USAL — جامعة العلوم والآداب اللبنانية — UNIVERSITY OF SCIENCES & ARTS IN LEBANON	**Computer Sciences Dept.** **Midterm Exam (Fall 2023/2024)** **MATH204 – Probability and Statistics I**	**Code: EXM-FR 01/1** **ED:2.1**

Stu. Name: **ID Number:**	**Inst. Name: Dr. Ayman El Zein** **Section: B**	**Booklet Needed: No** **Exam Date: 23/11/2023**
Exam Time: 1h 15 min	**Grade: …………/100**	

<u>Exercise 1:</u> (27 pts)

We roll a die three times. At each time, we note the obtained number. At the end of the experiment, we obtain a number of 3 digits.

For instance, if we obtain first a 3, then a 2 and finally a 6, then the obtained number is 326.

a. How many numbers can be obtained at the end of the experiment?
b. How many of these numbers is even?
c. How many of these numbers is greater than 400?
d. How many of these numbers is less than 200?
e. How many of these numbers is at least equal to 352?
f. How many of these numbers contains a 1 followed by a 6?

<u>Solution:</u>

a. $6 \times 6 \times 6 = 216$ **(5)**
b. $6 \times 6 \times 3 = 108$: The last number must be 2, 4 or 6. **(5)**
c. $3 \times 6 \times 6 = 108$: The first number must be 4, 5 or 6. **(5)**
d. $1 \times 6 \times 6 = 36$: The first number must be 1. **(5)**
e. $(1 \times 1 \times 5) + (1 \times 1 \times 6) + (3 \times 6 \times 6) = 119$: The number has three possibilities: $35x$, where x is one of the numbers $2, 3, 4, 5$ or 6 - $36x$, where x is one of the numbers $1, 2, 3, 4, 5$ or 6 - xyz, where x is one of the numbers $4, 5$ or 6 and y and z may be any number. **(2)**
f. $(1 \times 1 \times 6) + (6 \times 1 \times 1) = 12$: The number has two possibilities: $16x$ or $x16$. **(5)**

Exercise 2: (23 pts)

An urn contains:

Four red balls numbered $0, 0, 1, 2$

Three blue balls numbered $0, 1, 1$

And, two white balls numbered $1, 2$

We draw randomly and simultaneously two balls from the urn. Consider the following events:

A: "The two balls have the same color"

B: "The two balls have the same number"

C: "The sum of the two numbers of the balls is 2"

D: "The product of the two numbers of the balls is 0"

Calculate $P(A)$, $P(B)$, $P(C)$, $P(D)$ and $P(A \cup B)$.

Solution:

$|\Omega| = \binom{9}{2}$,

$|A| = \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$,

$|B| = \binom{3}{2} + \binom{4}{2} + \binom{2}{2}$,

$|C| = \binom{4}{2} + \binom{3}{1}\binom{2}{1}$,

$|D| = \binom{3}{1}\binom{6}{1} + \binom{3}{2}$,

$|A \cap B| = \binom{2}{2} + \binom{2}{2}$

Then, $P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{4}{2}+\binom{3}{2}+\binom{2}{2}}{\binom{9}{2}}$ **(5)**

$P(B) = \frac{\binom{3}{2}+\binom{4}{2}+\binom{2}{2}}{\binom{9}{2}}$ **(5)**

$P(C) = \frac{\binom{4}{2}+\binom{3}{1}\binom{2}{1}}{\binom{9}{2}}$ **(5)**

$P(D) = \frac{\binom{3}{1}\binom{6}{1}+\binom{3}{2}}{\binom{9}{2}}$ **(5)**

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{\binom{4}{2}+\binom{3}{2}+\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{2}{2}-\binom{2}{2}-\binom{2}{2}}{\binom{9}{2}}$ **(3)**

Exercise 3: **(12 pts)**

A set of 3 red balls, 3 green balls, and 3 yellow balls has to be distributed into three boxes labeled A, B and C.

a. What is the total number of ways to distribute these balls.
b. Calculate the probability that the box A contains the 3 red balls.
c. Calculate the probability that the box A contains the 3 red balls, the box B contains the 3 green balls, and the box C contains the 3 yellow balls.
d. Calculate the probability that each box contains 3 balls of same color.

