_1, \cdots, c_n) = \int_a^b [g(x) - f(x)]^2 w(x) dx$$

$$\frac{\partial \varphi}{\partial c_i} = 0 \Rightarrow \sum_{j=0}^{n} \left[\int_a^b g_i(x) g_j(x) w(x) dx \right] c_j = \int_a^b f(x) g_i(x) w(x) dx$$

ideal situation to have ftns $g_0, g_1, \cdots, g_n$ that have the orthogonality property: $\int_a^b g_i(x) g_j(x) w(x) dx = 0 \quad (i \neq j)$

Chebyshev polynomials

$$\int_{-1}^{1} T_i(x) T_j(x) (1-x^2)^{-\frac{1}{2}} dx = \begin{cases} 0 & i \neq j \\ \pi/2 & i = j > 0 \\ \pi & i = j = 0 \end{cases}$$

Nonlinear Example

(x_k, y_k) fitted by $y = e^{cx}$

$$\varphi(c) = \sum_{k=0}^{m} (e^{cx_k} - y_k)^2$$

$$0 = \frac{\partial \varphi}{\partial c} = \sum_{k=0}^{m} 2\,(e^{c x_k} - y_k)\, e^{c x_k} x_k$$

let $z = \ln y$. $\quad y = e^{cx} \Rightarrow z = cx$

$$\varphi(c) = \sum_{k=0}^{m} (c x_k - z_k)^2, \quad z_k = \ln y_k$$

$$c = \sum_{k=0}^{m} z_k x_k \Big/ \sum_{k=0}^{m} x_k^2$$

This value of c is not the solution of the original problem but may be satisfactory in some applications.

Linear and Nonlinear Examples

$$y = a \sin(bx)$$

$$\sum_{k=0}^{m} [a \sin(b x_k) - y_k]^2 \Rightarrow \begin{cases} \sum_{k=0}^{m} 2\,[a \sin(b x_k) - y_k] \sin(b x_k) = 0 \\ \sum_{k=0}^{m} 2\,[a \sin(b x_k) - y_k]\, a x_k \cos(b x_k) = 0 \end{cases}$$

solve for $a \Rightarrow$

$$\frac{\sum_{k=0}^{m} y_k \sin b x_k}{\sum_{k=0}^{m} (\sin b x_k)^2} = \frac{\sum_{k=0}^{m} x_k y_k \cos b x_k}{\sum_{k=0}^{m} x_k \sin b x_k \cos b x_k}$$

Additional Details on SVD

$$A = U D V^T$$

A: $m \times n$; U: $m \times m$ orthogonal; D: $m \times n$ diagonal; V^T: $n \times n$ orthogonal

The singular values of a matrix A are the positive square roots of the eigenvalues of $A^T A$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

$$U^T A V = D = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}_{m \times n}$$

$U^T U = I_m$, $V^T V = I_n$

$A v_i = \sigma_i u_i$, $\sigma_i = \|A v_i\|_2$

$$\|Ax - b\|_2^2 = \|U^T(Ax - b)\|_2^2 = \|U^T A x - U^T b\|_2^2$$
$$= \|U^T A (V V^T) x - U^T b\|_2^2$$
$$= \|(U^T A V)(V^T x) - U^T b\|_2^2$$
$$= \|D V^T x - U^T b\|_2^2 = \|Dy - c\|_2^2$$
$$= \sum_{i=1}^{r} (\sigma y_i - c_i)^2 + \sum_{i=r+1}^{m} c_i^2$$

where $y = V^T x$, $c = U^T b$

$y_i = c_i / \sigma_i$, $x = Vy$, $c_i = u_i^T b$, $x = Vy$

If $y_i = \sigma_i^{-1} c_i$ for $1 \le i \le r$

$$x_{LS} = \sum_{i=1}^{n} y_i v_i = \sum_{i=1}^{r} \sigma_i^{-1} c_i v_i = \sum_{i=1}^{r} \sigma_i^{-1} (u_i^T b) v_i$$

and $\|A x_{LS} - b\|_2^2 = \sum_{i=r+1}^{m} c_i = \sum_{i=r+1}^{m} (u_i^T b)^2$

Thm SVD Least square Thm

A : $m \times n$, rank r , $A = UDV^T$

The least squares sol of $Ax = b$ is $x_{LS} = \sum_{i=1}^{n} (\sigma_i^{-1} c_i) v_i$ where $c_i = v_i^T b$. If there exist many least-squares solutions to the given system, then the one of least 2-norm is x as described above

Ex1) Find the least-squares solution of this nonsquare system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Sol) SVD $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sqrt{6} & 0 & \frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$

