Detecting dependence

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Model selection tools for u-data

Now, that the u-scaled data are produced. we are looking for methods to find classes of suitable models, which are

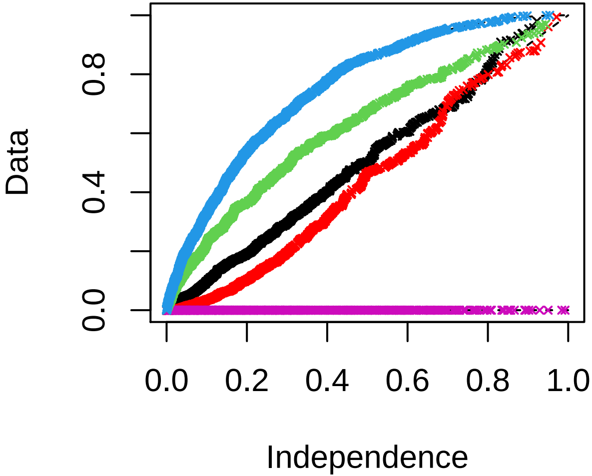
- Dependence structure
  - Kendall plot
  - Dependence measures
- Existence of tail dependence
- Exchangeability of model
- State of radial symmetry

Dependence structure via Kendall plot

In order to see dependence structure of a suitable model for the data, we use the well-known kendall plot:

```
#library(lcopula)
#m1 <- matrix(runif(2000),ncol=2)#Independence
#K.plot(m1, main="Kendall plot")
#m2 <- rCopula(n=1000,claytonCopula(param=-0.5,dim=2))
#K.plot(m2,add=TRUE,col="red") #Negative dependence
#m3 <- rCopula(n=1000,claytonCopula(param=-1,dim=2))
#K.plot(m3,add=TRUE,col="magenta") #Perfect negative dependence
#m4 <- rCopula(n=1000,claytonCopula(param=iTau(claytonCopula(0.3),0.5),dim=2))
#K.plot(m4,add=TRUE,col="green3") #Positive dependence
#m5<- rCopula(n=1000,claytonCopula(param=iTau(claytonCopula(0.3),1),dim=2))
#K.plot(m5,add=TRUE,col="blue") #Perfect positive dependence
```

Kendall plot



Dependence measures

The dependence structure of the suitable model for the data can be achieved via measures of central dependence.

Measures of centeral dependence

- Pearson correlation coefficient
- Spearman rank correlation coefficient
- Kendall's Tau coefficient

Dependence structure via Pearson correlation coefficient

Theoretical model based

For any random vector (X, Y), we have:

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Empirical estimator

For any set of joint observation (x_i, y_i) we have:

$$r_{xy} = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n \sum x_i^2 - (\sum x_i)^2} \sqrt{n \sum y_i^2 - (\sum y_i)^2}}$$

Dependence structure via Pearson correlation coefficient

```r
data(danube, package = "lcopula")
cor(danube)
```

```
##           donau       inn
## donau 1.0000000 0.7374098
## inn   0.7374098 1.0000000
```

Dependence structure via Spearman rank correlation coefficient

The Spearman correlation between two variables is equal to the Pearson correlation between the rank values of those two variables; while Pearson's correlation assesses linear relationships, Spearman's correlation assesses monotonic relationships (whether linear or not).

Theoretical model based

$$\rho_{X,Y} = \frac{Cov(rank(X), rank(Y))}{\sigma_{rank(X)}\sigma_{rank(Y)}}.$$

Based on concordance measures:

$$\rho_{X,Y} = 3Q(F, \Pi) = 3Q(C, \Pi) = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

Dependence structure via Spearman rank correlation coefficient

The empirical estimator is

$$r_{xy} = \frac{n\sum rank(x_i)rank(y_i) - \sum rank(x_i)\sum rank(y_i)}{\sqrt{n\sum rank(x_i)^2 - (\sum rank(x_i))^2}\sqrt{n\sum rank(y_i)^2 - (\sum rank(y_i))^2}}$$

```
#rho for data
data(danube, package = "lcopula")
cor(danube,method="spearman")[1,2]
```

```
## [1] 0.7374098
```

```
#rho for copula models in "Copula" package
gumbel.cop <- gumbelCopula(3)
rho(gumbel.cop)
```

```
## [1] 0.848167
```

Dependence structure via Kendall's Tau coefficient

Theoretical model based

Based on concordance measures:

$$\tau_{X,Y} = Q(F,F) = Q(C,C) = 4\int_0^1\int_0^1 C(u,v)\, dC(u,v) - 1.$$

Empirical estimator

The Kendall's τ coefficient is defined as:

$$\tau = \frac{(number\ \ of\ \ concordant\ \ pairs) - (number\ \ of\ \ discordant\ \ pairs)}{\binom{n}{2}}.$$

Any pair of observations (x_i, y_i) and (x_j, y_j), where $i < j$, are said to be concordant if the ranks for both elements (more precisely, the sort order by x and by y) agree.