Solution:

a. $|\Omega| = \binom{9}{3,3,3} = 1680$ **(3)**
b. $\frac{\binom{3}{3}\binom{6}{3,3}}{\binom{9}{3,3,3}} = \frac{1}{84}$ **(3)**
c. $\frac{\binom{3}{3}\binom{3}{3}\binom{3}{3}}{\binom{9}{3,3,3}} = \frac{1}{1680}$ **(3)**
d. $\frac{\binom{3}{3}\binom{3}{3}\binom{3}{3}\times 3!}{\binom{9}{3,3,3}} = \frac{3}{560}$ **(3)**

Exercise 4: **(14 pts)**

Three events A, B and C are such that $P(A) = 0.4$, $P(B) = 0.7$, $P(C) = 0.2$, $P\left(\overline{A} \cap \overline{B}\right) = 0.18$ and $P(A \cup C) = 0.5$.

a. Show that A and B are independent events.
b. Calculate $P(A \cap \overline{B})$ and $P(A|C)$.

Solution:

a. $P(A \cap B) = P(A) + P(B) - P(A \cup B) = P(A) + P(B) - \left(1 - P\left(\overline{A \cup B}\right)\right) = P(A) + P(B) - 1 + P\left(\overline{A} \cap \overline{B}\right) = 0.4 + 0.7 - 1 + 0.18 = 0.28 = P(A) \times P(B)$, so A and B are independent. **(6)**
b. $P(A) = P(A \cap B) + P(A \cap \overline{B})$, then $P\left(A \cap \overline{B}\right) = P(A) - P(A \cap B) = 0.4 - 0.28 = 0.12$ **(4)**

$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)+P(C)-P(A \cup C)}{P(C)} = \frac{0.4+0.2-0.5}{0.2} = 0.5$ **(4)**

Exercise 5: (24 pts)

A. The faces of a coin are numbered 1 and 2. We flip the coin two successive times. Consider the following events:
S_2: "The sum of the two appeared numbers is 2"
S_3: "The sum of the two appeared numbers is 3"
S_4: "The sum of the two appeared numbers is 4"
Calculate $P(S_2)$, $P(S_3)$ and $P(S_4)$.

B. A box contains 3 red cards and 5 black cards. A game consists of flipping the coin whose faces are numbered 1 and 2 two successive times, then to draw, simultaneously, a number of cards from the box equal to the sum of the two numbers. We consider the following event:
R: "Exactly one red card is drawn from the box"

a. Calculate $P(R|S_2)$ and deduce $P(R \cap S_2)$.
b. Calculate $P(R)$.
c. Knowing that exactly one red ball is drawn, what is the probability that the two faces of the flipped coin were numbered 2?

Solution:

A. $P(S_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}(1+1)$ **(4)**
$P(S_3) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}(1+2 \text{ or } 2+1)$ **(4)**
$P(S_4) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}(2+2)$ **(4)**

B.

a. $P(R|S_2) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$ **(3)**
$P(R \cap S_2) = P(R|S_2)P(S_2) = \frac{15}{28} \times \frac{1}{4} = \frac{15}{112}$ **(3)**

b. $P(R) = P(R \cap S_2) + P(R \cap S_3) + P(R \cap S_4) = \frac{15}{112} + \frac{\binom{3}{1}\binom{5}{2}}{\binom{8}{3}} \times \frac{1}{2} + \frac{\binom{3}{1}\binom{5}{3}}{\binom{8}{4}} \times \frac{1}{4} = \frac{15}{112} + \frac{15}{56} + \frac{3}{28} = \frac{54}{112} = \frac{27}{56}$ **(3)**

c. $P(S_4|R) = \frac{P(S_4 \cap R)}{P(R)} = \frac{\frac{3}{28}}{\frac{27}{56}} = \frac{2}{9}$ **(3)**

Bonus exercise: **(≤ 5)**

A game consists of flipping a coin, successively, many times. The game stops when obtaining a tail for the first time. What is the minimum number of flips for which the probability to attain is less than 1%?

Solution:

Let n be the number of flips when stopping, and let S_n be the event of stopping after n flips. Then, $P(S_n) = \frac{1}{2} \times \frac{1}{2} \times \ldots \times \frac{1}{2} = \frac{1}{2^n}$

We have:

$$P(S_n) < \frac{1}{100}$$

$$\frac{1}{2^n} < \frac{1}{100}$$

$$2^n > 100$$

$$n \ln 2 > \ln 100$$

$$n > \frac{\ln 100}{\ln 2} = 6.6$$

Thus, the minimum number of flips is 7.