$r = \text{rank}(A) = 2$

$\sigma_1 = \sqrt{3}, \ \sigma_2 = 1$

$c_1 = u_1^T b = \begin{bmatrix} \frac{1}{3}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\sqrt{6}$

$c_2 = u_2^T b = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \sqrt{2}$

$x_{LS} = (\sigma_1^{-1}c_1)v_1 + (\sigma_2^{-1}c_2)v_2$

$= \frac{1}{\sqrt{3}}\left(\frac{1}{3}\sqrt{6}\right)\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} + \sqrt{2}\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix}$ //

Using the SVD

For any system of linear equation $Ax=b$, we want to define a unique <u>minimal solution</u>

$\rho = \inf\{\|Ax-b\|_2 : x \in \mathbb{R}^n\}$

minimal sol. $\{x : \|Ax-b\|_2 = \rho\}$

If the system is consistent $\Rightarrow \rho = 0$

If the system is inconsistent

$\Rightarrow$ we want Ax to be as close as possible to b : that is $\|Ax-b\|_2 = \rho$.

pseudo-inverse of A

$$D = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \\ \hline & & m-n & & & \end{bmatrix} \quad D^+ = \left[\begin{array}{cccccc|c} \frac{1}{\sigma_1} & & & & & & \\ & \frac{1}{\sigma_2} & & & & & \\ & & \ddots & & & & m-n \\ & & & \frac{1}{\sigma_r} & & & \\ & & & & 0 & & \\ & & & & & \ddots \; 0 & \end{array}\right]$$

$m \times n$ $\qquad$ $n \times m$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad D^+ = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$A = UDV^T$: SVD

$A^+ = VD^+U^T$ unique if we impose the order $\sigma_1 \geq \sigma_2 \geq \cdots$

Thm Minimal Solution Thm

$Ax = b$, $A : m \times n$

The minimal sol of the system is A^+b

pf) $x \in \mathbb{R}^n$, let $y = V^Tx$ and $c = U^Tb$

$$\rho = \inf_x \|Ax - b\|_2 = \inf_x \|UDV^Tx - b\|_2$$
$$= \inf_x \|U^T(UDV^Tx - b)\|_2 = \inf_x \|DV^Tx - U^Tb\|_2 = \inf_y \|Dy - c\|_2$$

$$\|Dy - c\|_2^2 = \sum_{i=0}^{r} (\sigma_i y_i - c_i)^2 + \sum_{i=r+1}^{m} c_i^2$$

To minimize this, define $y_i = c_i / \sigma_i$, $1 \leq i \leq r$

to get the minimum norm, $y_i = 0$ for $r+1 \leq i \leq m$

$\Rightarrow y = D^+c$ $\qquad \therefore x = Vy = VD^+c = VD^+U^Tb = A^+b.$ //

Thm Penrose Properties of the pseduo inverse

$A = AA^+A$, $A^+ = A^+AA^+$, $AA^+ = (AA^+)^T$, $A^+A = (A^+A)^T$

Thm Spectral Thm for Symmetric Matrices

A : symmetric real $\Rightarrow$ $\exists$ orthogonal Q s.t.

Q^TAQ is a diagonal

9.4 Fourier Series

orthogonal set on $[-\pi, \pi]$

$$W = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos Nx, \sin Nx\}$$

inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$

$$\langle \cos mx, \cos nx\rangle = \pi \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n \neq 0) \end{cases}$$

$$\langle \sin mx, \sin nx\rangle = \pi \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n \neq 0) \end{cases}$$

$$\langle \cos mx, \sin nx\rangle = 0$$

Fourier series of f on $[-\pi, \pi]$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad (n \geq 1)$$

Fourier Convergence Theorem

The Fourier Series (5) is convergent if f is a continuous periodic function on $[-\pi, \pi]$ with period 2π or if f and f' are piecewise continuous. Where f is continuous, the sum of the Fourier series equals $f(c)$ at all numbers c. When f is discontinuous at a point c, the sum of the Fourier series is the average of the left and right limits; namely, $\frac{1}{2}[f(c^+) + f(c^-)]$, where $f(c^+) = \lim_{x \to c^+} f(x)$ and $f(c^-) = \lim_{x \to c^-} f(x)$.

Cosine Series and Sine Series

even ftn $f_e(-x) = f_e(x)$

cosine series $f_e(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

odd ftn $f_o(-x) = -f_o(x)$

sine series $f_o(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Periodic on $[-P/2, P/2]$

change the interval from $[-\pi, \pi] \rightarrow [-P/2, P/2]$

$x_{new} = [P/2\pi]\, x_{old}$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{P}x\right) + b_n \sin\left(\frac{2\pi n}{P}x\right)\right]$$

$$a_0 = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\,dx, \quad a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi n}{P}x\right)dx$$

$(n \geq 1)$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi n}{P}x\right)dx$$

$$p_N(x) = \frac{1}{2} - \frac{1}{\pi}\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{N}\sin Nx\right]$$ ■

Fourier Series Examples

Here are some common Fourier series, which are periodic on $[0, 2L]$ with $P = 2L$. (See Figures 9.7–9.9 over $[0, 2\pi]$ with approximations p_2, p_4, and p_{10}.)