Dependence structure via Kendall's Tau coefficient

```
#tau for data
data(danube, package = "lcopula")
cor(danube,method="kendall")[1,2]
```

```
## [1] 0.5484731
```

```
#tau for copula models in "Copula" package
gumbel.cop <- claytonCopula(3)
tau(gumbel.cop)
```

```
## [1] 0.6
```

```
#tau for copula models in "VineCopula" package
BiCopPar2Tau(family = 3, par = 3)
```

```
## [1] 0.6
```

```
#family =3 is the clayton family
```

Existence of tail dependence

The lower tail dependence is defined as

$$\lambda_\ell = \lim_{q \to 0} \mathrm{P}\left(X_2 \leq F_2^{\leftarrow}(q) \mid X_1 \leq F_1^{\leftarrow}(q)\right),$$

where $F^{\leftarrow}(q) = \inf\left\{x \in \mathbb{R} : F(x) \geq q\right\}$, that is, the inverse of the cumulative probability distribution function for q.

The upper tail dependence is defined analogously as

$$\lambda_u = \lim_{q \to 1} \mathrm{P}\left(X_2 > F_2^{\leftarrow}(q) \mid X_1 > F_1^{\leftarrow}(q)\right).$$

Existence of tail dependence (example 1)

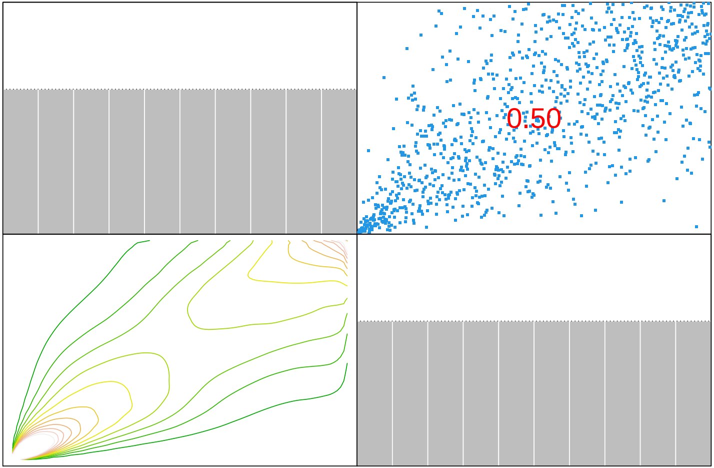
To see the existence of lower/upper tail dependence in the data we have to use the scatter plot & contour plot:

Example: Clayton copula has lower tail dependence.

```
#options(warn=-1) #warnings turned off
#library(copula)
#m1<- rCopula(n=1000,claytonCopula(
#  param=iTau(claytonCopula(0.3),0.5),dim=2))
#library(VineCopula)
#library(kdecopula)
#udata<-pobs(m1)
#pairs.copuladata(udata, method="kendall",
#                  margins = "unif", col=4, cex=4)
#options(warn=0) #warnings turned on
```

Existence of tail dependence (example 1)

Example: Clayton copula has lower tail dependence.



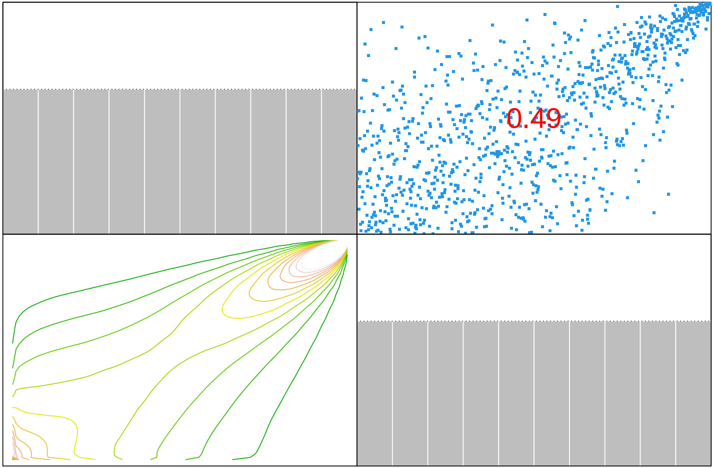
Existence of tail dependence (example 2)

Example: Joe copula has upper tail dependence.

```
#options(warn=-1) #warnings turned off
#library(copula)
#m2<- rCopula(n=1000,joeCopula(
#  param=iTau(joeCopula(2),0.5),dim=2))
#library(VineCopula)
#library(kdecopula)
#udata<-pobs(m2)
#pairs.copuladata(udata, method="kendall",
#                 margins = "unif", col=4, cex=4)
#options(warn=0) #warnings turned on
```

Existence of tail dependence (example 2)

Example: Joe copula has upper tail dependence.



