

Detecting dependence

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Model selection tools for u-data

Now, that the u-scaled data are produced. we are looking for methods to find classes of suitable models, which are

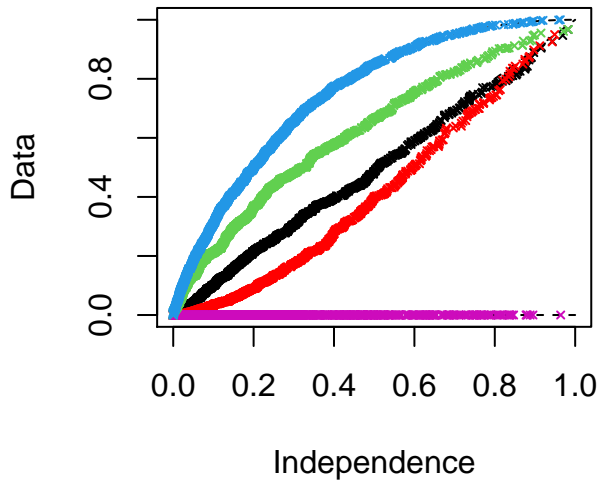
- ▶ Dependence structure
 - ▶ Kendall plot
 - ▶ Dependence measures
- ▶ Existence of tail dependence
- ▶ Exchangeability of model
- ▶ State of radial symmetry

Dependence structure via Kendall plot

In order to see dependence structure of a suitable model for the data, we use the well-known kendall plot:

```
#library(lcopula)
#m1 <- matrix(runif(2000),ncol=2)#Independence
#K.plot(m1, main="Kendall plot")
#m2 <- rCopula(n=1000,claytonCopula(param=-0.5,dim=2))
#K.plot(m2,add=TRUE,col="red") #Negative dependence
#m3 <- rCopula(n=1000,claytonCopula(param=-1,dim=2))
#K.plot(m3,add=TRUE,col="magenta") #Perfect negative dependence
#m4 <- rCopula(n=1000,claytonCopula(param=iTau(claytonCopula(0.3),0.5),dim=2))
#K.plot(m4,add=TRUE,col="green3") #Positive dependence
#m5<- rCopula(n=1000,claytonCopula(param=iTau(claytonCopula(0.3),1),dim=2))
#K.plot(m5,add=TRUE,col="blue") #Perfect positive dependence
```

Kendall plot



Dependence measures

The dependence structure of the suitable model for the data can be achieved via measures of central dependence.

Measures of central dependence

- ▶ Pearson correlation coefficient
- ▶ Spearman rank correlation coefficient
- ▶ Kendall's Tau coefficient

Dependence structure via Pearson correlation coefficient

Theoretical model based

For any random vector (X, Y) , we have:

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Empirical estimator

For any set of joint observation (x_i, y_i) we have:

$$r_{xy} = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n \sum x_i^2 - (\sum x_i)^2} \sqrt{n \sum y_i^2 - (\sum y_i)^2}}$$

Dependence structure via Pearson correlation coefficient

```
data(danube, package = "lcopula")  
cor(danube)
```

```
##           donau       inn  
## donau 1.0000000 0.7374098  
## inn   0.7374098 1.0000000
```

Dependence structure via Spearman rank correlation coefficient

The Spearman correlation between two variables is equal to the Pearson correlation between the rank values of those two variables; while Pearson's correlation assesses linear relationships, Spearman's correlation assesses monotonic relationships (whether linear or not).

Theoretical model based

$$\rho_{X,Y} = \frac{\text{Cov}(\text{rank}(X), \text{rank}(Y))}{\sigma_{\text{rank}(X)}\sigma_{\text{rank}(Y)}}.$$

Based on concordance measures:

$$\rho_{X,Y} = 3Q(F, \Pi) = 3Q(C, \Pi) = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

Dependence structure via Spearman rank correlation coefficient

The empirical estimator is

$$r_{xy} = \frac{n \sum \text{rank}(x_i) \text{rank}(y_i) - \sum \text{rank}(x_i) \sum \text{rank}(y_i)}{\sqrt{n \sum \text{rank}(x_i)^2 - (\sum \text{rank}(x_i))^2} \sqrt{n \sum \text{rank}(y_i)^2 - (\sum \text{rank}(y_i))^2}}$$

```
#rho for data  
data(danube, package = "lcopula")  
cor(danube, method="spearman") [1,2]
```

```
## [1] 0.7374098
```

```
#rho for copula models in "Copula" package  
gumbel.cop <- gumbelCopula(3)  
rho(gumbel.cop)
```

```
## [1] 0.848167
```

Dependence structure via Kendall's Tau coefficient

Theoretical model based

Based on concordance measures:

$$\tau_{X,Y} = Q(F, F) = Q(C, C) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1.$$

Empirical estimator

The Kendall's τ coefficient is defined as:

$$\tau = \frac{(\text{number of concordant pairs}) - (\text{number of discordant pairs})}{\binom{n}{2}}.$$

Any pair of observations (x_i, y_i) and (x_j, y_j) , where $i < j$, are said to be concordant if the ranks for both elements (more precisely, the sort order by x and by y) agree.