Sawtooth Wave $$ST(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right) \tag{19}$$

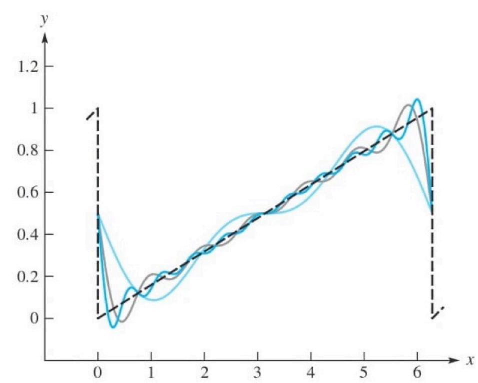

FIGURE 9.7 Sawtooth Wave on $[0, 2\pi]$ with approximations p_2, p_6, and p_{10}

Complex Fourier Series

f : period P on $[-P/2,\ P/2]$

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inw_0 x} \quad \text{where } w_0 = 2\pi/P$$

frequency

$$d_n = \frac{1}{P}\int_{-P/2}^{P/2} f(x) e^{-inw_0 x}\, dx$$

Root of Unity

$$1 = x^n = r^n e^{in\theta} = e^{in\theta} = e^{i2\pi k} \quad (k \geq 0)$$

$$\therefore \quad \theta = 2\pi(k/n)$$

$$x = e^{i2\pi(k/n)} \quad \text{for } k = 0, 1, 2, \cdots, n-1$$

n th Roots of Unity

$$\omega_n^k = e^{i2\pi(k/n)} = \begin{cases} \cos 0 + i\sin 0 = 1 & (k=0) \\ \cos\left(2\pi\frac{k}{n}\right) + i\sin\left(2\pi\frac{k}{n}\right) \neq 1 & (1 \leq k \leq n-1) \end{cases}$$

n	$k=0$	$k=1$	$k=2$	$k=3$
1	$\omega_1^0 = e^{i2\pi(0/1)} = 1$			
2	$\omega_2^0 = e^{i2\pi(0/2)} = 1$	$\omega_2^1 = e^{i2\pi(1/2)} = -1$		
3	$\omega_3^0 = e^{i2\pi(0/3)} = 1$	$\omega_3^1 = e^{i2\pi(1/3)} = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$	$\omega_3^2 = e^{i2\pi(2/3)} = -\frac{1}{2} - i\frac{1}{2}\sqrt{3}$	
4	$\omega_4^0 = e^{i2\pi(0/4)} = 1$	$\omega_4^1 = e^{i2\pi(1/4)} = i$	$\omega_4^2 = e^{i2\pi(2/4)} = -1$	$\omega_4^3 = e^{i2\pi(3/4)} = -i$

Three cases are shown in Figure 9.10.

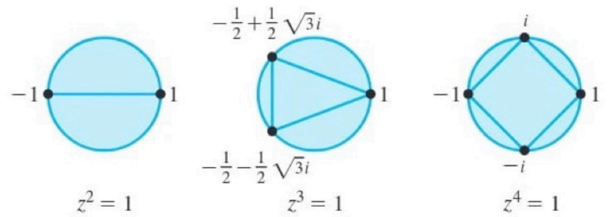

FIGURE 9.10
n-th roots of unity

Discrete Fourier Transform (DFT)

DFT of the vector $x = [x_0 \ x_1 \ \cdots x_{n-1}]^T$ to the vector $y = [y_0 \ y_1 \ \cdots \ y_{n-1}]^T$

$$y_j = \sum_{k=0}^{n-1} \omega_n^{jk} x_k \qquad (0 \leq j \leq n-1)$$

$$y = F_n x \qquad : \text{matrix notation}$$

where $F_n = (F_{jk})_{n \times n} = (\omega_n^{jk})_{0 \leq j \leq n-1,\ 0 \leq k \leq n-1}$

$$\boldsymbol{F}_1 = \omega_1^0 = 1, \qquad \boldsymbol{F}_2 = \begin{bmatrix} \omega_2^0 & \omega_2^0 \\ \omega_2^0 & \omega_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{F}_4 = \begin{bmatrix} \omega_4^0 & \omega_4^0 & \omega_4^0 & \omega_4^0 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ \omega_4^0 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ \omega_4^0 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