Test for tail independence

This is the tail dependence test described in Reiss and Thomas (2007) section 13.3. It is, unusually, a test whose null hypothesis is that the two random variables, X and Y, are dependent. So, for example, if a significance level alpha = 0.01 test is desired, then the null huypothesis (dependence) is rejected for values of the statistic with p-values less than 0.01.

Reference

Reiss, R.-D. and Thomas, M. (2007) Statistical Analysis of Extreme Values: with applications to insurance, finance, hydrology and other fields. Birkh"auser, 530pp., 3rd edition.

Test for tail independence (example 1)

Example: Joe copula has upper tail dependence.

```
library(copula)
m2<- rCopula(n=1000,joeCopula(
  param=iTau(joeCopula(2),0.5),dim=2))
library(extRemes)
taildep.test(m2)
```

```
##
##  Reiss-Thomas (13.35)
##
## data:  m2
## statistic = Inf, threshold = -0.5, m = 221.0, n = 1000.0
## rate (%) = 22.1, p-value = 1
## alternative hypothesis: less
```

Test for tail independence (example 2)

Example: Normal copula has no dependence in the tails.

```
library(copula)
m2<- rCopula(n=1000,normalCopula(0.5,dim=2))
library(extRemes)
taildep.test(m2)
```

```
##
##  Reiss-Thomas (13.35)
##
## data:  m2
## statistic = -4.8966, threshold = -0.5, m = 197.0, n = 1(
## exceedance rate (%) = 19.7, p-value = 4.876e-07
## alternative hypothesis: less
```

Exchangeability of model

A copula C is exchangeable if

$$C(u,v) = C(u,v) \qquad \textit{for all } u,v \in (0,1).$$

The hypothesis

$$H_0 : (X,Y) \ \textit{Exchangeable} \quad \textit{vs} \quad H_1 : (X,Y) \ \textit{Non-exchangeable}$$

Exchangeability of model (example 1)

Example: Joe copula is an exchangable model.

```
options(warn=-1) #warnings turned off
library(copula)
m2<- rCopula(n=1000,joeCopula(
  param=iTau(joeCopula(2),0.5),dim=2))
exchTest(m2)
```

```
##
##  Test of exchangeability for bivariate copulas with argument 'm' set to
##  0
##
## data:  m2
## statistic = 0.024319, p-value = 0.2403
```

```
options(warn=0) #warnings turned on
```

Exchangeability of model (example 2)

Example: khoudraji copula can construct non-exchangable model.

```
options(warn=-1) #warnings turned off
library(copula)
s <- c(0.6,0.95)
copula1 <- gumbelCopula
copula2 <- claytonCopula
param <- function(tau) c(iTau(copula1(), tau),iTau(copula2(), tau))
kho <- khoudrajiCopula(copula1 = copula1(param(0.65)[1])
                       ,copula2 = copula2(param(0.65)[2]),shapes=s)
U <- rCopula(1000,copula=kho)
exchTest(U)
```

```
##
##  Test of exchangeability for bivariate copulas with argument 'm' set to
##  0
##
## data:  U
## statistic = 0.17023, p-value = 0.0004995
```

```
options(warn=0) #warnings turned on
```

State of radial symmetry

A copula C is radially symmetric if and only if $C(u, v) = \bar{C}(u, v)$ for all $u, v \in (0, 1)$, where $\bar{C}(u, v)$ is the so-called survival copula defined as

$$\bar{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

$$H_0 : C(u, v) = \bar{C}(u, v) \qquad vs \qquad H_1 : C(u, v) \neq \bar{C}(u, v)$$

There is relation betweendail dependence measures and radial symmetry.

$$\lambda_l = lim_{q \to 0} \frac{C(q, q)}{q} \qquad \text{and} \qquad \lambda_u = lim_{q \to 0} \frac{\bar{C}(q, q)}{q}.$$

The equality of C and $\bar{C}$ states that $\lambda_l = \lambda_u$ for the set of observations!.

State of radial symmetry (example 1)

Example 1: Frank copula observations are radially symmetric

```
options(warn=-1) #warnings turned off
frank.cop <- rCopula(200, frankCopula(3))
radSymTest(frank.cop)
```

```
##
##  Test of radial symmetry based on the empirical copula
##
## data:  frank.cop
## statistic = 0.035975, p-value = 0.495
```

```
options(warn=0) #warnings turned on
```

State of radial symmetry (example 2)

Example 2: Gumbel copula observations are not radially symmetric

```
options(warn=-1) #warnings turned off
gumbel.cop <- rCopula(500, gumbelCopula(2, dim=3))
radSymTest(gumbel.cop)
```

```
##
##  Test of radial symmetry based on the empirical copula
##
## data:  gumbel.cop
## statistic = 0.2812, p-value = 0.0004995
```

```
options(warn=0) #warnings turned on
```

See my homepage

To use the copula course materials, go to the web-page

https://hamb8066.github.io/homepage

and click on teaching section. Choose the "Copula Theory and Applications (Msc)".