Dependence structure via Kendall's Tau coefficient

```
#tau for data  
data(danube, package = "lcopula")  
cor(danube, method="kendall")[1,2]
```

```
## [1] 0.5484731
```

```
#tau for copula models in "Copula" package  
gumbel.cop <- claytonCopula(3)  
tau(gumbel.cop)
```

```
## [1] 0.6
```

```
#tau for copula models in "VineCopula" package  
BiCopPar2Tau(family = 3, par = 3)
```

```
## [1] 0.6
```

```
#family =3 is the clayton family
```

Existence of tail dependence

The lower tail dependence is defined as

$$\lambda_\ell = \lim_{q \rightarrow 0} P \left(X_2 \leq F_2^{\leftarrow}(q) \mid X_1 \leq F_1^{\leftarrow}(q) \right),$$

where $F^{\leftarrow}(q) = \inf \{x \in \mathbb{R} : F(x) \geq q\}$, that is, the inverse of the cumulative probability distribution function for q .

The upper tail dependence is defined analogously as

$$\lambda_u = \lim_{q \rightarrow 1} P \left(X_2 > F_2^{\leftarrow}(q) \mid X_1 > F_1^{\leftarrow}(q) \right).$$

Existence of tail dependence (example 1)

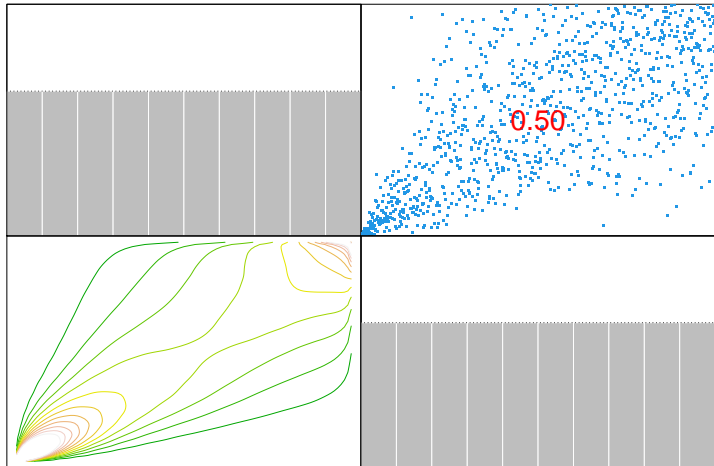
To see the existence of lower/upper tail dependence in the data we have to use the scatter plot & contour plot:

Example: Clayton copula has lower tail dependence.

```
#options(warn=-1) #warnings turned off  
#library(copula)  
#m1<- rCopula(n=1000,claytonCopula(  
#   param=iTau(claytonCopula(0.3),0.5),dim=2))  
#library(VineCopula)  
#library(kdecopula)  
#udata<-pobs(m1)  
#pairs.copuladata(udata, method="kendall",  
#                  margins = "unif", col=4, cex=4)  
#options(warn=0) #warnings turned on
```

Existence of tail dependence (example 1)

Example: Clayton copula has lower tail dependence.



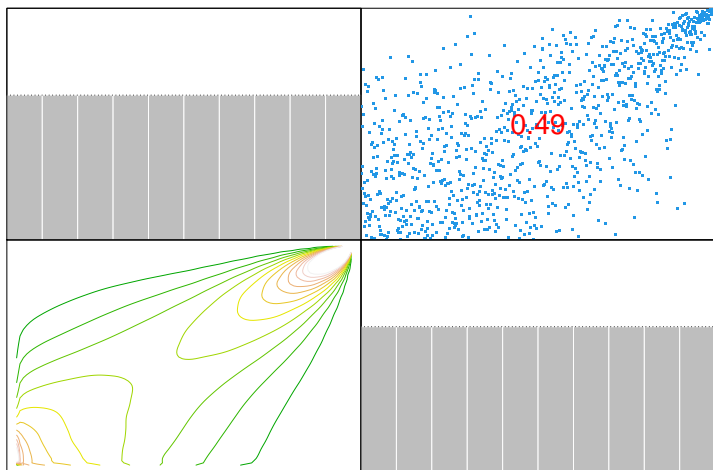
Existence of tail dependence (example 2)

Example: Joe copula has upper tail dependence.

```
#options(warn=-1) #warnings turned off
#library(copula)
#m2<- rCopula(n=1000,joeCopula(
#  param=iTau(joeCopula(2),0.5),dim=2))
#library(VineCopula)
#library(kdecopula)
#udata<-pobs(m2)
#pairs.copuladata(udata, method="kendall",
#                  margins = "unif", col=4, cex=4)
#options(warn=0) #warnings turned on
```

Existence of tail dependence (example 2)

Example: Joe copula has upper tail dependence.