For $n = 4$, we have

$$\boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{F}_4 \boldsymbol{x}$$

where

$$\boldsymbol{F}_4 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ \hline 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{array}\right]$$

The inverse DFT

$$x_k = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{-jk} y_j \qquad (0 \leq k \leq n-1)$$

$$x = F_n^{-1} y \qquad \text{where } F_n^{-1} = \frac{1}{n} F_n^*$$

$$F_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^{-1} & \omega_4^{-2} & \omega_4^{-3} \\ 1 & \omega_4^{-2} & \omega_4^{-4} & \omega_4^{-6} \\ 1 & \omega_4^{-3} & \omega_4^{-6} & \omega_4^{-9} \end{bmatrix} = \frac{1}{4}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ \hline 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{array}\right]$$

Moreover, we can show that

$$F_4 P_4 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{array}\right] = \begin{bmatrix} F_2 & D_2 F_2 \\ F_2 & -D_2 F_2 \end{bmatrix}$$

Thus, F_4 can be rearranged into diagonally scaled blocks of F_2, which holds for any even n. Here we use the permutation matrix

$$P_4 = \begin{bmatrix} e_1 & e_3 & e_2 & e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the diagonal matrix

$$D_2 = \begin{bmatrix} \omega_4^0 & 0 \\ 0 & \omega_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

In general, P_n is a permutation matrix that groups even-numbered columns and odd-numbered columns. and $D_{n/2} = \mathrm{Diag}(1, \omega_n^1, \omega_n^2, \cdots, \omega_n^{(n/2)-1})$

We can find F_n by applying $F_{n/2}$ to the even and to the odd subsequences. Then scaling the results by $\pm D_{n/2}$, where necessary.

This recursive DFT is called the Fast Fourier Transform (FFT).

Thm. Recursive Computation of DFT

The Discrete Fourier Transformation of an n-point sequence can be computed by two $n/2$-point Discrete Fourier Transformation (n even)

Fast Fourier Transforms

computation $O(n^2) \to O(n\log_2 n)$

when $n = 2^r$

The **Cooley-Tukey Fast Fourier (FFT) algorithm** rearranges the input values in bit-reversal order and then builds the output transformation. The basic idea is breaking a transform of length N into two transforms of length $N/2$ using the identity

Danielson-Lanczo Identity

$$\sum_{n=0}^{N-1} a_n e^{-i2\pi nk/N} = \sum_{n=0}^{N/2-1} a_{2n} e^{-i2\pi(2n)k/N} + \sum_{n=0}^{N/2-1} a_{2n+1} e^{-i2\pi(2n+1)k/N}$$
$$= \sum_{n=0}^{N/2-1} a_n^{\text{even}}\, e^{-i2\pi nk/(N/2)} + e^{-i2\pi k/N} \sum_{n=0}^{N/2-1} a_n^{\text{odd}}\, e^{-i2\pi nk/(N/2)}$$

This is also called the **Danielson-Lanczo Lemma**. It can be visualized via the **Fourier matrix**

Fourier Matrix

$$\boldsymbol{F}_n = (F_{jk})_{n\times n}$$

with entries

$$F_{jk} = e^{i2\pi(jk)/n} \equiv \omega_n^{jk} \qquad (0 \leqq j \leqq n-1,\ 0 \leqq k \leqq n-1)$$

where $\omega_n = e^{(i2\pi)/n}$. Multiplying by $1/\sqrt{n}$ makes the matrix unitary.

For example, we can show that

$$\boldsymbol{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{array}\right] \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In matrix form, we can write

$$\boldsymbol{F}_4 = \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{D}_2 \\ \boldsymbol{I}_2 & -\boldsymbol{D}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_2 \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{shuffle} \end{bmatrix}$$

where

$$\boldsymbol{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & i^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So in general, we have

$$\boldsymbol{F}_{2n} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{D}_n \\ \boldsymbol{I}_n & -\boldsymbol{D}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_n \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{shuffle} \end{bmatrix}$$

and repeating we obtain

$$\begin{bmatrix} \boldsymbol{F}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_n \end{bmatrix} = \left[\begin{array}{cc|cc} \boldsymbol{I}_{n/2} & \boldsymbol{D}_{n/2} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_{n/2} & -\boldsymbol{D}_{n/2} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{n/2} & \boldsymbol{D}_{n/2} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{n/2} & -\boldsymbol{D}_{n/2} \end{array}\right] \left[\begin{array}{cc|cc} \boldsymbol{F}_{n/2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{F}_{n/2} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{F}_{n/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{F}_{n/2} \end{array}\right] \left[\begin{array}{c} \text{Shuffle:} \\ \text{even-odd} \\ 0, 2\ (\text{mod } 4) \\ \hline \text{even-odd} \\ 1, 3\ (\text{mod } 4) \end{array}\right]$$