Test for tail independence

This is the tail dependence test described in Reiss and Thomas (2007) section 13.3. It is, unusually, a test whose null hypothesis is that the two random variables, X and Y , are dependent. So, for example, if a significance level $\alpha = 0.01$ test is desired, then the null hypothesis (dependence) is rejected for values of the statistic with p-values less than 0.01.

Reference

Reiss, R.-D. and Thomas, M. (2007) Statistical Analysis of Extreme Values: with applications to insurance, finance, hydrology and other fields. Birkhäuser, 530pp., 3rd edition.

Test for tail independence (example 1)

Example: Joe copula has upper tail dependence.

```
library(copula)
m2<- rCopula(n=1000,joeCopula(
  param=iTau(joeCopula(2),0.5),dim=2))
library(extRemes)
taildep.test(m2)
```

```
##
```

```
## Reiss-Thomas (13.35)
```

```
##
```

```
## data: m2
```

```
## statistic = Inf, threshold = -0.5, m = 222.0, n = 1000.0
```

```
## rate (%) = 22.2, p-value = 1
```

```
## alternative hypothesis: less
```

Test for tail independence (example 2)

Example: Normal copula has no dependence in the tails.

```
library(copula)
m2<- rCopula(n=1000,normalCopula(0.5,dim=2))
library(extRemes)
taildep.test(m2)
```

```
##
```

```
## Reiss-Thomas (13.35)
```

```
##
```

```
## data: m2
```

```
## statistic = -4.275, threshold = -0.5, m = 186.0, n = 1000
```

```
## rate (%) = 18.6, p-value = 9.557e-06
```

```
## alternative hypothesis: less
```

Exchangeability of model

A copula C is exchangeable if

$$C(u, v) = C(v, u) \quad \text{for all } u, v \in (0, 1).$$

The hypothesis

$$H_0 : (X, Y) \text{ Exchangeable} \quad \text{vs} \quad H_1 : (X, Y) \text{ Non-exchangeable}$$

Exchangeability of model (example 1)

Example: Joe copula is an exchangeable model.

```
options(warn=-1) #warnings turned off  
library(copula)  
m2<- rCopula(n=1000,joeCopula(  
  param=iTau(joeCopula(2),0.5),dim=2))  
exchTest(m2)
```

```
##  
## Test of exchangeability for bivariate copulas with argument 'm' set to  
## 0  
##  
## data: m2  
## statistic = 0.024403, p-value = 0.2263
```

```
options(warn=0) #warnings turned on
```

Exchangeability of model (example 2)

Example: khoudrapi copula can construct non-exchangeable model.

```
options(warn=-1) #warnings turned off
library(copula)
s <- c(0.6,0.95)
copula1 <- gumbelCopula
copula2 <- claytonCopula
param <- function(tau) c(iTau(copula1(), tau), iTau(copula2(), tau))
kho <- khoudrapiCopula(copula1 = copula1(param(0.65)[1])
                      , copula2 = copula2(param(0.65)[2]), shapes=s)
U <- rCopula(1000, copula=kho)
exchTest(U)

##
## Test of exchangeability for bivariate copulas with argument 'm' set to
## 0
##
## data: U
## statistic = 0.14356, p-value = 0.0004995
options(warn=0) #warnings turned on
```

State of radial symmetry

A copula C is radially symmetric if and only if $C(u, v) = \bar{C}(u, v)$ for all $u, v \in (0, 1)$, where $\bar{C}(u, v)$ is the so-called survival copula defined as

$$\bar{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

$$H_0 : C(u, v) = \bar{C}(u, v) \quad \text{vs} \quad H_1 : C(u, v) \neq \bar{C}(u, v)$$

There is relation between tail dependence measures and radial symmetry.

$$\lambda_l = \lim_{q \rightarrow 0} \frac{C(q, q)}{q} \quad \text{and} \quad \lambda_u = \lim_{q \rightarrow 0} \frac{\bar{C}(q, q)}{q}.$$

The equality of C and \hat{C} states that $\lambda_l = \lambda_u$ for the set of observations!.

State of radial symmetry (example 1)

Example 1: Frank copula observations are radially symmetric

```
options(warn=-1) #warnings turned off  
frank.cop <- rCopula(200, frankCopula(3))  
radSymTest(frank.cop)
```

```
##  
## Test of radial symmetry based on the empirical copula  
##  
## data: frank.cop  
## statistic = 0.033675, p-value = 0.6019
```

```
options(warn=0) #warnings turned on
```


State of radial symmetry (example 2)

Example 2: Gumbel copula observations are not radially symmetric

```
options(warn=-1) #warnings turned off  
gumbel.cop <- rCopula(500, gumbelCopula(2, dim=3))  
radSymTest(gumbel.cop)
```

```
##  
## Test of radial symmetry based on the empirical copula  
##  
## data:  gumbel.cop  
## statistic = 0.21547, p-value = 0.0004995
```

```
options(warn=0) #warnings turned on
```

See my homepage

To use the copula course materials, go to the web-page

<https://hamb8066.github.io/homepage>

and click on teaching section. Choose the “Copula Theory and Applications (Msc)”.